Tracial approximation in simple C^*-algebras

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Abstract

We revisit the notion of tracial approximation for unital simple C^*-algebras. We show that a unital simple separable infinite dimensional C^*-algebra A is asymptotically tracially in the class of C^*-algebras with finite nuclear dimension if and only if A is asymptotically tracially in the class of nuclear $\mathcal{Z}$-stable C^*-algebras.

1 Introduction

Nuclear dimension for C^*-algebras was first introduced in [72]. Over the time, this notion becomes increasingly important in the study of C^*-algebras in the connection with the Elliott program [16], the program of classification of separable simple amenable C^*-algebras by the Elliott invariant, a set of K-theory related invariant. The part of the Toms-Winter conjecture (see [72, Conjecture 9.3]) states that a unital simple nuclear separable C^*-algebra A has finite nuclear dimension if and only if A is $\mathcal{Z}$-stable, i.e., $A \otimes \mathcal{Z} \cong A$, where $\mathcal{Z}$ is the Jiang-Su algebra, a unital separable and infinite dimensional simple C^*-algebra which has $K_0(\mathcal{Z}) = \mathbb{Z}$ (as an ordered group), $K_1(\mathcal{Z}) = \{0\}$ and a unique tracial state (see [31]). This part of the Toms-Winter conjecture is now a theorem (see [69], [10], see also [50]).

On the other hand tracial rank was introduced in [41] (and see also [40]). C^*-algebras with tracial rank zero are also called C^*-algebras which are tracially AF. Amenable tracially AF-algebras and C^*-algebras of tracial rank one were classified in [42] and [43] with the presence of UCT (these classification results were preceded by [17] and [18], respectively). These had been generalized to the classification of the class of amenable simple C^*-algebras which have rationally generalized tracial rank at most one satisfying the UCT (see [26], and [27], see also [44], [70], [45], and [47]). In [19], it is proved that all unital separable simple C^*-algebras with finite nuclear dimension in the UCT class in fact have rationally generalized tracial rank at most one (using [60]). In other words, all unital separable simple C^*-algebras with finite nuclear dimension satisfying the UCT are classified (up to isomorphism) by their Elliott invariant. This can also be restated, by the proof of Toms-Winter conjecture as mentioned above, that all unital separable amenable simple $\mathcal{Z}$-stable C^*-algebras satisfying the UCT are classified.

The beginning point of this paper is to search a tracial version of a part of Toms-Winter conjecture, i.e., a separable amenable simple unital C^*-algebra is $\mathcal{Z}$-stable if and only if it has finite nuclear dimension (which is now a theorem). We revisit a version of tracial approximation (see Definition 3.1 and Proposition 3.10 below). The main results include the following statement: A unital separable infinite dimensional simple C^*-algebra A which is asymptotically tracially in $\mathcal{N}_{\mathcal{Z}}$ (the class of all nuclear $\mathcal{Z}$-stable C^*-algebras) if and only if A is asymptotically tracially in $\mathcal{N}_n$ (the class of all C^*-algebras with nuclear dimension at most n) for some integer $n \geq 0$ (see Theorem 9.3 below). It is also shown that a unital separable simple C^*-algebra A which is asymptotically tracially in $\mathcal{C}_{\mathcal{Z},s}$ (the class of all separable $\mathcal{Z}$-stable C^*-algebras), is

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either purely infinite, or has stable rank one (see Theorem 9.1). Moreover, A has strict comparison (for positive elements). Furthermore, it is shown that if A is a unital separable simple C^*-algebra which is asymptotically tracially in $\mathcal{N}$ (the class of all nuclear C^*-algebras) and A is asymptotically tracially in $\mathcal{C}_{\mathcal{Z},s}$, then A is asymptotically tracially in $\mathcal{N}_{\mathcal{Z}}$ (see Theorem 8.7). As one expects, in the case that A is a unital separable nuclear simple C^*-algebra, then A is asymptotically tracially in $\mathcal{N}_n$ if and only if it is asymptotically tracially in $\mathcal{C}_{\mathcal{Z},s}$, and, if and only if A has finite nuclear dimension and $\mathcal{Z}$-stable. A number of other related results are also obtained. In Example 9.17, a large number of unital non-exact separable simple C^*-algebras which are asymptotically tracially in $\mathcal{N}_n$ are presented. It should be mentioned that if a unital simple C^*-algebra A is asymptotically tracially in the class of finite dimensional C^*-algebras then A has tracial rank zero, and if A is asymptotically tracially in the class of C^*-algebras which are 1-dimensional NCCW complexes then A has generalized tracial rank at most one.

The organization of this paper is as follows. Section 2 serves as a preliminary. We fix some frequently used notations and concepts there. Section 3 studies some basic properties of asymptotical tracial approximation. Section 4 gives some useful properties that are preserved by asymptotical tracial approximation. One of the results is that, if A is a unital separable simple C^*-algebra which is asymptotically tracially in the class of exact C^*-algebras, then every 2-quasitrace of A is a trace (see Corollary 4.7). Section 5 is a preparation for Section 6 which gives a sufficient and necessary condition for a c.p.c. generalized inductive limit to have finite nuclear dimension (Theorem 6.5). Section 7 shows that every unital infinite dimensional separable simple C^*-algebra which is asymptotically tracially in $\mathcal{N}_n$ is asymptotically tracially in $\mathcal{N}_{\mathcal{Z}}$ (see Theorem 7.18). In Section 8, we show that a separable simple unital infinite dimensional C^*-algebra which is asymptotically tracially in $\mathcal{N}$ and is also asymptotically tracially in $\mathcal{C}_{\mathcal{Z},s}$, then it is asymptotically tracially in $\mathcal{N}_{\mathcal{Z}}$ (Theorem 8.7). In Section 9, we summarize and combine some of the results. Theorem 9.11 shows that asymptotical tracial approximation behaves well under the spatial tensor products. As a consequence, a variety of examples can be produced. For example, if A is any unital separable simple C^*-algebra and B is a unital infinite dimensional separable simple C^*-algebra which is asymptotically tracially in the class of $\mathcal{Z}$-stable C^*-algebras, then the spatial tensor product $A \otimes B$ is asymptotically tracially in the class of $\mathcal{Z}$-stable C^*-algebras. If both A and B are asymptotically tracially in $\mathcal{N}_n$, then the spatial tensor product $A \otimes B$ is also asymptotically tracially in $\mathcal{N}_n$ (see Corollary 9.12).

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2 Preliminary

Notation 2.1. Let X be a normed space and $0 \le r \le s$ be real numbers. Set $B_{r,s}(X) := \{x \in X : r \le \|x\| \le s\}$. Denote by X^1 the closed unit ball $B_{0,1}(X)$ of X.

Let $a, b \in X$ and let $\epsilon > 0$, we write $a \approx_\epsilon b$ if $\|a - b\| < \epsilon$. Let $Y, Z \subset X$ and let $\epsilon > 0$, we say Y is an ϵ-net of Z, and denoted by $Z \subset_\epsilon Y$, if, for all $z \in Z$, there is $y \in Y$ such that $z \approx_\epsilon y$.

Notation 2.2. Let A and B be C^*-algebras, let $\varphi : A \to B$ be a map, let $\mathcal{F} \subset A$, and let $\epsilon > 0$. The map φ is called $(\mathcal{F}, \epsilon)$-multiplicative, or called ϵ-multiplicative on $\mathcal{F}$, if for any

$x, y \in \mathcal{F}$, $\varphi(xy) \approx_\epsilon \varphi(x)\varphi(y)$. If, in addition, for any $x \in \mathcal{F}$, $\|\varphi(x)\| \approx_\epsilon \|x\|$, then we say φ is an $(\mathcal{F}, \epsilon)$-approximate embedding.

Notation 2.3. Let A and B be C^*-algebras. The spatial tensor product of A and B is denoted by $A \otimes B$.

Notation 2.4. Let $\mathbb{N} = \{1, 2, \cdots\}$ be the set of natural numbers. Let M_k denote the algebra of k by k complex matrices ($k \in \mathbb{N}$). Let $\{e_{i,j}^{(k)}\}$ denote the canonical matrix units of M_k ($1 \leq i, j \leq k$ and $k \in \mathbb{N}$). If F is a finite dimensional C^*-algebra, then $\mathcal{G}^F$ denotes the standard generating set of F, i.e., the union of canonical matrix units of each direct summand of F. Note that the standard generating set of F is in the unit ball of F and is also a linear generating set .

Notation 2.5. Let A be a C^*-algebra and let $S, T \subset A$ be subsets of A. Set $S \cdot T := \{st : s \in S, t \in T\}$ and set $S^\perp := \{a \in A : as = 0 = sa, \forall s \in S\}$. Let $\overline{S}^{\|\cdot\|}$ be the norm closure of S. Denote by $\mathrm{Her}_A(S)$ (or just $\mathrm{Her}(S)$) the hereditary C^*-subalgebra of A generated by S. Let $C^*(S)$ be the C^*-subalgebra of A generated by S. Denote by A_+ the set of all positive elements in A, by $A_+^1 := A_+ \cap A^1$, and by A_{sa} the set of all self-adjoint elements in A. Denote by $\mathcal{M}(A)$ the multiplier algebra of A. For $x \in A$, the spectrum of x is denoted by $\mathrm{sp}_A(x)$, or just $\mathrm{sp}(x)$.

Notation 2.6. Let A_i be C^*-algebras ($i \in \mathbb{N}$). Set $\prod_{i=1}^\infty A_i := \{\{a_1, a_2, \cdots\} : a_i \in A_i, \sup_{i\in\mathbb{N}} \|a_i\| < \infty\}$, and set $\bigoplus_{i=1}^\infty A_i := \{\{a_1, a_2, \cdots\} : a_i \in A_i, \lim_{i\to\infty} \|a_i\| = 0\}$. Denote by $\pi_\infty : \prod_{i=1}^\infty A_i \to \prod_{i=1}^\infty A_i / \bigoplus_{i=1}^\infty A_i$ the quotient map. We also use the notation $l^\infty(A) := \prod_{i=1}^\infty A$ and $c_0(A) := \bigoplus_{i=1}^\infty A$. Define $\iota : A \to l^\infty(A)$ by $\iota(a) = \{a, a, \cdots\}$, the constant sequence, for all $a \in A$. Define $\iota_A = \pi_\infty \circ \iota$.

Let $h : B \to \prod_{i=1}^\infty A_i / \bigoplus_{i=1}^\infty A_i$ be a *-homomorphism. The map h is called a *strict embedding,* if for any $b \in B$, there exists $\{b_1, b_2, \cdots\} \in \prod_{i=1}^\infty A_i$ such that $h(b) = \pi_\infty(\{b_1, b_2, \cdots\})$ and $\|b\| = \liminf_{i\to\infty} \|b_i\|$. If $C \subset \prod_{i=1}^\infty A_i / \bigoplus_{i=1}^\infty A_i$ is a C^*-subalgebra and the embedding map $\iota : C \hookrightarrow \prod_{i=1}^\infty A_i / \bigoplus_{i=1}^\infty A_i$ is a strict embedding, then we say C is strictly embedded.

(1) Note that, if $C \subset l^\infty(A)/c_0(A)$ is full in $l^\infty(A)/c_0(A)$, then C is strictly embedded (see also Proposition 2.7).

(2) For a C^*-algebra A, the map ι_A defined above is a strict embedding, and the map $\hat{\iota} : A \to l^\infty(A)/c_0(A)$, $a \mapsto \pi_\infty(\{a, 0, a, 0, a, 0, \cdots\})$ is not.

Proposition 2.7. *Let A_1, $A_2, \cdots$ be C^*-algebras and let A be a simple C^*-algebra. Let $h : A \to \prod_{i=1}^\infty A_i / \bigoplus_{i=1}^\infty A_i$ be an embedding. If for some nonzero element $a \in A \backslash \{0\}$, there exists $a_i \in A_i$ ($i \in \mathbb{N}$) such that $h(a) = \pi_\infty(\{a_1, a_2, \cdots\})$ and $\liminf_{i\to\infty} \|a_i\| > 0$ hold, then h is a strict embedding.*

Proof. If h is not a strict embedding, then we can choose $c \in A$ and natural numbers $i_1 < i_2 < \cdots$, and $c_i \in A_i$ ($i \in \mathbb{N}$) such that $h(c) = \pi_\infty(\{c_1, c_2, \cdots\})$ and $\lim_{n\to\infty} \|c_{i_n}\| < \|c\|$.

Let $\pi_1 : \prod_{i=1}^\infty A_i / \bigoplus_{i=1}^\infty A_i \to \prod_{n=1}^\infty A_{i_n} / \bigoplus_{n=1}^\infty A_{i_n}$ be the quotient map induced by the quotient map $\pi_0 : \prod_{i=1}^\infty A_i \to \prod_{n=1}^\infty A_{i_n}$. By the assumption of this proposition, $\|\pi_1 \circ h(a)\| = \|\pi_\infty(\{a_{i_1}, a_{i_2}, \cdots\})\| = \liminf_{n\to\infty} \|a_{i_n}\| > 0$. It follows that $\pi_1 \circ h$ is a nonzero *-homomorphism. Since A is simple, $\pi_1 \circ h$ is an embedding. However, by the choice of c, we have $\|\pi_1 \circ h(c)\| = \|\pi_\infty(\{c_{i_1}, c_{i_2}, \cdots\})\| = \lim_{n\to\infty} \|c_{i_n}\| < \|c\|$, which is contradicted to that $\pi_1 \circ h$ is an embedding. Thus h is a strict embedding.

$\square$

Notation 2.8. Let $\epsilon > 0$. Define a continuous function $f_\epsilon : [0, +\infty) \to [0, 1]$ by

$$f_\epsilon(t) = \begin{cases} 0 & t \in [0, \epsilon], \\ 1 & t \in [2\epsilon, \infty), \\ \text{linear} & t \in [\epsilon, 2\epsilon]. \end{cases}$$

Notation 2.9. Let $\varphi : A \to B$ be a linear map. The map φ is positive, if $\varphi(A_+) \subset B_+$ and φ is completely positive, abbreviated as c.p., if $\varphi \otimes \mathrm{id} : A \otimes M_n \to B \otimes M_n$ are positive for all $n \in \mathbb{N}$. If φ is positive linear and $\|\varphi\| \leq 1$, then it is called positive contractive, abbreviated as p.c., if φ is c.p. and $\|\varphi\| \leq 1$, then φ is completely positive contractive, abbreviated as c.p.c.. If φ is c.p.c. and $\varphi(1_A) = 1_B$, then φ is call unital completely positive, abbreviated as u.c.p..

The following lemma is a well known corollary of Stinespring's theorem (cf. [35, Lemma 7.11]):

Lemma 2.10. *Let $\varphi : A \to B$ be a c.p.c. map from C^*-algebra A to C^*-algebra B. Then $\|\varphi(xy) - \varphi(x)\varphi(y)\| \leq \|\varphi(xx^*) - \varphi(x)\varphi(x^*)\|^{1/2}\|y\|$ for all $x, y \in A$.*

The following lemma is taken from [36, Lemma 3.5].

Lemma 2.11. *Let A, B, C be C^*-algebras, let $a \in A_{sa}$, and let $\epsilon > 0$. Suppose that $\psi : A \to B$ and $\varphi : B \to C$ are c.p.c. maps and $\|\varphi \circ \psi(a^2) - \varphi \circ \psi(a)^2\| \leq \epsilon$. Then, for all $b \in B$,*

$$\|\varphi(\psi(a)b) - \varphi(\psi(a))\varphi(b)\| \leq \epsilon^{1/2}\|b\| \text{ and } \|\varphi(b\psi(a)) - \varphi(b)\varphi(\psi(a))\| \leq \epsilon^{1/2}\|b\|.$$

Proof. We will only show the first inequality. The proof of the second is similar. We have

$$0 \leq \varphi(\psi(a)^2) - \varphi(\psi(a))^2 \leq \varphi(\psi(a^2)) - \varphi(\psi(a))^2 \leq \epsilon.$$

Thus $\|\varphi(\psi(a)^2) - \varphi(\psi(a))^2\| \leq \varepsilon$. By Lemma 2.10 we have $\|\varphi(\psi(a)b) - \varphi(\psi(a))\varphi(b)\| \leq \epsilon^{1/2}\|b\|$. □

Some versions of the following statements are well known (which can also be derived by using Lemma 2.11 in the case of c.p.c. maps).

Lemma 2.12. *For any C^*-algebras A and B, any p.c. map (resp. c.p.c. map) $\varphi : A \to B$, any projection $p \in A$, any $\delta \in (0, 1/8)$, if $\|\varphi(p) - \varphi(p)^2\| \leq \delta$, then there exists a p.c. map (resp. c.p.c. map) $\psi : A \to B$ satisfying*
(1) $\psi(p)$ is a projection in $C^*(\varphi(p))$, and
(2) $\|(\varphi - \psi)|_{pAp}\| < 5\delta^{1/2}$.

Proof. If $\|\varphi(p) - \varphi(p)^2\| \leq \delta < 1/8$, one has $\mathrm{sp}(\varphi(p)) \subset [0, \eta] \cup [1 - \eta, 1]$, where $\eta = \frac{2\delta}{1+\sqrt{1-4\delta}} < \frac{4\delta}{2+\sqrt{2}}$. Then

$$h(t) = \begin{cases} 0, & \text{for } t \in [0, \eta], \\ 1/t^{1/2}, & \text{for } t \in [1 - \eta, 1] \end{cases}$$

is a continuous function on $\mathrm{sp}(\varphi(p))$. Let $c := h(\varphi(p))$. Define a positive linear map (resp. c.p. map) $\psi : A \to B$ by $x \mapsto c\varphi(pxp)c$ for all $x \in A$. Then $e := \psi(p) = h(\varphi(p))^2\varphi(p)$ is a projection in $C^*(\varphi(p))$. It follows from [58, Corollary 1] that ψ is a p.c. map (resp. c.p.c map). For $x \in (pAp)^1_{sa}$, by Kadison's generalized Schwarz inequality ([32, Theorem 1]),

$$\|(1 - c)\varphi(x)\|^2 = \|(1 - c)\varphi(x)^2(1 - c)\| \leq \|(1 - c)\varphi(x^2)(1 - c)\| \leq \|(1 - c)\varphi(p)(1 - c)\| < \eta.$$

Then, for $x \in (pAp)^1_{sa}$, one estimates

$$\|\varphi(x) - \psi(x)\| = \|\varphi(x) - c\varphi(x)c\| \leq \|(1 - c)\varphi(x)\| + \|c\|\|\varphi(x)(1 - c)\| < \eta^{1/2}(1 + \frac{1}{\sqrt{1 - \eta}}).$$

Therefore, $\|(\varphi - \psi)|_{pAp}\| < 2\eta^{1/2}(1 + \frac{1}{\sqrt{1-\eta}}) = \frac{4}{\sqrt{2+\sqrt{2}}}(1 + \frac{1}{\sqrt{1-\eta}})\delta^{1/2} < 5\delta^{1/2}$.

□

Definition 2.13. Let A be a C^*-algebra and let $M_\infty(A)_+ := \bigcup_{n\in\mathbb{N}} M_n(A)_+$. For $x \in M_n(A)$, we identify x with $\mathrm{diag}(x,0) \in M_{n+m}(A)$ for all $m \in \mathbb{N}$. Let $a \in M_n(A)_+$ and $b \in M_m(A)_+$. Define $a\oplus b := \mathrm{diag}(a,b) \in M_{n+m}(A)_+$. If $a, b \in M_n(A)$, we write $a \lesssim_A b$ if there are $x_i \in M_n(A)$ such that $\lim_{i\to\infty} \|a - x_i^* b x_i\| = 0$. If such $\{x_i\}$ does not exist, then we write $a \not\lesssim_A b$. We write $a \sim b$ if $a \lesssim_A b$ and $b \lesssim_A a$ hold. The Cuntz relation $\sim$ is an equivalence relation. We also write $a \lesssim b$ and $a \sim b$, when A is given and there is no confusion. Set $W(A) := M_\infty(A)_+/\sim_A$. Let $\langle a\rangle$ denote the equivalence class of a. We write $\langle a\rangle \le \langle b\rangle$ if $a \lesssim_A b$. $(W(A), \le)$ is a partially ordered abelian semigroup. $W(A)$ is called almost unperforated, if for any $\langle a\rangle, \langle b\rangle \in W(A)$, and for any $k \in \mathbb{N}$, if $(k+1)\langle a\rangle \le k\langle b\rangle$, then $\langle a\rangle \le \langle b\rangle$ (see [54]).

Let $k \in \mathbb{N}$ be an integer. We write $k\langle a\rangle \lessapprox \langle b\rangle$ if $\mathrm{Her}(b)$ contains k mutually orthogonal elements $b_1, b_2, \cdots, b_k$ such that $a \lesssim b_i$, $i = 1, 2, \cdots, k$.

If $B \subset A$ is a hereditary C^*-subalgebra, $a, b \in B_+$, then $a \lesssim_A b \Leftrightarrow a \lesssim_B b$.

Definition 2.14. Denote by $QT(A)$ the set of 2-quasitraces of A with $\|\tau\| = \tau(1_A) = 1$ (see [2, II 1.1, II 2.3]) and by $T(A)$ the set of all tracial states on A. We will also use $T(A)$ as well as $QT(A)$ for the extensions on $M_k(A)$ for each k. For $\tau \in QT(A)$, define a lower semi-continuous function $d_\tau : M_k(A)_+ \to \mathbb{C}$, $a \mapsto \lim_{n\to\infty} \tau(f_{1/n}(a))$. The function d_τ is called the dimension function induced by τ.

Definition 2.15. Let A be a unital C^*-algebra. We say that A has strict comparison (for positive elements), if, for all $a, b \in M_k(A)_+$, $a \lesssim b$, whenever $d_\tau(a) < d_\tau(b)$ holds for all $\tau \in QT(A)$.

3 Asymptotical tracial approximation

Definition 3.1 (Asymptotical tracial approximation)**.** Let A be a unital simple C^*-algebra, let $\mathcal{P}$ be a class of C^*-algebras. We say A is asymptotically tracially in $\mathcal{P}$, if for any finite subset $\mathcal{F} \subset A$, any $\epsilon > 0$, and any $a \in A_+\backslash\{0\}$, there exist a C^*-algebra B in $\mathcal{P}$, c.p.c. maps $\alpha : A \to B$, $\beta_n : B \to A$, and $\gamma_n : A \to A$ $(n \in \mathbb{N})$, such that

(1) $x \approx_\epsilon \gamma_n(x) + \beta_n \circ \alpha(x)$ for all $x \in \mathcal{F}$ and for all $n \in \mathbb{N}$,

(2) α is an $(\mathcal{F}, \epsilon)$-approximate embedding,

(3) $\lim_{n\to\infty} \|\beta_n(xy) - \beta_n(x)\beta_n(y)\| = 0$ and $\lim_{n\to\infty} \|\beta_n(x)\| = \|x\|$ for all $x, y \in B$, and

(4) $\gamma_n(1_A) \lesssim_A a$ for all $n \in \mathbb{N}$.

Remark 3.2. Let us point out that in the definition above, we may assume that $\mathcal{F}$ is a finite subset of A_+^1, $\epsilon \in (0,1)$, and $\|a\| = 1$, without loss of generality.

Asymptotical tracial approximation may also be defined for non-unital C^*-algebras as well as for non-simple C^*-algebras. These will be discussed in a subsequent paper.

Suppose that $\mathcal{P}$ has the property that, if $A \in \mathcal{P}$, then $M_n(A) \in \mathcal{P}$ for all integer $n \ge 1$. Then, it is easy to see that, if A is asymptotically tracially in $\mathcal{P}$, then $M_n(A)$ is also asymptotically tracially in $\mathcal{P}$ (cf. [39, Theorem 3.7.3]). Also see Theorem 9.11.

There are a number of properties in the C^*-algebra theory which are known to be preserved by asymptotic approxiamtion. We would like to exploit this further by studying asymptotic tracial approximation as define in 3.1. We show that some of these properties are even preserved by asymptotic tracial approximation. Section 4 gives an incomplete list of them. Theorem 9.5 further reinforce this point of view.

One may notice that, in Definition 3.1, B is not a C^*-subalgebra of A. This is different from the conventional tracial approximation. On the other hand, Proposition 3.10 shows the similarity between asymptotic tracial approximation and the conventional tracial approximation. Proposition 3.10 also justifies the terminology. This may also partially explain our motivation. It is worth to point out that the condition (2) in Proposition 3.10 shows that the map γ_n is approximately orthogonal to $\beta_n \circ \alpha$. Moreover, as in the conventional tracial approximation, if A does not have (SP) property, then asymptotic tracial approximation becomes local approximation, a fact that will be used several times in this paper.

Remark 3.3. Let $\mathcal{P}_0$ be the class of finite dimensional C^*-algebras and let $\mathcal{P}_1$ be the class of C^*-algebras of 1-dimensional NCCW complexes (see [15] for definition of 1-dimensional NCCW complexes) respectively. Since C^*-algebras in $\mathcal{P}_0$ as well as in $\mathcal{P}_1$ are semiprojective (see [15]), we will show in Proposition 3.11 that A is asymptotically tracially in $\mathcal{P}_0$ is equivalent to that A has tracial rank zero (or A is tracially AF), and A is asymptotically tracially in $\mathcal{P}_1$ is equivalent to that A has generalized tracial rank one.

Definition 3.4. Denote by $\mathcal{E}$ the class of exact C^*-algebras and by $\mathcal{N}$ the class of nuclear C^*-algebras. For each $n \in \mathbb{N} \cup \{0\}$, let $\mathcal{N}_n$ be the class of C^*-algebras with nuclear dimension at most n (see 5.11 below). Let $\mathcal{C}_{\mathcal{Z}}$ be the class of $\mathcal{Z}$-stable C^*-algebras, let $\mathcal{C}_{\mathcal{Z},s}$ (and $\mathcal{C}_{\mathcal{Z},s,s}$) be the class of separable (and simple) $\mathcal{Z}$-stable C^*-algebras, let $\mathcal{N}_{\mathcal{Z}}$ be the class of nuclear $\mathcal{Z}$-stable C^*-algebras, let $\mathcal{N}_{\mathcal{Z},s}$ (and $\mathcal{N}_{\mathcal{Z},s,s}$) be the class of separable (and simple) nuclear $\mathcal{Z}$-stable C^*-algebras, respectively.

Example 3.5. Let A be a unital separable residually finite dimensional C^*-algebra, i.e. there exists a sequence of finite dimensional representations $\{\pi_i\}$ of A such that $\{\pi_i\}$ separates the points in A. Let us recall the construction in [14] below. For instance, A can be the full group C^*-algebra of the free group of two generators. Let n_i be the dimension of π_i ($i \in \mathbb{N}$), let $m_1 = 1$ and let $m_i = \prod_{j=1}^{i-1}(n_j + 1)$ for $i \geq 2$. For each $i \in \mathbb{N}$, define an injective *-homomorphism

$$h_i : A \otimes M_{m_i} \to A \otimes M_{m_{i+1}}, \quad x \mapsto x \oplus (\pi_i \otimes \mathrm{id}_{M_{m_i}})(x).$$

Let $B := \lim_{i\to\infty}(A \otimes M_{m_i}, h_i)$, then B is simple separable unital with tracial rank zero ([14], see also [39, Example 3.7.7]). In particular, B is asymptotically tracially in $\mathcal{N}_0$. In fact, Dădărlăt showed that, for any unital infinite dimensional simple AF-algebra C, one can produce a unital separable simple non-exact C^*-algebra B with tracial rank zero such that $K_0(B) = K_0(C)$ as ordered groups (see [14, Proposition 9]). In [51], Niu and Wang showed that, for some choices of A, B can be constructed to be a simple separable unital exact C^*-algebra with tracial rank zero but not $\mathcal{Z}$-stable (so it is asymptotically tracially in $\mathcal{N}_0$ but not $\mathcal{Z}$-stable). However, we will see later that B is asymptotically tracially in $\mathcal{C}_{\mathcal{Z},s}$. Actually, every simple separable unital infinite dimensional C^*-algebra which is asymptotically tracially in $\mathcal{N}_n$ is asymptotically tracially in $\mathcal{N}_{\mathcal{Z},s,s}$ (see Theorem 7.18).

Definition 3.6. A class of C^*-algebras $\mathcal{P}$ is said to have property (H), if, for any $B \in \mathcal{P}$ and any nonzero projection $e \in B$, $eBe \in \mathcal{P}$.

The following lemma is well known.

Lemma 3.7. *For any $\varepsilon > 0$, there exists $\delta > 0$ such that, for any unital C^*-algebras A, and B, any C^*-algebra C, and, any p.c. maps (resp. c.p.c. maps) $\varphi : A \to C$ and $\psi : B \to C$, if $\|\varphi(1_A) - \varphi(1_A)^2\| < \delta$ and $\|(\varphi(1_A) + \psi(1_B)) - (\varphi(1_A) + \psi(1_B))^2\| < \delta$, then there exist p.c. maps (resp. c.p.c. maps) $\bar{\varphi} : A \to C$ and $\bar{\psi} : B \to C$, satisfying*

(1) $\bar{\varphi}(1_A)$, $\bar{\psi}(1_B)$ are projections and $\bar{\varphi}(1_A) \perp \bar{\psi}(1_B)$, and

(2) $\|\varphi - \bar{\varphi}\| < \varepsilon$ and $\|\psi - \bar{\psi}\| < \varepsilon$.

Moreover, if $\varphi(1_A)$ is a projection, one can take $\bar{\varphi} = \varphi$.

Proof. Let $\varepsilon > 0$. Put $\varepsilon_1 = \min\{\varepsilon/(144\sqrt{2}), 1/4\}$. There exists a universal constant $\delta \in (0, \varepsilon_1/16)$ such that if $\|\varphi(1_A) - \varphi(1_A)^2\| < \delta$ and $\|(\varphi(1_A) + \psi(1_B)) - (\varphi(1_A) + \psi(1_B))^2\| < \delta$, then

$$\|\psi(1_B) - \psi(1_B)^2\| < (\varepsilon_1/5)^2 \text{ and } \|\varphi(1_A)\psi(1_B)\| < (\varepsilon_1/5)^2. \tag{e 3.1}$$

By Lemma 2.12 and (e 3.1), there exist p.c. maps (resp. c.p.c. maps) $\bar{\varphi} : A \to C$ and $\hat{\psi} : B \to C$, such that

(i) $\bar{\varphi}(1_A)$ and $\hat{\psi}(1_B)$ are projections, and

(ii) $\|\varphi - \bar{\varphi}\| < \varepsilon_1$, and $\|\psi - \hat{\psi}\| < \varepsilon_1$.

Note that if $\varphi(1_A)$ is a projection, then one can simply take $\varphi = \bar{\varphi}$. By (e 3.1) and (ii), one has $\|\bar{\varphi}(1_A)\hat{\psi}(1_B)\| < (\varepsilon_1/5)^2 + 2\varepsilon_1 < 3\varepsilon_1$. Then

$$\|\hat{\psi}(1_B) - (1 - \bar{\varphi}(1_A))\hat{\psi}(1_B)(1 - \bar{\varphi}(1_A))\| < 6\varepsilon_1. \tag{e 3.2}$$

Thus $((1-\bar{\varphi}(1_A))\hat{\psi}(1_B)(1-\bar{\varphi}(1_A)))^2 \approx_{18\varepsilon_1} (1-\bar{\varphi}(1_A))\hat{\psi}(1_B)(1-\bar{\varphi}(1_A))$. Then (see [39, Lemma 2.5.5], for example) there is a projection $q \in C^*((1 - \bar{\varphi}(1_A))\hat{\psi}(1_B)(1 - \bar{\varphi}(1_A)))$ such that

$$\|q - \hat{\psi}(1_B)\| < 36\varepsilon_1. \tag{e 3.3}$$

Therefore (see [39, Lemma 2.5.1], for example) there exists a unitary $u \in \tilde{C}$ (or in C, when C is unital) such that $\|1_{\tilde{C}} - u\| < 36\sqrt{2}\varepsilon_1 \leq \varepsilon/4$ and $u^*\hat{\psi}(1_B)u = q$. Define $\bar{\psi} : B \to C$ by $\bar{\psi}(x) := u^*\hat{\psi}(x)u$ for all $x \in A$. One then verifies that $\bar{\varphi}$ and $\bar{\psi}$ meet the requirements. □

Proposition 3.8. *Let $\mathcal{P}$ be a class of C^*-algebras. Let A be a unital simple C^*-algebra which is asymptotically tracially in $\mathcal{P}$. Then the following conditions hold: For any unital hereditary C^*-subalgebra $B \subset A$, any finite subset $\mathcal{F} \subset B$, any $\epsilon > 0$ and any $b \in B_+\backslash\{0\}$, there exist a C^*-algebra $\bar{C}$ in $\mathcal{P}$, a unital hereditary C^*-subalgebra C of $\bar{C}$, c.p.c. maps $\alpha : B \to C$, $\beta_n : C \to B$, and $\gamma_n : B \to B \cap \beta_n(C)^\perp$ ($n \in \mathbb{N}$), such that*

(1) the map α is u.c.p., $\beta_n(1_C)$, $\gamma_n(1_B)$ are projections, and $1_B = \beta_n(1_C) + \gamma_n(1_B)$ for all $n \in \mathbb{N}$,

(2) $x \approx_\epsilon \gamma_n(x) + \beta_n \circ \alpha(x)$ for all $x \in \mathcal{F}$ and for all $n \in \mathbb{N}$,

(3) α is an $(\mathcal{F}, \epsilon)$-approximate embedding,

(4) $\lim_{n\to\infty} \|\beta_n(xy) - \beta_n(x)\beta_n(y)\| = 0$ and $\lim_{n\to\infty} \|\beta_n(x)\| = \|x\|$ for all $x, y \in C$, and

(5) $\gamma_n(1_B) \lesssim_B b$ for all $n \in \mathbb{N}$.

If, in addition, $\mathcal{P}$ has property (H), then C is in $\mathcal{P}$, whence every unital hereditary C^*-subalgebra of A is also asymptotically tracially in $\mathcal{P}$.

Proof. Without loss of generality, we may assume that $1_B \in \mathcal{F} \subset B^1$ and $\epsilon < 1$. Let $\delta_1 < \varepsilon/64$ be the universal constant (in place of δ) in Lemma 3.7 associated with $\varepsilon/64$ (in the place of ε). Let $\delta := \frac{1}{128}\min\{\varepsilon, (\delta_1/5)^2, 1\}$.

Since A is asymptotically tracially in $\mathcal{P}$, there exist a C^*-algebra $\bar{C}$ in $\mathcal{P}$ and c.p.c maps $\bar{\alpha} : A \to \bar{C}$, $\bar{\beta}_n : \bar{C} \to A$, and $\bar{\gamma}_n : A \to A$ ($n \in \mathbb{N}$) such that

($1'$) $x \approx_\delta \bar{\gamma}_n(x) + \bar{\beta}_n \circ \bar{\alpha}(x)$ for all $x \in \mathcal{F}$ and for all $n \in \mathbb{N}$,

($2'$) $\bar{\alpha}$ is an $(\mathcal{F}, \delta)$-approximate embedding,

($3'$) $\lim_{n\to\infty} \|\bar{\beta}_n(xy) - \bar{\beta}_n(x)\bar{\beta}_n(y)\| = 0$, $\lim_{n\to\infty} \|\bar{\beta}_n(x)\| = \|x\|$ for all $x, y \in \bar{C}$, and

($4'$) $\bar{\gamma}_n(1_A) \lesssim_A b$, for all $n \in \mathbb{N}$.

Since $\|\bar{\alpha}(1_B) - \bar{\alpha}(1_B)^2\| < \delta$ (see ($2'$)), by Lemma 2.12, there exists a c.p.c. map $\alpha : A \to \bar{C}$ such that $\alpha(1_B)$ is a projection and

$$\|\bar{\alpha}(x) - \alpha(x)\| < \frac{\delta_1}{8}\|x\| \text{ for all } x \in 1_B A 1_B = B. \tag{e 3.4}$$

Let $C := \alpha(1_B)\bar{C}\alpha(1_B)$ be a unital hereditary C^*-subalgebra of $\bar{C}$. We may view α as a map from B to C. Then, by ($2'$), (e 3.4), and by the choice of δ, α is an $(\mathcal{F}, \varepsilon)$-approximate embedding. Thus (3) in the proposition holds.

By ($3'$), we have $\lim_{n\to\infty} \|\bar{\beta}_n(1_C) - \bar{\beta}_n(1_C)^2\| = 0$. Then, by Lemma 2.12, there exist c.p.c. maps $\hat{\beta}_n : C \to A$ such that $\hat{\beta}_n(1_C)$ are projections and

$$\|\bar{\beta}_n|_C - \hat{\beta}_n\| \to 0 \ \text{(as } n \to \infty). \tag{e 3.5}$$

By (e 3.4) and (e 3.5), without loss of generality, we may assume that, for all $n \in \mathbb{N}$,

$$\bar{\beta}_n \circ \bar{\alpha}(x) \approx_{\delta_1/8} \hat{\beta}_n \circ \alpha(x) \text{ for all } x \in \mathcal{F}. \tag{e 3.6}$$

Then, from (e 3.6) and ($1'$),

$$\|(\bar{\gamma}_n(1_B) + \hat{\beta}_n \circ \alpha(1_B)) - (\bar{\gamma}_n(1_B) + \hat{\beta}_n \circ \alpha(1_B))^2\| < \delta_1.$$

By Lemma 3.7 and the choice of δ_1, for each $n \in \mathbb{N}$, there exists a c.p.c. map $\hat{\gamma}_n : A \to A$ such that

(i) $\hat{\gamma}_n(1_B)$ is a projection and $\hat{\gamma}_n(1_B) \perp \hat{\beta}_n \circ \alpha(1_A)$, and

(ii) $\|\hat{\gamma}_n - \bar{\gamma}_n\| < \varepsilon/64$.

By (ii), (e 3.6) and ($1'$), we have $1_B \approx_{\varepsilon/32} \hat{\gamma}_n(1_B) + \hat{\beta}_n \circ \alpha(1_B)$. Then there exist unitaries $u_n \in A$ ($n \in \mathbb{N}$) such that $\|1_A - u_n\| < \sqrt{2}\varepsilon/32$ and $u_n^*(\hat{\gamma}_n(1_B) + \hat{\beta}_n \circ \alpha(1_B))u_n = 1_B$ (see [39, Lemma 2.5.1]). Define c.p.c. maps $\gamma_n : B \to B$ by $\gamma_n(x) := u_n^*\hat{\gamma}_n(x)u_n$, and define c.p.c. maps $\beta_n : C \to B$ by $\beta_n(x) := u_n^*\hat{\beta}_n(x)u_n$. Then (1) in the proposition holds. By ($3'$), (e 3.5) and the fact that u_n are unitaries, condition (4) in the proposition holds.

By (ii) and the fact that $\|1_A - u_n\| < \sqrt{2}\varepsilon/32$, we have

$$\|\gamma_n - \bar{\gamma}_n\| < \varepsilon/4 \text{ and } \|\beta_n - \hat{\beta}_n\| < \varepsilon/4. \tag{e 3.7}$$

Then, by (e 3.7), (e 3.6) and ($1'$), condition (2) in the proposition holds.

By the fact that $\gamma_n(1_B)$ is a projection, (e 3.7), and [54, Proposition 2.2], we have $\gamma_n(1_B) \sim f_{1/4}(\gamma_n(1_B)) \lesssim \bar{\gamma}_n(1_B) \overset{(4')}{\lesssim} b$. Thus (5) in the proposition holds. The proposition follows.

□

The following lemma is well known.

Lemma 3.9. *Let A be a C^*-algebra, $a \in A_+$ and let $p \in A$ be a projection with $p \lesssim_A a$. Then there exists $s \in A$ such that $p = s^*as$. Moreover, if $\{a_n\} \in l^\infty(A)_+$ and $\{p_n\} \in l^\infty(A)$ is a projection such that $\pi_\infty(\{p_n\}) \lesssim_{l^\infty(A)/c_0(A)} \pi_\infty(\{a_n\})$, then there exists $\{s_n\} \in l^\infty(A)$ such that $s_n^*s_n = p_n$ and $s_ns_n^* \in \mathrm{Her}(a_n)$ for all large n.*

Proof. Since $p \lesssim_A a$, there exists $r \in A$ such that $\|p - r^*ar\| < 1/2$. Then $\|p - pr^*arp\| < 1/2$. Therefore pr^*arp is an invertible positive element in pAp. Hence, by functional calculus there exists $b \in (pAp)_+$ with $\|b\| \le \sqrt{2}$ such that $p = bpr^*arpb$. Choose $s = rpb$. Then $p = s^*as$.

For "Moreover" part, by what has been proved, there is $t \in l^\infty(A)/c_0(A)$ such that $\pi_\infty(\{p_n\}) = t^*\iota_A(a)t$. Then there exists $\{t_n\} \in l^\infty(A)$ such that $\|p_n - (t_n)^*at_n\| < 1/2$ for all large n. Thus, by what has been proved, there is $r_n \in A$ ($\|r_n\| \le \sqrt{2}\|t_n\|$) such that $p_n = r_n^*ar_n$ for all large n. Let $s_n = a_n^{1/2}r_n$. Then $s_n^*s_n = p_n$ and $q_n := s_ns_n^* = a_n^{1/2}r_nr_n^*a_n^{1/2} \in \mathrm{Her}(a_n)$.

□

The following proposition provides another picture of Definition 3.1.

Proposition 3.10. *Let A be a simple unital C^*-algebra and $\mathcal{P}$ be a class of separable C^*-algebras with Property (H). If A is asymptotically tracially in $\mathcal{P}$, then the following holds:*

For any finite subset $\mathcal{F}\subset A$, any $\epsilon>0$, and any $a\in A_+\backslash\{0\}$, there exists a C^*-subalgebra $B\subset l^\infty(A)/c_0(A)$ with unit 1_B which is strictly embedded such that B in $\mathcal{P}$, and (recall notations defined in Notation 2.6)

(1) $1_B\iota_A(x)\approx_\epsilon \iota_A(x)1_B$ for all $x\in\mathcal{F}$,

(2) $1_B\iota_A(x)1_B\in_\epsilon B$ and $\|1_B\iota_A(x)1_B\|\geq\|x\|-\epsilon$ for all $x\in\mathcal{F}$, and

(3) $\iota_A(1_A)-1_B\lesssim_{l^\infty(A)/c_0(A)}\iota_A(a)$.

If $\mathcal{P}$ is a class of separable nuclear C^*-algebras, then converse also holds.

Proof. Assume that A is asymptotically tracially in $\mathcal{P}$. Let $\mathcal{F}\subset A$ be a finite subset with $1_A\in\mathcal{F}$, let $\epsilon\in(0,1)$, let $a\in A_+$ with $\|a\|=1$, and let $\delta:=\epsilon^2/4$. By Proposition 3.8, there exist a unital C^*-algebra B in $\mathcal{P}$, c.p.c. maps $\alpha:A\to B$, $\beta_n:B\to A$, and $\gamma_n:A\to A\cap\beta_n(B)^\perp$ $(n\in\mathbb{N})$ such that

($1'$) α is u.c.p., $\beta_n(1_B)$ and $\gamma_n(1_A)$ are projections, and $1_A=\beta_n(1_B)+\gamma_n(1_A)$ for all $n\in\mathbb{N}$,

($2'$) $x\approx_\delta\gamma_n(x)+\beta_n\circ\alpha(x)$ for all $x\in\mathcal{F}$ and for all $n\in\mathbb{N}$,

($3'$) α is an $(\mathcal{F},\delta)$-approximate embedding,

($4'$) $\lim_{n\to\infty}\|\beta_n(xy)-\beta_n(x)\beta_n(y)\|=0$ and $\lim_{n\to\infty}\|\beta_n(x)\|=\|x\|$ for all $x,y\in B$, and

($5'$) $\gamma_n(1_A)\lesssim_A f_{1/2}(a)$ for all $n\in\mathbb{N}$.

Note that ($4'$) induces a strict embedding $\beta:B\to l^\infty(A)/c_0(A)$, $x\mapsto\pi_\infty(\{\beta_n(x)\})$, and that ($2'$) shows that, for any $x\in\mathcal{F}$,

$$\begin{aligned}
\|\beta(1_B)\iota_A(x)-\iota_A(x)\beta(1_B)\| &= \limsup_{n\to\infty}\|\beta_n\circ\alpha(1_A)x-x\beta_n\circ\alpha(1_A)\|\\
&\leq 2\delta+\limsup_{n\to\infty}\|\beta_n\circ\alpha(1_A)(\gamma_n(x)+\beta_n\circ\alpha(x))-(\gamma_n(x)+\beta_n\circ\alpha(x))\beta_n\circ\alpha(1_A)\|\\
&= 2\delta+\limsup_{n\to\infty}\|\beta_n\circ\alpha(1_A)\beta_n(\alpha(x))-\beta_n(\alpha(x))\beta_n\circ\alpha(1_A)\|\\
&= 2\delta+\limsup_{n\to\infty}\|\beta_n(\alpha(1_A)\alpha(x)-\alpha(x)\alpha(1_A))\|\leq 4\delta<\varepsilon.
\end{aligned}$$

Thus (1) of the proposition holds. For any $x\in\mathcal{F}$,

$$\begin{aligned}
\|\beta(1_B)\iota_A(x)\beta(1_B)-\beta\circ\alpha(x)\| &= \limsup_{n\to\infty}\|\beta_n(1_B)x\beta_n(1_B)-\beta_n\circ\alpha(x)\|\\
&\leq \delta+\limsup_{n\to\infty}\|\beta_n(1_B)(\gamma_n(x)+\beta_n\circ\alpha(x))\beta_n(1_B)-\beta_n\circ\alpha(x)\|\\
&= \delta+\limsup_{n\to\infty}\|\beta_n(1_B)\beta_n\circ\alpha(x)\beta_n(1_B)-\beta_n\circ\alpha(x)\|=\delta<\varepsilon.
\end{aligned}$$

Thus $\beta(1_B)\iota_A(x)\beta(1_B)\in_\epsilon\beta(B)$. By the estimation above, ($4'$), and by ($3'$), we also have

$$\|\beta(1_B)\iota_A(x)\beta(1_B)\|\geq\|\beta\circ\alpha(x)\|-\delta=\|\alpha(x)\|-\delta\geq\|x\|-2\delta\geq\|x\|-\epsilon.$$

Thus (2) of the proposition holds.

By ($1'$), ($5'$), and Lemma 3.9, there exist partial isometries $s_n\in A$ such that $1_A-\beta_n(1_B)=s_n^*s_n$ and $s_ns_n^*\in\mathrm{Her}_A(f_{1/2}(a))$ for all large n. Let $s=\{s_n\}\in l^\infty(A)$. Then $\pi_\infty(s)^*\pi_\infty(s)=\iota_A(1_A)-\beta(1_B)$ and, since $f_{1/4}(a)f_{1/2}(a)=f_{1/2}(a)$,

$$\begin{aligned}
\pi_\infty(s)\pi_\infty(s)^* &= \pi_\infty(\{s_ns_n^*\})=\pi_\infty(\{f_{1/4}(a)s_ns_n^*f_{1/4}(a)\}) && \text{(e 3.8)}\\
&= f_{1/4}(\iota_A(a))\pi_\infty(ss^*)f_{1/4}(\iota_A(a))\in\mathrm{Her}_{l^\infty(A)/c_0(A)}(\iota_A(a)), && \text{(e 3.9)}
\end{aligned}$$

which implies that (3) of the proposition holds. This proves the first part of the proposition.

For the second part, let us assume that $\mathcal{P}$ is a class of separable nuclear C^*-algebras and consider the converse. Let $\mathcal{F}\subset A^1$ be a finite subset, let $\epsilon>0$, and let $a\in A_+$ with $\|a\|=1$.

Let $\delta := \frac{\epsilon}{100}$ and let $\bar{\mathcal{F}} := \mathcal{F} \cup (\mathcal{F} \cdot \mathcal{F})$. Suppose that (1), (2) and (3) hold for $\bar{\mathcal{F}}$, δ, a, and some unital separable nuclear C^*-algebra $B \in \mathcal{P}$. By (2) and the fact that B is nuclear, and by a consequence of Arveson's extension theorem (see [39, Theorem 2.3.13]), there exists a c.p.c. map $\alpha' : l^\infty(A)/c_0(A) \to B$ such that

$$1_B \iota_A(x) 1_B \approx_{2\delta} \alpha'(1_B \iota_A(x) 1_B) \text{ for all } x \in \bar{\mathcal{F}}. \tag{e 3.10}$$

Define a c.p.c. map $\alpha : A \to B$ by $x \mapsto \alpha'(1_B \iota_A(x) 1_B)$. For $x, y \in \mathcal{F}$, by (e 3.10) and (2), we have $\|\alpha(x)\| \geq \|1_B \iota_A(x) 1_B\| - 2\delta \geq \|x\| - 3\delta$, and

$$\alpha(x)\alpha(y) \overset{(e\,3.10)}{\approx_{4\delta}} 1_B \iota_A(x) 1_B \iota_A(y) 1_B \overset{(1)}{\approx_{\delta}} 1_B \iota_A(xy) 1_B \overset{(e\,3.10)}{\approx_{2\delta}} \alpha(xy).$$

Thus (2) in Definition 3.1 holds. Since B is nuclear and separable, by the Choi-Effors Lifting Theorem (see [11, Theorem 3.10]), there exists a c.p.c. map $\beta : B \to l^\infty(A)$ such that $\pi_\infty \circ \beta = \mathrm{id}_B$. Let $\beta_n : B \to A$ be the n-th component of β. Applying Lemma 2.12, we may also assume that $\beta_n(1_B)$ is a projection for all large n. Since β is a strict embedding, $\{\beta_n\}$ satisfies (3) in Definition 3.1.

Define a c.p.c. map $\gamma_n : A \to A$ by $x \mapsto (1_A - \beta_n(1_B))x(1_A - \beta_n(1_B))$. Note that $\gamma_n(1_A)$ is a projection for all large n, and $\pi_\infty(\{\gamma_n(1_A)\}) = \iota(1_A) - 1_B$. By (3) and Lemma 3.9, we may also assume, for all large n, $\gamma_n(1_A) \lesssim_A a$. Hence (4) in Definition 3.1 holds for all large n.

By (1), for all $x \in \mathcal{F}$,

$$\begin{aligned}
\iota_A(x) &\approx_{2\delta} (\iota_A(1_A) - 1_B)\iota_A(x)(\iota_A(1_A) - 1_B) + (1_B \iota_A(x) 1_B) \\
&\approx_{2\delta} (\iota_A(1_A) - 1_B)\iota_A(x)(\iota_A(1_A) - 1_B) + \alpha(x) \\
&= \pi_\infty(\{1_A - \beta_n(1_B)\})\iota_A(x)\pi_\infty(\{1_A - \beta_n(1_B)\}) + \alpha(x) \\
&= \pi_\infty(\{(1_A - \beta_n(1_B))x(1_A - \beta_n(1_B))\} + \beta \circ \alpha(x)) \\
&= \pi_\infty(\{\gamma_n(x) + \beta_n \circ \alpha(x)\}).
\end{aligned}$$

Therefore $x \approx_{4\delta} \gamma_n(x) + \beta_n \alpha(x)$ for all large n. Hence (1) in Definition 3.1 holds for all large n. It follows that A is asymptotically tracially in $\mathcal{P}$.

□

Proposition 3.11. *Let $\mathcal{P}_0$ be the class of finite dimensional C^*-algebras and $\mathcal{P}_1$ be the class of 1-dimensional NCCW complexes. Suppose that A is a unital simple C^*-algebra. Then A is asymptotically tracially in $\mathcal{P}_0$ if and only A has tracial rank zero, and, A is asymptotically tracially in $\mathcal{P}_1$ if and only if A has generalized tracial rank at most one.*

Proof. It is clear, from the definition (see Theorem 6.13 and Lemma 5.5 of [41] and Definition 9.2, Remark 9.3 and 9.5 of [26]), if A has tracial rank zero, then A is asymptotically tracially in $\mathcal{P}_0$, and, if A has generalized tracial rank at most one, then A is asymptotically tracially in $\mathcal{P}_1$. We will show the converse of these statements.

Suppose that A is asymptotically tracially in $\mathcal{P}_i$ ($i = 0, 1$). Let $\mathcal{F} \subset A$, $\varepsilon > 0$ and $a \in A_+ \backslash \{0\}$. By Proposition 3.10, there is $B \in \mathcal{P}_i$ such that (1), (2) and (3) in Proposition 3.10 hold for $\varepsilon/4$ (in place of ε).

Let $\iota : B \to l^\infty(A)/c_0(A)$ be the embedding. Since C^*-algebras in both $\mathcal{P}_0$ and $\mathcal{P}_1$ are semiprojective (see Theorem 6.22 of [15]), there is a homomorphism $\varphi : B \to A$ such that, with $p = \varphi(1_B)$,

(i) $px \approx_\varepsilon xp$ for all $x \in \mathcal{F}$,
(ii) $pxp \in_{\varepsilon/2} \varphi(B)$ and $\|pxp\| \geq \|x\| - \varepsilon$ for all $x \in \mathcal{F}$, and
(iii) $1 - p \lesssim a$ (see Lemma 3.9).

If $i = 0$, then B is finite dimensional, so is $\varphi(B)$. Therefore, by (i), (ii) and (iii) above, A has tracial rank zero.

If $i = 1$, then $\varphi(B)$ is a quotient of a B. Then, by Lemma 3.20 of [26], there is a unital C^*-algebra $B_0 \subset \varphi(B)$ which is in $\mathcal{P}_1$ such that

(ii') $pxp \in_\varepsilon B_0$ for all $x \in \mathcal{F}$. Therefore, A has generalized tracial rank at most one (see).

□

The proof of the following is standard and can be found in [41, Theorem 5.3] (see also [39, Lemma 3.6.5]).

Proposition 3.12. *Let $\mathcal{P}$ be a class of unital C^*-algebras which satisfy property (H). Let A be a unital simple C^*-algebra which satisfies the first part of the conclusion of Proposition 3.10 (associated with $\mathcal{P}$). Then any unital hereditary C^*-subalgebra C of A also satisfies the first part of the conclusion of Proposition 3.10 associated with $\mathcal{P}$, i.e., for any finite subset $\mathcal{F} \subset C$, any $\epsilon > 0$, and any $a \in C_+ \backslash \{0\}$, there exists a C^*-subalgebra $B \subset l^\infty(C)/c_0(C)$ with unit 1_B which is strictly embedded such that B in $\mathcal{P}$, and*

(1) $1_B \iota_A(x) \approx_\epsilon \iota_A(x) 1_B$ for all $x \in \mathcal{F}$,

(2) $1_B \iota_A(x) 1_B \in_\epsilon B$ and $\|1_B \iota_A(x) 1_B\| \geq \|x\| - \epsilon$ for all $x \in \mathcal{F}$, and

(3) $\iota_A(1_A) - 1_B \lesssim_{l^\infty(A)/c_0(A)} \iota_A(a)$.

4 Properties passing by asymptotical tracial approximations

In this section, it will be shown that, for certain classes of C^*-algebras $\mathcal{P}$, if a unital simple C^*-algebra A is asymptotically tracially in $\mathcal{P}$, then A is actually in $\mathcal{P}$.

Definition 4.1. Recall that a unital C^*-algebra A is finite, if for any nonzero projection $p \in A$, $1_A \lesssim_A p$ implies $p = 1_A$. A is called stably finite, if $A \otimes M_n$ is finite for all $n \in \mathbb{N}$.

Proposition 4.2. *Let A be a unital separable simple C^*-algebra.*

(a) Let $\mathcal{P}_f$ be the class of unial finite C^*-algebras. If A is asymptotically tracially in $\mathcal{P}_f$, then $A \in \mathcal{P}_f$.

(b) Let $\mathcal{P}_{sf}$ be the class of unial stably finite C^*-algebras. If A is asymptotically tracially in $\mathcal{P}_{sf}$, then $A \in \mathcal{P}_{sf}$.

(c) Let $\mathcal{Q}$ be the class of separable quasidiagonal C^*-algebras. If A is asymptotically tracially in $\mathcal{Q}$, then $A \in \mathcal{Q}$.

Proof. For (a), assuming otherwise and that there is a projection $p \in A$ and there is $v \in A$ such that $v^*v = 1_A$ and $vv^* := p \neq 1_A$. Since A is asymptotically tracially in $\mathcal{P}_f$, and $\mathcal{P}_f$ has property (H), then by Proposition 3.8, for any $\varepsilon > 0$, with $\mathcal{F} = \{1_A, p, v, v^*, 1-p\}$, there is a u.c.p. map $\alpha : A \to B$ for some unital finite C^*-algebra B which is an $(\mathcal{F}, \varepsilon)$-approximate embedding.

With sufficiently small ε, we may assume that there is a projection $e \in B$ such that

$$\|\alpha(1_A) - \alpha(p)\| \geq 1 - 1/64, \tag{e 4.1}$$

$$\alpha(v)^*\alpha(v) \approx_{1/64} \alpha(1_A) = 1_B \text{ and } \alpha(v)\alpha(v)^* \approx_{1/64} \alpha(p) \approx_{1/64} e. \tag{e 4.2}$$

It follows from (e 4.2) that 1_B and e are equivalent in B, and from (e 4.1) that $\|1_B - e\| \geq 1/2$, which contradicts the assumption that B is finite. In other words, A is in $\mathcal{P}_f$.

For (b), note that B in $\mathcal{P}_{sf}$ implies $M_n(B)$ in $\mathcal{P}_{sf}$ for all $n \in \mathbb{N}$. Therefore (b) follows from (a) and Remark 3.2.

For (c), let $\mathcal{F} \subset A^1$ be a finite subset and let $\varepsilon > 0$. By Proposition 3.8, there is a unital quasidiagonal C^*-algebra B and a c.p.c. map $\alpha : A \to B$ such that

$$\|\alpha(a)\| \geq (1 - \varepsilon/4)\|a\| \text{ and } \|\alpha(ab) - \alpha(a)\alpha(b)\| < \varepsilon/4 \text{ for all } a, b \in \mathcal{F}. \tag{e 4.3}$$

Since B is quasidiagonal, by [65, Theorem 1], there is a c.p.c. map $\beta : B \to F$ (for some finite dimensional C^*-algebra F) such that $\|\beta(y)\| \geq \|y\| - \varepsilon/16$ and $\|\beta(xy) - \beta(x)\beta(y)\| < \varepsilon/16$ for all $x, y \in \alpha(\mathcal{F})$. Let $\varphi = \beta \circ \alpha$. Then φ is a c.p.c. map from A to F. For all $a \in \mathcal{F}$,

$$\|\varphi(a)\| = \|\beta \circ \alpha(a)\| \geq \|\alpha(a)\| - \varepsilon/16 \geq (1 - \varepsilon/4)\|a\| - \varepsilon/16 \geq \|a\| - \varepsilon. \tag{e 4.4}$$

Moreover, for all $a, b \in \mathcal{F}$.

$$\varphi(ab) = \beta(\alpha(ab)) \approx_{\varepsilon/4} \beta(\alpha(a)\alpha(b)) \approx_{\varepsilon/16} \beta(\alpha(a))\beta(\alpha(b)) = \varphi(a)\varphi(b). \tag{e 4.5}$$

It follows from [65, Theorem 1] that A is quasidiagonal.

□

The following is taken from the proof of [38, Lemma 2.4].

Lemma 4.3 (cf. [38, Lemma 2.4]). *Let A be a separable non-elementary simple C^*-algebra. Then there exists a sequence $\{d_n\}$ in A_+ such that $\|d_n\| = 1$, $(n+1)\langle d_{n+1}\rangle \lessapprox \langle d_n\rangle$ (recall the Definition 2.13) ($n \in \mathbb{N}$), and, for any $x \in A_+ \setminus \{0\}$, there exists $N \in \mathbb{N}$ such that $\langle d_N\rangle \leq \langle x\rangle$.*

Proof. The proof is contained in the proof of [38, Lemma 2.4]. Let $\{x_n\}$ be a dense sequence of the unit sphere of A, let $z_n = (x_n^* x_n)^{1/2}$ and $y_n = f_{1/2}(z_n)$, $n \in \mathbb{N}$. The proof of [38, Lemma 2.4] shows that, for any $x \in A_+ \setminus \{0\}$, (we may assume that $\|x\| = 1$) there exists N such that $y_N \lesssim x$. Indeed, as exactly in the proof of [38, Lemma 2.4], there is an integer N such that $\|x - z_N\|$ is sufficiently small, and, with $1/8 > \varepsilon > 0$,

$$\|f_\varepsilon(x) - f_\varepsilon(z_N)\| < \varepsilon/4.$$

By [54, Proposition 2.2],

$$y_N \lesssim f_{1/4}(z_N) \lesssim f_{\varepsilon/2}(f_\varepsilon(z_N)) \lesssim f_\varepsilon(x) \lesssim x. \tag{e 4.6}$$

Now let $d_1 = y_1/\|y_1\|$. There are 2 mutually orthogonal nonzero elements $z_{1,1}, z_{1,2} \in \mathrm{Her}(d_1)_+$ (as in the proof of [38, Lemma 2.4]). By [38, Lemma 2.3], for example, there is $d_2 \in \mathrm{Her}(d_1)_+$ such that $\|d_2\| = 1$ and $d_2 \lesssim y_2$, $z_{1,1}, z_{1,2}$. It follows that $2\langle d_2\rangle \lessapprox \langle d_1\rangle$.

Suppose $d_1, d_2, \cdots, d_n$ have been chosen so that $\|d_j\| = 1$, $d_j \lesssim y_j$ ($j = 1, 2, \cdots, n$), and $(j+1)\langle d_{j+1}\rangle \lessapprox \langle d_j\rangle$ ($j = 1, 2, \cdots, n-1$). There are $n+1$ mutually orthogonal nonzero elements $z_{n,1}, z_{n,2}, \cdots, z_{n,n+1} \in \mathrm{Her}(d_n)_+$ (as in the proof of [38, Lemma 2.4]). By [38, Lemma 2.3], for example, there is $d_{n+1} \in \mathrm{Her}(d_n)_+$ such that $\|d_{n+1}\| = 1$ and $d_{n+1} \lesssim y_{n+1}$, $z_{n,i}$, $i = 1, 2, \cdots, n+1$. It follows that $(n+1)\langle d_{n+1}\rangle \lessapprox \langle d_n\rangle$.

By the induction, we obtain a sequence $\{d_n\}$ such that $\|d_n\| = 1$, $d_n \lesssim y_n$, and $n\langle d_{n+1}\rangle \lessapprox \langle d_n\rangle$, $n \in \mathbb{N}$. By (e 4.6), for any $x \in A_+ \setminus \{0\}$, there is N such that $d_N \lesssim y_N \lesssim x$.

□

Proposition 4.4. *Let $\mathcal{P}$ be the class of separable purely infinite simple C^*-algebra. Suppose that A is a unital simple C^*-algebra which is asymptotically tracially in $\mathcal{P}$. Then A is a purely infinite simple C^*-algebra.*

Proof. We may assume that A is not elementary. Let $a \in A_+ \setminus \{0\}$. It suffices to show that $1_A \lesssim a$ ([13], see also [48]). We may assume that $\|a\| = 1$. By applying Lemma 4.3 to $\mathrm{Her}(a)$, we obtain two nonzero mutually orthogonal elements a_0 and a_1 with $\|a_0\| = 1$ and $\|a_1\| = 1$ such that $a_0 + a_1 \lesssim a$. Let $b = f_{1/2}(a_1)$ and let $\varepsilon := 1/2^{10}$. Since A is asymptotically tracially in $\mathcal{P}$, by Proposition 3.10, there exists a unital C^*-subalgebra $B \subset l^\infty(A)/c_0(A)$ which is strictly embedded such that B in $\mathcal{P}$, and

(1) $1_B\iota_A(b) \approx_\epsilon \iota_A(b)1_B$,

(2) $1_B\iota_A(b)1_B \in_\epsilon B$, $\|1_B\iota_A(b)1_B\| \ge \|b\| - \epsilon$, and

(3) $\iota_A(1_A) - 1_B \lesssim_{l^\infty(A)/c_0(A)} \iota_A(f_{1/2}(a_0))$.

By (2), there exists an element $b_1 \in B_+$ such that

$$\|1_B\iota_A(b)1_B - b_1\| < \varepsilon = 1/2^{10}. \tag{e 4.7}$$

Since B is purely infinite, by [55, Proposition 4.1.1], there is $x \in B$ such that $x^* f_{1/2}(b_1)x = 1_B$. There exists a sequence of projections $p_n \in A$ such that $\pi_\infty(\{p_n\}) = 1_B$, where $\pi_\infty : l^\infty(A) \to l^\infty(A)/c_0(A)$ is the quotient map. Then we obtain $\{x_n\}, \{b_{1,n}\} \in l^\infty(A)$ (with $\pi_\infty(\{x_n\}) = x$ and $b_1 = \pi_\infty(\{b_{1,n}\})$) such that

$$\lim_{n\to\infty} \|x_n^* f_{1/2}(b_{1,n})x_n - p_n\| = 0 \text{ and } \limsup \|p_n b p_n - b_{1,n}\| \le \varepsilon. \tag{e 4.8}$$

Then (e 4.8) (see [54, Proposition 2.2] again) implies that, for large n,

$$p_n \lesssim f_{1/2}(b_{1,n}) \text{ and } f_{1/2}(b_{1,n}) \lesssim p_n b p_n. \tag{e 4.9}$$

On the other hand, by (3) and Lemma 3.9, $1 - p_n \lesssim f_{1/2}(a_0)$ for all large n. It follows that, for all sufficiently large n,

$$1_A = (1 - p_n) + p_n \lesssim f_{1/2}(a_0) + b = f_{1/2}(a_0) + f_{1/2}(a_1) \lesssim a. \tag{e 4.10}$$

□

Remark 4.5. Let A be a unital separable simple C^*-algebra and let $\mathcal{P}$ be the class of unital purely infinite simple C^*-algebras. Suppose that A satisfies the conclusion of the first part of Proposition 3.10 with $\mathcal{P}$ above. Then the proof of Proposition 4.4 shows that A is purely infinite.

Theorem 4.6. *Let $\mathcal{T}$ be the class of unital C^*-algebras B such that every 2-quasitrace of B is a trace. Suppose that A is a unital separable C^*-algebra satisfying the following conditions: For any $\varepsilon > 0$, any $\eta > 0$, and any finite subset $\mathcal{F} \subset A$, there exist a unital C^*-algebra B in $\mathcal{T}$, and c.p.c maps $\alpha : A \to B$, $\beta_n : B \to A$, and $\gamma_n : A \to A$ ($n \in \mathbb{N}$) such that*

(1) $c \approx_\eta \gamma_n(c) + \beta_n \circ \alpha(c)$ for all $c \in \mathcal{F}$ and $n \in \mathbb{N}$,

(2) α is an $(\mathcal{F}, \eta)$-approximate embedding,

(3) $\lim_{n\to\infty} \|\beta_n(b_1b_2) - \beta_n(b_1)\beta_n(b_2)\| = 0$ and $\lim_{n\to\infty} \|\beta_n(b_1)\| = \|b_1\|$ for all $b_1, b_2 \in B$, and

(4) $\sup\{\tau(\gamma_n(1_A)) : \tau \in QT(A)\} < \varepsilon$ for all $n \in \mathbb{N}$.

Then $A \in \mathcal{T}$.

In particular, if A is a unital separable simple C^*-algebra which is asymptotically tracially in $\mathcal{T}$, then $A \in \mathcal{T}$.

Proof. Let $\tau \in QT(A)$. Fix $x, y \in A_{sa}$ and fix $1/2 > \varepsilon > 0$. Choose $0 < \delta < \varepsilon$ which satisfies the condition in [2, II. 2.6].

Fix $0 < \eta < \delta$. Choose $\mathcal{F} = \{1_A, x, y, x + y\}$. Let B, α, β_n and γ_n be as above associated with ε, η and $\mathcal{F}$. By Lemma 3.7, we may also assume, without loss of generality,

(5) $\|\gamma_n(a)\beta_n \circ \alpha(a) - \beta_n \circ \alpha(a)\gamma_n(a)\| < \delta$ for all $a \in \mathcal{F}$ and all $n \in \mathbb{N}$.

Let ω be a free ultra filter on $\mathbb{N}$. Let $J := \{\{a_n\} \in l^\infty(A) : \lim_\omega \|a_n\| = 0\}$. Note that J is an ideal of $l^\infty(A)$. Let $\pi_\omega : l^\infty(A) \to l^\infty(A)/J$ be the quotient map. Let $\tau_\omega : l^\infty(A)/J \to \mathbb{C}$ be defined by $\tau_\omega(\pi_\omega(\{a_n\})) := \lim_{n\to\omega} \tau(a_n)$ for all $\{a_n\} \in l^\infty(A)$. Note that $\tau_\omega \in QT(l^\infty(A)/J)$ (see the paragraph above [2, Corollary II.2.6]).

Define an injective *-homomorphism from $\beta : B \to A_\omega$ by $\beta(x) = \pi_\omega(\{\beta_1(x), \beta_2(x), \cdots\})$ for all $x \in B$. Then $\tau_\omega \circ \beta$ is a 2-quasitrace on B (with $\|\tau_\omega \circ \beta\| \le 1$). Since B is in $\mathcal{T}$,

$$\begin{aligned}
\lim_{i\to\omega} \tau \circ \beta_i(\alpha(x) + \alpha(y)) &= \tau_\omega \circ \beta(\alpha(x) + \alpha(y)) = \tau_\omega \circ \beta(\alpha(x)) + \tau_\omega \circ \beta(\alpha(y)) && \text{(e 4.11)}\\
&= \lim_{i\to\omega} \tau \circ \beta_i(\alpha(x)) + \lim_{i\to\omega} \tau \circ \beta_i(\alpha(y)) && \text{(e 4.12)}\\
&= \lim_{i\to\omega} (\tau \circ \beta_i(\alpha(x)) + \tau \circ \beta_i(\alpha(y))). && \text{(e 4.13)}
\end{aligned}$$

Therefore there exists $m \in \mathbb{N}$, such that

$$\tau \circ \beta_m(\alpha(x) + \alpha(y)) \approx_\delta \tau \circ \beta_m(\alpha(x)) + \tau \circ \beta_m(\alpha(y)). \tag{e 4.14}$$

Note that, for any $a \in A_{sa}$, $\|\tau(a)\| \le \|\tau\|\|a\| \le \|a\|$ (see [2, II.2.5, (iii)]). Then

$$\begin{aligned}
\tau(x+y) &\approx_\eta \tau(\beta_m \circ \alpha(x+y) + \gamma_m(x+y))\\
\text{(by (5) and [2, II.2.6])} &\approx_{\|x+y\|\varepsilon} \tau(\beta_m \circ \alpha(x+y)) + \tau(\gamma_m(x+y))\\
\text{by (4)} &\approx_{\|x+y\|\varepsilon} \tau(\beta_m \circ \alpha(x+y))\\
\text{(by (e 4.14))} &\approx_\eta \tau \circ \beta_m(\alpha(x)) + \tau \circ \beta_m(\alpha(y))\\
&\approx_{(\|x\|+\|y\|)\varepsilon} \tau \circ \beta_m(\alpha(x)) + \tau(\gamma_m(x)) + \tau \circ \beta_m(\alpha(y)) + \tau(\gamma_m(y))\\
\text{(by (5) and [2, II.2.6])} &\approx_{(\|x\|+\|y\|)\varepsilon} \tau(\beta_m(\alpha(x)) + \gamma_m(x)) + \tau(\beta_m(\alpha(y)) + \gamma_m(y))\\
\text{(by (1))} &\approx_{2\eta} \tau(x) + \tau(y). && \text{(e 4.15)}
\end{aligned}$$

Let $\varepsilon,\ \eta \to 0$. We have $\tau(x+y) = \tau(x) + \tau(y)$. It follows that τ is linear. In other words, τ is a trace.

To see the last part, assume that A is a unital separable simple C^*-algebra which is asymptotically tracially in $\mathcal{T}$. We may assume that A is infinite dimensional. Then, for any $\varepsilon > 0$, by Lemma 4.3 (cf. [39, 3.5.7]), there is a nonzero positive element $a \in A$ with $\|a\| = 1$ such that $\sup\{d_\tau(a) : \tau \in QT(A)\} < \varepsilon$. By the Definition 3.1 and applying what has been proved, we conclude that every 2-quasitrace of A is a trace.

□

Corollary 4.7. *If A is asymptotically tracially in $\mathcal{E}$, in particularly, in $\mathcal{N}$, then $QT(A) = T(A)$.*

The proof of the following is taken from the proof of [39, 3.6.10] (see also [40, Theorem 3.4], [21, 3.3], and [20, 4.3]). Recall that a C^*-algebra A is called has (SP) property, if every nonzero hereditary C^*-subalgebra of A contains a nonzero projection.

Theorem 4.8 (cf. [39, Theorem 3.6.10])**.** *Let $\mathcal{S}$ be the class of unital C^*-algebras with stable rank one. Suppose that A is a unital simple C^*-algebra satisfying the following condition: For any finite subset $\mathcal{F} \subset A$, any $\epsilon > 0$, and any $a \in A_+\backslash\{0\}$, there exists a unital C^*-subalgebra $B \subset l^\infty(A)/c_0(A)$ which is strictly embedded such that B in $\mathcal{S}$, and*

(1) $1_B\iota_A(x) \approx_\epsilon \iota_A(x)1_B$ for all $x \in \mathcal{F}$,

(2) $1_B\iota_A(x)1_B \in_\epsilon B$ and $\|1_B\iota_A(x)1_B\| \ge \|x\| - \epsilon$ for all $x \in \mathcal{F}$, and

(3) $\iota_A(1_A) - 1_B \lesssim_{l^\infty(A)/c_0(A)} \iota_A(a)$.

Then A in $\mathcal{S}$. Consequently, if A is asymptotically tracially in $\mathcal{S}$, then A in $\mathcal{S}$.

Proof. Note that C^*-algebras in $\mathcal{S}$ are stably finite (see [39, Proposition 3.3.4]). One may assume that A is infinite dimensional. Let $x \in A$. It will be shown that, for any $\varepsilon \in (0, 1/2)$, there exists an invertible element $y \in A$ such that $\|x - y\| < \varepsilon$. One may assume that $\|x\| \le 1$ and x is not invertible. As A is stably finite (see part (b) of Proposition 4.2), one may assume that x is not one-sided invertible. To show that x is a norm limit of invertible elements, it suffices to show that ux is a norm limit of invertible elements for some unitary $u \in A$. Thus, by [39, Lemma 3.6.9] (also see [53, Lemma 3.5]), one may assume that there exists a nonzero element $c_1 \in A_+$ such that $c_1x = xc_1 = 0$.

First consider the case that A has (SP) property. Then, by [39, Lemma 3.6.6], there are nonzero mutually orthogonal projections $p_1, p_2 \in \mathrm{Her}(c_1)$. Consider $A_1 = (1-p_1)A(1-p_1)$. Since A is simple and has (SP) property, there is a nonzero projection $p_1' \in A_1$ such that $p_1' \lesssim p_1$ (see, for example, [39, Lemma 3.5.6]). Note $x \in A_1$. Since $\mathcal{S}$ has property (H) (see [9, Corollary 3.6]), by Proposition 3.12, A_1 has the same property that A has, namely, there is a projection $q \in l^\infty(A_1)/c_0(A_1)$ and a C^*-subalgebra B of $l^\infty(A_1)/c_0(A_1)$ with $B \in \mathcal{S}$ and with $1_B = q$ such that

(1′) $\|q\iota_{A_1}(x) - \iota_{A_1}(x)q\| < \varepsilon/32$,

(2′) $q\iota_{A_1}(x)q \in_{\varepsilon/32} B$, and

(3′) $\iota_{A_1}(1_{A_1}) - q \lesssim_{l^\infty(A_1)/c_0(A_1)} \iota_{A_1}(p_1') \lesssim_{l^\infty(A)/c_0(A)} \iota_A(p_1)$.

Write $x_1 = q\iota_{A_1}(x)q$ and $x_2 = (\iota_{A_1}(1_{A_1}) - q)\iota_{A_1}(x)(\iota_{A_1}(1_{A_1} - q)$. Then, by (1′), one has

$$\|\iota_{A_1}(x) - (x_1 + x_2)\| < \varepsilon/16. \tag{e 4.16}$$

Since $B \in \mathcal{S}$, there is an invertible element $y_1 \in B$ such that

$$\|x_1 - y_1\| < \varepsilon/16. \tag{e 4.17}$$

By (3′), there is $v \in l^\infty(A)/c_0(A)$ such that $v^*v = \iota_{A_1}(1_{A_1}) - q = \iota_A(1_A - p_1) - q$ and $vv^* \le \iota_A(p_1)$. Set $y_2 := x_2 + (\varepsilon/16)v + (\varepsilon/16)v^* + (\varepsilon/16)(\iota_A(p_1) - vv^*)$. Note that $y_3 := x_2 + (\varepsilon/16)v + (\varepsilon/16)v^*$ has the form

$$\begin{pmatrix} x_2 & (\varepsilon/16)v^* \\ (\varepsilon/16)v & 0 \end{pmatrix}.$$

One checks that y_3 is invertible in $\mathrm{Her}_{l^\infty(A)/c_0(A)}((\iota_A(1_A - p_1) - q) + vv^*)$. Therefore y_2 is invertible in $\mathrm{Her}_{l^\infty(A)/c_0(A)}(\iota_A(1_A) - q)$. Hence $y_1 + y_2$ is invertible in $l^\infty(A)/c_0(A)$. Moreover,

$$\|x_2 - y_2\| < \varepsilon/8. \tag{e 4.18}$$

Finally, one has (by (e 4.16), (e 4.17) and (e 4.18))

$$\begin{aligned} \|\iota_A(x) - (y_1 + y_2)\| &\le \|\iota_A(x) - (x_1 + x_2)\| + \|x_1 - y_1\| + \|x_2 - y_2\| \\ &< \varepsilon/16 + \varepsilon/16 + \varepsilon/8 = \varepsilon/4. \end{aligned} \tag{e 4.19}$$

Let $z \in l^\infty(A)/c_0(A)$ be such that $z(y_1 + y_2) = (y_1 + y_2)z = 1_{l^\infty(A)/c_0(A)}$. Let $\{z(n)\}, \{y(n)\} \in l^\infty(A)$ such that $\pi_\infty(\{z(n)\}) = z$ and $\pi_\infty(\{y(n)\}) = y_1 + y_2$. Then, for all large n, $\|z(n)y(n) - 1_A\| < 1/2$ and $\|y(n)z(n) - 1\| < 1/2$. It follows that $y(n)$ is invertible for all sufficiently large n. By (e 4.19), for all sufficiently large n,

$$\|x - y(n)\| < \varepsilon.$$

This proves the case that A has (SP) property.

If A does not have (SP) property, one does not choose p_1 and p_2. However, there is $a \in A_+ \setminus \{0\}$ such that $\mathrm{Her}(a)$ has no nonzero projection. Replacing p_1 by a above. Since $\gamma_n(1_A)$ is a projection, $\gamma_n(1_A) \lesssim a$ implies that there is $s \in A$ such that $s^*s = \gamma_n(1_A)$ and $ss^* \in \mathrm{Her}(a)$ (see Lemma 3.9) which forces $\gamma_n(1_A) = 0$. Thus, in this case, one may assume that $\gamma_n = 0$. Argument becomes simpler. Indeed, choosing $A_1 = A$, then $x \approx_{\varepsilon/16} x_1 \approx_{\varepsilon/16} y_1$.

The last part of the statement follows the first part and Proposition 3.10. $\square$

Lemma 4.9. *Let $\mathcal{W}$ be the class of unital C^*-algebras whose Cuntz semigroup is almost unperforated (recall Definition 2.13). Let A be a unital simple C^*-algebra which is asymptotically tracially in $\mathcal{W}$ and $a, b, c \in A_+\backslash\{0\}$. Suppose that there exists $n \in \mathbb{N}$ satisfying $(n+1)\langle a\rangle \le n\langle b\rangle$. Then, for any $\varepsilon > 0$, there exist $a_1, a_2 \in A_+$ and a projection $p \in A$ such that*

(1) $a \approx_\epsilon a_1 + a_2$,

(2) $a_1 \lesssim_A b$, and

(3) $a_2 \le \|a\|p \lesssim_A c$.

Proof. Without loss of generality, one may assume that $a, b, c \in A^1_+\backslash\{0\}$ and $\epsilon < 1/2$. Let $\{e_{i,j}\}$ be a set of matrix units of M_{n+1}. Then $a \otimes \sum_{i=1}^{n+1} e_{i,i} \lesssim_{A\otimes M_{n+1}} b \otimes \sum_{i=1}^{n} e_{i,i}$. Let $r = \sum_{i,j=1}^{n+1} r_{i,j} \otimes e_{i,j} \in A \otimes M_{n+1}$ such that $a \otimes \sum_{i=1}^{n+1} e_{i,i} \approx_{\epsilon/64} r^*(b \otimes \sum_{i=1}^{n} e_{i,i})r$. Set

$$\mathcal{F} := \{a, b\} \cup \{r_{i,j}, r^*_{i,j} : i, j = 1, 2, \cdots, n+1\}.$$

Let $M := 1 + \|r\|$ and choose $\delta_1 := \frac{\epsilon}{64M^2(n+1)^4}$. Note that

$$r^*(b \otimes \sum_{i=1}^{n+1} e_{i,i})r \approx_{\epsilon/64} r^*((b - \delta_1)_+ \otimes \sum_{i=1}^{n+1} e_{i,i})r. \tag{e 4.20}$$

Note that $\mathcal{W}$ has property (H) (see the line following Definition 2.13). Since A is asymptotically tracially in $\mathcal{W}$, by Proposition 3.8, for any $\delta > 0$, there exist a unital C^*-algebra B with almost unperforated $W(B)$, c.p.c. maps $\alpha : A \to B$, $\beta_i : B \to A$, and $\gamma_i : A \to A \cap \beta_i(B)^\perp$ ($i \in \mathbb{N}$) such that

($1'$) α is a u.c.p. map, $\beta_i(1_B)$ and $\gamma_i(1_A)$ are projections, and $1_A = \beta_i(1_B) + \gamma_i(1_A)$ for all $i \in \mathbb{N}$,

($2'$) $x \approx_\delta \gamma_i(x) + \beta_i \circ \alpha(x)$ for all $x \in \mathcal{F}$ and all $i \in \mathbb{N}$,

($3'$) α is an $(\mathcal{F}, \delta)$-approximate embedding,

($4'$) $\lim_{i\to\infty} \|\beta_i(xy) - \beta_i(x)\beta_i(y)\| = 0$ and $\lim_{n\to\infty} \|\beta_i(x)\| = \|x\|$ for all $x, y \in B$, and

($5'$) $\gamma_i(1_A) \lesssim_A c$ for all $i \in \mathbb{N}$.

By ($3'$) and (e 4.20), for some sufficiently small δ ($< (\frac{\epsilon}{128M(n+1)^2})^4$), one has

$$\alpha(a) \otimes \sum_{i=1}^{n+1} e_{i,i} \approx_{\varepsilon/16} (\sum_{i,j=1}^{n+1} \alpha(r_{i,j}) \otimes e_{i,j})^*((\alpha(b) - \delta_1)_+ \otimes \sum_{i=1}^{n} e_{i,i})(\sum_{i,j=1}^{n+1} \alpha(r_{i,j}) \otimes e_{i,j}).$$

By [54, Proposition 2.2], with $R := (\sum_{i,j=1}^{n+1} \alpha(r_{i,j}) \otimes e_{i,j})$, in $B \otimes M_{n+1}$,

$$\begin{aligned}(\alpha(a) - \epsilon/8)_+ \otimes \sum_{i=1}^{n+1} e_{i,i} &= ((\alpha(a) \otimes \sum_{i=1}^{n+1} e_{i,i}) - \epsilon/8)_+ \\ &\lesssim R^*((\alpha(b) - \delta_1)_+ \otimes \sum_{i=1}^{n} e_{i,i})R \lesssim (\alpha(b) - \delta_1)_+ \otimes \sum_{i=1}^{n} e_{i,i}.\end{aligned}$$

Since $W(B)$ is almost unperforated, one obtains $(\alpha(a) - \epsilon/8)_+ \lesssim_B (\alpha(b) - \delta_1)_+$. Hence there exists $s \in B$ such that

$$(\alpha(a) - \epsilon/8)_+ \approx_{\epsilon/64} s^*(\alpha(b) - \delta_1)_+ s.$$

Then, by ($4'$), there exists $N \in \mathbb{N}$ such that

$$(\beta_N(\alpha(a)) - \epsilon/8)_+ \approx_{\epsilon/32} \beta_N((\alpha(a) - \epsilon/8)_+) \approx_{\epsilon/32} \beta_N(s^*(\alpha(b) - \delta_1)_+ s) \tag{e 4.21}$$

$$\approx_{\epsilon/32} \beta_N(s^*)(\beta_N(\alpha(b)) - \delta_1)_+ \beta_N(s). \tag{e 4.22}$$

Applying [54, Proposition 2.2], one has

$$(\beta_N(\alpha(a)) - \epsilon/4)_+ \lesssim_A \beta_N(s^*)(\beta_N(\alpha(b)) - \delta_1)_+ \beta_N(s) \lesssim_A (\beta_N(\alpha(b)) - \delta_1)_+. \qquad (\text{e}\,4.23)$$

Since $\beta_N(\alpha(b)) + \gamma_N(b) \approx_\delta b$, with $\delta < \delta_1$, applying [54, Proposition 2.2] again (noting $\gamma_N(b) \perp \beta_N(B)$), one has

$$(\beta_N(\alpha(b)) - \delta_1)_+ \leq (\beta_N(\alpha(b)) - \delta)_+ \leq ((\beta_N(\alpha(b)) - \delta)_+ + \gamma_N(b)) - \delta)_+ \lesssim_A b. \qquad (\text{e}\,4.24)$$

Choose $a_1 := (\beta_N(\alpha(a)) - \epsilon/4)_+$, $a_2 := \gamma_N(a)$ and $p := \gamma_N(1_A)$. Then, by (e 4.23) and (e 4.24), one has $a_1 \lesssim_A b$. Note that (5′) shows $a_2 \leq \|a\|p \lesssim_A c$. Thus a_1, a_2, p satisfy (2) and (3) of the lemma. By (2′),

$$a \approx_\delta \gamma_N(a) + \beta_N(\alpha(a)) \approx_{\epsilon/4} \gamma_N(a) + (\beta_N(\alpha(a)) - \epsilon/4)_+ = a_2 + a_1.$$

So (1) of the lemma is also satisfied and the lemma follows.

□

Theorem 4.10. *Let A be a unital simple C^*-algebra which is asymptotically tracially in $\mathcal{W}$ (see Lemma 4.9). Then $A \in \mathcal{W}$.*

Proof. Let $a, b \in M_m(A)_+ \setminus \{0\}$ with $\|a\| = 1 = \|b\|$ for some integer $m \geq 1$. Let $n \in \mathbb{N}$ and assume $(n+1)\langle a\rangle \leq n\langle b\rangle$. To prove the theorem, it suffices to prove that $a \lesssim b$.

Note that, if $B \in \mathcal{W}$, then, for each integer m, $M_m(B) \in \mathcal{W}$. It follows that $M_m(A)$ is asymptotically tracially in $\mathcal{W}$. To simplify notation, without loss of generality, one may assume $a, b \in A_+$.

First consider the case that A has (SP) property. By Lemma 4.3, $\mathrm{Her}(f_{1/4}(b))_+$ contains $2n+1$ nonzero mutually orthogonal elements $b_0, b_1, \cdots, b_{2n}$ such that $\langle b_i\rangle = \langle b_0\rangle$, $i = 1, 2, \cdots, 2n$. Since A has (SP) property, choose a nonzero projection $e_0 \in \mathrm{Her}(b_0)$. Replacing b by $g(b)$ for some $g \in C_0((0,1])$, one may assume that $be_0 = e_0b = e_0$. Put $c = b - e_0$. Keep in mind that $b = c + e_0$, $c \perp e_0$, and $2n\langle e_0\rangle \leq c = b - e_0$. One has

$$(2n+2)\langle a\rangle \leq 2n\langle b\rangle = 2n(\langle b - e_0\rangle + \langle e_0\rangle) \leq 2n\langle c\rangle + \langle c\rangle = (2n+1)\langle c\rangle. \qquad (\text{e}\,4.25)$$

By Lemma 4.9, for any $\varepsilon \in (0, 1/2)$, there exist $a_1, a_2 \in A_+$ such that
(i) $a \approx_{\epsilon/2} a_1 + a_2$,
(ii) $a_1 \lesssim_A c$, and
(iii) $a_2 \leq \|a\|p \lesssim_A e_0$.
By (i), (ii) and (iii), and applying [54, Proposition 2.2], one obtains (recall $be_0 = e_0b = e_0$)

$$(a - \varepsilon)_+ \lesssim a_1 + a_2 \lesssim c + e_0 = b. \qquad (\text{e}\,4.26)$$

Since this holds for every $\varepsilon \in (0, 1/2)$, one concludes that $a \lesssim b$.

If A does not have (SP) property, choose $b_0 \in A_+ \setminus \{0\}$ such that $\mathrm{Her}(b_0)$ has no nonzero projections. From $(n+1)\langle a\rangle \leq n\langle b\rangle$, Lemma 4.9 implies that $a \approx_\varepsilon a_1 + a_2$, $a_1 \lesssim b$ and $a_2 \leq p \lesssim b_0$. Projectionlessness of $\mathrm{Her}(b_0)$ forces $p = 0$, whence $a_2 = 0$. Thus one arrives

$$(a - \varepsilon)_+ \lesssim a_1 \lesssim b. \qquad (\text{e}\,4.27)$$

It follows $a \lesssim b$ and the lemma follows.

□

5 Order zero maps and nuclear dimension

Definition 5.1 ([71, Definition 2.3])**.** Recall that a c.p. map $\varphi : A \to B$ has order zero, if, for any $a, b \in A_+$ with $a \cdot b = 0$, one has $\varphi(a) \cdot \varphi(b) = 0$.

We would like to recall the following theorem.

Theorem 5.2 ([71, Theorem 3.3])**.** *Let A and B be C^*-algebras, and let $\varphi : A \to B$ be a completely positive order zero map. Let $C := C^*(\varphi(A)) \subset B$. Then there exists a positive element $h \in \mathcal{M}(C) \cap C'$ with $\|h\| = \|\varphi\|$ and a *-homomorphism $\pi_\varphi : A \to \mathcal{M}(C) \cap \{h\}'$ such that $\varphi(a) = \pi_\varphi(a)h$ for all $a \in A$. If, in addition, A is unital, then $h = \varphi(1_A) \in C$.*

Proposition 5.3. *Let $\varphi : A \to B$ be a c.p. order zero map. Let h and π_φ be as in Theorem 5.2. If A is simple, then the map $a \otimes x \mapsto \pi_\varphi(a) \cdot x$ defines an isomorphism $\gamma : A \otimes C^*(h) \cong C^*(\varphi(A))$. Moreover, for all $a \in A$, $\|\varphi(a)\| = \|\varphi\| \cdot \|a\|$.*

Proof. If $\|\varphi\| = 0$, then $h = 0$ and there is nothing to prove. Assume that $\|\varphi\| \neq 0$. Since A is simple, π_φ is injective and $\pi_\varphi(A)$ is also simple.

By (the proof of) [71, Corollary 4.1], γ gives a *-homomorphism from $A \otimes C^*(h)$ to $C^*(\varphi(A))$. Since $\varphi(A) \subset \gamma(A \otimes C^*(h))$, γ is surjective.

Let us show that γ is injective. Since A is simple, $\ker\gamma = A \otimes I$, where I is an ideal of $C^*(h)$ (see [5, Proposition 2.16.(2) and Proposition 2.17(2)]). Let $f(h) \in I$ for some $f \in C_0(\mathrm{sp}(h)\backslash\{0\})$. Then $a \otimes f(h) \in A \otimes I = \ker\gamma$ for all $a \in A$, which implies that $\pi_\varphi(a)f(h) = 0$ for all $a \in A$. It follows that $\varphi(a)f(h) = \pi_\varphi(a)hf(h) = f(h)\pi_\varphi(a)h = f(h)\varphi(a)$ and $\varphi(a)f(h) = \pi_\varphi(a)f(h)h = 0$. Thus $f(h) \perp C^*(\varphi(A)) = C$. Since $f(h) \in \mathcal{M}(C)$, this implies $f(h) = 0$. Thus $I = \{0\}$. In other words, γ is injective.

Moreover, recall, from Theorem 5.2, $\|\varphi\| = \|h\|$. Then, for $a \in A$, $\|\varphi(a)\| = \|h \cdot \pi_\varphi(a)\| = \|\gamma(\pi_\varphi(a) \otimes h)\| = \|\pi_\varphi(a) \otimes h\| = \|\pi_\varphi(a)\| \cdot \|h\| = \|a\| \cdot \|\varphi\|$.

□

Remark 5.4. (1) For the case that A is a matrix algebra, the proposition above was obtained in the proof of [36, Proposition 5.1].

(2) Consider $\varphi : \mathbb{C}\oplus\mathbb{C} \to \mathbb{C}\oplus\mathbb{C}$, $(x, y) \mapsto (x, y/2)$. Then φ is an injective norm one c.p.c. order zero map, but φ is not an isometry since $\|\varphi((1, 2))\| = 1 < 2 = \|(1, 2)\|$. Thus the last statement of Proposition 5.3 would fail without the assumption that A is simple.

The following proposition shows the existence of inverse *-homomorphism for norm one c.p. order zero map from simple C^*-algebras.

Proposition 5.5. *Let A be a simple C^*-algebra, B be another C^*-algebra, and let $\varphi : A \to B$ be a nonzero c.p. order zero map. Then there exists a *-homomorphism $\psi : C^*(\varphi(A)) \to A$ such that $\psi \circ \varphi = \|\varphi\| \cdot \mathrm{id}_A$ and $\varphi \circ \psi|_{\varphi(A)} = \|\varphi\| \cdot \mathrm{id}_{\varphi(A)}$.*

Proof. We will use the same notation as in Proposition 5.3, such as h, π_φ, and the isomorphism: $\gamma : A \otimes C^*(h) \to C^*(\varphi(A))$, $a \otimes x \mapsto \pi_\varphi(a) \cdot x$.

Note that $C^*(h) \cong C_0(\mathrm{sp}(h)\backslash\{0\})$ and $\|\varphi\| = \|h\|$. Define a *-homomorphism $\psi' : A \otimes C^*(h) \to A$ by $\psi'(a \otimes f(h)) = f(\|h\|)a$ for all $a \in A$ and $f \in C_0(\mathrm{sp}(h)\backslash\{0\})$, and define $\psi = \psi' \circ \gamma^{-1} : C^*(\varphi(A)) \to A$. Then, with the identity function $\imath : \mathrm{sp}(h) \to \mathrm{sp}(h)$, for any $a \in A$,

$$\psi \circ \varphi(a) = \psi' \circ \gamma^{-1}(\pi_\varphi(a)h) = \psi'(a \otimes \imath) = a\|h\| = \|\varphi\|a.$$

Therefore, for $a \in A$, $\varphi \circ \psi(\varphi(a)) = \varphi \circ \psi' \circ \gamma^{-1}(\pi_\varphi(a)h) = \varphi \circ \psi'(a \otimes h) = \varphi(\|\varphi\|a) = \|\varphi\|\varphi(a)$. The proposition follows. □

Proposition 5.6. *Let A be a C^*-algebra, F be a (nonzero) finite dimensional C^*-algebra, and let $\alpha : F \to A$ be an injective c.p. order zero map. Then there exists a c.p. map $\beta : A \to F$ such that $\beta \circ \alpha = \mathrm{id}_F$.*

Moreover, if α is an isometry, one may choose β to be a c.p.c. map.

Proof. Write $F = M_{k_1} \oplus \cdots \oplus M_{k_n}$ $(n, k_1, \cdots, k_n \in \mathbb{N})$ and $\alpha_i := \alpha|_{M_{k_i}} : M_{k_i} \to A$ $(i = 1, 2, \cdots, n)$. Then, by Proposition 5.5, there exists a *-homomorphism $\beta_i : C^*(\alpha_i(M_{k_i})) \to M_{k_i}$ such that $\frac{1}{\|\alpha_i\|}\beta_i \circ \alpha_i = \mathrm{id}_{M_{k_i}}$. Then the map $\bar{\beta} : C^*(\alpha_1(M_{k_1})) \oplus \cdots \oplus C^*(\alpha_n(M_{k_n})) \to F = M_{k_1} \oplus \cdots \oplus M_{k_n}$ defined by $\bar{\beta}((x_1, \cdots, x_n)) = (\frac{\beta_1(x_1)}{\|\alpha_1\|}, \cdots, \frac{\beta_n(x_n)}{\|\alpha_n\|})$ is a c.p. map. Since α is a c.p. order zero map, $C^*(\alpha_i(M_{k_i}))$ are mutually orthogonal $(i = 1, 2, \cdots, n)$. Thus $C^*(\alpha_1(M_{k_1})) \oplus \cdots \oplus C^*(\alpha_n(M_{k_n}))$ is a C^*-subalgebra of A. By Arveson's extension theorem, $\bar{\beta}$ has a c.p. extension $\beta : A \to F$ with $\beta \circ \alpha = \mathrm{id}_F$. Moreover, if α is an isometry, then $\bar{\beta}$ is a *-homomorphism. Hence the extension β can be chosen to be a c.p.c. map. □

Definition 5.7. Let $F = M_{k_1} \oplus \cdots \oplus M_{k_n}$ be a finite dimensional C^*-algebra. Let A be a C^*-algebra and $\varphi : F \to A$ be a linear map. Define

$$|||\varphi||| := \max\{\|\varphi|_{M_{k_i}}\| : i = 1, 2, \cdots, n\}.$$

Definition 5.8. Let A be a C^*-algebra and F be a finite dimensional C^*-algebra and let $\varphi : F \to A$ be a c.p. map. Fix $n \in \mathbb{N}$. Recall that the map φ is called n-decomposable (see [36, Definition 2.2]), if F can be written as $F = F_0 \oplus \cdots \oplus F_n$ (where F_i is a finite dimensional C^*-algebra) such that $\varphi|_{F_i}$ is a c.p. order zero map $(i = 0, 1, \cdots, n)$. If, in addition, each $\varphi|_{F_i}$ is assumed to be contractive, then φ is called *piecewise contractive* n-decomposable map.

Remark 5.9. Note that Theorem 5.2 implies the kernel of a c.p. order zero map is always an ideal (also see [37, Lemma 2.7]). Thus, for a c.p. order zero map $\varphi : F \to A$, where F is finite dimensional, one can write $F = \ker\varphi \oplus F_1$, where F_1 is an ideal of F. Note that $\varphi|_{F_1}$ is injective.

Proposition 5.10. *Let A and C be C^*-algebras, and B be a finite dimensional C^*-algebra. Suppose that $\alpha : A \to B$ and $\beta : B \to C$ are c.p. maps such that β is n-decomposable. Then there exist $\bar{n} \le n \in \mathbb{N} \cup \{0\}$, a finite dimensional C^*-algebra $\bar{B} = \bar{B}_0 \oplus \cdots \oplus \bar{B}_{\bar{n}}$ which is a summand of B, a c.p. map $\bar{\alpha} : A \to \bar{B}$, and a c.p. $\bar{n}$-decomposable map $\bar{\beta} : \bar{B} \to C$ such that*

(1) $\bar{\beta} \circ \bar{\alpha} = \beta \circ \alpha$,

(2) $\|\bar{\alpha}\| \le \min\{\|\beta \circ \alpha\|, \|\alpha\| \cdot |||\beta|||\}$, and

(3) $\bar{\beta}|_{\bar{B}_i}$ is a c.p.c. order zero isometry $(i = 0, 1, \cdots, \bar{n})$.

Proof. Let $\bar{n}$ be the minimal integer such that β is $\bar{n}$-decomposable. Then we can write $B = B_0 \oplus \cdots \oplus B_{\bar{n}}$ (where each B_i is a direct summand of B) such that $\beta|_{B_i}$ is a nonzero c.p. order zero map. By Remark 5.9, we can write $B_i = \ker(\beta|_{B_i}) \oplus \bar{B}_i$, where $\bar{B}_i$ is direct summand of B_i. Then $\beta|_{\bar{B}_i}$ is a nonzero injective c.p. order zero map $(i = 0, 1, \cdots, \bar{n})$. Define $\bar{B} := \bar{B}_0 \oplus \cdots \oplus \bar{B}_{\bar{n}}$. Note that $\bar{B}$ is a direct summand of B.

Write $\bar{B} = M_{k_1} \oplus \cdots \oplus M_{k_m}$, where $m, k_1, \cdots, k_m \in \mathbb{N}$. Let $P_j : B \to M_{k_j}$ be the projection map. Set $\alpha^{(j)} = P_j \circ \alpha$ and $\beta^{(j)} = \beta|_{M_{k_j}}$ $(j = 1, 2, \cdots, m)$. Note that each $\beta^{(j)}$ is a c.p. order zero map. Define $\bar{\alpha}^{(j)} := \|\beta^{(j)}\|\alpha^{(j)}$ and $\bar{\beta}^{(j)} := \frac{1}{\|\beta^{(j)}\|}\beta^{(j)}$ $j = 1, 2, \cdots, m$. By Proposition 5.3, each $\bar{\beta}^{(j)}$ is a c.p.c. order zero isometry. Note that

$$\beta \circ \alpha(x) = \sum_{j=1}^{m} \beta^{(j)} \circ \alpha^{(j)}(x) = \sum_{j=1}^{m} \bar{\beta}^{(j)} \circ \bar{\alpha}^{(j)}(x) \text{ for all } x \in A. \tag{e 5.1}$$

Define c.p. maps $\bar{\alpha} : A \to \bar{B} = M_{k_1} \oplus \cdots \oplus M_{k_m}$ by $x \mapsto (\bar{\alpha}^{(1)}(x), \cdots, \bar{\alpha}^{(m)}(x))$ and $\bar{\beta} : \bar{B} = M_{k_1} \oplus \cdots \oplus M_{k_m} \to A$ by $(x_1, \cdots, x_m) \mapsto \sum_{j=1}^{m} \bar{\beta}^{(j)}(x_j)$.

Write, for each i, $\bar{B}_i = \bigoplus_{S_i} M_{k_j}$, where S_i is a subset of $\{1, 2, \cdots, m\}$. Again, since $\beta|_{\bar{B}_i}$ is a c.p. order zero map, $\beta(M_{k_j}) \perp \beta(M_{k_{j'}})$, if $j \neq j'$ and $j, j' \in S_i$ for each $i \in \{0, 1, \cdots, \bar{n}\}$. In other words, $\bar{\beta}|_{\bar{B}_i}$ is a sum of mutually orthogonal c.p.c. order zero isometries. Hence $\bar{\beta}|_{\bar{B}_i}$ is still a c.p.c. order zero isometry. Therefore (3) holds.

For any $x \in A$, by (e 5.1), we have $\bar{\beta} \circ \bar{\alpha}(x) = \sum_{j=1}^{m} \bar{\beta}^{(j)}(\bar{\alpha}^{(j)}(x)) = \beta \circ \alpha(x)$. Thus (1) holds.

Let $a \in A_+^1$. Recall that $\bar{\beta}^{(j)}$ is a c.p.c order zero isometry ($j = 1, 2, \cdots, m$). We have

$$\|\bar{\alpha}^{(j)}(a)\| = \|\bar{\beta}^{(j)}(\bar{\alpha}^{(j)}(a))\| \leq \|\sum_{j=1}^{m} \bar{\beta}^{(j)}(\bar{\alpha}^{(j)}(a))\| \overset{(\text{e}\,5.1)}{=} \|\beta \circ \alpha(a)\| \leq \|\beta \circ \alpha\|.$$

Thus $\|\bar{\alpha}(a)\| = \max\{\|\bar{\alpha}^{(j)}(a)\| : j = 1, 2, \cdots, m\} \leq \|\beta \circ \alpha\|$, which implies $\|\bar{\alpha}\| \leq \|\beta \circ \alpha\|$. Also note that

$$\begin{aligned} \|\bar{\alpha}\| &= \max\{\|\bar{\alpha}^{(j)}\| : j = 1, 2, \cdots, m\} = \max\{\|\alpha^{(j)}\| \cdot \|\beta^{(j)}\| : j = 1, 2, \cdots, m\} \\ &\leq \max\{\|\alpha^{(j)}\| : j = 1, 2, \cdots, m\} \cdot |||\beta||| \leq \|\alpha\| \cdot |||\beta|||. \end{aligned} \tag{e 5.2}$$

So (2) holds. □

Definition 5.11 ([59, Definition 2.2] and [72, Definition 2.1])**.** Let A and B be C^*-algebras and let $h : A \to B$ be a *-homomorphism. Recall that h has *nuclear dimension at most n,* and denote by $\dim_{\text{nuc}} h \leq n$, if the following conditions hold:

For any finite subset $\mathcal{F} \subset A$ and any $\epsilon > 0$, there exist finite dimensional C^*-algebras $F_0, \cdots, F_n$ and, c.p. maps $\varphi : A \to F_0 \oplus \cdots \oplus F_n$ and $\psi : F_0 \oplus \cdots \oplus F_n \to B$ such that

(1) $\psi \circ \varphi(x) \approx_\epsilon h(x)$ for all $x \in \mathcal{F}$,

(2) $\|\varphi\| \leq 1$, and

(3) $\psi|_{F_i}$ is a c.p.c. order zero map, $i = 0, 1, \cdots, n$.

We say A has nuclear dimension at most n, and denoted by $\dim_{\text{nuc}} A \leq n$, if $\dim_{\text{nuc}} \text{id}_A \leq n$.

The following may be known to experts.

Proposition 5.12. *Let $h : A \to B$ be a *-homomorphism of C^*-algebras and $n \in \mathbb{N} \cup \{0\}$. Then $\dim_{\text{nuc}} h \leq n$ if and only if the following holds: For any finite subset $\mathcal{F} \subset A$ and any $\epsilon > 0$, there exist a finite dimensional C^*-algebra F, c.p. maps $\varphi : A \to F$ and $\psi : F \to B$ such that*

(1) $\psi \circ \varphi(x) \approx_\epsilon h(x)$ for all $x \in \mathcal{F}$, and

(2) ψ is n-decomposable (see Definition 5.8).

Proof. The "only if" part is trivial. For the "if" part, let $\mathcal{F} \subset A$ be a finite subset and let $\epsilon > 0$. Set $\delta := \frac{\epsilon}{3 + \max\{\|x\| : x \in \mathcal{F}\}}$. Choose $e \in A_+^1$ such that $exe \approx_\delta x$ for all $x \in \mathcal{F}$.

By our assumption, there exist a finite dimensional C^*-algebra F, c.p. maps $\varphi : A \to F$ and $\psi : F \to B$ such that

($1'$) $\psi \circ \varphi(x) \approx_\delta h(x)$ for all $x \in \{eye : y \in \mathcal{F}\} \cup \{e^2\}$, and

($2'$) ψ is n-decomposable.

Define a c.p. map $\tilde{\varphi} : A \to F$ by $x \mapsto \frac{1}{1+\delta}\varphi(exe)$ for all $x \in A$. Then, for any $a \in A_1^+$,

$$\|\psi \circ \tilde{\varphi}(a)\| = \frac{1}{1+\delta}\|\psi \circ \varphi(eae)\| \leq \frac{1}{1+\delta}\|\psi \circ \varphi(e^2)\| \overset{(\text{by } (1'))}{\leq} \frac{1}{1+\delta}(\|h(e^2)\| + \delta) \leq 1.$$

It follows

$$\|\psi \circ \tilde{\varphi}\| \leq 1. \tag{e 5.3}$$

By Proposition 5.10, there exist a finite dimensional C^*-algebra $\bar{F}$, and c.p. maps $\bar{\varphi} : A \to \bar{F}$ and $\bar{\psi} : \bar{F} \to B$, such that

(1′′) $\bar\psi\circ\bar\varphi=\psi\circ\tilde\varphi$,
(2′′) $\|\bar\varphi\|\le\|\psi\circ\tilde\varphi\|\le 1$, and
(3′′) $\bar\psi$ is a piecewise contractive n-decomposable c.p. map.
Then, by (2′′) and (3′′), $\bar\psi$, $\bar\varphi$ and $\bar F$ satisfy (2) and (3) of Definition 5.11. For all $x\in\mathcal{F}$,

$$\bar\psi\circ\bar\varphi(x)\overset{(\text{by }(1''))}{=}\psi\circ\tilde\varphi(x)=\frac{1}{1+\delta}\psi\circ\varphi(exe)\overset{(\text{by }(1'))}{\approx_{\frac{\delta}{1+\delta}}}\frac{1}{1+\delta}h(exe)\approx_{\frac{\delta}{1+\delta}}\frac{1}{1+\delta}h(x)\approx_{\frac{\delta}{1+\delta}\|x\|}h(x).$$

By the choice of δ, we have $h(x)\approx_\epsilon\bar\psi\circ\bar\varphi(x)$. Then, by Definition 5.11, we have $\dim_{\rm nuc}h\le n$. □

Corollary 5.13. *Let A be a C^*-algebra and let $n\in\mathbb{N}$. Then $\dim_{\rm nuc}A\le n$ if and only if the following conditions hold: For any finite subset $\mathcal{F}\subset A$ and any $\epsilon>0$, there exist a finite dimensional C^*-algebra F and c.p. maps $\varphi:A\to F$ and $\psi:F\to A$ such that*
(1) $\psi\circ\varphi(x)\approx_\epsilon x$ for all $x\in\mathcal{F}$ and
(2) ψ is n-decomposable.

Proposition 5.14. *Let A,B, and C be C^*-algebras, $\{e_\lambda\}_{\lambda\in\Lambda}$ be an approximate identity of A, $\varphi:A\to B$, $\psi:B\to C$ be c.p. maps with $\|\psi\circ\varphi\|\le 1$, and let $\epsilon>0$. Suppose that $a\in A_+$ such that $\|\psi\circ\varphi(a^2)-\psi\circ\varphi(a)^2\|\le\epsilon$ and $b\in B$ such that b commutes with $\{\varphi(e_\lambda)\}_{\lambda\in\Lambda}$. Then*

$$\limsup_\lambda\|\psi(\varphi(a)b)-\psi(\varphi(a))\psi(\varphi(e_\lambda)b)\|\le\epsilon^{1/2}\|b\|\ \text{ and}$$
$$\limsup_\lambda\|\psi(b\varphi(a))-\psi(\varphi(e_\lambda)b)\psi(\varphi(a))\|\le\epsilon^{1/2}\|b\|.\qquad\text{(e 5.4)}$$

Proof. We will show that the first inequality holds. The second one holds by taking conjugate of the first one. Put $M:=\|a\|+\|a\|^2$. Let $\theta>0$. Choose $\delta>0$ be such that

$$\left(\delta(2\|\psi\|+\|\psi\|\|\varphi\|+1)+\sqrt{(2M+1)(\|\psi\|+1)\delta+\epsilon}\right)\|b\|<\epsilon^{1/2}\|b\|+\theta.\qquad\text{(e 5.5)}$$

Let $\lambda_1\in\Lambda$ such that, for any $\lambda\ge\lambda_1$, any $x\in\{a,a^2\}$,

$$e_\lambda^{1/2}xe_\lambda^{1/2}\approx_\delta x\ \text{ and }\ \psi\circ\varphi(e_\lambda^{1/2}xe_\lambda^{1/2})\approx_\delta\psi\circ\varphi(x).\qquad\text{(e 5.6)}$$

Fix $\lambda\ge\lambda_1$. Note, for any $x\in A_+$, we have $0\le\varphi(e_\lambda^{1/2}xe_\lambda^{1/2})\le\|x\|\varphi(e_\lambda)$. Thus $\varphi(e_\lambda^{1/2}xe_\lambda^{1/2})\in{\rm Her}_B(\varphi(e_\lambda))$. Note that $\{E_n:=(1/n+\varphi(e_\lambda))^{-1}\varphi(e_\lambda)\}_{n\in\mathbb{N}}$ forms an approximate identity for ${\rm Her}_B(\varphi(e_\lambda))$. It follows that $\lim_{n\to\infty}\|\varphi(e_\lambda^{1/2}xe_\lambda^{1/2})-E_n^{1/2}\varphi(e_\lambda^{1/2}xe_\lambda^{1/2})E_n^{1/2}\|=0$. Therefore there exists $\eta>0$ such that, for $x\in\{a,a^2\}$,

$$\varphi(e_\lambda^{1/2}xe_\lambda^{1/2})\approx_\delta\varphi(e_\lambda)^{1/2}(\eta+\varphi(e_\lambda))^{-1/2}\cdot\varphi(e_\lambda^{1/2}xe_\lambda^{1/2})\cdot(\eta+\varphi(e_\lambda))^{-1/2}\varphi(e_\lambda)^{1/2}.\qquad\text{(e 5.7)}$$

Define the following c.p. maps:

$$\tilde\varphi:A\to B,\quad x\mapsto(\eta+\varphi(e_\lambda))^{-1/2}\cdot\varphi(e_\lambda^{1/2}xe_\lambda^{1/2})\cdot(\eta+\varphi(e_\lambda))^{-1/2}\ \text{ and}\qquad\text{(e 5.8)}$$
$$\tilde\psi:B\to C,\quad x\mapsto\psi(\varphi(e_\lambda)^{1/2}x\varphi(e_\lambda)^{1/2}).\qquad\text{(e 5.9)}$$

We claim that $\|\tilde\varphi\|\le 1$. Indeed, for any $x\in A_+^1$,

$$\begin{aligned}\|\tilde\varphi(x)\|&=\|(\eta+\varphi(e_\lambda))^{-1/2}\cdot\varphi(e_\lambda^{1/2}xe_\lambda^{1/2})\cdot(\eta+\varphi(e_\lambda))^{-1/2}\|\\&\le\|(\eta+\varphi(e_\lambda))^{-1/2}\cdot\varphi(e_\lambda)\cdot(\eta+\varphi(e_\lambda))^{-1/2}\|=\|\varphi(e_\lambda)\cdot(\eta+\varphi(e_\lambda))^{-1}\|\le 1.\end{aligned}\qquad\text{(e 5.10)}$$

We also claim that $\|\tilde{\psi}\| \le 1$. Indeed, for any $x \in B_+^1$, $\|\tilde{\psi}(x)\| = \|\psi(\varphi(e_\lambda)^{1/2}x\varphi(e_\lambda)^{1/2})\| \le \|\psi(\varphi(e_\lambda))\| \le \|\psi\circ\varphi\|\|e_\lambda\| \le 1$. Thus $\|\tilde{\psi}\| \le 1$.

Note that, by (e 5.7) and (e 5.6), for $x \in \{a, a^2\}$, we have

$$\tilde{\psi}\circ\tilde{\varphi}(x) \approx_{\|\psi\|\delta} \psi\circ\varphi(e_\lambda^{1/2}xe_\lambda^{1/2}) \approx_\delta \psi\circ\varphi(x). \tag{e 5.11}$$

Then we have, applying (e 5.11),

$$\begin{aligned} \tilde{\psi}\circ\tilde{\varphi}(a)^2 & \approx_{M(\|\psi\|+1)\delta} & \tilde{\psi}\circ\tilde{\varphi}(a)\cdot\psi\circ\varphi(a) \\ (\text{by (e 5.11) and } \|\psi\circ\varphi\|\le 1) & \approx_{M(\|\psi\|+1)\delta} & \psi\circ\varphi(a)^2 \approx_\varepsilon \psi\circ\varphi(a^2) \\ (\text{by (e 5.11)}) & \approx_{(\|\psi\|+1)\delta} & \tilde{\psi}\circ\tilde{\varphi}(a^2). \end{aligned} \tag{e 5.12}$$

Then

$$\begin{aligned} \psi(\varphi(a))\cdot\psi(\varphi(e_\lambda)b) & = & \psi\circ\varphi(a)\psi(\varphi(e_\lambda)^{1/2}b\varphi(e_\lambda)^{1/2}) \\ & = & \psi\circ\varphi(a)\tilde{\psi}(b) \\ (\text{by (e 5.11)}) & \approx_{(\|\psi\|+1)\delta\|b\|} & \tilde{\psi}\circ\tilde{\varphi}(a)\tilde{\psi}(b) \\ (\text{by (e 5.12) and Lemma 2.11}) & \approx_{\|b\|\sqrt{(2M+1)(\|\psi\|+1)\delta+\epsilon}} & \tilde{\psi}(\tilde{\varphi}(a)b) \\ & = & \psi(\varphi(e_\lambda)^{1/2}\tilde{\varphi}(a)b\varphi(e_\lambda)^{1/2}) \\ & = & \psi(\varphi(e_\lambda)^{1/2}\tilde{\varphi}(a)\varphi(e_\lambda)^{1/2}b) \\ (\text{by (e 5.7)}) & \approx_{\|\psi\|\|b\|\delta} & \psi\left(\varphi(e_\lambda^{1/2}ae_\lambda^{1/2})b\right) \\ (\text{by (e 5.6)}) & \approx_{\|\psi\|\|\varphi\|\|b\|\delta} & \psi(\varphi(a)b). \end{aligned} \tag{e 5.13}$$

By (e 5.5), we have $\psi(\varphi(a))\cdot\psi(\varphi(e_\lambda)b) \approx_{\epsilon^{1/2}+\theta} \psi(\varphi(a)b)$. Thus

$$\limsup_\lambda \|\psi(\varphi(a)b) - \psi(\varphi(a))\psi(\varphi(e_\lambda)b)\| \le (\epsilon^{1/2}+\theta)\|b\|.$$

Let $\theta \to 0$. The proposition then follows. □

Theorem 5.15. *Let A and B be C^*-algebras, $h : A \to B$ be a *-homomorphism, and let $n \in \mathbb{N}\cup\{0\}$. Then $\dim_{\mathrm{nuc}} h \le n$ if and only if the following condition holds: For any finite subset $\mathcal{G} \subset A_+$ and any $\epsilon > 0$, there exist a C^*-algebra C, a finite subset $\tilde{\mathcal{G}} \subset C_+$, a finite dimensional C^*-algebra F and, c.p. maps $\varphi : C \to F$ and $\psi : F \to B$ such that*

(1) $h(\mathcal{G}) \subset_\epsilon \psi\circ\varphi(\tilde{\mathcal{G}})$,

(2) ψ is n-decomposable, and

(3) $\psi\circ\varphi(xy) \approx_\epsilon \psi\circ\varphi(x)\cdot\psi\circ\varphi(y)$ for all $x, y \in \tilde{\mathcal{G}}\cup(\tilde{\mathcal{G}}\cdot\tilde{\mathcal{G}})$.

Proof. For the "only if" part, let $C = A$, let $\tilde{\mathcal{G}} = \mathcal{G}$ and let $M := \{\|z\| : z \in \tilde{\mathcal{G}}\cup(\tilde{\mathcal{G}}\cdot\tilde{\mathcal{G}})\}$. Put $\theta := \min\{1, \frac{\epsilon}{2(M+1)}\}$. Since $\dim_{\mathrm{nuc}} h \le n$, by Definition 5.11, we can choose a finite dimensional C^*-algebra F and, c.p.c. maps $\varphi : A \to F$ and $\psi : F \to B$ such that

$(1')$ $h(x) \approx_\theta \psi\circ\varphi(x)$ for all $x \in \mathcal{G}\cup(\mathcal{G}\cdot\mathcal{G})$ and

$(2')$ ψ is n-decomposable.

Then, by $(1')$, we have

$(3')$ $\psi\circ\varphi(xy) \approx_\theta h(xy) = h(x)h(y) \approx_{(2M+\theta)\theta} \psi\circ\varphi(x)\psi\circ\varphi(y)$ for all $y \in \mathcal{G}\cup(\mathcal{G}\cdot\mathcal{G})$.

Note that, by the choice of θ, we have $(2M+1+\theta)\theta \le \epsilon$. Thus the "only if" part holds.

For the "if" part, let $\mathcal{G} \subset A_+^1$ be a finite subset and let $\epsilon > 0$. There exists $\delta_1 > 0$ such that, for all $x \in A_+^1$ and for all $y \in A$ with $\|y\| \le 2$, if $yx \approx_{\delta_1} xy$, then $x^{1/2}yx^{1/2} \approx_{\epsilon/4(n+1)} yx$. Choose $\delta := \min\{\frac{1}{100}, (\frac{\epsilon}{32(n+3)})^2, (\frac{\delta_1}{12})^2\}$. Let $e \in A_+^1$ be such that

$$exe \approx_\delta x \ \text{ for all } \ x \in \mathcal{G}\cup(\mathcal{G}\cdot\mathcal{G}). \tag{e 5.14}$$

By our assumption, there exist a C^*-algebra C, a finite subset $\tilde{\mathcal{G}} \subset C_+$, and a finite dimensional C^*-algebra $\tilde{F}$ and, c.p. maps $\tilde{\varphi} : C \to \tilde{F}$ and $\tilde{\psi} : \tilde{F} \to B$, such that

(1″) $h(\mathcal{G} \cup \{e\}) \subset_\delta \tilde{\psi} \circ \tilde{\varphi}(\tilde{\mathcal{G}})$,

(2″) $\tilde{\psi}$ is n-decomposable, and

(3″) $\tilde{\psi} \circ \tilde{\varphi}(xy) \approx_\delta \tilde{\psi} \circ \tilde{\varphi}(x) \cdot \tilde{\psi} \circ \tilde{\varphi}(y)$ for all $x, y \in \tilde{\mathcal{G}} \cup (\tilde{\mathcal{G}} \cdot \tilde{\mathcal{G}})$.

By Proposition 5.10, there exist $\bar{n} \le n$, a finite dimensional C^*-algebra $F = F_0 \oplus \cdots \oplus F_{\bar{n}}$, and c.p. maps $\varphi : C \to F$ and $\psi : F \to B$, such that

(1‴) $\psi \circ \varphi = \tilde{\psi} \circ \tilde{\varphi}$ and

(2‴) $\psi|_{F_i}$ is a c.p.c. order zero isometry, $i = 0, 1, \cdots, \bar{n}$.

By (1″), for each $x \in h(\mathcal{G} \cup \{e\})$, there exists $\alpha(x) \in \tilde{\mathcal{G}}$ such that $x \approx_\delta \tilde{\psi} \circ \tilde{\varphi}(\alpha(x))$. Then, by (1‴), we have

$$\psi \circ \varphi(\alpha(x)) \approx_\delta x \ \text{ for all } \ x \in h(\mathcal{G} \cup \{e\}). \tag{e 5.15}$$

Note that $\mathcal{G} \subset A_+^1$. Then, by (e 5.15),

$$\|\psi \circ \varphi(\alpha(y))\| \le 1 + \delta \ \text{ for all } \ y \in h(\mathcal{G} \cup \{e\}). \tag{e 5.16}$$

Combining (3″), (1‴), (e 5.15) and (e 5.16), for any $x, y \in h(\mathcal{G})$, we have

$$\psi \circ \varphi(\alpha(x)\alpha(y)) \approx_\delta \psi \circ \varphi(\alpha(x))\psi \circ \varphi(\alpha(y)) \approx_{(1+\delta)\delta} x \cdot \psi \circ \varphi(\alpha(y)) \approx_\delta xy. \tag{e 5.17}$$

In particular,

$$\|\psi \circ \varphi(\alpha(x)\alpha(y))\| \le 1 + (3 + \delta)\delta \ \text{ for all } \ x, y \in h(\mathcal{G}). \tag{e 5.18}$$

Define a c.p. map $\bar{\varphi} : C \to F$ by $x \mapsto \frac{1}{1+(3+\delta)\delta}\varphi(\alpha(h(e))x\alpha(h(e)))$. Then, for any $x \in C_+^1$, by (e 5.18), we have

$$\|\psi \circ \bar{\varphi}(x)\| = \frac{\|\psi(\varphi(\alpha(h(e))x\alpha(h(e))))\|}{1 + (3 + \delta)\delta} \le \frac{\|\psi(\varphi(\alpha(h(e))^2))\|}{1 + (3 + \delta)\delta} \le 1.$$

Thus

$$\|\psi \circ \bar{\varphi}\| \le 1. \tag{e 5.19}$$

Let $x \in h(\mathcal{G})$. Then

$$\begin{aligned}
\psi \circ \bar{\varphi}(\alpha(x)) &= \frac{1}{1 + (3 + \delta)\delta}\psi \circ \varphi(\alpha(h(e))\alpha(x)\alpha(h(e))) \\
\text{(by (3″) and (1‴))} \quad &\approx_\delta \frac{1}{1 + (3 + \delta)\delta}\psi \circ \varphi(\alpha(h(e)))\psi \circ \varphi(\alpha(x)\alpha(h(e))) \\
\text{(by (3″), (1‴) and (e 5.16))} \quad &\approx_\delta \frac{1}{1 + (3 + \delta)\delta}\psi \circ \varphi(\alpha(h(e)))\psi \circ \varphi(\alpha(x))\psi \circ \varphi(\alpha(h(e))) \\
\text{(by (e 5.15))} \quad &\approx_{\frac{(1+\delta)^2\delta+(1+\delta)\delta+\delta}{1+(3+\delta)\delta}} \frac{h(e)xh(e)}{1 + (3 + \delta)\delta} \\
\text{(by (e 5.14))} \quad &\approx_\delta \frac{x}{1 + (3 + \delta)\delta} \approx_{4\delta} x.
\end{aligned} \tag{e 5.20}$$

Also, for $x \in h(\mathcal{G})$, we have

$$\begin{aligned}
\psi\circ\bar\varphi(\alpha(x)^2) &= \frac{1}{1+(3+\delta)\delta}\psi\circ\varphi(\alpha(h(e))\alpha(x)^2\alpha(h(e)))\\
\text{(by (3$''$) and (1$'''$))} \quad &\approx_{\delta} \frac{1}{1+(3+\delta)\delta}\psi\circ\varphi(\alpha(h(e))\alpha(x))\cdot\psi\circ\varphi(\alpha(x)\alpha(h(e)))\\
\text{(by (3$''$), (1$'''$), (e 5.18), (e 5.16))} \quad &\approx_{2\delta} \frac{1}{1+(3+\delta)\delta}\psi\circ\varphi(\alpha(h(e)))\cdot\psi\circ\varphi(\alpha(x))^2\cdot\psi\circ\varphi(\alpha(h(e)))\\
\text{(by (e 5.15))} \quad &\approx_{4\delta(1+\delta)} \frac{h(e)x^2h(e)}{1+(3+\delta)\delta}\\
\text{(by (e 5.14))} \quad &\approx_{\delta} \frac{x^2}{1+(3+\delta)\delta} \approx_{4\delta} x^2. \qquad \text{(e 5.21)}
\end{aligned}$$

By (e 5.21) and (e 5.20), we have

$$\psi\circ\bar\varphi(\alpha(x)^2) \approx_{12\delta(1+\delta)} x^2 \approx_{10\delta(2+\delta)} \psi\circ\bar\varphi(\alpha(x))^2 \ \text{ for all } \ x\in h(\mathcal{G}). \qquad \text{(e 5.22)}$$

Let p_i be the unit of F_i, $i=0,1,\cdots,\bar n$. Then each p_i is a central projection of F. We now apply Proposition 5.14. Recall $\bar\varphi: C\to F$ and $\psi: F\to B$ are c.p. maps such that $\|\psi\circ\bar\varphi\|\le 1$ (see (e 5.19)). Thus, by (e 5.22) and Proposition 5.14, there exists a positive element $c\in C_+^1$ such that, for $x\in h(\mathcal{G})$, the following hold (note, $(\delta(32+22\delta))^{1/2}<6\delta^{1/2}$):

$$\begin{aligned}
\psi(\bar\varphi(\alpha(x)))\cdot\psi(\bar\varphi(c)p_i) &\approx_{6\delta^{1/2}} \psi(\bar\varphi(\alpha(x))p_i)\\
&= \psi(p_i\bar\varphi(\alpha(x))) \approx_{6\delta^{1/2}} \psi(\bar\varphi(c)p_i)\cdot\psi(\bar\varphi(\alpha(x))). \qquad \text{(e 5.23)}
\end{aligned}$$

Note that $\psi(\bar\varphi(c)p_i)=\psi(\bar\varphi(c)^{1/2}p_i\bar\varphi(c)^{1/2})$ is a positive element, and, by (e 5.19),

$$\|\psi(\bar\varphi(c)p_i)\| = \|\psi(\bar\varphi(c)^{1/2}p_i\bar\varphi(c)^{1/2})\| \le \|\psi(\bar\varphi(c))\| \le \|c\| \le 1. \qquad \text{(e 5.24)}$$

Also note that $\|\psi(\bar\varphi(\alpha(x)))\|\le 1+\delta\le 2$ for all $x\in h(\mathcal{G})$. By (e 5.23), $\psi(\bar\varphi(c)p_i)$ approximately commutes with $\{\psi(\bar\varphi(\alpha(x))): x\in h(\mathcal{G})\}$ within $12\delta^{1/2}$, and, by the choice of δ and δ_1, we have

$$\begin{aligned}
\psi(\bar\varphi(c)p_i)^{1/2}\cdot\psi(\bar\varphi(\alpha(x)))\cdot\psi(\bar\varphi(c)p_i)^{1/2} &\approx_{\frac{\epsilon}{4(n+1)}} \psi(\bar\varphi(\alpha(x)))\cdot\psi(\bar\varphi(c)p_i)\\
&\approx_{6\delta^{1/2}} \psi(\bar\varphi(\alpha(x))p_i) \ \text{ for all } \ x\in h(\mathcal{G}). \qquad \text{(e 5.25)}
\end{aligned}$$

By (2$'''$) and by Proposition 5.6, there exists c.p.c. maps $\beta_i: B\to F_i$ such that

$$\beta_i\circ\psi|_{F_i} = \mathrm{id}_{F_i}, \ \ i=0,1,\cdots,\bar n. \qquad \text{(e 5.26)}$$

Define c.p. maps ($i=0,1,\cdots,\bar n$) $\gamma_i: A\to F_i$ by $x\mapsto \beta_i\left(\psi(\bar\varphi(c)p_i)^{1/2}\cdot h(x)\cdot\psi(\bar\varphi(c)p_i)^{1/2}\right)$ and define c.p. map $\gamma: A\to F=F_0\oplus\cdots\oplus F_{\bar n}$ by $x\mapsto(\gamma_0(x),\cdots,\gamma_{\bar n}(x))$. For $x\in\mathcal{G}$,

$$\begin{aligned}
\psi\circ\gamma(x)=\sum_{i=0}^{\bar n}\psi\circ\gamma_i(x) &= \sum_{i=0}^{\bar n}\psi\circ\beta_i\left(\psi(\bar\varphi(c)p_i)^{1/2}\cdot h(x)\cdot\psi(\bar\varphi(c)p_i)^{1/2}\right)\\
\text{(By (e 5.20), (e 5.24))} \quad &\approx_{10(n+1)\delta} \sum_{i=0}^{\bar n}\psi\circ\beta_i\left(\psi(\bar\varphi(c)p_i)^{1/2}\cdot\psi(\bar\varphi(\alpha(h(x))))\cdot\psi(\bar\varphi(c)p_i)^{1/2}\right)\\
\text{(By (e 5.25))} \quad &\approx_{\frac{\epsilon}{4}+6(n+1)\delta^{\frac12}} \sum_{i=0}^{\bar n}\psi\circ\beta_i\left(\psi(\bar\varphi(\alpha(h(x)))p_i)\right)\\
\text{((e 5.26), $\bar\varphi(\alpha(h(x)))p_i\in F_i$)} \quad &= \sum_{i=0}^{\bar n}\psi(\bar\varphi(\alpha(h(x)))p_i)= \psi(\bar\varphi(\alpha(h(x))))\\
\text{(By (e 5.20))} \quad &\approx_{10\delta} h(x). \qquad \text{(e 5.27)}
\end{aligned}$$

Note, by the choice of δ ($\le (\frac{\epsilon}{32(n+3)})^2$), we have $10(n+1)\delta + \frac{\epsilon}{4} + 6(n+1)\delta^{\frac{1}{2}} + 10\delta \le \epsilon$. Thus there exist a c.p. map $\gamma : A \to F$ and a c.p. n-decomposable map $\psi : F \to A$ such that $h(x) \approx_\epsilon \psi \circ \gamma(x)$ for all $x \in \mathcal{G}$. Finally, by Proposition 5.12, $\dim_{\mathrm{nuc}} h \le n$.

□

Proposition 5.16. *Let A and B be C^*-algebras, $h : A \to B$ be a *-homomorphism and let $\iota : h(A) \hookrightarrow B$ be the embedding. Then $\dim_{\mathrm{nuc}} h = \dim_{\mathrm{nuc}} \iota$.*

Proof. First, we note $\dim_{\mathrm{nuc}} h = \dim_{\mathrm{nuc}}(\iota \circ h) \le \dim_{\mathrm{nuc}} \iota$.

Next, if $\dim_{\mathrm{nuc}} h = \infty$, then we are done. Hence we may assume that $\dim_{\mathrm{nuc}} h = n$ for some $n \in \mathbb{N} \cup \{0\}$. Let $\mathcal{G} \subset h(A)_+$ be a finite subset and let $\epsilon > 0$. Then there exists a finite subset $\tilde{\mathcal{G}} \subset A_+$ such that

$$\iota(\mathcal{G}) = \mathcal{G} = h(\tilde{\mathcal{G}}). \tag{e 5.28}$$

Choose $M = \max\{\|x\| + 1 : x \in \tilde{\mathcal{G}}\}$ and $\delta := \frac{\min\{\epsilon,1\}}{2(M+1)^2}$. Since $\dim_{\mathrm{nuc}} h \le n$, there exist a finite dimensional C^*-algebra F, and c.p. maps $\varphi : A \to F$ and $\psi : F \to B$ such that

(1) $\psi \circ \varphi(x) \approx_\delta h(x) = \iota(h(x))$ for all $x \in \tilde{\mathcal{G}} \cup (\tilde{\mathcal{G}} \cdot \tilde{\mathcal{G}}) \cup (\tilde{\mathcal{G}} \cdot \tilde{\mathcal{G}} \cdot \tilde{\mathcal{G}}) \cup (\tilde{\mathcal{G}} \cdot \tilde{\mathcal{G}} \cdot \tilde{\mathcal{G}} \cdot \tilde{\mathcal{G}})$, and

(2) ψ is n-decomposable.

Then (e 5.28) and (1) show

$$\iota(\mathcal{G}) \subset_\epsilon \psi \circ \varphi(\tilde{\mathcal{G}}). \tag{e 5.29}$$

By (1), for all $x \in \tilde{\mathcal{G}} \cup (\tilde{\mathcal{G}} \cdot \tilde{\mathcal{G}})$, we have

$$\|\psi \circ \varphi(x)\| \le \delta + \|h(x)\| \le \delta + M^2. \tag{e 5.30}$$

Therefore, using (1) and (e 5.30), we have

$$\psi \circ \varphi(xy) \approx_\delta h(xy) = h(x)h(y) \approx_{(M^2+\delta)\delta} h(x)\psi \circ \varphi(y) \approx_{\delta(M^2+\delta)} \psi \circ \varphi(x)\psi \circ \varphi(y).$$

Then, by the choice of δ, we have

$$\psi \circ \varphi(xy) \approx_\epsilon \psi \circ \varphi(x)\psi \circ \varphi(y) \ \text{ for all } \ x, y \in \tilde{\mathcal{G}} \cup (\tilde{\mathcal{G}} \cdot \tilde{\mathcal{G}}). \tag{e 5.31}$$

Then (e 5.29), (2), together with (e 5.31), show that (with A in place of C), the conditions of Theorem 5.15 are satisfied. Therefore we have $\dim_{\mathrm{nuc}} \iota \le n = \dim_{\mathrm{nuc}} h$.

□

The following corollary shows that the image of a *-homomorphism of finite nuclear dimension must be exact.

Corollary 5.17. *Let A and B be C^*-algebras. If $h : A \to B$ is a *-homomorphism with $\dim_{\mathrm{nuc}} h < \infty$, then $h(A)$ is exact.*

Proof. By Proposition 5.16, the embedding $\iota : h(A) \hookrightarrow B$ satisfies $\dim_{\mathrm{nuc}} \iota = \dim_{\mathrm{nuc}} h < \infty$. Thus ι is a nuclear map. It follows that $h(A)$ is exact (see [55, 6.1.11]).

□

By [33, Theorem 2.8], every separable exact C^*-algebra admits an embedding into the Cuntz algebra $\mathcal{O}_2$. By [72, Theorem 7.4], one has $\dim_{\mathrm{nuc}} \mathcal{O}_2 = 1$. Thus every embedding of separable exact C^*-algebra into $\mathcal{O}_2$ has nuclear dimension at most 1. Therefore, it seems to be interesting to observe the following statement.

Proposition 5.18. *Let $h : A \to B$ be a *-homomorphism such that $h(A)$ is a hereditary C^*-subalgebra of B. Then $\dim_{\mathrm{nuc}} h = \dim_{\mathrm{nuc}} h(A)$. Moreover, if B is separable and $h(A)$ is a full hereditary C^*-subalgebra of B, then $\dim_{\mathrm{nuc}} h = \dim_{\mathrm{nuc}} B$.*

Proof. First, let us assume that h is surjective. Then the embedding $\iota : h(A) \to B$ is the identity map id_B. By Proposition 5.16, we have $\dim_{\mathrm{nuc}} h = \dim_{\mathrm{nuc}} \iota = \dim_{\mathrm{nuc}} \mathrm{id}_B = \dim_{\mathrm{nuc}} B$.

Now we assume that $C := h(A)$ is a hereditary C^*-subalgebra of B. Then by [7, Proposition 1.6] (also see [59, Proposition 2.4]), $\dim_{\mathrm{nuc}} h = \dim_{\mathrm{nuc}} h^C$, where $h^C : A \to C$ is the homomorphism defined by $h^C(a) := h(a)$ for all $a \in A$ (but $h : A \to B$). Now since h^C is surjective, by what we have proved, $\dim_{\mathrm{nuc}} h = \dim_{\mathrm{nuc}} h^C = \dim_{\mathrm{nuc}} h(A)$. Moreover, if B is separable and $h(A)$ is a full hereditary C^*-subalgebra of B, by [72, Corollary 2.8], then $\dim_{\mathrm{nuc}} B = \dim_{\mathrm{nuc}} h(A) = \dim_{\mathrm{nuc}} h$.

□

Corollary 5.19. *Let A be a C^*-algebra and $I \subset A$ be a closed ideal. If the quotient map $\pi : A \to A/I$ has finite nuclear dimension, then A/I also has finite nuclear dimension.*

6 A criterion for generalized inductive limits becoming finite nuclear dimension

Definition 6.1 ([3] Generalized inductive system)**.** Let A_n be a sequence of C^*-algebras and $\varphi_{m,n} : A_m \to A_n$ be a map ($m < n$). We say $(A_n, \varphi_{m,n})$ forms a generalized inductive system if the following hold: For any $k \in \mathbb{N}$, any $x, y \in A_k$, any $\lambda \in \mathbb{C}$, and any $\epsilon > 0$, there exists $M \in \mathbb{N}$ such that, for any $n > m \geq M$,

(1) $\|\varphi_{m,n}(\varphi_{k,m}(x) + \varphi_{k,m}(y)) - (\varphi_{k,n}(x) + \varphi_{k,n}(y))\| \leq \epsilon$,
(2) $\|\varphi_{m,n}(\lambda\varphi_{k,m}(x)) - \lambda\varphi_{k,m}(x)\| \leq \epsilon$,
(3) $\|\varphi_{m,n}(\varphi_{k,m}(x)^*) - \varphi_{k,n}(x)^*\| \leq \epsilon$,
(4) $\|\varphi_{m,n}(\varphi_{k,m}(x)\varphi_{k,m}(y)) - \varphi_{k,n}(x)\varphi_{k,n}(y)\| \leq \epsilon$, and
(5) $\sup_r \|\varphi_{k,r}(x)\| < \infty$.

The system is called p.c. (or c.p.c.), if all $\varphi_{m,n}$ are p.c. maps (or c.p.c. maps).

If $(A_n, \varphi_{m,n})$ forms a generalized inductive system, then the following is a C^*-algebra which we call it the generalized inductive limit of $(A_n, \varphi_{m,n})$:

$$\lim_n (A_n, \varphi_{m,n}) := \overline{\{\pi_\infty(\{\varphi_{n,1}(a), \varphi_{n,2}(a), \cdots\}) : n \in \mathbb{N}, a \in A_n\}}^{\|\cdot\|} \subset \prod_{n=1}^{\infty} A_n / \bigoplus_{n=1}^{\infty} A_n,$$

where $\varphi_{m,n} := 0$ for $m > n$, and $\varphi_{n,n} := \mathrm{id}_{A_n}$. For $i \in \mathbb{N}$, define (see [3, 2.1.2, 2.1.3]) $\varphi_{i,\infty} : A_i \to \lim_n (A_n, \varphi_{m,n})$ by $x \mapsto \pi_\infty(\{\varphi_{i,1}(x), \varphi_{i,2}(x), \cdots\})$.

Notation 6.2. Given a sequence of C^*-algebras A_n and a sequence of maps $\varphi_n : A_n \to A_{n+1}$, for $m < n$, define $\varphi_{m,n}$ to be the composition of $\varphi_m, \varphi_{m+1}, \cdots, \varphi_{n-1}$:

$$\varphi_{m,n} := \varphi_{n-1} \circ \varphi_{n-2} \circ \cdots \circ \varphi_m : A_m \to A_n,$$

and define $\varphi_{m,n} := 0$ for $m > n$, and define $\varphi_{n,n} := \mathrm{id}_{A_n}$. We say (A_n, φ_n) forms a generalized inductive system, if $(A_n, \varphi_{m,n})$ forms a generalized inductive system. Accordingly $\lim_n (A_n, \varphi_{m,n})$ will be denoted by $\lim_n (A_n, \varphi_n)$.

Lemma 6.3. *Let A_n be C^*-algebras and $\varphi_n : A_n \to A_{n+1}$ be p.c. maps (or c.p.c. maps), $n = 1, 2, \cdots$. Let $\varphi_{m,n}$ be defined as in Notation 6.2. If, for any $k \in \mathbb{N}$, any $\epsilon > 0$, and any $x, y \in A^1_{k+}$, there exists $m > k$ such that, for all $n > m$, $\varphi_{k,n}(x)\varphi_{k,n}(y) \approx_\epsilon \varphi_{m,n}(\varphi_{k,m}(x)\varphi_{k,m}(y))$, then (A_n, φ_n) forms a p.c. (or c.p.c.) generalized inductive system.*

Proof. Since φ_n are p.c. (or c.p.c.) maps, (1), (2), (3) and (5) in Definition 6.1 are satisfied. It remains to show that (4) in Definition 6.1 holds.

Let $k \in \mathbb{N}$, let $a, b \in A_{k+}^1$ and let $\epsilon > 0$. By the assumption, there exists $M \in \mathbb{N}$ such that, for any $i \geq M$, $\varphi_{k,i}(a) \cdot \varphi_{k,i}(b) \approx_{\epsilon/2} \varphi_{M,i}(\varphi_{k,M}(a) \cdot \varphi_{k,M}(b))$. Then, for any $n > m \geq M$, we have

$$\begin{aligned} \varphi_{m,n}(\varphi_{k,m}(a) \cdot \varphi_{k,m}(b)) &\approx_{\epsilon/2} \varphi_{m,n}(\varphi_{M,m}(\varphi_{k,M}(a) \cdot \varphi_{k,M}(b))) \\ &= \varphi_{M,n}(\varphi_{k,M}(a) \cdot \varphi_{k,M}(b)) \approx_{\epsilon/2} \varphi_{k,n}(a) \cdot \varphi_{k,n}(b). \end{aligned}$$

Thus (4) in Definition 6.1 holds for any $a, b \in A_{k+}$. Since A_{k+}^1 generates A_k as linear space, then (4) in Definition 6.1 holds for any $a, b \in A_k$. Lemma follows.

□

Lemma 6.4. *Let $(A_i, \varphi_{j,i})$ be a p.c. generalized inductive system of C^*-algebras. Then, for any $n, k \in \mathbb{N}$, any finite subset $\mathcal{F} \subset A_k$, and any $\epsilon > 0$, there exists $M > k (\in \mathbb{N})$ such that, for any $j > i \geq M$, any $m_1, m_2 \leq n \in \mathbb{N}$, and any $x_1, x_2, \cdots, x_{m_1}, y_1, y_2, \cdots, y_{m_2} \in \mathcal{F}$,*

$$\varphi_{i,j}\left(\prod_{r=1}^{m_1} \varphi_{k,i}(x_r) \cdot \prod_{r=1}^{m_2} \varphi_{k,i}(y_r)\right) \approx_\epsilon \varphi_{i,j}\left(\prod_{r=1}^{m_1} \varphi_{k,i}(x_r)\right) \cdot \varphi_{i,j}\left(\prod_{r=1}^{m_2} \varphi_{k,i}(y_r)\right).$$

Proof. It suffices to show that, for any $k, n \in \mathbb{N}$, any $\varepsilon > 0$, and any finite subset $\mathcal{F} \in A_k^1$, there exists $M > 0$ such that $(1 \leq l \leq n)$, for $j > i > M$,

$$\varphi_{i,j}(\prod_{r=1}^{l} \varphi_{k,i}(x_r)) \approx_\varepsilon \prod_{r=1}^{l} \varphi_{k,j}(x_r) \ \text{ for all } \ x_1, x_2, \cdots, x_l \in \mathcal{F}. \tag{e 6.1}$$

This follows from Definition 6.1 and the induction on n immediately. The case $n = 2$ follows from (4) in Definition 6.1. Assume the above holds for $2, 3, \cdots, n-1$. Then, for $\delta = \varepsilon/3$, there exists $M_0 > 0$ such that, for any $j > i > M_0$,

$$\varphi_{i,j}(\prod_{r=1}^{l'} \varphi_{k,i}(x_r)) \approx_\delta \prod_{r=1}^{l'} \varphi_{k,j}(x_r) \ \text{ for all } \ x_r \in \mathcal{F}\,(1 \leq r \leq l' \leq n-1). \tag{e 6.2}$$

For all $x_r \in \mathcal{F}$, with $y = \prod_{r=1}^{l'} \varphi_{k,i}(x_r)$ and $z = \varphi_{k,i}(x_{l'+1})$ $(1 \leq l' < l'+1 \leq n)$, there is $M_1 > 0$ such that, for $K > j \geq M_1$, $\varphi_{j,K}(\varphi_{i,j}(y)\varphi_{i,j}(z)) \approx_\delta \varphi_{i,K}(y)\varphi_{i,K}(z)$. Then

$$\varphi_{j,K}(\prod_{r=1}^{l'+1} \varphi_{k,j}(x_r)) \overset{(e\,6.2)}{\approx}_\delta \varphi_{j,K}(\varphi_{i,j}(y)\varphi_{i,j}(z)) \approx_\delta \varphi_{i,K}(y)\varphi_{i,K}(z) \overset{(e\,6.2)}{\approx}_\delta \prod_{r=1}^{l'+1} \varphi_{k,K}(x_r).$$

□

We end this section with a sufficient and necessary condition for a c.p.c. generalized inductive limit having finite nuclear dimension.

Theorem 6.5. *Let $n \in \mathbb{N} \cup \{0\}$. Let $(A_i, \varphi_{i,j})$ be a c.p.c. generalized inductive system of C^*-algebras. Let $A = \lim_i (A_i, \varphi_{i,j})$. Then $\dim_{\mathrm{nuc}} A \leq n$ if and only if the following hold:*

For any $i \in \mathbb{N}$, any finite subset $\mathcal{G} \subset A_i$, and any $\epsilon > 0$, there exist a finite dimensional C^*-algebra F, a c.p. map $\alpha : A_i \to F$, and an n-decomposable c.p. map $\beta : F \to A$ such that

$$\varphi_{i,\infty}(x) \approx_\epsilon \beta \circ \alpha(x) \ \text{ for all } \ x \in \mathcal{G}.$$

Proof. For the "only if" part, let us assume that $\dim_{\rm nuc} A \le n$. Let $i \in \mathbb{N}$, let $\mathcal{G} \subset A_i$ be a finite subset, and let $\epsilon > 0$. There exist a finite dimensional C^*-algebra F, a c.p.c. map $\alpha' : A \to F$, and an n-decomposable c.p. map $\beta : F \to A$ such that, for all $x \in \mathcal{G}$, $\varphi_{i,\infty}(x) \approx_\epsilon \beta \circ \alpha'(\varphi_{i,\infty}(x))$. Define a c.p.c. map $\alpha := \alpha' \circ \varphi_{i,\infty}$. Then, for all $x \in \mathcal{G}$, $\varphi_{i,\infty}(x) \approx_\epsilon \beta \circ \alpha'(\varphi_{i,\infty}(x)) = \beta \circ \alpha(x)$.

For the "if" part, we will apply Theorem 5.15 to show that $\dim_{\rm nuc} {\rm id}_A \le n$.

Let $\bar{\mathcal{G}} \subset A_+$ be a finite subset and let $\epsilon > 0$. Choose $N := 1 + \max\{\|x\| : x \in \bar{\mathcal{G}}\}$ and choose $\delta := \min\{1, \frac{\epsilon}{4(N+1)}\}$. There exist $k \in \mathbb{N}$ and a finite subset $\hat{\mathcal{G}} \subset (A_k)_{sa}$ such that

$$\{x^{1/2} : x \in \bar{\mathcal{G}}\} \subset_\delta \varphi_{k,\infty}(\hat{\mathcal{G}}). \tag{e 6.3}$$

Since $A = \lim_i(A_i, \varphi_{i,j})$ is a generalized inductive system, there exists $M_1 > k \in \mathbb{N}$ such that, for any $j > i \ge M_1 \in \mathbb{N}$ and any $x \in \hat{\mathcal{G}}$,

$$\|\varphi_{i,j}(\varphi_{k,i}(x)) - \varphi_{k,j}(x)\| \le \delta. \tag{e 6.4}$$

Hence, for any $i \ge M_1 \in \mathbb{N}$, we have

$$\|\varphi_{i,\infty}(\varphi_{k,i}(x)) - \varphi_{k,\infty}(x)\| \le \delta \text{ for all } x \in \hat{\mathcal{G}}. \tag{e 6.5}$$

By Lemma 6.4, there exists $M > M_1 \in \mathbb{N}$, such that, for any $j > M$, any $1 \le m_1, m_2 \le 4 \in \mathbb{N}$, and any $x_1, x_2, \cdots, x_{m_1}, y_1, y_2, \cdots, y_{m_2} \in \hat{\mathcal{G}}$,

$$\varphi_{M,j}\left(\prod_{r=1}^{m_1} \varphi_{k,M}(x_r) \cdot \prod_{r=1}^{m_2} \varphi_{k,M}(y_r)\right) \approx_\delta \varphi_{M,j}\left(\prod_{r=1}^{m_1} \varphi_{k,M}(x_r)\right) \cdot \varphi_{M,j}\left(\prod_{r=1}^{m_2} \varphi_{k,M}(y_r)\right). \tag{e 6.6}$$

Let $\mathcal{G} := \{\varphi_{k,M}(x)^2 : x \in \hat{\mathcal{G}}\} \subset (A_M)_+$. Then, by (e 6.6), we have

$$\varphi_{M,j}(xy) \approx_\delta \varphi_{M,j}(x)\varphi_{M,j}(y) \text{ for all } x, y \in \mathcal{G} \cup (\mathcal{G} \cdot \mathcal{G}) \text{ and for all } j > M. \tag{e 6.7}$$

Consequently, we have

$$\varphi_{M,\infty}(xy) \approx_\delta \varphi_{M,\infty}(x)\varphi_{M,\infty}(y) \text{ for all } x, y \in \mathcal{G} \cup (\mathcal{G} \cdot \mathcal{G}). \tag{e 6.8}$$

Let $N_1 := 1 + \max\{\sup_{j>M}\{\|\varphi_{M,j}(x)\| : x \in \mathcal{G} \cup (\mathcal{G} \cdot \mathcal{G})\}\}$. By the assumption of the theorem, there exists a finite dimensional C^*-algebra F, a c.p. map $\alpha : A_M \to F$, and an n-decomposable c.p. map $\beta : F \to A$ such that

$$\varphi_{M,\infty}(x) \approx_{\frac{\delta}{N_1}} \beta \circ \alpha(x) \text{ for all } x \in \mathcal{G} \cup (\mathcal{G} \cdot \mathcal{G}) \cup (\mathcal{G} \cdot \mathcal{G} \cdot \mathcal{G}) \cup (\mathcal{G} \cdot \mathcal{G} \cdot \mathcal{G} \cdot \mathcal{G}). \tag{e 6.9}$$

For any $a \in \bar{\mathcal{G}}$, by (e 6.3), there exists $x_a \in \hat{\mathcal{G}}$ such that $a^{1/2} \approx_\delta \varphi_{k,\infty}(x_a)$. Then

$$\begin{aligned}
a = (a^{1/2})^2 \ &\approx_{(2N+\delta)\delta} \ \varphi_{k,\infty}(x_a)^2 && \text{(by (e 6.3))}\\
&\approx_{(2N+\delta)\delta} \ \varphi_{M,\infty}(\varphi_{k,M}(x_a))^2 && \text{(by (e 6.5))}\\
&\approx_\delta \ \varphi_{M,\infty}(\varphi_{k,M}(x_a)^2) && \text{(by (e 6.8))}\\
&\approx_\delta \ \beta \circ \alpha(\varphi_{k,M}(x_a)^2) \in \beta \circ \alpha(\mathcal{G}). && \text{(by (e 6.9))}
\end{aligned}$$

Thus $\bar{\mathcal{G}} \subset_\epsilon \beta \circ \alpha(\mathcal{G})$. For $x, y \in \mathcal{G} \cup (\mathcal{G} \cdot \mathcal{G})$, by (e 6.9), (e 6.8) and (e 6.9) again,

$$\beta \circ \alpha(xy) \approx_\delta \varphi_{M,\infty}(xy) \approx_\delta \varphi_{M,\infty}(x)\varphi_{M,\infty}(y) \approx_{2\delta} \beta \circ \alpha(x)\beta \circ \alpha(y). \tag{e 6.10}$$

Therefore $\beta \circ \alpha(xy) \approx_\epsilon \beta \circ \alpha(x)\beta \circ \alpha(y)$ for all $x, y \in \mathcal{G} \cup (\mathcal{G} \cdot \mathcal{G})$. Then, by Theorem 5.15, we have $\dim_{\rm nuc} {\rm id}_A \le n$. Consequently, $\dim_{\rm nuc} A \le n$.

$\square$

7 Simple C^*-algebra of finite tracial nuclear dimension

Definition 7.1. Let A and B be C^*-algebras and let $\varphi : A \to B$ be a map. Let $\epsilon \geq 0$. If, for any $a_1, a_2 \in A_+^1$ with $a_1 a_2 = 0$, we have $\|\varphi(a_1)\varphi(a_2)\| \leq \epsilon$, then we say φ is an ϵ-almost order zero map.

Definition 7.2. Let A be a C^*-algebra and let F be a finite dimensional C^*-algebra. Let $\varphi : F \to A$ be a c.p. map and let $n \in \mathbb{N} \cup \{0\}$ be an integer. The map φ is called (n, ϵ)-dividable if F can be written as $F = F_0 \oplus \cdots \oplus F_n$ (where F_i are ideals of F) such that $\varphi|_{F_i}$ is a c.p.c. ϵ-almost order zero map for $i = 0, 1, \cdots, n$.

The next two propositions follow from the projectivity of the cone of finite dimensional C^*-algebras.

Proposition 7.3. *For any finite dimensional C^*-algebra F and any $\epsilon > 0$, there exists $\delta > 0$ such that, for any C^*-algebra A and any c.p.c. map $\varphi : F \to A$ which is δ-almost order zero, there exists a c.p.c. order zero map $\psi : F \to A$ satisfying $\|\varphi - \psi\| \leq \epsilon$.*

Proof. Let F be fixed. If such δ described in the proposition does not exists, then, there exist $\varepsilon_0 > 0$, a sequence of C^*-algebra A_n, and c.p.c. maps $\varphi_n : F \to A_n$ such that φ_n is $1/n$-almost order zero, and, for any $n \in \mathbb{N}$ and any c.p.c. order zero map $\psi : F \to A_n$, we have $\|\varphi_n - \psi\| > \epsilon_0$.

Define a c.p.c. order zero map $\Phi : F \to \prod_{n=1}^{\infty} A_n / \bigoplus_{n=1}^{\infty} A_n$ by $x \mapsto \pi_\infty(\{\varphi_n(x)\})$. Then, by [67, Proposition 1.2.4], Φ has a c.p.c. order zero lift $\Psi : F \to \prod_{n=1}^{\infty} A_n$. Let ψ_n be the components of Ψ corresponding to A_n. Since $\lim_{n\to\infty} \|\varphi_n(x) - \psi_n(x)\| = 0$ for all $x \in F$, and the unit ball of F is compact, there exists n_0 such that $\|\varphi_{n_0} - \psi_{n_0}\| < \epsilon_0/2$. This leads to a contradiction. Thus δ does exist and the proposition follows.

□

Proposition 7.4. *Let $F = M_{k_0} \oplus M_{k_1} \oplus \cdots \oplus M_{k_r}$ be a finite dimensional C^*-algebra with a standard generating set $\mathcal{G}^F$ (see Notation 2.4) in the unit ball of F.*

(1) For any $\epsilon > 0$, there exists $\delta_1(\varepsilon) > 0$ such that, for any $n \in \mathbb{N} \cup \{0\} (n \leq r)$, any C^*-algebra A and any $(n, \delta_1(\varepsilon))$-dividable c.p. map $\varphi : F \to A$, there exists a piecewise contractive n-decomposable c.p. map $\psi : F \to A$ satisfying $\|\varphi - \psi\| \leq \epsilon$.

(2) For any $\varepsilon > 0$, there exists $\delta_2(\varepsilon) > 0$ such that, for any $n \in \mathbb{N} \cup \{0\}$ $(n \leq r)$, any $\sigma > 0$, and any (n, σ)-dividable c.p. map $\alpha : F \to A$ (for any C^*-algebra A), and any c.p.c. map $\beta : B := C^*(\alpha(F)) \to C$ (for any C^*-algebra C) which is $(\alpha(\mathcal{G}^F), \delta_2(\varepsilon))$-multiplicative, $\beta \circ \alpha$ is an $(n, \sigma{+}\varepsilon)$-dividable c.p. map.

Proof. For (1), by Proposition 7.3, there exists $\delta_1(\varepsilon) > 0$ such that, for any C^*-algebra A and any $\delta_1(\varepsilon)$-almost order zero c.p.c. map $\chi : F \to A$, there exists a c.p.c. order zero map $\psi : F \to A$ satisfying $\|\chi - \psi\| \leq \frac{\epsilon}{r+1}$.

Now let $\varphi : F \to A$ be an $(n, \delta_1(\varepsilon))$-dividable c.p. map, i.e. F can be written as $F = F_0 \oplus \cdots \oplus F_n$ such that each $\varphi|_{F_j}$ is a c.p.c. $\delta_1(\varepsilon)$-almost order zero map. Then $n \leq r$.

Let $\pi_j : F \to F_j$ be the quotient map. Note that $\mathrm{id}_F = \sum_{j=0}^{n} \pi_j$. Then $\varphi \circ \pi_j$ is a c.p.c. $\delta_1(\varepsilon)$-almost order zero map on F, $j = 0, 1, \cdots, n$. By the choice of $\delta_1(\varepsilon)$, there are c.p.c. order zero maps $\psi_0, \psi_1, \cdots, \psi_n : F \to A$ such that $\|\psi_j - \varphi \circ \pi_j\| \leq \frac{\epsilon}{r+1}$. Therefore $\psi := \sum_{j=0}^{n} \psi_j \circ \pi_j : F \to A$ is piecewise contractive n-decomposable c.p. map and $\|\psi - \varphi\| = \|(\psi - \varphi) \circ (\sum_{j=0}^{n} \pi_j)\| = \|\sum_{j=0}^{n} (\psi - \varphi \circ \pi_j) \circ \pi_j\| \leq \epsilon$.

For (2), write $F = F_0 \oplus F_1 \oplus \cdots \oplus F_n$ such that $\alpha|_{F_i}$ is a σ-almost order zero map $(0 \leq i \leq n)$. One observes that if β is $(\alpha(\mathcal{G}^F), \delta)$-multiplicative, then

$$\|\beta(\alpha(a)\alpha(b)) - \beta \circ \alpha(a)\beta \circ \alpha(b)\| < \delta \text{ for all } a, b \in \mathcal{G}^F. \tag{e 7.1}$$

Since $\mathcal{G}^F$ is a standard generating set (see Notation 2.4) and the unit ball of F is compact, for any $\varepsilon > 0$, one can find a universal constant $\delta_2(\varepsilon) > 0$ independent of α (but dependent of F) such that

$$\sup_{\|a\|,\|b\|\le 1} \|\beta(\alpha(a)\alpha(b)) - \beta\circ\alpha(a)\beta\circ\alpha(b)\| < \varepsilon, \tag{e 7.2}$$

if β is $(\alpha(\mathcal{G}^F), \delta_2(\varepsilon))$-multiplicative. Thus $\beta\circ\alpha|_{F_i}$ is a $(\sigma+\varepsilon)$-almost order zero map $(0 \le i \le n)$. $\square$

Definition 7.5. Let F be a finite dimensional C^*-algebra and let $\epsilon > 0$. Define

$$\Delta(F, \epsilon) := \min\{\delta_1(\varepsilon), \delta_2(\varepsilon), \delta_2(\delta_1(\varepsilon)), 1/2\} > 0,$$

where $\delta_1(\varepsilon)$, $\delta_2(\varepsilon)$ and $\delta_2(\delta_1(\varepsilon))$ are as given in Proposition 7.4.

Definition 7.6. Let A be a unital simple C^*-algebra and let $n \in \mathbb{N}\cup\{0\}$. We say that id_A has tracial nuclear dimension no more than n, if, for any finite subset $\mathcal{F} \subset A$, any $\epsilon > 0$, and any $a \in A_+\backslash\{0\}$, there exist a finite dimensional C^*-algebra F, a c.p.c. map $\alpha : A \to F$, a nonzero piecewise contractive n-decomposable c.p. map $\beta : F \to A$, and a c.p.c. map $\gamma : A \to A\cap\beta(F)^\perp$, such that

(1) $x \approx_\epsilon \gamma(x) + \beta\circ\alpha(x)$ for all $x \in \mathcal{F}$, and

(2) $\gamma(1_A) \lesssim_A a$.

If id_A has tracial nuclear dimension no more than n, we write $\mathrm{Trdim_{nuc}}\,\mathrm{id}_A \le n$.

Note that, for any simple unital C^*-algebra A, we have $\mathrm{Trdim_{nuc}}\,\mathrm{id}_A \le \dim_{\mathrm{nuc}}\mathrm{id}_A = \dim_{\mathrm{nuc}} A$. Later, we will show that $\mathrm{Trdim_{nuc}}\,\mathrm{id}_A \le n$ is equivalent to the statement that A is asymptotically tracially in $\mathcal{N}_n$.

Proposition 7.7. *Let A be a unital simple separable C^*-algebra and let $n \in \mathbb{N}\cup\{0\}$. Assume that $\mathrm{Trdim_{nuc}}\mathrm{id}_A \le n$. Then, for any finite subset $\mathcal{F} \subset A$, any $\epsilon > 0$, and any $a \in A_+\backslash\{0\}$, there exist a finite dimensional C^*-algebra F, a c.p.c. map $\alpha : A \to F$, a nonzero piecewise contractive n-decomposable c.p. map $\beta : F \to A$, and a c.p.c. map $\gamma : A \to A\cap\beta(F)^\perp$ such that*

(1) $x \approx_\epsilon \gamma(x) + \beta\circ\alpha(x)$ for all $x \in \mathcal{F}$,

(2) $\gamma(1_A) \lesssim_A a$, and

(3) $\|\beta\circ\alpha(x)\| \ge \|x\| - \epsilon$ for all $x \in \mathcal{F}$.

Proof. Let $\mathcal{F} \subset A$ be a finite subset, let $\epsilon > 0$ and let $a \in A_+\backslash\{0\}$. Let $\mathcal{F} \subset X_1 \subset X_2 \subset \cdots \subset A$ be finite subsets such that $\cup_{m\ge1} X_m$ is norm dense in A. Since $\mathrm{Trdim_{nuc}}\,\mathrm{id}_A \le n$, for each $m \in \mathbb{N}$, there exist a finite dimensional C^*-algebra F_m, a c.p.c. map $\alpha_m : A \to F_m$, a nonzero piecewise contractive n-decomposable c.p. map $\beta_m : F_m \to A$, and a c.p.c. map $\gamma_m : A \to A\cap\beta_m(F_m)^\perp$ such that, for all m,

(i) $x \approx_{\frac{\epsilon}{m}} \gamma_m(x) + \beta_m\circ\alpha_m(x)$ for all $x \in X_m$ and

(ii) $\gamma_m(1_A) \lesssim_A a$.

Define a c.p.c. map $\Gamma : A \to l^\infty(A)/c_0(A)$ by $x \mapsto \pi_\infty(\{\gamma_1(x), \gamma_2(x), \cdots\})$ and define a c.p. map $\Phi : A \to l^\infty(A)/c_0(A)$ by $x \mapsto \pi_\infty(\{\beta_1\circ\alpha_1(x), \beta_2\circ\alpha_2(x), \cdots\})$. Since $\gamma_m(A)\perp(\beta_m\circ\alpha_m(A))$, we have $\Gamma(A)\perp\Phi(A)$. Note that, by (i), we have $\iota_A = \Gamma + \Phi$. It follows that Γ and Φ are *-homomorphisms.

If Φ is a zero map, then $\iota_A = \Gamma$. Thus there exists $m_0 \in \mathbb{N}$ such that $\|1_A - \gamma_{m_0}(1_A)\| < 1/2$. Therefore $\gamma_{m_0}(1_A)$ is invertible in A. Then $\gamma_{m_0}(1_A)\perp\beta_{m_0}(F_{m_0})$ implies $\beta_{m_0}(F_{m_0}) = \{0\}$, which is contradict to that β_{m_0} is a nonzero map. Hence Φ can not be a zero map. In other words, $\Phi(1_A)$

is a nonzero projection which has norm one. Thus there exist natural numbers $m_1 < m_2 < \cdots$ such that

$$\|\beta_{m_i} \circ \alpha_{m_i}(1_A)\| \geq 1 - 1/i, \ i = 1, 2, \cdots . \tag{e 7.3}$$

Define a *-homomorphism $\Psi : A \to l^\infty(A)/c_0(A)$ by $x \mapsto \pi_\infty(\{\beta_{m_1} \circ \alpha_{m_1}(x), \beta_{m_2} \circ \alpha_{m_2}(x), \cdots\})$. By (e 7.3) and Proposition 2.7, Ψ is a strict embedding. Therefore there exists $s \in \mathbb{N}$ such that

$$\|\beta_{m_s} \circ \alpha_{m_s}(x)\| \geq \|x\| - \epsilon \ \text{ for all } \ x \in \mathcal{F}. \tag{e 7.4}$$

Set $F := F_{m_s}$, $\alpha := \alpha_{m_s}$, $\beta := \beta_{m_s}$, and $\gamma := \gamma_{m_s}$. The proposition follows.

□

Remark 7.8. Note that condition (3) in Proposition 7.7 implies that β is nonzero. Therefore, in the light of Proposition 7.7, in Definition 7.6, we may replace the condition that $\beta \neq 0$ by condition (3) in Proposition 7.7.

The following proposition is extracted from the proof of [72, Proposition 2.5] (see also [36, Lemma 3.7, Proposition 3.8]).

Proposition 7.9. *Let A be a C^*-algebra, let $n \in \mathbb{N} \cup \{0\}$, and let $0 < \epsilon < \frac{1}{2^{16}}$. Let $a_0, a_1 \in A_+$ be norm one positive elements. Suppose that F is a finite dimensional C^*-algebra, $\alpha : A \to F$ is a c.p.c. map, and $\beta : F \to A$ is a piecewise contractive n-decomposable c.p. map. If $\beta \circ \alpha(a_1)a_0 \approx_\epsilon \beta \circ \alpha(a_1)$, then there exist a C^*-subalgebra $\bar{F} \subset F$, a c.p.c. map $\bar{\alpha} : A \to \bar{F}$, and a piecewise contractive n-decomposable c.p. map $\bar{\beta} : \bar{F} \to \mathrm{Her}_A(a_0)$ such that, for any $x \in A_+$ with $x \leq a_1$, $\|\beta \circ \alpha(x) - \bar{\beta} \circ \bar{\alpha}(x)\| \leq 10(n+1)\epsilon^{1/8}$.*

Proof. Write $F = F_0 \oplus \cdots \oplus F_n$ such that each $\beta|_{F_i}$ is a c.p.c. order zero map ($i = 0, 1, \cdots, n$). Let $\chi(x) : [0, 1] \to \{0, 1\}$ be the characteristic function of the interval $[\epsilon^{1/2}, 1]$. Since F is a finite dimensional C^*-algebra, $p := \chi(\alpha(a_1))$ is a projection in F. Note that $p \leq \frac{1}{\epsilon^{1/2}}\alpha(a_1)$. Let $\bar{F} := pFp$. Then $\beta|_{\bar{F}}$ is still a piecewise contractive n-decomposable c.p. map. Moreover, for each i, pF_ip is a C^*-subalgebra of F_i with unit $p_i := p1_{F_i}p$. Thus $\beta|_{pF_ip}$ is also a c.p.c. order zero map. Moreover,

$$\begin{aligned}\|\beta|_{pF_ip}(p1_{F_i})(1 - a_0)\|^2 = \|(1 - a_0)\beta(p1_{F_i})^2(1 - a_0)\| &\leq \|(1 - a_0)\beta(p)(1 - a_0)\| \\ &\leq \frac{1}{\epsilon^{1/2}}\|(1 - a_0)\beta(\alpha(a_1))(1 - a_0)\| \leq \epsilon^{1/2} \leq \frac{1}{2^8}.\end{aligned}$$

Then, by [36, Lemma 3.6], there exists a c.p.c. order zero map $\bar{\beta}_i : pF_ip \to \mathrm{Her}_A(a_0)$ satisfying

$$\|\beta|_{pF_ip}(x) - \bar{\beta}_i(x)\| \leq 8\epsilon^{1/8} \ \text{ for all } \ x \in (pF_ip)^1_+. \tag{e 7.5}$$

Define $\bar{F} := pFp = pF_0p \oplus \cdots \oplus pF_np$, and define a c.p.c. map $\bar{\alpha} : A \to \bar{F}$, $x \mapsto p\alpha(x)p$, and define a c.p. map $\bar{\beta} : \bar{F} \to \mathrm{Her}_A(a_0)$, $x \mapsto \sum_{i=0}^n \bar{\beta}_i(p_ixp_i)$. Note that $\bar{\beta}|_{pF_ip} = \bar{\beta}_i$. Thus $\bar{\beta}$ is a piecewise contractive n-decomposable c.p. map. It follows, for $x \in A_+$ with $x \leq a_1 \leq 1$,

$$\|(1 - p)\alpha(x)\| = \|(1 - p)\alpha(x)^2(1 - p)\|^{1/2} \leq \|(1 - p)\alpha(a_1)(1 - p)\|^{1/2} \leq \epsilon^{1/4}. \tag{e 7.6}$$

Then

$$
\begin{aligned}
\beta\circ\alpha(x)-\bar\beta\circ\bar\alpha(x) &= \sum_{i=0}^{n}\beta(1_{F_i}\alpha(x)1_{F_i})-\bar\beta_i(p_i\bar\alpha(x)p_i)\\
&= \sum_{i=0}^{n}\beta(1_{F_i}\alpha(x)1_{F_i})-\bar\beta_i(1_{F_i}p\alpha(x)p1_{F_i})\\
\text{(by (e 7.5))}\quad &\approx_{8(n+1)\epsilon^{1/8}} \sum_{i=0}^{n}\beta(1_{F_i}\alpha(x)1_{F_i})-\beta(1_{F_i}p\alpha(x)p1_{F_i})\\
\text{(by (e 7.6) and } \beta|_{F_i} \text{ are c.p.c. maps)}\quad &\approx_{2(n+1)\epsilon^{1/4}} \sum_{i=0}^{n}\beta(1_{F_i}\alpha(x)1_{F_i})-\beta(1_{F_i}\alpha(x)1_{F_i})=0.
\end{aligned}
$$

□

Proposition 7.10. *Let A be a unital simple C^*-algebra with $\mathrm{Trdim_{nuc}}\,\mathrm{id}_A\le n$ for some integer $n\ge 0$. Then, for any finite subset $\mathcal{F}\subset A$, any $\epsilon>0$ and any $a\in A_+\backslash\{0\}$, there exist c.p.c. maps $\varphi: A\to A$ and $\gamma: A\to A\cap\varphi(A)^{\perp}$, a finite dimensional C^*-algebra F, a c.p.c. map $\alpha: A\to F$, and a piecewise contractive n-decomposable c.p. map $\beta: F\to \mathrm{Her}_A(\varphi(1_A))$ such that*

(1) $x\approx_\epsilon \gamma(x)+\varphi(x)$ for all $x\in\mathcal{F}$,

(2) $\varphi(1_A)$ and $\gamma(1_A)$ are projections and $1_A=\gamma(1_A)+\varphi(1_A)$,

(3) $\gamma(1_A)\lesssim_A a$,

(4) $\|\varphi-\beta\circ\alpha\|\le\epsilon$, and

(5) φ is an $(\mathcal{F},\epsilon)$-approximate embedding.

Proof. Without loss of generality, one may assume that $\mathcal{F}\subset A^1$. Let $\delta:=\min\{\frac{1}{2^{10}},(\frac{\epsilon}{10})^2\}$. Since $\mathrm{Trdim_{nuc}id_A}\le \mathrm{n}$, there exist a finite dimensional C^*-algebra F_1, a c.p.c. map $\alpha: A\to F_1$, a piecewise contractive n-decomposable c.p. map $\beta': F_1\to A$, and a c.p.c. map $\gamma': A\to A\cap\beta'(F_1)^{\perp}$ such that

($1'$) $x\approx_\delta \gamma'(x)+\beta'\circ\alpha(x)$ for all $x\in\mathcal{F}\cup(\mathcal{F}\cdot\mathcal{F})\cup\{1_A\}$,

($2'$) $\gamma'(1_A)\lesssim_A a$, and

($3'$) $\|\beta'\circ\alpha(x)\|\ge\|x\|-\delta$ for all $x\in\mathcal{F}$.

Since ($1'$) holds for $x=1_A$ and $\gamma'(1_A)\in\beta'(F_1)^{\perp}$, one has

$$\beta'\circ\alpha(1_A)\approx_\delta(\beta'\circ\alpha(1_A))^2. \tag{e 7.7}$$

It follows from Lemma 2.12 that there is a c.p. map $\varphi: A\to A$ such that, $p:=\varphi(1_A)$ is a projection in $C^*(\beta'\circ\alpha(1_A))$ and

$$\|\varphi(x)-\beta'\circ\alpha(x)\|\le 5\delta^{1/2}|x\| \ \text{ for all } \ x\in A. \tag{e 7.8}$$

By ($1'$) again, one has $\gamma'(1_A)\approx_\delta\gamma'(1_A)^2$. Applying Lemma 2.12 again, one also obtains a c.p.c. map $\gamma: A\to A$ such that $q:=\gamma(1_A)$ is a projection in $C^*(\gamma'(1_A))$ and

$$\|\gamma(x)-\gamma'(x)\|\le 5\delta^{1/2}\|x\| \ \text{ for all } \ x\in A. \tag{e 7.9}$$

Since $\gamma'(1_A)\beta'\circ\alpha(1_A)=0$, it follows that $qp=0$. By ($1'$), (e 7.8), (e 7.9), and the choice of δ, $p+q=1_A$. It follows that $\beta'\circ\alpha(A)\subset pAp$. Let $F=\overline{\alpha(1_A)F_1\alpha(1_A)}^{\|\cdot\|}$ and $\beta=\beta'|_{F_1}$. Then F is a finite dimensional C^*-algebra and β maps F into $\mathrm{Her}(\varphi(1_A))=pAp$. Note that β is also a piecewise contractive n-decomposable c.p. map.

By ($1'$), (e 7.8), and (e 7.9), and by the choice of δ, one checks that (1) and (4) hold. Since $p+q=1_A$, (2) also holds. Since $\gamma(1_A)\in C^*(\gamma'(1_A))$, by ($2'$), one concludes that (3) holds.

By $(1')$, since the image of γ' is in $B \cap \varphi(F_1)^{\perp}$, one has

$$\begin{aligned}\gamma'(x)\gamma'(y) + \beta' \circ \alpha(x)\beta' \circ \alpha(y) &= (\gamma'(x) + \beta' \circ \alpha(x))(\gamma'(y) + \beta' \circ \alpha(y)) \\ &\approx_{\delta(1+\delta)} x(\gamma'(y) + \beta' \circ \alpha(y)) \\ &\approx_{\delta} xy \approx_{\delta} \gamma'(xy) + \beta' \circ \alpha(xy) \quad \text{for all } x, y \in \mathcal{F}.\end{aligned} \tag{e 7.10}$$

Using the fact that the image of γ' is in $B \cap \varphi(F_1)^{\perp}$ again, one obtains

$$\beta' \circ \alpha(x)\beta' \circ \alpha(y) \approx_{\delta(2+\delta)} \beta' \circ \alpha(xy) \text{ for all } x, y \in \mathcal{F}. \tag{e 7.11}$$

In other words, $\beta' \circ \alpha$ is $(\mathcal{F}, \delta(2+\delta))$-multiplicative. By (e 7.8) and the choice of δ, one checks that φ is $(\mathcal{F}, \epsilon)$-multiplicative. Finally, for any $x \in \mathcal{F}$, by (e 7.8) and $(3')$,

$$\|\varphi(x)\| \approx_{5\delta^{1/2}} \|\beta' \circ \alpha(x)\| = \|\beta \circ \alpha(x)\| \approx_{\delta} \|x\|. \tag{e 7.12}$$

Hence (5) holds.

□

Proposition 7.11. *Let A be a simple unital C^*-algebra and let $n \in \mathbb{N} \cup \{0\}$. If A is asymptotically tracially in $\mathcal{N}_n$, then* $\mathrm{Trdim_{nuc}id}_A \le n$.

Proof. Let $\mathcal{F} \subset A^1$ be a finite subset, let $\epsilon > 0$ and let $a \in A_+ \backslash \{0\}$. We may assume that $1_A \in \mathcal{F}$. Let $\delta := \frac{\min\{1,\epsilon\}}{n+5}$. Since A is asymptotically tracially in $\mathcal{N}_n$, by Proposition 3.8, there exist a unital C^*-algebra B with $\dim_{\mathrm{nuc}} B \le n$, and c.p.c. maps $\beta_i : B \to A$, u.c.p. maps $\alpha' : A \to B$, and $\gamma_i : A \to A \cap \beta_i(B)^{\perp}$ $(i \in \mathbb{N})$ such that

(1) $x \approx_{\delta} \gamma_i(x) + \beta_i \circ \alpha'(x)$ for all $x \in \mathcal{F}$ and for all $i \in \mathbb{N}$,
(2) α' is an $(\mathcal{F}, \delta)$-approximate embedding,
(3) $\lim_{i\to\infty} \|\beta_i(xy) - \beta_i(x)\beta_i(y)\| = 0$ and $\lim_{i\to\infty} \|\beta_i(x)\| = \|x\|$ for all $x, y \in B$, and
(4) $\gamma_i(1_A) \lesssim_A a$ for all $i \in \mathbb{N}$.

Since $\dim_{\mathrm{nuc}} B \le n$, there exist a finite dimensional C^*-algebra F and a c.p.c. map $\varphi : B \to F$, and a piecewise contractive n-decomposable c.p. map $\psi : F \to B$, such that

$$x \approx_{\delta} \psi \circ \varphi(x) \text{ for all } x \in \alpha'(\mathcal{F}). \tag{e 7.13}$$

By condition (3), there exists $m \in \mathbb{N}$ such that $\|\beta_m \circ \alpha'(x)\| > \|\alpha'(x)\| - \delta$ for all $x \in \mathcal{F}$ and $\beta_m \circ \psi : F \to \mathrm{Her}_A(\beta_m(B))$ is an $(n, \Delta(F, \delta))$-dividable c.p. map, where $\Delta(-,-)$ is defined in Definition 7.5. Then, by the definition of $\Delta(F, \delta)$ and Proposition 7.4, there exists a piecewise contractive n-decomposable c.p. map $\beta : F \to \mathrm{Her}_A(\beta_m(B))$ such that

$$\|\beta - \beta_m \circ \psi\| \le \delta. \tag{e 7.14}$$

Set $\gamma = \gamma_m$ and $\alpha = \varphi \circ \alpha'$. Then, by (1), (e 7.13), and (e 7.14), we have

$$x \approx_{2\delta} \gamma(x) + \beta \circ \alpha(x) \text{ for all } x \in \mathcal{F}.$$

Moreover, $\gamma(A) \perp \beta(F)$ and (by (4)) $\gamma(1_A) \lesssim a$.

It remains to show that $\beta \neq 0$. By (2) and the choice of m, we have $\|\beta_m \circ \alpha'(1_A)\| \ge 1 - 2\delta$. Then

$$\|\beta \circ \varphi \circ \alpha'(1_A)\| \overset{(\mathrm{e}\,7.14)}{\approx_{\delta}} \|\beta_m \circ \psi \circ \varphi \circ \alpha'(1_A)\| \overset{(\mathrm{e}\,7.13)}{\approx_{\delta}} \|\beta_m \circ \alpha'(1_A)\| \ge 1 - 2\delta.$$

Thus $\beta \neq 0$.

□

The proof of the following proposition is almost the same as the proof for finite nuclear dimension case, see [72, Proposition 2.5].

Proposition 7.12. *Let A be a simple unital C^*-algebra with* $\mathrm{Trdim_{nuc}}\,\mathrm{id_A} \le n$ *for some integer n and let $B \subset A$ be a unital hereditary C^*-subalgebra. Then* $\mathrm{Trdim_{nuc}}\,\mathrm{id}_B \le n$.

Proof. Let $\mathcal{F} \subset B_+^1$ be a finite subset with $1_B \in \mathcal{F}$, let $\epsilon > 0$ and let $b \in B_+\backslash\{0\}$. Choose $\eta > 0$ such that

$$((1+\eta)\eta)^{1/2} < 1/2^{16} \quad \text{and} \quad 10(n+1)((1+\eta)\eta)^{1/16} + 2\eta^{1/2} < \varepsilon. \tag{e 7.15}$$

Since $\mathrm{Trdim_{nuc}}\,\mathrm{id}_A \le n$, there exist a finite dimensional C^*-algebra F, a c.p.c. map $\alpha : A \to F$, a piecewise contractive n-decomposable c.p. map $\beta : F \to A$, and a c.p.c. map $\gamma : A \to A \cap \beta(F)^\perp$ such that

(1) $x \approx_\eta \gamma(x) + \beta \circ \alpha(x)$ for all $x \in \mathcal{F}$,

(2) $\gamma(1_A) \lesssim_A b$, and

(3) $\|\beta \circ \alpha(x)\| \ge \|x\| - \eta$ for all $x \in \mathcal{F}$ (see Remark 7.8).

Since $\gamma(A) \perp \beta(F)$, by (1), $\|\beta \circ \alpha(1_B)\| \le (1+\eta)$. It follows that $\beta \circ \alpha(1_B)^2 \le (1+\eta)\beta \circ \alpha(1_B)$. Therefore

$$\begin{aligned}
\|(1_A - 1_B)\beta \circ \alpha(1_B)\|^2 &= \|(1_A - 1_B)\beta \circ \alpha(1_B)^2(1_A - 1_B)\| && \text{(e 7.16)}\\
&\le (1+\eta)\|(1_A - 1_B)\beta \circ \alpha(1_B)(1_A - 1_B)\| && \text{(e 7.17)}\\
&\le (1+\eta)\|(1_A - 1_B)(\beta \circ \alpha(1_B) + \gamma(1_B))(1_A - 1_B)\| && \text{(e 7.18)}\\
&\le (1+\eta)(\|(1_A - 1_B)1_B(1_A - 1_B)\| + \eta) = (1+\eta)\eta. && \text{(e 7.19)}
\end{aligned}$$

Since γ is a c.p.c. map, a similar but simpler estimate shows that

$$\|(1_A - 1_B)\gamma(x)\|^2 \le \eta \ \text{ for all } \ x \in \mathcal{F}. \tag{e 7.20}$$

By the choice of η and by Proposition 7.9 (letting $a_0 = a_1 = 1_B$), there exists a C^*-subalgebra $\bar{F} \subset F$, a c.p.c. map $\bar{\alpha} : A \to \bar{F}$, and a piecewise contractive n-decomposable c.p. map $\bar{\beta} : \bar{F} \to \mathrm{Her}_A(1_B) = B$ such that, for any $x \in B_+^1$,

$$\|\beta \circ \alpha(x) - \bar{\beta}\bar{\alpha}(x)\| \le 10(n+1)((1+\eta)\eta)^{1/16}. \tag{e 7.21}$$

By (e 7.21) and (3) and the choice of η, for $x \in \mathcal{F}$, we have

$$\|\bar{\beta}\bar{\alpha}(x)\| \ge \|x\| - \epsilon.$$

Define a c.p.c. map $\bar{\gamma} : B \to B$, $x \mapsto 1_B\gamma(x)1_B$. Then $\bar{\gamma}(1_B) \lesssim_A \gamma(1_B) \le \gamma(1_A) \lesssim_A b$. Since B is hereditary C^*-subalgebra of A, we have $\bar{\gamma}(1_B) \lesssim_B b$.

Finally, for $x \in \mathcal{F}$, by (e 7.20), $\gamma(x) \approx_{2\eta^{1/2}} 1_B\gamma(x)1_B = \bar{\gamma}(x)$ for all $x \in \mathcal{F}$. Therefore

$$x \approx_\eta \gamma(x) + \beta \circ \alpha(x) \approx_{10(n+1)((1+\eta)\eta)^{1/16} + 2\eta^{1/2}} \bar{\gamma}(x) + \bar{\beta}\bar{\alpha}(x) \ \text{ for all } \ x \in \mathcal{F}.$$

Note that $10(n+1)((1+\eta)\eta)^{1/16} + 2\eta^{1/2} < \varepsilon$. It follows that $\mathrm{Trdim_{nuc}}\,\mathrm{id}_B \le n$.

□

Proposition 7.13 (cf. [22, Proposition 3.4])**.** *Let A be a unital C^*-algebra and let $X \subset A_+$ be a finite subset. Suppose that, for each $x \in X$, $f_{1/2}(x)$ is full in A. Then, there exist $\sigma(\mathcal{G}_X) > 0$ and a finite subset $\mathcal{G}_X \subset A$ such that, for any unital C^*-algebra B and any u.c.p. map $\psi : A \to B$ which is $(\mathcal{G}_X, \sigma(\mathcal{G}_X))$-multiplicative, $f_{1/2}(\psi(x))$ is a full element of B for each $x \in X$.*

The following lemma is a construction of simple generalized inductive limit of C^*-algebras.

Lemma 7.14. *Let $\{A_i\}$ be a sequence of unital separable C^*-algebras and let $\varphi_i : A_i \to A_{i+1}$ be u.c.p. maps ($i \in \mathbb{N}$). Let $X_i = \{x_{i,1}, x_{i,2}, \cdots\} \subset A^1_{i+}$ be a countable dense subset of A^1_{i+}, $X_{i,k} := \{x_{i,1} \cdots, x_{i,k}\}$ ($i, k \in \mathbb{N}$), and $Y_k := \cup_{1\le i\le k}\varphi_{i,k}(X_{i,k})$. Then (A_i, φ_i) forms a generalized inductive system and $\lim_i(A_i, \varphi_i)$ is simple, if the following hold for any $k \in \mathbb{N}$:*

(1) $f_{1/2}(a)$ is full in A_k for all $a \in \mathcal{F}_k := Y_k \cap B_{\frac{3}{4},1}(A_k)$ (recall Notation 2.1), and

(2) φ_k is ϵ_k-multiplicative on $Y_k \cup (\cup_{1\le j\le k}\varphi_{j,k}(\mathcal{G}_{\mathcal{F}_k}))$, where

$$\epsilon_k := \frac{1}{4^k} \min_{1\le j\le k}\{1, \sigma(\mathcal{G}_{\mathcal{F}_j})\}$$

(see Proposition 7.13 for $\mathcal{G}_{\mathcal{F}_j}$ and $\sigma(\mathcal{G}_{\mathcal{F}_j})$, see Notation 6.2 for $\varphi_{j,k}$).

Proof. First we show that (A_i, φ_i) forms a generalized inductive limit. Let $k \in \mathbb{N} \cup \{0\}$, $y_1, y_2 \in A^1_{k+}\backslash\{0\}$ and $\epsilon > 0$. Then there exist $t_1, t_2 \in \mathbb{N}$ such that $y_1 \approx_{\epsilon/4} x_{k,t_1}$ and $y_2 \approx_{\epsilon/4} x_{k,t_2}$. Note that $\sum_{i=1}^\infty \epsilon_i < \infty$. Thus there is $m > \max\{k, t_1, t_2\}$ such that $\sum_{i=m}^\infty \epsilon_i < \epsilon/4$. Then, for all $j > m$, by the choice of Y_j, we have $\varphi_{k,j}(x_{k,t_1})$, $\varphi_{k,j}(x_{k,t_2}) \in Y_j$. By (2), for all $i \ge m$, φ_i is ϵ_i-multiplicative on $\{\varphi_{k,i}(x_{k,t_1}), \varphi_{k,i}(x_{k,t_2})\}$. Hence $\varphi_{m,j}$ is $\sum_{i=m}^{j-1}\epsilon_i$-multiplicative on $\{\varphi_{k,m}(x_{k,t_1}), \varphi_{k,m}(x_{k,t_2})\}$. Then, for all $j \ge m$,

$$\begin{aligned}
\varphi_{k,j}(y_1)\cdot\varphi_{k,j}(y_2) &= \varphi_{m,j}(\varphi_{k,m}(y_1))\cdot\varphi_{m,j}(\varphi_{k,m}(y_2)) \\
&\approx_{\epsilon/4} \varphi_{m,j}(\varphi_{k,m}(x_{k,t_1}))\cdot\varphi_{m,j}(\varphi_{k,m}(x_{k,t_2})) \\
&\approx_{\sum_{i=m}^{j-1}\epsilon_i} \varphi_{m,j}(\varphi_{k,m}(x_{k,t_1})\cdot\varphi_{k,m}(x_{k,t_2})) \\
&\approx_{\epsilon/4} \varphi_{m,j}(\varphi_{k,m}(y_1)\cdot\varphi_{k,m}(y_2)).
\end{aligned}$$

By the choice of m, we have $\varphi_{k,j}(y_1)\cdot\varphi_{k,j}(y_2) \approx_\epsilon \varphi_{m,j}(\varphi_{k,m}(y_1)\cdot\varphi_{k,m}(y_2))$ for all $j \ge m$. By Lemma 6.3, (A_i, φ_i) forms a generalized inductive system.

Now we show that $A := \lim_i(A_i, \varphi_i)$ is simple. It suffices to show that every norm one positive element of A is full. Let $a \in A_+$ with $\|a\| = 1$. Then there exist $k, s \in \mathbb{N}$ such that $\|a - \varphi_{k,\infty}(x_{k,s})\| < 1/4$. Let $r > \max\{k, s\}$ be such that $\|\varphi_{k,r}(x_{k,s})\| \ge 3/4$. Then we have $\varphi_{k,r}(x_{k,s}) \in \mathcal{F}_r := Y_r \cap B_{\frac{3}{4},1}(A_r)$. Condition (2) shows that, for all $j > r$, $\varphi_{r,j}$ is $\sum_{i=r}^{j-1}\epsilon_i$-multiplicative on $\mathcal{G}_{\mathcal{F}_r}$. By the choice of ϵ_i ($i \in \mathbb{N}$), the map $\varphi_{r,\infty}$ is $\sigma(\mathcal{G}_{\mathcal{F}_r})$-multiplicative on $\mathcal{G}_{\mathcal{F}_r}$. Then, by Proposition 7.13, $f_{1/2}(\varphi_{k,\infty}(x_{k,s})) = f_{1/2}(\varphi_{r,\infty}(\varphi_{k,r}(x_{k,s})))$ is a full element of A. Since $\|a - \varphi_{k,\infty}(x_{k,s})\| < 1/4$, by [54, Proposition 2.2], $f_{1/2}(\varphi_{k,\infty}(x_{k,s})) = c^*ac$ for some $c \in A$. Thus a is also a full element of A. Since a is arbitrary, so A is simple.

□

The following is a construction of simple separable unital finite nuclear dimension C^*-algebras using generalized inductive limits.

Lemma 7.15. *Let $n \in \mathbb{N} \cup \{0\}$. Let $\{A_i\}$ be a sequence of unital separable C^*-algebras and $\varphi_i : A_i \to A_{i+1}$ be u.c.p. maps ($i \in \mathbb{N}$). Let $X_i = \{x_{i,1}, x_{i,2}, \cdots\} \subset A^1_{i+}$ be a countable dense subset of A^1_{i+}, let $X_{i,k} := \{x_{i,1}, x_{i,2}, \cdots, x_{i,k}\}$, and let $Y_k := \cup_{1\le j\le k}\varphi_{j,k}(X_{j,k})$ ($i, k \in \mathbb{N}$). Let $F_0 = \mathbb{C}$ and let $\beta_0 : F_0 \to A_1$ be the zero map. Then (A_i, φ_i) forms a generalized inductive limit and $A := \lim_i(A_i, \varphi_i)$ is simple with $\dim_{\mathrm{nuc}} A \le n$, if the following hold for all $k \in \mathbb{N}$:*

(1) For all $a \in \mathcal{F}_k := Y_k \cap B_{\frac{3}{4},1}(A_k)$, $f_{1/2}(a)$ is full in A_k,

(2) there exist a finite dimensional C^*-algebra F_k, a c.p.c. map $\alpha_k : A_k \to F_k$, and a piecewise contractive n-decomposable c.p. map $\beta_k : F_k \to A_{k+1}$ such that $\varphi_k(x) \approx_{\frac{1}{k}} \beta_k \circ \alpha_k(x)$ for all $x \in Y_k$, and

(3) φ_k is ϵ_k-multiplicative on

$$Y_k \cup \left(\cup_{1\le j\le k}\varphi_{j,k}(\mathcal{G}_{\mathcal{F}_j})\right) \cup \left(\cup_{1\le j\le k}\varphi_{j,k}(\beta_{j-1}(\mathcal{G}^{F_{j-1}}))\right),$$

where

$$\epsilon_k := \frac{1}{4^k}\min_{1\le j\le k}\{1, \sigma(\mathcal{G}_{\mathcal{F}_j}), \Delta(F_{j-1}, \frac{1}{j})\} \text{ and}$$

$\mathcal{G}^{F_{j-1}}$ is the standard generating set of F_{j-1} in F^1_{j-1} (see Proposition 7.13 for $\mathcal{G}_{\mathcal{F}_j}$ and $\sigma(\mathcal{G}_{\mathcal{F}_j})$, see Definition 7.5 for $\Delta(-,-)$, and see Notation 6.2 for $\varphi_{j,k}$).

Proof. By Lemma 7.14, (A_i, φ_i) forms a generalized inductive system and $A := \lim_{i\to\infty}(A_i, \varphi_i)$ is a simple C^*-algebra.

To show $\dim_{\mathrm{nuc}} A \le n$, let $i \in \mathbb{N}$, $\epsilon > 0$, and $\mathcal{F} \subset A^1_{i+}$ be a finite subset. By the definition of Y_j, there exists $m \ge i + 1 + \frac{4}{\epsilon}$ such that $\varphi_{i,m}(\mathcal{F}) \subset_{\epsilon/4} Y_m$.

By (3), $\varphi_{m+1,\infty}$ is $\sum_{j=m+1}^{\infty}\epsilon_j$-multiplicative on $\beta_m(\mathcal{G}^{F^1_m})$. By the choice of ϵ_j, one has $\sum_{j=m+1}^{\infty}\epsilon_j \le \Delta(F_m, \frac{1}{m})$. Then $\varphi_{m+1,\infty}\circ\beta_m$ is an $(n, \delta_1(\frac{1}{m}))$-dividable map (see Definition 7.5 and part (2) of Proposition 7.4). By Proposition 7.4, there exists a piecewise contractive n-decomposable c.p. map $\beta : F_m \to A$ such that

$$\|\beta - \varphi_{m+1,\infty}\circ\beta_m\| \le 1/m. \tag{e 7.22}$$

For any $x \in \mathcal{F}$, there exists $y \in Y_m$ such that $\varphi_{i,m}(x) \approx_{\epsilon/4} y$. Then

$$\begin{aligned}\varphi_{i,\infty}(x) &= \varphi_{m,\infty}(\varphi_{i,m}(x)) \approx_{\frac{\epsilon}{4}} \varphi_{m,\infty}(y) \overset{(2)}{\approx_{\frac{\epsilon}{4}}} \varphi_{m+1,\infty}\circ\beta_m\circ\alpha_m(y) \overset{(\mathrm{e}\,7.22)}{\approx_{\frac{\epsilon}{4}}} \beta\circ\alpha_m(y)\\ &\approx_{\frac{\epsilon}{4}} \beta\circ\alpha_m\circ\varphi_{i,m}(x).\end{aligned} \tag{e 7.23}$$

Then, by Theorem 6.5 (with $\alpha_m\circ\varphi_{i,m}$ in place of α), $\dim_{\mathrm{nuc}} A \le n$.

□

Theorem 7.16. *Let $n \in \mathbb{N}\cup\{0\}$. Let A be a simple separable unital infinite dimensional C^*-algebra and $\mathrm{Trdim_{nuc}}\,\mathrm{id}_A \le n$. Then A is asymptotically tracially in $\mathcal{N}_{n,s,s}$ (recall Definition 3.4 for the class $\mathcal{N}_{n,s,s}$).*

Proof. Let $\mathcal{F} \subset B_{\frac{3}{4},1}(A_+)$ be a finite subset with $1_A \in \mathcal{F}$, let $\epsilon \in (0,1)$, and let $a \in A_+\backslash\{0\}$ with $\|a\| = 1$. Since A is simple, unital and infinite dimensional, A is non-elementary. Thus there exist a sequence of norm one positive elements $a_0, a_1, \cdots, a_n, \cdots$ in $\mathrm{Her}_A(f_{1/2}(a))_+\backslash\{0\}$ such that $a_i \perp a_j$, $i \ne j$ (see Lemma 4.3).

Let $A_0 := A$. Let $\mathcal{F}_0 := \mathcal{F}$ and let $\epsilon_0 := \epsilon/8$. Since $\mathrm{Trdim_{nuc}id}_{A_0} \le n$, by Proposition 7.10, there exist two c.p.c. maps $\varphi_0 : A_0 \to A_0$, $\gamma_0 : A_0 \to A_0 \cap \varphi_0(A_0)^{\perp}$, and a finite dimensional C^*-algebra F_0, and a c.p.c. map $\alpha_0 : A_0 \to F_0$, and a piecewise contractive n-decomposable c.p. map $\beta_0 : F_0 \to \mathrm{Her}_{A_0}(\varphi_0(1_{A_0}))$ such that

(0,1) $x \approx_{\epsilon_0} \gamma_0(x) + \varphi_0(x)$ for all $x \in \mathcal{F}_0$,

(0,2) $\varphi_0(1_{A_0})$ and $\gamma_0(1_{A_0})$ are projections, and $1_{A_0} = \gamma_0(1_{A_0}) + \varphi_0(1_{A_0})$,

(0,3) $\gamma_0(1_{A_0}) \lesssim_{A_0} a_0$.

(0,4) $\|\varphi_0 - \beta_0\circ\alpha_0\| \le \epsilon_0$, and

(0,5) φ_0 is an $(\mathcal{F}_0, \epsilon_0)$-approximate embedding.

Define $A_1 := \mathrm{Her}_{A_0}(\varphi_0(1_{A_0}))$. Note that A_1 is a simple separable unital non-elementary C^*-algebra, and there exists $\bar{a}_1 \in A_{1+}\backslash\{0\}$ such that $\bar{a}_1 \lesssim_A a_1$. There exists a norm one c.p.c. order zero map $\chi_1 : M_1 = \mathbb{C} \to A_1$. Let $Z_1 \subset \chi_1(M^1_1)$ be a finite subset which is a $\frac{1}{4}\Delta(M_1, 1)$-net of $\chi_1(M^1_1)$.

Let $X_1 = \{x_{1,1}, x_{1,2}, \cdots\} \subset A^1_{1+}$ be a countable dense subset of A^1_{1+} and let $X_{1,k} := \{x_{1,j} : 1 \le j \le k\}$, $k \in \mathbb{N}$. Set $Y_1 := \cup_{1\le i\le 1}\varphi_{i,1}(X_{i,1})= X_{1,1}$ (with $\varphi_{1,1} = \mathrm{id}_{A_1}$), $\bar{Z}_1 := Z_1$, and $Y_1' = Y_1 \cap B_{\frac{3}{4},1}(A_1)$. Note $f_{1/2}(b) \neq 0$ and (since A is simple) therefore is full in A_1 for all $b \in Y_1'$. Let $\mathcal{G}_1 := \mathcal{G}_{Y_1'}$ and $\sigma(\mathcal{G}_{Y_1'})$ be as in Proposition 7.13 associated with the set Y_1' (in place of X). Define

$$\mathcal{F}_1 := \varphi_0(\mathcal{F}_0) \cup Y_1 \cup \mathcal{G}_1 \cup \beta_0(\mathcal{G}^{F_0}) \cup \bar{Z}_1 \text{ and}$$
$$\epsilon_1 := \frac{1}{4}\min\{\sigma(\mathcal{G}_{Y_1'}), \Delta(F_0,1), \Delta(M_1,1), \epsilon/4, \}.$$

By Proposition 7.12, $\mathrm{Trdim_{nuc}id}_{A_1} \le n$. By Proposition 7.10, there exist two c.p.c. maps $\varphi_1 : A_1 \to A_1$, $\gamma_1 : A_1 \to A_1 \cap \varphi_1(A_1)^{\perp}$, a finite dimensional C^*-algebra F_1, a c.p.c. map $\alpha_1 : A_1 \to F_1$, and a piecewise contractive n-decomposable c.p. map $\beta_1 : F_1 \to \mathrm{Her}_{A_1}(\varphi_1(1_{A_1}))$ such that

(1,1) $x \approx_{\epsilon_1} \gamma_1(x) + \varphi_1(x)$ for all $x \in \mathcal{F}_1$,
(1,2) $\varphi_1(1_{A_1})$ and $\gamma_1(1_{A_1})$ are projections and $1_{A_1} = \gamma_1(1_{A_1}) + \varphi_1(1_{A_1})$,
(1,3) $\gamma_1(1_{A_1}) \lesssim_{A_1} \bar{a}_1$,
(1,4) $\|\varphi_1 - \beta_1 \circ \alpha_1\| \le \epsilon_1$, and
(1,5) φ_1 is an $(\mathcal{F}_1, \epsilon_1)$-approximate embedding.

Assume that, for $1 \le k \in \mathbb{N}$, we have constructed, for each $1 \le j \le k$, a hereditary C^*-subalgebra $A_j := \mathrm{Her}_{A_{j-1}}(\varphi_{j-1}(1_{A_{j-1}})) \subset A$, $\bar{a}_j \in A_{j+}\backslash\{0\}$ with $\bar{a}_j \lesssim_A a_j$, and $X_j = \{x_{j,1}, x_{j,2}, \cdots\} \subset A^1_{j+}$, $Y_j := \cup_{1\le i\le j}\varphi_{i,j}(X_{i,j})$ (see Notation 6.2 for $\varphi_{j,k}$), $Y_j' := Y_j \cap B_{\frac{3}{4},1}(A_j)$, $\sigma(\mathcal{G}_{Y_j'}) > 0$ and $\mathcal{G}_j := \mathcal{G}_{Y_j'}$ as in Proposition 7.13 associated with Y_j' (in place of X), a finite subset $Z_j \subset \chi_{k+1}(M^1_{k+1})$ which is a $\frac{1}{4}\Delta(M_j, 1/j)$-net of $\chi_j(M_j^1)$, $\bar{Z}_j := \cup_{1\le i\le j}\varphi_{i,j}(Z_i)$, and a norm one c.p.c. order zero map $\chi_j : M_j \to A_j$, a finite subset

$$\mathcal{F}_j := \varphi_{0,j}(\mathcal{F}_0) \cup Y_j \cup (\cup_{1\le i\le j}\varphi_{i,j}(\mathcal{G}_i)) \cup (\cup_{1\le i\le j}\varphi_{i,j}(\beta_{i-1}(\mathcal{G}^{F_{i-1}}))) \cup \bar{Z}_j \subset A_j, \qquad \text{(e 7.24)}$$

and

$$\epsilon_j = \frac{1}{4^j}\min_{1\le i\le j}\{\sigma(\mathcal{G}_{Y_i'}), \Delta(F_{i-1}, \frac{1}{i}), \Delta(M_i, \frac{1}{i}), \epsilon/4\} > 0, \ 1 \le j \le k \text{ (and } \varepsilon_0 = \varepsilon/8),$$

and there exist two c.p.c. maps $\varphi_j : A_j \to A_j$, $\gamma_j : A_j \to A_j \cap \varphi_j(A_j)^{\perp}$, a finite dimensional C^*-algebra F_j, a c.p.c. map $\alpha_j : A_j \to F_j$, and a piecewise contractive n-decomposable c.p. map $\beta_j : F_j \to \mathrm{Her}_{A_j}(\varphi_j(1_{A_j}))$ such that

$(j,1)$ $x \approx_{\epsilon_j} \gamma_j(x) + \varphi_j(x)$ for all $x \in \mathcal{F}_j$,
$(j,2)$ $\varphi_j(1_{A_j})$ and $\gamma_j(1_{A_j})$ are projections and $1_{A_j} = \gamma_j(1_{A_j}) + \varphi_j(1_{A_j})$,
$(j,3)$ $\gamma_j(1_{A_j}) \lesssim_{A_j} \bar{a}_j$,
$(j,4)$ $\|\varphi_j - \beta_j \circ \alpha_j\| \le \epsilon_j$, and
$(j,5)$ φ_j is an $(\mathcal{F}_j, \epsilon_j)$-approximate embedding.

Define $A_{k+1} := \mathrm{Her}_{A_k}(\varphi_k(1_{A_k}))$. Note that there exists $\bar{a}_{k+1} \in (A_{k+1})_+\backslash\{0\}$ such that $\bar{a}_{k+1} \lesssim_A a_{k+1}$. Also note that A_{k+1} is simple, separable, unital and non-elementary. Then, by [34, Proposition 4.10], there exists a norm one c.p.c. order zero map $\chi_{k+1} : M_{k+1} \to A_{k+1}$. Let $Z_{k+1} \subset \chi_{k+1}(M^1_{k+1})$ be a finite subset which is a $\frac{1}{4}\Delta(M_{k+1}, \frac{1}{k+1})$-net of $\chi_{k+1}(M^1_{k+1})$. Let $X_{k+1} = \{x_{k+1,1}, x_{k+1,2}, \cdots\} \subset (A_{k+1})^1_+$ be a countable dense subset of $(A_{k+1})^1_+$, and let $X_{k+1,i} := \{x_{k+1,j} : 1 \le j \le i\}$, $i \in \mathbb{N}$. Let $Y_{k+1} := \cup_{1\le j\le k+1}\varphi_{j,k+1}(X_{j,k+1})$ and $\bar{Z}_{k+1} := \cup_{1\le j\le k+1}\varphi_{j,k+1}(Z_j)$. Note that $f_{1/2}(b)$ is full in A_k. Set $Y'_{k+1} := Y_{k+1} \cap B_{\frac{3}{4},1}(A_{k+1})$, $\sigma(\mathcal{G}_{Y'_{k+1}}) > 0$, and finite subset $\mathcal{G}_{k+1} := \mathcal{G}_{Y'_{k+1}}$ be as in Proposition 7.13 associated with Y'_{k+1} (in place of X). Define

$$\mathcal{F}_{k+1} := \varphi_{0,k+1}(\mathcal{F}_0) \cup Y_{k+1} \cup (\cup_{1\le i\le k+1}\varphi_{i,k+1}(\mathcal{G}_i)) \cup (\cup_{1\le i\le k+1}\varphi_{i,k+1}(\beta_{i-1}(\mathcal{G}^{F_{i-1}}))) \cup \bar{Z}_{k+1}$$
$$\text{and } \epsilon_{k+1} := \frac{1}{4^{k+1}}\min_{1\le j\le k+1}\{\sigma(\mathcal{G}_{Y'_{k+1}}), \Delta(F_{j-1}, \frac{1}{j}), \Delta(M_j, \frac{1}{j}), \epsilon/4\} > 0. \qquad \text{(e 7.25)}$$

(Note $\mathcal{F}_{k+1}$ is a finite set.)

Note also $\mathrm{Trdim}_{\mathrm{nuc}}\mathrm{id}_{A_{k+1}} \le n$ (by Proposition 7.12). Then, by Proposition 7.10, there exist two c.p.c. maps $\varphi_{k+1} : A_{k+1} \to A_{k+1}$, $\gamma_{k+1} : A_{k+1} \to A_{k+1} \cap \varphi_{k+1}(A)^{\perp}$, a finite dimensional C^*-algebra F_{k+1}, and a c.p.c. map $\alpha_{k+1} : A_{k+1} \to F_{k+1}$, a piecewise contractive n-decomposable c.p. map $\beta_{k+1} : F_{k+1} \to \mathrm{Her}_{A_{k+1}}(\varphi_{k+1}(1_{A_{k+1}}))$ such that

$(k+1,1)$ $x \approx_{\epsilon_{k+1}} \gamma_{k+1}(x) + \varphi_{k+1}(x)$ for all $x \in \mathcal{F}_{k+1}$,

$(k+1,2)$ $\varphi_{k+1}(1_{A_{k+1}})$ and $\gamma_{k+1}(1_{A_{k+1}})$ are projections, and $1_{A_{k+1}} = \gamma_{k+1}(1_{A_{k+1}}) + \varphi_{k+1}(1_{A_{k+1}})$,

$(k+1,3)$ $\gamma_{k+1}(1_{A_{k+1}}) \lesssim_{A_{k+1}} \bar{a}_{k+1}$,

$(k+1,4)$ $\|\varphi_{k+1} - \beta_{k+1} \circ \alpha_{k+1}\| \le \epsilon_{k+1}$, and

$(k+1,5)$ φ_{k+1} is an $(\mathcal{F}_{k+1}, \epsilon_{k+1})$-approximate embedding.

Then, by induction, for each $k \in \mathbb{N}$, we obtain a hereditary C^*-subalgebra $A_k \subset A$, $\bar{a}_k \in A_{k+}\backslash\{0\}$ with $\bar{a}_k \lesssim_A a_k$, a norm one c.p.c. order zero map $\chi_k : M_k \to A_k$, a finite subset $\mathcal{F}_k \subset A_k$ satisfying (e 7.25), and $\epsilon_k > 0$ satisfying (e 7.25), and, there exist two c.p.c. maps $\varphi_k : A_k \to A_k$, $\gamma_k : A_k \to A_k \cap \varphi_k(A_k)^{\perp}$, a finite dimensional C^*-algebra F_k, a c.p.c. map $\alpha_k : A_k \to F_k$, and a piecewise contractive n-decomposable c.p. map $\beta_k : F_k \to \mathrm{Her}_{A_k}(\varphi_k(1_{A_k}))$ such that conditions $(k,1)$ to $(k,5)$ hold.

By Lemma 7.15 (see (k+1, 4) and (k+1, 5)), (A_k, φ_k) forms a generalized inductive system and $\bar{A} := \lim_k(A_k, \varphi_k)$ is a simple separable unital C^*-algebra which has nuclear dimension at most n.

Let us now show that $\bar{A}$ is infinite dimensional. For $4 \le k \in \mathbb{N}$ and for all $m \ge k$, by $(m,5)$ and the choice of $\bar{Z}_m$ and ϵ_m, the map $\varphi_{k,\infty}$ is $\frac{1}{4}\Delta(M_k, \frac{1}{k})$-multiplicative on Z_k. Since Z_k is $\frac{1}{4}\Delta(M_k, \frac{1}{k})$-net of $\chi_k(M_k^1)$, the composition $\varphi_{k,\infty} \circ \chi_k : M_k \to \bar{A}$ is $\Delta(M_k, \frac{1}{k})$-almost order zero. Then, by Proposition 7.4, and the definition of $\Delta(M_k, \frac{1}{k})$, there exists a c.p.c. order zero map $\bar{\chi}_k : M_k \to \bar{A}$ such that $\|\bar{\chi}_k - \varphi_{k,\infty} \circ \chi_k\| \le \frac{1}{k}$. By $(m,5)$, for $m \ge k$, we have $\|\varphi_{k,\infty} \circ \chi_k(1_{M_k})\| \ge 1 - \frac{1}{k} - \sum_{i=k}^{\infty} \epsilon_i \ge 1/2$, whence $\|\bar{\chi}_k\| \ge \|\varphi_{k,\infty} \circ \chi_k\| - \frac{2}{k} \ge 1 - \sum_{i=k}^{\infty} \epsilon_i - \frac{2}{k} > 0$. Thus $\bar{\chi}_k$ is nonzero. Since $\bar{A}$ admits nonzero c.p.c. order zero map $\bar{\chi}_k : M_k \to \bar{A}$ (for all $k \ge 4$), $\bar{A}$ must be infinite dimensional.

Note that $1_{\bar{A}} = \pi_\infty(\{\varphi_1(1_{A_1}), \varphi_2(1_{A_2}), \cdots\})$ and

$$\pi_\infty(1_A) - 1_{\bar{A}} = \pi_\infty(\{\gamma_1(1_{A_1}), \sum_{i=1}^{2} \gamma_i(1_{A_i}), \cdots\}).$$

Since, $\gamma_j(1_{A_j}) \lesssim \bar{a}_j \lesssim a_j$, and $a_i \perp a_j (i \neq j)$, for all $k \in \mathbb{N}$, $\sum_{i=1}^{k} \gamma_i(1_{A_i}) \lesssim \sum_{i=1}^{k} a_k \lesssim f_{1/2}(a)$. It follows

$$\pi_\infty(1_A) - 1_{\bar{A}} \lesssim_{l^\infty(A)/c_0(A)} a. \tag{e 7.26}$$

For $x \in \mathcal{F}$ and $k \in \mathbb{N}$, $x1_{A_k} \approx_{\epsilon_0} (\gamma_0(x) + \varphi_0(x))1_{A_k} = \varphi_0(x)1_{A_k} \approx_{\epsilon_1} (\gamma_1(\varphi_0(x)) + \varphi_1(\varphi_0(x)))1_{A_k} = \varphi_{0,2}(x)1_{A_k} \approx_{\epsilon_2} \cdots \approx_{\epsilon_{k-1}} \varphi_{0,k-1}(x)1_{A_k} = \varphi_{0,k-1}(x)$. Similarly, we have $1_{A_k}x \approx_{\sum_{i=0}^{k-1}\epsilon_i} \varphi_{0,k-1}(x)$. Thus $1_{A_k}x \approx_{\sum_{i=0}^{k-1}\epsilon_i} x1_{A_k}$. Note that $2\sum_{i=0}^{\infty} \epsilon_i < \epsilon$. Hence

$$1_{\bar{A}}\iota_A(x) \approx_\epsilon \iota_A(x)1_{\bar{A}}. \tag{e 7.27}$$

Moreover, $1_{A_k}x1_{A_k} \approx_{2\sum_{i=0}^{k-1}\epsilon_i} \varphi_{0,k-1}(x)$ implies

$$1_{\bar{A}}\iota_A(x)1_{\bar{A}} \approx_\epsilon \pi_\infty(\{\varphi_{0,1}(x), \varphi_{0,2}(x), \cdots\}) \in \bar{A}. \tag{e 7.28}$$

By Proposition 3.10 (see (e 7.27), (e 7.28) and (e 7.26)) A is asymptotically tracially in $\mathcal{N}_{n,s,s}$.

$\square$

Corollary 7.17. *Let A be a simple separable infinite dimensional unital C^*-algebra, then the following are equivalent:*

(1) A is asymptotically tracially in $\mathcal{N}_n$ for some $n \in \mathbb{N} \cup \{0\}$,

(2) $\mathrm{Trdim_{nuc}id}_A \le n$ for some $n \in \mathbb{N} \cup \{0\}$, and

(3) A is asymptotically tracially in $\mathcal{N}_{n,s,s}$ for some $n \in \mathbb{N} \cup \{0\}$.

Proof. Note that (3) $\Rightarrow$ (1) is automatic. (1) $\Rightarrow$ (2) follows from Proposition 7.11, and that (2) $\Rightarrow$ (3) follows from Theorem 7.16.

□

Theorem 7.18. *Let $n \in \mathbb{N} \cup \{0\}$. Let A be a simple separable unital infinite dimensional C^*-algebra and A is asymptotically tracially in $\mathcal{N}_n$. Then A is asymptotically tracially in $\mathcal{N}_{\mathcal{Z},s,s}$.*

Proof. This follows from Theorem 7.16 and [69, Theorem 7.1].

□

8 $\mathcal{Z}$-stable generalized inductive limits

The following notation is taken from [68] with a modification.

Notation 8.1. (cf. [68, Notation 2.2]) Let A be a unital C^*-algebra, $n \in \mathbb{N}$, $\epsilon \ge 0$, and let $\mathcal{F} \subset A$ be a finite subset. If $\psi : M_n \to A$ is a c.p.c. map and $v \in A^1$ such that

(i) $\|v^*v - (1_A - \psi(1_{M_n}))\| \le \epsilon$,

(ii) $\|vv^*\psi(e_{1,1}^{(n)}) - vv^*\| \le \epsilon$,

(iii) $\|[\psi(y), x]\| \le \epsilon$ for all $x \in \mathcal{F}$ and for all $y \in M_n^1$,

(iv) $\|[v, x]\| \le \epsilon$ for all $x \in \mathcal{F}$, and

(v) ψ is c.p.c. ϵ-almost order zero map (recall Definition 7.1),

then we say ψ and v satisfy the relation $\check{\mathcal{R}}_A(n, \mathcal{F}, \epsilon)$ or the pair (φ, v) satisfies the relation $\check{\mathcal{R}}_A(n, \mathcal{F}, \epsilon)$.

Lemma 8.2. *Let A be a unital C^*-algebra, $n \in \mathbb{N}$, $\epsilon > 0$, and let $\mathcal{F} \subset A$ be a finite subset. Suppose that a c.p.c. map $\psi : M_n \to A$ and $v \in A^1$ satisfy the relation $\check{\mathcal{R}}_A(n, \mathcal{F}, \epsilon)$. Suppose also that B is a unital C^*-algebra, $\varphi : A \to B$ is a u.c.p. map and $0 < \delta < \Delta(M_n, \epsilon)$ is a positive number (see Definition 7.5 for the definition of $\Delta(-,-)$). If φ is δ-multiplicative on $\mathcal{F} \cup \psi(\mathcal{G}^{M_n}) \cup \{v, v^*, vv^*\}$ (recall that $\mathcal{G}^{M_n}$ is the standard generating set of M_n, see Notation 2.4), then $\varphi \circ \psi$ and $\varphi(v)$ satisfy the relation $\check{\mathcal{R}}_B(n, \varphi(\mathcal{F}), 2\epsilon + 3\delta^{1/2})$.*

Proof. We verify this as follows.

(1) $\|\varphi(v)^*\varphi(v) - (1_B - \varphi \circ \psi(1_{M_n}))\| \approx_\delta \|\varphi(v^*v) - (\varphi(1_A) - \varphi \circ \psi(1_{M_n}))\| \le \varepsilon$ (see (i) of Notation 8.1).

$$\begin{aligned}(2)\ \|\varphi(v)\varphi(v)^*\varphi \circ \psi(e_{1,1}^{(n)}) - \varphi(v)\varphi(v)^*\| &\approx_{2\delta} \|\varphi(vv^*)\varphi \circ \psi(e_{1,1}^{(n)}) - \varphi(vv^*)\| \\ &\overset{(\text{Lemma 2.10})}{\approx_{\delta^{1/2}}} \|\varphi(vv^*\psi(e_{1,1}^{(n)})) - \varphi(vv^*)\| \\ &\overset{((\text{iii}) \text{ of Notation 8.1})}{\le} \varepsilon. \end{aligned} \tag{e 8.1}$$

(3) Let $x \in \mathcal{F}$ and $y \in M_n^1$, Then, by Lemma 2.10, $\varphi \circ \psi(y)\varphi(x) \approx_{\delta^{1/2}} \varphi(\psi(y)x)$. Similarly, $\varphi(x)\varphi \circ \psi(y) \approx_{\delta^{1/2}} \varphi(x\psi(y))$. Thus $\|[\varphi \circ \psi(y), \varphi(x)]\| \le \varepsilon + 2\delta^{1/2}$ (using (iii) of Notation 8.1).

(4) Let $x \in \mathcal{F}$, then $\|\varphi(v)\varphi(x) - \varphi(x)\varphi(v)\| \approx_{2\delta} \|\varphi(vx - xv)\| \le \varepsilon$ (using (iv) of Notation 8.1).

(5) By Definition 7.5 and (v) of Notation 8.1, $\varphi \circ \psi$ is 2ε-almost order zero map.

Thus $\varphi \circ \psi$, $\varphi(v)$ satisfy the relation $\check{\mathcal{R}}_B(n, \varphi(\mathcal{F}), 2\epsilon + 3\delta^{1/2})$.

□

Also recall the following proposition (with a mild modification):

Proposition 8.3 (cf. [68, Proposition 2.3]). *Let A be a separable unital C^*-algebra. Then A is $\mathcal{Z}$-stable if and only if the following condition holds: For any $n \in \mathbb{N}$, any finite subset $\mathcal{F} \subset A_+$ and any $0 < \epsilon < 1$, there are $m \in \mathbb{N}$, a c.p.c. map $\psi : M_{mn} \to A$ and $v \in A^1$ satisfying the relation $\check{\mathcal{R}}_A(mn, \mathcal{F}, \varepsilon)$.*

Proof. Note that if A is $\mathcal{Z}$-stable, then $\mathcal{Z}$ (hence the dimension drop algebra $\mathcal{Z}_{n,n+1}$) is unitally embedded into $(l^\infty(A)/c_0(A)) \cap A'$ (see [31, Theorem 8.7], see also [62, Theorem 2.2]). It follows from "(iv) $\Rightarrow$ (iii)" of [57, Proposition 5.1] that there is an order zero map $\Psi : M_n \to (l^\infty(A)/c_0(A)) \cap A'$ and $V \in (l^\infty(A)/c_0(A))^1$ satisfy condition (i), (ii) and (v) with $\varepsilon = 0$. There is a c.p.c. map $\Psi : M_n \to l^\infty(A)$ and there is a $\{v_n\} \in (l^\infty(A))^1$ such that, $\pi_\infty \circ \Psi = \Phi$ (see [67, Proposition 1.2.4]) and $\pi_\infty(\{v_n\}) = V$. Then the "only if" part follows.

For the "if" part, let $n \in \mathbb{N}$, let $\mathcal{F} \subset A_+$ be a finite subset, and let $0 < \epsilon < 1$. Choose $N := 1 + \max\{\|x\| : x \in \mathcal{F}\}$ and $\delta := \min\{\Delta(M_n, \epsilon/2N), (\varepsilon/4)^2\}$ (see Definition 7.5 for the definition of $\Delta(-,-)$). Then, by our assumption, there are $m \in \mathbb{N}$, a c.p.c. δ-almost order zero map $\psi : M_{mn} \to A$ and $v \in A^1$ satisfying the relation $\check{\mathcal{R}}_A(mn, \mathcal{F}, \delta)$.

Let $h : M_n \hookrightarrow M_{mn}$ be a unital embedding such that $e_{1,1}^{(mn)} \le h(e_{1,1}^{(n)})$. Then $\psi \circ h : M_n \to A$ is a c.p.c. δ-almost order zero map. By the choice of δ and the definition of $\Delta(M_n, \epsilon/2N)$, there exists a c.p.c. order zero map $\varphi : M_n \to A$ such that $\|\psi \circ h - \varphi\| \le \epsilon/2N$. Then one has

$$\begin{aligned}
\|vv^*\varphi(e_{1,1}^{(n)}) - vv^*\| &\approx_{\epsilon/2N} \|vv^*\psi \circ h(e_{1,1}^{(n)}) - vv^*\| = \|vv^*(1_A - \psi \circ h(e_{1,1}^{(n)}))^2 vv^*\|^{1/2} \\
&\le \|vv^*(1_A - \psi \circ h(e_{1,1}^{(n)}))vv^*\|^{1/2} \le \|vv^*(1_A - \psi(e_{1,1}^{(mn)}))vv^*\|^{1/2} \\
&\le \|vv^*(1_A - \psi(e_{1,1}^{(mn)}))\|^{1/2} \le \delta^{1/2}.
\end{aligned}$$

Thus φ, v satisfy (ii) in the relation $\check{\mathcal{R}}_A(n, \mathcal{F}, \varepsilon)$. One easily checks that φ and v also satisfy the rest terms in the relation $\check{\mathcal{R}}_A(n, \mathcal{F}, \varepsilon)$. Since φ is an order zero c.p.c. map, [68, Proposition 2.3] applies and A is $\mathcal{Z}$-stable.

□

Lemma 8.4. *Let A_i be a unital separable C^*-algebra and let $\varphi_i : A_i \to A_{i+1}$ be u.c.p. maps ($i \in \mathbb{N}$). Let $X_i = \{x_{i,1}, x_{i,2}, \cdots\} \subset A^1_{i+}$ be a countable dense subset of A^1_{i+}, let $X_{i,k} := \{x_{i,1}, x_{i,2}, \cdots, x_{i,k}\}$, and let $Y_k := \cup_{1 \le i \le k} \varphi_{i,k}(X_{i,k})$ ($i, k \in \mathbb{N}$). Set $A_0 = A_1$, $Y_0 = \{0\} \subset A_0$ and $\varphi_0 := \mathrm{id}_{A_0} : A_0 \to A_1$.*

Then the system (A_i, φ_i) forms a generalized inductive system and $A := \lim_i(A_i, \varphi_i)$ is a simple and $\mathcal{Z}$-stable C^*-algebra, if the following conditions hold for any $n \in \mathbb{N}$:

(1) $f_{1/2}(x)$ is full in A_n for all $x \in \mathcal{F}_n := Y_n \cap B_{\frac{3}{4},1}(A_n)$,

(2) there exist a c.p.c. map $\psi_n : M_{n!} \to A_n$ and $v_n \in A^1_n$ such that ψ_n and v_n satisfy the relation $\check{\mathcal{R}}_{A_n}(n!, \varphi_{n-1}(Y_{n-1}), \frac{1}{n!})$, and

(3) φ_n is ϵ_n-multiplicative on

$$Y_n \cup \left(\cup_{1 \le j \le n} \varphi_{j,n}(\mathcal{G}_{\mathcal{F}_j})\right) \cup \left(\cup_{1 \le j \le n} \left(\varphi_{j,n} \circ \psi_j(\mathcal{G}^{M_{j!}}) \cup \{\varphi_{j,n}(v_j), \varphi_{j,n}(v_j)^*, \varphi_{j,n}(v_j v_j^*)\}\right)\right),$$

where

$$\epsilon_n := \frac{1}{4^n} \min_{1 \le j \le n} \{1, \sigma(\mathcal{G}_{\mathcal{F}_j}), \Delta(M_{j!}, \frac{1}{j!})\} \text{ and}$$

$\mathcal{G}^{M_{j!}}$ is the standard generating set of $M_{j!}$ (see Proposition 7.13 for $\mathcal{G}_{\mathcal{F}_j}$ and $\sigma(\mathcal{G}_{\mathcal{F}_j})$, and see Notation 6.2 for $\varphi_{j,k}$).

Proof. By Lemma 7.14, (A_i, φ_i) forms a generalized inductive system and $A := \lim_i(A_i, \varphi_i)$ is a simple C^*-algebra. We will show that A is $\mathcal{Z}$-stable.

Let $\varepsilon > 0$, $n \in \mathbb{N}$ and let $\mathcal{F} \subset A_+^1$ be a finite subset. Then there exists $n_1 > n \in \mathbb{N}$ such that

$$\mathcal{F} \subset_{\frac{\epsilon}{16}} \varphi_{n_1,\infty}(Y_{n_1}). \tag{e 8.2}$$

Choose $n_2 > n_1$ such that $\frac{2}{n_2!} + 3(\sum_{i=n_2}^{\infty} \epsilon_i)^{1/2} < \varepsilon/8$. By our assumption, there exist a c.p.c. map $\psi_{n_2} : M_{n_2!} \to A_{n_2}$ and $v_{n_2} \in A_{n_2}^1$ such that

(1′) the pair (ψ_{n_2}, v_{n_2}) satisfies the relation $\check{\mathcal{R}}_{A_{n_2}}(n_2!, \varphi_{n_2-1}(Y_{n_2-1}), \frac{1}{n_2!})$, and

(2′) for any $k \geq n_2$, the map φ_k (from A_k to A_{k+1}) is ϵ_k-multiplicative on

$$Y_k \cup \varphi_{n_2,k} \circ \psi_{n_2}(\mathcal{G}^{M_{n_2!}}) \cup \{\varphi_{n_2,k}(v_{n_2}), \varphi_{n_2,k}(v_{n_2})^*, \varphi_{n_2,k}(v_{n_2}v_{n_2}^*)\}.$$

By (2′), for any $k \geq n_2$, $\varphi_{n_2,k}$ is $(Y_{n_2} \cup \psi_{n_2}(\mathcal{G}^{M_{n_2!}}) \cup \{v_{n_2}, v_{n_2}^*, v_{n_2}v_{n_2}^*\}, \sum_{i=n_2}^{k} \epsilon_i)$-multiplicative. Therefore $\varphi_{n_2,\infty}$ is $(Y_{n_2} \cup \psi_{n_2}(\mathcal{G}^{M_{n_2!}}) \cup \{v_{n_2}, v_{n_2}^*, v_{n_2}v_{n_2}^*\}, \sum_{i=n_2}^{\infty} \epsilon_i)$-multiplicative. Note that $\sum_{i=n_2}^{\infty} \epsilon_i < \Delta(M_{n_2!}, \frac{1}{n_2!})$. Then, by Lemma 8.2, the pair $(\varphi_{n_2,\infty} \circ \psi_{n_2}, \varphi_{n_2,\infty}(v_{n_2}))$ satisfies the relation

$$\check{\mathcal{R}}_A(n_2!, \varphi_{n_2,\infty}(\varphi_{n_2-1}(Y_{n_2-1})), \frac{2}{n_2!} + 3(\sum_{i=n_2}^{\infty} \epsilon_i)^{1/2}).$$

By (e 8.2), we have $\mathcal{F} \subset_{\frac{\epsilon}{16}} \varphi_{n_2,\infty}(\varphi_{n_2-1}(Y_{n_2-1}))$. Also note $\frac{2}{n_2!} + 3(\sum_{i=n_2}^{\infty} \epsilon_i)^{1/2} < \varepsilon/8$. Therefore the pair $(\varphi_{n_2,\infty}, \varphi_{n_2,\infty}(v_{n_2}))$ satisfies the relation $\check{\mathcal{R}}_A(n_2!, \mathcal{F}, \epsilon)$. Thus, by Proposition 8.3, A is $\mathcal{Z}$-stable.

□

Lemma 8.5. *Let A be a unital simple C^*-algebra which is asymptotically tracially in $\mathcal{C}_{\mathcal{Z},s}$ (see Definition 3.4). Then, for any finite subset $\mathcal{F} \subset A$, any $\epsilon > 0$, any $n \in \mathbb{N}$, and any $a \in A_+ \backslash \{0\}$, the following conditions hold.*

There exist a separable unital C^*-algebra B and a u.c.p. map $\alpha : A \to B$ such that

(1) α is an $(\mathcal{F}, \epsilon/2)$-approximate embedding, and

for any finite subset $\mathcal{G} \subset B$, there exist three c.p.c. maps $\beta : B \to A$, $\gamma : A \to (\beta \circ \alpha(A))^\perp$, $\psi : M_n \to \mathrm{Her}_A(\beta \circ \alpha(1_A))$, and $v \in \mathrm{Her}_A(\beta \circ \alpha(1_A))^1$ such that

(2) $\beta \circ \alpha(1_A)$, $\gamma(1_A)$ are projections and $1_A = \beta \circ \alpha(1_A) + \gamma(1_A)$,

(3) $x \approx_\epsilon \beta \circ \alpha(x) + \gamma(x)$ for all $x \in \mathcal{F}$,

(4) β is a $(\mathcal{G}, \epsilon)$-approximate embedding.

(5) $\gamma(1_A) \lesssim_A a$, and

(6) ψ and v satisfy the relation $\check{\mathcal{R}}_{\mathrm{Her}_A(\beta \circ \alpha(1_A))}(n, \beta \circ \alpha(\mathcal{F}), \varepsilon)$.

If, in addition, A is assumed to be asymptotically tracially in $\mathcal{N}$, then B above can be chosen to be nuclear.

Proof. Let $\mathcal{F} \subset A$ be a finite subset. Without loss of generality, we may assume that $\|x\| \leq 1$ for all $x \in \mathcal{F}$. Let $\epsilon \in (0, 1)$, let $n \in \mathbb{N}$, and let $a \in A_+ \backslash \{0\}$. Since A is simple, unital and asymptotically tracially in $\mathcal{C}_{\mathcal{Z},s}$, A is non-elementary. Then there exist $a_0, a_1 \in \mathrm{Her}_A(a)_+ \backslash \{0\}$ such that $a_0 a_1 = 0$. Let $\delta := \min\{(\epsilon/8)^2, \Delta(M_n, \epsilon/4), 1/2\}$.

By [62, Corollary 3.1], $\mathcal{C}_{\mathcal{Z},s}$ has property (H). Then, by Proposition 3.8, there exist a unital separable $\mathcal{Z}$-stable C^*-algebra $\bar{B}$ and c.p.c. maps $\bar{\alpha} : A \to \bar{B}$, $\bar{\beta}_i : \bar{B} \to A$, and $\bar{\gamma}_i : A \to A \cap (\bar{\beta}_i \circ \bar{\alpha}(A))^\perp$ $(i \in \mathbb{N})$ such that

(1′) $\bar{\alpha}(1_A) = 1_{\bar{B}}$, $\bar{\beta}_i(1_{\bar{B}})$ and $\bar{\gamma}_i(1_A)$ are projections, $1_A = \bar{\beta}_i(1_{\bar{B}}) + \bar{\gamma}_i(1_A)$ for all $i \in \mathbb{N}$,

(2′) $x \approx_\delta \bar{\gamma}_i(x) + \bar{\beta}_i \circ \bar{\alpha}(x)$ for all $x \in \mathcal{F}$ for all $i \in \mathbb{N}$,

(3′) $\bar{\alpha}$ is an $(\mathcal{F}, \delta)$-approximate embedding,

(4′) $\lim_{i\to\infty} \|\bar{\beta}_i(xy) - \bar{\beta}_i(x)\bar{\beta}_i(y)\| = 0$ and $\lim_{i\to\infty} \|\bar{\beta}_i(x)\| = \|x\|$ for all $x, y \in \bar{B}$, and

(5′) $\bar{\gamma}_i(1_A) \lesssim_A a_0$ for all $i \in \mathbb{N}$.

Since $\bar{B}$ is $\mathcal{Z}$-stable, by Proposition 8.3, there is a c.p.c. order zero map $\bar{\psi} : M_n \to \bar{B}$ and there is $\bar{v} \in \bar{B}^1$ such that

(6′) the pair $(\bar{\psi},\ \bar{v})$ satisfies the relation $\check{\mathcal{R}}_{\bar{B}}(n, \bar{\alpha}(\mathcal{F}), \varepsilon/8)$.

Set $B := \bar{B}$ and $\alpha := \bar{\alpha}$. Then, by (3′), (1) holds.

Let $\bar{\mathcal{G}} \subset \bar{B}$ be a finite subset containing $\bar{\psi}(\mathcal{G}^{M_n}) \cup \{\bar{v}, \bar{v}^*, \bar{v}\bar{v}^*\}$. By (4′) and (6′), for a sufficiently large $k \in \mathbb{N}$, the following (7′) and (8′) hold:

(7′) The map $\bar{\beta}_k$ is an $(\bar{\alpha}(\mathcal{F}) \cup \bar{\mathcal{G}}, \delta)$-approximate embedding.

(8′) The pair $(\bar{\beta}_k \circ \bar{\psi},\ \bar{\beta}_k(\bar{v}))$ satisfies the relation $\check{\mathcal{R}}_{\mathrm{Her}_A(\bar{\beta}_k(1_{\bar{B}}))}(n, \bar{\beta}_k(\bar{\alpha}(\mathcal{F})), \varepsilon/4)$.

Set $\mathcal{G} := \bar{\mathcal{G}}$, $\beta := \bar{\beta}_k$, $\gamma := \gamma_k$, $\psi := \bar{\beta}_k \circ \bar{\psi}$, $v := \bar{\beta}_k(\bar{v})$. Then, by (1′), (2) above holds, by (2′), (3) holds, by (7′), (4) holds, by (5′), (5) holds, and, by (8′), (6) holds. This proves the first part of the lemma.

If, in addition, A is also assumed to be asymptotically tracially in $\mathcal{N}$, then, by Proposition 3.8, $\mathrm{Her}_A(\bar{\beta}_k(1_{\bar{B}}))$ is simple and asymptotically tracially in $\mathcal{N}$. There exists $a_2 \in \mathrm{Her}_A(\bar{\beta}_k(1_{\bar{B}}))_+ \backslash \{0\}$ such that $a_2 \lesssim_A a_1$. Since $\mathrm{Her}_A(\bar{\beta}_k(1_{\bar{B}}))$ is asymptotically tracially in $\mathcal{N}$, by Proposition 3.8, there exist a unital nuclear C^*-algebra B and c.p.c. maps $\hat{\alpha} : \mathrm{Her}_A(\bar{\beta}_k(1_{\bar{B}})) \to B$, $\hat{\beta}_i : B \to \mathrm{Her}_A(\bar{\beta}_k(1_{\bar{B}}))$, and $\hat{\gamma}_i : \mathrm{Her}_A(\bar{\beta}_k(1_{\bar{B}})) \to \mathrm{Her}_A(\bar{\beta}_k(1_{\bar{B}})) \cap \hat{\beta}_i(B)^{\perp}$ $(i \in \mathbb{N})$ such that

(1″) $\hat{\alpha}$ is a u.c.p. map, $\hat{\beta}_i(1_B)$ and $\hat{\gamma}_i(\bar{\beta}_k \circ \bar{\alpha}(1_A))$ are projections, $\bar{\beta}_k(1_{\bar{B}}) = \hat{\beta}_i(1_B) + \hat{\gamma}_i(\bar{\beta}_k(1_{\bar{B}}))$ for all $i \in \mathbb{N}$,

(2″) $x \approx_{\delta} \hat{\gamma}_i(x) + \hat{\beta}_i \circ \hat{\alpha}(x)$ for all $x \in \bar{\beta}_k \circ \bar{\alpha}(\mathcal{F})$ and for all $i \in \mathbb{N}$,

(3″) $\hat{\alpha}$ is a $(\bar{\beta}_k \circ \bar{\alpha}(\mathcal{F}) \cup \bar{\beta}_k \circ \bar{\psi}(\mathcal{G}^{M_n}) \cup \{\bar{\beta}_k(v), \bar{\beta}_k(v)^*, \bar{\beta}_k(vv)^*\}, \delta)$-approximate embedding,

(4″) $\lim_{i\to\infty} \|\hat{\beta}_i(xy) - \hat{\beta}_i(x)\hat{\beta}_i(y)\| = 0$ and $\lim_{i\to\infty} \|\hat{\beta}_i(x)\| = \|x\|$ for all $x, y \in B$, and

(5″) $\hat{\gamma}_i(\bar{\beta}_k \circ \bar{\alpha}(1_A)) \lesssim_{\mathrm{Her}_A(\bar{\beta}_k(1_{\bar{B}}))} a_2$ for all $i \in \mathbb{N}$.

Let $\alpha := \hat{\alpha} \circ \bar{\beta}_k \circ \bar{\alpha}$. Then, since $\delta < \varepsilon/8$, by (3′) and (3″), (1) of the lemma holds. Let $\mathcal{G} \subset B$ be a finite subset. By (4″), there exits a large $m \in \mathbb{N}$ such that

(6″) $\hat{\beta}_m$ is a $(\mathcal{G} \cup \hat{\alpha} \circ \bar{\beta}_k \circ \bar{\alpha}(\mathcal{F}) \cup \hat{\alpha} \circ \bar{\beta}_k(\bar{\psi}(\mathcal{G}^{M_n})) \cup \{\hat{\alpha} \circ \bar{\beta}_k(v), \hat{\alpha} \circ \bar{\beta}_k(v)^*, \hat{\alpha} \circ \bar{\beta}_k(vv)^*\}, \delta)$-approximate embedding.

Then, by the choice of $\bar{\mathcal{G}}$, and by (7′), (3″), and (6″), the map $\hat{\beta}_m \circ \hat{\alpha} \circ \bar{\beta}_k$ is 3δ-multiplicative on $\bar{\alpha}(\mathcal{F}) \cup \bar{\psi}(\mathcal{G}^{M_n}) \cup \{\bar{v}, \bar{v}^*, \bar{v}\bar{v}^*\}$. Moreover, by (8′) and Lemma 8.2, we have

(7″) the pair $(\hat{\beta}_m \circ \hat{\alpha} \circ \bar{\beta}_k \circ \bar{\psi},\ \hat{\beta}_m \circ \hat{\alpha} \circ \bar{\beta}_k(\bar{v}))$ satisfies the relation $\check{\mathcal{R}}_{\mathrm{Her}_A(\hat{\beta}_m \circ \hat{\alpha} \circ \bar{\beta}_k(1_{\bar{B}}))}(n, \hat{\beta}_m \circ \hat{\alpha} \circ \bar{\beta}_k(\bar{\alpha}(\mathcal{F})), 5\delta)$.

Define $\beta := \hat{\beta}_m$, $\gamma := \bar{\gamma}_k + \hat{\gamma}_m \circ \bar{\beta}_k \circ \bar{\alpha}$, $\psi := \hat{\beta}_m \circ \hat{\alpha} \circ \bar{\beta}_k \circ \bar{\psi}$ and $v := \hat{\beta}_m \circ \hat{\alpha} \circ \bar{\beta}_k(\bar{v})$. Since $\bar{\gamma}_k(A) \perp \hat{\gamma}_m \circ \bar{\beta}_k \circ \bar{\alpha}(A)$, we have that $\gamma := \bar{\gamma}_k + \hat{\gamma}_m \circ \bar{\beta}_k \circ \bar{\alpha}$ is also a c.p.c. map. Then, by (1′) and (1″), (2) holds, by (2′) and (2″), (3) holds, by (6″), (4) holds, by (5′), (5″), and by the fact that $a_2 \lesssim_A a_1$ and $a_0 \perp a_1$, and $a_0 + a_1 \lesssim_A a$, (5) holds. Finally, by (7″), (6) holds.

□

The following lemma is well known.

Lemma 8.6. *Let A be a C^*-algebra and B be a nuclear C^*-algebra. If there exist c.p.c. maps $\alpha : A \to B$ and $\beta : B \to A$ such that $\beta \circ \alpha = \mathrm{id}_A$, then A is also nuclear.*

Proof. Let $\mathcal{F} \subset A$ be a finite subset and let $\varepsilon > 0$. Since B is nuclear, there exist a finite dimensional C^*-algebra F and two c.p.c. maps $\varphi : B \to F$, and $\psi : F \to B$ such that $\alpha(x) \approx_{\varepsilon/(\|\beta\|+1)} \psi \circ \varphi(\alpha(x))$ for all $x \in \mathcal{F}$. Note that $\varphi \circ \alpha : A \to F$ and $\beta \circ \psi : F \to A$ are c.p.c. maps. For any $x \in \mathcal{F}$, $x = \beta(\alpha(x)) \approx_{\varepsilon} \beta(\psi \circ \varphi(\alpha(x)))$. Thus A is nuclear. □

Theorem 8.7. *Let A be a simple separable unital C^*-algebra. Assume that A is asymptotically tracially in $\mathcal{C}_{\mathcal{Z},s}$. Then, for any finite subset $\mathcal{F} \subset A$, any $\epsilon > 0$, and any $a \in A_+ \backslash \{0\}$, there exists a unital C^*-subalgebra $B \subset l^{\infty}(A)/c_0(A)$ which is strictly embedded such that B in $\mathcal{C}_{\mathcal{Z},s,s}$, and*

(1) $1_B\iota_A(x)\approx_\epsilon \iota_A(x)1_B$ *for all* $x\in\mathcal{F}$,

(2) $1_B\iota_A(x)1_B\in_\epsilon B$ *and* $\|1_B\iota_A(x)1_B\|\geq\|x\|-\epsilon$ *for all* $x\in\mathcal{F}$, *and*

(3) $\iota_A(1_A)-1_B\lesssim_{l^\infty(A)/c_0(A)}\iota_A(a)$.

Moreover, if, in addition, A *is also asymptotically tracially in* $\mathcal{N}$, *then* A *is asymptotically tracially in* $\mathcal{N}_{\mathcal{Z},s,s}$ *(see Definition 3.4).*

Proof. Let $\mathcal{F}\subset B_{\frac{4}{5},1}(A_+)$ be a finite subset, let $\epsilon\in(0,1)$ and let $a\in A_+\backslash\{0\}$. We may assume that A is infinite dimensional. Since A is also simple and unital, we further assume that A is non-elementary. Then there exists a sequence of mutually orthogonal norm one positive elements $\{a_n\}$ in $\mathrm{Her}_A(a)_+\backslash\{0\}$.

Choose $A_0=A$ and $Y_0=\{0\}\subset A_0$. Let $\varphi_0=\mathrm{id}_{A_0}:A_0\to A_0$. Set $\epsilon_0:=\epsilon/100$, $\psi_1:M_1(=\mathbb{C})\to\mathrm{Her}_A(\varphi_0(1_{A_0}))$ the unital *-homomorphism, and set $v_1=1_A\in\mathrm{Her}_A(\varphi_0(1_{A_0}))(=A)$. Let $B_0=\mathbb{C}$, let $\beta_0:B_0\to A_0$ be the zero map, and let $W_{0,i}=\{0\}$ for all $i\in\mathbb{N}$.

We claim that, for each $k\in\mathbb{N}$, we can make the following choices:

(k - 1) A hereditary C^*-subalgebra $A_k=\mathrm{Her}_A(\varphi_{k-1}(1_{A_{k-1}}))\subset A$,

(k - 2) a positive element $\bar{a}_k\in(A_k)_+\backslash\{0\}$ such that $\bar{a}_k\lesssim_A a_k$,

(k - 3) a countable dense subset $X_k=\{x_{k,1}:=1_{A_k},x_{k,2},\cdots\}\subset A_k^1$, and finite subsets $X_{k,i}=\{1_{A_k}\}\cup\{x_{k,j}:1\leq j\leq i\}\subset A_k^1$ $(i\in\mathbb{N})$,

(k - 4) a finite subset $Y_k=\cup_{1\leq j\leq k}\varphi_{j,k}(X_{j,k})\cap B_{\frac{3}{4},1}(A_k)\subset A_k$,

(k - 5) a finite subset $G_{Y_k}\subset A_k$, and a positive number $\sigma(G_{Y_k})>0$ as in Proposition 7.13,

(k - 6) a finite subset $\mathcal{F}_k\subset A_k$ such that (see Notation 6.2 for notation $\varphi_{j,k}$)

$$\begin{aligned}\mathcal{F}_k\ &=\ \varphi_{1,k}(\mathcal{F})\cup Y_k\cup\big(\cup_{1\leq j\leq k}\varphi_{j,k}(\mathcal{G}_{Y_j})\big)\cup(\cup_{1\leq j\leq k}\varphi_{j,k}(\beta_{j-1}(W_{j-1,k})))\\ &\quad\cup(\cup_{1\leq j\leq k}\big(\varphi_{j,k}(\psi_j(\mathcal{G}^{M_{j!}}))\cup\{\varphi_{j,k}(v_j),\varphi_{j,k}(v_j)^*,\varphi_{j,k}(v_jv_j^*)\}\big),\end{aligned}\tag{e 8.3}$$

(k - 7) a positive number

$$\epsilon_k=\frac{1}{4^k}\min_{1\leq j\leq k}\{\sigma(\mathcal{G}_{Y_j}),\Delta(M_{j!},\frac{1}{j!}),\frac{\varepsilon}{100}\},\tag{e 8.4}$$

(k - 8) a unital C^*-algebra B_k, and a c.p.c. map $\alpha_k:A_k\to B_k$, such that B_k is a separable unital (if, in addition, A is asymptotically tracially in $\mathcal{N}$, B_k is also nuclear) C^*-algebra, and α_k is an $(\mathcal{F}_k,\epsilon_k/2)$-approximate embedding,

(k - 9) a countable dense subset $W_k=\{w_{k,1},w_{k,2},\cdots\}\subset B_k^1$, and finite subsets $W_{k,i}=\{w_{k,1},w_{k,2},\cdots,w_{k,i}\}\subset B_k$ $(i\in\mathbb{N})$,

(k - 10) a finite subset $\mathcal{G}_k=\alpha_k(\mathcal{F}_k)\subset B_k$,

(k - 11) a c.p.c. map $\beta_k:B_k\to A_k$, and a c.p.c. map $\gamma_k:A_k\to(\beta_k\circ\alpha_k(A_k))^\perp$ such that the following (k - 12) - (k - 15) hold:

(k - 12) $\beta_k\circ\alpha_k(1_A)$ and $\gamma_k(1_{A_k})$ are projections, and $1_{A_k}=\beta_k\circ\alpha_k(1_{A_k})+\gamma_k(1_{A_k})$,

(k - 13) $x\approx_{\epsilon_k}\beta_k\circ\alpha_k(x)+\gamma_k(x)$ for all $x\in\mathcal{F}_k$,

(k - 14) β_k is a $(\mathcal{G}_k,\epsilon_k)$-approximate embedding,

(k - 15) $\gamma_k(1_A)\lesssim_A\bar{a}_k$,

(k - 16) a c.p.c. map $\psi_{k+1}:M_{(k+1)!}\to\mathrm{Her}_A(\beta_k\circ\alpha_k(1_{A_k}))$, and an element $v_{k+1}\in\mathrm{Her}_A(\beta_k\circ\alpha_k(1_{A_k}))$ such that the pair $(\psi_{k+1},\ v_{k+1})$ satisfies the relation $\check{\mathcal{R}}_{\mathrm{Her}_A(\beta_k\circ\alpha_k(1_A))}(k!,\beta_k\circ\alpha_k(\mathcal{F}_k),\frac{1}{(k+1)!})$, and

(k - 17) a c.p.c. map $\varphi_k:=\beta_k\circ\alpha_k:A_k\to A_k$.

We make our choices recursively. For the case $k=1$:

(1 - 1) Define $A_1:=\mathrm{Her}_A(\varphi_0(1_{A_0}))=A$.

(1 - 2) Choose $\bar{a}_1:=a_1$.

(1 - 3) Choose a countable dense subset $X_1=\{x_{1,1},x_{1,2},\cdots\}\subset A_{1+}^1$, and let $X_{1,i}:=\{1_{A_1}\}\cup\{x_{1,j}:1\leq j\leq i\}\subset A_1^1$ $(i\in\mathbb{N})$.

(1 - 4) Set $Y_1 := X_{1,1} \cap B_{\frac{3}{4},1}(A_{1+})$.

(1 - 5) Choose $\mathcal{G}_{Y_1}$ and $\sigma(\mathcal{G}_{Y_1})$ as in Proposition 7.13.

(1 - 6) Choose

$$\begin{aligned} \mathcal{F}_1 &= \varphi_{1,1}(\mathcal{F}) \cup Y_1 \cup \left(\cup_{1\le j\le 1}\varphi_{j,1}(\mathcal{G}_{Y_j})\right) \cup \left(\cup_{1\le j\le 1}\varphi_{j,1}(\beta_{j-1}(W_{j-1,1}))\right) \\ &\quad \cup \left(\cup_{1\le j\le 1}\left(\varphi_{j,1}(\psi_j(\mathcal{G}^{M_{j!}})) \cup \{\varphi_{j,1}(v_j), \varphi_{j,1}(v_j)^*, \varphi_{j,1}(v_jv_j^*)\}\right)\right). \end{aligned} \tag{e 8.5}$$

(1 - 7) Choose

$$\epsilon_1 := \frac{1}{4^1}\min_{1\le j\le 1}\{\sigma(\mathcal{G}_{Y_j}), \Delta(M_{j!}, \frac{1}{j!}), \frac{\varepsilon}{100}\}. \tag{e 8.6}$$

Since A_1 is asymptotically tracially in $\mathcal{C}_{\mathcal{Z}}$ (and is asymptotically tracially in $\mathcal{N}$), by Lemma 8.5, we can further make the following choices:

(1 - 8) There exist a separable unital (nuclear, in case that A is asymptotically tracially in $\mathcal{N}$) C^*-algebra B_1 and a c.p.c. map $\alpha_1 : A_1 \to B_1$ such that α_1 is an $(\mathcal{F}_1, \epsilon_1/2)$-approximate embedding, and,

(1 - 9) a countable dense subset $W_1 = \{w_{1,1}, w_{1,2}, \cdots\} \subset B_1^1$ and finite subsets $W_{1,i} := \{w_{1,1}, w_{1,2}, \cdots, w_{1,i}\}$ $(i \in \mathbb{N})$,

(1 - 10) and a finite subset $\mathcal{G}_1 := \alpha_1(\mathcal{F}_1)$, and

(1 - 11) there exist a c.p.c. map $\beta_1 : B_1 \to A_1$ and a c.p.c. map $\gamma_1 : A_1 \to (\beta_1 \circ \alpha_1(A_1))^{\perp}$ such that

(1 - 12) $\beta_1 \circ \alpha_1(1_A)$ and $\gamma_1(1_{A_1})$ are projections, and $1_{A_1} = \beta_1 \circ \alpha_1(1_{A_1}) + \gamma_1(1_{A_1})$,

(1 - 13) $x \approx_{\epsilon_1} \beta_1 \circ \alpha_1(x) + \gamma_1(x)$ for all $x \in \mathcal{F}_1$,

(1 - 14) β_1 is a $(\mathcal{G}_1, \epsilon_1)$-approximate embedding,

(1 - 15) $\gamma_1(1_A) \lesssim_A \bar{a}_1$, and

(1 - 16) there exist a c.p.c. map $\psi_2 : M_{2!} \to \mathrm{Her}_A(\beta_1\circ\alpha_1(1_{A_1}))$, and an element $v_2 \in \mathrm{Her}_A(\beta_1\circ\alpha_1(1_{A_1}))^1$ such that the pair $(\psi_2,\ v_2)$ satisfies the relation $\check{\mathcal{R}}_{\mathrm{Her}_A(\beta_1\circ\alpha_1(1_A))}(2!, \beta_1 \circ \alpha_1(\mathcal{F}_1), \frac{1}{2!})$.

(1 - 17) Define $\varphi_1 := \beta_1 \circ \alpha_1$ which is a c.p.c. map from A_1 to A_1.

Assume, for $k \ge 1 \in \mathbb{N}$, we have made the choices (j - 1)-(j - 17) for all $1 \le j \le k$. Then, for $k+1$, we make the following choices:

(k+1 - 1) Define $A_{k+1} := \mathrm{Her}_A(\varphi_k(1_{A_k}))$.

(k+1 - 2) Choose $\bar{a}_{k+1} \in (A_{k+1})_+\backslash\{0\}$ such that $\bar{a}_{k+1} \lesssim_A a_{k+1}$.

(k+1 - 3) Choose a countable subset $X_{k+1} = \{x_{k+1,1}, x_{k+1,2}, \cdots\} \subset (A_{k+1})_+^1$ which is dense in $(A_{k+1})_+^1$ and choose $X_{k+1,i} := \{1_{A_{k+1}}\} \cup \{x_{k+1,j} : 1 \le j \le i\}$ $(i \in \mathbb{N})$.

(k+1 - 4) Choose $Y_{k+1} := \cup_{1\le j<k+1}\varphi_{j,k+1}(X_{j,k+1}) \cap B_{\frac{3}{4},1}(A_{k+1})$.

(k+1 - 5) Let $\mathcal{G}_{Y_{k+1}}$ and $\sigma(\mathcal{G}_{Y_{k+1}})$ be as in Proposition 7.13.

(k+1 - 6) Let

$$\begin{aligned} \mathcal{F}_{k+1} &= \varphi_{1,k+1}(\mathcal{F}) \cup Y_{k+1} \cup \left(\cup_{1\le j\le k+1}\varphi_{j,k+1}(\mathcal{G}_{Y_j})\right) \cup \left(\cup_{1\le j\le k+1}\varphi_{j,k+1}(\beta_{j-1}(W_{j-1,k+1}))\right) \\ &\quad \cup \left(\cup_{1\le j\le k+1}\varphi_{j,k+1}(\psi_j(\mathcal{G}^{M_{j!}})) \cup \{\varphi_{j,k+1}(v_j), \varphi_{j,k+1}(v_j)^*, \varphi_{j,k+1}(v_jv_j^*)\}\right). \end{aligned}$$

(k+1 - 7) Let

$$\epsilon_{k+1} = \frac{1}{4^{k+1}}\min_{1\le j\le k+1}\{\sigma(\mathcal{G}_{Y_j}), \Delta(M_{j!}, \frac{1}{j!}), \frac{\varepsilon}{100}\}. \tag{e 8.7}$$

Since A_{k+1} is asymptotically tracially in $\mathcal{C}_{\mathcal{Z}}$ (and is asymptotically tracially in $\mathcal{N}$), by Lemma 8.5, we can further make the following choices:

(k+1 - 8) There exist a separable unital (nuclear, in the case that A is asymptotically tracially in $\mathcal{N}$) C^*-algebra B_{k+1}, and a c.p.c. map $\alpha_{k+1} : A_{k+1} \to B_{k+1}$ such that α_{k+1} is an $(\mathcal{F}_{k+1}, \epsilon_{k+1}/2)$-approximate embedding, and, (k+1 - 9) a countable dense subset $W_{k+1} = \{w_{k+1,1}, w_{k+1,2}, \cdots\} \subset B_{k+1}^1$, and finite subsets $W_{k+1,i} = \{w_{k+1,1}, w_{k+1,2}, \cdots, w_{k+1,i}\}$ $(i \in \mathbb{N})$,

(k+1 - 10) and for $\mathcal{G}_{k+1} = \alpha_{k+1}(\mathcal{F}_{k+1}) \subset B_{k+1}$,

(k+1 - 11) there exist a c.p.c. map $\beta_{k+1} : B_{k+1} \to A_{k+1}$ and a c.p.c. map $\gamma_{k+1} : A_{k+1} \to (\beta_{k+1} \circ \alpha_{k+1}(A_{k+1}))^{\perp}$ such that

(k+1 - 12) $\beta_{k+1} \circ \alpha_{k+1}(1_A)$ and $\gamma_{k+1}(1_{A_{k+1}})$ are projections, and $1_{A_{k+1}} = \beta_{k+1} \circ \alpha_{k+1}(1_{A_{k+1}}) + \gamma_{k+1}(1_{A_{k+1}})$,

(k+1 - 13) $x \approx_{\epsilon_{k+1}} \beta_{k+1} \circ \alpha_{k+1}(x) + \gamma_{k+1}(x)$ for all $x \in \mathcal{F}_{k+1}$,

(k+1 - 14) β_{k+1} is a $(\mathcal{G}_{k+1}, \epsilon_{k+1})$-approximate embedding,

(k+1 - 15) $\gamma_{k+1}(1_A) \lesssim_A \bar{a}_{k+1}$, and

(k+1 - 16) there exist a c.p.c. map $\psi_{k+2} : M_{(k+2)!} \to \mathrm{Her}_A(\beta_{k+1} \circ \alpha_{k+1}(1_{A_{k+1}}))$ and an element $v_{k+2} \in \mathrm{Her}_A(\beta_{k+1} \circ \alpha_{k+1}(1_{A_{k+1}}))^1$ such that the pair (ψ_{k+2}, v_{k+2}) satisfies the relation $\check{\mathcal{R}}_{\mathrm{Her}_A(\beta_{k+1} \circ \alpha_{k+1}(1_A))}((k+2)!, \beta_{k+1} \circ \alpha_{k+1}(\mathcal{F}_{k+1})_{\frac{1}{(k+2)!}})$.

(k+1 - 17) Define $\varphi_{k+1} := \beta_{k+1} \circ \alpha_{k+1}$.

Therefore, by induction, for each $k \in \mathbb{N}$, we have made choices (k - 1) - (k - 17).

For each $k \in N$, by (k+1 - 1), we may view φ_k as a map from A_k to A_{k+1}.

Since A_k is simple, $f_{1/2}(x)$ is full in A_k for each $x \in \mathcal{F}_k \cap B_{3/4,1}(A_k)$. Then, by (k - 4), (k - 8), (k - 14), (k - 6), (k - 7), and by Lemma 8.4, we conclude that (A_k, φ_k) forms a generalized inductive limit which is simple, separable, unital, and $\mathcal{Z}$-stable. We denote this generalized inductive limit by $\bar{A}$.

If in addition A is also asymptotically tracially in $\mathcal{N}$, then each B_k are chosen to be nuclear as mentioned above. We claim that $\bar{A}$ is nuclear.

Denote the map $\alpha_{k+1} \circ \beta_k : B_k \to B_{k+1}$ by θ_k $(k \in \mathbb{N})$. Let $k \in \mathbb{N}$, let $z_1, z_2 \in B_k^1$ and let $\delta > 0$. Then there are $i_1, i_2 \in \mathbb{N}$ such that $z_1 \approx_{\delta/8} w_{k,i_1}$ and $z_2 \approx_{\delta/8} w_{k,i_2}$. Let $K \in \mathbb{N}$ such that $K > \max\{k, i_1, i_2, \frac{8}{\delta}\}$ and $\frac{1}{4^K} < \frac{\delta}{8}$. Note that $\theta_{i,j} = \alpha_j \circ \varphi_{j,i+1} \circ \beta_i$ for $j > i \in N$ (see Notation 6.2 for the notation $\theta_{i,j}$), then by (K - 10) and (K - 6), $\theta_{k,K}(w_{k,i_1}), \theta_{k,K}(w_{k,i_2}) \in \mathcal{G}_K$. For any $j \geq K$, keep using (i - 14) and (i - 8) for $j \geq i \geq K$, we have

$$\theta_{K,j}(\theta_{k,K}(w_{k,i_1})\theta_{k,K}(w_{k,i_2})) \approx_{2\sum_{i=K}^{j} \frac{1}{4^j}} \theta_{K,j}(\theta_{k,K}(w_{k,i_1}))\theta_{K,j}(\theta_{k,K}(w_{k,i_2})).$$

Note that $2\sum_{i=K}^{\infty} \frac{1}{4^j} < \delta/2$. Then, for any $j \geq K$,

$$\begin{aligned} \theta_{K,j}(\theta_{k,K}(z_1)\theta_{k,K}(z_2)) &\approx_{\delta/4} \theta_{K,j}(\theta_{k,K}(w_{k,i_1})\theta_{k,K}(w_{k,i_2})) \\ &\approx_{\delta/2} \theta_{K,j}(\theta_{k,K}(w_{k,i_1}))\theta_{K,j}(\theta_{k,K}(w_{k,i_2})) \\ &\approx_{\delta/4} \theta_{K,j}(\theta_{k,K}(z_1))\theta_{K,j}(\theta_{k,K}(z_2)) = \theta_{k,j}(z_1)\theta_{k,j}(z_2). \end{aligned}$$

Then, by Lemma 6.3, (B_k, θ_k) forms a generalized inductive limit. Since θ_k is a c.p.c. map for all $k \in \mathbb{N}$, by [3, Proposition 5.1.3], $\lim_{k\to\infty}(B_k, \alpha_{k+1} \circ \beta_k)$ is a nuclear C^*-algebra.

Recall that $\beta_k : B_k \to A_k$ and $\alpha_k : A_k \to B_k$ are c.p.c. maps, and $\varphi_k = \beta_k \circ \alpha_k$ (see (k - 17)). By the commutative diagram

$$\begin{tikzcd} A_1 \arrow[r, "\varphi_1"] \arrow[d, "\alpha_1"'] & A_2 \arrow[r, "\varphi_2"] \arrow[d, "\alpha_2"'] & A_3 \arrow[r, "\varphi_3"] \arrow[d, "\alpha_3"'] & \cdots\cdots & \bar{A} \\ B_1 \arrow[r, "\theta_1"] \arrow[ur, "\beta_1"] & B_2 \arrow[r, "\theta_2"] \arrow[ur, "\beta_2"] & B_3 \arrow[r, "\theta_3"] & \cdots\cdots & B \end{tikzcd}$$

we obtain two c.p.c. maps $\alpha : \bar{A} \to B$ and $\beta : B \to \bar{A}$ such that $\beta \circ \alpha = \mathrm{id}_{\bar{A}}$. By Lemma 8.6, $\bar{A}$ is also nuclear. This proves the claim.

Now back to the general case. We embed $\bar{A}$ into $l^\infty(A)/c_0(A)$ as follows. Let $x \in A_k$. Define

$$\iota(\varphi_{k,\infty}(x)) = \pi_\infty(\{0, 0, \cdots, 0, \varphi_{k,k}(x), \varphi_{k,k+1}(x), \cdots\}),$$

where $\pi_\infty : l^\infty(A) \to l^\infty(A)/c_0(A)$ is the quotient map. By (k - 17), (k - 8), and (k - 14),

$$\liminf_{n\to\infty} \|\varphi_{k,k+n}(x)\| \geq (1 - 4\sum_{j=k}^{\infty} \varepsilon_j)\|x\| \geq (1/2)\|x\| \text{ for all } x \in \mathcal{F}_k. \tag{e 8.8}$$

It follows (see Proposition 2.7) that ι defines a strict embedding from $\bar{A}$ into $l^\infty(A)/c_0(A)$. Note that

$$1_{\bar{A}} = \pi_\infty(\{\varphi_1(1_{A_1}), \varphi_2(1_{A_2}), \cdots\}) \text{ and}$$
$$1_A - 1_{\bar{A}} = \pi_\infty(\{\gamma_1(1_{A_1}), \sum_{i=1}^{2} \gamma_i(1_{A_i}), \cdots\}).$$

For all $k \in \mathbb{N}$, by (k - 15) and by the fact that $a_i \perp a_j$ $(i \neq j)$, we have $\sum_{i=1}^{k} \gamma_i(1_{A_i}) \lesssim \sum_{i=1}^{k} a_k \lesssim a$. It follows that

$$\iota_A(1_A) - 1_{\bar{A}} \lesssim_{l^\infty(A)/c_0(A)} \iota_A(a). \tag{e 8.9}$$

For $x \in \mathcal{F}$ and $k \geq 2 \in \mathbb{N}$, using (j - 13), (j - 12), (j - 17), and (j - 1) for $1 \leq j \leq k \in \mathbb{N}$, repeatedly, we have

$$\begin{aligned} x1_{A_k} &\approx_{\epsilon_1} (\gamma_1(x) + \varphi_1(x))1_{A_k} = \varphi_1(x)1_{A_k} = \varphi_{1,2}(x)1_{A_k} \\ &\approx_{\epsilon_2} (\gamma_2(\varphi_{1,2}(x)) + \varphi_3(\varphi_{1,2}(x)))1_{A_k} = \varphi_{1,3}(x)1_{A_k} \\ &\approx_{\epsilon_3} \cdots \approx_{\epsilon_{k-1}} \varphi_{1,k}(x)1_{A_k} = \varphi_{1,k}(x). \end{aligned} \tag{e 8.10}$$

Similarly, we have $1_{A_k}x \approx_{\sum_{i=1}^{k-1}\epsilon_i} \varphi_{1,k}(x)$. Thus $1_{A_k}x \approx_{2\sum_{i=1}^{k-1}\epsilon_i} x1_{A_k}$. Note that $2\sum_{i=1}^{\infty}\epsilon_i < \epsilon$. Hence

$$1_{\bar{A}}\iota_A(x) \approx_\epsilon \iota_A(x)1_{\bar{A}} \quad \text{for all } x \in \mathcal{F}. \tag{e 8.11}$$

By (e 8.11) and (e 8.10), we also have

$$1_{\bar{A}}\iota_A(x)1_{\bar{A}} \approx_\epsilon \iota_A(x)1_{\bar{A}} \approx_\epsilon \pi_\infty(\{\varphi_{1,k}(x)\}) \in \iota(\bar{A}) \quad \text{for all } x \in \mathcal{F}. \tag{e 8.12}$$

This proves the first part of the theorem. If, in addition, A is asymptotically tracially in $\mathcal{N}$, by the claim above, $\bar{A} \in \mathcal{N}_{\mathcal{Z},s,s}$. Since C^*-algebras in $\mathcal{N}_{\mathcal{Z},s,s}$ have property (H) (see [62, Corollary 3.1]), by Proposition 3.10, A is asymptotically tracially in $\mathcal{N}_{\mathcal{Z},s,s}$.

□

9 Simple C^*-algebras which are asymptotically tracially in $\mathcal{C}_{\mathcal{Z},s}$ or in $\mathcal{N}_n$

Theorem 9.1. *Let A be a simple separable unital C^*-algebra which is asymptotically tracially in $\mathcal{C}_{\mathcal{Z},s}$. Then, either A has stable rank one, or A is purely infinite. Moreover, if A is asymptotically tracially in $\mathcal{C}_{\mathcal{Z},s}$ and is not purely infinite, A has strict comparison for positive elements.*

Proof. Suppose that A is a unital separable simple C^*-algebra which is asymptotically tracially in $\mathcal{C}_{\mathcal{Z},s}$. Let $\mathcal{P}_1$ be the class of unital separable simple $\mathcal{Z}$-stable C^*-algebras which are purely infinite and let $\mathcal{P}_2$ be the class of unital separable simple $\mathcal{Z}$-stable C^*-algebras which have stable rank one. Then either (I) or (II) hold:

(I): For any finite subset $\mathcal{F} \subset A$, any $\epsilon > 0$, and any $a \in A_+\backslash\{0\}$, there exists a unital C^*-subalgebra $B \subset l^\infty(A)/c_0(A)$ which is strictly embedded such that B in $\mathcal{P}_1$, and

(1) $1_B\iota_A(x) \approx_\epsilon \iota_A(x)1_B$ for all $x \in \mathcal{F}$,

(2) $1_B\iota_A(x)1_B \in_\epsilon B$ and $\|1_B\iota_A(x)1_B\| \geq \|x\| - \epsilon$ for all $x \in \mathcal{F}$, and

(3) $\iota_A(1_A) - 1_B \lesssim_{l^\infty(A)/c_0(A)} \iota_A(a)$.

(II): The same statement holds as in (I) but replacing $\mathcal{P}_1$ by $\mathcal{P}_2$.

We may assume that A is infinite dimensional. By Lemma 4.3, there is a sequence of nonzero positive elements $\{d_n\} \subset A_+$ such that $d_{n+1} \lesssim d_n$ for all $n \in \mathbb{N}$, and, for any $x \in A_+ \setminus \{0\}$, there exists N such that $d_n \lesssim x$ for all $n \geq N$. Let $\mathcal{F}_n \subset A$ be an increasing sequence of finite subsets of A whose union is dense in A. Since A is asymptotically tracially in $\mathcal{C}_{\mathcal{Z},s}$, by Theorem 8.7, there exists a sequence of decreasing positive numbers $\{\varepsilon_n\}$ with $\lim_{n\to\infty}\varepsilon_n = 0$ and a sequence of unital C^*-algebras $B_k \in \mathcal{C}_{\mathcal{Z},s}$ such that

(1′) $\|1_{B_k}\iota_A(x) - \iota_A(x)1_{B_k}\| < \varepsilon_k$ for all $x \in \mathcal{F}_k$;

(2′) $1_{B_k}\iota_A(x)1_{B_k} \in_{\varepsilon_k} B_k$ for all $x \in \mathcal{F}_k$, and

(3′) $\iota_A(1_A) - 1_{B_k} \lesssim_{l^\infty(A)/c_0(A)} \iota_A(d_k)$.

If there are infinitely many B_k which are purely infinite, then, since, for any $a \in A_+ \setminus \{0\}$, there is K such that $d_K \lesssim a$, (I) holds.

Otherwise, by [56, Theorem 6.7], (II) holds. It follows from the proof of Proposition 4.4 (see also Remark 4.5) that, if (I) holds, A is purely infinite. On the other hand, if (II) holds, by Theorem 4.8, A has stable rank one. This completes the proof of the first part of the theorem.

For the last part, by [56, Theorem 4.5] and by Theorem 4.10, $W(A)$ is almost unperforated. Then, by the proof of [56, Corollary 4.6], A has strict comparison. Note that the proof of [56, Corollary 4.6] refers to the proof of [54, Theorem 5.2], where quasitraces are used (see also [54, Theorem 4.3] and [2, Theorem II.2.2], as well as [52, Proposition 2.1]). □

Corollary 9.2. *Let A be a simple separable unital C^*-algebra which is asymptotically tracially in $\mathcal{N}_n$ for some integer $n \geq 0$. Then, either A has stable rank one, or A is purely infinite. Moreover, if A is not purely infinite, A has strict comparison for positive elements.*

Proof. We note, by Corollary 7.17, that A is asymptotically tracially in $\mathcal{N}_{n,s,s}$, where $\mathcal{N}_{n,s,s}$ is the class of unital separable simple C^*-algebras with nuclear dimension at most n. By [69], C^*-algebras in $\mathcal{N}_{n,s,s}$ are nuclear simple $\mathcal{Z}$-stable C^*-algebras. Thus Theorem 9.1 applies. □

Theorem 9.3. *Let A be a simple separable infinite dimensional unital C^*-algebra. Then the following are equivalent:*

(1) A is asymptotically tracially in $\mathcal{N}_n$ for some $n \in \mathbb{N} \cup \{0\}$,

(2) A is asymptotically tracially in $\mathcal{N}_{n,s,s}$ for some $n \in \mathbb{N} \cup \{0\}$,

(3) A is asymptotically tracially in $\mathcal{N}_{\mathcal{Z},s,s}$,

(4) A is asymptotically tracially in $\mathcal{N}$ and is asymptotically tracially in $\mathcal{C}_{\mathcal{Z},s}$.

Proof. (1) ⇒ (2) follows from Theorem 7.16, (2) ⇒ (3) follows from [69, Corollary 7.3], (3) ⇒ (4) is trivial, (4) ⇒ (1) follows from Theorem 8.7 and [10, Theorem A]. □

Lemma 9.4 (see [46, Lemma 8.2])**.** *Let A be a unital separable nuclear simple C^*-algebra which is asymptotically tracially $\mathcal{N}_{d,s}$ (for some integer $d \geq 0$). Then, for any integer $k \geq 1$, there is a sequence of order zero c.p.c. maps $L_n : M_k \to A$ such that $\{L_n(e)\}$ is a central sequence of A for a minimal projection $e \in M_k$ and such that, for every integer $m \geq 1$,*

$$\lim_{n\to\infty} \max_{\tau\in T(A)} \{|\tau(L_n(e)^m) - 1/k|\} = 0. \tag{e 9.1}$$

Proof. The proof follows the same lines of that of [46, Lemma 8.2] with some minor modifications. Fix $k \in \mathbb{N}$. Fix a dense subset $\{x_1, x_2, \cdots\}$ of the unit ball of A and let $\mathcal{F}_n = \{x_1, x_2, \cdots, x_n\}$ with $1_A = x_1$ ($n \in \mathbb{N}$). Let $\gamma_n > 0$ be in the fifth line of the proof of [46, Lemma 8.2]. By

Lemma 4.3, there is a sequence $\{a_n\}$ of $A_+ \setminus \{0\}$ such that $0 < d_\tau(a_n) < 1/4n^2$ ($n \in \mathbb{N}$). By Corollary 7.17, A is asymptotically tracially in $\mathcal{N}_{d,s,s}$. Therefore, by Proposition 3.8, for each $n \in \mathbb{N}$, there exists a C^*-algebra $B_n \in \mathcal{N}_{d,s,s}$ and c.p.c maps $\alpha_n : A \to B_n$, $\beta_{n,j} : B_n \to A$, and $\gamma_{n,j} : A \to A \cap \beta_{n,j}(B_n)^\perp$ ($j \in \mathbb{N}$), such that

(1) $\gamma_{n,j}(1_A)$ and $p_{n,j} := \beta_{n,j}(1_{B_n})$ are projections, $1 = \gamma_{n,j}(1_A) + \beta_{n,j}(1_{B_n})$, and $\alpha_n(1_A) = 1_{B_n}$,

(2) $x_i \approx_{\gamma_n/2^n} \gamma_{n,j}(x_i) + \beta_{n,j} \circ \alpha_n(x_i)$ for all $1 \le i \le n$ and all $j \in \mathbb{N}$,

(3) α_n is an $(\mathcal{F}_n, 1/2^n)$-approximate embedding,

(4) $\lim_{j\to\infty} \|\beta_{n,j}(xy) - \beta_{n,j}(x)\beta_{n,j}(y)\| = 0$ and $\lim_{j\to\infty} \|\beta_{n,j}(x)\| = \|x\|$ for all $x, y \in B_n$, and

(5) $\gamma_{n,j}(1_A) \lesssim_A a_n$ for all $j \in \mathbb{N}$.

Note that one also has

(6) $\|p_{n,j}x - xp_{n,j}\| < 1/2^{n-1}$ for all $x \in \mathcal{F}_n$.

By [69, Lemma 5.11] (since $B \in \mathcal{N}_{d,s,s}$), for each n, there is an order zero c.p.c. map $\Psi_n : M_k \to B_n$ such that

$$\|[\Psi_n(c), \alpha_n(x)]\| < 1/2^n \text{ for all } c \in M_k^1 \text{ and } x \in \mathcal{F}_n, \text{ and} \tag{e 9.2}$$
$$\inf\{\tau(\Psi_n(1_{M_k})) : \tau \in T(B_n)\} > 1 - 1/4n. \tag{e 9.3}$$

Consider, for each m, $\Psi_{n,m} = \beta_{n,m} \circ \Psi_n : M_k \to p_{n,m}Ap_{n,m}$. Note that, by (4), for each $n \in \mathbb{N}$, there exists $\bar{m}(n) \in \mathbb{N}$ such that, for all $m \ge \bar{m}(n)$, $\Psi_{n,m}$ is a $\Delta(M_k, \gamma_n/2^n)$-almost order zero map (recall Definition 7.5 for $\Delta(-,-)$), and

$$\|[\beta_{n,m} \circ \Psi_n(c), \beta_{n,m} \circ \alpha_n(x)]\| < \gamma_n/2^n \text{ for all } c \in M_k^1 \text{ and } x \in \mathcal{F}_n. \tag{e 9.4}$$

Claim: For fixed n, there is $m(n) > \bar{m}(n)$ such that, for all $m > m(n)$,

$$\inf\{\tau(\Psi_{n,m}(1_{M_k})) : \tau \in T(A)\} \ge 1 - 1/2n.$$

Otherwise, there would be a subsequence $\{m(l)\}$ and $\tau_{m(l)} \in T(A)$ such that

$$\tau_{m(l)} \circ \beta_{n,m(l)} \circ \Psi_n(1_{M_k}) < 1 - 1/2n.$$

Let t_0 be a weak*-limit of the sequence of contractive positive linear functionals $\{\tau_{m(l)} \circ \beta_{n,m(l)}\}$ of B_n. Then $t_0(\Psi_n(1_{M_k})) \le 1 - 1/2n$. On the other hand, by (5) and (1), $t_0(1_{B_n}) \ge 1 - 1/4n^2$. Moreover, by (4), t_0 is a positive tracial functional with $\|t_0\| \ge 1 - 1/4n^2$. It follows from (e 9.3) that $t_0(\Psi_n(1_{M_k})) \ge (1 - 1/4n^2)(1 - 1/4n) > 1 - 1/2n$. This proves the claim.

For all $c \in M_k^1$ and $x \in \mathcal{F}_n$, one has

$$\begin{aligned} \|[\beta_{n,m(n)} \circ \Psi_n(c), x]\| &\overset{\text{(by (2))}}{\approx_{\gamma_n/2^{n-1}}} \|[\beta_{n,m(n)} \circ \Psi_n(c), \gamma_{n,m(n)}(x) + \beta_{n,m(n)} \circ \alpha_n(x)]\| \\ &\overset{\text{(by (1))}}{=} \|[\beta_{n,m(n)} \circ \Psi_n(c), \beta_{n,m(n)} \circ \alpha_n(x)]\| \overset{\text{(by (e 9.4))}}{<} \gamma_n/2^n. \end{aligned} \tag{e 9.5}$$

Since $\Psi_{n,m(n)}$ is a $\Delta(M_k, \gamma_n/2^n)$-almost order zero map, by the choice of $\Delta(M_k, \gamma_n/2^n)$ (see Definition 7.5 and Proposition 7.4), one obtains a sequence of order zero c.p.c. maps $\Phi_n : M_k \to A$ such that

$$\|\Phi_n - \beta_{n,m(n)} \circ \Psi_n\| \le \gamma_n/2^n \text{ for all } n \in \mathbb{N}. \tag{e 9.6}$$

By (e 9.5), as well as the claim, for $n \ge 3$, one has

$$\|\Phi_n(c)x - x\Phi_n(c)\| < \min\{1/4n, \gamma_n\} \text{ for all } c \in M_k^1 \text{ and } x \in \mathcal{F}_n, \text{ and} \tag{e 9.7}$$
$$\inf\{\tau(\Phi_n(1_{M_k})) : \tau \in T(A)\} \ge 1 - 1/n. \tag{e 9.8}$$

There is a homomorphism $\varphi_n : C_0((0,1]) \otimes M_k \to A$ such that $\Phi_n(c) = \varphi_n(\iota \otimes a)$ for all $c \in M_k$, where $\iota(t) = t$ for all $t \in (0,1]$. Let $c_n = \iota^{1/n}$. Define $L_n(c) = \varphi_n(c_n \otimes c)$ for all $c \in M_k$. It is an order zero c.p.c. map from M_k to A. Choose a minimal projection $e_1 \in M_k$. Then

$$(L_n(e_1))^m = \varphi_n(c_n^m \otimes e_1) = \varphi_n(\iota \otimes e_1)^{m/n}) = \Phi_n(e_1)^{m/n}. \tag{e 9.9}$$

One then verifies that, for this L_n, (e 9.1) holds exactly the same way as the proof of [46, Lemma 8.2]. $\square$

Theorem 9.5. *Every unital separable simple nuclear C^*-algebra which is asymptotically tracially in $\mathcal{N}_n$ is $\mathcal{Z}$-stable and has nuclear dimension at most 1.*

On the other hand, every unital separable simple nuclear C^*-algebra which is asymptotically tracially in $\mathcal{C}_{\mathcal{Z},s}$ also has nuclear dimension at most 1.

Proof. Let A be a unital separable simple nuclear C^*-algebra which is asymptotically tracially in $\mathcal{N}_n$ for some non-negative integer n. By Theorem 9.3, A is asymptotically tracially in $\mathcal{N}_{n,s,s}$. By Corollary 9.2, and by [6, Corollary 9.9], we may assume that A has stable rank one and has strict comparison. We first prove that A is $\mathcal{Z}$-stable. The proof of this is exactly the same as that of [46, Theorem 8.3] but using Lemma 9.4 (By the exactly the same argument for the proof of (ii) implies (iii) in [49], using Lemma 9.4 instead of [49, Lemma 3.3], one concludes that any c.p. map from A to A can be excised in small central sequence. As in [49], this implies that A has property (SI). Using Lemma 9.4, the same proof that (iv) implies (i) in [49] shows that A is $\mathcal{Z}$-stable).

Then, by [10, Theorem A], A has finite nuclear dimension. It follows from [10, Theorem B] that A has in fact nuclear dimension at most 1.

Finally, the last statement follows the first part of the statement and part (4) of Theorem 9.3. $\square$

Corollary 9.6 (cf. Cor. 9.6 of [26])**.** *Every unital separable simple nuclear C^*-algebra which has generalized tracial rank at most one is $\mathcal{Z}$-stable.*

Lemma 9.7. *Let A, B, C_i, D_i be C^*-algebras ($i \in \mathbb{N}$), and let $\alpha_i : A \to C_i$, $\beta_i : B \to D_i$ be c.p.c. maps such that*

$$\alpha : A \to \prod_{i=1}^{\infty} C_i / \bigoplus_{i=1}^{\infty} C_i, \quad a \mapsto \pi_\infty(\{\alpha_i(a)\}_i) \text{ and } \beta : B \to \prod_{i=1}^{\infty} D_i / \bigoplus_{i=1}^{\infty} D_i, \quad b \mapsto \pi_\infty(\{\beta_i(b)\}_i)$$

*are *-homomorphisms. Then the following map is also a *-homomorphism:*

$$\gamma : A \otimes B \to \prod_{i=1}^{\infty} (C_i \otimes D_i) / \bigoplus_{i=1}^{\infty} (C_i \otimes D_i), \quad a \otimes b \mapsto \pi_\infty(\{\alpha_i(a) \otimes \beta_i(b)\}_i). \tag{e 9.10}$$

If, in addition, both α and β are strict embeddings, so is γ.

Proof. Note that $\alpha_i \otimes \beta_i : A \otimes B \to C_i \otimes D_i$, $a \otimes b \mapsto \alpha_i(a) \otimes \beta_i(b)$ are c.p.c. maps. Thus γ is also a c.p.c. map. Fix $\check{a} \in A$, $\check{b} \in B$. Since α and β are *-homomorphisms, we have

$$\lim_{i \to \infty} \|\alpha_i(\check{a}\check{a}^*) - \alpha_i(\check{a})\alpha_i(\check{a})^*\| + \|\beta_i(\check{b}\check{b}^*) - \beta_i(\check{b})\beta_i(\check{b})^*\| = 0. \tag{e 9.11}$$

Then

$$
\begin{aligned}
\gamma(\check{a}\otimes\check{b})\cdot\gamma(\check{a}\otimes\check{b})^* &= \pi_\infty(\{\alpha_i(\check{a})\otimes\beta_i(\check{b})\}_i)\cdot\pi_\infty(\{\alpha_i(\check{a})\otimes\beta_i(\check{b})\}_i)^* && \text{(e 9.12)}\\
&= \pi_\infty(\{(\alpha_i(\check{a})\alpha_i(\check{a})^*)\otimes(\beta_i(\check{b})\beta_i(\check{b})^*)\}_i) && \text{(e 9.13)}\\
\text{(by (e 9.11))} &= \pi_\infty(\{(\alpha_i(\check{a}\check{a})^*)\otimes(\beta_i(\check{b}\check{b})^*)\}) = \gamma((\check{a}\check{a}^*)\otimes(\check{b}\check{b}^*)) && \text{(e 9.14)}\\
&= = \gamma((\check{a}\otimes\check{b})\cdot(\check{a}\otimes\check{b})^*). && \text{(e 9.15)}
\end{aligned}
$$

Similarly, we have $\gamma(\check{a}\otimes\check{b})^*\cdot\gamma(\check{a}\otimes\check{b}) = \gamma((\check{a}\otimes\check{b})^*\cdot(\check{a}\otimes\check{b}))$ (see, for example, [8, Proposition 1.5.7.(ii)]). Thus $\check{a}\otimes\check{b}$ lies in the multiplicative domain of γ. Since the linear span of elementary tensor products is dense in $A\otimes B$, we see that $A\otimes B$ lies in the multiplicative domain of γ. In other words, γ is a *-homomorphism.

Assume in addition both α and β are strict embeddings. If γ is not a strict embedding, then there exist $z_0\in A\otimes B$, $\epsilon>0$, and a subsequence $\{m_i\}_{i\in\mathbb{N}}\subset\mathbb{N}$ such that

$$
\limsup_{i\to\infty}\|\alpha_{m_i}\otimes\beta_{m_i}(z_0)\| \le \|z_0\|-\epsilon. \qquad \text{(e 9.16)}
$$

By what has been proved, the following map is also a *-homomorphism:

$$
\tilde{\gamma}: A\otimes B\to\prod_{i=1}^{\infty}(C_{m_i}\otimes D_{m_i})/\bigoplus_{i=1}^{\infty}(C_{m_i}\otimes D_{m_i}),\ \ a\otimes b\mapsto\pi_\infty(\{\alpha_{m_i}(a)\otimes\beta_{m_i}(b)\}_i). \qquad \text{(e 9.17)}
$$

By (e 9.16), $\tilde{\gamma}$ is not an isometry. Thus $\tilde{\gamma}$ could not be injective. By [5, Lemma 2.12(ii)], $\ker\tilde{\gamma}$ (which is an ideal of $A\otimes B$) contains a nonzero elementary tensor product $a_0\otimes b_0$. Then

$$
\begin{aligned}
0=\|\tilde{\gamma}(a_0\otimes b_0)\| &= \limsup_{i\to\infty}(\|\alpha_{m_i}(a_0)\otimes\beta_{m_i}(b_0)\|)\\
&= \limsup_{i\to\infty}\|\alpha_{m_i}(a_0)\|\cdot\|\beta_{m_i}(b_0)\|\\
(\alpha,\ \beta \text{ are strict embeddings}) &= \lim_{i\to\infty}\|\alpha_{m_i}(a_0)\|\cdot\lim_{i\to\infty}\|\beta_{m_i}(b_0)\| = \|a_0\|\cdot\|b_0\|,
\end{aligned}
$$

which is contradict to the assumption that $a_0\otimes b_0\neq 0$. Hence γ is a strict embedding.

□

Lemma 9.8. *Let A_1 and A_2 be C^*-algebras and let $\mathcal{F}\subset A_1\otimes A_2$ be a finite subset. Then, for any $\epsilon>0$, there exist finite subsets $\mathcal{G}_i\subset A_i$ ($i=1,2$) and $\delta>0$ such that, for any C^*-algebras B_1 and B_2, and, for any c.p.c. maps $\alpha_i: A_i\to B_i$ which are $(\mathcal{G}_i,\delta)$-approximate embeddings, the map $\alpha_1\otimes\alpha_2: A_1\otimes A_2\to B_1\otimes B_2$ is an $(\mathcal{F},\epsilon)$-approximate embedding.*

Proof. Without loss of generality, we may assume that $0<\epsilon<1$. Let $M:=1+\max\{\|x\|: x\in\mathcal{F}\}$. Let $\mathcal{F}_i\subset A_i$ ($i=1,2$) and n be some integer such that $\mathcal{F}_i=\mathcal{F}_i^*$ ($i=1,2$) and $\mathcal{F}\subset_{\frac{\epsilon}{8M^2}}\mathcal{F}^{1,2}$, where $\mathcal{F}^{1,2}:=\{\sum_{i=1}^n x_i\otimes y_i: x_i\in\mathcal{F}_1 \text{ and } y_i\in\mathcal{F}_2\}$. Let $M_1:=1+\max\{\|x\|: x\in\mathcal{F}^{1,2}\}$.

Keeping Lemma 2.10 in mind, it is straightforward to see that there exists $\delta_0>0$ such that, for any c.p.c. maps $\alpha_i: A_i\to B_i$ ($i=1,2$, B_i are C^*-algebras), if α_i is $(\mathcal{F}_i,\delta_0)$-multiplicative ($i=1,2$), then $\alpha_1\otimes\alpha_2: A_1\otimes A_2\to B_1\otimes B_2$ is $(\mathcal{F}^{1,2},\frac{\epsilon}{8})$-multiplicative, and, hence $\alpha_1\otimes\alpha_2: A_1\otimes A_2\to B_1\otimes B_2$ is $(\mathcal{F},\epsilon)$-multiplicative. Let $\mathcal{F}_i\subset\mathcal{F}_{i,1}\subset\mathcal{F}_{i,2}\subset\cdots$ be finite subsets of $C^*(\mathcal{F}_i)$ such that $\cup_{j\in\mathbb{N}}\mathcal{F}_{i,j}$ is dense in $C^*(\mathcal{F}_i)$ ($i=1,2$).

Now let us assume the lemma does not hold. Then there exists a sequence of C^*-algebras $B_{i,m}$ and c.p.c. maps $\alpha_{i,m}: A_i\to B_{i,m}$ such that $\alpha_{i,m}$ is an $(\mathcal{F}_{i,m},\delta_0/m)$-approximate embedding ($i=1,2$, $m\in\mathbb{N}$), and $\alpha_{1,m}\otimes\alpha_{2,m}: A_1\otimes A_2\to B_{1,m}\otimes B_{2,m}$ is not an $(\mathcal{F},\epsilon)$-approximate embedding ($m\in\mathbb{N}$). However, since $\mathcal{F}_i\subset\mathcal{F}_{i,m}$, by the choice of δ_0, and by the fact that

$\alpha_{1,m} \otimes \alpha_{2,m}$ is $(\mathcal{F}, \epsilon)$-approximate multiplicative, for each m, there must be some $z_m \in \mathcal{F} \subset \mathcal{F}^{1,2}$ such that

$$\|\alpha_{1,m} \otimes \alpha_{2,m}(z_m)\| < \|z_m\| - \epsilon. \tag{e 9.18}$$

Since $\mathcal{F}^{1,2}$ is a finite subset, by (e 9.18), there exists $z_0 \in \mathcal{F}^{1,2}$ and an increasing sequence $\{m_j\} \subset \mathbb{N}$ such that

$$\|\alpha_{1,m_j} \otimes \alpha_{2,m_j}(z_0)\| < \|z_0\| - \epsilon \text{ for all } j \in \mathbb{N}. \tag{e 9.19}$$

Note that the map $\bar{\alpha}_1 : A_1 \to \prod_{m=1}^{\infty} B_{1,m}/\bigoplus_{m=1}^{\infty} B_{1,m}$ defined by $a \mapsto \pi_\infty(\{\alpha_{1,m}(a)\})$ and the map $\bar{\alpha}_2 : A_2 \to \prod_{m=1}^{\infty} B_{2,m}/\bigoplus_{m=1}^{\infty} B_{2,m}$ defined by $a \mapsto \pi_\infty(\{\alpha_{2,m}(a)\})$ are strict embeddings. Then, by Lemma 9.7, the following is also a strict embedding:

$$\gamma : A_1 \otimes A_2 \to \prod_{m=1}^{\infty}(B_{1,m} \otimes B_{2,m})/\bigoplus_{m=1}^{\infty}(B_{1,m} \otimes B_{2,m}), \;\; a \otimes b \mapsto \pi_\infty(\{\alpha_{1,m}(a) \otimes \alpha_{2,m}(b)\}).$$

But this contradicts with (e 9.19). The lemma then follows.

□

Notation 9.9. Let $\mathcal{X}_1$, $\mathcal{X}_2$ be two classes of C^*-algebras. Denote $\mathcal{X}_1 \otimes \mathcal{X}_2 := \{A \otimes B : A \in \mathcal{X}_1, B \in \mathcal{X}_2\}$, where each $A \otimes B$ is the spatial tensor product.

Recall the following result (see [5, Lemma 2.15], also see [55, Lemma 4.1.9]):

Lemma 9.10 (Kirchberg's Slice Lemma)**.** *Let A and B be C^*-algebras, and let D be a nonzero hereditary C^*-subalgebra of the spatial tensor product $A \otimes B$. Then there exists a nonzero element $z \in A \otimes B$ such that $z^*z = a \otimes b$ for some $a \in A$, $b \in B$, and $zz^* \in D$.*

Theorem 9.11. *Let $\mathcal{X}_1$, $\mathcal{X}_2$ be two classes of C^*-algebras. Let A and B be unital simple separable infinite dimensional C^*-algebras. Assume that A is asymptotically tracially in $\mathcal{X}_1$ and B is asymptotically tracially in $\mathcal{X}_2$. Then the spatial tensor product $A \otimes B$ is asymptotically tracially in $\mathcal{X}_1 \otimes \mathcal{X}_2$.*

Proof. Let $\mathcal{F} \subset A \otimes B$, let $c \in (A \otimes B)_+ \backslash \{0\}$, and let $1/4 > \epsilon > 0$. By Kirchberg's Slice Lemma (see Lemma 9.10), there exists $a \in A_+ \backslash \{0\}$ and there exists $b \in B_+ \backslash \{0\}$, such that

$$a \otimes b \lesssim_{A \otimes B} c. \tag{e 9.20}$$

Note that A and B are non-elementary. Then one may choose $\check{a}, \hat{a} \in \mathrm{Her}_A(a)_+ \backslash \{0\}$ and $\check{b}, \hat{b} \in \mathrm{Her}_A(b)_+ \backslash \{0\}$ such that $\check{a} \perp \hat{a}$, $\check{a} \sim_A \hat{a}$, $\check{b} \perp \hat{b}$, and $\check{b} \sim_A \hat{b}$ (see Lemma 4.3, for example).

Since A and B are simple and unital, there exist $k \in \mathbb{N}$, $r_1, r_2, \cdots, r_k \in A$, and $s_1, s_2, \cdots, s_k \in B$ such that $1_A = \sum_{i=1}^k r_i^* \hat{a} r_i$ and $1_B = \sum_{i=1}^k s_i^* \hat{b} s_i$. Since A and B are simple and infinite dimensional, so are $\mathrm{Her}_A(\check{a})$ and $\mathrm{Her}_B(\check{b})$. Then (see Lemma 4.3) there exist mutually orthogonal positive elements $a_1, a_2, \cdots, a_{k+1} \in \mathrm{Her}_A(\check{a})_+ \backslash \{0\}$ and mutually orthogonal positive elements $b_1, b_2, \cdots, b_{k+1} \in \mathrm{Her}_A(\check{b})_+ \backslash \{0\}$ such that $a_1 \sim_A a_2 \sim_A \cdots \sim_A a_{k+1}$ and $b_1 \sim_B b_2 \sim_B \cdots \sim_B b_{k+1}$ (recall Definition 2.13 for the definition of "$\sim$").

Let $N \in \mathbb{N}$, let $\tilde{\mathcal{F}}_1 \subset A$, and let $\tilde{\mathcal{F}}_2 \subset B$ be finite subsets such that

$$\mathcal{F} \subset_{\epsilon/2} \left\{ \sum_{i=1}^{N} x_i \otimes y_i : x_i \in \tilde{\mathcal{F}}_1, y_i \in \tilde{\mathcal{F}}_2 \right\}. \tag{e 9.21}$$

By Lemma 9.8, there exist finite subsets $\bar{\mathcal{F}}_1 \subset A$, $\bar{\mathcal{F}}_2 \subset B$, and $\delta_0 > 0$ such that, for any C^*-algebras D_1, D_2 and any c.p.c. maps $\eta_1 : A \to D_1$, and $\eta_2 : B \to D_2$, if η_i is an $(\bar{\mathcal{F}}_1, \delta_0)$-approximate embedding ($i = 1, 2$), then $\eta_1 \otimes \eta_2 : A \otimes B \to D_1 \otimes D_2$ is an $(\mathcal{F}, \epsilon)$-approximate embedding.

Let $\mathcal{F}_i := \tilde{\mathcal{F}}_i \cup \bar{\mathcal{F}}_i$, $i = 1, 2$. Let $M := 1 + \max\{\|x\| : x \in \mathcal{F}_1 \cup \mathcal{F}_2\}$. Choose $\delta := \min\{\delta_0, \frac{\epsilon}{3(N+1)(M+1)^2}\}$. Note $2\delta^2 < \delta < M/4$.

Since A is asymptotically tracially in $\mathcal{X}_1$, there exist a C^*-algebra C_1 in $\mathcal{X}_1$, and c.p.c maps $\alpha : A \to C_1$, $\beta_i : C_1 \to A$, and $\gamma_i : A \to A$ ($i \in \mathbb{N}$) such that

(1) $x \approx_\delta \gamma_i(x) + \beta_i \circ \alpha(x)$ for all $x \in \mathcal{F}_1$ and for all $i \in \mathbb{N}$,

(2) α is an $(\mathcal{F}_1, \delta)$-approximate embedding,

(3) $\lim_{i\to\infty} \|\beta_i(xy) - \beta_i(x)\beta_i(y)\| = 0$ and $\lim_{n\to\infty} \|\beta_i(x)\| = \|x\|$ for all $x, y \in C_1$, and

(4) $\gamma_i(1_A) \lesssim_A a_1 (\sim a_{k+1})$ for all $i \in \mathbb{N}$.

Since B is asymptotically tracially in $\mathcal{X}_2$, there exist a C^*-algebra C_2 in $\mathcal{X}_2$, and c.p.c maps $\varphi : B \to C_2$, $\psi_i : C_2 \to B$, and $\theta_i : B \to B$ ($i \in \mathbb{N}$) such that

(1′) $x \approx_\delta \theta_i(x) + \psi_i \circ \varphi(x)$ for all $x \in \mathcal{F}_2$ and for all $i \in \mathbb{N}$,

(2′) φ is an $(\mathcal{F}_2, \delta)$-approximate embedding,

(3′) $\lim_{i\to\infty} \|\psi_i(xy) - \psi_i(x)\psi_i(y)\| = 0$ and $\lim_{n\to\infty} \|\psi_i(x)\| = \|x\|$ for all $x, y \in C_2$, and

(4′) $\theta_i(1_B) \lesssim_B b_1$ $(\sim b_{k+1})$ for all $i \in \mathbb{N}$.

Note that $C_1 \otimes C_2$ is in $\mathcal{X}_1 \otimes \mathcal{X}_2$. Now define a c.p.c. map

$$\rho := \alpha \otimes \varphi : A \otimes B \to C_1 \otimes C_2. \tag{e 9.22}$$

By (2), (2′) and by the choice of δ, δ_0, $\mathcal{F}_1$, $\bar{\mathcal{F}}_1$, $\mathcal{F}_2$ and $\bar{\mathcal{F}}_2$, the map ρ is an $(\mathcal{F}, \epsilon)$-approximate embedding. Hence (2) of Definition 3.1 holds.

For $i \in \mathbb{N}$, define a c.p.c. map

$$\omega_i := \beta_i \otimes \psi_i : C_1 \otimes C_2 \to A \otimes B. \tag{e 9.23}$$

Define c.p.c. maps $\beta : C_1 \to l^\infty(A)/c_0(A)$ by $x \mapsto \pi_\infty(\{\beta_1(x), \beta_2(x), \cdots\})$ and $\psi : C_2 \to l^\infty(B)/c_0(B)$ by $x \mapsto \pi_\infty(\{\psi_1(x), \psi_2(x), \cdots\})$, respectively. Then, by (3) and (3′), β and ψ are strict embeddings. By Lemma 9.7, the map $\omega : C_1 \otimes C_2 \to l^\infty(A \otimes B)/c_0(A \otimes B)$ defined by $x \otimes y \mapsto \pi_\infty(\{\beta_1(x) \otimes \psi_1(y), \beta_2(x) \otimes \psi_2(y), \cdots\})$ is also a strict embedding: This is equivalent to say that (3) of Definition 3.1 holds.

Note that by (1) and (1′) above, for $i \in \mathbb{N}$, one has

$$\begin{aligned} 1_A \otimes 1_B \quad \approx_{2\delta(1+\delta)} \quad & (\beta_i \circ \alpha(1_A) + \gamma_i(1_A)) \otimes (\psi_i \circ \varphi(1_B) + \theta_i(1_B)) \\ = \quad & \beta_i \circ \alpha(1_A) \otimes \psi_i \circ \varphi(1_B) \\ & + \gamma_i(1_A) \otimes \psi_i \circ \varphi(1_B) + \beta_i \circ \alpha(1_A) \otimes \theta_i(1_B) + \gamma_i(1_A) \otimes \theta_i(1_B). \end{aligned}$$

Thus

$$\|\gamma_i(1_A) \otimes \psi_i \circ \varphi(1_B) + \beta_i \circ \alpha(1_A) \otimes \theta_i(1_B) + \gamma_i(1_A) \otimes \theta_i(1_B)\| \le 1 + 2\delta + 2\delta^2 < 1 + 3\delta.$$

It follows that the map defined below

$$\begin{aligned} \sigma_i : A \otimes B \quad & \to \quad A \otimes B, \\ x \otimes y \quad & \mapsto \quad \frac{1}{1+3\delta} (\gamma_i(x) \otimes \psi_i \circ \varphi(y) + \beta_i \circ \alpha(x) \otimes \theta_i(y) + \gamma_i(x) \otimes \theta_i(y)) \end{aligned}$$

is c.p.c. map ($i \in \mathbb{N}$). By (1) and (1′), for $x \in \mathcal{F}_1$ and $y \in \mathcal{F}_2$, and for any $i \in \mathbb{N}$, one has

$$x \otimes y \quad \approx_{2\delta(M+\delta)} \quad (\beta_i \circ \alpha(x) + \gamma_i(x)) \otimes (\psi_i \circ \varphi(y) + \theta_i(y)) \tag{e 9.24}$$

$$= \quad \beta_i \circ \alpha(x) \otimes \psi_i \circ \varphi(y) + \gamma_i(x) \otimes \psi_i \circ \varphi(y) \tag{e 9.25}$$

$$+\beta_i \circ \alpha(x) \otimes \theta_i(y) + \gamma_i(x) \otimes \theta_i(y) \tag{e 9.26}$$

$$\approx_{3M^2\delta} \quad \omega_i \circ \rho(x \otimes y) + \sigma_i(x \otimes y). \tag{e 9.27}$$

Then, for $\sum_{j=1}^N x_j \otimes y_j$ with $x_j \in \mathcal{F}_1$, $y_j \in \mathcal{F}_2$ $(j = 1, 2, \cdots, N)$, and, for any $i \in \mathbb{N}$, one has

$$\sum_{j=1}^N x_j \otimes y_j \approx_{3NM(M+1)\delta} \sum_{j=1}^N \omega_i \circ \rho(x_j \otimes y_j) + \sigma_i(x_j \otimes y_j) \tag{e 9.28}$$

$$= \omega_i \circ \rho\left(\sum_{j=1}^N x_j \otimes y_j\right) + \sigma_i\left(\sum_{j=1}^N x_j \otimes y_j\right). \tag{e 9.29}$$

Thus, by the choice of δ, (1) of Definition 3.1 holds.

Claim: For all $i \in \mathbb{N}$, $\sigma_i(1_A \otimes 1_B) \lesssim c$ in $A \otimes B$. Indeed, one has

$$\gamma_i(1_A) \otimes \psi_i \circ \varphi(1_B) \lesssim a_1 \otimes 1_B = a_1 \otimes (\sum_{i=1}^k s_i^* \hat{b} s_i) \lesssim \sum_{i=1}^k a_i \otimes \hat{b}, \text{ and} \tag{e 9.30}$$

$$\beta_i \circ \alpha(1_A) \otimes \theta_i(1_B) \lesssim 1_A \otimes b_1 = (\sum_{i=1}^k r_i^* \hat{a} r_i) \otimes b_1 \lesssim \sum_{i=1}^k \hat{a} \otimes b_i, \text{ and} \tag{e 9.31}$$

$$\gamma_i(1_A) \otimes \theta_i(1_B) \lesssim a_1 \otimes b_1 \sim a_{k+1} \otimes b_{k+1}. \tag{e 9.32}$$

Then

$$\begin{aligned}
(1+3\delta)\sigma_i(1_A \otimes 1_B) &= \gamma_i(1_A) \otimes \psi_i \circ \varphi(1_B) + \beta_i \circ \alpha(1_A) \otimes \theta_i(1_B) \\
&\quad + \gamma_i(1_A) \otimes \theta_i(1_B) \\
(\hat{a} \perp \check{a}, \hat{b} \perp \check{b}, i \neq j) &\lesssim (\sum_{i=1}^k a_i \otimes \hat{b}) + (\sum_{i=1}^k \hat{a} \otimes b_i) + a_{k+1} \otimes b_{k+1} \\
&\lesssim (\sum_{i=1}^k a_i \otimes \hat{b}) + (\sum_{i=1}^{k+1} \hat{a} \otimes b_i) \\
&\lesssim \check{a} \otimes \hat{b} + \hat{a} \otimes \check{b} \leq (\check{a} + \hat{a}) \otimes (\check{b} + \hat{b}) \lesssim a \otimes b \lesssim c.
\end{aligned}$$

This proves the claim. Then (4) of Definition 3.1 holds. It follows that $A \otimes B$ is asymptotically tracially in $\mathcal{X}_1 \otimes \mathcal{X}_2$.

□

Corollary 9.12. *Let A and B be unital separable simple C^*-algebras which are asymptotically tracially in $\mathcal{N}_n$. Then the spatial tensor product $A \otimes B$ is asymptotically tracially in $\mathcal{N}_1$.*

Proof. Note that $\mathcal{N}_n \otimes \mathcal{N}_n \subset \mathcal{N}_{2n+1}$ (see [72, Proposition 2.3(ii)]). Therefore, by Theorem 9.11, $A \otimes B$ is asymptotically tracially in $\mathcal{N}_{2n+1}$. By Theorem 9.3, $A \otimes B$ is asymptotically tracially in $\mathcal{N}_{2n+1,s,s}$. It follows from [10, Corollary C] that $A \otimes B$ is asymptotically tracially in $\mathcal{N}_{1,s,s}$.

□

Corollary 9.13. *Let A be a unital separable simple C^*-algebra and let B be a unital separable simple C^*-algebra which is asymptotically tracially in $\mathcal{C}_{\mathcal{Z},s}$. Then the spatial tensor product $A \otimes B$ is asymptotically tracially in $\mathcal{C}_{\mathcal{Z},s}$.*

Corollary 9.14. *Let A be a unital separable simple C^*-algebra which is asymptotically tracially in $\mathcal{N}$ and let B be a unital separable simple C^*-algebra which is asymptotically tracially in $\mathcal{N}_{\mathcal{Z}}$. Then the spatial tensor product $A \otimes B$ is asymptotically tracially in $\mathcal{N}_1$.*

Corollary 9.15. *Let A be a unital separable simple nuclear C^*-algebra and B be a unital separable simple C^*-algebra which is asymptotically tracially in $\mathcal{N}_n$. Then $A \otimes B$ is asymptotically tracially in $\mathcal{N}_1$.*

Remark 9.16. (1) There are unital separable nuclear simple C^*-algebras which are not asymptotically tracially in $\mathcal{N}_n$ for any $n \geq 0$.

Let A be one of Villadsen's examples of unital simple AH-algebras which has stable rank $r > 1$ (see [64]). Then A is nuclear and it is finite. However, if A were asymptotically tracially in $\mathcal{N}_n$ for some integer $n \geq 0$, then, by Theorem 9.1, A would have stable rank one as it cannot be purely infinite.

(2) There are unital separable nuclear simple C^*-algebras which have stable rank one but are not asymptotically tracially in $\mathcal{N}_n$ for any $n \geq 0$.

Let A be another construction of Villadsen's AH-algebra (see [63]) which is a unital separable nuclear simple C^*-algebra and has stable rank one. However, A does not have strict comparison for projections, this fact together with Theorem 9.1 and Theorem 8.7 implies that A is not asymptotically tracially in $\mathcal{N}_n$ for any $n \geq 0$.

(3) There are unital separable nuclear simple C^*-algebras which have stable rank one and unperforated K_0 group, but are not asymptotically tracially in $\mathcal{N}_n$ for any $n \geq 0$.

Let A be Toms's construction (see [61, Corollary 1.1]). Then A is a unital separable nuclear simple C^*-algebra with stable rank one which has unperforated K_0 group, but the Cuntz semigroup of A is not almost unperforated. Then, by Theorem 9.1 and Theorem 8.7, A is not asymptotically tracially in $\mathcal{N}_n$ for any $n \geq 0$.

Example 9.17. Let B be a unital separable simple C^*-algebra which has tracial rank zero but not exact (see [14], for example). Let C be any unital nuclear separable simple C^*-algebra. Consider $A = C \otimes B$. Since B is a non-exact C^*-subalgebra of A, it follows that A is not exact (see [55, 6.1.10(i)]) (thus non-nuclear) either. By Theorem 7.18, B is asymptotically tracially in $\mathcal{N}_{\mathcal{Z},s,s}$. By Corollary 9.13, A is asymptotically tracially in $\mathcal{C}_{\mathcal{Z},s}$. Since C is nuclear and B is asymptotically tracially in $\mathcal{N}$, then, by Theorem 9.11, we have that A is asymptotically tracially in $\mathcal{N}$. Then, by Theorem 8.7, A is asymptotically tracially in $\mathcal{N}_{\mathcal{Z},s,s}$. This provides a great number of examples of unital separable simple C^*-algebras which are asymptotically tracially in $\mathcal{N}_{\mathcal{Z},s,s}$ but not exact. For example, one may choose C to be a unital simple AH-algebra. Moreover, though $C \otimes B$ are not exact, they are "regular" in the sense that they have almost unperforated Cuntz semigroups and has strict comparison.

In a subsequent paper, we will show that unital separable simple C^*-algebras which are not exact but can exhaust all possible Elliott invariants.

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