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On the resolution of reductive monoids and multiplicativity of γ-factors

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abstract

Conjectures of Braverman and Kazhdan, refined by Ngo, give a framework for studying general Langlands L-functions. Under suitable assumptions, we use this framework to prove that gamma-factors of general Langlands L-functions are invariant under parabolic induction. We also isolate a subspace of the Schwartz space of a reductive monoid, and we prove that this subspace contains the basic function. We also discuss the resolution of singularities and their rationality for reductive monoids, which are among the basic objects in the program.

0. Introduction

Every theory of L-functions must satisfy the axiom of multiplicativity/inductivity, which simply requires that γ-factors for induced representations are equal to those of the inducing representations. This axiom is a theorem for Artin L-functions and the L-functions obtained from the Langlands-Shahidi method [Sha10], and is a main tool

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in computing γ-factors, root numbers, and L-functions. On the other hand, its proof in the cases obtained from Rankin-Selberg methods is quite involved and complicated. It is also central in proving equality of these factors when they are defined by different methods and in establishing the local Langlands correspondence (LLC) [Sha12,Sha17, HT01,Hen00,GT11,CST17]. Its importance as a technical tool in proving certain cases of functoriality [CKPSS04,Kim03,KS02] is now well established.

In this paper we will provide a proof of multiplicativity for γ-factors defined by the method of Braverman-Kazhdan/Ngo [BK02,BK10,BNS16,Ngô20] and L. Lafforgue [Laf14] in general under the assumption that the ρ-Fourier transforms on the group G and the inducing Levi subgroup L commute with the ρ-Harish-Chandra transform, a generalized Satake transform sending $C_c^\infty(G(k)) \to C_c^\infty(L(k))$, defined in Section 5.2, where ρ is a finite dimensional representation of the L-group of G by means of which the γ-factors are defined.

Within our proof, we define a space $\mathsf{S}^\rho(G)$ of ρ-Schwartz functions for every ρ as

$$\mathsf{S}^\rho(G) := C_c^\infty(G(k)) + \mathcal{J}^\rho(C_c^\infty(G(k))) \subset C^\infty(G(k)). \tag{0.1}$$

This definition is crucial since the ρ-Schwartz functions defined in this way will be uniformly bi-K-finite (see equation (5.16) and Lemma 5.5), making the descent to the inducing level possible, an important step in the proof of multiplicativity. While the γ-factor can be defined as the kernel of the Fourier transform, it is the full functional equation that allows our descent to the inducing level in a transparent fashion, using our subspace of ρ-Schwartz functions.

In [BK10], Braverman and Kazhdan defined their Schwartz space as a "saturation" of ours. But our Schwartz space, which is denoted by V_ρ in [BK10], covers a significant part of theirs and in particular, contains the ρ-basic function as we prove in Proposition 5.3. This is done using the extended Satake transform to almost compact functions [Li17] and the fact that it commutes with the Fourier transform induced from tori which is now defined in general (Section 6 and in particular diagram (6.8)).

The commutativity assumption allows us to extend the ρ-Harish-Chandra transform to $\mathsf{S}^\rho(G)$, commuting with J^ρ and J^{ρ_L}, respectively, where ρ_L is the restriction of ρ to the L-group of L. This construction of $\mathsf{S}^\rho(G)$ agrees with that of Braverman-Kazhdan in the case of doubling method [BK02,GPSR87,Li18,LR05,PSR86,Sha18,JLZ20,GL20], since G being the interior of the defining monoid embeds as a unique open orbit into the Braverman-Kazhdan space (cf. [Li18]). Our proof is a generalization of Godement-Jacquet for GL_n, Theorem 3.4 of [GJ72].

Our commutativity axiom, which implies multiplicativity and multiplicativity itself give rise to an inductive scheme that allows for a definition of Fourier transform J^ρ by building from the case of conjugacy classes of Levi subgroups L of G. In fact, Theorem 5.4 gives the γ factors $\gamma(s, \pi, \rho, \psi)$, π an irreducible constituent of $\mathrm{Ind}(\sigma)$, equal to the inducing γ-factor $\gamma(s, \sigma, \rho_L, \psi)$, which in turn is defined through convolution by J^{ρ_L}. For example, for GL_2, the Levi subgroups consist of split tori for which a canonical

Fourier transform exists (cf. [Ngô20]; see (6.2) here) and GL_2 itself, which is equivalent to understanding supercuspidal γ-factors. We refer to section 5.4 for a more detailed discussion of this inductive construction.

In the case of GL_2, Laurent Lafforgue [Laf14] has defined a candidate distribution which is shown formally to commute with the Harish-Chandra transform and evidence exists that it may give the correct supercuspidal factors as observed by Jacquet, but it is still unknown if this is the right distribution. Work in this direction for tamely ramified representations is being pursued by the second author.

Although our definition of the space $\mathsf{S}^\rho(G)$ depends on the knowledge of how J^ρ acts on $C_c^\infty(G(k))$, this seems to be the most efficient way of defining $\mathsf{S}^\rho(G)$ at present and sufficient for our purposes as a working definition, allowing us to begin making some initial steps toward understanding the general theory. As observed earlier after equation (0.1), this is essential in proving the uniform K-finiteness of ρ-Schwartz functions.

One hopes that the geometry of M^ρ will provide some insight into what this Fourier transform ought to be. In fact, the geometric techniques used to study the basic functions on reductive monoids via arc spaces in the function field setting [BNS16] tell us that the nature of the singularities of the monoid very much controls the asymptotics of the basic function. Taking cue from this, it is natural to consider the geometry of the singularities in the p-adic case as well. As a first step, we may classify the singularities of our monoids via the theory of spherical varieties and we find that there is a good and explicit choice of G-equivariant resolution of singularities [Bri89,Per14]. The resolution is moreover rational and so we may pass without trouble between differential forms on the monoid and its resolution. The geometric aspects of this theory are discussed in part in Section 3 of the present paper. Since our Schwartz spaces are, at least tentatively, linked by the definition of the Fourier transform J^ρ via $\mathsf{S}^\rho(G) = C_c^\infty(G(k)) + J^\rho(C_c^\infty(G(k))$, we are able, at least speculatively, to unite the themes of this paper. Here is the outline of the paper.

Section 1 is a quick review of the method for $GL(n)$ as developed in [GJ72]. Renner's construction of reductive monoids is briefly discussed in Section 2 which concludes with a treatment of the cases of symmetric powers for $GL(2)$, describing all the objects involved in those cases. Section 3 covers the geometric aspects studied in the paper. This includes the resolution of the singularities of reductive monoids, leading to a proof of rationality of these singularities. This allows a transfer of measures from the monoid to its resolution as discussed in Section 4 and can be applied to the integration of basic functions on corresponding toric varieties in Example 4.1. Multiplicativity is stated and proved in Section 5, concluding with the example of $GL(n)$ in 5.3 and a discussion of the inductive nature of Fourier transforms in 5.4. In proving multiplicativity, we have found it easier to work with the full functional equation rather than the definition given by convolutions. The cases of a tori and unramified data are addressed in Section 6. The paper is concluded with a brief discussion of the doubling construction of Piatetski-Shapiro and Rallis with relevant references cited.

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1. The case of standard representation for GL_n

We recall that the Godement–Jacquet [GJ72] theory for standard L-functions of GL_n, which this method aims to generalize, can be presented briefly through the definition of the corresponding γ-factors.

Let F be a p-adic field and $G = GL_n$. Let π be an irreducible admissible representation of $GL_n(F)$. Given a Schwartz function ϕ on $M_n(F)$, i.e., $\phi \in C_c^\infty(M_n(F))$, a smooth function of compact support on $M_n(F)$, one can define a zeta-function

$$Z(\phi, f, s) = \int_{GL_n(F)} \phi(x) f(x) |\det x|^s dx,$$

where $f(x) = \langle \pi(x)v, \tilde{v}\rangle$, $v \in \mathsf{H}(\pi)$ and $\tilde{v} \in \mathsf{H}(\tilde{\pi})$, is a matrix coefficient and $s \in \mathbb{C}$. Here $\tilde{\pi}$ is the contragredient of π, and $\mathsf{H}(\pi)$ and $\mathsf{H}(\tilde{\pi})$ denote the spaces of π and $\tilde{\pi}$, respectively. Let

$$\hat{\phi}(x) := \int_{M_n(F)} \phi(y)\psi(tr(xy))dy$$

be the Fourier transform of ϕ with respect to the (additive) character $\psi = 1$ of F.

If $\check{f}(g) = f(g^{-1})$, $g \in GL_n(F)$, then we can consider $Z(\hat{\phi}, \check{f}, s)$. The Godement–Jacquet theory defines a γ-factor $\gamma^{\mathrm{std}}(\pi, s)$ which depends only on π and s and is a rational function of q^{-s}, satisfying

$$Z(\hat{\phi}, \check{f}, (1-s) + \frac{n-1}{2}) = \gamma^{\mathrm{std}}(\pi, s) Z(\phi, f, s + \frac{n-1}{2}) \tag{1.1}$$

for all ϕ and f.

It is not hard to see that if we introduce the Int(G)-invariant kernel

$$\Phi_\psi(g) = \psi(tr(g))|\det g|^n dg$$

of the Fourier transform, then

$$\Phi_\psi * (f|\det|^{s+\frac{n-1}{2}}) = \gamma^{\mathrm{std}}(\pi, s) f|\det|^{s+\frac{n-1}{2}} \tag{1.2}$$

by virtue of irreducibility of π and the Schur's lemma.

This formulation for the γ-factor is a quick and convenient way of introducing them which is amenable to generalization. We can therefore write

$$\gamma^{\mathrm{std}}(\pi, s) = \Phi_\psi(\pi) = \int_{GL_n(F)} \Phi_\psi(g)\pi(g)dg,$$

pointing to the significance of the kernel Φ_ψ in defining the γ-factors.

2. The general case; monoids and Renner's construction

To treat the general case we need to generalize $M_n(F)$. Let k be an algebraically closed field of characteristic zero. A monoid M is an affine algebraic variety over k with an associative multiplication and an identity 1. For our purposes, we also need M to be normal, i.e., $k[M]$ is integrally closed in $k(M)$. We can always find a normalization in case M is not normal, i.e., an epimorphism (in the category of monoids) $M' \to M$ such that integral closure of $k[M]$ in $k(M)$ equals $k[M']$ as we realize $k[M] \hookrightarrow k[M']$.

We thus let M be a normal monoid and let $G = G(M) = M^*$, be the units of M. We say M is *reductive* if G is. We now like to attach a monoid to a finite dimensional representation ρ of $\widehat{G} = {}^LG$, L-group of G, $\rho : \widehat{G} \to GL(V_\rho)$, where G is a split reductive group. Let $T \subset G$ be a maximal torus and write

$$\rho \mid \hat{T} = \sum_{\lambda \in W(\rho)} \lambda,$$

where $W(\rho)$ is the set of weights of ρ. Let $\Lambda = \mathrm{Hom}(\mathbb{G}_m, T)$ be the set of cocharacters of T or characters of $\widehat{T}$ and set $\Lambda_{\mathbb{R}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$. Next, denote by $\Omega(\rho)$ the convex span in $\Lambda_{\mathbb{R}}$ of weights of ρ and let $\xi(\rho)$ be the cone in $\Lambda_{\mathbb{R}}$ generated by rays through $\Omega(\rho)$.

Let $\sigma^\vee = \xi(\rho)^\vee \cap X^*(T)$, be the "rational" dual cone to $\xi(\rho) \cap X_*(T)$ and $k[\sigma^\vee]$ the group algebra of $\sigma^\vee$. One can then identify $\sigma^\vee$ as a subset of $k[\sigma^\vee]$ by $\mu \in \sigma^\vee$ going to χ_μ, defined by

$$\chi_\mu(\eta) = 0 \quad \text{unless } \eta = \mu, \ \eta \in \sigma^\vee,$$

$\chi_\mu(\mu) = 1$, and $\chi_{\mu_1} \cdot \chi_{\mu_2} = \chi_{\mu_1+\mu_2}$, where the sum is the one on the semigroup $\sigma^\vee$. We note that this is valid for any semigroup S and

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$$k[S] = (\chi_s | s \in S).$$

Now, assume G has a character

$$\nu : G \longrightarrow \mathbb{G}_m$$

such that

$$\mathbb{C}^* \xrightarrow{\nu^\vee} \hat{G} \xrightarrow{\rho} GL(V_\rho)$$

sends $z \in \mathbb{C}^*$ to $z \cdot \mathrm{Id}$. This means that $\langle \nu, \omega \rangle = 1$ for any weight ω of ρ. In fact, for $z \in \mathbb{C}^*$,

$$z^{\langle \nu, \omega \rangle} = \omega(\nu^\vee(z)) = \rho(\nu^\vee(z)) = z$$

and thus $\langle \nu, \omega \rangle = 1$. Then $\nu \in \sigma^\vee$ and its existence implies that $\xi(\rho)$ is strictly convex, i.e., has no lines in it. In fact, the cone $\xi(\rho)$ is contained in the open half–space of vectors $x \in \Lambda_{\mathbb{R}}$, $\Lambda_{\mathbb{R}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$, $\Lambda = \mathrm{Hom}(\mathbb{G}_m, T)$, satisfying $\langle \nu, x \rangle > 0$. It is therefore strictly convex (cf. [Ngô20], Proposition 5.1).

By the theory of toric varieties [CLS11], the strictly convex cone $\xi(\rho)$ determines (uniquely) a *normal* toric variety, i.e., a normal affine torus embedding $j : T \subset M_T$. Here M_T is the monoid for T attached to $\rho|\hat{T}$. More precisely, $M_T = \mathrm{Spec}(k[\sigma^\vee])$ by Theorem 1.3.5 and Proposition 1.3.8, pg. 39 of [CLS11]. By definition 3.19 of [Ren05], $k[\sigma^\vee]$ is generated by $X(M_T)$, the characters of M_T and thus $X(M_T) = \sigma^\vee$, the semigroup defining M_T. The embedding $j : T \subset M_T$, defines $j^* : X(M_T) \to X(T)$, a semigroup morphism, into the character group of T.

The dominant characters in $X(T)$ all lie in $X(M_T)$ and are those that extend to semigroup morphisms $M_T \to \mathbb{A}^1 = \mathbb{G}_a$ (Proposition 3.20 of [Ren05]).

Finally we observe that ν is integral and dominant and thus $\nu \in X(M_T)$.

To proceed, we remark that the Weyl group $W = W(G, T)$ acts on T, M_T, $X(T)$ and $X(M_T)$ in the usual manner. Moreover, the dual rational cone $\sigma^\vee$ may be identified with $X(M_T)$, both semigroups, since its group algebra generated by elements of $X(M_T)$ or $\sigma^\vee$, is $k[M_T]$ as we discussed earlier.

Let $\lambda \in X(T)$ be a dominant (and integral) character. Then $\lambda|T_{\mathrm{der}}$ defines an irreducible finite dimensional (rational) representation μ°_λ of G_{der}, $T_{\mathrm{der}} = T \cap G_{\mathrm{der}}$, of highest weight $\lambda|T_{\mathrm{der}}$. Since

$$\mu^\circ_\lambda | Z(G) \cap G_{\mathrm{der}} = \lambda | Z(G) \cap G_{\mathrm{der}},$$

we can extend μ°_λ to an irreducible rational representation $\mu_\lambda = \mu^\circ_\lambda \otimes (\lambda|Z(G))$ of

$$G = (G_{\mathrm{der}} \times Z(G))/G_{\mathrm{der}} \cap Z(G).$$

Definition 2.1. μ_λ is called the irreducible representation of G of highest weight λ, where λ is a dominant rational character of T, extending the notion from the standard semisimple setting to the reductive one.

This in particular is valid for dominant elements in $X(M_T)$. We note that $\nu \in X(M_T)$ is one such.

Now choose $\{\lambda_i\}_{i=1}^s$ so that $\bigsqcup_{i=1}^s W \cdot \lambda_i \subset X(M_T)$ generates $X(M_T)$. Let $(\mu_{\lambda_i}, V_{\lambda_i})$ be the representation attached to λ_i. Set $\mu = \bigoplus_{i=1}^s \mu_{\lambda_i}$ and $V = \bigoplus_{i=1}^s V_{\lambda_i}$. The character ν will be among these λ_i. We may assume $\lambda_1 = \nu$. Define $M_1 = \mu(G) \subset \mathrm{End}(V)$. We let M be a normalization of M_1.

2.1. *The case of symmetric powers of GL_2*

As an example in this section we consider the symmetric power representations of $GL_2(\mathbb{C})$ and describe these objects in this case.

Let $G = GL_2$ and $\rho = \mathrm{Sym}^n : GL_2(\mathbb{C}) \to GL_{n+1}(\mathbb{C})$, the n-th symmetric power of the standard representation of $GL_2(\mathbb{C})$. Write $\mathbb{C}^{n+1} = \langle e_1, \ldots, e_{n+1}\rangle$ with the basis $e_1, \ldots, e_{n+1}$. Let $\{\mu_i\}$ denote the weights of Sym^n. Then we can order them as

$$\mu_i(\mathrm{diag}(x, y)) = x^i y^{n-i} \quad ((x, y) \in (\mathbb{C}^*)^2),$$

$i = 0, \ldots, n$. We have

$$\xi(\mathrm{Sym}^n) \cap X_*(T) = \mathbb{Z}_{\geq 0} - \mathrm{span}\ \{(n-k, k) | k = 0, \ldots, n\}$$

inside $\mathbb{R}^2$ which equals

$$\mathbb{Z}_{\geq 0} - \mathrm{span}\ \{(m, l) |\ m, l \geq 0,\ m + l \in n\mathbb{Z}\}.$$

The dual cone to $\{(m, l) |\ m, l \geq 0, m + l \in n\mathbb{Z}\}$ is

$$\{(a, b) \in \frac{1}{n}\mathbb{Z} \times \frac{1}{n}\mathbb{Z} \mid a, b \geq 0\, a - b \in \mathbb{Z}\}.$$

Thus the dual to $\xi(\mathrm{Sym}^n) \cap X_*(T)$ is the $\mathbb{Z}_{\geq 0}$ − span of $\{(1, 0), (0, 1), (\frac{1}{n}, \frac{1}{n})\}$. It is a lattice in the shaded area, corresponding to $\sigma^\vee$,

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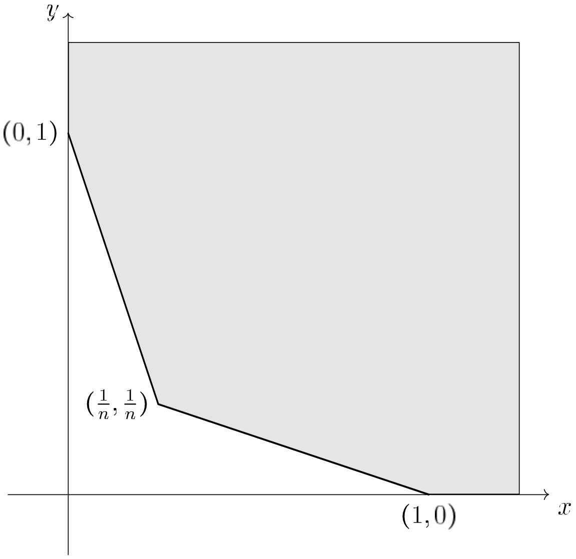

We use x, y, and z to denote $(1,0), (0,1)$ and $(\frac{1}{n}, \frac{1}{n})$ in the semigroup algebra $k[\sigma^\vee]$ as before, i.e., $x = \chi_{(1,0)}$ and so on, then

$$k(x, y, z) = k[X, Y, Z]/(XY - Z^n).$$

The corresponding toric variety is

$$M_T = \operatorname{Spec} k[X, Y, Z]/(XY - Z^n) \subseteq k^3,$$

the variety defined by the zeros of $XY - Z^n = 0$, and

$$\begin{aligned} T &= M_T \cap (k^*)^3 \\ &= \{(t_1, t_2^n t_1^{-1}, t_2) \,|\, t_i \in k^*, \ i = 1, 2\} \end{aligned}$$

<u>The monoid M for Sym^n</u> (Renner's construction): The dual cone in

$$X^*(T) \otimes_{\mathbb{Z}} \mathbb{Q} = X_*(\hat{T}) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is generated by $(1,0), (\frac{1}{n}, \frac{1}{n})$ and $(0,1)$. The vectors $(1,0)$ and $(0,1)$ are W-conjugate and therefore we have as our dominant weights $\lambda_i \in \{(1, 0), (\frac{1}{n}, \frac{1}{n})\}$. They correspond, respectively, to std, the standard representation, and $\nu = \det^{1/n}$ (to be explained) and thus

$$\begin{aligned} \mu : G &\longrightarrow \operatorname{End}(V_{\mathrm{std}} \oplus V_\nu) = M_2 \times \mathbb{A}^1 \\ g &\longrightarrow (g, (\det g)^{1/n}). \end{aligned}$$

Then

$$\begin{aligned} M &= \overline{\mu(G)} \\ &= \overline{\{(g, a) \mid \deg g = a^n\}} \\ &\cong \operatorname{Spec} k[X_1, \ldots, X_5] \,/\, (X_1X_4 - X_2X_3 = X_5^n). \end{aligned}$$

<u>The character ν for Sym^n:</u>

Recall that the fibered product of GL_2 and $\mathbb{G}_m$ giving the units of the monoid for Sym^n is (cf. [Sha17])

$$G = GL_2 \times_{\mathbb{G}_m} \mathbb{G}_m = \{(g, a) \mid \det g = a^n\} = \begin{cases} GL_2 & n = \text{odd} \\ SL \times GL & n = \text{even} \end{cases}.$$

We then have the commuting diagram

$$\begin{array}{ccc} G = GL_2 \times_{\mathbb{G}_m} \mathbb{G}_m & \xrightarrow{\mathrm{Proj}_1} & GL_2 \\ \mathrm{Proj}_2 \big\downarrow & & \big\downarrow \det \\ \mathbb{G}_m & \longrightarrow & \mathbb{G}_m \\ & x \longrightarrow x^n. & \end{array}$$

Thus

$$\begin{array}{ccc} (g, a) & \longrightarrow & g \\ \downarrow & & \downarrow \\ a & \longrightarrow & \det g = a^n \end{array}$$

and the left vertical arrow, the Proj_2, gives

$$\nu : (g, a) \longrightarrow (\det g)^{\frac{1}{n}} = a$$

for which

$$z \xrightarrow{\nu^\vee} \operatorname{diag}(z^{1/n}, \ldots, z^{1/n}) \xrightarrow{\mathrm{Sym}^n} z \cdot I_{n+1}.$$

3. Some geometry of reductive monoids as spherical varieties

In this section, we give an exposition of some of the geometry of reductive monoids in terms of their idempotents and their relation to their structure as spherical varieties, and outline the construction of resolutions of their singularities. Renner's classification of reductive monoids uses the "extension principle" [Ren05]. The extension principle follows in the spirit of many similar classification results for spherical varieties that rely on the existence of an open $B \times B^{op}$-orbit where B is a Borel subgroup of G, that is in Renner's case adapted to account for the monoid structure. By Renner's classification, the category of Reductive monoids is equivalent to the category of tuples $(G, T, \overline{T})$, where T is any maximal torus in G and $\overline{T}$ is a Weyl-group stable toric variety. A morphism of data $(G, T, \overline{T}) \to (G', T', \overline{T'})$ is given by a pair (φ, τ) where $\varphi : G \to G'$ is a morphism of reductive groups and $\tau : \overline{T} \to \overline{T'}$ a morphism of toric varieties such that the restriction of each morphism to the maximal torus agrees $\varphi|_T = \tau|_T$. In the following, we reframe these results in terms of the theory of spherical varieties, in order to state the existence of a G-equivariant resolution of singularities. A nice general treatment over any characteristic are in [BK07], [Rit03]. See also [Rit98].

Let G be a split reductive group defined over a characteristic zero field k. Let X be a variety defined over k with a regular action $\alpha : G \times X \to X$. In this case we say X is a G-variety. Let $\mathcal{O}_X$ be the sheaf of regular functions on X. If X is affine, we will identify $\mathcal{O}_X$ with the coordinate algebra $k[X]$. In this case α induces as usual a co-action map $\alpha^* : k[X] \to k[G] \otimes k[X]$ by $(\alpha^* f)(x, g) = f(g^{-1}x) = \sum_i h_i(g) f_i(x)$ with $h_i \in k[G]$, where the latter is a finite sum. Thus each f determines a finite dimensional G-module. Because G is reductive and we are in characteristic 0, each finite dimensional G-module decomposes as a finite sum of irreducible representations indexed by their highest weight vector with weight λ. As such we may decompose $k[X] = \bigoplus k[X]_\lambda$, indexed by the λ that appear in $k[X]$.

Definition 3.1. A G-variety X is *spherical* if X has an open B-orbit for some (hence any) Borel B in G.

Suppose X is spherical. Then as above, by highest weight theory, each dominant integral character λ of $T(k)$ that appears in $k[X]$ has a highest weight vector f_λ. The line $k \cdot f_\lambda$ is the unique line stabilized by B on which B acts through the character $\lambda : f_\lambda(bx) = \lambda(b) f_\lambda$. In other words f_λ is a semi-invariant. Suppose f_1 and f_2 are semi-invariants that are λ-eigenfunctions appearing in $k[X]$. Then the rational function f_1/f_2 is B-invariant. As the B-orbit in G is dense, this implies f_1/f_2 is constant. Hence for spherical varieties, each λ that appears can only appear with multiplicity one.

Theorem 3.2. *Let X be an affine G-variety for a reductive group G. Then the ring of G-invariants $k[X]^G$ is finitely generated, say $k[X]^G = k[f_1, \ldots, f_n]$. Then $k[X]^G \hookrightarrow k[X]$ defines a surjective quotient which is moreover a categorical quotient $q : X \to X//G$. Each fiber of q contains a unique closed G-orbit in X, and $X//G$ is normal if X is.*

Definition 3.3. A spherical variety X is *simple* if it has a unique closed G-orbit.

We are interested in reductive monoids, which have open G-orbit and are spherical with respect to $G \times G$ with an open dense Borel $= B \times B^{op}$ orbit, where B^{op} is the Borel subgroup opposite to B.

Proposition 3.4. *Suppose X is affine and has an open G-orbit. Then X has a unique closed orbit.*

Proof. The reductive quotient $q : X \to X//G$ is constant on orbits, in particular on the open orbit. Hence $X//G = \{pt\}$. The fiber $q^{-1}(pt)$ contains a unique closed orbit by Theorem 3.2. □

Therefore such X are simple. Once again let us consider an affine simple X as a $G \times G$ variety. Decomposing $_\lambda k[X]_\lambda \cong V_\lambda$ where V_λ is the highest weight module for λ. The (B, λ) eigenfunction f_λ is U-invariant. Thus one may consider taking $U \times U^{op}$-invariants $k[X]^{U\times U^{op}}$ are therefore generated as a vector space by the f_λ. Using the following

Theorem 3.5. *Let G be reductive with maximal unipotent subgroup U, and let X be a G-variety. Then $k[X]^U$ is finitely generated. Moreover $X/U = \mathrm{spec}\ k[X]^U$ is normal if X is.*

Proof. One first establishes the theorem for G/U i.e. $k[G]^U$ is finitely generated and in fact G/U is a geometric quotient (a so called horospherical variety). One has a map $\Phi : X/U \cong X \times^G G/U$ where the quotient is by the diagonal action. On coordinate rings: a U-invariant f defines a G-invariant function $(\Phi^* f)(x, gU) = f(gx)$. Thus by Theorem 3.2 $k[X \times G/U]^G \cong k[X]^U$ is finitely generated. □

We can conclude that the variety $X/(U \times U^{op})$ is a $T \cong U\backslash BB^{op}/U^{op} \hookrightarrow U^{op}\backslash G/U$ variety, on which T acts on f_λ through the character λ. In other words, we have a ring $(V_\lambda \otimes V_\lambda^*)^{(U\times U^{op})}$ graded by $\Lambda_X = \{\lambda \in X^*(T) : k[X]_\lambda \neq 0\}$. By Theorem 3.5, this is a finitely generated monoid. Each summand has a diagonalizable action by the torus T. Moreover, if X is normal the associated toric variety is normal, hence the cone of weights of X defining the toric variety is saturated.

Recall that a G-variety is simple if it contains a unique closed orbit. When X is affine, it is enough that G embeds as an open subvariety to imply X is simple. We state without proof the following:

To classify a general spherical variety one needs the following additional data.

Definition 3.6. Let $V(X)$ denote the G-stable discrete valuations on $k[G]$.

Definition 3.7. Let $\mathrm{Div}_B(X)$ denote the set of B-stable prime divisors of X.

Definition 3.8. Let Z be a G-orbit in X. Then $\mathrm{Div}_B(X : Z)$ is the set of B-stable prime divisors containing Z.

Definition 3.9. Let $\mathsf{B}(X)$ denote the set of irreducible G-stable divisors.

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Proposition 3.10. *For a divisor $D \in Div_B(X)$, either*

(1) The B-orbit B D is open and dense in the open orbit of X.
(2) D is G-stable.

Briefly (although see [Kno91] for details) a simple spherical variety X is determined by its weight monoid, plus the data of which B-stable boundary divisors containing the unique closed orbit Z are G-stable and which are not. More precisely, each divisor D defines a valuation by first restricting $D \cap G$ which defines a valuation on the multiplicative group of rational functions $k(G)^\times$ on G (the valuation $v_D \cap G$ is the order of vanishing of a rational function f/g on $D \cap G$). This defines a so-called *colored cone*, defined by the $\mathbb{Q}_{\geq 0}$ span of the finite number of valuations v_D as above. The colors comprise the valuations that are B-stable but not G-stable (or equivalently, their corresponding divisors have a B-open-orbit).

Thus the set $D(X : Z)$ gives the set of *colors* of the simple spherical variety X. The cone generated by $\mathcal{B}(X)$ and the natural image of $D(X : Z)$ in the set of $K[G]$ valuations is the *colored cone* $\mathcal{C}$ determined by the data $(V(X), \mathcal{B}(X))$ that determines up to isomorphism the spherical variety X. For reductive monoids, this cone is equivalent to the one constructed in the earlier section via highest weight theory.

Example 3.11. For a reductive monoid M, there is a beautiful description of the boundary $\partial M = M \setminus G$ in terms of $B \times B^{op}$-stable boundary divisors in the form of an extended Bruhat decomposition: Let $R = \overline{N_G(T)} \subset M$ be the Zariski closure of the normalizer of a maximal torus T in G. Let $I(M)$ be the set of idempotents in M, and note that reductive monoids are *regular* (in Renner's sense), that is, we can decompose $M = G\,I(M)$. Then we can construct the *Renner monoid* (sometimes called the Rook monoid) $\mathcal{R} := R/T$. Because reductive monoids are regular, $\mathcal{R}$ makes sense as a finite monoid whose unit group is the Weyl group W, and having the property that

$$M = \coprod_{x \in \mathcal{R}} BxB.$$

From this description, it may be deduced that the set $D(M : Z)$, with $Z = \{0\}$ the unique closed orbit in M, is given by the codimension one orbits $\overline{Bs_\alpha B^{op}}$ for $s_\alpha \in \mathcal{R}$ the simple reflection in the Weyl group determined by the simple root α. The $G \times G$ orbit structure is given by the decomposition

$$M = \coprod_{e \in J} GeG,$$

where J denotes the subset of $I(M)$ contained in $\overline{T}$ such that $Be = eBe$. More details on the semigroup structure in terms of the structure of its set of idempotents are given in sections 4 and 7 of [Ren05]. Let J^1 denote the maximal idempotents and suppose $e \in J^1$.

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We can find a one parameter subgroup $\lambda : \mathbb{G}_m \to G$, unique up to G-conjugation, such that $\lim_{t\to 0}\lambda(t) = e$. We may assume the image of λ is in the maximal torus T and as in Proposition 7 of [Rit03] λ defines a valuation v_λ of $k(G)$. Each singleton $\{v_\lambda\} \subset V(G)$ defines an elementary embedding X^λ ([Kno91]) of G such that BeB^{op} is open in $X^\lambda \supseteq G$. One checks as in [Rit03] that in fact v_λ is an invariant valuation. The set of all v_λ so constructed gives an explicit identification between J^1 and $\mathsf{B}(M)$.

Thus, for a monoid with reductive group G embedded as its unit group, the colors of M are all $B \times B^{op}$ stable irreducible divisors of G, and thus the monoid is determined purely by the data $\mathsf{B}(M)$ or equivalently $\mathsf{C}(M)$. We state for convenience this form of the classification.

Theorem 3.12. *Let G be a reductive group. The irreducible, normal algebraic monoids M with unit group G are the strictly convex polyhedral cones in $X_*(T) \otimes \mathbb{Q}$ generated by $D(M)$ and a finite set of elements in $V(G)$.*

Definition 3.13. A spherical variety X is *toroidal* if $D(X)$ is empty.

Every spherical variety has an open dense toroidal subvariety in codimension 2.

Proposition 3.14. *Suppose the spherical variety X is toroidal and let $\partial X = X \setminus G$. Let P_X be the $G \times G$ stabilizer of ∂X. Then P is a parabolic and moreover satisfies the local structure theorem, i.e. there is a Levi $L \subset P$, depending only on G and a closed L-variety Z such that*

$$P_u \times Z \to X \setminus \partial X$$

is an isomorphism. Moreover, Z is a toric variety under $L/[L, L]$.

As a consequence of the above isomorphism, the L orbits of Z correspond to G orbits in X. Note that when $X = M$ is a reductive monoid, this is precisely Renner's extension theorem [Ren05] with $P_M = B \times B^{op}$ and $Z = \overline{T}$. The proposition implies that the singularities of X are those determined by the cone of the toric variety Z.

Let us recall Renner's extension theorem for normal reductive monoids. It states that a morphism of reductive monoids $M \to M'$ is given by the data $(G, T, \overline{T})$ and $(G', T', \overline{T'})$ and a morphism $\varphi : M \to M'$ is equivalent to $\phi : G \to G'$ and $\tau|\overline{T} : \overline{T} \to \overline{T'}$. Briefly, in M one has an analogue of the open cell which is the image of an open embedding $U^{op} \times \overline{T} \times U \to U^{op}\overline{T}U$ which has codimension ≥ 2 in M. Thus one gets an equivariant morphism $(u', t, u) \to \varphi(u')\tau(t)\varphi(u) \in M'$. By normality of M, the codimension ≥ 2 condition extends the map uniquely to $M \to M'$, and one verifies this is in fact a morphism of monoids.

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Remark 3.15. The above map in Renner uses an open Bruhat cell analogue in the context of monoids, which yields a structure theorem parallel to Proposition 3.14. More generally, G-equivariant dominant morphisms $\varphi : Y \to Y^{1}$ of spherical varieties are in bijection with linear maps $\varphi_* : X_*(Y) \otimes \mathbb{Q} \to X_*(Y^{1}) \otimes \mathbb{Q}$, where $X_*(Y)$ and $X_*(Y^{1})$ are the lattices of co-weights of the underlying group G, such that the image of the colored fan C_Y is contained within $\mathsf{C}_{Y^{1}}$.

In view of the above structure theory, the singularities of a spherical G variety X are determined by those of its associated toric variety $X//U \cong \overline{T}$. Smooth toric varieties are classified as follows.

Theorem 3.16. *An affine toric variety $\overline{T}$ is smooth if and only if the extremal rays of the rational polyhedral cone generated by its weight lattice is a basis for the character lattice $X^*(T)$.*

As a general toric variety is glued from affine toric varieties, it is given by a fan consisting of rational polyhedral cones. Thus a toric variety is smooth if and only if its fan consists of rational polyhedral cones whose extremal rays generate (as a $\mathbb{Z}$-module) $X^*(T)$. Moreover any piecewise linear morphism of rational polyhedral cones $\mathsf{C} \to \mathsf{C}^{1}$ defines a T-equivariant morphism of toric varieties, thus, by Theorem 3.16, one obtains an algorithm giving a resolution of singularities of a toric variety $\overline{T}$.

Theorem 3.17. *Let $\sigma^\vee = \mathsf{C}(\overline{T}) \cap X^*(T)$ = the monoid of weights of $\overline{T}$. Starting from an extremal ray $\mathsf{C}(\overline{T})$, successively subdividing the cone such that each resulting cone is smooth, defines a smooth fan consisting of smooth cones $\widetilde{\sigma}^\vee$ such that the inclusion map $\widetilde{\sigma}^\vee \hookrightarrow \sigma^\vee$ defines a canonical T-equivariant resolution of singularities $\widetilde{T} \to \overline{T}$.*

Finally, one may define a G-equivariant resolution from Remark 3.15 and Theorems 3.16 and 3.17:

Theorem 3.18. *Let M be a reductive monoid. Then there exists a smooth spherical G-variety $\widetilde{M}$ that is toroidal, and a proper G-equivariant morphism $\varphi : \widetilde{M} \to M$.*

Proof. Take the colored cone C_M determined by M. Deleting all colors, construct a fan $\widetilde{\mathsf{C}}_M$ generated by a subdivision of C_M into smooth colorless cones (which always exists, see [CLS11]). Then $\widetilde{T} \to \overline{T} \subset M$ is a resolution of toric varieties, and $\widetilde{M}$ has an affine chart given by the open cell $U^{\mathrm{op}} \times \widetilde{T} \times U$ by Proposition 3.14, and therefore is smooth. The natural inclusion of $\widetilde{\mathsf{C}}_M$ into C_M defines a dominant G-equivariant morphism $\widetilde{M} \to M$, giving us our resolution. □

Example 3.19. The case of $G = GL_2$ and $\rho = \mathrm{Sym}^n$ monoids are determined by their respective toric varieties $\{xy - z^n = 0\} \cong \mathbb{A}^2/C_n$, which are realized finite quotients of $\mathbb{A}^2$ by cyclic group C_n of order n. These are the well known finite quotient surface

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singularities of type A_n and their resolutions are given by $[\frac{n}{2}]$ blow-ups at the origin. Likewise, the resolution of M^ρ is given by $[\frac{n}{2}]$-blow-ups at 0 of M^ρ.

Next we discuss the rationality of singularities. Recall that X has rational singularities if a (and hence any) resolution $r : \tilde{X} \to X$ has vanishing higher cohomology: $R^i \mathcal{O}_{\tilde{X}} = 0$ if $i > 0$. Moreover, recall that for a resolution $r : \tilde{X} \to X$ the fiber over the singular locus $\tilde{X}^{\mathrm{sing}} = r^{-1}(X^{\mathrm{sing}}) = E$ is the exceptional divisor. It can be computed explicitly as in [Rit03] which follows from the following result, which is also deduced in [BK07], [Rit03].

Theorem 3.20. *Let X be a normal variety with rational singularities. Let $j : X^{sm} \hookrightarrow X$ denote the embedding of the smooth locus of X into X. If we define $\omega_X := j_*(\omega_{X^{sm}})$, by extending (uniquely, by normality) algebraic top-dimensional differentials form to X. If $r : \tilde{X} \to X$ is any resolution of singularities, then $r^*\omega_X$ extends over E to an algebraic differential form on $\tilde{X}$.*

4. Integration on singular varieties

Let ω be a top-differential form on a reductive monoid M. Taking the resolution as above $r : \tilde{M} \to M$ we get a well defined differential form $r^*\omega$ on $\tilde{M}$. Restricting r to the open set on $\tilde{M}$ over which r is an isomorphism, we have upon passing to k-points

$$\int_{\tilde{X}(k)} |r^*\omega| = \int_{\tilde{X}(k)} |\mathrm{Jac}(r)||\omega| = \int_{X(k)} |\omega|,$$

where Jac(r) is the Jacobian of r and $|\omega|$ is the measure constructed by Weil from the top-differential form ω on X.

Example 4.1. As a first basic example, we may consider the basic function on toric varieties. The basic function on a toric variety $\overline{T}$ (defined by cone σ) is supported on the dual cone $\sigma^\vee$. Suppose the generators of the weight monoid of T are $e_1, \ldots, e_n$. The e_i are co-characters $k^\times \to T(k)$. For each co-character λ in the weight monoid, $\lambda(w)T(O_k)$ is an open neighborhood on which the value of the basic function $f^\rho_{\overline{T}}$ is $\#\{(a_i) \in \mathbb{Z}^n_{\geq 0} | \sum_i a_i e_i = \lambda\}$. It is known [Stu95,Cas17] that the function

$$\lambda \to \#\{(a_i) \in \mathbb{Z}^n_{\geq 0} \mid \sum_i a_i e_i = \lambda\} : \sigma \cap X_*(T) \to \mathbb{Z}_{\geq 0}$$

as an integer valued function of the weight monoid is *quasi-polynomial*, i.e., it is in the sub-algebra of functions on $X_*(T)$ generated by polynomials and periodic functions. More precisely, for each $x \in \lambda(w)T(O_k)$, $f^\rho_{\overline{T}}(x)$ is given by a root of unity times a polynomial in the valuation of det(x).

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It is also classical that locally in some coordinate system on $\tilde{T}(k)$, the map $r : \tilde{T}(k) \to T(k)$ is given by a monomial transformation, i.e., has the form $(t_1, \ldots, t_n) \to (t_1^{d_{11}} \cdots t_n^{d_{n1}}, \ldots, t_1^{d_{n1}} \cdots t_n^{d_{nn}})$. For example, the Sym^2 toric variety is the cone $\{xy - z^2 = 0\}$ and is resolved affine-locally by the classical "cylinder" resolution $(x, y, z) \to (xz, yz, z)$. Therefore the jacobian $\mathrm{Jac}(r)$ is in local coordinates a product of the form $|t_i|^{d_i}$. For the cylinder resolution it is $z \to |z|^2$. Thus as $w^N \to 0$ the Jacobian grows as the inverse of an exponential while the basic function is dominated by polynomial growth. Hence the pullback of the differential form $r^*(f^\rho\omega)$ makes sense as $\mathrm{val} \circ \det \to \infty$ on $\tilde{T}(k)$.

5. Multiplicativity

Every theory of L-functions is expected to satisfy "multiplicativity", i.e., the equality of γ-factors for parabolically inducing and induced data. Our goal in the rest of the paper is to establish, under natural assumptions on related Fourier transforms, the multiplicativity in the context of this theory in general. Our proof is a generalization of the standard case of Godement–Jacquet [GJ72]. We first need to connect Renner's construction to parabolic induction.

5.1. Renner's construction and parabolic induction

We start by observing that Renner's construction [Ren05] respects parabolic induction. More precisely, let $P = LN$ be a parabolic subgroup of G, with unipotent radical N and a Levi subgroup L which we fix by assuming $T \subset L$. Now L is a reductive group with a maximal torus T to which Renner's construction applies. Let $\rho_L = \rho|\hat{L}$, where $\hat{L}$ is the connected component of the L-group of L.

Now $\rho_L|\hat{T}$ gives the same weights as $\rho|\hat{T}$ does and thus ρ_L shares the same toric variety M_T coming from ρ. Let $W_L = W(L, T)$. Then each orbit $W\lambda_i$ breaks up to a disjoint union of orbits $W_L\lambda_j$ and thus

$$\bigoplus_{i=1}^{s} V_{\lambda_i}|_L = \bigoplus_{i=1}^{s}\bigoplus_{j=1}^{s_i} V^L_{\lambda_{j'}}$$

where

$$V_{\lambda_i}|_L = \bigoplus_{j=1}^{s_i} V^L_{\lambda_j}.$$

(5.1) *Conclusion: The monoid M^{ρ_L} attached to ρ_L by Renner's construction for L, $L = G(M^{\rho_L})$, is the same as the closure of L as a subgroup of G upon action on $V = \bigoplus_{i=1}^{s} V_{\lambda_i}$, i.e., $M^{\rho_L} = \overline{\mu(L)}$.*

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We also need to remark that the character $\nu : G \to \mathbb{G}_m$ discussed earlier, when restricted to L maybe considered as the corresponding character ν_L of L, i.e., $\nu_L := \nu|L$. In fact, $\nu^\vee$ and $\nu_L^\vee$ both take values in the centers of $\widehat{G}$ and $\widehat{L}$, respectively. The natural embedding of $\widehat{L} \subset \widehat{G}$ by means of the root data of $\widehat{L}$ and $\widehat{G}$ which are dual to root data of L and G who share the maximal torus T, identifies $Z(\widehat{G})$ as a subgroup of $Z(\widehat{L})$. We therefore have the commutative diagram

$$\begin{array}{ccccc} \mathbb{C}^* & \xrightarrow{\nu^\vee} & \widehat{G} & \xrightarrow{\rho} & GL_N(V_\rho) \\ & {\scriptstyle \nu_L^\vee} \searrow & \uparrow & \nearrow {\scriptstyle \rho_L} & \\ & & \widehat{L} & & \end{array}$$

and consequently $\rho_L \cdot \nu_L^\vee(z) = z \cdot \mathrm{Id}$ as needed.

<u>The shift in general.</u> To get the precise γ-factor in general one needs to shift s by $\langle \eta_G, \lambda \rangle$ or $\pi \otimes |\nu|^s$ should change to $\pi \otimes |\nu|^{s+\langle \eta_G, \lambda \rangle}$, with notation as in [Ngô20], where η_G is half the sum of positive roots in a Borel subgroup of G and λ the highest weight of ρ.

In our setting, we need to deal with the representation ρ_L of $\widehat{L}$ as well which is not necessarily irreducible. Let $\lambda_1, \ldots, \lambda_r$ be the highest weights of ρ_L. We may assume $\lambda_1 = \lambda$. The shift will then be $\langle \eta_L, \lambda_1 + \cdots + \lambda_r \rangle$.

Let us define $\delta_{G,\rho} = |\nu|^{\langle 2\eta_G, \lambda \rangle}$, $\delta_{L,\rho_L} = |\nu_L|^{\langle 2\eta_L, \lambda_1 + \cdots + \lambda_r \rangle}$ and set

$$\nu_{G/L} = \nu_{G/L,\rho} := \delta_{G,\rho} / \delta_{L,\rho_L}.$$

Finally, let δ_P be the modulus character of $P = NL$.

5.2. The ρ–Harish–Chandra transform

We now recall the ρ-Harish–Chandra transform, a generalization of the Satake transform. Given $\Phi \in C_c^\infty(G(k))$, define its Harish–Chandra transform $\Phi_P \in C_c^\infty(L(k))$ by

$$\Phi_P(l) = \delta_P^{-\frac{1}{2}}(l) \int_{N(k)} \Phi(nl)\,dn. \tag{5.1}$$

Next define the ρ-Harish–Chandra transform, ρ–HC in short, by

$$\Phi_P^\rho(l) = \nu_{G/L,\rho}^{\frac{1}{2}}(l)\, \Phi_P(l). \tag{5.2}$$

<u>Fourier transforms</u>

The conjectural Fourier transform (kernel) J^ρ is supposed to give the γ-factor $\gamma(s, \pi, \rho)$ for every irreducible admissible representation π of $G(k)$ through the convolution

$$J^\rho * (f|\nu|^{s+(\eta_G,\lambda)}) = \gamma(s,\pi,\rho) f|\nu|^{s+(\eta_G,\lambda)}, \tag{5.3}$$

where π is an irreducible admissible representation of $G(k)$ and $f(g) = \pi(g)v, v$ is a matrix coefficient of π. In the context of parabolic induction from $P = NL$ we will also have $\gamma(s,\sigma,\rho_L)$ defined by J^{ρ_L}.

Fourier transforms and Schwartz spaces

We will now assume we are given Fourier transforms J^ρ and J^{ρ_L}

$$J^\rho : C_c^\infty(G(k)) \longrightarrow C^\infty(G(k))$$

and

$$J^{\rho_L} : C_c^\infty(L(k)) \longrightarrow C^\infty(L(k)),$$

commuting with the ρ–Harish–Chandra transform Φ_P^ρ, i.e.,

$$(J^\rho\Phi)_P^\rho = J^{\rho_L}\Phi_P^\rho, \tag{5.4}$$

or equivalently we have the commuting diagram

$$\begin{array}{ccc} C_c^\infty(G(k)) & \xrightarrow{J^\rho} & J^\rho(C_c^\infty(G(k)) \subset C^\infty(G(k)) \\ {\scriptstyle \rho-\mathrm{HC}}\downarrow & & \downarrow{\scriptstyle \rho-\mathrm{HC}} \\ C_c^\infty(L(k)) & \xrightarrow{J^{\rho_L}} & J^{\rho_L}(C_c^\infty(L(k)) \subset C^\infty(L(k)). \end{array} \tag{5.5}$$

We will label this as "Assumption $(*)$".

We now define the Schwartz spaces $\mathcal{S}^\rho(G) = \mathcal{S}^\rho(G(k))$ and $\mathcal{S}^{\rho_L}(L) = \mathcal{S}^{\rho_L}(L(k))$ as follows:

$$\mathcal{S}^\rho(G) := C_c^\infty(G(k)) + J^\rho(C_c^\infty(G(k)) \subset C^\infty(G(k)) \tag{5.6}$$

and

$$\mathcal{S}^{\rho_L}(L) := C_c^\infty(L(k)) + J^{\rho_L}(C_c^\infty(L(k)) \subset C^\infty(L(k)). \tag{5.7}$$

As we pointed out in the introduction, these are subspaces of the conjectural ρ-Schwartz spaces and suitable to our purposes. Moreover, as we prove in Proposition 5.3, they contain the ρ and ρ_L-basic functions. We recall that the ρ-basic function ϕ^ρ is the unique one for which

$$Z(\phi^\rho, f_s) = L(s,\pi,\rho),$$

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where f_s is the normalized spherical matrix coefficient of $\pi \otimes |\nu|^s$ with Z defined as in (5.11) below.

Note that $\Phi \mapsto \Phi_P^\rho$ sends $C_c^\infty(G(k))$ into $C_c^\infty(L(k))$ and thus (5.5) implies that it also sends

$$J^\rho(C_c^\infty(G(k))) \longrightarrow J^{\rho_L}(C_c^\infty(L(k))).$$

We thus have:

Proposition 5.1. *Under Assumption (∗), the ρ–Harish–Chandra transform* Φ_P^ρ *sends* $\mathcal{S}^\rho(G)$ *into* $\mathcal{S}^{\rho_L}(L)$*. In particular, equation* (5.4) *is valid for our spaces of ρ-Schwartz functions on* $G(k)$ *and* $L(k)$*.*

We remark that our definition of Schwartz spaces $\mathcal{S}^\rho(G)$ and $\mathcal{S}^{\rho_L}(L)$, as well as Proposition 5.1, will be needed in our proof of multiplicativity in Theorem 5.4, e.g., we use uniform smoothness which follows from our definition, see the discussion after equation (5.16)

Remark 5.2. This definition of Schwartz spaces agrees with ideas of Braverman–Kazhdan [BK02,GL20] and with the case of standard representation of $GL_n(\mathbb{C})$. To wit consider $G = GL_1$, i.e., the Tate's setting, and check it for the $\Phi_0 = \mathrm{char}(O_k)$, i.e., the corresponding "basic function". Let $\Phi = \mathrm{char}(O_k^*) \in C_c^\infty(k^*)$. Now J^ρ is just the standard Fourier transform

$$J^\rho\Phi(y) = \hat{\Phi}(y) = \int_k \Phi(x)\psi(tr(xy))dx. \tag{5.8}$$

It can be easily checked that

$$\Phi_0 = \frac{1}{q-1}\mathrm{char}(P_k^{-1} \setminus O_k) + \frac{q}{q-1}\hat{\Phi} \tag{5.9}$$

$$\in C_c^\infty(k^*) + J^\rho(C_c^\infty(k^*)), \tag{5.10}$$

where P_k is the maximal ideal of O_k. Here the additive character ψ is unramified, the measure dx satisfies $dx(O_k) = 1$, and q is the cardinality of O_k/P_k.

This simple calculation allows us to prove the following general result:

Proposition 5.3. *The space* $\mathcal{S}^\rho(G)$ *contains the ρ-basic function.*

We give the proof in Section 6 after introducing diagrams (6.7) and (6.8).

Multiplicativity. As we discussed earlier every theory of L-functions must satisfy multiplicativity, an axiom that is a theorem for all the Artin L-functions and is the main tool in computing γ-factors and L-functions. To explain, let $P = NL$ be a parabolic subgroup of G with a Levi subgroup L, uniquely fixed such that $L \supset T$, the maximal torus of G fixed in our construction throughout. Let σ be an irreducible admissible representation of $L(k)$ and let ρ be a finite dimensional complex representation of $\widehat{G}$ and $\rho_L = \rho|\widehat{L}$ as before. For each irreducible admissible representation σ of $L(k)$, we can define the γ-factors $\gamma(s, \sigma, \rho_L)$ and $\gamma(s, \mathrm{Ind}_{P(k)}^{G(k)}\sigma, \rho)$. Multiplicativity states that:

$$\gamma(s, \mathrm{Ind}_{P(k)}^{G(k)}\sigma, \rho) = \gamma(s, \sigma, \rho_L). \tag{5.11}$$

Here we suppress the dependence of the factors on the non-trivial additive character of k.

We note that, since the induced representation $\mathrm{Ind}_{P(k)}^{G(k)}\sigma$ may not be irreducible, the γ-factor $\gamma(s, \mathrm{Ind}_{P(k)}^{G(k)}\sigma, \rho)$ is defined to be $\gamma(s, \pi, \rho)$, where π is any irreducible constituent of $\mathrm{Ind}_{P(k)}^{G(k)}\sigma$. The γ-factor will not depend on the choice of π as the proof below establishes.

A proof of (5.9) is usually fairly hard for γ-factors defined by Rankin–Selberg method [JPSS83,Sou93]. On the contrary, (5.9) is a general result within the Langlands–Shahidi [Sha10] method with a very natural proof.

Our aim here is to give a general proof of (5.9) within the Braverman–Kazhdan/Ngo and Lafforgue programs using (5.4) and Int(K)-invariance of J^ρ. It follows the arguments given in [GJ72]. To proceed, we emphasize that we are assuming the existence of the Fourier transforms J^ρ for functions in $C_c^\infty(G(k))$, commuting with the ρ_ HC transform, such that the functional equation

$$Z(J^\rho\Phi, \check{f}) = \gamma(\pi, \rho)Z(\Phi, f)$$

for every irreducible admissible representation π of $G(k)$ and every matrix coefficient f of π, is satisfied for every $\Phi \in \mathcal{S}^\rho(G)$. In particular, we assume $Z(\Phi, f)$ is not identically zero for all Φ and f. Here

$$Z(\Phi, f) = \int_{G(k)} \Phi(g)f(g)\delta_{G,\rho}^{1/2}(g)dg$$

and

$$Z(J^\rho\Phi, \check{f}) = \int_{G(k)} J^\rho\Phi(g)f(g^{-1})\delta_{G,\rho}^{1/2}(g)|\nu(g)|dg.$$

See also the papers [Get18] and [Luo19] that address some of the analytic difficulties in the archimedean setting. We now proceed to give a proof of (5.9) which we formally state as:

Theorem 5.4. *Let σ be an irreducible admissible representation of $L(k)$ and let $\Pi = \mathrm{Ind}_{P(k)}^{G(k)}\sigma$. Let ρ be an irreducible finite dimensional complex representation of $\widehat{G}$ and let $\rho_L = \rho|\widehat{L}$. Assume the validity of (5.4) (Assumption $(*)$) for Schwartz functions. Then*

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$$\gamma(s,\Pi,\rho)=\gamma(s,\sigma,\rho_L).$$

Proof. Let $\tilde{\Pi}=\mathrm{Ind}_{P(k)}^{G(k)}\tilde{\sigma}$ be the contragredient of Π. Choose $v\in\Pi$ and $\tilde{v}\in\tilde{\Pi}$. Then a matrix coefficient for Π can be written as

$$\begin{aligned} f(g) &= (\Pi(g)v,\tilde{v}) \\ &= \int_K (v(kg),\tilde{v}(k))_0\, dk, \end{aligned} \tag{5.12}$$

where $v(.)$ and $\tilde{v}(.)$ are values of the functions in Π and $\tilde{\Pi}$, respectively, and $(\sigma(l)w,\tilde{w})_0$ is a matrix coefficient for σ, $w\in\sigma$ and $\tilde{w}\in\tilde{\sigma}$. Let $\Phi\in\mathcal{S}_\rho(G)$ be a ρ-Schwartz function in $C^\infty(G(k))$. We absorb the complex number s in f by replacing π by $\pi\otimes|\nu|^s$ and then ignoring it throughout the proof.

Now we have the zeta function

$$Z(\Phi,f)=\int \Phi(g)f(g)\delta_{G,\rho}^{1/2}(g)dg. \tag{5.13}$$

As explained earlier, the shift $\delta_{G,\rho}^{1/2}$ allows us to get the precise L-function at s, rather than a shift of s, when Φ is the basic function of σ for a spherical representation σ. Using (5.12), (5.13) equals

$$\begin{aligned} Z(\Phi,f) &= \int_{K\times G(k)} \Phi(g)(v(kg),\tilde{v}(k))_0\,\delta_{G,\rho}^{1/2}(g)dk\,dg \\ &= \int_{K\times G(k)} \Phi(k^{-1}g)(v(g),\tilde{v}(k))_0\,\delta_{G,\rho}^{1/2}(g)dk\,dg. \end{aligned} \tag{5.14}$$

Write $g=nlh$, $n\in N(k)$, $l\in L(k)$, $h\in K$. Then

$$dg=\delta_P^{-1}(l)dn\,dl\,dh.$$

With notation as in [GJ72], define:

$$(h\cdot\Phi\cdot k^{-1})(x):=\Phi(k^{-1}xh). \tag{5.15}$$

Thus

$$Z(\Phi,f)=\int_{N(k)\times L(k)\times K\times K} (h\cdot\Phi\cdot k^{-1})(nl)\,\delta_P^{1/2}(l)(\sigma(l)v(h),\tilde{v}(k))_0\,\delta_{G,\rho}^{1/2}(l)\delta_P^{-1}(l)dn\,dl\,dh\,dk. \tag{5.16}$$

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Recall the HC–transform $(h \cdot \Phi \cdot k^{-1})_P$:

$$(h \cdot \Phi \cdot k^{-1})_P(l) = \delta_P^{-1/2}(l) \int_{N(k)} (h \cdot \Phi \cdot k^{-1})(nl)dn.$$

Then (5.16) equals

$$\int_{L(k)\times K\times K} \nu_{G/L,\rho}^{1/2}(l)(h \cdot \Phi \cdot k^{-1})_P(l)(\sigma(l)v(h), \tilde{v}(k))_0 \; \delta_{L,\rho_L}^{1/2}(l)dl\, dh\, dk$$

$$= \int_{L(k)\times K\times K} (h \cdot \Phi \cdot k^{-1})_P^{\rho}(l)(\sigma(l)v(h), \tilde{v}(k))_0 \; \delta_{L,\rho_L}^{1/2}(l)dl\, dh\, dk. \tag{5.17}$$

Since K is compact and v and $\tilde{v}$ are smooth functions, there exist matrix coefficients $f_i^L(l)$ of σ and continuous functions λ_i on $K \times K$ such that

$$(\sigma(l)v(h), \tilde{v}(k))_0 = \sum_i f_i^L(l)\lambda_i(h, k).$$

Similarly, there are Schwartz functions Φ_j in $\mathcal{S}^\rho(G)$ and continuous symmetric functions μ_j on $K \times K$ such that

$$h \cdot \Phi \cdot k^{-1} = \sum_j \Phi_j \mu_j(h, k). \tag{5.18}$$

This is clearly true if $\Phi \in C_c^\infty(G(k))$, since it will then be uniformly smooth. Otherwise, using (5.18) we have

$$(h \cdot \Phi \cdot k^{-1})^\wedge = \sum_j \hat{\Phi}_j \mu_j(h, k),$$

where $\hat{\Phi} := \mathcal{J}^\rho \Phi$ for simplicity. But Lemma 5.5, proved later, implies

$$k \cdot \hat{\Phi} \cdot h^{-1} = \sum_j \hat{\Phi}_j \mu_j(k, h)$$

for all h and k in K and thus (5.16) holds for all $\Phi \in \mathcal{S}^\rho(G)$. Consequently

$$(h \cdot \Phi \cdot k^{-1})_P^\rho(l) = \sum_j \Phi_{j,P}^\rho(l)\mu_j(h, k)$$

with $\Phi_{j,P}^\rho \in \mathcal{S}^{\rho_L}(L)$ by Proposition 5.1. Let

$$c_{ij} = \int_{K\times K} \lambda_i \mu_j(h, k)dhdk.$$

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Then we have

$$Z(\Phi, f) = \sum_{i,j} c_{ij} Z(\Phi^{\rho}_{j,P}, f^{L}_{i}) \tag{5.19}$$

For simplicity of notation, let for each $\Phi \in \mathcal{S}^{\rho}(G)$, $\hat{\Phi} := \mathcal{J}^{\rho}\Phi$. We now calculate

$$Z(\hat{\Phi}, \check{f}) = \int \hat{\Phi}(g)\check{f}(g)\delta^{1/2}_{G,\rho}(g)\,|\nu(g)|\,dg.$$

One needs to be careful since the involution $g \to g^{-1}$ will now play an important role. We should also point out that $|\nu(g)|$ needs to be inserted to take into account the appearance of $1-s$ in the left hand side of the functional equation as it is the case in equation (1.1) for GL_n. We note that $|\nu(g)| = |\nu(l)| = |\nu_L(l)|$ and that $-s$ will appear as $\tilde{\pi} \otimes |\nu()|^{-s}$, and thus included in $\tilde{\pi}$ as s did in π as $\pi \otimes |\nu()|^{s}$. Note that in the case of GL_n and $\rho = \mathrm{std}$, $\nu = \det$, and $\delta^{1/2}_{G,\mathrm{std}} = |\det|^{-\frac{n-1}{2}}$ as reflected in (1.1).

We have

$$\begin{aligned} Z(\hat{\Phi}, \check{f}) &= \int_{G(k)} \hat{\Phi}(g) f(g^{-1})\delta^{1/2}_{G,\rho}(g)\,|\nu(g)|\,dg \\ &= \int_{G(k)\times K} \hat{\Phi}(g)(v(kg^{-1}), \tilde{v}(k))_0\,\delta^{1/2}_{G,\rho}(g)\,|\nu(g)|\,dg\,dk. \end{aligned} \tag{5.20}$$

Changing g to gk, (5.20) equals

$$= \int_{G(k)\times K} \hat{\Phi}(gk)(v(g^{-1}), \tilde{v}(k))_0\,\delta^{1/2}_{G,\rho}(g)\,|\nu(g)|\,dg\,dk. \tag{5.21}$$

Write $g = h^{-1}ln$, $n \in N(k)$, $l \in L(k)$, $h \in K$. Then

$$dg = d(g^{-1}) = \delta_P(l)\,dh\,dl\,dn \tag{5.22}$$

and (5.21) equals

$$= \int_{K\times L(k)\times N(k)\times K} \hat{\Phi}(h^{-1}lnk)(v(l^{-1}h), \tilde{v}(k))_0\,\delta^{1/2}_{G,\rho}(l)\delta_P(l)\,|\nu(l)|\,dh\,dl\,dn\,dk. \tag{5.23}$$

$$= \int_{K\times L(k)\times K} (\delta_P^{1/2}(l)$$

$$\times \int_{N(k)} (k\cdot\hat{\Phi}\cdot h^{-1})(ln)dn)(v(l^{-1}h), \tilde{v}(k))_0\,\delta^{1/2}_{G,\rho}(l)\delta_P^{1/2}(l)\,|\nu_L(l)|\,dh\,dl\,dk$$

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$$= \int_{K\times L(k)\times K} \Big(\delta_P^{-1/2}(l) \times \int_{N(k)} (k\cdot\hat{\Phi}\cdot h^{-1})(nl)\,dn\Big)(v(l^{-1}h), \tilde{v}(k))_0\, \nu^{1/2}_{G/L,\rho}(l)\, \delta_P^{1/2}(l)\, \delta^{1/2}_{L,\rho_L}(l)\, |\nu_L(l)|\, dh\, dl\, dk$$

$$= \int_{K\times L(k)\times K} \nu^{1/2}_{G/L,\rho}(l)(k\cdot\hat{\Phi}\cdot h^{-1})_P(l)(v(l^{-1}h), \tilde{v}(k))_0\, \delta^{1/2}_{L,\rho_L}(l)\, |\nu_L(l)|\, \delta_P^{1/2}(l)\, dh\, dl\, dk,$$

which finally equals

$$\int_{K\times L(k)\times K} (k\cdot\hat{\Phi}\cdot h^{-1})^{\rho}_P(l)\, \delta_P^{-1/2}(l)(\sigma(l^{-1})v(h), \tilde{v}(k))_0\, \delta^{1/2}_{L,\rho_L}(l)\, |\nu_L(l)|\, \delta_P^{1/2}(l)\, dh\, dl\, dk. \tag{5.24}$$

Again, for simplicity for each $\Phi \in \mathsf{S}^{\rho}(G)$, let $\hat{\Phi}$ denote its ρ-Fourier transform

$$\hat{\Phi}(x) = (J^{\rho} * \check{\Phi})(x). \tag{5.25}$$

Here $\check{\Phi}(x) = \Phi(x^{-1})$. To proceed, we need:

Lemma 5.5. *Let k and h be in K. Then*

$$(k\cdot\Phi\cdot h^{-1})^{\wedge} = h\cdot\hat{\Phi}\cdot k^{-1}.$$

Proof. We have

$$\begin{aligned}(k\cdot\Phi\cdot h^{-1})^{\wedge}(x) &= \int_{M^{\rho}(k)} (k\cdot\Phi\cdot h^{-1})(y)J^{\rho}(xy)dy \\ &= \int \Phi(h^{-1}yk)J^{\rho}(xy)dy \\ &= \int \Phi(y)J^{\rho}(xhyk^{-1})dy,\end{aligned} \tag{5.26}$$

since h and k are in K whose modulus character is 1. Now using Int(G)-invariance of J^{ρ}, (5.26) equals

$$\begin{aligned}&= \int \Phi(y)J^{\rho}(k^{-1}xhy)dy \\ &= \hat{\Phi}(k^{-1}xh) \\ &= (h\cdot\hat{\Phi}\cdot k^{-1})(x),\end{aligned}$$

completing the proof. □

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Remark 5.6. In terms of J^ρ we have proved:

$$J^\rho * (k\cdot\Phi\cdot h^{-1})^\vee = h\cdot(J^\rho * \check{\Phi})\cdot k^{-1} \tag{5.27}$$

We now apply Lemma 5.5 to equation (5.24) to get:

$$Z(\hat{\Phi},\check{f}) = \int_{L(k)\times K\times K} ((h\cdot\Phi\cdot k^{-1})^\wedge)^\rho_P(l)(\sigma(l^{-1})v(h),\tilde{v}(k))_0\,\delta^{1/2}_{L,\rho_L}(l)\,|\nu_L(l)|\,dl\,dh\,dk. \tag{5.28}$$

But, using (5.18),

$$(h\cdot\Phi\cdot k^{-1})^\wedge = \sum_j \hat{\Phi}_j\cdot\mu_j(h,k)$$

and

$$\begin{aligned}(\sigma(l^{-1})v(h),\tilde{v}(k))_0 &= \sum_i f_i^L(l^{-1})\cdot\lambda_i(h,k)\\ &= \sum_i \check{f}_i^L(l)\cdot\lambda_i(h,k)\end{aligned}$$

and therefore (5.28) equals

$$\sum_{i,j} c_{ij}\int_{L(k)} (J^\rho\Phi_j)^\rho_P(l)\check{f}_i^L(l)\delta^{1/2}_{L,\rho_L}(l)\,|\nu_L(l)|\,dl. \tag{5.29}$$

We can now apply the commutativity of ρ–Harish–Chandra transform and Fourier transforms J^ρ and J^{ρ_L}, i.e., equation (5.4) to conclude that (5.27) equals

$$\sum_{i,j} c_{ij}\int_{L(k)} J^\rho_L(\Phi^\rho_{j,P})(l)\check{f}_i^L(l)\delta^{1/2}_{L,\rho_L}(l)\,|\nu_L(l)|\,dl. \tag{5.30}$$

But

$$Z(\hat{\Phi},\check{f}) = \gamma(\Pi,\rho)Z(\Phi,f)$$

by the functional equation for G. On the other hand the functional equation for L gives (5.30) as

$$\gamma(\sigma,\rho_L)\sum_{i,j} c_{ij}\int_{L(k)} \Phi^\rho_{j,P}(l)f_i^L(l)\delta^{1/2}_{L,\rho_L}(l)dl$$

which equals

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$$= \gamma(\sigma, \rho_L) \sum_{i,j} c \; Z(\Phi^{\rho}_{j,P}, f^L_i)$$

$$= \gamma(\sigma, \rho_L) Z(\Phi, f)$$

by (5.19). The equality

$$\gamma(\mathrm{Ind}_{P(k)}^{G(k)} \sigma, \rho) = \gamma(\sigma, \rho_L)$$

is now immediate. □

Remark 5.7. The commutativity relation (5.4), although originally stated for $\Phi \in C_c^\infty(G(k))$, extends to all of $\mathcal{S}^\rho(G)$, as stated in Proposition 5.1. In our proof of Theorem 5.4, we used the functional equation

$$Z(J^\rho \Phi, \check{f}) = \gamma(\pi, \rho) Z(\Phi, f)$$

with arbitrary $\Phi \in \mathcal{S}^\rho(G)$ for which we proved uniform smoothness, needed to reduce $Z(J^\rho\Phi, f^\vee)$ to a sum of Zeta functions for L and $\mathcal{S}^{\rho_L}(L)$. Although we could have started with $\Phi \in C_c^\infty(G(k))$, we would eventually need to use uniform smoothness for $Z(J^\rho\Phi, f^\vee)$ in which $J^\rho\Phi$ is no longer of compact support.

5.3. The case of GL_n

We now determine $\nu_{G/L}$ in the case $G = GL_n$ and $\rho = \mathrm{std}$, i.e., that of Godement–Jacquet [GJ72] and show that it agrees with calculations in Lemma 3.4.0 of [GJ72], after a suitable normalization. We thus assume $P = NL$ is the standard maximal parabolic subgroup of GL_n, containing the subgroup of upper triangular elements B, $N \subset B$, with $L = GL_{n'} \times GL_{n''}$, $n = n' + n''$. Recall that we need to determine $\nu_{G/L,\mathrm{std}} = \delta_{G,\mathrm{std}} / \delta_{L,\mathrm{std}}$. But for $g = \mathrm{diag}(g', g'') \in L$

$$\begin{aligned} |\det g'|^{-n''} \cdot \delta_G^{1/2}(g', g'') &= |\det g'|^{-n''} |\det g|^{\frac{1}{2}(n'+n''-1)} \\ &= |\det g'|^{\frac{1}{2}(n'-1)} |\det g''|^{\frac{1}{2}(n''-1)} \cdot |\det g'|^{-n''/2} |\det g''|^{n'/2} \\ &= \delta_L^{1/2}(g', g'') \cdot |\det g'|^{-n''/2} |\det g''|^{n'/2}. \end{aligned}$$

Thus

$$\nu_{G/L}^{1/2}(g', g'') = |\det g'|^{n''/2} |\det g''|^{n'/2}. \tag{5.31}$$

Moreover

$$\delta_P(g', g'') = |\det g'|^{n''} \cdot |\det g''|^{-n'} \tag{5.32}$$

and thus

$$\nu_{G/L}^{1/2}\,\delta_{\overline{P}}^{1/2}(g^{\text{I}},g^{\text{II}}) = |\det g^{\text{II}}|^{n^{\text{I}}}. \tag{5.33}$$

We now verify that Lemma 3.4.0 of [GJ72] is equivalent to our commutative diagram (5.5).

Let $J = J^{\text{std}}$ be the standard Fourier transform for $C_c^\infty(M_n(k))$ and J_L its restriction to $C_c^\infty(M_{n^{\text{I}}}(k)\times M_{n^{\text{II}}}(k))$. With notation as in pages 37–38 of [GJ72],

$$\varphi_\Phi(x,y) = \int_{k^{n^{\text{I}}n^{\text{II}}}} \Phi\begin{pmatrix} x & u\\ 0 & y\end{pmatrix} du, \tag{5.34}$$

where $u \in M_{n^{\text{I}}\times n^{\text{II}}}(k)$ and $\Phi \in C_c^\infty(M_n(k))$, is the analogue of our HC–transform. In fact, (5.34) can be written as

$$\begin{aligned}\varphi_\Phi(x,y) &= \int \Phi\left(\begin{pmatrix} I & uy^{-1}\\ 0 & I\end{pmatrix}\begin{pmatrix} x & 0\\ 0 & y\end{pmatrix}\right) du\\ &= |\det y|^{n^{\text{I}}}\int_{N(k)} \Phi(nl)dn,\\ &= \nu_{G/L}^{1/2}(l)\,\delta_P^{-1/2}(l)\int_{N(k)}\Phi(nl)\,dn\end{aligned} \tag{5.35}$$

by (5.33), where $N = M_{n^{\text{I}}\times n^{\text{II}}}$ and $l = \text{diag}(x,y)$.

In the notation of [GJ72], Lemma 3.4.0 of [GJ72] states that

$$\varphi_{\hat{\Phi}}(x,y) = \varphi_\Phi(x,y), \tag{5.36}$$

for $\Phi \in C_c^\infty(M_n(F))$ with $\hat{\Phi}$ its standard Fourier transform.

Then by (5.35) the right hand side of (5.36) equals

$$\begin{aligned}\varphi_\Phi(x,y) &= J_L\left(\int \Phi\begin{pmatrix} x & u\\ 0 & y\end{pmatrix} du\right)\\ &= J_L\left(|\det y|^{n^{\text{I}}}\int_{N(k)}\Phi(nl)\,dn\right)\\ &= J_L(\nu_{G/L}^{1/2}(l)\Phi_P(l))\\ &= J_L(\Phi_P^{\text{std}}(l))\end{aligned} \tag{5.37}$$

by (5.33), where

$$\Phi_P(l) = \delta_P^{-1/2}(l) \int_{N(k)} \Phi(nl)\, dn$$

as defined by (5.1).

Similarly from the left hand side of (5.36), using a change of variables as in (5.35), we have

$$\begin{aligned} \varphi_{\hat{\Phi}}(x, y) &= \int \hat{\Phi}\begin{pmatrix} x & u \\ 0 & y \end{pmatrix} du \\ &= |\det y|^{n'} \int_{N(k)} \hat{\Phi}(nl)\, dn \\ &= \nu_{G/L}^{1/2}(l)\, (\hat{\Phi})_P(l) \\ &= (J\Phi)_P^{\mathrm{std}}(l). \end{aligned} \tag{5.38}$$

Thus (5.36) is equivalent to (5.4) for GL_n and $\rho = \mathrm{std}$.

5.4. Inductive definition of J^ρ

In the introduction we mentioned that multiplicativity plus a definition of Fourier transform that acts through the correct scalar factors equal to the gamma factors on supercuspidal representations/characters, is enough to characterize the full Fourier transform. Indeed, if we assume that J^ρ is a good distribution in the sense of Braverman-Kazhdan [BK10], then we can identify J^ρ with a rational, scalar valued function $\pi \mapsto \gamma(\rho, \pi)$, where $\gamma(\rho, \pi)$ is defined by $J^\rho * \pi = \gamma(\rho, \pi)\pi$.

Our results on multiplicativity allow in principle for us to construct in an inductive fashion a distribution J^ρ on G by formally inducing from J^{ρ_L} for each conjugacy class of Levi subgroup $L \subset G$. In fact, our setup and definitions, culminating in Theorem 5.4, are normalized so as to make induction of representations adjoint to our ρ-Harish-Chandra transform, that is, we have an equality

$$(J^\rho, \mathrm{Ind}_L(\theta)) = (J^{\rho_L}, \theta).$$

Here θ is a supercuspidal character of a representation on L. The adjunction allows us to identify the J^ρ and J^{ρ_L} actions on the Bernstein components of $\mathrm{Ind}_L(\sigma) = \pi$ on $G(k)$ and the Bernstein component of σ on $L(k)$, respectively. In 5.3, we started with an assumption of knowledge of J^ρ and J^{ρ_L} and we showed that this is equivalent to an equality of gamma factors. However, the gamma factors determine the distribution uniquely, and so one can in principle characterize completely a distribution J^ρ by specifying its action on supercuspidal representations on $G(k)$, and postulating multiplicativity as an axiom. More concretely, if we inductively know J^{ρ_L} for conjugacy classes of parabolic subgroups

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L, we may formally induce to provide a definition of J^ρ with a correct action, at least on functions whose spectral decomposition consists solely of induced data from L:

$$(\mathrm{Ind}_L(J^{\rho_L}), f) = (J^\rho, HC(f)).$$

The distribution $\mathrm{Ind}_L(J^{\rho_L})$ can be a priori defined by the above in order to meet this adjunction, and in fact J^{ρ_L} will then be represented by the conjugation-invariant function

$$\mathrm{Ind}_L(J^{\rho_L}) : x \to |D_G(x)|^{-\frac{1}{2}} \sum_y |D_L(y)|^{\frac{1}{2}} J^{\rho_L}(y)$$

where the y are chosen representatives of $L(k)$-conjugacy classes of elements that are $G(k)$-conjugate to x, and D_G and D_L are the respective discriminant functions on G and L. (Here we are identifying J^{ρ_L} with the invariant function representing it.)

That $\mathrm{Ind}_L(J^{\rho_L})$ satisfies the first adjunction, and therefore multiplicativity, follows from the formula for the trace, and the expression of the distribution character $\Theta_\pi = \mathrm{Ind}_L(\Theta_\sigma)$ in terms of Θ_σ, adapted to the ρ-setting.

6. Example: the case of Tori and unramified data

We now consider the case of tori, which for present purposes we assume are split. Let T be a split torus over k. When T is a maximal split torus in a reductive group G, the upcoming discussion gives the first term of the inductive construction defining the Fourier transform for $L = T$, with minimal parabolic $P_0 = P = LN = TN$ which is a Borel subgroup. Let $\rho = \rho_T$ be a finite dimensional representation of $\hat{T}$. Our notation is justified if we assume $\rho_T = \rho|\hat{T}$, where ρ is a representation of $\hat{G}$. Let $n = \dim \rho_T$. Then

$$\rho_T : \hat{T} \to GL_n(\mathbb{C}).$$

Write

$$\rho_T = \mu_1 \oplus \cdots \mu_n, \tag{6.1}$$

where the μ_i, $1 \le i \le n$, are the weights of ρ_T. We note that they are not necessarily distinct. If we realize these weights of $\hat{T}$ as co-characters of T, we get a map $\tilde{\rho}_T : \mathbb{G}_m^n \to T$ (defined over k, as T is split), which being dual to ρ_T, is given by (cf. [Ngô20])

$$\tilde{\rho}_T(x_1, \ldots, x_n) = \mu_1(x_1) \cdots \mu_n(x_n).$$

We can extend this to a monoid homomorphism

$$\tilde{\rho}_T : \mathbb{A}^n \to M^{\rho_T},$$

where M^{ρ_T} is the corresponding toric variety. As in [Ngô20], define the trace function $h : \mathbb{A}^n \to \mathbb{A}$ by

$$h((x_i)) = \sum_i x_i$$

and set

$$h_\psi : k^n \to \mathbb{C}^*$$

by $x \to \psi(h(x))$, where ψ is our fixed non-trivial character of k.

Denote by J^{std} the kernel

$$J^{\mathrm{std}}(g) = \psi(\mathrm{tr}(g))|\det g|^n dg$$

for $g \in GL_n(k)$ as defined in Section 1, i.e., the standard Fourier transform on $M_n(k)$. We use again J^{std} for its restriction to $\mathbb{A}^n$, the monoid for $T_n = \mathbb{G}_m^n$.

In [Ngô20] Ngo defines the kernel J^{ρ_T} for the Fourier transform on T by

$$J^{\rho_T}(t) = \int_{\rho^{-1}(t)} h_\psi(x)dx, \tag{6.2}$$

which equals to

$$J^{\rho_T}(t) = \int_{x \in U(k)} h_\psi(xt)dx \tag{6.3}$$

where U is the kernel of $\tilde{\rho}_T$. In Proposition 6 of [Ngô20], Ngo regularizes this integration into a principal value integral.

The space of Schwartz functions on k^n are compactly supported functions in k^n that are restrictions of standard Schwartz functions on $M_n(k)$ to k^n. Their further restriction to $T_n(k)$ is $\mathcal{S}^{\mathrm{std}}(T_n)$ in our notation.

Let ρ_* be the push-forward of $\tilde{\rho}_T$. We will verify that the diagram

$$\begin{array}{ccc} \mathcal{S}^{\mathrm{std}}(T_n) & \xrightarrow{\rho_*} & \mathcal{S}^{\rho_T}(T) \\ {\scriptstyle J^{\mathrm{std}}}\downarrow & & \downarrow{\scriptstyle J^{\rho_T}} \\ \mathcal{S}^{\mathrm{std}}(T_n) & \xrightarrow{\rho_*} & \mathcal{S}^{\rho_T}(T) \end{array} \tag{6.4}$$

commutes, where $\mathcal{S}^{\rho_T}(T)$ is the image of $\mathcal{S}^{\mathrm{std}}(T_n)$ under ρ_*.

Let $\phi \in \mathsf{S}^{\mathrm{std}}(T_n)$ and define

$$\tilde{\phi}(\tilde{t}) = \int_{U(k)} \phi(u\tilde{t})du \tag{6.5}$$

where $\tilde{t} \in T(k)$. The commutativity of (6.4) is equivalent to

Lemma 6.1. *For $\phi \in \mathsf{S}^{std}(T_n)$, define $\tilde{\phi}$ by (6.5). Then*

$$\rho_*(J^{std} * \phi^\vee) = J^{\rho_T} * \tilde{\phi}^\vee$$

Proof. By definition, for $t \in T$,

$$\begin{aligned}
\rho_*(J^{\mathrm{std}} * \phi^\vee)(t) &= \int_{U(k)} (J^{\mathrm{std}} * \phi^\vee)(ut)du \\
&= \int_{U(k)} \left(\int_{T_n(k)} h_\psi(ut\tilde{t})\phi(\tilde{t})d\tilde{t}\right)du \\
&= \int_{T_n(k)} h_\psi(t\tilde{t})\left(\int_{U(k)} \phi(u^{-1}\tilde{t})du\right)d\tilde{t} \\
&= \int_{T_n(k)} h_\psi(t\tilde{t})\tilde{\phi}(\tilde{t})d\tilde{t} \qquad (6.6)\\
&= \int_{T(k)} \left(\int_{U(k)} h_\psi(ut\tilde{t})du\right)\tilde{\phi}(u\tilde{t})d\tilde{t} \\
&= \int_{T(k)} \left(\int_{U(k)} h_\psi(u\tilde{t})du\right)\tilde{\phi}(t^{-1}\tilde{t})d\tilde{t} \\
&= (J^{\rho_T} * \tilde{\phi}^\vee)(t),
\end{aligned}$$

using $T = T_n/U$ in (6.6), then the lemma follows. □

The push-forward ρ_* can be restricted to

$$\mathbb{C}[\hat{T}_n]^{W_n} \cong \mathsf{H}(T_n(k), T_n(\mathsf{O}_k))^{W_n}$$

leading to

$$\begin{array}{ccc}
\mathsf{H}(T_n(k), T_n(\mathsf{O}_k))^{W_n} & \xrightarrow{\rho_*} & \mathsf{H}(T(k), T(\mathsf{O}_k))^{W} \\
{\scriptstyle J^{\mathrm{std}}}\big\downarrow & & \big\downarrow{\scriptstyle J^{\rho_T}} \\
\mathsf{H}(T_n(k), T_n(\mathsf{O}))^{W_n} & \xrightarrow{\rho_*} & \mathsf{H}(T(k), T(\mathsf{O}_k))^{W},
\end{array} \tag{6.7}$$

where W_n is the Weyl group $W_n = W(Gl_n, T_n)$, $W := W(G, T)$ and $\mathcal{H}$ denotes the corresponding Hecke algebra. We recall that $\mathcal{H}(G) = C_c^\infty(G)$ and set $\mathcal{H}(G, K)$ to be the subset of $\mathcal{H}(G)$ consisting of K bi-invariant functions, for K an open compact subgroup of G. Identifying, via the corresponding Satake isomorphisms

$$\mathcal{H}^\circ(GL_n(k)) := \mathcal{H}(GL_n(k), GL_n(\mathcal{O}_k)) \cong \mathcal{H}(T_n(k), T_n(\mathcal{O}_k))^{W_n}$$

and

$$\mathcal{H}(G(k), G(\mathcal{O}_k)) \cong \mathcal{H}(T(k), T(\mathcal{O}_k))^W$$

$$\begin{array}{ccccccc} \mathcal{H}^\circ(GL_n(k)) & \xrightarrow{Sat} & \mathcal{H}(T_n(k), T_n(\mathcal{O}_k))^{W_n} & \xrightarrow{\rho_*} & \mathcal{H}(T(k), T(\mathcal{O}_k))^W & \xrightarrow{Sat^{-1}} & \mathcal{H}(G(k), G(\mathcal{O}_k)) \\ J^{\mathrm{std}} \downarrow & & \downarrow J^{std} & & \downarrow J^{\rho_T} & & J^\rho \downarrow \\ \mathcal{H}^\circ(GL_n(k)) & \xrightarrow{Sat} & \mathcal{H}(T_n(k), T_n(\mathcal{O}_k))^{W_n} & \xrightarrow{\rho_*} & \mathcal{H}(T(k), T(\mathcal{O}_k))^W & \xrightarrow{Sat^{-1}} & \mathcal{H}(G(k), G(\mathcal{O}_k)), \end{array} \tag{6.8}$$

in which J^{ρ_T} defines the Fourier transform

$$J^\rho : \mathcal{S}(G) \to \mathcal{S}(G)$$

restricted to $\mathcal{H}(G(k), G(\mathcal{O}_k))$. Consequently, at least on $\mathcal{H}(G(k), G(\mathcal{O}_k))$, the Fourier transform J^ρ and J^{ρ_T} commute with the Harish-Chandra transform.

We now complete the proof of Proposition 5.3.

Proof. Note that when $L = T$ is a maximal torus, the ρ-Harish-Chandra transform becomes (a twist of) the Satake transform, and in this case diagram (5.5) can be extended to the class of almost compact (ac) spherical functions as defined by Wen-Wei Li in [Li17], and we note that the ρ-basic function is amongst this class (see [Sak18]). The computation in section 5 (see (5.9) and (5.10)) giving

$$\Phi_0 = \frac{1}{q-1}\mathrm{char}(P_k^{-1} \setminus O_k) + \frac{q}{q-1}\hat{\Phi}$$

can be extended to show that the function $f^{\mathrm{std}} = \mathrm{char}(\mathbb{A}^n(\mathcal{O}_k) \cap T_n(k))$ is also a sum in $C_c^\infty(T_n(k)) + J^{\mathrm{std}}(C_c^\infty(T_n(k)))$, where $T_n(k) \hookrightarrow \mathbb{A}^n(k)$ is the standard embedding of a maximal torus $T_n \cong \mathbb{G}_m^n$ of GL_n into affine space.

Let Sat := std − HC be this extended Satake transform. Given a decomposition $\phi_{T_n}^{\mathrm{std}} = f_1 + J^{\mathrm{std}}(f_2)$, with $f_1, f_2 \in C_c^\infty(T_n(k))$, the commutativity of (5.5) implies that the standard basic function on $GL_n(k)$, $\phi^{\mathrm{std}} = \mathrm{Sat}^{-1}(f_1) + \mathrm{Sat}^{-1}(J^{\mathrm{std}_T}(f_2)) = \mathrm{Sat}^{-1}(f_1) + J^{\mathrm{std}}(\mathrm{Sat}^{-1}(f_2))$, lies in $\mathcal{S}^{\mathrm{std}}(GL_n(k))$ as defined in (5.6).

Note that here Sat is an isomorphism of K-spherical compactly supported functions on $G(k)$ and the Weyl-invariant compactly supported functions on $T(k)/T(\mathcal{O}_k)$. The basic function on $T_n(k)$, and the functions in its decomposition as $f_1 + J^{\mathrm{std}}(f_2)$ are invariant under permutations of the coordinates, and so the above maps are well-defined in the remarks above.

We deduce the case for general ρ from the standard case above as follows: Let T be a maximal torus in G with representation ρ of the dual group of G. One obtains a canonical map $\tilde{\rho}_T : T_n(k) \to T(k)$ that extends to a map $\mathbb{A}^n(k) \to M_T(k)$, with the target of this map being the toric variety constructed in section 2. The ρ-Schwartz space on $T(k)$ can be defined (see [Ngô20]) as the image of

$$C_c^\infty(\mathbb{A}^n(k)) \cap C^\infty(T_n(k)) \to \rho_*(C_c^\infty(\mathbb{A}^n(k)) \cap C^\infty(T_n(k))),$$

the pushforward by $\tilde{\rho}_T$. Then the torus basic function ϕ_T^ρ can be expressed as $\rho_*(\phi_{T_n}^{\text{std}})$. Moreover, this pushforward is compatible with the ρ_T-Fourier transform on tori, as in diagram (6.4). That is,

$$\begin{aligned}\phi_T^\rho = \rho_*(\phi_{T_n}^{\text{std}}) &= \rho_*(f_1 + J_T^{\text{std}}(f_2)) \\ &= \rho_*(f_1) + \rho_*(J_T^{\text{std}}(f_2)) \\ &= \rho_*(f_1) + J^{\rho_T}(\rho_*(f_2)),\end{aligned}$$

which shows that $\phi_T^\rho \in C_c^\infty(T(k)) + J^{\rho_T}(C_c^\infty(T(k)))$. Finally, the commutativity of diagram (6.8) allows us to lift this decomposition to a decomposition of the basic function as

$$\begin{aligned}\phi^\rho &= \text{Sat}^{-1}(\phi_T^\rho) \\ &= \text{Sat}^{-1}(\rho_*(f_1)) + \text{Sat}^{-1}(J^{\rho_T}(\rho_*(f_2))) \\ &= \text{Sat}^{-1}(\rho_*(f_1)) + J^\rho(\text{Sat}^{-1}(\rho_*(f_2)). \quad \square\end{aligned}$$

7. The case of standard *L*-functions for classical groups; the doubling method

We conclude by addressing multiplicativity in the case of standard L-functions, twisted by a character, for classical groups as developed by Piatetski-Shapiro and Rallis, which has been addressed further by a number of other authors [BK02,JLZ20,Li18,Sha18] within our present context. We refer to the local theory developed by Lapid and Rallis. We will be brief and only mention the relevant statements.

The ρ-Harish-Chandra transform is the one given in Proposition 1 of [LR05] as $\Psi(\omega, s)$ with notation as in [LR05]. Our commutativity equation (5.4) in this case is equation (17) in Lemma 9 of [LR05] in which $J^\rho = M_v^*(\omega, A, s)$, a normalized intertwining operator, while J^{ρ_L} acts as the operator induced from $M^*_W(\omega, B, s)$ with notation as in [LR05] in the context of doubling construction, or simply put $J^{\rho_L} = M^*_W(\omega, B, s)$.

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