Forum of Mathematics, Sigma (2022), **Vol. 10**:e14 1–50
doi:10.1017/fms.2021.79

RESEARCH ARTICLE

Nonamenable simple C^*-algebras with tracial approximation

Xuanlong Fu [1] and Huaxin Lin [2,3,*]

[1]Shanghai Center for Mathematical Sciences, Fudan University, 2005 Songhu Road, Shanghai, 200438 China;
E-mail: xuanlong.fu@utoronto.ca.
[2]Department of Mathematics, East China Normal University, Shanghai, China.
[3]Current: Department of Mathematics, University of Oregon, Eugene, OR 97402, USA; E-mail: hlin@uoregon.edu.
*Corresponding author

Received: 18 March 2021; **Revised:** 12 November 2021; **Accepted:** 28 November 2021

2020 Mathematics Subject Classification: *Primary* – 46L35, 46L05

Abstract
We construct two types of unital separable simple C^*-algebras: $A_z^{C_1}$ and $A_z^{C_2}$, one exact but not amenable, the other nonexact. Both have the same Elliott invariant as the Jiang–Su algebra – namely, $A_z^{C_i}$ has a unique tracial state,

$$\left(K_0\left(A_z^{C_i}\right), K_0\left(A_z^{C_i}\right)_+, \left[1_{A_z^{C_i}}\right]\right) = (\mathbb{Z}, \mathbb{Z}_+, 1),$$

and $K_1\left(A_z^{C_i}\right) = \{0\}$ $(i = 1, 2)$. We show that $A_z^{C_i}$ $(i = 1, 2)$ is essentially tracially in the class of separable $\mathcal{Z}$-stable C^*-algebras of nuclear dimension 1. $A_z^{C_i}$ has stable rank one, strict comparison for positive elements and no 2-quasitrace other than the unique tracial state. We also produce models of unital separable simple nonexact (exact but not nuclear) C^*-algebras which are essentially tracially in the class of simple separable nuclear $\mathcal{Z}$-stable C^*-algebras, and the models exhaust all possible weakly unperforated Elliott invariants. We also discuss some basic properties of essential tracial approximation.

1. Introduction

Simple unital projectionless amenable C^*-algebras were first constructed by Blackadar [2]. The C^*-algebra A constructed by Blackadar has the property that $K_0(A) = \mathbb{Z}$ with the usual order but with nontrivial $K_1(A)$. The Jiang–Su algebra $\mathcal{Z}$ given by Jiang and Su [27] is a unital infinite-dimensional separable amenable simple C^*-algebra with Elliott invariant exactly the same as that of the complex field $\mathbb{C}$, Let A be any σ-unital C^*-algebra. Then $K_i(A) = K_i(A \otimes \mathcal{Z})$ $(i = 0, 1)$ as abelian groups and $T(A) \cong T(A \otimes \mathcal{Z})$. If A is a separable simple C^*-algebra, then $A \otimes \mathcal{Z}$ has nice regularity properties. For example, $A \otimes \mathcal{Z}$ is either purely infinite or stably finite [42]. In fact, if $A \otimes \mathcal{Z}$ is not purely infinite, then it has stable rank one when A is not stably projectionless [42], or it almost has stable rank one when it is stably projectionless [38]. Also, $A \otimes \mathcal{Z}$ has weakly unperforated K_0-group [23]. Another important regularity property is that $A \otimes \mathcal{Z}$ has strict comparison [42] (see also Definition 2.6). If A has weakly unperforated $K_0(A)$, then A and $A \otimes \mathcal{Z}$ have the same Elliott invariant. In other words, A and $A \otimes \mathcal{Z}$ are not distinguishable from the Elliott invariant.

The Jiang–Su algebra $\mathcal{Z}$ is an inductive limit of 1-dimensional noncommutative CW complexes. In fact, $\mathcal{Z}$ is the unique infinite-dimensional separable simple C^*-algebra with finite nuclear dimension in the UCT class which has the same Elliott invariant as that of the complex field $\mathbb{C}$ (see [16, Corollary

4.12]). These properties give $\mathcal{Z}$ a prominent role in the study of structure of C^*-algebras, in particular in the study of classification of amenable simple C^*-algebras.

Attempts to construct a nonnuclear Jiang–Su-type C^*-algebra have been on the horizon for over a decade. In particular, after Dădărlat's construction of nonamenable models for non–type I separable unital AF-algebras [13], this should be possible. The construction in [13] generalised some earlier constructions of simple C^*-algebras of real rank zero such as that of Goodearl [25]. Jiang and Su's construction has a quite different feature. To avoid producing any nontrivial projections, unlike Dădărlat's construction, Jiang and Su did not use any finite-dimensional representations as a direct summand of connecting maps in the inductive systems. The construction used prime-dimension drop algebras, and connecting maps were highly inventive so that the traces eventually collapse to one. In fact, Rørdam and Winter took another approach [43] using a C^*-subalgebra of $C\big([0,1], M_{\mathfrak{p}} \otimes M_{\mathfrak{q}}\big)$, where $\mathfrak{p}$ and $\mathfrak{q}$ are relatively prime supernatural numbers. One possible attempt to construct a nonamenable Jiang–Su-type C^*-algebra would use $C\big([0,1], B_{\mathfrak{p}} \otimes B_{\mathfrak{q}}\big)$, where $B_{\mathfrak{p}}$ and $B_{\mathfrak{q}}$ are, respectively, nonamenable models for $M_{\mathfrak{p}}$ and $M_{\mathfrak{q}}$ constructed in [13]. However, one usually would avoid computation of the K-theory of tensor products of nonexact simple C^*-algebras such as $B_{\mathfrak{p}}$ and $B_{\mathfrak{q}}$. Moreover, Rørdam and Winter's construction depends on knowing the existence of the Jiang–Su algebra $\mathcal{Z}$. On the other hand, if one considers nonexact interval 'dimension drop algebras', besides controlling K-theory one has additional issues such as the fact that each fibre of the 'dimension drop algebra' is not simple (unlike the usual dimension drop algebras, whose fibres are simple matrix algebras).

We will present some nonexact (or exact but nonnuclear) unital separable simple C^*-algebras A_z^C which have the property that their Elliott invariants are the same as that of the Jiang–Su algebra $\mathcal{Z}$ – namely, $\left(K_0\big(A_z^C\big), K_0\big(A_z^C\big)_+, \left[1_{A_z^C}\right]\right) = (\mathbb{Z}, \mathbb{Z}_+, 1)$, $K_1\big(A_z^C\big) = \{0\}$ and A_z^C has a unique tracial state. Moreover, A_z^C has stable rank one and has strict comparison for positive elements. A_z^C has no (nonzero) 2-quasitrace other than the unique tracial state. Even though A_z^C may not be exact, it is essentially tracially approximated by $\mathcal{Z}$. In particular, it is essentially tracially approximated by unital simple C^*-algebras with nuclear dimension 1.

In this paper, we will also study the tracial approximation. We will make it precise what we mean by saying that A_z^C is essentially tracially approximated by $\mathcal{Z}$ (Definition 3.1, Lemma 8.1). We expect that regularity properties such as stable rank one, strict comparison for positive elements and almost unperforated Cuntz semigroups, as well as approximate divisibility, are preserved by tracial approximation. In fact, we show that if a unital separable simple C^*-algebra A is essentially tracially in $\mathcal{C}_{\mathcal{Z}}$, the class of $\mathcal{Z}$-stable C^*-algebras, then – as far as the usual regularity properties are concerned – A behaves just like C^*-algebras in $\mathcal{C}_{\mathcal{Z}}$. More precisely, we show that if A is simple and essentially tracially in $\mathcal{C}_{\mathcal{Z}}$, then A is tracially approximately divisible. If A is not purely infinite, then it has stable rank one (or almost has stable rank one, if A is not unital) and has strict comparison, and its Cuntz semigroup is almost unperforated. If A is essentially tracially in the class of exact C^*-algebras, then every 2-quasitrace of $\overline{aAa}$, for any a in the Pedersen ideal of A, is in fact a trace.

Using A_z^C, we present a large class of nonexact (or exact but nonnuclear) unital separable simple C^*-algebras which exhaust all possible weakly unperforated Elliott invariants. Moreover, every C^*-algebra in the class is essentially tracially in the class of unital separable simple C^*-algebras which are $\mathcal{Z}$-stable, and has nuclear dimension at most 1.

The paper is organised as follows: Section 2 serves as preliminaries, where some frequently used notations and definitions are listed. Section 3 introduces the notion of essential tracial approximation for simple C^*-algebras. In Section 4 we present some basic properties of essential tracial approximation. For example, we show that if A is a simple C^*-algebra and is essentially tracially approximated by C^*-algebras whose Cuntz semigroups are almost unperforated, then the Cuntz semigroup of A is almost unperforated (Theorem 4.3). In particular, A has strict comparison for positive elements. In Section 5 we study the separable simple C^*-algebras which are essentially tracially approximated by $\mathcal{Z}$-stable C^*-algebras. We show that such C^*-algebras are either purely infinite or almost have stable rank one (or do have stable rank one, if the C^*-algebras are unital). These simple C^*-algebras are tracially approximately divisible and have strict comparison for positive elements. In Section 6 we begin the construction of

A_z^C. In Section 7 we show that the construction in Section 6 can be made simple, and the Elliott invariant of A_z^C is precisely the same as that of a complex field, just as with the Jiang–Su algebra $\mathcal{Z}$. In Section 8 we show that A_z^C has all expected regularity properties. Moreover, A_z^C is essentially tracially approximated by $\mathcal{Z}$. Using A_z^C, we also produce, for each weakly unperforated Elliott invariant, a unital separable simple nonexact (or exact but nonnuclear) C^*-algebra B which has the said Elliott invariant, has stable rank one, is essentially tracially approximated by C^*-algebras with nuclear dimension at most 1, has almost unperforated Cuntz semigroup, has strict comparison for positive elements and has no 2-quasitraces which are not traces.

2. Preliminaries

In this paper, the set of all positive integers is denoted by $\mathbb{N}$. If A is unital, $U(A)$ is the unitary group of A. A linear map is said to be c.p.c., if it is a completely positive contraction.

Notation 2.1. Let A be a C^*-algebra and $\mathcal{F} \subset A$ be a subset. Let $\epsilon > 0$. Set $a, b \in A$ and write $a \approx_\epsilon b$ if $\|a - b\| < \epsilon$. We write $a \in_\varepsilon \mathcal{F}$ if there is $x \in \mathcal{F}$ such that $a \approx_\varepsilon x$.

Notation 2.2. Let A be a C^*-algebra and let $S \subset A$ be a subset of A. Denote by $\mathrm{Her}_A(S)$ (or just $\mathrm{Her}(S)$, when A is clear) the hereditary C^*-subalgebra of A generated by S. Denote by $A^{\mathbf{1}}$ the closed unit ball of A, by A_+ the set of all positive elements in A, by $A_+^{\mathbf{1}} := A_+ \cap A^{\mathbf{1}}$ and by A_{sa} the set of all self-adjoint elements in A. Denote by $\widetilde{A}$ (or $A^\sim$) the minimal unitisation of A. When A is unital, denote by $GL(A)$ the set of invertible elements of A and by $U(A)$ the unitary group of A.

Notation 2.3. Let $\epsilon > 0$. Define a continuous function $f_\epsilon : [0, +\infty) \to [0, 1]$ by

$$f_\epsilon(t) := \begin{cases} 0, & t \in [0, \epsilon/2], \\ 1, & t \in [\epsilon, \infty), \\ \text{linear}, & t \in [\epsilon/2, \epsilon]. \end{cases}$$

Definition 2.4. Let A be a C^*-algebra and set $M_\infty(A)_+ := \bigcup_{n\in\mathbb{N}} M_n(A)_+$. For $x \in M_n(A)$, we identify x with $\mathrm{diag}(x, 0) \in M_{n+m}(A)$ for all $m \in \mathbb{N}$. Set $a \in M_n(A)_+$ and $b \in M_m(A)_+$. We may write $a \oplus b := \mathrm{diag}(a, b) \in M_{n+m}(A)_+$. If $a, b \in M_n(A)$, we write $a \lesssim b$ if there are $x_i \in M_n(A)$ such that $\lim_{i\to\infty}\left\|a - x_i^* b x_i\right\| = 0$. We write $a \sim b$ if $a \lesssim b$ and $b \lesssim a$ hold. The Cuntz relation $\sim$ is an equivalence relation. Set $W(A) := M_\infty(A)_+/\sim$. Let $\langle a \rangle$ denote the equivalence class of a. We write $\langle a \rangle \le \langle b \rangle$ if $a \lesssim b$. $(W(A), \le)$ is a partially ordered abelian semigroup. Let $\mathrm{Cu}(A) = W(A \otimes \mathcal{K})$. $W(A)$ (resp., $\mathrm{Cu}(A)$) is called almost unperforated if, for any $\langle a \rangle, \langle b \rangle \in W(A)$ (resp., $\mathrm{Cu}(A)$) and any $k \in \mathbb{N}$, when $(k + 1)\langle a \rangle \le k\langle b \rangle$, we have $\langle a \rangle \le \langle b \rangle$ (see [40]).

Let $B \subset A$ be a hereditary C^*-subalgebra, and set $a, b \in B_+$. It is clear that $a \lesssim_B b$ implies $a \lesssim_A b$. Conversely, if $a \lesssim_A b$, then, for any $\varepsilon > 0$, there exists $x \in A$ such that $\|a - x^*bx\| < \varepsilon/4$. Choose $e \in B_+^{\mathbf{1}}$ such that $\|a - eae\| < \varepsilon/4$. Then $\left\|a - ex^*b^{1/4}b^{1/2}b^{1/4}xe\right\| < \varepsilon/2$. It follows that $a \lesssim_B b^{1/2} \sim_B b$. In other words, $a \lesssim_A b \Leftrightarrow a \lesssim_B b$.

Remark 2.5. It is known to some experts that the condition that $W(A)$ be almost unperforated is equivalent to the condition that $\mathrm{Cu}(A)$ be almost unperforated. To see this briefly, let us assume that $W(A)$ is almost unperforated and set $a, b \in (A \otimes \mathcal{K})_+$ such that $(k + 1)\langle a \rangle \le k\langle b \rangle$. Let $\left\{e_{i,j}\right\}$ be the system of matrix units for $\mathcal{K}$ and $E_n = \sum_{i=1}^n 1_{\tilde{A}} \otimes e_{i,i}$, and let $\varepsilon > 0$. Note that $E_n a E_n \in M_n(A)_+$ for all $n \in \mathbb{N}$. Moreover, $a \approx_{\varepsilon/8} E_n a E_n$ for some large $n \in \mathbb{N}$. It follows from [40, Proposition 2.2] that $(a - \varepsilon)_+ \lesssim (E_n a E_n - \varepsilon/4)_+$ and $(E_n a E_n - \varepsilon/4)_+ \lesssim (a - \varepsilon/8)_+$. By [40, Proposition 2.4], there exists $\delta > 0$ such that $(k + 1)\langle (a - \varepsilon/8)_+ \rangle \le k\langle (b - \delta)_+ \rangle$. Repeating Rørdam's results [40], one obtains that $\langle (b - \delta)_+ \rangle \le \langle E_m b E_m \rangle$ for some even larger m ($m \ge n$). Now one has $(k + 1)\langle (E_n a E_n - \varepsilon/4)_+ \rangle \le k\langle E_m b E_m \rangle$. By the last paragraph of Definition 2.4, this holds in $M_m(A)$. Since $W(A)$ is almost unperforated, $(a - \varepsilon)_+ \lesssim (E_n a E_n - \varepsilon/4)_+ \lesssim E_m b E_m$. Then $(a - \varepsilon)_+ \lesssim E_m b E_m \lesssim b$. It follows that $a \lesssim b$. Therefore $W(A)$ being almost unperforated implies that $\mathrm{Cu}(A)$ is almost unperforated.

To see the converse, just notice again that A is a hereditary C^*-subalgebra of $A \otimes \mathcal{K}$; then $\langle a \rangle \le \langle b \rangle$ in $\mathrm{Cu}(A) = W(A \otimes \mathcal{K})$ implies $\langle a \rangle \le \langle b \rangle$ in $W(A)$.

Definition 2.6. Denote by $QT(A)$ the set of 2-quasitraces of A with $\|\tau\| = 1$ (see [4, II 1.1, II 2.3]) and by $T(A)$ the set of all tracial states on A. We will also use $T(A)$ as well as $QT(A)$ for the extensions on $M_k(A)$ for each k. In fact, $T(A)$ and $QT(A)$ may be extended to lower semicontinuous traces and lower semicontinuous quasitraces on $A \otimes \mathcal{K}$ (see before [17, Proposition 4.2] and [7, Remark 2.27(viii)]).

Let A be a C^*-algebra. Denote by $\mathrm{Ped}(A)$ the Pedersen ideal of A (see [36, 5.6]). Suppose that A is a σ-unital simple C^*-algebra. Choose $b \in \mathrm{Ped}(A)_+$ with $\|b\| = 1$. Put $B := \overline{bAb} = \mathrm{Her}(b)$. Then by [8], $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$. For each $\tau \in QT(B)$, define a lower semicontinuous function $d_\tau : A \otimes \mathcal{K}_+ \to [0, +\infty]$, $x \mapsto \lim_{n\to\infty} \tau\big(f_{1/n}(x)\big)$. The function d_τ is called the dimension function induced by τ.

We say A has strict comparison (for positive elements) if, for any $a, b \in A \otimes \mathcal{K}_+$, the statement $d_\tau(a) < d_\tau(b)$ for all $\tau \in QT(B)$ implies that $a \lesssim b$.

3. Tracial approximation

Definition 3.1. Let $\mathcal{P}$ be a class of C^*-algebras that is closed under isomorphisms, and let A be a simple C^*-algebra. We say A is essentially tracially in $\mathcal{P}$ (abbreviated as 'e. tracially in $\mathcal{P}$') if, for any finite subset $\mathcal{F} \subset A$, any $\varepsilon > 0$ and any $s \in A_+ \setminus \{0\}$, there exist an element $e \in A_+^{\mathbf{1}}$ and a nonzero C^*-subalgebra B of A which is in $\mathcal{P}$ such that the following hold:

(1) $\|ex - xe\| < \varepsilon$ for all $x \in \mathcal{F}$.
(2) $(1-e)x \in_\varepsilon B$ and $\|(1-e)x\| \ge \|x\| - \varepsilon$ for all $x \in \mathcal{F}$.
(3) $e \lesssim s$.

Proposition 3.2. *Let $\mathcal{P}$ be a class of C^*-algebras and let A be a simple C^*-algebra. Then A is e. tracially in $\mathcal{P}$ if and only if the following hold: For any $\varepsilon > 0$, any finite subset $\mathcal{F} \subset A$, any $a \in A_+ \setminus \{0\}$ and any finite subset $\mathcal{G} \subset C_0((0,1])$, there exist an element $e \in A_+^{\mathbf{1}}$ and a nonzero C^*-subalgebra B of A such that B in $\mathcal{P}$, and the following hold:*

(1) $\|ex - xe\| < \varepsilon$ for all $x \in \mathcal{F}$.
(2) $g(1-e)x \in_\varepsilon B$ for all $g \in \mathcal{G}$ and $\|(1-e)x\| \ge \|x\| - \varepsilon$ for all $x \in \mathcal{F}$. and
(3) $e \lesssim a$.

Proof. The 'if' part follows easily by taking $\mathcal{G} = \{\iota\}$, where $\iota(t) = t$ for all $t \in [0,1]$.

We now show the 'only if' part.

Suppose that A is e. tracially in $\mathcal{P}$. Let $\varepsilon > 0$ and let $\mathcal{F} \subset A$ be a finite subset, and without loss of generality we may assume that $\mathcal{F} \subset A^{\mathbf{1}}$. Moreover, without loss of generality (omitting an error within $\varepsilon/16$, say), we may further assume that there is $e_A \in A_+^{\mathbf{1}}$ such that

$$e_A x = x = x e_A \text{ for all } x \in \mathcal{F}. \tag{e3.1}$$

Set $a \in A_+ \setminus \{0\}$, let $\varepsilon > 0$ and let $\mathcal{G} = \{g_1, g_2, \ldots, g_n\} \subset C_0((0,1])$ be a finite subset.

By the Weierstrass theorem, there are $m \in \mathbb{N}$ and polynomials $p_i(t) = \sum_{k=1}^m \beta_k^{(i)} t^k$ such that

$$|p_i(t) - g_i(t)| < \varepsilon/4 \text{ for all } t \in [0,1] \text{ and all } i \in \{1, 2, \ldots, n\}. \tag{e3.2}$$

Let $M = 1 + \max\left\{\left|\beta_k^{(i)}\right| : i = 1, 2, \ldots, n,\ k = 1, 2, \ldots, m\right\}$ and $\delta := \frac{\varepsilon}{32 m^3 M}$.

Now, since A is e. tracially in $\mathcal{P}$, there exist an element $e \in A_+^{\mathbf{1}}$ and a nonzero C^*-subalgebra $B \subset A$ such that B in $\mathcal{P}$, and the following hold:

(1) $\|ex - xe\| < \delta$ for all $x \in \mathcal{F} \cup \{e_A\}$.
(2′) $(1-e)x \in_\delta B$ and $\|(1-e)x\| \ge \|x\| - \delta$ for all $x \in \mathcal{F} \cup \{e_A\}$.
(3) $e \lesssim a$.

It remains to show that $g_i(1-e)x \in_{\varepsilon/2} B$ for all $x \in \mathcal{F}$, $i = 1, 2, \dots, n$.

Claim: For all $x \in \mathcal{F}$ and all $k \in \{1, 2, \dots, m\}$, we have $(1-e)^k x \in_{\frac{\varepsilon}{16mM}} B$. In fact,

$$(1-e)^k x \overset{(e3.1)}{=} (1-e)^k e_A^{k-1} x \approx^{(1)}_{k^2\delta} \overbrace{(1-e)e_A(1-e)e_A \cdots (1-e)e_A}^{k-1}(1-e)x \in^{(2')}_{k\delta} B. \tag{e3.3}$$

Note that $2k^2\delta \le 2m^2\delta < \varepsilon/16mM$. The claim follows.

By formula (e3.2) and the claim, for $x \in \mathcal{F}$ and $i \in \{1, 2, \dots, n\}$ we have

$$g_i(1-e)x \approx_{\varepsilon/4} p_i(1-e)x = \sum_{k=1}^{m} \beta_k^{(i)} (1-e)^k x \in_{\varepsilon/4} B. \tag{e3.4}$$

□

Remark 3.3.

(1) A similar notion as in Definition 3.1 could also be defined for nonsimple C^*-algebras. However, in the present paper we are interested in only the simple case.

(2) Note that in Proposition 3.2, $g(1-e)$ is an element in $\widetilde{A}$. But $g(1-e)x \in A$. In the case that A is unital, the condition $\|(1-e)x\| \ge \|x\| - \varepsilon$ for all $x \in \mathcal{F}$ in condition (2) of the definition 3.1 is redundant for most cases (we leave the discussion to [22]).

(3) The notion of tracial approximation was first introduced in [29] (see also [30]). Let $\mathcal{P}$ be a class of unital C^*-algebras – for example, the class of C^*-algebras which are isomorphic to C^*-algebras of the form $C([0,1], F)$, where F are finite-dimensional C^*-algebras. If, in Definition 3.1, $1-e$ can be chosen to be the unit of $B (\in \mathcal{P})$, then A is TAI or A has tracial rank at most 1 [30, 32]. In general, if A is unital simple and is TA$\mathcal{P}$ (see [14, Definition 2.2] and [18]), then A is e. tracially in $\mathcal{P}$. The difference is that we allow e to be a positive element rather than a projection.

To see this, let A be a unital simple C^*-algebra which is TA$\mathcal{P}$. Fix a finite subset $\mathcal{F} \subset A$ that contains 1_A. Fix $\varepsilon > 0$ and $a \in A_+ \backslash \{0\}$. By a well-known result due to Blackadar (see, for example, [3, II.8.5.6]), there is a unital separable simple C^*-subalgebra $C \subset A$ such that $\mathcal{F} \subset C$. Let $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ be an increasing sequence of finite subsets of C whose union is dense in C, and $\mathcal{F} \subset \mathcal{F}_1$. Since A is TA$\mathcal{P}$ in the sense of [14, Definition 2.2], there are nonzero projections $p_n \in A$ and C^*-algebras $B_n \subset A$ with B_n in $\mathcal{P}$, and p_n is the unit of B_n $(n \in \mathbb{N})$, which satisfies

(i) $\|p_n x - x p_n\| < \varepsilon/2n$ for all $x \in \mathcal{F}_n$,
(ii) $p_n x p_n \in_{\varepsilon/2n} B_n$ for all $x \in \mathcal{F}_n$ and
(iii) $1 - p_n \lesssim a$.

Assume that for each $n \in \mathbb{N}$, there is some $x \in \mathcal{F}$ such that $\|p_n x p_n\| \le \|p_n x\| < \|x\| - \varepsilon$. Then since $\mathcal{F}$ is a finite set, we can find $x_0 \in \mathcal{F}$ and an increasing sequence of natural numbers $\{n_m\}_{m\in\mathbb{N}}$ such that $\left\|p_{n_m} x_0 p_{n_m}\right\| < \|x_0\| - \varepsilon$ for all $m \in \mathbb{N}$. Define a c.p.c. linear map $\varphi : C \to l^\infty(A)/c_0(A)$ by $\varphi(x) := \pi\left(\{p_{n_1} x p_{n_1}, p_{n_2} x p_{n_2}, \dots\}\right)$, where $x \in C$ and $\pi : l^\infty(A) \to l^\infty(A)/c_0(A)$ is the quotient map. By condition (i) we see that φ is a homomorphism. Since $\varphi(1_A) = \pi\left(\{p_{n_1}, p_{n_2}, \dots\}\right) \ne 0$, φ is nonzero. Since C is simple, φ is injective and hence isometric. However, $\|\varphi(x_0)\| = \left\|\pi\left(\{p_{n_1} x_0, p_{n_2} x_0, \dots\}\right)\right\| \le \sup_{m\in\mathbb{N}} \left\|p_{n_m} x_0 p_{n_m}\right\| \le \|x_0\| - \varepsilon$: a contradiction. Therefore, there is $n_0 \in \mathbb{N}$ such that $\left\|p_{n_0} x\right\| \ge \|x\| - \varepsilon$ for all $x \in \mathcal{F}$. Set $e := 1_A - p_{n_0}$; then by (i)–(iii) and the choice of n_0, we have

(1′) $\|ex - xe\| < \varepsilon$ for all $x \in \mathcal{F}$,
(2′) $(1-e)x \in_\varepsilon B_{n_0}$ and $\|(1-e)x\| \ge \|x\| - \varepsilon$ for all $x \in \mathcal{F}$ and
(3′) $e \lesssim a$.

Hence A is e. tracially in $\mathcal{P}$.

We note also that in general, a C^*-algebra that is essentially tracially in $\mathcal{P}$ may not be TA$\mathcal{P}$ (see Remark 8.5).

(4) The current definition is also related to the notion of a 'centrally large subalgebra' ([37, Definition 4.1] and [1, Definition 2.1]) but not the same. The main difference is that the C^*-subalgebra B in [1, Definition 2.1] is fixed. In fact, for a simple unital C^*-algebra A and a class of C^*-algebras $\mathcal{P}$, if A has a centrally large subalgebra B with $B \in \mathcal{P}$, then A is essentially tracially in $\mathcal{P}$. On the other hand, in general, if A is essentially tracially in $\mathcal{P}$, one may not find a centrally large C^*-subalgebra B which is in $\mathcal{P}$ (for example, if $\mathcal{P}$ is the class of finite-dimensional C^*-algebras, then every unital infinite-dimensional simple AF-algebra is e. tracially in $\mathcal{P}$, but may not have centrally large finite-dimensional C^*-subalgebras [37, Theorem 6.8]).

(5) In [21], a notion of asymptotically tracial approximation is introduced, studying tracial approximation of certain properties which are closely related to weakly stable relations. It also mainly studies unital simple C^*-algebras with a rich structure of projections. This is different from Definition 3.1. However, if A is a unital (infinite-dimensional) simple C^*-algebra which is asymptotically tracially in the class $\mathcal{C}$ of 1-dimensional noncommutative CW complexes, then one can show that A is also essentially tracially in the same class $\mathcal{C}$. Moreover, many classes $\mathcal{P}$ of C^*-algebras are preserved by asymptotically tracial approximation [21, Section 4]. Some more discussion may be found in a forthcoming paper [22].

Definition 3.4. Let $\mathcal{P}$ be a class of C^*-algebras. The class $\mathcal{P}$ is said to have property (H) if, for any nonzero A in $\mathcal{P}$ and any nonzero hereditary C^*-subalgebra $B \subset A$, B is also in $\mathcal{P}$.

Proposition 3.5. *Let $\mathcal{P}$ be a class of C^*-algebras which has property (H). Suppose that A is a simple C^*-algebra which is e. tracially in $\mathcal{P}$. Then every nonzero hereditary C^*-subalgebra $B \subset A$ is also e. tracially in $\mathcal{P}$.*

Proof. Assume $\mathcal{P}$ has property (H) and A is e. tracially in $\mathcal{P}$. Let $B \subset A$ be a nonzero hereditary C^*-subalgebra of A. Set $\mathcal{F} \subset B$ and $s \in B_+ \setminus \{0\}$, and $\varepsilon \in (0, 1/4)$.

Without loss of generality, we may assume that $\mathcal{F} \subset B_+^{\mathbf{1}}$. Let $d \in B_+^{\mathbf{1}}$ be such that $dx \approx_{\varepsilon/32} x \approx_{\varepsilon/32} xd$ and $x \approx_{\varepsilon/32} dxd$ for all $x \in \mathcal{F}$.

Put $\varepsilon_1 = \varepsilon/32$. By [15, Lemma 3.3], there is $\delta_1 \in (0, \varepsilon_1)$ such that for any C^*-algebra E and any $x, y \in E_+^{\mathbf{1}}$, if $x \approx_{\delta_1} y$, then there is an injective homomorphism $\psi : \mathrm{Her}_E\big(f_{\varepsilon_1/2}(x)\big) \to \mathrm{Her}_E(y)$ satisfying $z \approx_{\varepsilon_1} \psi(z)$ for all $z \in \mathrm{Her}_E\big(f_{\varepsilon_1/2}(x)\big)^{\mathbf{1}}$.

Note that there is $\delta_2 \in (0, \delta_1)$ such that for any C^*-algebra E and any $x, y \in E_+^{\mathbf{1}}$, if $xy \approx_{\delta_2} yx$, then $x^{1/4}y \approx_{\delta_1/2} yx^{1/4}$, $x^{1/8}y^{1/2} \approx_{\delta_1/2} y^{1/2}x^{1/8}$ and $x^{1/8}y \approx_{\delta_1/2} yx^{1/8}$.

Let $\delta = \delta_2/2$. Let $\mathcal{G} = \{t, t^{1/4}, t^{1/8}\} \subset C_0((0, 1])$. Since A is e. tracially in $\mathcal{P}$, by Proposition 3.2 there exist a positive element $a \in A_+^{\mathbf{1}}$ and a nonzero C^*-subalgebra $C \subset A$ which is in $\mathcal{P}$ such that

(1) $\|ax - xa\| < \delta$ for all $x \in \mathcal{F} \cup \{d, d^{1/2}, d^2\}$,
(2) $g(1 - a)x \in_\delta C$ for all $g \in \mathcal{G}$ and $\|(1 - a)x\| \ge \|x\| - \delta$ for all $x \in \mathcal{F} \cup \{d, d^{1/2}, d^2\}$ and
(3) $a \lesssim s$.

By (2), there is $c \in C$ such that $c \approx_{\delta_1/2} (1 - a)^{1/4}d$. By (1) and the choice of δ_2, we have $c \approx_{\delta_1} d^{1/2}(1 - a)^{1/4}d^{1/2}$. Then by [15, Lemma 3.3] and the choice of δ_1, there is a monomorphism

$$\varphi : \mathrm{Her}_A\big(f_{\varepsilon_1/2}(c)\big) \to \mathrm{Her}_A\Big(d^{1/2}(1 - a)^{1/4}d^{1/2}\Big) \subset B$$

satisfying $\|\varphi(x) - x\| < \varepsilon_1$ for all $x \in \mathrm{Her}_C\big(f_{\varepsilon_1/2}(c)\big)^{\mathbf{1}}$. Define $D := \varphi\big(\mathrm{Her}_C\big(f_{\varepsilon_1/2}(c)\big)\big) \subset B$. Since C is in $\mathcal{P}$ and $\mathcal{P}$ has property (H), $D \cong \mathrm{Her}_C\big(f_{\varepsilon_1/2}(c)\big)$ is in $\mathcal{P}$. Set $b := dad \in B_+^{\mathbf{1}}$. Then by (1) and the choice of d, we have

$$\|bx - xb\| = \|dadx - xdad\| \approx_{4\varepsilon_1} \|adxd - dxda\| \approx_{2\varepsilon_1} \|ax - xa\| < \delta \text{ for all } x \in \mathcal{F}. \tag{e3.5}$$

By (2), for any $x \in \mathcal{F}$ there is $\bar{x} \in C$ such that $(1-a)^{1/4}x(1-a)^{1/4} \approx_{2\varepsilon_1} \bar{x}$. Then

$$\begin{aligned}(1-b)x = (1-dad)x &\approx_{3\varepsilon_1} (1-a)dxd \\ &\approx_{4\varepsilon_1} (1-a)^{1/8}d(1-a)^{1/8} \cdot (1-a)^{1/4}x(1-a)^{1/4} \cdot (1-a)^{1/8}d(1-a)^{1/8} \\ &\approx_{4\varepsilon_1} c\bar{x}c \approx_{2\varepsilon_1} (c-\varepsilon_1)_+\bar{x}(c-\varepsilon_1)_+ \\ &\approx_{\varepsilon_1} \varphi((c-\varepsilon_1)_+\bar{x}(c-\varepsilon_1)_+) \in D.\end{aligned} \tag{e3.6}$$

In other words,

$$(1-b)x \in_{\varepsilon} D. \tag{e3.7}$$

Therefore, for all $x \in \mathcal{F}$,

$$\begin{aligned}\|(1-b)x\| = \|(1-dad)x\| &\geq \left\|\left(1-ad^2\right)x\right\| - \delta \\ &\geq \|(1-a)x\| - 3\varepsilon_1 \geq \|x\| - \delta - 3\varepsilon_1 \geq \|x\| - \varepsilon.\end{aligned} \tag{e3.8}$$

By (3), we have $b = dad \lesssim_A s$. Note that $b, s \in B$. Since B is a hereditary C^*-subalgebra, we have $b \lesssim_B s$. By formulas (e3.5) and (e3.7), we see that B is also e. tracially in $\mathcal{P}$. ◻

4. Basic properties

Notation 4.1. Let $\mathcal{W}$ be the class of C^*-algebras A such that $W(A)$ is almost unperforated.

Let $\mathcal{Z}$ be the Jiang–Su algebra [27]. A C^*-algebra A is called $\mathcal{Z}$-stable if $A \otimes \mathcal{Z} \cong A$. Let $\mathcal{C}_{\mathcal{Z}}$ be the class of separable $\mathcal{Z}$-stable C^*-algebras.

Lemma 4.2. *Let A be a simple C^*-algebra which is e. tracially in $\mathcal{W}$, and set $a, b, c \in A_+ \backslash \{0\}$. Suppose that there exists $n \in \mathbb{N}$ satisfying $(n+1)\langle a\rangle \leq n\langle b\rangle$. Then for any $\varepsilon > 0$, there exist $a_1, a_2 \in A_+$ such that*

(1) $a \approx_{\epsilon} a_1 + a_2$,
(2) $a_1 \lesssim_A b$ *and*
(3) $a_2 \lesssim_A c$.

Proof. Without loss of generality, one may assume that $a, b, c \in A_+^{\mathbf{1}} \backslash \{0\}$ and $\epsilon < 1/2$. Then $(n+1)\langle a\rangle \leq n\langle b\rangle$ implies that there exists $r = \sum_{i,j=1}^{n+1} r_{i,j} \otimes e_{i,j} \in A \otimes M_{n+1}$ such that

$$a \otimes \sum_{i=1}^{n+1} e_{i,i} \approx_{\epsilon/128} r^* \left(b \otimes \sum_{i=1}^{n} e_{i,i}\right) r. \tag{e4.1}$$

Set $\mathcal{F} := \{a, b\} \cup \left\{r_{i,j}, r_{i,j}^* : i, j = 1, 2, \ldots, n+1\right\}$ and $M := 1 + \|r\|$. Let $\sigma = \frac{\varepsilon}{32M^2(n+1)^4}$. Since A is e. tracially in $\mathcal{W}$, by Proposition 3.2, for any $\delta \in \left(0, \frac{\varepsilon}{256M(n+1)^2}\right)$, there exist $f \in A_+^{\mathbf{1}} \setminus \{0\}$ and a C^*-subalgebra $B \subset A$ which has almost unperforated $W(B)$ such that

(1′) $\|fx - xf\| < \delta$ for $x \in \mathcal{F}$,
(2′) $(1-f)^{1/4}x, (1-f)^{1/2}a(1-f)^{1/2}, (1-f)^{1/4}x(1-f)^{1/4} \in_{\delta} B$ for all $x \in \mathcal{F}$ and
(3′) $f \lesssim c$.

Put $g = 1 - f$. Let

$$\mathcal{G} := \left\{g^{1/4}x, g^{1/2}xg^{1/2}, g^{1/4}xg^{1/4} : x \in \mathcal{F}\right\}.$$

Set $x \in \mathcal{G} \cap A_+$. By (2′), there is $\bar{x} \in B$ such that $\|x - \bar{x}\| < \delta$. Let $x' := (\bar{x} + \bar{x}^*)/2 \in B_{sa}$. Then $x \approx_\delta x'$. Then $x' + \delta \ge x \ge 0$, which implies $\|x'_-\| \le \delta$. Then $x \approx_\delta x' = x'_+ - x'_- \approx_\delta x'_+ \in B_+$. Therefore, there is a map $\alpha : \mathcal{G} \to B$ such that $\alpha(\mathcal{G} \cap A_+) \subset B_+$, and

$$x \approx_{2\delta} \alpha(x) \text{ for all } x \in \mathcal{G}. \tag{e4.2}$$

From (1′) and (2′), one can choose δ sufficiently small such that

$$a \approx_{\varepsilon/16} g^{1/2} a g^{1/2} + (1-g)^{1/2} a (1-g)^{1/2} \text{ and} \tag{e4.3}$$

$$\left(g^{1/2} a g^{1/2} - \varepsilon/8\right)_+ \approx_{\varepsilon/16} \left(\alpha\left(g^{1/2} a g^{1/2}\right) - \epsilon/8\right)_+. \tag{e4.4}$$

By (1′) and formula (e4.1) (with δ sufficiently small), one can also assume that

$$g^{1/2} a g^{1/2} \otimes \sum_{i=1}^{n+1} e_{i,i} \approx_{\varepsilon/64} R^*\left(g^{1/4} b g^{1/4} \otimes \sum_{i=1}^{n} e_{i,i}\right) R, \tag{e4.5}$$

where $R := \sum_{i,j=1}^{n+1} \left(g^{1/4} r_{i,j}\right) \otimes e_{i,j}$. By formulas (e4.5) and (e4.2) and $\delta < \frac{\varepsilon}{256M\,(n+1)^2}$, one has

$$\alpha\left(g^{1/2} a g^{1/2}\right) \otimes \sum_{i=1}^{n+1} e_{i,i} \approx_{\varepsilon/32} \bar{R}^*\left(\alpha\left(g^{1/4} b g^{1/4}\right) \otimes \sum_{i=1}^{n} e_{i,i}\right) \bar{R}, \tag{e4.6}$$

where $\bar{R} := \sum_{i,j=1}^{n+1} \alpha\left(g^{1/4} r_{i,j}\right) \otimes e_{i,j}$. Then by the choice of σ,

$$\alpha\left(g^{1/2} a g^{1/2}\right) \otimes \sum_{i=1}^{n+1} e_{i,i} \approx_{\varepsilon/16} \bar{R}^*\left(\left(\alpha\left(g^{1/4} b g^{1/4}\right) - \sigma\right)_+ \otimes \sum_{i=1}^{n} e_{i,i}\right) \bar{R}. \tag{e4.7}$$

By formula (e4.7) and [40, Proposition 2.2], one has

$$\left(\alpha\left(g^{1/2} a g^{1/2}\right) - \epsilon/8\right)_+ \otimes \sum_{i=1}^{n+1} e_{i,i} \lesssim \left(\alpha\left(g^{1/4} b g^{1/4}\right) - \sigma\right)_+ \otimes \sum_{i=1}^{n} e_{i,i}. \tag{e4.8}$$

Since $W(B)$ is almost unperforated, one obtains

$$\left(\alpha\left(g^{1/2} a g^{1/2}\right) - \epsilon/8\right)_+ \lesssim \left(\alpha\left(g^{1/4} b g^{1/4}\right) - \sigma\right)_+. \tag{e4.9}$$

By [40, Proposition 2.2] and formulas (e4.4), (e4.9) and (e4.2), it follows that

$$\left(g^{1/2} a g^{1/2} - \varepsilon/4\right)_+ \lesssim \left(\alpha\left(g^{1/2} a g^{1/2}\right) - \epsilon/8\right)_+ \tag{e4.10}$$

$$\lesssim \left(\alpha\left(g^{1/4} b g^{1/4}\right) - \sigma\right)_+ \lesssim g^{1/4} b g^{1/4} \lesssim b. \tag{e4.11}$$

By (1′) and the choice of δ,

$$a \approx_{\varepsilon/16} (1-f)^{1/2} a (1-f)^{1/2} + f^{1/2} a f^{1/2}. \tag{e4.12}$$

Choose

$$a_1 := \left(g^{1/2} a g^{1/2} - \varepsilon/2\right)_+ = \left((1-f)^{1/2} a (1-f)^{1/2} - \varepsilon/2\right)_+ \text{ and} \tag{e4.13}$$

$$a_2 := f^{1/2} a f^{1/2}. \tag{e4.14}$$

Then by formula (e4.11), one has $a_1 \lesssim_A b$. Note that (3′) implies $a_2 \lesssim_A c$. Thus a_1 and a_2 satisfy (2) and (3) of the lemma. By formula (e4.12),

$$a \approx_{\varepsilon/16} (1-f)^{1/2} a (1-f)^{1/2} + f^{1/2} a f^{1/2} \approx_{\epsilon/2} a_1 + a_2.$$

So (1) of the lemma also holds, and the lemma follows. □

Theorem 4.3. *Let A be a simple C^*-algebra which is e. tracially in $\mathcal{W}$ (see Notation 4.1). Then $A \in \mathcal{W}$.*

Proof. We may assume that A is nonelementary. Set $a, b \in M_m(A)_+ \setminus \{0\}$ with $\|a\| = 1 = \|b\|$ for some integer $m \geq 1$. Set $n \in \mathbb{N}$ and assume $(n+1)\langle a\rangle \leq n\langle b\rangle$. To prove the theorem, it suffices to prove that $a \lesssim b$.

Note that if $B \in \mathcal{W}$, then for each integer m, we have $M_m(B) \in \mathcal{W}$. It follows that $M_m(A)$ is e. tracially in $\mathcal{W}$. To simplify notation, without loss of generality one may assume $a, b \in A_+$.

By [21, Lemma 4.3], $\mathrm{Her}\big(f_{1/4}(b)\big)_+$ contains $2n+1$ nonzero mutually orthogonal elements $b_0, b_1, \ldots, b_{2n}$ such that $\langle b_i\rangle = \langle b_0\rangle$, $i = 1, 2, \ldots, 2n$. Without loss of generality, we may assume that $\|b_0\| = 1$. If b_0 is a projection, choose $e_0 = b_0$. Otherwise, by replacing b_0 by $g_1(b_0)$ for some continuous function $g_1 \in C_0((0,1])$, we may assume that there is a nonzero $e_0 \in A_+$ such that $b_0 e_0 = e_0 b_0 = e_0$. Replacing b by $g(b)$ for some $g \in C_0((0,1])$, one may assume that $bb_0 = b_0 b = b_0$. Put $c = b - b_0$. Note that

$$ce : 0 = (b - b_0)e_0 = be_0 - e_0 = b_0 e_0 - e_0 = 0 = e_0 c. \tag{e4.15}$$

Keep in mind that $b \geq c + e_0$, $c \perp e_0$ and $2n\langle b_0\rangle \leq \langle c\rangle = \langle b - b_0\rangle$. One has

$$(2n+2)\langle a\rangle \leq 2n\langle b\rangle \leq 2n(\langle b - b_0\rangle + \langle b_0\rangle) \leq 2n\langle c\rangle + \langle c\rangle = (2n+1)\langle c\rangle. \tag{e4.16}$$

By Lemma 4.2, for any $\varepsilon \in (0, 1/2)$ there exist $a_1, a_2 \in A_+$ such that

(i) $a \approx_{\epsilon/2} a_1 + a_2$,
(ii) $a_1 \lesssim_A c$ and
(iii) $a_2 \lesssim_A e_0$.

By (i)–(iii) and applying [40, Proposition 2.2] (recall $be_0 = e_0 b = e_0$), one obtains

$$(a - \varepsilon)_+ \lesssim a_1 + a_2 \lesssim c + e_0 \leq b. \tag{e4.17}$$

Since this holds for every $\varepsilon \in (0, 1/2)$, one concludes that $a \lesssim b$. □

Corollary 4.4. *Let A be a simple C^*-algebra which is e. tracially in $\mathcal{C}_{\mathcal{Z}}$. Then $W(A)$ is almost unperforated.*

Proof. It follows from [42, Theorem 4.5] and Theorem 4.3. □

Definition 4.5. Let A be a C^*-algebra. Let $\mathcal{T}$ denote the class of C^*-algebras A such that for every $a \in \mathrm{Ped}(A)_+ \setminus \{0\}$, every 2-quasitrace of $\overline{aAa}$ is a trace.

Set $A \in \mathcal{T}$ and let $B \subset A$ be a hereditary C^*-subalgebra. If $b \in \mathrm{Ped}(B)_+ \setminus \{0\}$, then $b \in \mathrm{Ped}(A)_+$ and $\overline{bBb} = \overline{bAb}$. It follows that every 2-quasitrace of $\overline{bBb}$ is a trace. Hence $\mathcal{T}$ has property (H).

Proposition 4.6. *Let A be a simple C^*-algebra which is e. tracially in $\mathcal{T}$. Then A is in $\mathcal{T}$.*

Proof. Fix $a \in \mathrm{Ped}(A)_+^{\mathbf{1}}$ and let $C = \mathrm{Her}(a)$. We will show that every 2-quasitrace of C is a trace. We may assume that C is nonelementary. Set $\tau \in QT(C)$. Fix $x, y \in C_{\mathrm{sa}}$ with $\|x\|, \|y\| \leq 1/2$. Set $\varepsilon \in (0, 1/2)$. Let $\mathcal{F} := \{x, y, x+y\}$. Let $n \in \mathbb{N}$ be such that $\varepsilon > 1/n$. By [21, Lemma 4.3], there exist mutually orthogonal norm 1 positive elements $c_1, c_2, \ldots, c_n \in A_+ \setminus \{0\}$ such that $c_1 \sim c_2 \sim \cdots \sim c_n$. Then $d_\tau(c_1) \leq 1/n < \varepsilon$.

Let $\delta \in (0, \varepsilon)$ be such that for any $d \in C_+^{\mathbf{1}}$ and $z \in C_{\mathrm{sa}}^{\mathbf{1}}$, if $\|[d, z]\| < \delta$, then

$$z \approx_\varepsilon (1-d)^{1/2} z (1-d)^{1/2} + d^{1/2} z d^{1/2} \tag{e4.18}$$

and

$$\tau(z) \approx_\varepsilon \tau\Big((1-d)^{1/2} z (1-d)^{1/2}\Big) + \tau\Big(d^{1/2} z d^{1/2}\Big). \tag{e4.19}$$

(Note [4, II.2.6] that $\big\|\big[(1-d)^{1/2} z (1-d)^{1/2}, d^{1/2} z d^{1/2}\big]\big\|$ can be sufficiently small depending on δ.) Note that $\mathcal{T}$ has property (H). Since A is simple and e. tracially in $\mathcal{T}$, by Proposition 3.5 C is also e. tracially in $\mathcal{T}$. There exist an element $e \in C_+^{\mathbf{1}}$ and a nonzero C^*-subalgebra $B \subset C$ such that B is in $\mathcal{T}$, and the following are true:

(1) $\|ez - ze\| < \delta$ for all $z \in \mathcal{F}$.
(2) $(1-e)^{1/2} z (1-e)^{1/2} \in_{\delta/2} B$ for all $z \in \mathcal{F}$.
(3) $e \lesssim c_1$.

We may choose $e_B \in \mathrm{Ped}(B)_+^{\mathbf{1}}$ such that

(2′) $(1-e)^{1/2} z (1-e)^{1/2} \in_\delta B_1 := \overline{e_B B e_B}$ for all $z \in \mathcal{F}$.

Note that for $z \in \mathcal{F}$, $e^{1/2} z e^{1/2}$ is self-adjoint. One has $\big(e^{1/2} z e^{1/2}\big)_+, \big(e^{1/2} z e^{1/2}\big)_- \in \mathrm{Her}_A(e)$. Then

$$\Big|\tau\Big(e^{1/2} z e^{1/2}\Big)\Big| = \Big|\tau\Big(\Big(e^{1/2} z e^{1/2}\Big)_+\Big) - \tau\Big(\Big(e^{1/2} z e^{1/2}\Big)_-\Big)\Big| \tag{e4.20}$$

$$\leq d_\tau\Big(\Big(e^{1/2} z e^{1/2}\Big)_+\Big) + d_\tau\Big(\Big(e^{1/2} z e^{1/2}\Big)_-\Big) \leq 2 d_\tau(e) \leq 2\varepsilon. \tag{e4.21}$$

Then by (1), the choice of δ and formulas (e4.18) and (e4.19), for $z \in \mathcal{F}$,

$$\tau(z) \approx_{2\varepsilon} \tau\Big((1-e)^{1/2} z (1-e)^{1/2}\Big) + \tau\Big(e^{1/2} z e^{1/2}\Big) \tag{e4.22}$$

$$(\text{by formula (e4.21)}) \approx_{2\varepsilon} \tau\Big((1-e)^{1/2} z (1-e)^{1/2}\Big). \tag{e4.23}$$

By (2′), there are $\bar{x}, \bar{y} \in (B_1)_{\mathrm{sa}}$ such that

$$(1-e)^{1/2} x (1-e)^{1/2} \approx_{2\delta} \bar{x}, \qquad (1-e)^{1/2} y (1-e)^{1/2} \approx_{2\delta} \bar{y}. \tag{e4.24}$$

Then

$$\begin{aligned}
\tau(x+y) &\overset{\text{formula (e 4.23)}}{\approx_{4\varepsilon}} \tau\Big((1-e)^{1/2}(x+y)(1-e)^{1/2}\Big) \\
&\overset{\text{formula (e 4.24)}}{\approx_{4\delta}} \tau(\bar{x}+\bar{y}) \\
(\tau \text{ is a trace on } B_1) &= \tau(\bar{x}) + \tau(\bar{y}) \\
&\overset{\text{formula (e 4.24)}}{\approx_{4\delta}} \tau\Big((1-e)^{1/2} x (1-e)^{1/2}\Big) + \tau\Big((1-e)^{1/2} y (1-e)^{1/2}\Big) \\
&\overset{\text{formula (e 4.23)}}{\approx_{4\varepsilon}} \tau(x) + \tau(y).
\end{aligned}$$

Since ε and δ are arbitrary small, we have $\tau(x+y) = \tau(x) + \tau(y)$, and therefore τ is a trace on C. □

Definition 4.7. Let A be a C^*-algebra. Recall that an element $a \in \mathrm{Ped}(A)_+$ is said to be infinite if there are nonzero elements $b, c \in \mathrm{Ped}(A)_+$ such that $bc = cb = 0$, $b + c \lesssim c$ and $c \lesssim a$. A is said to be finite if every element $a \in \mathrm{Ped}(A)_+$ is not infinite (see, for example, [33, Definition 1.1]). A is stably finite if $M_n(A)$ is finite for every integer $n \geq 1$.

 https://doi.org/10.1017/fms.2021.79

Recall that a simple C^*-algebra A is purely infinite if and only if every nonzero element in $\mathrm{Ped}(A)_+$ is infinite (see [33, Condition (vii), Theorem 2.2]). Let $\mathcal{PI}$ be the class of C^*-algebras such that every nonzero positive element in the Pedersen ideal is infinite.

Theorem 4.8. *Let A be a simple C^*-algebra which is e. tracially in $\mathcal{PI}$. Then A is purely infinite.*

Proof. Note that A has infinite dimension. Set $a \in \mathrm{Ped}(A)_+ \setminus \{0\}$ with $\|a\| = 1$.

Since $\overline{f_{1/4}(a)Af_{1/4}(a)}$ is an infinite-dimensional simple C^*-algebra, one may choose $c, d \in \overline{f_{1/4}(a)Af_{1/4}(a)}_+ \setminus \{0\}$ such that $cd = dc = 0$.

Since A is e. tracially in $\mathcal{PI}$, there exist a sequence of positive elements $e_n \in A_+$ with $\|e_n\| \le 1$ and a sequence of C^*-subalgebra $B_n \subset A$ such that B_n in $\mathcal{PI}$, and the following are true:

(1) $a \approx_{1/2^n} e_n^{1/2} a e_n^{1/2} + (1-e_n)^{1/2} a (1-e_n)^{1/2}$.
(2) $(1-e_n)^{1/2} a (1-e_n)^{1/2} \in_{1/2^n} B_n$ and $\|(1-e_n)^{1/2} a (1-e_n)^{1/2}\| \ge \|a\| - 1/2^n$.
(3) $e_n \lesssim c$.

By (2), there is $b_n \in B_{n+}$ such that $b_n \approx_{1/2^n} (1-e_n)^{1/2} a (1-e_n)^{1/2}$. Then by (1),

$$a \approx_{2/2^n} b_n + e_n^{1/2} a e_n^{1/2}. \tag{e4.25}$$

Note that $\inf_n\{\|b_n\|\} \ge \|a\|/2 > 0$. Choose $0 < \varepsilon < \|a\|/16$.

By [37, Lemma 1.7], for all sufficiently large n we have

$$0 \ne (b_n - 2\varepsilon)_+ \lesssim \left(b_n + e_n^{1/2} a e_n^{1/2} - 2\varepsilon\right)_+ \lesssim a. \tag{e4.26}$$

Note that $(b_n - 2\varepsilon)_+ \in \mathrm{Ped}(B_n)_+ \setminus \{0\}$. Then there are $d_1, d_2 \in \mathrm{Ped}(B_n)_+ \setminus \{0\}$ such that $d_1 \perp d_2$, $d_1 + d_2 \lesssim d_2 \lesssim (b_n - 2\varepsilon)_+$ and

$$d_1 + d_2 \lesssim (b_n - 2\varepsilon)_+ \lesssim a. \tag{e4.27}$$

It follows that a is infinite, and therefore A is purely infinite. □

Proposition 4.9 ([42, Corollary 5.1].). *Let A be a σ-unital simple C^*-algebrasuch that $W(A)$ is almost unperforated. If A is not purely infinite, then $\overline{aAa}$ has a nonzero 2-quasitrace for every $a \in \mathrm{Ped}(A)_+ \setminus \{0\}$. Consequently, A is stably finite.*

Proof. This is a theorem of Rørdam [42, Corollary 5.1]. Since we do not assume that A is exact and will use only 2-quasitraces, some more explanation is in order. The explanation, of course, follows exactly the same lines as the proof of [42, Corollary 5.1].

Set $a \in \mathrm{Ped}(A)_+^{\mathbf{1}}$ and $B := \overline{aAa}$. Then B is algebraically simple (see, for example, [3, II.5.4.2]). Assume that B has no nonzero 2-quasitraces.

Consider $W(B)$. Note that $W(B) \subset W(A)$, and $W(B)$ has the property that if $x \in W(B)$ and $y \in W(A)$ such that $y \le x$, then $y \in W(B)$. It follows that $W(B)$ is almost unperforated. Since B is algebraically simple, every element in $W(B)$ is a strong order unit.

Set $t, t' \in W(B)$ (with t a strong order unit). The statement (and the proof) of [40, Proposition 3.1] imply that if there is no state on $W(B)$ (with the strong order unit t), then there must be some integer $n \in \mathbb{N}$ and $u \in W(B)$ such that

$$nt' + u \le nt + u. \tag{e4.28}$$

Then by [40, Proposition 3.2] (see the proof also), as $W(B)$ is almost unperforated,

$$t' \le t. \tag{e4.29}$$

On the other hand, by [4, II.2.2], every lower semicontinuous dimension function on $W(B)$ is induced by a 2-quasitrace on B. Since B is assumed to have no nonzero 2-quasitraces, combining with [40, Proposition 4.1] (as well as the paragraph before it) shows that there is no state on $W(B)$. Therefore formula (e4.29) implies that for any $b, c \in B_+ \setminus \{0\}$, we have $b \lesssim c$. It follows that B is purely infinite and so is A.

To see the last part of the statement, suppose that there are $b, c \in \mathrm{Ped}(A)_+^{\mathbf{1}} \setminus \{0\}$ such that $bc = cb = 0$ and $b + c \lesssim c$. Let $a = b + c$ and $B = \overline{aAa}$. Note that $a \in \mathrm{Ped}(A)_+$. Then B has nonzero 2-quasitraces.

Therefore

$$d_\tau(c) \geq d_\tau(b+c) \text{ for all } \tau \in QT(B). \tag{e4.30}$$

On the other hand, for any $\tau \in QT(B)$ and any $1 > \varepsilon > 0$,

$$\tau(f_\varepsilon(b+c)) = \tau(f_\varepsilon(b) + f_\varepsilon(c)) = \tau(f_\varepsilon(b)) + \tau(f_\varepsilon(c)). \tag{e4.31}$$

Fix $1 > \varepsilon_0 > 0$ such that $f_{\varepsilon_0}(b) \neq 0$. Since B is algebraically simple, $\tau\big(f_{\varepsilon_0}(b)\big) > 0$ for all 2-quasitraces τ. Fix $\tau \in QT(B)$. Then, by equation (e4.31),

$$d_\tau(b+c) \geq \tau\big(f_{\varepsilon_0}(b)\big) + d_\tau(c) > d_\tau(c). \tag{e4.32}$$

This contradicts formula (e4.30). It follows that no such pairs b and c exist. Thus A is finite.

Since $M_n(A)$ has the same relevant property as A, we conclude that A is stably finite. □

Corollary 4.10. *Let A be a σ-unital simple C^*-algebrasuch that A is e. tracially in $\mathcal{W}$. Then A has strict comparison.*

Proof. By Theorem 4.3, $W(A)$ is almost unperforated. It follows from Remark 2.5 that $\mathrm{Cu}(A)$ is almost unperforated. Fix $e \in \mathrm{Ped}(A)_+ \setminus \{0\}$ and let $B := \mathrm{Her}(e)$. As in the proof of Proposition 4.9, every lower semicontinuous dimension function on $W(B)$ is induced by a 2-quasitrace of B. Set $a, b \in (A \otimes \mathcal{K})_+$ such that $d_\tau(a) < d_\tau(b)$ for all $\tau \in QT(B)$. By [17, Propositions 4.2, 4.6], $a \lesssim b$. □

5. Essentially tracially $\mathcal{Z}$-stable C^*-algebras

Recall from Notation 4.1 that $\mathcal{C}_{\mathcal{Z}}$ is the class of separable $\mathcal{Z}$-stable C^*-algebras.

Theorem 5.1. *Let A be a σ-unital simple C^*-algebra which is e. tracially in $\mathcal{C}_{\mathcal{Z}}$. Then A is either purely infinite or stably finite. Moreover, if A is not purely infinite, then it has strict comparison for positive elements.*

Proof. It follows from [42, Theorem 4.5] that every C^*-algebra B in $\mathcal{C}_{\mathcal{Z}}$ has almost unperforated $W(B)$. It follows from Theorem 4.3 and Remark 2.5 that $\mathrm{Cu}(A)$ is almost unperforated. By Proposition 4.9, if A is not purely infinite, then it is stably finite, and by the proof of Corollary 4.10, A has strict comparison for positive elements. □

Definition 5.2. Let A be a simple C^*-algebra. A is said to be tracially approximately divisible if for any $\varepsilon > 0$, any $\mathcal{F} = \{x_1, x_2, \ldots, x_m\} \subset A$, any element $e_F \in A_+^{\mathbf{1}}$ with $e_F y_i = y_i = y_i e_F$ for some $y_i \approx_{\varepsilon/4} x_i$, $1 \leq i \leq m$, any $s \in A_+ \setminus \{0\}$, and any integer $n \geq 1$, there are $\theta \in A_+^{\mathbf{1}}$, a C^*-subalgebra $D \otimes M_n \subset A$ and a c.p.c. map $\beta : A \to A$ such that the following are true:

(1) $x \approx_\varepsilon x' + \beta(x)$ for all $x \in \mathcal{F}$, where $\|x'\| \leq \|x\|$, $x' \in \mathrm{Her}(\theta)$.
(2) $\beta(x) \in_\varepsilon D \otimes 1_n$ and $e_F\beta(x) \approx_\varepsilon \beta(x) \approx_\varepsilon \beta(x)e_F$ for all $x \in \mathcal{F}$.
(3) $\theta \lesssim s$.

The notion of approximate divisibility for C^*-algebras was introduced in [6]. The term 'tracially approximate divisibility' appeared in [32] (for special cases, see [32, Definition 5.3, proof of Theorem 5.4], [29, Lemma 6.10], [15, Definition 10.1]).

(1) If A is a unital separable simple C^*-algebra which is approximately divisible, then it is tracially approximately divisible. To see this, recall that by [6, Theorem 1.4(d)], A has strict comparison. Let $\varepsilon > 0$, a finite subset $\mathcal{F} \subset A$, $a \in A_+ \setminus \{0\}$ and $n \in \mathbb{N}$ be given. We assume that $1 \in \mathcal{F}$. Choose an integer m such that $d_\tau(a) \geq 1/m$ for all $\tau \in QT(A)$ (recall that A is a unital separable simple C^*-algebra and $QT(A)$ is a simplex, by [5, II.4.4]). Choose an integer $k > mn$. It follows from [6, Corollary 2.10] that we may assume that $\mathcal{F} \subset_{\varepsilon/2} \bigoplus_{i=1}^s A_n \otimes 1_{M_{k_i}} \subset \bigoplus_{i=1}^s A_n \otimes M_{k_i}$, where A_n is a C^*-subalgebra of A and $k_i \geq k$. Write $k_i = l_i mn + r_i$, where $l_i, r_i \in \mathbb{N}$ and $0 \leq r_i < mn$, $i = 1, 2, \ldots, s$. Note that $1 \in \bigoplus_{i=1}^s A_n \otimes M_{k_i}$. So each A_n is unital. In each M_{k_i}, find a projection e_i with rank $l_i mn$, $i = 1, 2, \ldots, s$. Put $e = \oplus_{i=1}^s 1_{A_n} \otimes e_i$ and $\theta = 1 - e$. We will identify $M_{l_i mn}$ with $M_{l_i m} \otimes M_n$. Then we have
   (i) $\theta x \approx_\varepsilon x\theta$ for all $x \in \mathcal{F}$,
   (ii) $(1-\theta)x(1-\theta) = exe \in_\varepsilon \bigoplus_{i=1}^s A_n \otimes e_i \subset \bigoplus_{i=1}^s (A_n \otimes M_{l_i m}) \otimes 1_n$ and
   (iii) $\theta \lesssim a$, as $d_\tau(1-e) < 1/mn < d_\tau(a)$ for all $\tau \in QT(A)$.
   From this we conclude that A is tracially approximately divisible (see also Proposition 5.3).
(2) Note that the Jiang–Su algebra $\mathcal{Z}$ is not approximately divisible, as it has no nontrivial projections. However, by Theorem 5.9, it is tracially approximately divisible.
(3) In a subsequent paper [20, Theorem 4.11], at least in the separable case, we show that the converse of Proposition 5.3 also holds. In fact, in [20, Lemma 4.9] we show that a weaker version of Definition 5.2, without mentioning e_F, implies the conditions stated in Proposition 5.3. In other words, in Definition 5.2, any reference to e_F could be omitted. However, the proof is somewhat more involved; we refer the reader to [20] for further discussion.
(4) There is also a notion called 'tracially almost divisibility' (see [48, Definition 3.5]). That definition uses quasitraces, whereas Definition 5.2 does not mention quasitraces. They are quite different. However, it is not hard to show that tracially approximate divisibility implies tracially almost divisibility. In [20], we show a separable simple C^*-algebra A which is tracially approximately divisible, has strict comparison and stable rank one and has a nice description of its Cuntz semigroup. These imply, in particular, that A has the tracially almost divisible property defined in [48, Definition 3.5]. The converse, in general, does not hold even with strict comparison – for example, $A = C^*_{\text{red}}(F_\infty)$ (see [20, 7.3]).

Proposition 5.3 (compare [32, 5.3]). *Suppose that A is a simple C^*-algebra which satisfies the following conditions: For any $\varepsilon > 0$, any finite subset $\mathcal{F} \subset A$, any $s \in A_+ \setminus \{0\}$ and any integer $n \geq 1$, there are $\theta \in A_+^{\mathbf{1}}$ and a C^*-subalgebra $D \otimes M_n \subset A$ such that*

(i) *$\theta x \approx_\varepsilon x\theta$ for all $x \in \mathcal{F}$,*
(ii) *$(1-\theta)x \in_\varepsilon D \otimes 1_n$ for all $x \in \mathcal{F}$ and*
(iii) *$\theta \lesssim s$.*

Then A is tracially approximately divisible.

Proof. Let $\mathcal{F} \subset A$ a finite subset, $\varepsilon > 0$, $s \in A_+ \setminus \{0\}$ and an integer n be given. Suppose that there are a finite subset $\mathcal{F}'$ and an element $e_F \in A_+^{\mathbf{1}}$ such that $e_F y = y = y e_F$ for all $y \in \mathcal{F}'$, and if $x \in \mathcal{F}$, there is $y \in \mathcal{F}'$ such that $\|y - x\| < \varepsilon/4$. Without loss of generality, we may assume that $\mathcal{F} \subset A^{\mathbf{1}}$. We may further assume that $\mathcal{F}' \subset A^{\mathbf{1}}$.

Let $\delta \in (0, \varepsilon/8)$ be a positive number such that for any elements $z \in A^{\mathbf{1}}$ and $w \in A_+^{\mathbf{1}}$, $\|zw - wz\| < \delta$ implies that

$$\left\|(1-w)^{1/2}z - z(1-w)^{1/2}\right\| < \varepsilon/8. \tag{e5.1}$$

Put $\mathcal{F}_1 = \mathcal{F} \cup \{e_F\} \cup \mathcal{F}'$. Suppose that there are $\theta \in A_+^{\mathbf{1}}$ and D as in the statement of the proposition, such that (i), (ii) and (iii) hold for δ (in place of ε) and $\mathcal{F}_1$ (in place of $\mathcal{F}$).

Then in Definition 5.2(3) holds.

Define $\beta : A \to A$ by $\beta(a) := (1-\theta)^{1/2}a(1-\theta)^{1/2}$ for all $a \in A$. It is a c.p.c. map. For each $x \in \mathcal{F}_1$, define $x_1 := \theta^{1/2}x\theta^{1/2} \in \mathrm{Her}(\theta)$. Then $\|x_1\| \le \|x\|$. Note that by the choice of δ, for all $x \in \mathcal{F} \cup \mathcal{F}'$,

$$e_F\beta(x) = e_F(1-\theta)^{1/2}x(1-\theta)^{1/2} \approx_{\varepsilon/8} (1-\theta)^{1/2}e_F x(1-\theta)^{1/2} \approx_{\varepsilon/8} \beta(x) \approx_{\varepsilon/4} \beta(x)e_F. \quad \text{(e5.2)}$$

Moreover, for all $x \in \mathcal{F}_1$,

$$\beta(x) = (1-\theta)^{1/2}x(1-\theta)^{1/2} \approx_{\varepsilon/8} (1-\theta)x \in_\delta D \otimes 1_n. \quad \text{(e5.3)}$$

So Definition 5.2(2) holds. Also by the choice of δ, for all $x \in \mathcal{F}_1$,

$$x = \theta x + (1-\theta)x \approx_{\varepsilon/4} \theta^{1/2}x\theta^{1/2} + (1-\theta)^{1/2}x(1-\theta)^{1/2} = x_1 + \beta(x). \quad \text{(e5.4)}$$

Hence Definition 5.2(1) holds. Thus A is tracially approximately divisible. □

The following lemma is convenient folklore:

Lemma 5.4. *Let $\delta > 0$. There is an integer $N(\delta) \ge 1$ such that for any C^*-algebra A, any $e \in A_+^{\mathbf{1}}$ and any $x \in A$, if $x^*x \le e$ and $xx^* \le e$, then*

$$e^{1/n}x \approx_\delta x \approx_\delta xe^{1/n} \text{ for all } n \ge N(\delta). \quad \text{(e5.5)}$$

Proof. Let $\delta > 0$ be given. Choose $N(\delta) \ge 1$ such that

$$\max\left\{\left|\left(1-t^{1/n}\right)^2 t\right| : t \in [0,1]\right\} < \delta^2 \text{ for all } n \ge N(\delta). \quad \text{(e5.6)}$$

Then for any C^*-algebra A, any $e \in A_+^{\mathbf{1}}$ and any $x \in A$ satisfying $x^*x \le e$ and $xx^* \le e$,

$$\left\|\left(1-e^{1/n}\right)x\right\| = \left\|\left(1-e^{1/n}\right)xx^*\left(1-e^{1/n}\right)\right\|^{1/2} \le \left\|\left(1-e^{1/n}\right)e\left(1-e^{1/n}\right)\right\|^{1/2} < \delta \quad \text{(e5.7)}$$

for all $n \ge N(\delta)$. Similarly, we also have $\left\|x\left(1-e^{1/n}\right)\right\| < \delta$ for all $n \ge N(\delta)$. The lemma follows. □

Theorem 5.5. *If A is a simple C^*-algebra which is tracially approximately divisible, then every hereditary C^*-subalgebra of A is also tracially approximately divisible.*

Proof. Let B be a hereditary C^*-subalgebra of A, $\mathcal{F} \subset B^{\mathbf{1}}$ be a finite subset, $\varepsilon > 0$, $s \in B_+ \setminus \{0\}$ be a positive element and $n \ge 1$ be an integer. Suppose also that there exists a finite subset $\mathcal{F}' \subset B^{\mathbf{1}}$ such that $y \in_{\varepsilon/4} \mathcal{F}'$ for all $y \in \mathcal{F}$, and there exists an element $e_F \in B_+^{\mathbf{1}}$ such that $e_F x = x = xe_F$ for all $x \in \mathcal{F}'$. Let $g_0, g_1 \in C_0((0,1])$ be such that $0 \le g_0, g_1 \le 1$, $g_0(0) = 0$, $g_0(t) = 1$ for $t \in [1-\varepsilon/64, 1]$ and g_0 is linear on $[0, 1-\varepsilon/64]$; and $g_1(t) = 0$ if $t \in [0, 1-\varepsilon/64]$, $g_1(1) = 1$ and g_1 is linear on $[1-\varepsilon/64, 1]$. Put $b_0 := g_0(e_F)$ and $b_1 := g_1(e_F)$. Then

$$b_0b_1 = b_1 = b_1b_0, \qquad b_0 \ge e_F, \qquad \|b_0 - e_F\| < \varepsilon/64. \quad \text{(e5.8)}$$

Since for all $x \in \mathcal{F}'$ we have $e_F xx^* = xx^* = xx^*e_F$ and $e_F x^*x = x^*x = x^*xe_F$, by the spectral theory, we have $b_ixx^* = xx^* = xx^*b_i$ and $b_ix^*x = x^*x = x^*xb_i$, $i = 0, 1$. It follows that

$$b_ix = x = xb_i \text{ and } b_ix^* = x^* = x^*b_i \text{ for all } x \in \mathcal{F}', \quad i = 0, 1. \quad \text{(e5.9)}$$

Let $\mathcal{F}_1 = \{b_1\} \cup \mathcal{F}'$. Choose $\delta > 0$ in [15, Lemma 3.3] associated with $\varepsilon/64$ (in place of ε) and $\sigma = \varepsilon/64$. Set $\eta = \min\{\delta/4, \varepsilon/256\}$.

We choose $N := N(\eta) \ge 1$ as in Lemma 5.4.

Let $0 < \delta_1 < \eta/2$. Moreover, we choose δ_1 sufficiently small that if $C_1 \subset C_2$ is any pair of C^*-algebras and $c \in C_2$ with $0 \le c \le 1$ and $c \in_{\delta_1} C_1$, and if $0 \le c_1, c_2 \le 1$ and $c_1c_2 \approx_{\delta_1} c_2 \approx_{\delta_1} c_2c_1$, then

$$c^{1/N} \in_\eta (C_1)_+^{\mathbf{1}} \qquad \text{and} \qquad c_1c_2^{1/N}c_1 \approx_\eta c_2^{1/N}. \tag{e5.10}$$

Since A is tracially approximately divisible, there are $\theta_a \in A_+^{\mathbf{1}}$, a C^*-subalgebra $D_a \otimes M_n \subset A$ and a c.p.c. map $\beta : A \to A$ such that

(1) $x \approx_{\delta_1/2} x_1 + \beta(x)$ such that $\|x_1\| \le 1$ and $x_1 \in \mathrm{Her}(\theta_a)$ for all $x \in \mathcal{F}_1$,
(2) $\beta(x) \in_{\delta_1/2} D_a \otimes 1_n$ and $b_0\beta(x) \approx_{\delta_1/2} \beta(x) \approx_{\delta_1/2} \beta(x)b_0$ for all $x \in \mathcal{F}_1$ and
(3) $\theta_a \lesssim s$.

Choose $d(x) \in (D_a \otimes 1_n)^{\mathbf{1}}$ such that

$$\|\beta(x) - d(x)\| < \delta_1 \text{ for all } x \in \mathcal{F}_1. \tag{e5.11}$$

Let $b_2 = \beta(b_1)^{1/N}$. By equation (e5.9), $\beta(b_1) \ge \beta(x)^*\beta(x)$ and $\beta(b_1) \ge \beta(x)\beta(x)^*$ for all $x \in \mathcal{F}'$ (see, for example, [5, Corollary 4.1.3]). By condition (2) here, the choice of N and application of Lemma 5.4,

$$b_2\beta(x) = \beta(b_1)^{1/N}\beta(x) \approx_\eta \beta(x) \qquad \text{for all } x \in \mathcal{F}'. \tag{e5.12}$$

Recall that $\beta(b_1) \in_{\delta_1} D_a \otimes 1_n$. By the choice of δ_1, we may choose $d \in (D_a \otimes 1_n)_+$ such that

$$\|d - b_2\| < \eta. \tag{e5.13}$$

Then, with $b := b_0b_2b_0$, by the second part of formula (e5.10),

$$\|d - b\| < 2\eta \qquad \text{and} \qquad f_{\varepsilon/64}(d)d \approx_{\varepsilon/64} d \approx_{2\eta} b. \tag{e5.14}$$

By the choice of η, applying [15, Lemma 3.3] yields an isomorphism

$$\varphi : \overline{f_{\varepsilon/64}(d)(D_a \otimes M_n)f_{\varepsilon/64}(d)} \to \overline{bAb} \subset B$$

such that

$$\|\varphi(y) - y\| < \varepsilon/64\|y\| \text{ for all } y \in \overline{f_{\varepsilon/64}(d)(D \otimes 1_n)f_{\varepsilon/64}(d)}. \tag{e5.15}$$

Note that $\overline{f_{\varepsilon/64}(d)(D_a \otimes M_n)f_{\varepsilon/64}(d)} \cong D_1 \otimes M_n$ and $\overline{f_{\varepsilon/64}(d)(D_a \otimes 1_n)f_{\varepsilon/64}(d)} \cong D_1 \otimes 1_n$ for some C^*-subalgebra $D_1 \subset D_a$. Let $D_b = \varphi(D_1)$. Define a c.p.c. map $\alpha : B \to B$ by

$$\alpha(y) := b\beta(y)b \text{ for all } y \in B. \tag{e5.16}$$

Then, for all $x \in \mathcal{F}_1$, by formulas (e5.14) and (e5.11),

$$\alpha(x) = b\beta(x)b \approx_{2(2\eta+\varepsilon/64)} f_{\varepsilon/64}(d)d\beta(x)df_{\varepsilon/64}(d) \tag{e5.17}$$
$$\approx_{\delta_1} f_{\varepsilon/64}(d)dd(x)df_{\varepsilon/64}(d) \in_{\varepsilon/64} D_b \otimes 1_n \subset \overline{bAb} \subset B. \tag{e5.18}$$

If $y \in \mathcal{F}$, choose $x \in \mathcal{F}'$ such that $\|y - x\| < \varepsilon/4$. Then $\alpha(y) \approx_{\varepsilon/4} \alpha(x) \in_{\varepsilon/4} D_b \otimes 1_n$. Define $y_1 = b_0x_1b_0$. Then, by conditions (1) and (2) and equation (e5.12),

$$y \approx_{\delta_1/2+\varepsilon/4} b_0(x_1 + \beta(x))b_0 = y_1 + b_0\beta(x)b_0 \tag{e5.19}$$
$$\approx_{2\eta} y_1 + b_0b_2\beta(x)b_2b_0 \tag{e5.20}$$
$$\approx_{2\delta_1} y_1 + b_0b_2b_0\beta(x)b_0b_2b_0 = y_1 + \alpha(x) \tag{e5.21}$$
$$\approx_{\varepsilon/4} y_1 + \alpha(y) \quad \text{for all } y \in \mathcal{F}. \tag{e5.22}$$

Note that $\delta_1/2 + \varepsilon/4 + 2\eta + 2\delta_1 + \varepsilon/4 < \varepsilon$. Put $\delta_2 := 3\delta_1/2 + 2\eta$. Then $0 < \delta_2 < 5\varepsilon/256$. Also for all $y \in \mathcal{F}$,

$$e_F\alpha(y) \approx_{\varepsilon/4} e_F\alpha(x) = e_F b_0 b_2 b_0 \beta(x) b_0 b_2 b_0 \approx_{\delta_2} e_F\beta(x) \approx_{\varepsilon/64} b_0\beta(x) \approx_{\delta_1} \beta(x) \tag{e5.23}$$

$$\approx_{\delta_1} \beta(x)b_0 \approx_{\varepsilon/64} \beta(x)e_F \approx_{\delta_2+\varepsilon/4} \alpha(y)e_F \tag{e5.24}$$

(recall $\|b_0 - e_F\| < \varepsilon/64$). Put $\theta_b = b_0\theta_a b_0$. Then $y_1 \in \overline{\theta_b B \theta_b}$. Moreover,

$$\theta_b \lesssim \theta_a \lesssim s. \tag{e5.25}$$

From formulas (e5.22), (e5.18), (e5.24) and (e5.25), the theorem follows. □

Lemma 5.6. *Let A be a C^*-algebra and set $n \in \mathbb{N}$. Let $e_1, \ldots, e_n \in A_+$ be mutually orthogonal nonzero positive elements. Assume $d_1, \ldots, d_n \in A_+$ such that $d_i \lesssim e_i$ $(i = 1, \ldots, n)$, and $e_i d_j = 0$ whenever $i \le j$ and $i, j = 1, \ldots, n$. Then for any $a \in \overline{d_1Ad_1 + \cdots + d_nAd_n}$ and any $\varepsilon > 0$, there are nilpotent elements $x, y \in A$ such that $\|a - yx\| < \varepsilon$.*

Proof. Set $a \in \overline{d_1Ad_1 + \cdots + d_nAd_n}$ and fix $\varepsilon > 0$. Then there exist $a_1, \ldots, a_n \in A$ and $\delta > 0$ such that $a \approx_\varepsilon f_\delta(d_1)a_1 f_\delta(d_1) + \cdots + f_\delta(d_n)a_n f_\delta(d_n)$. Set $x_1, \ldots, x_n \in A$ such that $x_i^* x_i = f_\delta(d_i)$ and $x_i x_i^* \in \overline{e_iAe_i}$, $i = 1, \ldots, n$ (see [40, Proposition 2.4]). For $i, j \in \{1, \ldots, n\}$ and $i \le j$, $e_i d_j = 0$ implies $x_j^* x_j x_i x_i^* = 0$, thus

$$x_j x_i = 0 \quad (i \le j). \tag{e5.26}$$

Claim 1: $(x_1 + x_2 + \cdots + x_n)^{n+1} = 0$.

Proof of Claim 1: Note that $(x_1 + x_2 + \cdots + x_n)^{n+1}$ is a sum of n^{n+1} terms with the form $x_{k_1}x_{k_2}\cdots x_{k_{n+1}}$ $(k_1, \ldots, k_{n+1} \in \{1, \ldots, n\})$. Assume $x_{k_1}x_{k_2}\cdots x_{k_n+1} \neq 0$; then $x_{k_i}x_{k_{i+1}} \neq 0$ $(i = 1, \ldots, n)$. By equation (e5.26), it follows that $k_{i+1} \le k_i - 1$ $(i = 1, \ldots, n)$. In particular, $k_{n+1} \le k_n - 1$. Then $k_{n+1} \le k_n - 1 \le k_{n-1} - 2$. An induction implies that $k_{n+1} \le k_1 - n \le 0$, which gives a contradiction. Thus all n^{n+1} terms of the form $x_{k_1}x_{k_2}\cdots x_{n+1}$ are zero. It follows that $(x_1 + x_2 + \cdots + x_n)^{n+1} = 0$.

Claim 2: $\left(f_\delta(d_1)a_1x_1^* + \cdots + f_\delta(d_n)a_nx_n^*\right)^{n+1} = 0$.

Proof of Claim 2: Let $y_i = f_\delta(d_i)a_ix_i^*$ $(i = 1, \ldots, n)$. For $i \le j$, using equation (e5.26), we have

$$y_iy_j = f_\delta(d_i)a_ix_i^* f_\delta(d_j)a_jx_j^* = f_\delta(d_i)a_ix_i^*\left(x_j^*x_j\right)a_jx_j^* = f_\delta(d_i)a_i\left(x_jx_i\right)^* x_ja_jx_j^* = 0. \tag{e5.27}$$

Then, as in the proof of Claim 1, we have $(y_1 + \cdots + y_n)^{n+1} = 0$. Claim 2 follows.

Let $x = x_1 + \cdots + x_n$ and let $y = y_1 + \cdots + y_n = f_\delta(d_1)a_1x_1^* + \cdots + f_\delta(d_n)a_nx_n^*$. Then by Claims 1 and 2, both x and y are nilpotent elements. For $i, j \in \{1, \ldots, n\}$ and $i \neq j$, $e_ie_j = 0$ implies $x_ix_i^*x_jx_j^* = 0$, thus $x_i^*x_j = 0$. Then $yx = f_\delta(d_1)a_1f_\delta(d_1) + \cdots + f_\delta(d_n)a_nf_\delta(d_n) \approx_\varepsilon a$. □

Recall that a non-unital C^*-algebra is said to almost have stable rank one if for every hereditary C^*-subalgebra $B \subset A$, B lies in the closure of invertible elements of $\widetilde{B}$ [38, Definition 3.1].

Theorem 5.7. *Let A be a simple C^*-algebra which is tracially approximately divisible. Suppose that A is stably finite and $W(A)$ is almost unperforated. Then A has stable rank one if it is unital, or almost has stable rank one if it is not unital.*

Proof. We assume that A is infinite-dimensional. Fix an element $x \in A$ and fix $\varepsilon > 0$. We may assume that x is not invertible. Since A is finite, x is not one-sided invertible. To show that x is a norm limit of invertible elements, it suffices to show that ux is a norm limit of invertible elements for some unitary $u \in \widetilde{A}$. Note that since A is simple, $\widetilde{A}$ is prime. Thus, by [39, Proposition 3.2, Lemma 3.5], we may assume that there is $a' \in \widetilde{A}_+\backslash\{0\}$ and $a'x = xa' = 0$. There is $e \in A_+$ such that $a'ea' \neq 0$. Put $a = a'ea'$.

Let $B_0 = \{z \in A : az = za = 0\}$. Then $x \in B_0$, and B_0 is a hereditary C^*-subalgebra of A. There is $e_b' \in B_{0+}$ with $\|e_b\| = 1$ such that $e_b' x e_b' \approx_{\varepsilon/64} x$. So $f_{\varepsilon/64}(e_b') x f_{\varepsilon/64}(e_b') \approx_{\varepsilon/16} x$. Put $e_b = f_{\varepsilon/64}(e_b')$ and $B = \mathrm{Her}(e_b)$. Without loss of generality, we may further assume that $x \in B$.

Since we assume that A is infinite-dimensional, $\overline{aAa}$ contains nonzero positive elements a_0, a_1 such that $a_0 a_1 = 0$.

Since A is simple, there is $c \in A$ such that $e_b c (a_1)^{1/2} \neq 0$ (see the proof of [12, 1.8]).

Note that since $e_b \in \mathrm{Ped}(B)$, we have $\mathrm{Ped}(B) = B$ (see, for example, [3, II.5.4.2]). It follows that there are $y_1, y_2, \ldots, y_m \in B$ such that

$$\sum_{i=1}^{m} y_i^* e_b c a_1 c^* e_b y_i = e_b. \tag{e5.28}$$

It then follows that $\langle e_b \rangle \le m \langle a_1 \rangle$. Put $n = 2m$.

For any $z_1, z_2, \ldots, z_n \in B_+$ which are n mutually orthogonal and mutually equivalent positive elements,

$$n\langle z_1 \rangle \le \langle e_b \rangle \le m \langle a_1 \rangle.$$

Since $W(A)$ is almost unperforated,

$$z_1 \lesssim a_1. \tag{e5.29}$$

Since B is a hereditary C^*-subalgebra of A, by Theorem 5.5, B is also tracially approximately divisible. There are $b \in B_+^{\mathbf{1}}$, a C^*-subalgebra $D \otimes M_n \subset B$ and a c.p.c. map $\beta : A \longrightarrow A$ such that

(1) $x \approx_{\varepsilon/8} x_0 + \beta(x)$, where $x_0 \in \overline{bAb}$,
(2) $\beta(x) \in_{\varepsilon/8} D \otimes 1_n$ and
(3) $b \lesssim a_0$.

Thus, there is $x_1 \in D \setminus \{0\}$ such that

$$\|x - (x_0 + x_1 \otimes 1_n)\| < \varepsilon/4. \tag{e5.30}$$

Choose a positive element $d \in D$ such that

$$\|d x_1 d - x_1\| < \varepsilon/4. \tag{e5.31}$$

By the choice of n, we have $d \otimes e_{1,1} \lesssim a_1$, where $\{e_{i,j}\}$ forms a system of matrix units for M_n.

Define $g_1 := a_0$, $g_2 := a_1$, $g_{2+i} := d \otimes e_{i,i}$ $(i = 1, \ldots, n-1)$.

Define $h_1 := b$, $h_{1+i} := d \otimes e_{i,i}$ $(i = 1, \ldots, n)$.

Note that $h_i \lesssim g_i$ $(i = 1, \ldots, n+1)$ and $\overline{g_i h_j} = 0$, if $i \le j$ and $i, j = 1, \ldots, n+1$. Note that $x_0 + d x_1 d \otimes 1_n \in \overline{h_1 A h_1} + \overline{h_2 A h_2} + \cdots + \overline{h_{n+1} A h_{n+1}}$. Then by Lemma 5.6, there are nilpotent elements $v, w \in A$ such that $x_0 + d x_1 d \otimes 1_n \approx_{\varepsilon/4} vw$. Choose $\delta > 0$ such that $vw \approx_{\varepsilon/4} (v + \delta)(w + \delta)$. Since v, w are nilpotent elements, $v + \delta$ and $w + \delta$ are invertible. Then, combining formulas (e5.30) and (e5.31),

$$x \approx_{\varepsilon/4} x_0 + x_1 \otimes 1_n \approx_{\varepsilon/4} x_0 + d x_1 d \otimes 1_n \approx_{\varepsilon/2} (v + \delta)(w + \delta) \in GL\left(\widetilde{A}\right). \tag{e5.32}$$

Therefore we have shown that $x \in \overline{GL\left(\widetilde{A}\right)}$. Thus, in the case that A is unital, A has stable rank one. Since, by Theorem 5.5, this works for every hereditary C^*-subalgebra of A, A almost has stable rank one in the case that A is not unital. □

Remark 5.8. Under the assumption of Theorem 5.7, if $x \in A$ is not invertible, then there is a unitary $u \in \tilde{A}$ such that $(ux)e = e(ux) = 0$ for some $e \in A_+ \setminus \{0\}$. The proof shows that ux can be approximated by products of two nilpotents in A. The idea of the proof is taken from the proof of [15, Lemma 11.1], which originates from that of [38, Lemma 2.1] and [39].

In a subsequent paper [20], we will show that a separable simple C^*-algebra which is tracially approximately divisible has strict comparison for positive elements. So there is a redundancy in the assumption of Theorem 5.7.

Theorem 5.9. *Let A be a simple C^*-algebra. If A is essentially tracially in $\mathcal{C}_{\mathcal{Z}}$, then it is tracially approximately divisible.*

Proof. We assume that A is infinite-dimensional. Let A be a simple C^*-algebra which is e. tracially in $\mathcal{C}_{\mathcal{Z}}$. By [42, Theorem 4.5], every $\mathcal{Z}$-stable C^*-algebra B has almost unperforated $W(B)$ (see Remark 2.5). Then, by Theorem 4.3, $W(A)$ is almost unperforated. Let $\varepsilon > 0$, $\mathcal{F} \subset A^{\mathbf{1}}$ a finite subset, $a \in A_+ \setminus \{0\}$ and $n \geq 1$ an integer be given. Since A is infinite dimensional, choose $a_1, a_2 \in \mathrm{Her}(a)_+ \setminus \{0\}$ such that $a_1 a_2 = a_2 a_1 = 0$.

There are $e_A \in A_+^{\mathbf{1}}$ and $\delta > 0$ such that

$$f_\delta(e_A)x \approx_{\varepsilon/4} x \approx_{\varepsilon/4} x f_\delta(e_A) \text{ for all } x \in \mathcal{F}. \tag{e5.33}$$

Note that by Theorem 5.5, $A_1 := \overline{f_{\delta/2}(e_A) A f_{\delta/2}(e_A)}$ is also a (σ-unital) simple C^*-algebra which is e. tracially in $\mathcal{C}_{\mathcal{Z}}$ (as $\mathcal{C}_{\mathcal{Z}}$ has property (H); see [46, Corollary 3.1]).

Note also that $f_{\delta/2}(e_A) a f_{\delta/2}(e_A) \lesssim a$. To simplify notation, by replacing x by $f_\delta(e_A) x f_\delta(e_A)$ for all $x \in \mathcal{F}$, a by $f_{\delta/2}(e_A) a f_{\delta/2}(e_A)$ and a_i by $f_{\delta/2}(e_A) a_i f_{\delta/2}(e_A)$ $(i = 1, 2)$, without loss of generality we may assume that $x, a, a_1, a_2 \in A_1$. We may also assume, without loss of generality,

$$e_1 x = x = x e_1 \text{ for all } x \in \mathcal{F} \tag{e5.34}$$

for some strictly positive element $e_1 \in A_1^1$. Note that $f_{\delta/2}(e_A) \in \mathrm{Ped}(A)$. Therefore A_1 is algebraically simple and $f_{\delta/2}(e_A)$ is a strictly positive element of A_1. There are an integer $l \geq 1$ and $x_i \in A_1$, $i = 1, 2, \ldots, l$, such that

$$\sum_{i=1}^{l} x_i^* a_1 x_i = e_1. \tag{e5.35}$$

Set $\mathcal{F}_1 = \mathcal{F} \cup \{e_1\}$. Choose $0 < \eta < \varepsilon/2$ such that if $\theta' \in A_+^{\mathbf{1}}$ with $\|\theta' x - x\theta'\| < \eta$, then

$$(\theta')^{1/2} x \approx_{\varepsilon/2} x (\theta')^{1/2} \text{ for all } x \in \mathcal{F}_1. \tag{e5.36}$$

There exist $\theta_1 \in A_+^{\mathbf{1}}$ and a $\mathcal{Z}$-stable C^*-subalgebra B of A_1 such that

(i) $\|\theta_1 x - x\theta_1\| < \eta/64$ and $\left\|(1-\theta_1)^{1/2} x - x(1-\theta_1)^{1/2}\right\| < \eta/64$ for all $x \in \mathcal{F}_1$,
(ii) $(1-\theta_1)^{1/2} x (1-\theta_1)^{1/2}, (1-\theta_1)^{1/2} x, x(1-\theta_1)^{1/2}, (1-\theta_1)x, x(1-\theta_1), (1-\theta_1)x(1-\theta_1) \in_{\eta/64} B$ for all $x \in \mathcal{F}_1$ and
(iii) $\theta_1 \lesssim a_2$.

Let

$$\mathcal{F}_2 = \Big\{(1-\theta_1)^{1/2} x (1-\theta_1)^{1/2}, (1-\theta_1)^{1/2} x, x(1-\theta_1)^{1/2}, (1-\theta)x, x(1-\theta_1), (1-\theta_1)x(1-\theta_1) : x \in \mathcal{F}\}.$$

For each $f \in \mathcal{F}_2$, fix $b(f) \in B$ such that $\|b(f)\| \leq 1$ and

$$\|f - b(f)\| < \eta/32. \tag{e5.37}$$

Let $\mathcal{G} = \{b(f) : f \in \mathcal{F}_2\}$. We write $B = C \otimes \mathcal{Z}$. Since $\mathcal{Z}$ is strongly self-absorbing, without loss of generality we may assume that there is a finite subset $\mathcal{G}_1 \subset C$ such that $\mathcal{G} = \{y \otimes 1_{\mathcal{Z}} : y \in \mathcal{G}_1\} \subset C \otimes 1_{\mathcal{Z}}$. To further simplify notation, without loss of generality we may assume that there exists a strictly positive element $e_C \in C$ such that

$$e_b y = y = y e_b \text{ for all } y \in \mathcal{G}_1, \tag{e5.38}$$

where $e_b = e_C \otimes 1_{\mathcal{Z}}$.

For any integer n, choose m such that $m > l$ and n divides m. Let $\psi : M_m \to \mathcal{Z}$ be an order 0 c.p.c. map such that

$$1_{\mathcal{Z}} - \psi(1_m) \lesssim_{\mathcal{Z}} \psi\big(e_{1,1}\big) \tag{e5.39}$$

(see [43, Proposition 5.1(iv) implying (ii)]). Define $\varphi : M_m \to B$ by $\varphi(c) := e_C \otimes \psi(c)$ for all $c \in M_m$. Set

$$\theta_2 := e_b - \varphi(1_m) = e_C \otimes 1_{\mathcal{Z}} - e_C \otimes \psi(1_m) = e_C \otimes (1_{\mathcal{Z}} - \psi(1_m)) \lesssim_B e_C \otimes \psi\big(e_{1,1}\big). \tag{e5.40}$$

Note that

$$\theta_2 g = g\theta_2 \text{ for all } g \in \mathcal{G}. \tag{e5.41}$$

It follows from formulas (e5.37) and (e5.41) that for any $y \in \mathcal{F}_2$,

$$\theta_2 y \approx_{\varepsilon/32} y\theta_2. \tag{e5.42}$$

Define $D := \overline{e_C c e : C \otimes \psi\big(e_{1,1}\big)}$ and D' the C^*-subalgebra generated by

$$\{e_C c e : C \otimes \psi(z) : c \in C \text{ and } z \in M_m\}. \tag{e5.43}$$

Recall that ψ gives a homomorphism $H : C^*(\psi(1_m)) \otimes M_m \to \mathcal{Z}$ such that $H(\iota \otimes g) = \psi(g)$ for all $g \in M_m$, where $\iota(t) = t$ for $t \in \mathrm{sp}(\psi(1_m))$ (see [49, Corollary 4.1]). It follows that $D' \cong D \otimes M_m$. Define $\beta_1 : A \to A$ by

$$\beta_1(y) := \big(1_{\widetilde{A}} - \theta_2\big)^{1/2} y \big(1_{\widetilde{A}} - \theta_2\big)^{1/2} \text{ for all } y \in A \tag{e5.44}$$

(where $1_{\widetilde{A}}$ denotes the identity of $\widetilde{A}$ when A is not unital and is the identity of A if A has one). Note also that $(1-\theta)^{1/2}$ is an element which has the form $1 + f_1(\theta)$ for $f_1(t) = (1-t)^{1/2} - 1 \in C_0((0,1])$**1**. If $g = y \otimes 1_{\mathcal{Z}} \in \mathcal{G}$, then (noting that $y \in \mathcal{G}_1 \subset C$, and seeing equation (e5.38)),

$$\beta_1(g) = (1-\theta_2)g = g - e_C \otimes (1_{\mathcal{Z}} - \psi(1_m))g \tag{e5.45}$$

$$= (e_C \otimes 1_{\mathcal{Z}})g - e_C \otimes (1_{\mathcal{Z}} - \psi(1_m))g \tag{e5.46}$$

$$= (e_C \otimes \psi(1_m))(y \otimes 1_{\mathcal{Z}}) = \Big(e_C^{1/2} y e_C^{1/2}\Big) \otimes \psi(1_m) \in D \otimes 1_m. \tag{e5.47}$$

Define a c.p.c. map $\beta : A \to A$ by

$$\beta(x) := \beta_1\Big((1-\theta_1)^{1/2} x (1-\theta_1)^{1/2}\Big) \text{ for all } x \in A. \tag{e5.48}$$

For $x \in \mathcal{F}$, let $f = (1-\theta_1)^{1/2} x (1-\theta_1)^{1/2}$. Then, by formula (e5.37),

$$\beta(x) = \beta_1\Big((1-\theta_1)^{1/2} x (1-\theta_1)^{1/2}\Big) \approx_{\eta/32} \beta_1(b(f)) \in D \otimes 1_m. \tag{e5.49}$$

Put $\theta = \theta_1 + (1-\theta_1)^{1/2}\theta_2(1-\theta_1)^{1/2}$. We have

$$0 \le \theta \le \theta_1 + (1-\theta_1)^{1/2}(1-\theta_1)^{1/2} = 1. \tag{e5.50}$$

For $x \in \mathcal{F}$, let $f' = (1-\theta_1)x$. Recall that we assume that $b(f') = y' \otimes 1_{\mathcal{Z}}$ for some $y' \in C^{\mathbf{1}}$. Then for $x \in \mathcal{F}$, applying formulas (e5.37) and (e5.41) repeatedly, we have

$$(1-\theta)x = (1-\theta_1)x - (1-\theta_1)^{1/2}\theta_2(1-\theta_1)^{1/2}x \tag{e5.51}$$

$$\approx_{\eta/32} (1-\theta_1)x - (1-\theta_1)x\theta_2 = (1-\theta_1)x(1-\theta_2) \tag{e5.52}$$

$$\approx_{\eta/32} b(f')(1-\theta_2) = (1-\theta_2)^{1/2}b(f')(1-\theta_2)^{1/2} \tag{e5.53}$$

$$= \beta_1(b(f')) \approx_{\eta/32} \beta_1\Big((1-\theta_1)^{1/2}x(1-\theta_1)^{1/2}\Big) = \beta(x). \tag{e5.54}$$

From equations (e5.49) and (e5.54), we have

$$(1-\theta)x \in_{\eta/8} D \otimes 1_m \text{ for all } x \in \mathcal{F}. \tag{e5.55}$$

Recall that $(1-\theta_1)^{1/2}x, x(1-\theta_1)^{1/2}, (1-\theta_1)^{1/2}x(1-\theta_1)^{1/2} \in \mathcal{F}_2$. Hence, for $x \in \mathcal{F}$, by (i) above and formula (e5.42),

$$\theta x = \Big(\theta_1 + (1-\theta_1)^{1/2}\theta_2(1-\theta_1)^{1/2}\Big)x \approx_{2\eta/64} x\theta_1 + (1-\theta_1)^{1/2}\theta_2 x(1-\theta_1)^{1/2} \tag{e5.56}$$

$$\approx_{\eta/32} x\theta_1 + (1-\theta_1)^{1/2}x(1-\theta_1)^{1/2}\theta_2 \approx_{\eta/32} x\theta_1 + \theta_2(1-\theta_1)^{1/2}x(1-\theta_1)^{1/2} \tag{e5.57}$$

$$\approx_{\eta/32} x\theta_1 + (1-\theta_1)^{1/2}x\theta_2(1-\theta_1)^{1/2} \tag{e5.58}$$

$$\approx_{\eta/64} x\theta_1 + x(1-\theta_1)^{1/2}\theta_2(1-\theta_1)^{1/2} = \theta x. \tag{e5.59}$$

Note that by formulas (e5.40) and (e5.35), in $W(A)$ we have

$$m\langle\theta_2\rangle = m\langle e_C \otimes (1_{\mathcal{Z}} - \psi(1_m))\rangle \tag{e5.60}$$

$$\le m\big\langle e_C \otimes \psi\big(e_{1,1}\big)\big\rangle \le \langle e_C \otimes \psi(1_m)\rangle \le \langle e_C \otimes 1_{\mathcal{Z}}\rangle \le l\langle a_1\rangle. \tag{e5.61}$$

Therefore (recall that $W(A)$ is almost unperforated), since $l < m$,

$$\theta_2 \lesssim a_1. \tag{e5.62}$$

It follows (noting that $a_1a_2 = a_2a_1 = 0$) that

$$\theta = \theta_1 + (1-\theta_1)^{1/2}\theta_2(1-\theta_1)^{1/2} \lesssim a_2 + a_1 \lesssim a. \tag{e5.63}$$

Finally, the theorem follows from formulas (e5.59), (e5.55) and (e5.63), the fact that $D \otimes 1_n$ is embedded into $D \otimes 1_m$ unitally (as n divides m) and Proposition 5.3. □

Corollary 5.10. *Let A be a simple C^*-algebra which is e. tracially in $\mathcal{C}_{\mathcal{Z}}$. If A is not purely infinite, then it has stable rank one if it is unital and almost has stable rank one if it is not unital.*

Proof. By Theorem 5.9, A is tracially approximately divisible. By Theorem 5.1, if A is not purely infinite, then it has strict comparison for positive elements. It follows then from Theorem 5.7 that A has stable rank one if it is unital and almost has stable rank one if it is not unital. □

Remark 5.11. For the rest of this paper, we will present nonamenable examples of C^*-algebras which are possibly stably projectionless and are essentially tracially in the class $\mathcal{C}_{\mathcal{Z}}$, the class of $\mathcal{Z}$-stable C^*-algebras.

6. Construction of A_z^C

In this section we first fix a separable residually finite-dimensional (RFD) C^*-algebra C, which may not be exact.

Let B be the unitisation of $C_0((0,1],C)$. Since $C_0((0,1],C)$ is contractible, $V(B) = \mathbb{N}\cup\{0\}$, $K_0(B)=\mathbb{Z}$ and $K_1(B)=\{0\}$.

Let us make the convention that B includes the case that $C=\{0\}$ – that is, $B=\mathbb{C}$.

Let $\mathfrak{p} = p_1^{r_1}\cdot p_2^{r_2}\cdot\cdots$ be a supernatural number, where $p_1,p_2,\ldots$ is a sequence (possibly finite) of distinct prime numbers and $r_i\in\mathbb{N}\cup\{\infty\}$. Denote by $\mathbb{D}_{\mathfrak{p}}$ the subgroup of $\mathbb{Q}$ generated by finite sums of rational numbers of the form $\frac{m}{p_j^i}$, where $m\in\mathbb{Z}$ and $i\in\mathbb{N}\cap\left[1,r_j\right]$.

Denote by $M_{\mathfrak{p}}$ the UHF-algebra associated with the supernatural number $\mathfrak{p}$.

The following is a result of Dădărlat [13]:

Theorem 6.1. *Fix a supernatural number $\mathfrak{d}$. There is a unital simple C^*-algebra $A_{\mathfrak{d}}$ which is an inductive limit of $M_{m(n)}(B)$ with injective and unital connecting maps such that*

$$\left(K_0(A_{\mathfrak{d}}),K_0(A_{\mathfrak{d}})_+,\left[1_{A_{\mathfrak{d}}}\right]\right) = (\mathbb{D}_{\mathfrak{d}},\mathbb{D}_{\mathfrak{d}+},1),$$

$K_1(A_{\mathfrak{d}})=\{0\}$, and $A_{\mathfrak{d}}$ has a unique tracial state and tracial rank zero.

Proof. This is taken from [13]; we retain the notation used there. For the supernatural number $\mathfrak{d}$, there is a standard Bratteli system $\left\{B,\underline{\pi}\right\}$ given by Glimm. We use Dădărlat's restricted system as defined in [13, Definition 3]. Let $D=AF\left(\underline{\pi}\right)$. Then D is the UHF-algebra with $(K_0(D),K_0(D)_+,[1_D]) = (\mathbb{D}_{\mathfrak{d}},\mathbb{D}_{\mathfrak{d}+},1)$. Set $A_{\mathfrak{d}} := B\left(\underline{\pi}\right)$ as in [13, Proposition 8]. Note that [13, Definition 3(ii)] implies that the connecting maps in the restricted system are injective (see also [13, proof of Proposition 8]). The proof of [13, Proposition 9] shows that $B\left(\underline{\pi}\right)$ is a unital simple C^*-algebra of real rank zero and stable rank one, $\left(K_0(A_{\mathfrak{d}}),K_0(A_{\mathfrak{d}})_+,\left[1_{A_{\mathfrak{d}}}\right]\right) = (\mathbb{D}_{\mathfrak{d}},\mathbb{D}_{\mathfrak{d}+},1)$, and has a unique tracial state. Note also that since $K_1(B)=0$, $K_1\left(B\left(\underline{\pi}\right)\right)=0$. So $K_1(A_{\mathfrak{d}})=0$. The fact that $A_{\mathfrak{d}}$ has tracial rank zero is also known and, for example, follows from [28, Theorem 3.7.9]. □

We will review the construction of $A_{\mathfrak{d}}$ and introduce some notation for our construction.

Definition 6.2. Fix a supernatural number $\mathfrak{d}$. Choose a Bratteli system $AF\left(\underline{\pi}\right)$ (see [13, Definition 2]) for $M_{\mathfrak{d}}$ given by Glimm. Recall that $B=C_0((0,1],C)^\sim$. Following Dădărlat's construction (see [13, Definition 3, proof of Proposition 8]), one may write $A_{\mathfrak{d}} = \lim_{n\to\infty}\left(M_{d_n'}(B),\delta_n'\right)$, $d_{n+1}' = d_n\cdot d_n'$, where $d_n,d_n'>1$ are integers, $\delta_n' : M_{d_n'}(B)\to M_{d_{n+1}'}(B)$ is defined by

$$\delta_n'(f) := \begin{pmatrix} f & 0\\ 0 & \gamma_n(f)\end{pmatrix} \text{ for all } f\in M_{d_n'}(B) \tag{e6.1}$$

and $\gamma_n : B\to M_{d_n-1}$ is a unital homomorphism, a d_n-1-dimensional representation (we then use γ_n for the extension $\gamma_n\otimes \mathrm{id}_{d_n'} : M_{d_n'}(B)\to M_{(d_n-1)d_n'}$) which also has the form described in the proof of [13, Proposition 8]. By that proof, this can always be done.

In the Bratteli system $AF\left(\underline{\pi}\right)$, we may also assume, by passing to a subsequence, that

$$\lim_{n\to\infty} d_n = \infty. \tag{e6.2}$$

Also, we assume for any n that $\{\gamma_m : m\geq n\}$ is a separating sequence of finite-dimensional representations. For a more specific construction of $A_{\mathfrak{d}}$, readers are referred to [13, Definition 3, Proposition 8, Section 3].

It is important that for any $\tau \in T(B)$,

$$\lim_{n\to\infty}\left|\tau\circ\delta_n'(a)-\tau(\gamma_n(a))\right|=0 \text{ for all } a\in M_{d_n'}(B). \tag{e6.3}$$

(Note that by τ we mean $\tau\otimes \mathrm{tr}_{d_n'}$, where $\mathrm{tr}_{d_n'}$ is the tracial state of $M_{d_n'}$.)

Consider $\delta_{m,n}' := \delta_{n-1}'\circ\delta_{n-2}'\circ\cdots\circ\delta_m' : M_{d_m'}(B)\to M_{d_n'}(B)$. Then we may write

$$\delta_{m,n}'(f)=\begin{pmatrix} f & 0\\ 0 & \gamma_{m,n}(f)\end{pmatrix} \text{ for all } f\in M_{d_m'}(B), \tag{e6.4}$$

where $\gamma_{m,n} : B\to M_{d_n'/d_m'-1}$ is a finite-dimensional representation. (In the rest of the paper, we also use $\gamma_{m,n} := \gamma_{m,n}\otimes \mathrm{id}_N : M_N(B)\to M_{(d_n'/d_m'-1)N}$ for all integers $N\geq 1$.) Therefore, if we fix a finite subset $\mathcal{F}_m\subset M_{d_m'}(B)$, we may assume that for any $a\in\mathcal{F}_m\setminus\{0\}$, we have $\gamma_{m,n}(a)\neq 0$ for some large $n\geq m$. Choose a function $g\in C([0,\|a\|)_+$ such that $0\leq g(t)\leq 1$ for all $t\in[0,\infty)$, $g(t)=1$ if $t\in[\|a\|-\|a\|/2m,\|a\|)$ and $g(t)=0$ if $t\in[0,\|a\|-\|a\|/m]$. We may assume that $\gamma_{m,n}(g(|a|))\neq 0$ for all $a\in\mathcal{F}_m\setminus\{0\}$. It follows that $\left\|\gamma_{m,n}(|a|)\right\|\geq(1-1/m)\|a\|$ for all $a\in\mathcal{F}_m\setminus\{0\}$. Thus we may assume that for any $a\in\mathcal{F}_m$ and all $n>m$,

$$\left\|\gamma_{m,n}(a)\right\|\geq(1-1/m)\|a\|. \tag{e6.5}$$

In what follows, $A_{\mathfrak{z}}=\lim_{n\to\infty}\left(M_{d_n'}(B),\delta_n'\right)$ is the C^*-algebra in Theorem 6.1 and δ_n' is as described in formula (e6.1) such that formula (e6.5) holds for $n\geq m+1$.

We wish to construct a unital simple C^*-algebra A_z^C with a unique tracial state such that $K_0\left(A_z^C\right)=\mathbb{Z}$ and $K_1\left(A_z^C\right)=\{0\}$.

The strategy is to have a Jiang–Su-style inductive limit of some C^*-subalgebras of $C\big([0,1],M_p(B)\otimes M_q(B)\big)$ for some nonnuclear RFD algebra B, or perhaps some C^*-subalgebra of $C\big([0,1],M_{pq}(B)\big)$. However, there are several difficulties to be resolved. One should avoid using $M_p(B)\otimes M_q(B)$ as building blocks, since there are different C^*-tensor products and potential difficulties in computing the K-theory. Other issues include the fact that each fibre $M_m(B)$ is not simple.

We begin with the following building blocks:

Definition 6.3. For a pair of integers $m,k\geq 1$, define

$$E_{m,k} := \{f\in C([0,1],M_{mk}(B)) : f(0)\in M_m(B)\otimes 1_k \text{ and } f(1)\in 1_m\otimes M_k\}.$$

Note that here one views $M_m(B)\otimes 1_k, 1_m\otimes M_k\subset M_m(B)\otimes M_k=M_{mk}(B)$ as unital C^*-subalgebras.

Fix integers $m,n\geq 1$. Let $D(m,k)=M_m(B)\oplus M_k$. Define $\varphi_0 : D(m,k)\to M_m(B)\otimes 1_k$ by $\varphi_0((a,b)) := a\otimes 1_k$ for all $(a,b)\in D(m,k)$ and $\varphi_1 : D(m,k)\to M_k$ by $\varphi_1((a,b))=1_m\otimes b$.

Then

$$E_{m,k}\cong\{(f,g)\in C([0,1],M_{mk}(B))\oplus D(m,k) : f(0)=\varphi_0(g) \text{ and } f(1)=\varphi_1(g)\}. \tag{e6.6}$$

Denote by $\pi_e : E_{m,k}\to D(m,k)$ the quotient map which maps (f,g) to g. Denote by $\pi_0 : E_{m,k}\to M_m(B)\otimes 1_k$ the homomorphism defined by $\pi_0((f,g)) := \varphi_0(g)=f(0)$ and by $\pi_1 : E_{m,k}\to 1_m\otimes M_k$ the homomorphism defined by $\pi_1((f,g)) := \varphi_1(g)=f(1)$.

Lemma 6.4. *If m and k are relatively prime, then $E_{m,k}$ has no proper projections.*

Proof. Recall that $B=C_0((0,1],C)^\sim$, the unitisation of $C_0((0,1],C)$. Let τ_B be the tracial state on $M_m(B)$ induced by the quotient map $B\to B/C_0((0,1],C)\cong\mathbb{C}$, and let tr_k be the tracial state of M_k. Let $\tau=\tau_B\otimes\mathrm{tr}_k$.

Let $e\in E_{m,k}$ be a nonzero projection. Note that $E_{m,k}\subset C([0,1],M_{mk}(B))$. Note also that $K_0(B)=\mathbb{Z}$ and 1_B is the only nonzero projection of B. Then for each $x\in[0,1]$, $e(x)$ is a nonzero projection in

$M_{mk}(B)$. One easily shows that $\tau(e(x))$ is a constant function on $[0,1]$. Let $\tau(e(x)) = r \in (0,1]$. But $\tau(e(0)) \in \{i/m : i = 0,1,\dots,m\}$ and $\tau(e(1)) \in \{j/k, i = 0,1,\dots,k\}$. Since m and k are relatively prime, $\tau(e(0)) = \tau(e(1)) = 1$. Hence $\tau(e(x)) = 1$ for all $x \in [0,1]$. This is possible only when $e = 1_m \otimes 1_k$. □

Lemma 6.5. *Suppose that m and k are relatively prime. Then*

$$\big(K_0(E_{m,k}), K_0(E_{m,k})_+, [1_{E_{m,k}}]\big) = (\mathbb{Z}, \mathbb{N}\cup\{0\}, 1) \qquad \textit{and} \qquad K_1(E_{m,k}) = \{0\}.$$

Proof. Let

$$I = \{f \in E_{m,k} : f(0) = f(1) = 0\}.$$

Then $I \cong C_0((0,1)) \otimes M_{mk}(B) = S(M_{mk}(B))$. It follows that

$$K_0(I) = K_1(M_{mk}(B)) = \{0\} \qquad \text{and} \qquad K_1(I) = K_0(M_{mk}(B)) = \mathbb{Z}. \tag{e6.7}$$

Consider the short exact sequence

$$0 \to I \xrightarrow{\iota_I} E_{m,k} \xrightarrow{\pi_e} D(m,k) \to 0, \tag{e6.8}$$

where $\iota_I : I \to E_{m,k}$ is the embedding and $\pi_e : E_{m,k} \to D(m,k)$ is the quotient map. One obtains the following six-term exact sequence:

$$\begin{array}{ccccc} K_0(I) & \xrightarrow{\iota_{I*0}} & K_0(E_{m,k}) & \xrightarrow{\pi_{e*0}} & K_0(D(m,k)) \\ \uparrow\delta_1 & & & & \downarrow\delta_0 \\ K_1(D(m,k)) & \xleftarrow{\pi_{e*1}} & K_1(E_{m,k}) & \xleftarrow{\iota_{I*1}} & K_1(I), \end{array} \tag{e6.9}$$

which becomes

$$\begin{array}{ccccc} 0 & \xrightarrow{\iota_{I*0}} & K_0(E_{m,k}) & \xrightarrow{\pi_{e*0}} & \mathbb{Z}\oplus\mathbb{Z} \\ \uparrow\delta_1 & & & & \downarrow\delta_0 \\ 0 & \xleftarrow{\pi_{e*1}} & K_1(E_{m,k}) & \xleftarrow{\iota_{I*1}} & \mathbb{Z}. \end{array} \tag{e6.10}$$

Note that

$$\mathrm{im}(\pi_{e*0}) = \{(x,y) \in K_0(D(m,k)) : \varphi_{0*0}(x) = \varphi_{1*0}(y)\}.$$

The lemma follows from a straightforward computation. □

Set $\tau \in T(C([0,1], M_{mk}(B)))$. By, for example, [26, Theorem 2.1] and the Choquet and Fubini theorems,

$$\tau(f) = \int_{\partial_e T(C([0,1]))\times\partial_e T(M_{mk}(B))} f\, d(\mu\times\mu_B)$$

for all $f \in \mathrm{Aff}(T(C([0,1], M_{mk}(B))))_{\mathrm{sa}}$, where μ is a probability Borel measure on $[0,1]$ and μ_B is a probability Borel measure on $\partial_e T(M_{mk}(B))$. By the Fubini theorem again, we may write $\tau(f) = \int_{[0,1]} \sigma_t(f(t))d\mu$, where σ_t is a tracial state of $M_{mk}(B)$. Let I be the ideal in the proof of Lemma 6.5. Then $I \cong C_0((0,1)) \otimes M_{mk}(B)$. Now set $\tau \in T(E_{m,k})$ such that $\|\tau|_I\| \neq 0$. Since $(1/\|\tau_I\|)\tau|_I$ can be extended to a tracial state of $C([0,1], M_{mk}(B))$, we may write $\tau|_I(f) = \int_{(0,1)} \sigma_t(f(t))d\mu$ for all $f \in C_0((0,1)) \otimes M_{mk}(B)$, where σ_t is a tracial state of $M_{mk}(B)$ and μ is a Borel measure on $(0,1)$

(with $\|\mu\| = \|\tau|_I\| \le 1$). Since $E_{m,k}/I = M_m(B) \oplus M_k$, as in [31, 2.5], one may write

$$\tau(f) = \int_0^1 \sigma_t(f(t))d\nu \text{ for all } f \in E_{m,k}, \tag{e6.11}$$

where σ_0 is a tracial state on $M_m(B)$, σ_1 is a tracial state on M_k, ν is a probability Borel measure on $[0,1]$, $\nu|_{(0,1)} = \mu|_{(0,1)}$, and if $\|\tau|_I\| = 0$, then $\nu|_{(0,1)} = 0$.

Notation 6.6. Let $\gamma : B \to M_r$ be a finite-dimensional representation with rank r – that is, γ is a finite direct sum of irreducible representations $\gamma_j : j = 1, 2, \dots, l$, each of which has rank r_j $(1 \le j \le l)$, such that $r = \sum_{j=1}^{l} r_j$. We will also use γ for $\gamma \otimes \mathrm{id}_m : M_m(B) \to M_{rm}$ for all integers $m \ge 1$. In what follows we may also write M_L for $M_L(\mathbb{C} \cdot 1_B)$ for all integers $L \ge 1$. In this way, γ (or $\gamma \otimes \mathrm{id}_m$) is a homomorphism from $M_m(B)$ into $M_{rm} \subset M_{rm}(B)$.

Let $\xi_0, \xi_1, \xi_2, \dots, \xi_{k-1} : [0,1] \to [0,1]$ be continuous paths. Define a homomorphism

$$H : C([0,1], M_{mn}(B)) \to C\big([0,1], M_{((k-1)r+1)mn}(B)\big)$$

by

$$H(f)(t) := \begin{pmatrix} f \circ \xi_0(t) & 0 & \cdots & 0 \\ 0 & \gamma(f \circ \xi_1(t)) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \gamma(f \circ \xi_{k-1}(t)) \end{pmatrix} \text{ for all } f \in C([0,1], M_{mn}(B))$$

and $t \in [0,1]$. Note that H can be also defined on $E_{m,n} \subset C([0,1], M_{mn}(B))$. But in general, H does not map $E_{m,n}$ into $E_{m,n}$. However, with some restrictions on the boundary (restriction on ξ_is), it is possible that H maps $E_{m,n}$ into $E_{m,n}$.

For the convenience of the construction, let us add some notation and terminology.

Set $f, g \in M_n(B)$. We write $f =^s g$ if there is a scalar unitary $w \in M_n$ such that $w^* f w = g$. Also, if $f, g \in C([0,1], M_n(B))$, we write $f =^s g$ if there is a unitary $w \in C([0,1], M_n)$ such that $w^* f w = g$.

6.7. We will construct $A = \lim_{n\to\infty}(A_n, \varphi_m)$. The construction will be by induction. Fix B as in Notation 6.6. Set $A_1 = E_{3,5}$.

Denote by $\bar{3}$ the supernatural number 3^∞. Write $A_{\bar{3}} = \lim_{n\to\infty}\big(M_{d'_n}(B), \delta'_n\big)$ (see Theorem 6.1), where

$$\delta'_n(f) = \begin{pmatrix} f & 0 \\ 0 & \gamma_n(f) \end{pmatrix} \text{ for all } f \in M_{d'_n}(B), \tag{e6.12}$$

as in formula (e6.1), which also has the properties in equations (e6.2) and (e6.3) (with $d_n = 3^l$ for some integer $l \ge 1$). Hence, without loss of generality, by passing to a subsequence we may assume, for all n,

$$\frac{1}{d_n - 1} < 1/3^n. \tag{e6.13}$$

Recall that $B = C_0((0,1], C)^\sim$. For each $t \in [0,1]$, denote by $\theta_t : B \to B$ the homomorphism defined, for all $f \in B$, by

$$\theta_t(f)(x) := f((1-t)x) \text{ for all } x \in (0,1]. \tag{e6.14}$$

Note also that for any integer $l \ge 1$, we will use θ_t for $\theta_t \otimes \mathrm{id}_l : M_l(B) \to M_l(B)$. Thus, if $f \in M_l(B)$,

$$\theta_1(f) = f(0) \in M_l. \tag{e6.15}$$

It should be noted that $\theta_0 = \mathrm{id}_{M_l(B)}$.

We state the inductive step as the following lemma:

Lemma 6.8. *For $A_m = E_{p_m,q_m}$ with $(p_m, q_m) = 1$, we have $(5, p_m) = 1$ and $(3, q_m) = 1$. There exist $A_{m+1} = E_{p_{m+1},q_{m+1}}$, where $(p_{m+1}, q_{m+1}) = 1$, $(5, p_{m+1}) = 1$ and $(3, q_{m+1}) = 1$, and a unital injective homomorphism $\varphi_m : A_m \to A_{m+1}$ of the form*

$$\varphi_m(f)(t) = u^* \begin{pmatrix} \Theta_m(f)(t) & 0 & \cdots & 0 \\ 0 & \gamma_m(f \circ \xi_1(t)) \otimes 1_5 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \gamma_m(f \circ \xi_k(t)) \otimes 1_5 \end{pmatrix} u \tag{e6.16}$$

for all $f \in A_m$, where $u \in U\big(C([0,1], M_{p_{m+1}q_{m+1}})\big)$, $\Theta_m : A_m \to C\big([0,1], M_{5kp_mq_m}(B)\big)$ is a homomorphism, $k \geq 1$ is an integer, $t \in [0,1]$ and $\gamma_m : M_{p_mq_m}(B) \to M_{R(m)p_mq_m}$ is a finite-dimensional representation, where $R(m) \geq 1$ is an integer. Moreover, the following are true:

(1) *Each $\xi_i : [0,1] \to [0,1]$ is a continuous map which has one of the following three forms:*

$$\xi_i(t) = \begin{cases} (2/3)t & \text{if } t \in [0, 3/4], \\ 1/2 & \text{if } t \in (3/4, 1], \end{cases} \tag{e6.17}$$

$$\xi_i(t) = 1/2 \text{ for all } t \in [0,1], \tag{e6.18}$$

$$\xi_i(t) = \begin{cases} 1/2 + (2/3)t & \text{if } t \in [0, 3/4], \\ 1 & \text{if } t \in (3/4, 1], \end{cases} \tag{e6.19}$$

and each type of ξ_i appears in equation (e6.16) at least once.

(2) $5k/5kR(m) = 1/R(m) < 1/3^m$.

(3) *For a fixed finite subset $\mathcal{F}_m \subset A_m \setminus \{0\} \subset C\big([0,1], M_{p_mq_m}(B)\big)$,*

$$\|\gamma_m(f(t))\| > (1 - 1/2m)\|f\| \neq 0 \text{ for some } t \in [0,1].$$

(4) *We have*

$$\Theta_m(f) = \operatorname{diag}\Big(\theta^{(1)}(f), \ldots, \theta^{(k)}(f)\Big), \tag{e6.20}$$

where $\theta^{(i)} : E_{p_m,q_m} \to C\big([0,1], M_{5p_mq_m}(B)\big)$ is defined – if $\xi_i(3/4) = 1/2$ – for each $f \in E_{p_m,q_m}$ by

$$\theta^{(i)}(f)(t) := \begin{cases} f(\xi_i(t)) & \text{if } t \in [0, 3/4], \\ \theta_{4(t-3/4)}(f(\xi_i(t))) & \text{if } t \in (3/4, 1], \end{cases}$$

and where $\theta_t : M_{p_mq_m}(B) \to M_{p_mq_m}(B)$ (recall that $B = C_0((0,1], C)^\sim$) is a unital homomorphism defined by

$$\theta_t(f)(x) := f((1-t)x) \text{ for all } x \in (0,1] \text{ and all } t \in [0,1] \tag{e6.21}$$

and, if $\xi_i(3/4) = 1$, for each $f \in E_{p_m,q_m}$,

$$\theta^{(i)}(f)(t) = f(\xi_i(t)), \quad t \in [0,1].$$

Proof. To avoid the potential complication of computing relative primality of integers, we will have a three-stage construction.

Stage 1: Write $A_m = E_{p_m,q_m}$, where $(p_m, q_m) = 1$. Also $(5, p_m) = 1$ and $(3, q_m) = 1$.

Fix any finite subset $\mathcal{F}_m \subset E_{p_m,q_m} \setminus \{0\}$. One can choose a finite subset $S \subset [0,1]$ such that, for any $f \in \mathcal{F}_m$, there is $s \in S$, $\|f(s)\| > (1-1/2m)\|f\| \neq 0$. Note that $\mathcal{F}' = \{f(s) : f \in \mathcal{F}_m \text{ and } s \in S\} \setminus \{0\}$ is a finite subset of $M_{p_m q_m}(B)$. By passing to a subsequence, we may assume (replacing γ_m by $\gamma_{m,n}$ as mentioned in formula (e6.5)) that

$$\|\gamma_m(g)\| > (1-1/2m)\|g\| \neq 0 \text{ for all } g \in \mathcal{F}'. \tag{e6.22}$$

It follows that for any $f \in \mathcal{F}_m$,

$$\|\gamma_m(f(s'))\| \geq (1-1/2m)\|f\| \neq 0 \text{ for some } s' \in S \subset [0,1]. \tag{e6.23}$$

Define $\psi'_m : M_{p_m}(B) \otimes M_{q_m} \to M_{d_m p_m}(B) \otimes M_{5q_m}$ by $\psi'_m := \delta_m \otimes s$, where

$$\delta_m(a) = \begin{pmatrix} a & 0 \\ 0 & \gamma_m(a) \end{pmatrix} \text{ for all } a \in M_{p_m}(B), \text{ and } s(c) = c \otimes 1_5 \text{ for all } c \in M_{q_m}. \tag{e6.24}$$

Define $\psi_m : E_{p_m,q_m} \to E_{d_m p_m, 5q_m}$ by

$$\psi_m(f)(t) := \psi'_m(f(t)) \text{ for all } f \in E_{p_m,q_m} \text{ and } t \in [0,1]. \tag{e6.25}$$

Set $f \in E_{p_m,q_m}$. Then $f(0) = b \otimes 1_{q_m}$, where $b \in M_{p_m}(B)$. Thus,

$$\psi_m(f)(0) = \psi'_m(f(0)) = \delta_m(b) \otimes \left(1_{q_m} \otimes 1_5\right) \in M_{d_m p_m}(B) \otimes 1_{5q_m}. \tag{e6.26}$$

On the other hand, $f(1) = 1_{p_m} \otimes c$, where $c \in M_{q_m}$. Thus

$$\psi_m(f)(1) = \psi'_m(f(1)) = 1_{d_m p_m} \otimes (c \otimes 1_5) \in 1_{d_m p_m} \otimes M_{5q_m}. \tag{e6.27}$$

So indeed, ψ_m maps E_{p_m,q_m} into $E_{d_m p_m, 5q_m}$.

Note that for $t \in [0,1]$, we have for all $f \in E_{p_m,q_m}$ (writing γ_m for $\gamma_m \otimes \mathrm{id}_{M_{q_m}}$)

$$\psi_m(f)(t) = \psi'_m(f(t)) = \begin{pmatrix} f(t) & 0 \\ 0 & \gamma_m(f(t)) \end{pmatrix} \otimes 1_5. \tag{e6.28}$$

Recall that $\gamma_m : M_{p_m q_m}(B) \to M_{R(m) p_m q_m}$ is a unital homomorphism with $R(m) = d_m - 1$. Note that by formula (e6.13), we may assume that $R(m) > 3^m$.

Stage 2: We will use a modified construction of Jiang and Su and define φ_m on $[0, 3/4]$.

Choose a (first) pair of different prime numbers k_0 and k_1 such that

$$k_0 > 15q_m \qquad \text{and} \qquad k_1 > 15k_0 d_m p_m. \tag{e6.29}$$

In particular, $k_0, k_1 \neq 3, 5$.

Recall that $(3, q_m) = 1$, $(5, p_m) = 1$ and $d_m = 3^{l_m}$ for some $l_m \geq 1$. Therefore, $(k_0 d_m p_m, k_1 5 q_m) = 1$. Let $p_{m+1} = k_0 d_m p_m$, $q_{m+1} = k_1 5 q_m$ and $k = k_0 k_1$. Then $(p_{m+1}, q_{m+1}) = 1$, $(5, p_{m+1}) = 1$ and $(3, q_{m+1}) = 1$. Write

$$k = r_0 + m(0) q_{m+1} \qquad \text{and} \qquad k = r_1 + m(1) p_{m+1}, \tag{e6.30}$$

where $m(0), r_0, m(1), r_1 \geq 1$ are integers and

$$0 < r_0 < q_{m+1}, \quad r_0 \equiv k (\mathrm{mod}\, q_{m+1}), \tag{e6.31}$$
$$0 < r_1 < p_{m+1}, \quad r_1 \equiv k (\mathrm{mod}\, p_{m+1}). \tag{e6.32}$$

Moreover, by formula (e6.29),

$$\begin{aligned} k - r_1 - r_0 &> k - q_{m+1} - p_{m+1} = k - k_1 5q_m - k_0 d_m p_m \\ &= k_1(k_0 - 5q_m) - k_0 d_m p_m \\ &> k_1(10q_m) - k_0 d_m p_m > 0. \end{aligned} \tag{e6.33}$$

We will construct paths ξ_i. At $t = 0$, define

$$\xi_i(0) := \begin{cases} 0 & \text{if } 1 \le i \le r_0, \\ 1/2 & \text{if } r_0 < i \le k. \end{cases} \tag{e6.34}$$

Note that since

$$r_0 5q_m \equiv k5q_m \equiv k_0 k_1 5q_m \equiv 0(\mathrm{mod}\, q_{m+1}), \tag{e6.35}$$

$r_0 5q_m = t_0 q_{m+1}$ for some integer $t_0 \ge 1$. Note also that if $f \in E_{d_m p_m, 5q_m}$, then $f(0) = b \otimes 1_{5q_m}$ for some $b \in M_{d_m p_m}(B)$. Hence $f(0) \otimes 1_{r_0} = b \otimes 1_{r_0 5q_m} = \left(b \otimes 1_{t_0}\right) \otimes 1_{q_{m+1}}$ for any $f \in E_{d_m p_m, 5q_m}$. On the other hand, for any $f \in E_{d_m p_m, 5q_m}$,

$$\mathrm{diag}\big(f(\xi_{r_0+1}(0)), \ldots, f(\xi_k(0))\big) =^s (f(1/2)) \otimes 1_{m(0)q_{m+1}}. \tag{e6.36}$$

In fact, there is a scalar unitary $s_0 \in M_{m(0)q_{m+1}d_m p_m 5q_m}$ such that

$$s_0^* \mathrm{diag}(\overbrace{b, b, \ldots, b}^{k-r_0}) s_0 = b \otimes 1_{m(0)q_{m+1}} \text{ for all } b \in M_{d_m p_m 5q_m}(B)$$

(recall that $f(1/2) \in M_{d_m p_m 5q_m}(B)$). Therefore, there exists a unitary $v_0 \in U\left(M_{p_{m+1}q_{m+1}}\right)$ such that for all $f \in E_{d_m p_m, 5q_m}$,

$$\rho_0(f) := v_0^* \begin{pmatrix} f(\xi_1(0)) & 0 & \cdots & 0 \\ 0 & f(\xi_2(0)) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & f(\xi_k(0)) \end{pmatrix} v_0 \tag{e6.37}$$

is in $M_{p_{m+1}}(B) \otimes 1_{q_{m+1}}$. So ρ_0 defines a homomorphism from $E_{d_m p_m, 5q_m}$ into $M_{p_{m+1}}(B) \otimes 1_{q_{m+1}}$.

At $t = 3/4$, define

$$\xi_i(3/4) := \begin{cases} 1/2 & \text{if } 1 \le i \le k - r_1, \\ 1 & \text{if } k - r_1 < i \le k. \end{cases} \tag{e6.38}$$

As in the case at 0, by formula (e6.32),

$$r_1 d_m p_m \equiv k d_m p_m \equiv k_0 k_1 d_m p_m \equiv 0(\mathrm{mod}\, p_{m+1}).$$

So one may write $r_1 d_m p_m = t_1 p_{m+1}$ for some integer $t_1 \ge 1$. Set $f \in E_{d_m p_m, 5q_m}$. Then $f(1) = 1_{d_m p_m} \otimes c$ for some $c \in M_{5q_m}$. It follows that $1_{r_1} \otimes f(1) = 1_{p_{m+1}} \otimes \left(1_{t_1} \otimes c\right)$. Also,

$$\mathrm{diag}\big(f(\xi_1(3/4)), \ldots, f\big(\xi_{k-r_1}(3/4)\big)\big) =^s 1_{m(1)p_{m+1}} \otimes f(1/2).$$

In fact, there is a scalar unitary $s_1 \in M_{m(1)p_{m+1}d_m p_m 5q_m}$ such that

$$s_1^*\mathrm{diag}(\overbrace{b, b, \ldots, b}^{k-r_1})s_1 = 1_{m(1)p_{m+1}} \otimes b \text{ for all } b \in M_{d_m p_m 5q_m}(B).$$

Thus there is a unitary $v_{3/4} \in U\big(M_{p_{m+1}q_{m+1}}\big)$ such that for $f \in E_{d_m p_m, 5q_m}$,

$$\rho_{3/4}(f) := v_{3/4}^* \begin{pmatrix} f(\xi_1(3/4)) & 0 & \cdots & 0 \\ 0 & f(\xi_2(3/4)) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & f(\xi_k(3/4)) \end{pmatrix} v_{3/4} \tag{e6.39}$$

defines a homomorphism from $E_{d_m p_m, 5q_m}$ to $1_{p_{m+1}} \otimes M_{q_{m+1}}(B)$.

To connect $\xi_i(0)$ and $\xi_i(3/4)$ continuously, on $[0, 3/4]$, let us define (see formula (e6.33))

$$\xi_i(t) := \begin{cases} 2t/3 & \text{if } 1 \le i \le r_0, \\ 1/2 & \text{if } r_0 < i \le k - r_1, \\ 1/2 + 2t/3 & \text{if } k - r_1 < i \le k. \end{cases} \tag{e6.40}$$

Let $v \in C\big([0, 3/4], M_{p_{m+1}q_{m+1}}\big)$ be a unitary such that $v(0) = v_0$ and $v(3/4) = v_{3/4}$. Now, on $[0, 3/4]$, define, for all $f \in E_{p_m, q_m}$,

$$\varphi_m(f)(t) := v(t)^* \begin{pmatrix} \psi_m(f) \circ \xi_1(t) & 0 & \cdots & 0 \\ 0 & \psi_m(f) \circ \xi_2(t) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \psi_m(f) \circ \xi_k(t) \end{pmatrix} v(t). \tag{e6.41}$$

Stage 3: We connect 3/4 to 1, recalling that $\pi_1\big(E_{p_{m+1}, q_{m+1}}\big) = 1_{p_{m+1}} \otimes M_{q_{m+1}}$.

We first extend ξ_i by defining

$$\xi_i(t) = \xi_i(3/4) \text{ for all } t \in (3/4, 1], \quad i = 1, 2, \ldots, k. \tag{e6.42}$$

Recall equation (e6.28); at $t = 3/4$, for each i and for $f \in E_{p_m, q_m}$,

$$\psi_m(f)(\xi_i(3/4)) = \begin{pmatrix} f(\xi_i(3/4)) & 0 \\ 0 & \gamma_m(f(\xi_i(3/4))) \end{pmatrix} \otimes 1_5. \tag{e6.43}$$

For $k - r_1 < i \le k$, define, for $t \in (3/4, 1]$ and $f \in E_{p_m, q_m}$,

$$\tilde{\psi}_{m,i}(f)(t) := \begin{pmatrix} f(\xi_i(3/4)) & 0 \\ 0 & \gamma_m(f(\xi_i(3/4))) \end{pmatrix} \otimes 1_5 = \begin{pmatrix} f(1) & 0 \\ 0 & \gamma_m(f(1)) \end{pmatrix} \otimes 1_5. \tag{e6.44}$$

Recall that $\gamma_m(f(1)) = f(1) \otimes 1_{d_m - 1}$. Therefore there exists a scalar unitary $s_3 \in M_{p_m q_m 5 d_m}$ such that

$$s_3^*\big(\tilde{\psi}_{m,i}(f)(t)\big)s_3 = 1_{d_m} \otimes f(1) \otimes 1_5 \text{ for all } f \in E_{p_m, q_m}, \quad t \in [3/4, 1].$$

Note that $f(1)$ has the form $1_{p_m} \otimes c$ for some $c \in M_{q_m}$. So there is a scalar unitary $s_4 \in M_{t_1 p_{m+1} q_m 5}$ such that

$$s_4^*\mathrm{diag}\big(\tilde{\psi}_{m,k-r_1+1}(f)(t), \ldots, \tilde{\psi}_{m,k}(f)(t)\big)s_4 = 1_{r_1 d_m p_m} \otimes c \otimes 1_5 = 1_{t_1 p_{m+1}} \otimes c \otimes 1_5. \tag{e6.45}$$

Now recall formula (e6.14) for the definition of θ_t. For $1 \le i \le k - r_1$, define, for $t \in (3/4, 1]$,

$$\tilde{\psi}_{m,i}(f)(t) := \begin{pmatrix} \theta_{4(t-3/4)}(f(\xi_i(t))) & 0 \\ 0 & \gamma_m(f(\xi_i(t))) \end{pmatrix} \otimes 1_5 \text{ for all } f \in E_{p_m,q_m}. \tag{e6.46}$$

Note that $\theta_1(f(\xi_i(3/4))) = \theta_1(f(1/2)) \in M_{p_m q_m}$ (see formulas (e6.38) and (e6.15)). Recall that $\varphi_m(f)(3/4) \in 1_{p_{m+1}} \otimes M_{q_{m+1}}(B)$. Note that for $1 \le i \le k - r_1$,

$$\tilde{\psi}_{m,i}(f)(1) = \begin{pmatrix} \theta_1(f(1/2)) & 0 \\ 0 & \gamma_m(f(1/2)) \end{pmatrix} \otimes 1_5 \text{ for all } f \in E_{p_m,q_m}.$$

Moreover (see also equation (e6.30)), there is a scalar unitary $s_5 \in M_{m(1)p_{m+1}d_m p_m q_m 5}$ such that

$$s_5^* \mathrm{diag}\big(\tilde{\psi}_{m,1}(f)(1), \ldots, \tilde{\psi}_{m,k-r_1}(f)(1)\big) s_5 = 1_{m(1)p_{m+1}} \otimes \begin{pmatrix} \theta_1(f(1/2)) & 0 \\ 0 & \gamma_m(f(1/2)) \end{pmatrix} \otimes 1_5.$$

Thus, for $t = 1$, there is a unitary $v_1 \in 1_{p_{m+1}} \otimes M_{q_{m+1}}$ such that

$$\rho_1(f) := v_1^* \begin{pmatrix} \tilde{\psi}_{m,1}(f)(1) & 0 & \cdots & 0 \\ 0 & \tilde{\psi}_{m,2}(f)(1) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \tilde{\psi}_{m,k}(f)(1) \end{pmatrix} v_1 \tag{e6.47}$$

defines a homomorphism from E_{p_m,q_m} to $1_{p_{m+1}} \otimes M_{q_{m+1}}$. There is a continuous path of unitaries $\{v(t) : t \in [3/4, 1]\} \subset M_{p_{m+1}q_{m+1}}$ such that $v(3/4)$ is as defined and $v(1) = v_1$ – so now $v \in C\big([0,1], M_{p_{m+1}q_{m+1}}\big)$ with $v(0) = v_0$ and $v(1) = v_1$, and $v(3/4)$ is consistent with the previous definition. Now define, for $t \in (3/4, 1]$,

$$\varphi_m(f)(t) := v(t)^* \begin{pmatrix} \tilde{\psi}_{m,1}(f)(t) & 0 & \cdots & 0 \\ 0 & \tilde{\psi}_{m,2}(f)(t) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \tilde{\psi}_{m,k}(f)(t) \end{pmatrix} v(t) \tag{e6.48}$$

for all $f \in E_{p_m,q_m}$. Note that by formulas (e6.43), (e6.46), (e6.47) and (e6.39),

$$\varphi_m(f)(1) = \rho_1(f) \text{ for all } f \in E_{p_m,q_m}. \tag{e6.49}$$

Hence φ_m is a unital injective homomorphism from E_{p_m,q_m} to $E_{p_{m+1},q_{m+1}}$. (Note that injectivity follows from the fact that $\cup_{i=1}^k \xi_i([0,1]) = [0,1]$, as $r_0 \ge 1$ and $k - r_1 > 0$.)

For convenience of notation and for later use, let us define $\tilde{\psi}_{m,i} : E_{p_m,q_m} \to C\big([0,1], M_{d_m p_m 5 q_m}(B)\big)$ by

$$\tilde{\psi}_{m,i}(f)(t) := \begin{cases} \psi_m(f \circ \xi_i(t)) & \text{if } t \in [0, 3/4], \\ \tilde{\psi}_{m,i}(f)(t) & \text{if } t \in (3/4, 1], \end{cases} \tag{e6.50}$$

for all $f \in E_{p_m,q_m}$. Then we may write, for all $t \in [0,1]$ and all $f \in E_{p_m,q_m}$,

$$\varphi_m(f)(t) = v(t)^* \begin{pmatrix} \tilde{\psi}_{m,1}(f)(t) & 0 & \cdots & 0 \\ 0 & \tilde{\psi}_{m,2}(f)(t) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \tilde{\psi}_{m,k}(f)(t) \end{pmatrix} v(t). \tag{e6.51}$$

Define $\theta^{(i)'} : E_{p_m,q_m} \to C\big([0,1], M_{p_mq_m}(B)\big)$, for each $f \in E_{p_m,q_m}$, by

$$\theta^{(i)'}(f)(t) := \begin{cases} f(\xi_i(t)) & \text{if } t \in [0,3/4], \\ \theta_{4(t-3/4)}(f(\xi_i(t))) & \text{if } t \in (3/4,1], \end{cases} \tag{e6.52}$$

if $\xi_i(3/4) = 1/2$; and if $\xi_i(3/4) = 1$, define

$$\theta^{(i)'}(f)(t) := f(\xi_i(t)) \text{ for all } t \in [0,1]. \tag{e6.53}$$

Define $\theta^{(i)}(f) := \theta^{(i)'}(f) \otimes 1_5$ for $f \in E_{p_m,q_m}$ and $\Theta_m : E_{p_m,q_m} \to C\big([0,1], M_{5kp_mq_m}(B)\big)$, for each $f \in E_{p_m,q_m}$, by

$$\Theta_m(f) := \operatorname{diag}\Big(\theta^{(1)}(f), \ldots, \theta^{(k)}(f)\Big) \text{ for all } t \in [0,1]. \tag{e6.54}$$

By formulas (e6.48), (e6.28) and (e6.46), as well as the definition of Θ_m, and by conjugating another unitary in $C\big([0,1], M_{p_{m+1}q_{m+1}}\big)$, we may write

$$\varphi_m(f) = u^* \begin{pmatrix} \Theta_m(f) & 0 & \cdots & 0 \\ 0 & \gamma_m(f \circ \xi_1) \otimes 1_5 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \gamma_m(f \circ \xi_k) \otimes 1_5 \end{pmatrix} u \text{ for all } f \in A_m. \tag{e6.55}$$

So φ_m does have the form of formula (e6.16). Condition (1) of the lemma follows from the definition of ξ_i and formulas (e6.40), (e6.42) and (e6.33). Condition (2) follows from formulas (e6.13) and (e6.28) (and two lines after it). Condition (3) follows from formula (e6.23). Finally, condition (4) follows from the definition of Θ_m. □

Definition 6.9. From 6.7 and Lemma 6.8, inductively, we define $A_1 = E_{3,5}$, $A_m = E_{p_m,q_m}$ and homomorphism s $\varphi_m : A_m \to A_{m+1} = A_{p_{m+1},q_{m+1}}$ as described in Lemma 6.8. Then we define $A = \lim_{n\to\infty}(A_m, \varphi_m)$.

Remark 6.10. It should be noted that if $f(0), f(1) \in M_{p_mq_m}$, then $\Theta_m(f(0))$ and $\Theta_m(f(1))$ are also scalar matrices.

7. Conclusion of the construction

Definition 7.1. Let $\{\xi_i : 1 \le i \le m\}$ be a collection of maps described in Lemma 6.8(1). Note that each of the three types occurs at least once. Such a collection is said to be full. Let $C_2 := \Big\{\xi_i^{(1)} \circ \xi_j^{(2)} : 1 \le i \le m_1, 1 \le j \le m_2\Big\}$ be a collection of compositions of two maps in Lemma 6.8(1). This collection is called full if $\Big\{\xi_j^{(2)} : 1 \le j \le m_2\Big\}$ is a full collection, and for each fixed $\xi_j^{(2)}$, $\Big\{\xi_i^{(1)} : \xi_i^{(1)} \circ \xi_j^{(2)} \in C_2\Big\}$ is also a full collection. Inductively, a collection of n compositions of maps in Lemma 6.8(1),

$$C_n = \Big\{\xi_{j(1)}^{(1)} \circ \xi_{j(2)}^{(2)} \circ \cdots \circ \xi_{j(n)}^{(n)} : 1 \le j(i) \le m_i : i = 1, 2, \ldots, n\Big\}$$

is called full if $\Big\{\xi_{j(n)}^{(n)} : 1 \le j(n) \le m_n\Big\}$ is a full collection, and for each fixed $\xi_{j(n)}^{(n)}$, the collection

$$\Big\{\xi_{j(1)}^{(1)} \circ \xi_{j(2)}^{(2)} \circ \cdots \circ \xi_{j(n-1)}^{(n-1)} : \xi_{j(1)} \circ \xi_{j(2)} \circ \cdots \circ \xi_{j(n-1)} \circ \xi_{j(n)} \in C_n\Big\}$$

is full.

Lemma 7.2. *Let* $\Xi = \xi_{j(1)} \circ \xi_{j(2)} \circ \cdots \circ \xi_{j(n)} : [0,1] \to [0,1]$ *be a composition of n maps, where each* $\xi_{j(k)} : [0,1] \to [0,1]$ *(*$1 \le k \le n$*) is one of the three types of continuous maps given in Lemma 6.8(1). Then for any* $x, y \in [0,1]$,

$$|\Xi(x) - \Xi(y)| \le (2/3)^n. \tag{e7.1}$$

Moreover, if $\{\Xi_j : 1 \le j \le l\}$ *is a full collection of compositions of n maps as before, then*

$$\cup_{j=1}^{l} \Xi_j([0,1]) = [0,1], \tag{e7.2}$$

and for each $t \in [0,1]$, $\{\Xi_j(t) : 1 \le j \le l\}$ *is* $(2/3)^n$*-dense in* $[0,1]$.

Proof. Note that for each i and for any $x, y \in [0,1]$, we have $|\xi_i(x) - \xi_i(y)| \le (2/3)|x-y|$. Then, by induction, for all $x, y \in [0,1]$,

$$|\Xi(x) - \Xi(y)| = \left|\xi_{j(1)} \circ \xi_{j(2)} \circ \cdots \circ \xi_{j(n)}(x) - \xi_{j(1)} \circ \xi_{j(2)} \circ \cdots \circ \xi_{j(n)}(y)\right| \tag{e7.3}$$

$$\le (2/3)\left|\xi_{j(2)} \circ \cdots \circ \xi_{j(n)}(x) - \xi_{j(2)} \circ \cdots \circ \xi_{j(n)}(y)\right| \tag{e7.4}$$

$$\le \cdots \le (2/3)^n |x-y| \le (2/3)^n. \tag{e7.5}$$

One already observes that $\cup_{j \in S} \xi_j([0,1]) = [0,1]$ if $\{\xi_j : j \in S\}$ is a full collection. An induction shows that if $\{\Xi_j : 1 \le j \le l\}$ is a full collection, then

$$\cup_{j=1}^{l} \Xi_j([0,1]) = [0,1].$$

To show the last statement, fix $t \in [0,1]$. Set $x \in [0,1]$. Then for some $y \in [0,1]$ and $j \in \{1, 2, \ldots, l\}$,

$$\Xi_j(y) = x.$$

Now, by the first part of the statement, for any $t \in [0,1]$,

$$\left|\Xi_j(t) - \Xi_j(y)\right| \le (2/3)^n.$$

It follows that

$$\left|\Xi_j(t) - x\right| = \left|\Xi_j(t) - \Xi_j(y)\right| \le (2/3)^n. \qquad \square$$

Theorem 7.3. *The inductive limit A defined in Definition 6.9 can be made into a unital simple* C^**-algebra* A_z^C *such that*

$$\left(K_0(A_z), K_0(A_z)_+, [1_{A_z}], K_1(A_z)\right) = (\mathbb{Z}, \mathbb{Z}_+, 1, \{0\}). \tag{e7.6}$$

If C is not exact, then A_z^C *is not exact.*

Proof. For convenience, one makes an additional requirement in the construction. Let $\mathcal{F}_{m,1} \subset \mathcal{F}_{m,2}, \ldots, \mathcal{F}_{m,n}, \ldots$ be an increasing sequence of finite subsets of A_m such that $\cup_n \mathcal{F}_{m,n}$ is dense in A_m.

One requires $\varphi_m(\mathcal{F}_{m,m+1}) \subset \mathcal{F}_{m+1,1}$ and $\varphi_m(\mathcal{F}_{m,m+n}) \subset \mathcal{F}_{m+1,n}$, $m, n = 1, 2, \ldots$.

This is done inductively as follows: Choose any increasing sequence of finite subsets $\mathcal{F}_{1,1} \subset \mathcal{F}_{1,2}, \ldots, \subset A_1$ such that $\cup_n \mathcal{F}_{1,n}$ is dense in A_1. Specify $\mathcal{F}_1 = \mathcal{F}_{1,1} \setminus \{0\}$. Choose A_2 and define $\varphi_1 : A_1 \to A_2$ as in the construction of Lemma 6.8.

Choose an increasing sequence of finite subsets $\mathcal{F}_{2,1}, \mathcal{F}_{2,2}, \ldots$ of A_2 such that $\varphi_1(\mathcal{F}_{1,n}) \subset \mathcal{F}_{2,n}$ ($n = 1, 2, \ldots$) such that $\cup_n \mathcal{F}_{2,n}$ is dense in A_2. Specify $\mathcal{F}_2 = \mathcal{F}_{2,1} \setminus \{0\}$.

Once $\mathcal{F}_{m,1}, \mathcal{F}_{m,2}, \ldots$ are determined, specify $\mathcal{F}_m = \mathcal{F}_{m,1} \setminus \{0\}$. Then construct A_{m+1} and $\varphi_m : A_m \to A_{m+1}$ as in Lemma 6.8. Choose $\mathcal{F}_{m+1,1}, \mathcal{F}_{m+1,2}, \ldots$ so that $\varphi_m(\mathcal{F}_{m,m+1}) \subset \mathcal{F}_{m+1,1}$ and $\varphi_m(\mathcal{F}_{m,m+n}) \subset$

 https://doi.org/10.1017/fms.2021.79

$\mathcal{F}_{m+1,n}$, as well as $\mathcal{F}_{m+1,n} \subset \mathcal{F}_{m+1,n+1}$. Moreover, $\cup_n \mathcal{F}_{m+1,n}$ is dense in A_{m+1}. Choose $\mathcal{F}_{m+1} = \mathcal{F}_{m+1,1} \setminus \{0\}$. Thus the requirement can be made.

Let us now prove that A is simple. For this, we will prove the following claim:

Claim: For any fixed i, and $g \in A_i \backslash \{0\}$, there exists $n > i$ such that $\varphi_{i,n}(g)$ is full in A_n. Without loss of generality, we may assume that $\|g\| = 1$. There are j and $f \in \mathcal{F}_{i,j}$ such that $\|f - g\| < 1/64$.

To simplify notation, without loss of generality we may write $i = 1$. Set $\varphi_{j,j'} = \varphi_{j'-1} \circ \cdots \circ \varphi_j$ for $j' > j$. Then $\varphi_{1,j'}(f) \in \varphi_{j'}(\mathcal{F}_{j'-1,j'}) \subset \mathcal{F}_{j',1}$. Recall also that each φ_j is unital and injective. To further simplify the notation, without loss of generality we may write $\mathcal{F}_{i,j} \setminus \{0\} = \mathcal{F}_m = \mathcal{F}_{m,1} \setminus \{0\}$. We assume that $m > 128$. By construction, for some $t \in (0,1)$,

$$\|\gamma_m(f(t))\| > (1 - 1/2m)\|f\| \neq 0. \tag{e7.7}$$

By continuity, there is $n(m) \geq 1$ such that for any $(2/3)^{n(m)-1}$-dense set S of $[0,1]$,

$$\|\gamma_m(f(s))\| \geq (1 - 1/2m)\|f\| \neq 0 \text{ for some } s \in S. \tag{e7.8}$$

For any $f \in C([0,1], M_{p_m q_m}(B))$ and i, denote $h(t) = \gamma_m(f \circ \xi_i(t)) \otimes 1_5$ (for $t \in [0,1]$). Then, for any $k > m$ and j ($t \in [0,1]$),

$$\gamma_k(h \circ \xi_j(t)) = \gamma_k(\gamma_m(f \circ \xi_i \circ \xi_j(t)) \otimes 1_5) = \gamma_m(f \circ \xi_i \circ \xi_j(t)) \otimes 1_{5R(k)}, \tag{e7.9}$$

where $R(k)$ is the rank of γ_k and ξ_i and ξ_j are as defined in Lemma 6.8(1). Denote

$$\bar{\gamma}_{k+1}(f)(t) = \gamma_{k+1}(f(t)) \otimes 1_5 \text{ for all } f \in C([0,1], M_{p_k q_k}(B)) \text{ (and } t \in [0,1]). \tag{e7.10}$$

Therefore, from Lemma 6.8 and formula (e6.16) (also keep in mind Remark 6.10), we may write, for each $f \in A_m = E_{p_m, q_m}$,

$$\varphi_{m,m+2}(f) = w_1^* \begin{pmatrix} H_0(f) & & & 0 \\ & \bar{\gamma}_m\left(f \circ \xi_1^{(2)}\right) \otimes 1_{R(m+1)} & & \\ & & \ddots & \\ 0 & & & \bar{\gamma}_m\left(f \circ \xi_{l(m+1)}^{(2)}\right) \otimes 1_{R(m+1)} \end{pmatrix} w_1, \tag{e7.11}$$

where $H_0 : A_m \to C([0,1], M_{L_0 p_m q_m})$ is a homomorphism (for some integer $L_0 \geq 1$), $w_1 \in C([0,1], M_{p_{m+2} q_{m+2}})$ is a unitary, $R(m+1)$ is the rank of $\bar{\gamma}_{m+1}$ and $\left\{\xi_j^{(2)} : 1 \leq j \leq l(m+1)\right\}$ is a full collection of compositions of two ξ_is (maps in Lemma 6.8(1)).

Therefore, by induction, for any $n > n(m) + m$ one may write, from the construction of Lemma 6.8 (see equation (e6.16)), for all $f \in A_m = E_{p_m, q_m}$,

$$\varphi_{m,n}(f) = w^* \begin{pmatrix} H(f) & & & 0 \\ & \bar{\gamma}_m(f \circ \Xi_1) \otimes 1_{R(n,1)} & & \\ & & \ddots & \\ 0 & & & \bar{\gamma}_m(f \circ \Xi_l) \otimes 1_{R(n,l)} \end{pmatrix} w, \tag{e7.12}$$

where $H : A_m \to C([0,1], M_{L p_m q_m}(B))$ is a homomorphism (for some integer $L \geq 1$), Ξ_j is a composition of $n-m$ maps in Lemma 6.8(1) such that the collection $\{\Xi_j : 1 \leq j \leq l\}$ is full, $R(n,j) \geq 1$ is an integer, $j = 1, 2, \ldots, l$, and $w \in C([0,1], M_{p_n q_n})$ is a unitary.

It follows from Lemma 7.2 that

$$|\Xi_i(x) - \Xi_i(y)| < (2/3)^{m-n} \text{ for all } x, y \in [0,1], \quad 1 \le i \le l, \qquad \text{and} \qquad \cup_i^l \Xi_i([0,1]) = [0,1]. \tag{e7.13}$$

Fix any $t \in [0,1]$ and $x \in [0,1]$; by Lemma 7.2, there are $y \in [0,1]$ and $j \in \{1, 2, \ldots, l\}$ such that $\Xi_j(y) = x$. Then

$$\left|\Xi_j(t) - x\right| = \left|\Xi_j(t) - \Xi_j(y)\right| < (2/3)^{n-m} < (2/3)^{n(m)}. \tag{e7.14}$$

It follows from the choice of $n(m)$ and formula (e7.8) that, for $f \in \mathcal{F}_m$,

$$\left\|\gamma_m\big(f \circ \Xi_j(t)\big)\right\| \ge (1 - 1/m)\|f\| \ge \left(\frac{63}{64}\right)^2 \text{ for all } t \in [0,1]. \tag{e7.15}$$

Since $\left\|f\big(\Xi_j(t)\big) - g\big(\Xi_j(t)\big)\right\| < 1/64$, this implies that

$$\left\|\gamma_m\big(g\big(\Xi_j(t)\big)\big)\right\| \ge \frac{63^2 - 64}{64^2} \text{ for all } t \in [0,1]. \tag{e7.16}$$

Since for each $t \in [0,1]$, we have $\gamma_m(g \circ \Xi_i(t)) \in M_{p_m q_m}$, $i = 1, 2, \ldots, l$, we know that $\varphi_{m,n}(g)(t)$ is not in any closed ideal of $M_{p_n q_n}(B)$ for each $t \in [0,1]$. Therefore $\varphi_{m,n}(g)$ is full in $E_{p_n,q_n} = A_n$. This proves the claim.

It follows from the claim that A_z^C is simple. To see this, let $I \subset A_z^C$ be an ideal such that $I \ne A_z^C$ and put $C_n = \varphi_{n,\infty}(A_n)$. Then $C_n \subset C_{n+1}$ for all n. Set $a \in C_m \setminus \{0\}$. By the claim, there is $n' > m$ such that a is full in $C_{n'}$, and therefore a is full in every C_n for $n \ge n'$. In other words, $a \notin C_n \cap I$ for all n. It follows that $C_m \cap I = \{0\}$, as $C_m \subset C_n$ for all $n \ge m$. It is then standard to show that $I = \{0\}$. Thus A_z^C is simple.

Since, by Lemma 6.5, we have for each m that

$$\big(K_0(A_m), K_0(A_m)_+, \left[1_{A_m}\right], K_1(A_m)\big) = (\mathbb{Z}, \mathbb{Z}_+, 1, \{0\}),$$

one concludes (as each φ_n is unital) that

$$\left(K_0\left(A_z^C\right), K_0\left(A_z^C\right)_+, \left[1_{A_z^C}\right], K_1\left(A_z^C\right)\right) = (\mathbb{Z}, \mathbb{Z}_+, 1, \{0\}). \tag{e7.17}$$

Finally, if C is not exact, then B is not exact, since B has quotients of the form $\mathbb{C} \oplus C$, which is not exact.

Define $\Phi : B \to C([0,1], M_{15}(B))$ by

$$\Phi(f)(t) := \theta_t(f) \otimes 1_{15} \text{ for all } f \in B \text{ and } t \in [0,1], \tag{e7.18}$$

where $\theta_t : B \to B$ is defined in formula (e6.14). Note that for $f \in B$,

$$\Phi(f)(0) = \theta_0(f) \otimes 1_{15} = f \otimes 1_{15} \in M_3(B) \otimes 1_5 \qquad \text{and} \qquad \Phi(f)(1) = f(0) \otimes 1_{15} \in \mathbb{C} \cdot 1_{15}. \tag{e7.19}$$

One then obtains a unitary $u \in C([0,1], M_{15})$ such that

$$u^*\Phi(f)u \in E_{3,5}. \tag{e7.20}$$

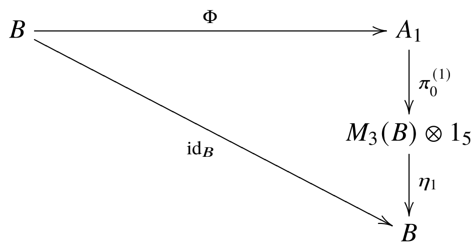
Define $\Psi(f) := u^*\Phi(f)u$ for all $f \in B$. Then Ψ is a unital injective homomorphism. In other words, B is embedded unitally into $A_1 = E_{3,5}$. Since each $\varphi_m : A_m \to A_{m+1}$ is unital and injective, B is embedded into A_z^C. Since B is not exact, neither is A_z^C (see, for example, [47, Proposition 2.6]). □

Proposition 7.4. *If C is exact but not nuclear, then A_z^C is exact and not nuclear.*

Proof. Note that since C is nonnuclear and exact, so is B. Note also that $A_n = E_{p_n,q_n}$ is a C^*-subalgebra of the exact C^*-algebra $C\big([0,1], M_{p_nq_n}(B)\big)$. So each A_n is exact. By [47, 2.5.5], A_z^C is exact.

Let $\Phi : B \to A_1 = E_{3,5}$ be as in the end of the proof of Theorem 7.3. Let $\pi_0^{(1)} : A_1 \to M_3(B) \otimes M_5$ be the evaluation at 0, namely $\pi_0^{(1)}(f) = f(0)$ for all $f \in A_1$. Let $\eta_1 : M_3(B) \otimes 1_5 \to B$ be given by defining $\eta_1\big((b_{i,j})_{3\times 3} \otimes 1_5\big) = b_{1,1}$, where $b_{i,j} \in B$, $1 \le i, j \le 3$. Then η_1 is a norm 1 c.p.c. map. Define $\pi_0^{(1,1)} : A_1 \to B$ by $\pi_0^{(1,1)}(f) := \eta_1 \circ \pi_0^{(1)}$. Note that $\pi_0^{(1,1)} \circ \Phi$ is an isomorphism. In fact, $\pi_0^{(1,1)} \circ \Phi(b) = \theta_0(b) = b$ (see equation (e7.19)) for all $b \in B$.

The foregoing is illustrated in the following diagram:



(e7.21)

We will use the same diagram in the n-stage.

In Lemma 6.8(4), let us denote ξ_1 such that $\xi_1(t) = 2t/3$ for $t \in [0, 3/4]$ and $\theta^{(1)}(f(0)) = f(0)$ for all $f \in E_{3,5}$ (note that we do not change the connecting map, but only for convenience in equation (e7.22)). So by formulas (e6.16), (e6.20), (e6.52) and (e6.53), we may write

$$\varphi_1(f)(= \varphi_{1,2}(f)) = u_1^* \mathrm{diag}\Big(\theta^{(1,2)}(f), H_1'(f)\Big)u_1 \text{ for all } f \in A_1, \tag{e7.22}$$

where $\theta^{(1,2)} := \theta^{(1)'} : A_1 \to C([0,1], M_{15}(B))$ and $\theta^{(1,2)}(f)(0) = f(0)$ for $f \in A_1$, and $H_1' : A_1 = E_{3,5} \to C\big([0,1], M_{p_1q_1}(B)\big)$ is a homomorphism. Note that the image of H_1' is in a corner of $C\big([0,1], M_{p_1q_1}(B)\big)$, and $u_1 \in U\big(C\big([0,1], M_{p_2q_2}\big)\big)$. Similarly, again by formulas (e6.16), (e6.20), (e6.52) and (e6.53), we may also write

$$\varphi_{1,3}(f) = u_2^* \mathrm{diag}\Big(\theta^{(1,3)}(f), H_2'(f)\Big)u_2 \text{ for all } f \in A_1, \tag{e7.23}$$

where $\theta^{(1,3)}(f)(0) = f(0)$ for $f \in A_1$, $H_2' : A_1 \to C\big([0,1], M_{p_2q_2}(B)\big)$ is a homomorphism and $u_2 \in U\big(C\big([0,1], M_{p_3q_3}\big)\big)$. By induction, for any $n > 1$ we may write

$$\varphi_{1,n}(f) = u_n^* \mathrm{diag}\Big(\theta^{(1,n)}(f), H_n'(f)\Big)u_n \text{ for all } f \in A_1, \tag{e7.24}$$

where $\theta^{(1,n)}(f)(0) = f(0)$, $H_n' : A_1 \to C\big([0,1], M_{p_nq_n}(B)\big)$ is a homomorphism and $u_n \in C\big([0,1], M_{p_{n+1}q_{n+1}}\big)$. (One should be warned that $u_n^* \mathrm{diag}\Big(\theta^{(1,n)}, 0, \ldots, 0\Big)u_n$ is not in A_n.)

Now we prove that A_z^C is not nuclear. We follow the proof of [13, Proposition 6]. Assume otherwise: For any finite subset $\mathcal{F} \subset B$ and $\varepsilon > 0$, if A_z^C were nuclear, then $\varphi_{1,\infty} \circ \Phi$ would be nuclear. Therefore there would be a finite-dimensional C^*-algebra D and c.p.c. maps $\alpha : B \to D$ and $\beta : D \to A_z^C$ such that

$$\Big\|\varphi_{1,\infty} \circ \Phi(b) - \beta \circ \alpha(b)\Big\| < \varepsilon/2 \text{ for all } b \in \mathcal{F}. \tag{e7.25}$$

Since A_z^C is assumed to be nuclear, by the Effros–Choi lifting theorem [11], there exist an integer $n \ge 1$ and a unital c.p.c. map $\beta_n : D \to A_n$ such that

$$\Big\|\beta(x) - \varphi_{n,\infty} \circ \beta_n(x)\Big\| < \varepsilon/2 \text{ for all } x \in \alpha(\mathcal{F}). \tag{e7.26}$$

Thus

$$\left\|\varphi_{n,\infty}\big(\varphi_{1,n}\circ\Phi(b)-\beta_n\circ\alpha(b)\big)\right\|<\varepsilon. \tag{e7.27}$$

As $\varphi_{n,\infty}$ is an isometry, this implies that

$$\left\|\varphi_{1,n}\circ\Phi(b)-\beta_n\circ\alpha(b)\right\|<\varepsilon \text{ for all } b\in B. \tag{e7.28}$$

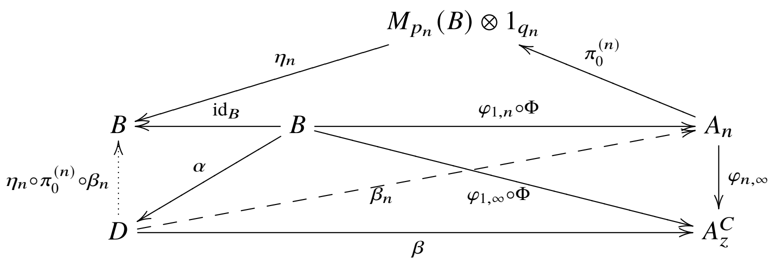
Let $\pi_0^{(n)}: E_{p_n,q_n}\to M_{p_n}(B)\otimes 1_{q_n}$ be the evaluation at 0 defined by $\pi_0^{(n)}(a):=a(0)$. We have, by equation (e7.24),

$$\pi_0^{(n)}\big(\varphi_{1,n}\circ\Phi(b)\big)=\mathrm{diag}\big(\theta_0(b)\otimes 1_{15}, H_n'(\Phi(f))(0)\big) \text{ for all } b\in B. \tag{e7.29}$$

Recall that $\theta_0(b)=b$. Now a rank one projection p corresponding the first $(1,1)$ corner is in $M_{p_n}(B)\otimes 1_{q_n}$. Put $q=u_n(0)^*pu_n(0)$. We now use the n-stage diagram (e7.21). Let $\eta_n: M_{p_n}(B)\otimes 1_{q_n}\to B$ be defined by $\eta_n(x):=u_n(0)qxqu_n(0)^*$ for all $x\in M_{p_n}(B)\otimes 1_{q_n}$ which is a unital c.p.c. map $\eta_n\Big(u_n(0)^*\Big(\big(b_{i,j}\big)_{p_n\times p_n}\otimes 1_{q_n}\Big)u_n(0)\Big)=b_{1,1}$. Note that $\eta_n\circ\pi_0^{(n)}\circ\varphi_{1,n}\circ\Phi=\mathrm{id}_B$. By formula (e7.28),

$$\left\|b-\eta_n\circ\pi_0^{(n)}\circ\beta_n\circ\alpha(b)\right\|=\left\|\eta_n\circ\pi_0^{(n)}\big(\varphi_{1,n}\circ\Phi(b)-\beta_n\circ\alpha(b)\big)\right\|<\varepsilon \text{ for all } b\in B. \tag{e7.30}$$

This would imply that B is nuclear. Therefore A_z^C is not nuclear. The foregoing could be illustrated by the following diagram, which is only approximately commutative below the top triangle:



□

Theorem 7.5. *The inductive limit A_z^C in Theorem 7.3 has a unique tracial state.*

Proof. First we note each unital C^*-algebra $A_m=E_{p_m,q_m}$ has at least one tracial state, say τ_m. Note that $\varphi_{m,\infty}$ is an injective homomorphism. So we may view τ_m as a tracial state of $\varphi_{m,\infty}(A_m)$. Extend τ_m to a state t_m on A_z^C. Choose a weak*-limit of $\{t_m\}$, say t. Then t is a state of the unital C^*-algebra A_z^C. Note that $\varphi_{m,\infty}(A_m)\subset\varphi_{n,\infty}(A_n)$ if $n>m$. Then for each pair $x,y\in\varphi_{m,\infty}(A_m)$, and for any $n>m$, $t_n(xy)=t_n(yx)$. It follows that t is a tracial state of A_z^C. In other words, A_z^C has at least one tracial state.

Claim: For each k, each $a\in A_k$ with $\|a\|\le 1$ and each $\varepsilon>0$, there exists $N>k$ such that, for all $n\ge N$,

$$\left|\tau_1\big(\varphi_{k,n}(a)\big)-\tau_2\big(\varphi_{k,n}(a)\big)\right|<\varepsilon \text{ for all } \tau_1,\tau_2\in T(A_n). \tag{e7.31}$$

Fix $a\in A_k$. To simplify the notation, without loss of generality we may assume that $k=1$.

Choose $m>1$ such that

$$1/3^{m-1}<\varepsilon/4. \tag{e7.32}$$

Put $g = \varphi_{1,m}(a)$. There is $\delta > 0$ such that

$$\|g(x) - g(y)\| < \varepsilon/4 \text{ for all } x, y \in [0,1] \text{ with } |x - y| < \delta. \tag{e7.33}$$

Recall that here we view γ_m as a map from $M_{p_m q_m}(B)$ to $M_{R(m)p_m q_m}$. Note that for each $f \in A_m$, since $\gamma_m(f(t))$ is a scalar matrix for all $t \in [0,1]$, we have that $\gamma_m(f(t))(x)$, as an element in $M_{R(m)p_m q_m}(B)$, is a constant matrix (for $x \in (0,1]$) in $M_{R(m)p_m q_m}(C_0((0,1], C)^\sim)$. Hence (see equation (e7.10) for $\bar{\gamma}_m$), for $t \in [3/4, 1]$ – recalling that $\xi_i(t) = \xi_i(3/4)$ for all t in $[3/4, 1]$ –

$$\theta_{4(t-3/4)}\big(\bar{\gamma}_m\big(f\big(\Xi_j \circ \xi_i\big)\big)(3/4)\big) = \bar{\gamma}_m\big(f\big(\Xi_j \circ \xi_i\big)(3/4)\big) = \bar{\gamma}_m\big(f\big(\Xi_j \circ \xi_i\big)\big)(t). \tag{e7.34}$$

(Recall the definition of θ_t in formula (e6.21)). Therefore (see the definition of $\theta^{(i)}$ in formula (e6.52)), for any i with $\xi_i(3/4) \neq 1$,

$$\theta^{(i)}\big(\bar{\gamma}_m\big(f \circ \Xi_j\big)\big)(t) = \begin{cases} \bar{\gamma}_m\big(f \circ \Xi_j \circ \xi_i\big)(t) & \text{if } t \in [0, 3/4], \\ \theta_{4(t-3/4)}\big(\bar{\gamma}_m\big(f \circ \Xi_j \circ \xi_i\big)(3/4)\big) & \text{if } t \in (3/4, 1] \end{cases} \tag{e7.35}$$

$$= \bar{\gamma}_m\big(f\big(\Xi_j \circ \xi_i\big)\big)(t). \tag{e7.36}$$

For those i such that $\xi_i(3/4) = 1$, one also has

$$\theta^{(i)}\big(\bar{\gamma}_m\big(f \circ \Xi_j\big)\big) = \bar{\gamma}_m\big(f \circ \Xi_j \circ \xi_i\big). \tag{e7.37}$$

It follows (recall Lemma 6.8(4) for the definition of Θ_{m+1}, and also keep Remark 6.10 in mind) that

$$\Theta_{m+1}(\varphi_m(f)) = u^* \begin{pmatrix} \Theta'_{m+1}(f) & 0 & \cdots & 0 \\ 0 & \bar{\gamma}_m\Big(f \circ \xi_1^{(2)}\Big) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \bar{\gamma}_m\Big(f \circ \xi_{k'}^{(2)}\Big) \end{pmatrix} u \text{ for all } f \in A_m, \tag{e7.38}$$

where $u \in C\big([0,1], M_{5k_2 p_{m+1} q_{m+1}}\big)$ is a unitary (the integer k_2 is the integer k in Lemma 6.8 for φ_{m+1}), $\Theta'_{m+1} : A_m \to C\big([0,1], M_{T(0)p_m q_m}(B)\big)$ is a homomorphism for some integer $T(0) \geq 1$ and $\Big\{\xi_j^{(2)} : 1 \leq j \leq k'\Big\}$ is a full collection of compositions of two maps in Lemma 6.8(1). Moreover, by Lemma 6.8(2),

$$T(0)/5k'R(m) < 1/3^m. \tag{e7.39}$$

Then, combining with equation (e7.9), we may write $\varphi_{m,m+2} : A_m \to A_{m+2}$ as

$$\varphi_{m,m+2}(f) = u_1^* \begin{pmatrix} H_{m+1}(f) & 0 & \cdots & 0 \\ 0 & \bar{\gamma}_m\Big(f \circ \xi_1^{(2)}\Big) \otimes 1_{r(1)} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \bar{\gamma}_m\Big(f \circ \xi_{l(m)}^{(2)}\Big) \otimes 1_{r(l(m))} \end{pmatrix} u_1 \tag{e7.40}$$

for all $f \in A_m$, where $u_1 \in C\big([0,1], M_{p_{m+2}q_{m+2}}\big)$ is a unitary, $H_{m+1} : A_m \to C\big([0,1], M_{T(1)p_m q_m}(B)\big)$ is a homomorphism for some integer $T(1) \geq 1$, $\Big\{\xi_j^{(2)} : 1 \leq j \leq l\Big\}$ is a full collection of compositions of two maps in Lemma 6.8(1) and $r(l(j)) \geq 1$ is an integer, $j = 1, 2, \ldots, l(m)$. Moreover,

$$T(1)/5R(m)\left(\sum_{j=1}^{l(m)} r(l(j))\right) < 1/3^m. \tag{e7.41}$$

Therefore, by Lemma 6.8 – noting equations (e6.16), (e6.20) and (e7.36) – and the proof of Theorem 7.3 (see equation (e7.9)), as well as equation (e7.38) repeatedly, one may write, for each $n > m$ and all $f \in A_m$,

$$\varphi_{m,n}(f) = w^* \begin{pmatrix} H_{m,n}(f) & & & 0 \\ & \bar{\gamma}_m(f \circ \Xi_1) & & \\ & & \ddots & \\ 0 & & & \bar{\gamma}_m(f \circ \Xi_L) \end{pmatrix} w, \tag{e7.42}$$

where $w \in C\big([0,1], M_{p_n q_n}\big)$ is a unitary, $H_{m,n} : A_m \to C\big([0,1], M_{L(0)p_m q_m}(B)\big)$ is a homomorphism for some integer $L(0) \geq 1, \Xi_j : [0,1] \to [0,1]$ is a composition of $n-m$ many ξ_is and $\{\Xi_j : 1 \leq j \leq L\}$ is a full collection. Moreover,

$$L(0)/5LR(m) < 1/3^m. \tag{e7.43}$$

We choose N such that $(2/3)^{N-m} < \delta$ and choose any $n \geq N$.

Set $\tau_i \in T(A_n)$ $(i = 1, 2)$. Then, for any $f \in A_n$,

$$\tau_i(f) = \int_0^1 \sigma_i(t)(f(t))d\mu_i, \quad i = 1, 2, \tag{e7.44}$$

where $\sigma_i(t)$ is a tracial state of $M_{p_n q_n}(B)$ for all $t \in (0,1)$, $\sigma_i(0)$ is a tracial state of $M_{p_n}(B) \otimes 1_{q_n}$, $\sigma_i(1)$ is a tracial state of $1_{p_n} \otimes M_{q_n}$ and μ_i is a probability Borel measure on $[0,1]$, $i = 1, 2$. For each $t \in [0,1]$ and for $f(t) \in M_{p_n q_n} \subset M_{p_n q_n}(B)$,

$$\sigma_i(t)(f(t)) = \mathrm{tr}(f(t)), \quad i = 1, 2, \tag{e7.45}$$

where tr is the normalised trace on $M_{p_n q_n}$ (see equation (e6.11)). For each $j \in \{1, 2, \ldots, L\}$, by Lemma 7.2,

$$\left|\Xi_j(x) - \Xi_j(y)\right| < (2/3)^{n-m} < \delta \text{ for all } x, y \in [0,1]. \tag{e7.46}$$

By the choice of δ,

$$\left\|g \circ \Xi_j(x) - g \circ \Xi_j(y)\right\| < \varepsilon/4 \text{ for all } x, y \in [0,1]. \tag{e7.47}$$

For each $f \in A_m$, write

$$H'(f)(t) = \begin{pmatrix} H_{m+1}(f)(t) & 0 \\ 0 & 0 \end{pmatrix} \text{ for all } t \in [0,1]. \tag{e7.48}$$

Then one has, for each $f \in A_m$ and $i = 1, 2$,

$$\tau_i\big(\varphi_{m,n}(f)\big) = \int_0^1 \sigma_i(t)\big(\varphi_{m,n}(f)\big)d\mu_i \tag{e7.49}$$

$$= \int_0^1 \sigma_i(t)(H'(f)(t))d\mu_i + \int_0^1 \mathrm{tr}\left(\bigoplus_{j=1}^{L}(f \circ \Xi_j(t))\right)d\mu_i. \tag{e7.50}$$

By formula (e7.47), recalling that $\|g\| \le 1$,

$$\int_0^1 \left|\mathrm{tr}\left(\bigoplus_{j=1}^{L}(g \circ \Xi_j(1/2)\right) - \bigoplus_{j=1}^{L}(g \circ \Xi_j(t))\right|d\mu_i < (\varepsilon/4)\int_0^1 d\mu_i = \varepsilon/4. \tag{e7.51}$$

By formula (e7.43),

$$\int_0^1 |\sigma_i(t)(H'(g)(t))|d\mu_i < (1/3)^m < \varepsilon/4. \tag{e7.52}$$

Recall that $\varphi_{1,n}(a) = \varphi_{m,n}(g)$. Thus, by formulas (e7.49), (e7.50), (e7.51) and (e7.52),

$$\left|\tau_i\big(\varphi_{1,n}(a)\big) - \sum_{j=1}^{L}\mathrm{tr}\big(g\big(\Xi_j(1/2)\big)\big)\right| < \varepsilon/2, \quad i = 1, 2. \tag{e7.53}$$

Therefore,

$$\big|\tau_1\big(\varphi_{1,n}(a)\big) - \tau_2\big(\varphi_{1,n}(a)\big)\big| < \varepsilon. \tag{e7.54}$$

This proves the claim.

To complete the proof, set $s_1, s_2 \in T\big(A_z^C\big)$. Set $a \in A_z^C$ and $\varepsilon > 0$. Then there is $f \in A_k$ for some $k \ge 1$ such that

$$\big\|a - \varphi_{k,\infty}(f)\big\| < \varepsilon/3. \tag{e7.55}$$

Let $\tau_{i,n} = s_i \circ \varphi_{n,\infty}$. Then, by the claim, there exists $N \ge k$ such that for all $n > N$,

$$\big|\tau_{1,n}\big(\varphi_{k,n}(f)\big) - \tau_{2,n}\big(\varphi_{k,n}(f)\big)\big| < \varepsilon/3. \tag{e7.56}$$

It follows that

$$\big|s_1\big(\varphi_{k,\infty}(f)\big) - s_2\big(\varphi_{k,\infty}(f)\big)\big| \le \varepsilon/3. \tag{e7.57}$$

Therefore,

$$\begin{aligned}|s_1(a) - s_2(a)| \le{}& \big|s_1(a) - s_1\big(\varphi_{k,\infty}(f)\big)\big| \\ &+ \big|s_1\big(\varphi_{k,\infty}(f)\big) - s_2\big(\varphi_{k,\infty}(f)\big)\big| + \big|s_2(a) - s_2\big(\varphi_{k,\infty}(f)\big)\big| < \varepsilon.\end{aligned}$$

It follows that $s_1(a) = s_2(a)$. Thus A_z^C has a unique tracial state. □

Remark 7.6. Recall that the construction allows $B = \mathbb{C}$ (with $C = \{0\}$). In that case, of course, $A_z^C = \mathcal{Z}$. Note that when $B = \mathbb{C}$, we have $\theta_t(b) = b$ for all $b \in M_{p_m q_m}$. In other words, $\theta_t = \mathrm{id}_B$.

Let

$$Z_{p_m,q_m} = \big\{f \in C\big([0,1], M_{p_m q_m}\big) : f(0) \in M_{p_m} \otimes 1_{q_m} \text{ and } f(1) \in 1_{p_m} \otimes M_{q_m}\big\}. \tag{e7.58}$$

In general (when $C \neq \{0\}$), one has $Z_{p_m,q_m} \subset E_{p_m,q_m}$, as we view $\mathbb{C} \subset B$ and $M_{p_m q_m} \subset M_{p_m q_m}(B)$. Let $\varphi_m^z = \varphi_m|_{Z_{p_m,q_m}}$. Then, since $v \in C\big([0,1], M_{p_{m+1}q_{m+1}}\big)$ (see the line before formula (e6.48)), $\varphi_m^z\big(Z_{p_m,q_m}\big) \subset Z_{p_{m+1},q_{m+1}}$. Thus, one obtains a unital C^*-subalgebra (of A_z^C)

$$B_z = \lim_{n\to\infty} \big(Z_{p_m,q_m}, \varphi_m^z\big). \tag{e7.59}$$

Then $B_z \cong \mathcal{Z}$ [27].

8. Regularity properties of A_z^C

In this section, let A_z^C be the C^*-algebra in Theorem 7.3.

Lemma 8.1. *The inductive system can be arranged so that A_z^C has the following properties:*

(1) *A_z^C has a unital C^*-subalgebra $B_z \cong \mathcal{Z}$.*
(2) *For any finite subset $\mathcal{F} \subset A_m$ and $\varepsilon > 0$, there is $e \in (A_{m+1})_+^{\mathbf{1}} \setminus \{0\}$ such that the following hold:*
   (i) *$e(t) \in M_{p_{m+1}q_{m+1}}$ for all $t \in [0,1]$ and $e(1) = 0$.*
   (ii) *$\|ex - xe\| < \varepsilon$ for all $x \in \varphi_m(\mathcal{F})$.*
   (iii) *$\varphi_{m+1,\infty}\big((1-e)^\beta \varphi_m(f)\big) \in_\varepsilon B_z$ for all $f \in \mathcal{F}$, and for any $\beta \in (0,\infty)$,*

$$\Big\|\varphi_{m+1,\infty}\Big((1-e)^\beta \varphi_m(y)\Big)\Big\| \geq (1-\varepsilon)\|\varphi_m(y)\| \text{ for all } y \in \mathcal{F}_m. \tag{e8.1}$$

   (iv) *$d_\tau(e) < 1/3^m$ for all $\tau \in T(A_{m+1})$.*

(Recall that $\mathcal{F}_m$ was constructed in the proof of Theorem 7.3.)

Proof. We will keep the notation used in the proof of Lemma 6.8.

For (i), we note that the C^*-subalgebra $B_z = \lim_{n\to\infty}\Big(Z_{p_m,q_m}, \varphi_m|_{Z_{p_m,q_m}}\Big)$ has been identified in Remark 7.6, where

$$Z_{p_m,q_m} = \big\{f \in C\big([0,1], M_{p_m q_m}\big) : f(0) \in M_{p_m} \otimes 1_{q_m} \text{ and } f(1) \in 1_{p_m} \otimes M_{q_m}\big\}. \tag{e8.2}$$

There is $\delta \in (0, \varepsilon/2)$ such that if $|t - t'| < 2\delta$,

$$\|\varphi_m(f)(t) - \varphi_m(f)(t')\| < \varepsilon/4 \text{ for all } f \in \mathcal{F}. \tag{e8.3}$$

In particular, there is $t_1 \in (0,1)$ $(1 - t_1 < \delta)$ such that

$$\|\varphi_m(f)(t) - \varphi_m(f)(1)\| < \varepsilon/4 \text{ for all } f \in \mathcal{F} \text{ and } t \in (t_1, 1). \tag{e8.4}$$

Choose a continuous function $g \in C([0,1])$ such that $0 \leq g \leq 1$, $g(t) = 1$ for all $t \in [0, t_1]$ and $g(t) = (1-t)/(1-t_1)$ for $t \in (t_1, 1]$. Let $e_0(t) = g(t) \cdot 1_{A_m}$ for all $t \in [0,1]$. Note that $e_0(0) = 1_{p_m q_m} \in M_{p_m}(B) \otimes 1_{q_m}$ and $e_0(1) = 0 \in 1_{p_m} \otimes M_{q_m}$. So $e_0 \in A_m$. Moreover, e_0 is in the center of A_m. Define $\sigma_0 : M_{p_m}(B) \otimes M_{q_m} \to M_{d_m p_m}(B) \otimes M_{5q_m}$ by $\sigma_0' \otimes s$, where

$$\sigma_0'(a) = \begin{pmatrix} \theta_1(a) & 0 \\ 0 & 0 \end{pmatrix} \text{ for all } a \in M_{p_m}(B) \text{ and } s(c) = c \otimes 1_5 \text{ for all } c \in M_{q_m}, \tag{e8.5}$$

where $\theta_1 : M_{p_m}(B) \to M_{p_m} \subset M_{p_m}(B)$ is defined by $\theta_1(c)(x) := c(0)$ for $c \in M_{p_m}(B) = M_{p_m}(C_0((0,1], C)^\sim)$ and for all $x \in [0,1]$, and the "0" in the lower corner of the matrix has the size of $(d_m - 1)p_m \times (d_m - 1)p_m$. Then define $\sigma_1 : A_m \to C\big([0,1], M_{d_m p_m 5q_m}(B)\big)$ by

$$\sigma_1(f)(t) := \sigma_0(f(t)) \text{ for all } f \in E_{p_m,q_m} \text{ and } t \in [0,1]. \tag{e8.6}$$

It follows that for all fixed $t \in [0,1]$,

$$\sigma_1(e_0)(t) = \sigma_0(e_0(t)) = \sigma_0\big(g(t)\cdot 1_{A_m}\big) = \sigma_0\big(g(t)\cdot 1_{p_m}\otimes 1_{q_m}\big) \tag{e8.7}$$

$$= \begin{pmatrix} (\theta_1(g(t)\cdot 1_{p_m}) & 0 \\ 0 & 0\end{pmatrix}\otimes 1_{5q_m} = \begin{pmatrix} g(t)\cdot 1_{p_m} & 0 \\ 0 & 0\end{pmatrix}\otimes 1_{5q_m} = \begin{pmatrix} g(t)\cdot 1_{p_mq_m} & 0 \\ 0 & 0\end{pmatrix}\otimes 1_5, \tag{e8.8}$$

where the last "0" in the last matrix has the size $(d_m-1)p_mq_m \times (d_m-1)p_mq_m$. Thus

$$\sigma_1(e_0)(0) = b\otimes 1_{5q_m} \qquad \text{and} \qquad \sigma_1(e_0(1)) = 0, \tag{e8.9}$$

where $b = \begin{pmatrix} 1_{p_m} & 0 \\ 0 & 0\end{pmatrix}$. It follows that $\sigma_1(e_0) \in E_{d_mp_m,5q_m}$. Note that for each $\tau \in T(A_m)$, by formula (e 6.13).

$$d_\tau(\sigma_1(e_0)) < 1/3^m. \tag{e8.10}$$

Let us recall the definition of $\tilde{\psi}_{m,i}$ in the proof of Lemma 6.8, $1\le i\le k$ (see formula (e6.50)). Then for all $f\in A_m$, by formulas (e6.50), (e6.44), (e6.46) and (e6.52), for each $t\in[0,1]$ we have

$$\tilde{\psi}_{m,i}(f)(t)\sigma_1(e_0)(t) = \begin{pmatrix} \theta^{(i)}(f)(t) & 0 \\ 0 & \gamma_m(f(t))\end{pmatrix}\otimes 1_5\cdot\begin{pmatrix} g(t)\cdot 1_{p_mq_m} & 0 \\ 0 & 0\end{pmatrix}\otimes 1_5 \tag{e8.11}$$

$$= \begin{pmatrix} \theta^{(i)}(f)(t)\cdot g(t)\cdot 1_{p_mq_m} & 0 \\ 0 & 0\end{pmatrix}\otimes 1_5 \tag{e8.12}$$

$$= \begin{pmatrix} g(t)\cdot 1_{p_mq_m} & 0 \\ 0 & 0\end{pmatrix}\otimes 1_5\cdot\begin{pmatrix} \theta^{(i)}(f)(t) & 0 \\ 0 & \gamma_m(f(t))\end{pmatrix}\otimes 1_5 \tag{e8.13}$$

$$= \sigma_1(e_0)(t)\tilde{\psi}_{m,i}(f)(t). \tag{e8.14}$$

In other words, for all $f\in E_{p_m,q_m}$,

$$\tilde{\psi}_{m,i}(f)\sigma_1(e_0) = \sigma_1(e_0)\tilde{\psi}_{m,i}(f), \quad i=1,2,\dots,k. \tag{e8.15}$$

Define $\alpha : [0,1]\longrightarrow[0,1]$ by

$$\alpha(t) := \begin{cases} \frac{t}{t_1} & \text{if } t\in[0,t_1], \\ 1 & \text{if } t\in(t_1,1].\end{cases} \tag{e8.16}$$

Then for all j, $f\circ\alpha\in E_{p_j,q_j}$ if $f\in E_{p_j,q_j}$. Moreover, by formula (e8.3),

$$\|\varphi_m(f)\circ\alpha - \varphi_m(f)\| < \varepsilon/4 \text{ for all } f\in\mathcal{F}. \tag{e8.17}$$

Therefore, for each $f\in A_m$, each $t\in[0,1]$ and each $\beta\in(0,\infty)$, with $l = d_mp_m5q_m$,

$$(1_l - \sigma_1(e_0))^\beta\tilde{\psi}_{m,i}(f)\circ\alpha(t) = \begin{pmatrix} (1-g(t))^\beta\cdot 1_{p_mq_m}\cdot\theta^{(i)}(f)\circ\alpha(t) & 0 \\ 0 & \gamma_m(f(t))\end{pmatrix}\otimes 1_5, \tag{e8.18}$$

for $i=1,2,\dots,k$. For $t\in[0,t_1]$, by the definition of g,

$$(1-g(t))^\beta\cdot 1_{p_mq_m}\cdot\theta^{(i)}(f)(t) = 0. \tag{e8.19}$$

For $t \in (t_1, 1]$,

$$(1-g(t))^\beta \cdot 1_{p_m q_m} \cdot \theta^{(i)}(f)\circ\alpha(t) = (1-g(t))^\beta \cdot \theta^{(i)}(f)(1) \in M_{p_m q_m}. \tag{e8.20}$$

Hence

$$(1_l - \sigma_1(e_0))^\beta \tilde{\psi}_{m,i}(f\circ\alpha) \in C\big([0,1], M_{d_m p_m 5 q_m}\big). \tag{e8.21}$$

Moreover, by formula (e7.8), for $f \in \mathcal{F}_m$ we have

$$\big\|(1_l - \sigma_1(e_0))^\beta \tilde{\psi}_{m,i}(f)\circ\alpha\big\| \geq (1-1/m)\|f\|. \tag{e8.22}$$

Using the same v as in formula (e6.48), with $\sigma_1(e_0)(t)$ repeating k times, define

$$e := v(t)^* \begin{pmatrix} \sigma_1(e_0)(t) & 0 & \cdots & 0 \\ 0 & \sigma_1(e_0)(t) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \sigma_1(e_0)(t) \end{pmatrix} v(t). \tag{e8.23}$$

With b as in the line after equation (e8.9), $b\otimes 1_{5q_m}\otimes 1_{r_0} = b\otimes 1_{r_0 5 q_m} = \big(b\otimes 1_{t_0}\big)\otimes 1_{q_{m+1}}$ (see formula (e6.35)). Since $\sigma_1(e_0)\in E_{d_m p_m, 5q_m}$, as in equation (e8.9),

$$e(0) = v_0^* \begin{pmatrix} b\otimes 1_{5q_m} & 0 & \cdots & 0 \\ 0 & b\otimes 1_{5q_m} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & b\otimes 1_{5q_m} \end{pmatrix} v_0 \in M_{p_{m+1}} \otimes 1_{q_{m+1}} \tag{e8.24}$$

(see formula (e6.37)). Combining with the fact that $e(1) = 0$, one concludes that $e \in E_{p_{m+1},q_{m+1}} = A_{m+1}$. Moreover, by equation (e8.7) and the fact that $v \in C\big([0,1], M_{p_{m+1}q_{m+1}}\big)$ (see discussion before formula (e6.48)), $e(t) \in M_{p_{m+1}q_{m+1}}$ for each $t \in [0,1]$ and $e(1) = 0$. So part (2)(i) of the statement of the lemma holds.

By equation (e8.15) and (e6.51), one computes that for all $f \in A_m$,

$$e\varphi_m(f) = v^* \begin{pmatrix} \sigma_1(e_0)\tilde{\psi}_{m,1}(f) & & 0 \\ & \ddots & \\ 0 & & \sigma_1(e_0)\tilde{\psi}_{m,k}(f) \end{pmatrix} v \tag{e8.25}$$

$$= v^* \begin{pmatrix} \tilde{\psi}_{m,1}(f)\sigma_1(e_0) & & 0 \\ & \ddots & \\ 0 & & \tilde{\psi}_{m,k}(f)\sigma_1(e_0) \end{pmatrix} v = \varphi_m(f)e. \tag{e8.26}$$

In other words, part (2)(ii) in the statement of the lemma holds. Now

$$(1-e)^\beta \varphi_m(f\circ\alpha) = v^* \begin{pmatrix} (1_l - \sigma_1(e_0))^\beta \tilde{\psi}_{m,1}(f)\circ\alpha & & 0 \\ & \ddots & \\ 0 & & (1_l - \sigma_1(e_0))^\beta \tilde{\psi}_{m,k}(f)\circ\alpha \end{pmatrix} v$$

for all $f \in A_m$, where $l = d_m p_m 5 q_m$. Note that $(1-e)^\beta\varphi_m(f)\circ\alpha \in E_{p_{m+1},q_{m+1}}$. It follows from formula (e8.21) that

$$(1-e)^\beta \varphi_m(f)\circ\alpha \in Z_{p_{m+1},q_{m+1}} \text{ for all } f \in \mathcal{F}. \tag{e8.27}$$

Then, by formula (e8.17),

$$(1-e)^{\beta}\varphi_m(f) \in_{\varepsilon/4} Z_{p_{m+1},q_{m+1}} \text{ for all } f \in \mathcal{F}. \tag{e8.28}$$

It follows that

$$\varphi_{m,\infty}\Big((1-e)^{\beta} f\Big) \in_{\varepsilon} B_z \text{ for all } f \in \mathcal{F}. \tag{e8.29}$$

Moreover, by formula (e8.22), formula (e8.1) also holds, so part (2)(iii) of the statement of the lemma holds. It follows from formula (e8.10) that part (2)(iv) also holds. □

Lemma 8.2. *Let*

$$E_{p,q} = \Big\{(f,c) : C\big([0,1], M_{pq}(B)\big) \oplus \big(M_p(B) \oplus M_q\big) : \pi_0(c) = f(0) \text{ and } \pi_1(c) = f(1)\Big\}, \tag{e8.30}$$

where $\pi_0 : M_p(B) \oplus M_q \to M_p(B) \otimes 1_q \subset M_{pq}(B)$ *defined by* $\pi_0(c_1 \oplus c_2) := c_1 \otimes 1_q$ *for all* $c_1 \in M_p(B)$ *and* $c_2 \in M_q$, *and* $\pi_1 : M_p(B) \oplus M_q \to 1_p \otimes M_q \subset M_{pq}(B)$ *defined by* $\pi_1((c_1 \oplus c_2)) := 1_p \otimes c_2$ *for all* $c_1 \in M_p(B)$ *and* $c_2 \in M_q$ *(see formula (e6.6)). Let*

$$L_{p,q} = \Big\{(f,c) : C\big([0,1], M_{pq}\big) \oplus M_p : \pi_0|_{M_p}(c) = f(0)\Big\}, \tag{e8.31}$$

where $\pi_0|_{M_p}(c) = c \otimes 1_q$ *for all* $c \in M_p$.

Suppose that $a, b \in E_{p,q_+}$ *are such that*

(1) $a(t) \in C\big([0,1], M_{pq}\big)$ *and* $a(1) = 0$ *and*
(2) *there is* $b_0 \in C\big([0,1], M_{pq}\big)_+$ *such that* $b_0(t) \le b(t)$ *for all* $t \in [0,1]$ *and*

$$a \lesssim b_0 \text{ in } L_{p,q} \tag{e8.32}$$

(i.e., there exists a sequence $x_n \in L_{p,q}$ *such that* $x_n^* b_0 x_n \to a$*).*

Then

$$a \lesssim b \text{ in } E_{p,q}. \tag{e8.33}$$

Proof. Let $1 > \varepsilon > 0$. Consider a continuous function $h_\delta \in E_{p,q}$:

$$h_\delta(t) = \begin{cases} 1_{M_{pq}(B)} & \text{if } t \in [0, 1-\delta], \\ 0 & \text{if } t \in (1-\delta/2, 1], \\ \text{linear} & \text{otherwise.} \end{cases} \tag{e8.34}$$

Since $a(1) = 0$, there exists $\delta_0 > 0$ such that $\Big\|a - h_{\delta_0}^{1/2} a \cdot h_{\delta_0}^{1/2}\Big\| < \varepsilon$.

Note that h_{δ_0} lies in the center of $C\big([0,1], M_{pq}(B)\big)$, and for any $f \in L_{p,q}$ and any $n \in \mathbb{N}$, we have $h_{\delta_0}^{1/n} \cdot f \in E_{p,q}$. Then since $a \lesssim b_0$ in $L_{p,q}$, one checks $h_{\delta_0}^{1/2} a h_{\delta_0}^{1/2} \lesssim h_{\delta_0}^{1/2} b_0 h_{\delta_0}^{1/2} \le b_0 \le b$ in $E_{p,q}$. This implies that $a \approx_\varepsilon h_{\delta_0}^{1/2} a h_{\delta_0}^{1/2} \lesssim b$ in $E_{p,q}$. Since this holds for any $1 > \varepsilon > 0$, one concludes that $a \lesssim b$ in $E_{p,q}$. □

Definition 8.3 (compare [19, 2.1.1.]). In the spirit of Definition 3.1, a simple C^*-algebra A is said to have essential tracial nuclear dimension at most n if A is essentially tracially in $\mathcal{N}_n$, the class of C^*-algebras with nuclear dimension at most n – that is, if for any $\varepsilon > 0$ and any finite subsets $\mathcal{F} \subset A$ and $a \in A_+ \setminus \{0\}$, there exist an element $e \in A_+^{\mathbf{1}}$ and a C^*-subalgebra $B \subset A$ which has nuclear dimension at most n such that

(1) $\|ex - xe\| < \varepsilon$ for all $x \in \mathcal{F}$,
(2) $(1-e)x \in_\varepsilon B$ and $\|(1-e)x\| \geq \|x\| - \varepsilon$ for all $x \in \mathcal{F}$ and
(3) $e \lesssim a$.

Let us denote by $\mathcal{N}_{\mathcal{Z},s,s}$ the class of separable nuclear simple $\mathcal{Z}$-stable C^*-algebras.

Theorem 8.4. *The unital simple C^*-algebra A_z^C is essentially tracially in $\mathcal{N}_{\mathcal{Z},s,s}$ and has essential tracial nuclear dimension at most 1, stable rank one and strict comparison for positive elements. Moreover, A_z^C has a unique tracial state and has no 2-quasitraces other than the unique tracial state, and*

$$\left(K_0\left(A_z^C\right), K_0\left(A_z^C\right)_+, \left[1_{A_z^C}\right], K_1\left(A_z^C\right)\right) = (\mathbb{Z}, \mathbb{Z}_+, 1, \{0\}). \tag{e8.35}$$

Recall that if C is exact and not nuclear, then A_z^C is exact and not nuclear (Theorem 7.3), and if C is not exact, then A_z^C is not exact (Proposition 7.4).

Proof. We will first show that A_z^C is essentially tracially in $\mathcal{N}_{\mathcal{Z},s,s}$. We will retain the notation from the construction of A_z^C.

Fix a finite subset $\mathcal{F}$ and an element $a \in A_{z\,+}^C$ with $\|a\| = 1$. To verify that A_z^C has the specified property, without loss of generality we may assume that there is a finite subset $\mathcal{G} \subset A_1^{\mathbf{1}}$ such that $\varphi_{1,\infty}(\mathcal{G}) = \mathcal{F}$. Since $\cup_{n=1}\mathcal{F}_{1,n}$ is dense in A_1 (see the proof of Theorem 7.3), we may assume that $\mathcal{G} \subset \mathcal{F}_{1,m}$ for some $m \geq 1$. By the first few lines of the proof of Theorem 7.3, we may assume that $\varphi_{1,m}(\mathcal{G}) \subset \varphi_{1,m}(\mathcal{F}_{1,m}) \subset \mathcal{F}_{m+1,1}$. Starting from $m+1$ instead of 1, to further simplify notation, without loss of generality we may assume that $\mathcal{G} \subset \mathcal{F}_{1,1} = \mathcal{F}_1 \cup \{0\}$. Without loss of generality, we may assume that there is $a' \in (A_1)_+^{\mathbf{1}}$ with $\|a'\| = 1$ such that

$$\|\varphi_{1,\infty}(a') - a\| < 1/4. \tag{e8.36}$$

It follows from [40, Proposition 2.2] that

$$\varphi_{1,\infty}\left(f_{1/4}(a')\right) = f_{1/4}\left(\varphi_{1,\infty}(a')\right) \lesssim a. \tag{e8.37}$$

Put $a_0' = f_{1/4}(a')$ $(\neq 0)$. Since A_z^C is simple, there are $x_1, x_2, \ldots, x_k \in A_z^C$ such that

$$\sum_{i=1}^k x_i^* \varphi_{1,\infty}(a_0') x_i = 1. \tag{e8.38}$$

It follows that for some large n_0, there are $y_1, y_2, \ldots, y_k \in A_{n_0}$ and $n_1 \geq n_0$ such that

$$\left\|\sum_{i=1}^k \varphi_{n_0,n_1}(y_i^*)\varphi_{1,n_1}(a_0')\varphi_{n_0,n_1}(y_i) - 1_{A_{n_1}}\right\| < 1/4. \tag{e8.39}$$

It follows that $a_0 := \varphi_{1,n_1}\left(a_0'\right)$ is a full element in A_{n_1}.

Set

$$d = \inf\left\{d_\tau(a_0) : \tau \in T\left(A_{n_1}\right)\right\}. \tag{e8.40}$$

Since a_0 is full in A_{n_1} and $a_0 \in \left(A_{n_1}\right)_+$, we have $d > 0$. Choose $m > n_1$ such that

$$d/4 > 1/3^{m-1}. \tag{e8.41}$$

By applying Lemma 8.1, we obtain $e \in (A_{m+1})_+^{\mathbf{1}} \setminus \{0\}$ such that

(i) $e(t) \in M_{p_{m+1}q_{m+1}}$ for all $t \in [0,1]$ and $e(1) = 0$,
(ii) $\|ex - xe\| < \varepsilon$ for all $x \in \varphi_m\left(\varphi_{1,m}(\mathcal{G})\right)$,

(iii) $\varphi_{m+1,\infty}\big((1-e)\varphi_m(\varphi_{1,m}(x))\big) \in_\varepsilon B_z$, and

$$\big\|\varphi_{m+1,\infty}\big((1-e)\varphi_m(\varphi_{1,m}(x))\big)\big\| \geq (1-1/m)\big\|\varphi_{1,m}(x)\big\|$$

for all $x \in \mathcal{F}_{1,1}$, and

(iv) $d_\tau(e) < 1/3^m$ for all $\tau \in T(A_{m+1})$.

Denote $a_1 = \varphi_{n_1,m}(a_0)$. It is full in A_m. Write, as in Theorem 6.8 and equation (e6.16),

$$\varphi_m(a_1) = u^* \begin{pmatrix} \Theta_m(a_1) & 0 & \cdots & 0 \\ 0 & \gamma_m(a_1 \circ \xi_1) \otimes 1_5 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \gamma_m(a_1 \circ \xi_k) \otimes 1_5 \end{pmatrix} u, \tag{e8.42}$$

where $u \in U\big(C\big([0,1], M_{p_{m+1}q_{m+1}}(B)\big)\big)$, $\Theta_m : A_m \to C\big([0,1], M_{kp_mq_m}(B)\big)$ is a homomorphism, $k \geq 1$ is an integer and $\gamma_m : B \to M_{R(m)}$ is a finite-dimensional representation. Moreover,

$$k/5kR(m) < 1/3^m. \tag{e8.43}$$

Let

$$b_0 = u^* \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \gamma_m(a_1 \circ \xi_1) \otimes 1_5 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \gamma_m(a_1 \circ \xi_k) \otimes 1_5 \end{pmatrix} u. \tag{e8.44}$$

Note that $b_0 \in C\big([0,1], M_{p_{m+1}q_{m+1}}\big)$. Moreover, since $a_1 \in E_{p_m,q_m}$, we have $a_1(0) = a_1' \otimes 1_{q_m}$ for some $a_1' \in M_{p_m}(B)$. Therefore,

$$\gamma_m(a_1(0)) = \gamma_m(a_1') \otimes 1_{q_m}. \tag{e8.45}$$

Put

$$c_0' = \begin{pmatrix} 0 & 0 \\ 0 & \gamma_m(a_1') \end{pmatrix} \tag{e8.46}$$

and

$$c_0 = \begin{pmatrix} 0 & 0 \\ 0 & \gamma_m(a_1(0)) \end{pmatrix} \otimes 1_5 = \begin{pmatrix} 0 & 0 \\ 0 & \gamma_m(a_1') \end{pmatrix} \otimes 1_{5q_m} = c_0' \otimes 1_{5q_m}. \tag{e8.47}$$

Note that $c_0' \in M_{d_m p_m}$. Put

$$c_i(t) = \begin{pmatrix} 0 & 0 \\ 0 & \gamma_m(a_1 \circ \xi_i(t)) \end{pmatrix} \otimes 1_5, \quad i = r_0+1, r_0+2, \ldots, k. \tag{e8.48}$$

Recall (see formula (e6.34)) that at $t = 0$,

$$\xi_i(0) = \begin{cases} 0 & \text{if } 1 \leq i \leq r_0, \\ 1/2 & \text{if } r_0 < i \leq k. \end{cases} \tag{e8.49}$$

Recall also (see the line after equation (e6.35)) that $r_0 5q_m = t_0 q_{m+1}$ for some integer $t_0 \geq 1$. Hence $(c_0' \otimes 1_{5q_m}) \otimes 1_{r_0} = c_0' \otimes 1_{r_0 5q_m} = (c_0' \otimes 1_{t_0}) \otimes 1_{q_{m+1}}$. On the other hand, since $k = r_0 + m(0)q_{m+1}$ (see equation (e6.30)), we have

$$\operatorname{diag}\big(c_{r_0+1}(0),\ldots,c_k(0)\big) =^s \left(\begin{pmatrix} 0 & 0 \\ 0 & \gamma_m(a_1(1/2)) \end{pmatrix} \otimes 1_5\right) \otimes 1_{m(0)q_{m+1}}. \tag{e8.50}$$

Note that $=^s$ is implemented by the same scalar unitary as in equation (e6.36) (see also the end of Notation 6.6 for the notation $=^s$). As in formula (e6.37) (and the line after it), since $b_0 \in C([0,1], M_{p_{m+1}q_{m+1}})$ (mentioned earlier), this implies that $b_0 \in L_{p_{m+1},q_{m+1}}$ (see equation (e8.31)).

Since $a_1 \geq 0$, $b_0(t) \leq a_1(t)$ for all $t \in [0,1]$ (see equation (e8.44)). Since φ_k is an injective unital homomorphism for all k, by equation (e8.40), we also have

$$\inf\{d_\tau(\varphi_m(a_1)) : \tau \in T(A_{m+1})\} = \inf\big\{d_\tau\big(\varphi_{n_1,m+1}(a_0)\big) : \tau \in T(A_m)\big\} \geq d. \tag{e8.51}$$

By formulas (e8.42), (e8.44), (e8.43) and (e8.41), we conclude that for each $t \in (0,1)$,

$$d_\sigma(e(t)) < d_\sigma(b_0(t)), \quad d_{\tau_0}(e(0)) < d_{\tau_0}(b_0(0)) \quad \text{and} \quad d_{\tau_1}(e(1)) < d_{\tau_1}(\varphi_m(a_1)(1)), \tag{e8.52}$$

where σ is the unique tracial state of $M_{p_{m+1}q_{m+1}}$, τ_0 is the unique tracial state of $M_{p_{m+1}} \otimes 1_{q_{m+1}}$ and τ_1 is the unique tracial state of $1_{q_{m+1}} \otimes M_{q_{m+1}}$. Note that $e(1) = 0$. It follows that for all $\tau \in T\big(L_{p_{m+1},q_{m+1}}\big)$,

$$d_\tau(e) < d_\tau(b_0). \tag{e8.53}$$

By, for example, [24, Theorem 3.18],

$$e \lesssim b_0 \text{ in } L_{p_{m+1},q_{m+1}}. \tag{e8.54}$$

By Lemma 8.2,

$$e \lesssim \varphi_m(a_1) \text{ in } E_{p_{m+1},q_{m+1}} = A_{m+1}. \tag{e8.55}$$

It follows that

$$e \lesssim \varphi_{m,\infty}(a_1) = f_{1/4}\big(\varphi_{1,\infty}(a')\big) \lesssim a. \tag{e8.56}$$

Combining this with (ii) and (iii) here, we conclude that A_z^C is essentially tracially in $\mathcal{N}_{\mathcal{Z},s,s}$. Since $B_z \cong \mathcal{Z}$, which has nuclear dimension 1 [44, Theorem 6.4], A_z^C has essential tracial nuclear dimension at most 1. Since $\mathcal{Z}$ is in $\mathcal{T}$, A_z^C is e. tracially in $\mathcal{T}$. By Proposition 4.6, every 2-quasitrace of A_z^C is a tracial state. By Corollary 5.10, A has stable rank one. □

Remark 8.5. Note that the proof of Theorem 8.4 actually shows that A_z^C is essentially tracially approximated by $\mathcal{Z}$ itself, as $B_z \cong \mathcal{Z}$. Let $\mathcal{P}$ be the class of separable nuclear C^*-algebras. Then A_z^C is essentially tracially in $\mathcal{P}$, since $\mathcal{Z} \in \mathcal{P}$. By Proposition 7.4 and Theorem 7.3, A_z^C is not nuclear if C is not nuclear. Since A_z^C has no nonzero projection other than the identity, for nonnuclear C it cannot be TA$\mathcal{P}$.

Theorem 8.6. *Let (G, G_+, g) be a countable weakly unperforated simple ordered group, F be a countable abelian group, Δ be a metrisable Choquet simplex and $\lambda : G \to \mathrm{Aff}_+(\Delta)$ be a homomorphism.*

Then there is a unital simple nonexact (or exact but nonnuclear) C^*-algebra A which is e. tracially in $\mathcal{N}_{\mathcal{Z},s,s}$ and has essential tracial nuclear dimension at most 1, *stable rank one and strict comparison for positive elements, such that*

$$(K_0(A), K_0(A)_+, [1_A], K_1(A), T(A), \rho_A) = (G, G_+, g, F, \Delta, \lambda). \tag{e8.57}$$

Proof. It follows from [24, Theorem 13.50] that there is a unital simple A_0 with finite nuclear dimension which satisfies the UCT such that

$$\left(K_0(A_0), K_0(A_0)_+, \left[1_{A_0}\right], K_1(A_0), T(A_0), \rho_{A_0}\right) = (G, G_+, g, F, \Delta, \lambda). \tag{e8.58}$$

Let A_z^C be the C^*-algebra in Theorem 7.3 with A_z^C nonexact (or exact but nonnuclear). Then define $A := A_0 \otimes A_z^C$. Note that since A_0 is a separable amenable C^*-algebrasatisfying the UCT, by [45, Theorem 2.14], the Künneth formula holds for the tensor product $A = A_0 \otimes A_z^C$. Since the only normalised 2-quasitrace of A_z^C is the unique tracial state, and

$$\left(K_0\left(A_z^C\right), K_0\left(A_z^C\right)_+, \left[1_{A_z^C}\right], K_1\left(A_z^C\right)\right) = (\mathbb{Z}, \mathbb{Z}_+, 1, 0),$$

one computes (applying the Künneth formula) that

$$(K_0(A), K_0(A)_+, [1_A], K_1(A), T(A), \rho_A) = (G, G_+, g, F, \Delta, \lambda). \tag{e8.59}$$

We will show that A is essentially tracially in $\mathcal{N}_{\mathcal{Z},s,s}$ and has e. tracial nuclear dimension at most 1. Once this is done, by Definition 8.3 A has essentially tracial nuclear dimension at most 1 and by Corollary 5.10 has stable rank one and strict comparison for positive elements.

To see that A is essentially tracially in $\mathcal{N}_{\mathcal{Z},s,s}$, set $1 > \varepsilon > 0$, let $\mathcal{F} \subset A^{\mathbf{1}} \setminus \{0\}$ be a finite subset and set $d \in A_+ \setminus \{0\}$. Without loss of generality, we may assume that $\|x\| \geq \varepsilon$ for all $x \in \mathcal{F}$. For each $x \in \mathcal{F}$, put $a(x) = g_{\varepsilon^2/32}(xx^*)$ and $b(x) = g_{\varepsilon^2/16}(xx^*)$, where $g_r(t) \in C_0((0,1])$ such that $g_r(t) = 0$ for $t \in \left[0, \|x\|^2 - r/2\right]$, $g_r(t) = 1$ for $t \in \left[\|x\|^2 - r/4, 1\right]$ and g_r is linear in $\left[\|x\|^2 - r/2, \|x\|^2 - r/4\right]$. Note that $b(x)xx^*b(x) = \left(g_{\varepsilon^2/16}(xx^*)\right)^2 xx^*$. Therefore,

$$b(x)xx^*b(x) \geq \left(\|xx^*\| - (\varepsilon/4)^2\right) b(x)^2. \tag{e8.60}$$

By Kirchberg's slice lemma (see, for example, [41, Lemma 4.1.9]), for each $x \in \mathcal{F}$ there are $c(x) \in (A_0)_+ \setminus \{0\}$, $d(x) \in A^C_{z\,+} \setminus \{0\}$ and $z(x) \in A_0 \otimes A_z^C$ such that $z(x)^*z(x) = c(x) \otimes d(x)$ and $z(x)z(x)^* \in \mathrm{Her}(a(x))$. We may assume that $\|c(x)\| = \|d(x)\| = 1$. This also implies that $\|z(x)z(x)^*\| = 1$. Put

$$\mathcal{F}' := \mathcal{F} \cup \left\{x^*x, xx^*, a(x), b(x), c(x), d(x), d(x)^{1/2}, z(x), z^*(x), z(x)^*z(x), z(x)z(x)^* : x \in \mathcal{F}\right\}.$$

Without loss of generality, we may assume that there are $n \in \mathbb{N}$, $M \geq 1$ and finite subsets $\mathcal{F}_0 \subset A$ and $\mathcal{F}_1 \subset A_z^C$ such that for all $y \in \mathcal{F}'$,

$$y \in_{\varepsilon^2/64} \mathcal{F}'' := \left\{\sum_{i=1}^n a_i \otimes b_i, a_i \in \mathcal{F}_0, b_i \in \mathcal{F}_1\right\}, \tag{e8.61}$$

$$c(x) \in \mathcal{F}_0,\ d(x), d(x)^{1/2} \in \mathcal{F}_1 \text{ for all } x \in \mathcal{F}, \tag{e8.62}$$

$$\|f_0\|,\ \|f_1\| \leq M \text{if } f_0 \in \mathcal{F}_0 \text{ and } f_1 \in \mathcal{F}_1. \tag{e8.63}$$

By Kirchberg's slice lemma, there are $a_0 \in (A_0)_+ \setminus \{0\}$ and $b_0 \in \left(A_z^C\right)_+ \setminus \{0\}$ such that $a_0 \otimes b_0 \lesssim d$.

Let us identify A_0 with $A_0 \otimes \mathcal{Z}$ (see [48, Corollary 7.3]). In $A_0 \otimes \mathcal{Z}$, choose $1_{A_0} \otimes a_z$ with $a_z \in \mathcal{Z}_+ \setminus \{0\}$ such that $1_{A_0} \otimes a_z \lesssim_{A_0} a_0$. Choose $b_z \in (B_z)_+ \setminus \{0\} \subset A_z^C$ such that $b_z \lesssim_{A_z^C} b_0$.

Note that $B_z \cong \mathcal{Z}_b \otimes B_z$, where $\mathcal{Z}_b \cong \mathcal{Z}$. Put $c_0 := \sigma(a_z) \otimes b_z \in B_z$, where $\sigma : 1_{A_0} \otimes \mathcal{Z} (\subset A_0) \to \mathcal{Z}_b \otimes 1_{B_z}$ is an isomorphism. Consider $D_0 := A_0 \otimes \sigma\left(1_{A_0} \otimes \mathcal{Z}\right) \otimes 1_{B_z} \subset A_0 \otimes B_z$. We may also write $D_0 = (A_0 \otimes \mathcal{Z}) \otimes \sigma\left(1_{A_0} \otimes \mathcal{Z}\right) \otimes 1_{B_z}$. There is a sequence of unitaries $v_n \in D_0$ such that

$$\lim_{n\to\infty} v_n^*\left(1_{A_0} \otimes \sigma(a_z) \otimes 1_{B_z}\right)v_n = 1_{A_0} \otimes a_z \otimes 1_{B_z}. \tag{e8.64}$$

It follows that

$$1_{A_0} \otimes c_0 = 1_{A_0} \otimes \sigma(a_z) \otimes b_z \sim 1_{A_0} \otimes a_z \otimes b_z \lesssim a_0 \otimes b_0 \lesssim d. \tag{e8.65}$$

By Remark 8.5, there exists $e_1 \in A_z^C$ with $0 \le e_1 \le 1$ such that for all $f \in \mathcal{F}_1$,

$$\|e_1 f - f e_1\| < \varepsilon^2/64(nM), \qquad \left(1_{A_z^C} - e_1\right) f \in_{\varepsilon^2/64(nM)} B_z, \tag{e8.66}$$

$$\left\|\left(1_{A_z^C} - e_1\right) f\right\| \ge \left(1 - \varepsilon^2/64(nM)\right)\|f\| \text{ and } e_1 \lesssim c_0. \tag{e8.67}$$

Put $B = A_0 \otimes B_z$. Then by [10, Theorem B], B is a separable simple $\mathcal{Z}$-stable C^*-algebra and has nuclear dimension at most 1.

Put $e = 1_{A_0} \otimes e_1$. Then $0 \le e \le 1$. For any $f'' \in \mathcal{F}''$, we have that $f'' = \sum_{i=1}^n a_i \otimes b_i$ for some $a_i \in \mathcal{F}_0$ and $b_i \in \mathcal{F}_1$. It follows from formula (e8.66) that

$$\|ef'' - f''e\| = \left\|e\left(\sum_{i=1}^n a_i \otimes b_i\right) - \left(\sum_{i=1}^n a_i \otimes b_i\right)e\right\| = \left\|\sum_{i=1}^n a_i \otimes (e_1 b_i - b_i e_1)\right\| < \varepsilon^2/64. \tag{e8.68}$$

Also by formula (e8.66), for $f'' \in \mathcal{F}''$,

$$(1-e)f'' = (1 - 1_{A_0} \otimes e_1)\left(\sum_{i=1}^n a_i \otimes b_i\right) = \sum_{i=1}^n a_i \otimes \left(1_{A_z^C} - e_1\right) b_i \in_{\varepsilon^2/64} A_0 \otimes B_z. \tag{e8.69}$$

It follows (recall formula (e8.61)) that for all $f \in \mathcal{F}$,

$$\|ef - fe\| < \varepsilon^2/16 \qquad \text{and} \qquad (1-e)f \in_{\varepsilon^2/16} A_0 \otimes B_z = B. \tag{e8.70}$$

Moreover, by formulas (e8.65) and (e8.67),

$$e \lesssim a_0 \otimes b_0 \lesssim d. \tag{e8.71}$$

To estimate $\|(1-e)x\|$ for $x \in \mathcal{F}$, we note that by formula (e8.67) (recall that $\|c(x)\| = \|d(x)\| = 1$), for $x \in \mathcal{F}$ we have

$$\left\|(1-e)(c(x) \otimes d(x))^{1/2}\right\| = \left\|c(x)^{1/2} \otimes \left(1_{A_z^C} - e_1\right) d(x)^{1/2}\right\| \tag{e8.72}$$

$$\ge \left(1 - \varepsilon^2/64\right)\|c(x)\|\|d(x)\| = \left(1 - \varepsilon^2/64\right). \tag{e8.73}$$

Then by formulas (e8.70) and (e8.73),

$$\begin{aligned}\|(1-e)z(x)z(x)^*(1-e)\| &= \left\|z(x)^*(1-e)^2 z(x)\right\| \\ &\ge \|(1-e)z(x)^* z(x)(1-e)\| - \varepsilon^2/8 \quad \text{(e8.74)} \\ &= \left\|(1-e)(c(x) \otimes d(x))^{1/2}\right\|^2 - \varepsilon^2/8 \\ &\ge \left(1 - \varepsilon^2/64\right)^2 - \varepsilon^2/8 > 1 - 5\varepsilon^2/32. \quad \text{(e8.75)}\end{aligned}$$

Since $b(x)z(x)z(x)^* = z(x)z(x)^*b(x) = z(x)z(x)^*$, we compute by formulas (e8.60) and (e8.75)) that for all $x \in \mathcal{F}$,

$$\|(1-e)xx^*(1-e)\| \geq \left\|(1-e)b(x)^{1/2}xx^*b(x)^{1/2}(1-e)\right\| \tag{e8.76}$$

$$\geq \left(\|x\|^2 - \varepsilon^2/16\right)\|(1-e)b(x)(1-e)\| \tag{e8.77}$$

$$\geq \left(\|x\|^2 - \varepsilon^2/16\right)\|(1-e)z(x)z(x)^*(1-e)\| \tag{e8.78}$$

$$\geq \left(\|x\|^2 - \varepsilon^2/16\right)\left(1 - 5\varepsilon^2/32\right) \geq \|x\|^2 - 7\varepsilon^2/32 \tag{e8.79}$$

$$\geq \|x\|^2 - 2\varepsilon\|x\| + \varepsilon^2. \tag{e8.80}$$

(Recall for the last inequality that $\|x\| \geq \varepsilon$.) It follows that for all $x \in \mathcal{F}$,

$$\|(1-e)x\| \geq \|x\| - \varepsilon. \tag{e8.81}$$

This, together with formula (e8.70), implies that $A_0 \otimes A_z^C$ is essentially tracially in $\mathcal{N}_{\mathcal{Z},s,s}$, since B is $\mathcal{Z}$-stable and has nuclear dimension no more than 1 (see [10, Theorem B]).

Now suppose that we choose A_z^C not exact. Since A_z^C embeds into $A_0 \otimes A_z^C$ and A_z^C is not exact, $A_0 \otimes A_z^C$ is also not exact (see, for example, [47, Proposition 2.6]). If A_z^C is exact but not nuclear, then $A_0 \otimes A_z^C$ is exact but not nuclear (by [9, Propositions 10.2.7, 10.1.7]). □

Remark 8.7.

(1) Let A_0 be a unital separable nuclear purely infinite simple C^*-algebra in the UCT class. Then the proof of Theorem 8.6 also shows that $A := A_0 \otimes A_z^C$ is a nonexact purely infinite simple C^*-algebra (if C is nonexact) which has essential tracial nuclear dimension 1 and $\mathrm{Ell}(A) = \mathrm{Ell}(A_0)$.

(2) If the RFD C^*-algebra C at the beginning of Section 6 is amenable, then $C_0((0,1], C)$ is a nuclear contractible C^*-algebra which satisfies the UCT. It follows that the unitisation B of $C_0((0,1], C)$ also satisfies the UCT. Therefore $D(m,k)$ and $I = C_0((0,1), M_{mk}(B))$ in formula (e6.8) satisfy the UCT. Thus $E_{m,k}$ is nuclear and satisfies the UCT. It follows that A_z^C is a unital amenable separable simple stable rank one C^*-algebra with a unique tracial state which also has strict comparison for positive elements and satisfies the UCT. By [34, Theorem 1.1], A_z^C is $\mathcal{Z}$-stable. By [35, Theorem 1.1], A_z^C has finite nuclear dimension. Then by [16, Corollary 4.11], A_z^C is classifiable by the Elliott invariant (see also [16, Remark 4.6]). Since A_z^C has the same Elliott invariant of $\mathcal{Z}$, it follows that $A_z^C \cong \mathcal{Z}$.

(3) On the other hand, we make no attempt at this time to classify C^*-algebras A_z^C constructed in Section 6 in the nonnuclear cases. We do not know whether $A_z^{C_1}$ is isomorphic to $A_z^{C_2}$ if C_1 and C_2 are nonisomorphic, nonnuclear C^*-subalgebras. In fact, as it stands, A_z^C may depend on the connecting maps used in the construction.

Conflict of interest: None.

Financial support. The first author was supported by the China Postdoctoral Science Foundation (grant KLH1414009) and partially supported by an NSFC grant (NSFC 11420101001) and a Discovery Grant of the NSERC of Canada. The second author was partially supported by an NSF grant (DMS-1954600). Both authors would like to acknowledge the support during their visits to the Research Center of Operator Algebras at East China Normal University, which is partially supported by the Shanghai Key Laboratory of PMMP, Science and Technology Commission of Shanghai Municipality (STCSM), grant 13dz2260400 and an NNSF grant (11531003).

References

[1] D. Archey, J. Buck and N. C. Phillips, 'Centrally large subalgebras and tracial Z-absorption', *Int. Math. Res. Not. IMRN* **6** (2018), 1857–1877.

[2] B. Blackadar, 'A simple C^*-algebra with no nontrivial projections', *Proc. Amer. Math. Soc.* **78**(4) (1980), 504–508.
[3] B. Blackadar, *Operator algebras. Theory of C^*-algebras and von Neumann algebras,* Encyclopaedia of Mathematical Sciences, **122**. (Springer-Verlag, Berlin, 2006). xx+517.
[4] B. Blackadar and D. Handelman, 'Dimension functions and traces on C^*-algebra', *J. Funct. Anal.* **45** (1982), 297–340.
[5] B. Blackadar and E. Kirchberg, 'Generalized inductive limits of finite-dimensional C^*-algebras', *Math. Ann.* **307**(3) (1997), 343–380.
[6] B. Blackadar, A. Kumjian and M. Rørdam, 'Approximately central matrix units and the structure of noncommutative tori', *K-Theory* **6** (1992), 267–284.
[7] E. Blanchard and E. Kirchberg, 'Non-simple purely infinite C^*-algebras: The Hausdorff case', *J. Funct. Anal.* **207**(2) (2004), 461–513.
[8] L. G. Brown, 'Stable isomorphism of hereditary subalgebras of C^*-algebras', *Pacific J. Math.* **71**(2) (1977), 335–348.
[9] N. P. Brown and N. Ozawa, '*C^*-Algebras and Finite-Dimensional Approximations*, *Graduate Studies in Mathematics* vol. **88** (American Mathematical Society, Providence, RI, 2008). xvi+509.
[10] J. Castillejos, S. Evington, A. Tikuisis, S. White and W. Winter, 'Nuclear dimension of simple C^*-algebras', *Invent. Math. 224* (2021), no. 1, 245–290.
[11] M. Choi and E. Effros, 'The completely positive lifting problem for C^*-algebras', *Ann. of Math. (2)* **104**(3) (1976) 585–609.
[12] J. Cuntz, 'The structure of multiplication and addition in simple C^*-algebras', *Math. Scand.* **40**(2) (1977), 215–233.
[13] M. Dădărlat, 'Nonnuclear subalgebras of AF algebras', *Amer. J. Math.* **122**(3) (2000), 581–597.
[14] G. Elliott and Z. Niu, 'On tracial approximation', *J. Funct. Anal.* **254**(2) (2008), 396–440.
[15] G. A. Elliott, G. Gong, H. Lin and Z. Niu, 'Simple stably projectionless $C*$-algebras of generalized tracial rank one', *J. Noncommut. Geom.* **14** (2020), 251–347.
[16] G. A. Elliott, G. Gong, H. Lin and Z. Niu, 'On the classification of simple amenable C*-algebras with finite decomposition rank, II', Preprint, 2015, arXiv:1507.03437.
[17] G. A. Elliott, L. Robert and L. Santiago, 'The cone of lower semicontinuous traces on a C*-algebra', *Amer. J. Math.* **133** (2011), 969–1005.
[18] Q. Fan and X. Fang, 'C^*-algebras of tracially stable rank one', *Acta Math. Sinica (Chin. Ser.)* **48**(5) (2005), 929–934.
[19] X. Fu, *Tracial Nuclear Dimension of C^*-Algebras*, Ph. D. thesis (East China Normal University, Shanghai, China, 2018).
[20] X. Fu, K. Li and H. Lin, 'Tracial approximate divisibility and stable rank one', Preprint, 2021, arXiv:2108.08970v2.
[21] X. Fu and H. Lin, 'On tracial approximation of simple C^*-algebras', *Canad. J. Math.*, forthcoming. doi:10.4153/S0008414X21000158.
[22] X. Fu and H. Lin, In preparation.
[23] G. Gong, X. Jiang and H. Su, 'Obstructions to $\mathcal{Z}$-stability for unital simple C^*-algebras', *Canad. Math. Bull.* **43**(4) (2000), 418–426.
[24] G. Gong, H. Lin and Z. Niu, 'A classification of finite simple amenable $\mathcal{Z}$-stable C^*-algebras, I: C^*-algebras with generalized tracial rank one', *C. R. Math. Acad. Sci. Soc. R. Can.* **42** (2020) 63–450.
[25] K. R. Goodearl, 'Notes on a class of simple C^*-algebras with real rank zero', *Publ. Mat.* **36**(2A) (1992), 637–654.
[26] C. Ivanescu and D. Kučerovský, 'Traces and Pedersen ideals of tensor products of nonunital C^*-algebras', *New York J. Math.* **25** (2019), 423–450.
[27] X. Jiang and H. Su, 'On a simple unital projectionless C^*-algebra', *Amer. J. Math.* **121**(2) (1999), 359–413.
[28] H. Lin, *An Introduction to the Classification of Amenable C^*-Algebras* (World Scientific Publishing Co. Inc., River Edge, NJ, 2001), xii+320.
[29] H. Lin, 'Tracially AF C*-algebras', *Trans. Amer. Math. Soc.* **353** (2001), 693–722.
[30] H. Lin, 'The tracial topological rank of C^*-algebras', *Proc. London Math. Soc. (3)* **83** (2001), 199–234.
[31] H. Lin, 'Traces and simple C^*-algebras with tracial topological rank zero', *J. Reine Angew. Math.* **568** (2004), 99–137.
[32] H. Lin, 'Simple nuclear C^*-algebras of tracial topological rank one', *J. Funct. Anal.* **251** (2007), 601–679.
[33] H. Lin and S. Zhang, 'On infinite simple C^*-algebras', *J. Funct. Anal.* **100** (1991), 221–231.
[34] H. Matui and Y. Sato, 'Strict comparison and $\mathcal{Z}$-absorption of nuclear C^*-algebras', *Acta Math.* **209**(1) (2012), 179–196.
[35] H. Matui and Y. Sato, 'Decomposition rank of UHF-absorbing C^*-algebras', *Duke Math. J.* **163**(14) (2014), 2687–2708.
[36] G. K. Pedersen, *C^*-Algebras and Their Automorphism Groups*, *London Mathematical Society Monographs* vol. **14** (Academic Press, London, 1979).
[37] N. C. Phillips, 'Large subalgebras', Preprint, 2014, arXiv:1408.5546v1.
[38] L. Robert, 'Remarks on $\mathcal{Z}$-stable projectionless C^*-algebras', *Glasg. Math. J.* **58**(2) (2016), 273–277.
[39] M. Rørdam, 'On the structure of simple C^*-algebras tensored with a UHF-algebra', *J. Funct. Anal.* **100** (1991), 1–17.
[40] M. Rørdam, 'On the structure of simple C^*-algebras tensored with a UHF-algebra. II', *J. Funct. Anal.* **107** (1992), 255–269.
[41] M. Rørdam, *Classification of nuclear, simple C^*-algebras. Classification of nuclear C^*-algebras. Entropy in operator algebras*, 1–145, Encyclopaedia Math. Sci. **126**, Springer, Berlin, 2002.
[42] M. Rørdam, 'The stable and the real rank of $\mathcal{Z}$-absorbing C^*-algebras', *Internat. J. Math.* **15**(10) (2004), 1065–1084.
[43] M. Rørdam and W. Winter, 'The Jiang-Su algebra revisited', *J. Reine Angew. Math.* **642** (2010), 129–155.
[44] Y. Sato, S. White and W. Winter, 'Nuclear dimension and $\mathcal{Z}$-stability', *Invent. Math.* **202**(2) (2015), 893–921.
[45] C. Schochet, 'Topological methods for $C*$-algebras, II: Geometric resolution and the Künneth formula', *Pacific J. Math.* **114** (1982), 443–458.

[46] A. Toms and W. Winter, ‘Strongly self-absorbing C^*-algebras’, *Trans. Amer. Math. Soc.* **359**(8) (2007), 3999–4029.
[47] S. Wassermann, *Exact C^*-Algebras and Related Topics, Lecture Notes Series* vol. **19** (Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, Korea, 1994). viii+92.
[48] W. Winter, ‘Nuclear dimension and $\mathcal{Z}$-stability of pure C^*-algebras’, *Invent. Math.* **187**(2) (2012), 259–342.
[49] W. Winter and J. Zacharias, ‘Completely positive maps of order zero’, *Münster J. Math.* **2** (2009), 311–324.