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A direct approach to nonuniqueness and failure of compactness for the SQG equation

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A direct approach to nonuniqueness and failure of compactness for the SQG equation

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Abstract

We give an alternative proof of the nonuniqueness of weak solutions to the surface quasigeostrophic equation (SQG) first shown in [3]. Our approach proceeds directly at the level of the scalar field. Furthermore, we prove that every smooth scalar field with compact support that conserves the integral can be realized as a weak limit of solutions to SQG.

1. Introduction

In this work, we are concerned with the two-dimensional surface quasi-geostrophic (SQG) equation, which is the following transport equation for a scalar field $\theta : \mathbb{R} \times \mathbb{T}^2 \to \mathbb{R}$

$$\partial_t \theta + \nabla \cdot (\theta u) = 0, \quad u := \nabla^\perp \Lambda^{-1} \theta, \tag{1}$$

where $\Lambda = (-\Delta)^{1/2}$, and $\nabla^\perp = (-\partial_2, \partial_1)$.

This equation is a basic example of an *active scalar* equation, so called because the drift velocity u depends at every time (nonlocally) on the scalar field θ that is being transported. The SQG equation arises as a model in geophysical fluid dynamics, where it has applications to both meteorological and oceanic flows [17, 28]. In this context, the field θ represents temperature

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or surface buoyancy in a certain regime of stratified fluid flow. The equation has been studied extensively in the mathematical literature due to its close analogy with the 3D incompressible Euler equations and the problem of blowup for initially classical solutions, which remains open as it does for the Euler equations. A survey of developments is given in the introduction to [3].

Fundamental to the study of the SQG equation are the following basic conservation laws:

(a) For all sufficiently smooth solutions, the *Hamiltonian* $\frac{1}{2}\int_{\mathbb{T}^2} |\Lambda^{-1/2}\theta(x,t)|^2 \, dx$ remains constant.

(b) For all sufficiently smooth solutions, the L^p norms $\|\theta(t)\|_{L^p(\mathbb{T}^2)}$ remain constant $1 \leqslant p \leqslant \infty$, as do the integrals $\int_{\mathbb{T}^2} F(\theta(x,t))dx$ for any smooth function F.

(c) For all weak solutions to SQG, the integral $\int_{\mathbb{T}^2} \theta(x,t)dx$ remains constant.

(To prove (a), multiply (1) by $\Lambda^{-1}\theta$ and integrate by parts. To prove (b), use $\nabla \cdot u = 0$ to check that $F(\theta)$ satisfies $\partial_t F(\theta) + \nabla \cdot (F(\theta)u) = 0$, then integrate. To prove (c), simply integrate in space.)

Note that, in contrast to (c), the nonlinear laws (a) and (b) require that the solution is 'sufficiently smooth'. If one expects that turbulent SQG solutions have a dual energy cascade as in the Batchelor–Kraichnan predictions of 2D turbulence [3, 7, 8], then one has motivation to consider weak solutions that are not smooth. A basic open question for the SQG equations is then: what function spaces represent the minimal amount of smoothness required for the conservation laws to hold? This question is exactly the concern of the (generalized) *Onsager conjectures* for the SQG equation. A closely related problem is to find the minimal regularity that guarantees uniqueness for the initial value problem.

Using Hölder spaces as a natural scale to measure spatial regularity, the generalized Onsager conjectures for SQG can be stated as follows. (The space-time regularity below is on a finite time interval).

(a) If $\theta \in C^0$, then the conservation of the Hamiltonian holds. However, for any $0 < \alpha < 1/2$ there exist weak solutions of class $\Lambda^{-1/2}\theta \in L_t^\infty C_x^\alpha$ that fail to conserve the Hamiltonian.

(b) Let $F \in C^\infty(\mathbb{R})$. Then if $\theta \in L_t^\infty C_x^\alpha$ for some $\alpha > 1/3$, the law $\int_{\mathbb{T}^2} F(\theta(x,t))dx \equiv$ const is satisfied. On the other hand, for any $\alpha < 1/3$, there exist solutions $\theta \in L_t^\infty C_x^\alpha$ that violate this law.

Some remarks about these conjectures are in order:

(a) These conjectures generalize the original Onsager conjecture [27], which concerned turbulent dissipation in the incompressible Euler equations and stated that the Hölder exponent 1/3 should mark the threshold regularity for conservation of energy for solutions to the incompressible Euler equations. See [16] for discussion of the significance of Onsager's conjecture in turbulence theory.

(b) The conjectured threshold exponents are derived from the fact that the conservation law for sufficiently regular solutions has been proven in both cases (a) and (b). Namely, [23] proves conservation of the Hamiltonian for solutions with $\theta \in L^3(I \times \mathbb{T}^2)$, while [1] proves the conservation law (b) for $\alpha > 1/3$. The proofs are variants of the kinematic argument of [9], which proved energy conservation for the Euler equations above Onsager's conjectured threshold. One can also formulate an Onsager conjecture for any function space that embeds into $L_t^2 \dot{H}_x^{-1/2}$ but not into the L^3 type spaces considered in [1, 23].

(c) Following the seminal work [14], advances in the method of convex integration have made possible the pursuit of Onsager's conjecture both for the Euler equations and more general fluid equations. In particular, Onsager's conjecture for the 3D Euler equations has been proven in [21] (see also [2, 20]), while the first progress towards the Onsager conjecture

(a) for SQG has been made in [3]. See [4, 13] for surveys and [24] for a discussion of generalized Onsager conjectures.

(d) To make sense of the conjecture in part (a), it must be noted that the SQG equation is well-defined for θ having negative regularity. Namely, for any smooth vector field $\phi(x)$ on $\mathbb{T}^2$, the quadratic form $\int_{\mathbb{T}^2} \theta \nabla^\perp \Lambda^{-1}\theta \cdot \phi(x)\mathrm{d}x$, initially defined for smooth θ, has a unique bounded extension to $\theta \in H^{-1/2}$. This fact, which relies on the anti-self-adjointness of the operator $\nabla^\perp \Lambda^{-1}$, allows the SQG nonlinearity to be well-defined in $\mathcal{D}'$ for θ of class $\theta \in L_t^2 H_x^{-1/2}$ (see [3, definition 1.1]).

The boundedness of the nonlinearity in a negative Sobolev space is key to constructing weak solutions to the SQG equations by compactness methods [29]. These solutions obtained in [25, 29] are known to exist for all time and to have L^p norms bounded uniformly in t by the initial data, but it remains unknown whether they are uniquely determined by their initial data. Rather, known proofs of uniqueness require close to Lipschitz regularity (see also [10] for a uniqueness result on SQG patch solutions).

The [29] construction of weak solutions is closely tied to the phenomenon of **weak compactness** for SQG solutions. Namely weak limits of sequences of solutions to SQG in the space L^∞ weak-* must also be solutions to the SQG equations. The proof (see e.g. [23]) relies on the boundedness of the nonlinearity in a negative Sobolev space. The significance of weak compactness in the present context is that weak compactness appears to present an obstruction to attacking Onsager's conjecture and to proving nonuniqueness using convex integration methods when it is present [15].

Another point of view on the difficulty in applying convex integration to SQG, which is noted in [23], is that the most basic ansatz appears to fail. Namely, if one considers a correction of the form $\Theta = \sum_k \mathrm{e}^{\mathrm{i}\lambda k \cdot x}\theta_k(t, x)$, where each wave of index k has a conjugate wave of index $-k$, the leading order change in the low frequency part of the nonlinear term θu vanishes due to the oddness of the multiplier

$$\sum_k \left(\theta_{-k}\,\mathrm{e}^{-\mathrm{i}\lambda k \cdot x}\right) \nabla^\perp \Lambda^{-1} \left(\theta_k\,\mathrm{e}^{\mathrm{i}\lambda k \cdot x}\right) \approx \sum_k |\theta_k|^2 k^\perp |k|^{-1}$$

$$= \frac{1}{2}\sum_k |\theta_k|^2 \left(k^\perp |k|^{-1} + (-k)^\perp |-k|^{-1}\right) = 0.$$

As a result, when this ansatz is used, there is apparently no low frequency term in the expansion of the nonlinearity that can be used in convex integration to cancel the error in solving the equation.

The main goals of the present paper are to provide an alternative approach to the nonuniqueness for SQG first shown in [3], thus offering a different point of view from which to pursue the Onsager conjecture for SQG, and to establish an 'h-principle' result that demonstrates the stark failure of weak compactness for SQG weak solutions. Our main results are the following.

Theorem 1 (Existence of weak solutions).   *For any* $0 < \alpha < 3/10$ *and* $0 < \beta < 1/4$ *there exists nontrivial weak solutions* θ *to SQG with compact support in time such that the potential function* $\Lambda^{-1/2}\theta$ *is in the Hölder*[1] *class* $C_t^0 C_x^\alpha \cap C_t^\beta C_x^0(\mathbb{R} \times \mathbb{T}^2)$.

Theorem 2 (h-principle).   *Fix any* $\alpha < 3/10, \beta < 1/4$, *and let f be a smooth scalar field with compact support in time that satisfies the conservation law* $\int_{\mathbb{T}^2} f(x, t)\mathrm{d}x = 0$ *as a function*

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[1] Note that, since the domain is compact, we automatically obtain solutions in spaces with lower integrability into which Hölder spaces embed.

of time. Then there exists a sequence of solutions to SQG, $\{\theta_n\}_{n\in\mathbb{N}}$, such that each scalar field θ_n has compact support in time and a corresponding potential function $\Lambda^{-1/2}\theta_n$ of Hölder class $\Lambda^{-1/2}\theta_n \in C_t^\beta C_x^0 \cap C_t^0 C_x^\alpha(\mathbb{R} \times \mathbb{T}^2)$, and such that $\Lambda^{-1/2}\theta_n \rightharpoonup \Lambda^{-1/2}f$ in the $L^\infty(\mathbb{R} \times \mathbb{T}^2)$ weak-$*$ topology.

The 'h-principle' result stated in theorem 2 is a new result to this paper, while theorem 1 was first obtained in the work [3]. (The main result of [3] is stated in terms of $\Lambda^{-1}\theta$ rather than $\Lambda^{-1/2}\theta$, but in terms of spatial regularity the results are equivalent).

The main contributions of our work are as follows:

(a) One of the main ideas of [3] to apply convex integration to the SQG equation is to recast the SQG equation in the following *momentum form* for the unknown $v = \Lambda^{-1}u = \nabla^\perp \Delta^{-1}\theta$:

$$\partial_t v + u \cdot \nabla v - (\nabla v)^T \cdot u = -\nabla p, \qquad \operatorname{div} v = 0, \qquad u = \Lambda v.$$

The authors are then able to perform convex integration at the level of the vector field v rather than the scalar field θ. The scalar field θ can then be recovered from v by setting $\theta = -\nabla^\perp \cdot v$.

In this work, we take a different approach, working directly at the level of the scalar field θ. To execute this approach, we require the error term in the construction to have the structure of a second order divergence, as opposed to the first order divergence form used in [23]. We also employ the 'constant trick' from [11, 23, 30] that the divergence of a function of t is zero.

Given that the Onsager conjecture for SQG remains open, it is useful to have more than one approach to constructing SQG solutions. Having a second approach gives a separate angle from which to consider the problem and opens the door to considering more general active scalar equations. This approach might also be useful for the still open Onsager conjecture for 2D Euler. Furthermore, it is desirable from a physical point of view to have an approach that works directly at the level of θ, since the variable θ has a clear physical meaning. We also note there is yet a different approach to convex integration for SQG that has been obtained independently in [5]. This work proceeds at the level of $\Lambda^{-1}\theta$ and considers the steady state SQG equation.

(b) While typically theorems on the failure of compactness can be obtained as essentially a byproduct of a successful convex integration scheme, the argument of [3] is not quite structured towards proving an h-principle result. Our proof of theorem 2 involves taking a different organizational strategy in the estimates compared to [3]. The proof of the h-principle also involves an additional approximation step compared to similar results in [22, 23], as the convex integration scheme implemented here is required to have compact frequency support.

(c) A key idea of [3] is to express the nonlinearity $u \cdot \nabla v - (\nabla v)^T \cdot u$ as the sum of the divergence of a two-tensor and the gradient of a scalar function to have good control over high-high to low interactions. Here we show that the desired divergence form can be obtained for the nonlinearity $\theta\nabla^\perp\Lambda^{-1}\theta$ of SQG, and we extend the derivation so that it applies to general odd multipliers.

(d) We employ a 'bilinear microlocal lemma' analogous to [23, lemma 4.1]. This lemma allows us to obtain a better estimate on the low frequency part of the error compared to the corresponding technique in [3], which relies on the use of sharp time cutoffs to approximate nonlinear phase functions by their initial conditions. Having such an improved estimate is needed for schemes that aim to improve the Hölder regularity. The estimate we obtain is compatible with an ideal case scenario for SQG, so that the only terms now

limiting the regularity below the conjectured threshold are the high-frequency interference terms.

(e) Finally, by utilizing to the greatest extent possible the framework of estimates developed in [18] and the additional tools of [23], our proof identifies precisely which points require technical novelty in SQG compared to previous work on Euler and non-odd active scalars.

The statement of theorem 2 on the h-principle helps clarify an important point about the method of convex integration, which is that the method appears so far to apply only in cases where an appropriate h-principle implying failure of compactness can be proven [15]. Namely, theorem 2 shows that even though weak compactness does hold for SQG in L^∞, which prevents the schemes of [23, 30] for active scalar equations from applying to SQG, there is still failure of compactness in the space $\Lambda^{-1/2}\theta \in L^\infty$, thus permitting the scheme of [3] and the present work to succeed. The statement of theorem 2 is modeled off h-principles proven in [22, 23], which also relate weak limits of solutions to conservation laws. In general, 'h-principles' in PDE are modeled off the result of Nash [26] that C^0 limits of isometric immersions of closed n-manifolds into $\mathbb{R}^{n+2}$ can realize any smooth, short immersion. For further examples of h-principles in fluids, we refer to [2, 6, 12].

2. SQG-Reynolds flows

To state the main lemma we will introduce a notion of an SQG-Reynolds flow. The key new features are that we work at the level of the scalar field θ, and the error tensor is required to have a double-divergence form. We will consistently employ the summation notation for repeated indices.

Definition 2.1. A scalar-valued function $\theta : \mathbb{R} \times \mathbb{T}^2 \to \mathbb{R}$ and a symmetric, traceless tensor field $R^{jl} : \mathbb{R} \times \mathbb{T}^2 \to \mathbb{R}^{2 \times 2}$ satisfy the **SQG-Reynolds equations** if

$$\partial_t \theta + u^l \nabla_l \theta = \nabla_j \nabla_l R^{jl}$$

$$\operatorname{div} u = \nabla_l u^l = 0$$

$$u^l = T^l \theta,$$

where the operator T is given as in (1) by the Fourier multiplier $m^l(\xi) := i\epsilon^{la}\xi_a/|\xi|$ for $\xi \in \hat{\mathbb{R}}^2$. Here ϵ^{la} is the Levi-Civita symbol in two dimensions. By convention this symbol is defined as follows

$$\epsilon^{11} = \varepsilon^{22} = 0, \qquad \epsilon^{12} = 1, \qquad \epsilon^{21} = -1. \tag{2}$$

Any solution to the SQG-Reynold's equation, (θ, u, R), is called an **SQG-Reynolds flow**. The symmetric and traceless tensor field R^{jl} is called the **stress tensor**.

At times we will write (θ, R) to refer to the SQG-Reynolds flow, implying that $u^l = T^l[\theta]$.

2.1. Frequency and energy levels

We will also use a notion of frequency-energy levels similar to those used with the Euler equations in [18] but with the difference that we assume θ to be frequency localized similar to [3]. A key point is that the relevant fields are measured relative to the size of the stress tensor R^{jl}.

Definition 2.2. Let (θ, u, R) be a solution of the SQG-Reynolds equation, $\Xi \geqslant 1$, and $Ð_u \geqslant Ð_R \geqslant 0$ be non-negative numbers. Define the **advective derivative** associated to $T^l\theta$ to be

$D_t := \partial_t + T^l \theta \nabla_l$. We say that (θ, u, R) has **frequency-energy levels** below $(\Xi, Ð_u, Ð_R)$ to order L in C^0 if θ and R are of class $C_t^1 C_x^L(\mathbb{R} \times \mathbb{T}^2)$ and the following statements hold

$$\text{supp } \hat{\theta} \subseteq \{\xi : |\xi| \leqslant \Xi\}$$

$$\|\nabla_{\vec{a}}\theta\|_{C^0}, \|\nabla_{\vec{a}}u\|_{C^0} \leqslant \Xi^{|\vec{a}|} e_u^{1/2} \qquad \text{for all } |\vec{a}| = 0, \ldots, L$$

$$\|\nabla_{\vec{a}}R\|_{C^0} \leqslant \Xi^{|\vec{a}|} Ð_R \qquad \text{for all } |\vec{a}| = 0, \ldots, L$$

$$\|\nabla_{\vec{a}}D_t\theta\|_{C^0}, \|\nabla_{\vec{a}}D_t u\|_{C^0} \leqslant \Xi^{|\vec{a}|}(\Xi e_u^{1/2}) e_u^{1/2} \qquad \text{for all } |\vec{a}| = 0, \ldots, L-1$$

$$\|\nabla_{\vec{a}}D_t R\|_{C^0} \leqslant \Xi^{|\vec{a}|}(\Xi e_u^{1/2}) Ð_R \qquad \text{for all } |\vec{a}| = 0, \ldots, L-1$$

The quantity $(\Xi e_u^{1/2})^{-1}$ is the **natural time scale**, the term $e_u^{1/2} := \Xi^{1/2} Ð_u^{1/2}$, and ∇ refers only to derivatives in the spatial variables. We also define a second quantity $e_R := \Xi Ð_R$ that, like e_u, we think of as having units of 'energy density'.

　　Note that these estimates differ from previous work on the Euler equations and non-odd active scalars [23], where there is no need to distinguish between an error parameter $(Ð_u, Ð_R)$ and an energy level (e_u, e_R) and also no need to require a C^0 bound on the active scalar or velocity field. The relation $e_u^{1/2} := \Xi^{1/2} Ð_u^{1/2}$ is dictated by the SQG scheme.

3. Main lemma

Lemma 3.1 (Main lemma).　　*For $L \geqslant 2$ there exists a constant $\widehat{C}$ such that the following holds: given an SQG-Reynolds flow (θ, u, R) with frequency and energy levels below $(\Xi, Ð_u, Ð_R)$ and a non-empty closed interval, J, with $\text{supp}_t R \subseteq J \subseteq \mathbb{R}$. Let*

$$N \geqslant \left(\frac{Ð_u}{Ð_R} \right).$$

Then there exists an SQG-Reynolds flow $(\overset{*}{\theta}, \overset{*}{u}, \overset{*}{R})$ of the form $\overset{*}{\theta} = \theta + \Theta$, $\overset{*}{u} = u + T[\Theta]$ with frequency and energy levels bounded by

$$(\overset{*}{\Xi}, \overset{*}{Ð}_u, \overset{*}{Ð}_R) = (\widehat{C}N\Xi, Ð_R, GÐ_R), \quad \text{where } G := \left(\frac{Ð_u^{1/4}}{Ð_R^{1/4} N^{3/4}} \right) \qquad (3)$$

to order L in C^0. Furthermore the new stress $\overset{*}{R}$ and the correction Θ are supported in time in a neighborhood of J

$$\text{supp}_t \overset{*}{R} \cup \text{supp}_t \Theta \subseteq N(J) := \left\{ t + h : t \in J, |h| \leqslant 3(\Xi e_u^{1/2})^{-1} \right\}. \qquad (4)$$

Additionally one may arrange that $\Lambda^{-1/2}\Theta$ has the form $\Lambda^{-1/2}\Theta = \nabla_i W^i$ that satisfies the following statements

$$\|\nabla_{\vec{a}}\Lambda^{-1/2}\Theta\|_{C^0} \leqslant \widehat{C}(N\Xi)^{|\vec{a}|} Ð_R^{1/2}, \quad \text{for } |\vec{a}| \leqslant 1$$

$$\|W\|_{C^0} \leqslant \widehat{C}(N\Xi)^{-1} Ð_R^{1/2}$$

$$\|\partial_t \Lambda^{-1/2}\Theta\|_{C^0} \leqslant \widehat{C}Ð_R^{1/2} \left(N\Xi e_u^{1/2} \right),$$

where $e_u^{1/2}$ is defined in definition 2.2.

We note that the low regularity of the solutions is due to the combination of the estimates in the main lemma and the way they are iterated in the proof. See section 5.3 for further discussion.

4. Proof of main lemma

4.1. Shape of the scalar corrections

Our correction Θ is a sum of scalar valued waves Θ_I that oscillate at a large frequency λ, similar to the corrections defined in [19, section 4.2] or [23, section 5.1].

$$\Theta := \sum_I \Theta_I(x, t), \quad \Theta_I(x, t) := P_\lambda^I \left(e^{i\lambda\xi_I} \theta_I \right) = e^{i\lambda\xi_I} \left(\theta_I + \delta\theta_I \right). \tag{5}$$

The **frequency parameter** $\lambda \in 2\pi\mathbb{N}$ is on the order of $B_\lambda N \Xi$. More precisely,

$$\lambda \in [B_\lambda N \Xi, 2B_\lambda N \Xi] \cap (2\pi\mathbb{N}),$$

where $B_\lambda \geqslant 1$ a very large constant associated to λ that is chosen in section 4.8. The term $\xi_I := \xi_I(x, t)$ is a nonlinear **phase function**, $\theta_I := \theta_I(x, t)$ is the **amplitude**, and $\delta\theta_I := \delta\theta_I(x, t)$ is a small correction to ensure that Θ_I is compactly supported in frequency space. The second equality for Θ_I and the explicit formula for $\delta\theta_I$ comes from an application of the microlocal lemma 4.1 while the other components of the correction are specified below.

4.1.1. The index set. The index I for the wave Θ_I is a tuple $I = (k, f) \in \mathbb{Z} \times F$ where $F := \{\pm(1, 2), \pm(2, 1)\}$. Furthermore we define the set $\mathbb{F} := F/(+, -) = \{(1, 2), (2, 1)\}$. The index k specifies the interval of time support for the wave $\Theta_I = \Theta_{(k,f)}$ and f represents an initial direction of oscillation. For each index $I = (k, f) \in \mathbb{Z} \times F$ there is a conjugate index $\bar{I} = (k, \bar{f}) \in \mathbb{Z} \times F$ such that in the summation (5) each wave Θ_I has a conjugate wave $\Theta_{\bar{I}} := \overline{\Theta}_I$ with an opposite sign phase function $\xi_{\bar{I}} := -\xi_I$ and matching amplitude $\theta_{\bar{I}} := \overline{\theta}_I$ so that the overall summation (5) is real-valued. We also define the notation

$$[k] := \begin{cases} 0 & \text{if } k \bmod 2 \equiv 0 \\ 1 & \text{if } k \bmod 2 \equiv 1. \end{cases}$$

4.1.2. Phase functions. For a given SQG-Reynold's flow θ and each index $I = (k, f)$, the phase functions ξ_I are solutions to the transport equation

$$\left(\partial_t + T^l[\theta]\nabla_l \right) \xi_I = 0, \qquad \xi_I(t(I), x) = \hat{\xi}_I(x),$$

where the initial data is $\hat{\xi}_I(x) = \hat{\xi}_{(k,f)}(x) := J^k f \cdot x$ for $J :=$ a $90°$ rotation. The initial time is $t(I) := k\tau$ where τ is the time step from section 4.1.3. Furthermore, for a small fixed constant $c_1 > 0$ the time step τ in section 4.1.3 will be sufficiently small to ensure the phase function gradients stay close to their initial data, and stay bounded away from 0

$$\left| \nabla\xi_I - \nabla\hat{\xi}_I \right| \leqslant c_1, \qquad |\nabla\xi_I| \geqslant 1 > 0. \tag{6}$$

The constant c_1 is determined later by calculation in (31). It is taken to be sufficiently small to ensure that the phase gradients remain close to their initial conditions and stay in a neighborhood where the angular frequency localizing operator described in section 4.1.4 is 1. In particular $c_1 < 1/4$.

4.1.3. Time cutoffs. Define a family of time cutoffs similar to what is done in [23, section 5.2]. For a given index $I = (k, f)$ define the **time step** $\tau := (\Xi e_u^{1/2})^{-1} b$ where b is a small fraction defined by $b := \left(\frac{\text{Đ}_u^{1/2}}{\text{Đ}_R^{1/2} B_\lambda^{3/2} N^{3/2}} \right)^{1/2} b_0$ and $b_0 \leqslant 1$ is a uniform constant chosen to make τ sufficiently small enough to satisfy the conditions of (6). Note that τ is smaller than the natural time scale given in Definition 2.2. Additionally, we define time cutoff functions as elements of a rescaled partition of unity in time

$$\phi_k(t) := \phi \left(\frac{t - k\tau}{\tau} \right), \quad \text{where} \sum_{u \in \mathbb{Z}} \phi^2(t - u) = 1.$$

By our choice of ϕ_k each scalar correction Θ_I will be supported in a time interval $[k\tau - \frac{2}{3}\tau, k\tau + \frac{2}{3}\tau]$. The size of the time interval is chosen to optimize the size of the resulting errors between the transport stress and high frequency interference stress.

4.1.4. Angular frequency localizing operator. The operator P_λ^I localizes to frequencies of order λ in a neighborhood around $\lambda \nabla \hat{\xi}_I$. This is identical to the frequency localizing operator used in [23, section 5.1]. More precisely, for a function $f \in C^0(\mathbb{T}^2)$ and an index I we define frequency localizing operators in the following way:

$$P_\lambda^I f := \int_{\mathbb{R}^2} f(x - h) \chi_\lambda^I(h) \text{d}h,$$

where $\hat{\chi}_\lambda^I(\xi) := \hat{\chi}_1^I(\lambda^{-1}\xi)$ and we take $\hat{\chi}_1^I \in C_c^\infty(B_{|\nabla \hat{\xi}_I|/2}(\nabla \hat{\xi}_I))$ to be a smooth bump function such that $\hat{\chi}_1^I(\xi) = 1$ if $|\xi - \nabla \hat{\xi}_I| \leqslant |\nabla \hat{\xi}_I|/4$. Moreover it follows from this definition that the corrections Θ_I and Θ have frequency support contained in the annulus of order λ. That is

$$\text{supp } \hat{\Theta}_I \cup \text{supp } \hat{\Theta} \subseteq \{\xi \in \hat{\mathbb{R}}^2 : \lambda/2 \leqslant |\xi| \leqslant 2\lambda\}.$$

4.1.5. Lifting function. A lifting function is used in the calculation of the low-frequency stress error in section 4.6 below. Its purpose is to scale the wave corrections so that they are larger than the low-frequency stress error that is approximately of size Đ_R. Let τ be the time step from definition 2.2 and let K be a large constant to be determined later in the calculation (32). We choose $e(t) : \mathbb{R} \to \mathbb{R}_{\geqslant 0}$ to be a function such that $e(t) \geqslant K\text{Đ}_R$ for all $t \in \{s + h | s \in J, |h| \leqslant \epsilon_t + (\Xi e_u^{1/2})^{-1}\}$, where ϵ_t is the flow mollification parameter defined below in (8), such that the bounds

$$\left\| \left(\frac{\text{d}}{\text{d}t} \right)^r e(t)^{1/2} \right\|_{C^0} \lesssim (\Xi e_u^{1/2})^r \text{Đ}_R^{1/2}, \quad 0 \leqslant r \leqslant 2$$

hold, and such that e has time support $\text{supp}_t\, e(t) \subseteq N(J) := \{t + h : t \in J, |h| \leqslant 3(\Xi e_u^{1/2})^{-1}\}$. A simple way to construct $e(t)$ is to define $N_0(J) = \{t + h : t \in J, |h| \leqslant 2(\Xi e_u^{1/2})^{-1}\}$, take the characteristic function of $N_0(J)$, mollify in time with a non-negative convolution kernel $\eta_{(\Xi e_u^{1/2})^{-1}}$ supported in $\{|t| \leqslant (1/3)(\Xi e_u^{1/2})^{-1}\}$, and multiply by $(K\text{Đ}_R)^{1/2}$ to construct $e^{1/2}(t)$:

$$e^{1/2}(t) = (K\text{Đ}_R)^{1/2} \eta_{(\Xi e_u^{1/2})^{-1}} * \mathbf{1}_{N_0(J)}(t).$$

We remark that the last time support condition holds because $\epsilon_t \leqslant (\Xi e_u^{1/2})^{-1}$, which is verified in section 4.8 below.

4.1.6. Amplitudes. Given an index $I = (k, f)$, the amplitude $\theta_I(x, t)$ has the form

$$\theta_I(x, t) = \lambda^{1/2} \gamma_I(x, t) \phi_k(t) e^{1/2}(t). \tag{7}$$

The factors $e(t)$ and $\phi_k(t)$ are the previously defined lifting function and time-cutoff. The functions $\gamma_I : \mathbb{T}^2 \times \mathbb{R} \to \mathbb{R}$ are called the **coefficients**, and are specified in section 4.6. In particular γ_I depends on the choice of initial directions $\mathbb{F}$ from section 4.1.1 and we construct it to be close in absolute value to 1 for the duration of the time interval $\phi_k(t)$ is supported on.

4.2. Shape of the velocity corrections

By applying the microlocal lemma 4.1 with the convolution operator $T^l P^I_\lambda$, we can calculate the drift velocity correction $U_I := U_I(x, t) : \mathbb{T}^2 \times \mathbb{R} \to \mathbb{R}^2$ from Θ_I:

$$U^l_I := T^l \Theta_I = T^l P^I_\lambda \left[e^{i\lambda \xi_I} \theta_I \right] = e^{i\lambda \xi_I} \left(\theta_I m^l(\nabla \xi_I) + \delta u^l_I \right).$$

We define

$$U^l_I := e^{i\lambda \xi_I} \left(u^l_I + \delta u^l_I \right), \qquad u^l_I := \theta_I m^l(\nabla \xi_I)$$

with an explicit error term δu^l_I given by the microlocal lemma 4.1. Here we have used that m^l is homogeneous of degree zero to obtain $m^l(\lambda \nabla \xi_I) = m^l(\nabla \xi_I)$, and have used that the frequency cutoff in the symbol of P^I_λ is equal to 1 at the point $\lambda \nabla \xi_I$, due to the condition $|\nabla \xi_I - \nabla \hat{\xi}_I| \leqslant c_2$ in (6).

4.3. The microlocal lemma

We borrow the following lemma from [23, section 4], which shows that a convolution operator applied to a highly oscillatory function is multiplication operator to leading order. In all of our applications the kernel $K^l(h)$ will be a Schwartz function essentially supported online scales of order $|h| \sim \lambda^{-1}$. The Fourier-transform of a function $K^l : \mathbb{R}^2 \to \mathbb{C}$ is normalized to be $\hat{K}^l(\xi) = \int_{\mathbb{R}^2} e^{-i\xi \cdot h} K^l(h) \mathrm{d}h$.

Lemma 4.1 (Microlocal lemma). *Suppose that $T^l[\Theta](x) = \int_{\mathbb{R}^2} \Theta(x - h) K^l(h) \mathrm{d}h$ is a convolution operator acting on functions $\Theta : \mathbb{T}^2 \to \mathbb{C}$ with a kernel $K^l : \mathbb{R}^2 \to \mathbb{C}$ in the Schwartz class. Let $\xi : \mathbb{T}^2 \to \mathbb{C}$ and $\theta : \mathbb{T}^2 \to \mathbb{C}$ be smooth functions and $\lambda \in \mathbb{Z}$ be an integer. Then for any input of the form $\Theta = e^{i\lambda \xi(x)} \theta(x)$ we have the formula*

$$T^l[\Theta](x) = e^{i\lambda \xi(x)} \left(\theta(x) \hat{K}^l(\lambda \nabla \xi(x)) + \delta[T\Theta](x) \right),$$

where the error in the amplitude term has the explicit form

$$\delta[T^l \Theta](x) = \int_0^1 \mathrm{d}r \frac{\mathrm{d}}{\mathrm{d}r} \int_{\mathbb{R}^2} e^{-i\lambda \nabla \xi(x) \cdot h} e^{iZ(r, x, h)} \theta(x - rh) K^l(h) \mathrm{d}h$$

$$Z(r, x, h) = r\lambda \int_0^1 h^a h^b \nabla_a \nabla_b \xi(x - sh)(1 - s) \mathrm{d}s$$

The proof of this lemma is given in [23, section 4].

4.4. Mollification of the stress tensor

For the stress tensor, R, we define its spatial mollification R_{ϵ_x} by $R^{jl}_{\epsilon_x} := \chi_{\epsilon_x} * \chi_{\epsilon_x} * R^{jl}$, with a Schwartz kernel at length scale ϵ_x so that $\chi_{\epsilon_x}(h) = \epsilon_x^{-2} \chi_1(\epsilon_x^{-1} h)$ where $\chi_1 : \mathbb{R}^2 \to \mathbb{R}$ is a

Schwartz kernel with integral 1 with a vanishing moment condition that is $\int_{\mathbb{R}^2} h^{\vec{a}} \chi_{\epsilon_x}(h) \mathrm{d}h = 0$ for all multi-indices $1 \leqslant |\vec{a}| \leqslant L$. We next define the mollified stress tensor R_ϵ by a mollification of R_{ϵ_x} along the coarse scale flow in time:

$$R_\epsilon^{jl} = \eta_{\epsilon_t} *_\Phi R^{jl} := \int_{\mathbb{R}} R_{\epsilon_x}^{jl}(\Phi_s(t,x)) \eta_{\epsilon_t}(s) \mathrm{d}s.$$

Where $\eta_{\epsilon_t}(s) = \epsilon_t^{-1} \eta_1(\epsilon_t^{-1} s)$ is a smooth, positive mollifying kernel with compact support in the interval $|s| \leqslant \epsilon_t \leqslant (\Xi e_u^{1/2})^{-1}$ and $\int_{\mathbb{R}} \eta_{\epsilon_t}(s) \mathrm{d}s = 1$, while $\Phi_s(t,x)$ is the flow map of $\partial_t + u \cdot \nabla$, which takes values in $\mathbb{R} \times \mathbb{T}^2$ and is defined as the unique solution to the ODE

$$\Phi_s(t,x) = (t+s, \Phi_s^i(t,x)), \qquad \Phi_0(t,x) = (t,x), \qquad \frac{\mathrm{d}}{\mathrm{d}s} \Phi_s^i(t,x) = u^i(\Phi_s(t,x)).$$

We choose the mollification parameters to be

$$\epsilon_x = c_0 N^{-\frac{3}{2L}} \Xi^{-1}, \qquad \epsilon_t = c_0 N^{-\frac{3}{2}} \Xi^{-3/2} \mathrm{D}_R^{-1/2}. \tag{8}$$

By this choice of parameters we may obtain bounds that are similar to the bounds given in [18]. We note however that these choices of parameters are different from those in [18] and are tailored to the SQG error terms. Finally, the constant c_0 will only depend on the Schwartz kernels η_1, χ_1. We state C^0 norm estimates for R_ϵ in section 4.7.1 below.

4.5. The error terms

Let (θ, u, R) be an SQG-Reynolds flow as defined in section 2.1. We will construct the new scalar field $\overset{*}{\theta} := \theta + \Theta$ and $\overset{*}{u} = T[\overset{*}{\theta}]$ by adding a high frequency correction Θ. The new Reynolds stress solves

$$\nabla_j \nabla_l \overset{*}{R}^{jl} = \nabla_j \nabla_l R_\epsilon^{jl} + \nabla_j \nabla_l \left(R^{jl} - R_\epsilon^{jl} \right) + \left(\partial_t \Theta + T^l \theta \nabla_l \Theta + T^l \Theta \nabla_l \theta \right)$$
$$+ T^l \Theta \nabla_l \Theta.$$

If we recall that the corrections are of the form $\Theta = \sum_I \Theta_I$ then we can decompose the new error into the following terms

$$\nabla_j \nabla_l \overset{*}{R}^{jl} = \nabla_j \nabla_l \left(R_{\mathrm{T}}^{jl} + R_{\mathrm{H}}^{jl} + R_{\mathrm{M}}^{jl} + R_{\mathrm{S}}^{jl} \right)$$

$$\text{where } \nabla_j \nabla_l R_{\mathrm{H}}^{jl} = \sum_{J,I \in \mathbb{Z} \times F, J \neq \bar{I}} T^l[\Theta_I] \nabla_l \Theta_J + T^l[\Theta_J] \nabla_l \Theta_I$$

$$\nabla_j \nabla_l R_{\mathrm{T}}^{jl} = \partial_t \Theta + T^l \theta \nabla_l \Theta + T^l \Theta \nabla_l \theta$$

$$\nabla_j \nabla_l R_{\mathrm{M}}^{jl} = \nabla_j \nabla_l \left(R^{jl} - R_\epsilon^{jl} \right)$$

$$\nabla_j \nabla_l R_{\mathrm{S}}^{jl} = \nabla_j \nabla_l R_\epsilon^{jl} + \nabla_l \sum_{I \in \mathbb{Z} \times F} T^l[\Theta_I] \Theta_{\bar{I}}.$$

We refer to the term R_{S} as the **low frequency stress error**, R_{T} as the **transport stress error**, R_{H} as the **high frequency interference stress error**, and R_{M} as the **mollification stress error**. These terms are symmetric $(2,0)$-tensors. Let $\mathcal{R}$ be the second order anti-divergence operator

defined in section 4.5.1, let B_λ be the bilinear anti-divergence operator defined in (18), and let $M_{[k]}$ be the constant matrix defined in (24). We can define the stress errors using these tools as

$$\overset{*}{R}{}^{jl} := R_{\mathrm{H}}^{jl} + R_{\mathrm{T}}^{jl} + R_{\mathrm{M}}^{jl} + R_{\mathrm{S}}^{jl} \tag{9}$$

$$R_{\mathrm{H}}^{jl} := \sum_{J,I \in \mathbb{Z} \times F, J \neq \bar{I}} \mathcal{R}^{jl} \left[T^i[\Theta_I] \nabla_i \Theta_J + T^i[\Theta_J] \nabla_i \Theta_I \right] \tag{10}$$

$$R_{\mathrm{T}}^{jl} := \mathcal{R}^{jl} \left[\partial_t \Theta + T^i \theta \nabla_i \Theta + T^i \Theta \nabla_i \theta \right] \tag{11}$$

$$R_{\mathrm{M}}^{jl} := R^{jl} - R_\epsilon^{jl} \tag{12}$$

$$R_{\mathrm{S}}^{jl} := \sum_{I=(k,f) \in \mathbb{Z} \times \mathbb{F}} B_\lambda^{jl}[\Theta_I, \Theta_{\bar{I}}] - \sum_{k \in \mathbb{Z}} \phi_k^2(t) \left(e(t) M_{[k]}^{jl} - R_\epsilon^{jl} \right). \tag{13}$$

The operator $\mathcal{R}$ above satisfies $\nabla_j \nabla_l \mathcal{R}^{jl}[U] = U$ whenever U has mean zero. Note that the argument of $\mathcal{R}$ in lines (10) and (11) has mean zero due to the frequency support of $\partial_t \Theta$ and the fact that the other terms can be written in divergence form. In line (13), we use that $\nabla_j \nabla_l B_\lambda^{jl}[\Theta_I, \Theta_{\bar{I}}] = \nabla_l \left[T^l[\Theta_I] \Theta_{\bar{I}} + T^l[\Theta_{\bar{I}}] \Theta_I \right]$ and that the spatial divergence of a function of t alone is 0. An interesting geometric point is that, in contrast to previous uses of this trick, the function of t that we have subtracted in (13) oscillates in time.

In the following sections we define the operators used above as well as other tools used in estimating the stress tensors. To prove the main lemma we will prove estimates on each of the stress errors. What we will show is that for the weighted norm $\overset{*}{H}[\cdot]$ defined in lemma 4.3.

$$\overset{*}{H}[R_{\mathrm{T}}] \lesssim (B_\lambda N \Xi)^{-3/2} Đ_R^{1/2} (\Xi e_u^{1/2}) b^{-1}, \qquad \overset{*}{H}[R_{\mathrm{S}}] \lesssim B_\lambda^{-1} N^{-1} Đ_R,$$

$$\overset{*}{H}[R_{\mathrm{H}}] \lesssim b Đ_R, \qquad \overset{*}{H}[R_{\mathrm{M}}] \leqslant \frac{G Đ_R}{1000}.$$

Note that the choice of b in section 4.1.3 optimizes between R_{T} and R_{H}, which are the largest errors.

4.5.1. Second order anti-divergence equation.
We use a second order anti-divergence operator to produce a symmetric and traceless (2, 0) tensor for the transport stress error and the high-high interference error. For all smooth, mean-zero $f : \mathbb{C}^2 \to \mathbb{C}$, we define the **second order anti-divergence operator** as the operator

$$\mathcal{R} := 2\mathcal{R}_1 - \mathcal{R}_2, \tag{14}$$

where $\mathcal{R}_1$ and $\mathcal{R}_2$ are the second order anti-divergence operators $\mathcal{R}_1^{jl} := \nabla^j \nabla^l \Delta^{-2}$, $\mathcal{R}_2^{jl} := \delta^{jl} \Delta^{-1}$. We note that $\mathcal{R}^{jl}[f]$ is a symmetric, traceless tensor that satisfies $\nabla_j \nabla_l \mathcal{R}^{jl}[f] = f$. The need for an order -2 operator, rather than the usual order -1 operator, arises because our error term has a second order divergence form.

4.5.2. A bilinear anti-divergence form.
We use a bilinear first order anti-divergence operator to produce a symmetric, traceless (2, 0) tensor for the low frequency stress error. The work of this section is motivated by [3, section 5.4.2]. Deriving and estimating this bilinear form is a key ingredient for working on SQG compared to non-odd active scalar equations.

Consider the problem of solving $\nabla_j R^{j\ell} = Q^l$ for the term

$$Q^l(x) := T^l[\Theta_I] \Theta_J + T^l[\Theta_J] \Theta_I.$$

Although it is not obvious that an anti-divergence for Q^l would have a good form, there is at least hope of finding an anti-divergence for Q^l as Q^l satisfies the necessary and sufficient condition of having integral 0 due to the anti-self-adjointness of T^l, which is equivalent to the multiplier m^l being odd. Since the anti-self-adjointness is most easily seen in frequency space, there is motivation to look at the problem in frequency space[2] to define R^{jl}.

Recall now that the corrections from section 4.1 are constructed with compact frequency support in the annulus $\{\xi : 2^{-1}\lambda \leqslant |\xi| < 2\lambda\}$. We start by replacing Θ_I, Θ_J with Schwartz approximations that we will also call Θ_I, Θ_J by an abuse of notation. By using a mollification in frequency space we may assume that the Schwartz approximations maintain frequency support in $\{\xi : 2^{-1}\lambda \leqslant |\xi| < 2\lambda\}$, and that they converge pointwise in physical space while also remaining uniformly bounded there. For these approximations we construct a bilinear anti-divergence as follows. Define a smooth bump function $\hat{\chi}_{\approx\lambda}$ that is 1 on the annulus $\{\xi : 2^{-1}\lambda \leqslant |\xi| < 2\lambda\}$ so that $\hat{\Theta}_I = \hat{\chi}_{\approx\lambda}\hat{\Theta}_I$ and $\hat{\Theta}_J = \hat{\chi}_{\approx\lambda}\hat{\Theta}_J$. Recall from definition 2.1 that m is the multiplier for T and take a Fourier transform of $Q^l(x)$.

$$\hat{Q}^l(\xi) = \int_{\hat{\mathbb{R}}^2} m^l(\xi - \eta)\hat{\Theta}_I(\xi - \eta)\hat{\Theta}_J(\eta) + \hat{\Theta}_I(\xi - \eta)m^l(\eta)\hat{\Theta}_J(\eta)\frac{\mathrm{d}\eta}{(2\pi)^2}$$

$$= \int_{\hat{\mathbb{R}}^2} m^l(\xi - \eta)\hat{\chi}_{\approx\lambda}\hat{\Theta}_I(\xi - \eta)\hat{\Theta}_J(\eta) + \hat{\Theta}_I(\xi - \eta)m^l(\eta)\hat{\chi}_{\approx\lambda}\hat{\Theta}_J(\eta)\frac{\mathrm{d}\eta}{(2\pi)^2}$$

Define $m_{\approx\lambda}(\cdot) := m(\cdot)\hat{\chi}_{\approx\lambda}(\cdot)$

$$= \int_{\hat{\mathbb{R}}^2} \left[m^l_{\approx\lambda}(\xi - \eta) + m^l_{\approx\lambda}(\eta)\right] \hat{\Theta}_I(\xi - \eta)\hat{\Theta}_J(\eta)\frac{\mathrm{d}\eta}{(2\pi)^2}$$

By the oddness of the multiplier $m^l_{\approx\lambda}$

$$= \int_{\hat{\mathbb{R}}^2} \left[m^l_{\approx\lambda}(\xi - \eta) - m^l_{\approx\lambda}(-\eta)\right] \hat{\Theta}_I(\xi - \eta)\hat{\Theta}_J(\eta)\frac{\mathrm{d}\eta}{(2\pi)^2}$$

$$= i\xi_j \int_{\hat{\mathbb{R}}^2} \int_0^1 - i\nabla^j m^l_{\approx\lambda}(u_\sigma)\mathrm{d}\sigma \, \hat{\Theta}_I(\zeta)\hat{\Theta}_J(\eta)\frac{\mathrm{d}\eta}{(2\pi)^2}$$

$$=: i\xi_j \int_{\hat{\mathbb{R}}^2} \hat{K}^{jl}_\lambda(\zeta, \eta)\hat{\Theta}_I(\zeta)\hat{\Theta}_J(\eta)\frac{\mathrm{d}\eta}{(2\pi)^2}, \tag{15}$$

where we have set

$$\zeta := \xi - \eta, \qquad u_\sigma := \sigma(\xi - \eta) - (1 - \sigma)\eta,$$

and we have used Taylor's remainder theorem in the last equality. In the last line we defined

$$\hat{K}^{jl}_\lambda(\zeta, \eta) := \int_0^1 - i\nabla^j m^l_{\approx\lambda}(u_\sigma)\hat{\chi}_{\approx\lambda}(\zeta)\hat{\chi}_{\approx\lambda}(\eta)\mathrm{d}\sigma. \tag{16}$$

Line (15) defines in frequency space a solution R^{jl} to $\nabla_j R^{jl} = Q^l$ that is not symmetric, whereas our ultimate goal is in fact to define a solution to $\nabla_j \nabla_l R^{jl} = \nabla_l Q^l$ that is symmetric. Therefore we proceed by taking the symmetric part of the above solution (noting that $\nabla_j \nabla_l R^{jl}_{\mathrm{asym}} = 0$ for the anti-symmetric part), and taking an inverse Fourier transform.

---

[2] See also [22] for a different anti-divergence operator derived using Taylor expansion in frequency space.

Define the symmetric part of this kernel as

$$\hat{K}^{jl}_{\lambda,\text{sym}} := \frac{\hat{K}^{jl}_\lambda + \hat{K}^{lj}_\lambda}{2}. \tag{17}$$

Note that $\hat{K}^{jl}_{\lambda,\text{sym}} = \hat{K}^{lj}_{\lambda,\text{sym}}$ is symmetric. Now consider

$$\hat{Q}^l_2(\xi) := i\xi_j \int_{\hat{\mathbb{R}}^2} \hat{K}^{jl}_{\lambda,\text{sym}}(\zeta,\eta)\hat{\Theta}_I(\zeta)\hat{\Theta}_J(\eta)\frac{\mathrm{d}\eta}{(2\pi)^2}$$

and take the inverse Fourier transform to obtain

$$Q^l_2(x) = \nabla_j \int_{\hat{\mathbb{R}}^2} e^{i\xi\cdot x} \int_{\hat{\mathbb{R}}^2} \hat{K}^{jl}_{\lambda,\text{sym}}(\zeta,\eta)\hat{\Theta}_I(\zeta)\hat{\Theta}_J(\eta)\frac{\mathrm{d}\eta}{(2\pi)^2}\frac{\mathrm{d}\xi}{(2\pi)^2}$$

$$= \nabla_j \int_{\hat{\mathbb{R}}^2\times\hat{\mathbb{R}}^2} e^{i(\zeta+\eta)\cdot x}\hat{K}^{jl}_{\lambda,\text{sym}}(\zeta,\eta)\hat{\Theta}_I(\zeta)\hat{\Theta}_J(\eta)\frac{\mathrm{d}\eta}{(2\pi)^2}\frac{\mathrm{d}\zeta}{(2\pi)^2}.$$

Observe that $\hat{K}^{j\ell}_{\lambda,\text{sym}}$ is compactly supported in frequency and smooth due to the factors of $\hat{\chi}_{\approx\lambda}$, making $\hat{K}^{j\ell}_{\lambda,\text{sym}}$ a Schwartz function. Thus we can define the corresponding physical space kernel $K^{jl}_{\lambda,\text{sym}}(h_1,h_2)$ as the inverse Fourier transform of $\hat{K}^{jl}_{\lambda,\text{sym}}(\zeta,\eta)$ so that

$$\hat{K}^{jl}_{\lambda,\text{sym}}(\zeta,\eta) = \int_{\hat{\mathbb{R}}^2\times\hat{\mathbb{R}}^2} e^{-i(\zeta,\eta)\cdot(h_1,h_2)} K^{jl}_{\lambda,\text{sym}}(h_1,h_2)\mathrm{d}h_1\,\mathrm{d}h_2.$$

Then

$$Q^l_2(x) = \nabla_j \int_{\mathbb{R}^2\times\mathbb{R}^2} e^{i(\zeta+\eta)\cdot x} \int_{\hat{\mathbb{R}}^2\times\hat{\mathbb{R}}^2} e^{-i(\zeta,\eta)\cdot(h_1,h_2)} K^{jl}_{\lambda,\text{sym}}$$

$$\times (h_1,h_2)\mathrm{d}h_1\,\mathrm{d}h_2\,\hat{\Theta}_I(\zeta)\hat{\Theta}_J(\eta)\frac{\mathrm{d}\eta}{(2\pi)^2}\frac{\mathrm{d}\zeta}{(2\pi)^2}$$

$$= \nabla_j \int_{\mathbb{R}^2\times\mathbb{R}^2} K^{jl}_{\lambda,\text{sym}}(h_1,h_2) \int_{\hat{\mathbb{R}}^2} e^{i\zeta\cdot(x-h_1)}\hat{\Theta}_I(\zeta)\frac{\mathrm{d}\zeta}{(2\pi)^2}$$

$$\times \int_{\hat{\mathbb{R}}^2} e^{i\eta\cdot(x-h_2)}\hat{\Theta}_J(\eta)\frac{\mathrm{d}\eta}{(2\pi)^2}\,\mathrm{d}h_1\,\mathrm{d}h_2$$

$$= \nabla_j \int_{\mathbb{R}^2\times\mathbb{R}^2} K^{jl}_{\lambda,\text{sym}}(h_1,h_2)\Theta_I(x-h_1)\Theta_J(x-h_2)\mathrm{d}h_1\,\mathrm{d}h_2$$

$$=: \nabla_j B^{jl}_\lambda[\Theta_I,\Theta_J](x),$$

Here we define the **bilinear anti-divergence operator** or **bilinear form** as

$$B^{jl}_\lambda[F_1,F_2](x) := \int_{\mathbb{R}^2\times\mathbb{R}^2} K^{jl}_{\lambda,\text{sym}}(h_1,h_2)F_1(x-h_1)F_2(x-h_2)\mathrm{d}h_1\,\mathrm{d}h_2, \tag{18}$$

for scalar fields $F_1, F_2 \in C^\infty(\mathbb{T}^2)$. This bi-convolution operator satisfies

$$T^l[\Theta_I]\nabla_l\Theta_J + T^l[\Theta_J]\nabla_l\Theta_I = \nabla_l\nabla_j B^{jl}_\lambda[\Theta_I,\Theta_J](x) \tag{19}$$

for all Schwartz approximations to Θ_I, Θ_J with Fourier support in $\{|\xi| \sim \lambda\}$. Since $\|K_{\lambda,\text{sym}}\|_{L^1(\mathbb{R}^2)}$ is finite, we may pass to the limit in (18) from Schwartz approximations using

the dominated convergence theorem to conclude that (19) holds (both in $\mathcal{D}'$ and classically) for the periodic functions Θ_I, Θ_J. We note that by scaling considerations we have the following inequality for $h = (h_1, h_2) \in \mathbb{R}^2 \times \mathbb{R}^2$

$$\lambda^m \||h|^m K_{\lambda,\text{sym}}^{j\ell}(h_1, h_2)\|_{L^1(\mathbb{R}^2 \times \mathbb{R}^2)} \lesssim_m \lambda^{-1}, \tag{20}$$

and we also observe for use in the calculations of section 4.6.2 that, by (16),

$$\hat{K}_{\lambda,\text{sym}}(\lambda \nabla \xi_I, -\lambda \nabla \xi_I) = -i(2\lambda)^{-1}(\nabla^j m^l + \nabla^l m^j)(\nabla \xi_I) \tag{21}$$

is homogeneous of degree -1 in λ and in $\nabla \xi_I$. In the above we have used that ∇m is degree -1 homogeneous, and that $\hat{\chi}_{\approx \lambda}(\lambda \nabla \xi_I) = 1$ while $\nabla \hat{\chi}_{\approx \lambda}(\lambda \nabla \xi_I) = 0$ due to the condition (6), which keeps the phase gradient within a small distance of its initial conditions.

4.6. Low frequency part of the error

The goal of this section is to explain how the wave corrections described in section 4.1 are used to cancel the Reynolds stress error in the low frequency stress error. The tools described in this section are similar to ones used in [23]. In this section we will define the coefficient functions γ.

4.6.1. The bilinear microlocal lemma. The lemma below is inspired from an analogous version used in [23] and stated in lemma 4.1. This tool tells us that the bilinear convolution operator acts like a multiplication operator to leading order when it is applied to our highly oscillatory scalar field corrections Θ.

Lemma 4.2 (Bilinear microlocal lemma). *For conjugate scalar field corrections $\Theta_I, \Theta_{\bar{I}}$ as defined in section 4.1 and a bilinear form B_λ as defined in (18) we have the following identity*

$$B_\lambda^{j\ell}[\Theta_I, \Theta_{\bar{I}}](x, t) = \theta_I^2(x, t) \hat{K}_{\lambda,\text{sym}}^{j\ell}[\lambda \nabla \xi_I, -\lambda \nabla \xi_I] + \delta B_I^{j\ell}(x, t),$$

where $\delta B^{j\ell}$ is a small error term that has the explicit form

$$\delta B_I^{j\ell}(x, t) := \int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{i\lambda \nabla \xi_I(x) \cdot h_1} e^{-i\lambda \nabla \xi_I(x) \cdot h_2}$$

$$\times K_{\lambda,sym}^{j\ell}[h_1, h_2] \theta_I(x) Y(x, h_1) \mathrm{d}h_1 \, \mathrm{d}h_2$$

$$+ \int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{i\lambda \nabla \xi_I(x) \cdot h_1} e^{-i\lambda \nabla \xi_I(x) \cdot h_2}$$

$$\times K_{\lambda,sym}^{j\ell}[h_1, h_2] \theta_I(x) Y(x, h_2) \mathrm{d}h_1 \, \mathrm{d}h_2$$

$$+ \int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{i\lambda \nabla \xi_I(x) \cdot h_1} e^{-i\lambda \nabla \xi_I(x) \cdot h_2}$$

$$\times K_{\lambda,sym}^{j\ell}[h_1, h_2] Y(x, h_1) \cdot Y(x, h_2) \mathrm{d}h_1 \, \mathrm{d}h_2$$

$$Y(x, h) := \int_0^1 \frac{\mathrm{d}}{\mathrm{d}r} \left[e^{iZ(r,x,h)} \theta_I(x - rh) \right] \mathrm{d}r$$

$$Z(r, x, h) := r\lambda \int_0^1 h^a h^b \nabla_a \nabla_b \xi_I(x - sh)(1 - s) \mathrm{d}s.$$

Proof. The proof is based on the Taylor expansion as in the microlocal lemma 4.1. The starting point for the computation is the approximation

$$
\begin{aligned}
B_\lambda^{j\ell}[\Theta_I, \Theta_{\bar{I}}] &= \int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{i\lambda\xi_I(x-h_1)} e^{-i\lambda\xi_I(x-h_2)} \theta_I(x-h_1)\theta_I(x-h_2) \\
&\qquad \times K_{\lambda,\mathrm{sym}}^{jl}[h_1, h_2]\mathrm{d}h_1\,\mathrm{d}h_2 \\
&= \int_{\mathbb{R}^2 \times \mathbb{R}^2} \left[e^{i\lambda(\xi_I(x-h_1)-\xi_I(x))}\theta_I(x-h_1) \right] \\
&\qquad \times \left[e^{-i\lambda(\xi_I(x-h_2)-\xi_I(x))}\theta_I(x-h_2) \right] K_{\lambda,\mathrm{sym}}^{jl}[h_1, h_2]\mathrm{d}h_1\,\mathrm{d}h_2 \\
&\approx \theta_I(x)^2 \int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{-i\lambda\nabla\xi_I \cdot h_1} e^{i\lambda\nabla\xi_I \cdot h_2} K_{\lambda,\mathrm{sym}}^{jl}[h_1, h_2]\mathrm{d}h_1\,\mathrm{d}h_2,
\end{aligned}
$$

where we recognize that the last integral is exactly $\widehat{K}_{\lambda,\mathrm{sym}}^{jl}[\lambda\nabla\xi_I(x), -\lambda\nabla\xi_I(x)]$. $\qquad\square$

4.6.2. Explicitly calculating the symmetric multiplier.
We explicitly calculate the $\hat{K}_{\lambda,\mathrm{sym}}^{j\ell}$ term. Starting from $m^l(p) = i\epsilon^{\ell a}p_a/|p|$ and (21), we have

$$
\begin{aligned}
\hat{K}_{\lambda,\mathrm{sym}}^{j\ell}(\lambda\nabla\xi_I, -\lambda\nabla\xi_I) &= -2^{-1}\lambda^{-1}i\left(\nabla^j m^\ell + \nabla^\ell m^j\right)(\nabla\xi_I) \\
&= (-2\lambda)^{-1}\left(\frac{\epsilon^{\ell a}\nabla_a\xi_I\nabla^j\xi_I + \epsilon^{ja}\nabla_a\xi_I\nabla^\ell\xi_I}{|\nabla\xi_I|^3}\right)
\end{aligned}
\tag{22}
$$

$$
\hat{K}_{\lambda,\mathrm{sym}}(\lambda\nabla\xi_I, -\lambda\nabla\xi_I) = (2\lambda)^{-1}\begin{bmatrix} -2\nabla_1\xi_I\nabla_2\xi_I & (\nabla_1\xi_I)^2 - (\nabla_2\xi_I)^2 \\ (\nabla_1\xi_I)^2 - (\nabla_2\xi_I)^2 & 2\nabla_1\xi_I\nabla_2\xi_I \end{bmatrix} |\nabla\xi_I|^{-3}.
$$

In particular, the result is traceless and homogeneous of degree -1 in λ and the phase gradients.

4.6.3. Solving the quadratic equation using linear phase functions.
Our goal now is to define coefficients γ_I such that the low frequency error term (13) is close to zero. The overarching theme of the following two sections is similar to other *geometric lemmas* that have appeared in convex integration [14], while the specific details are unique to the current SQG scheme.

We will treat the term R_ϵ as a lower order term, and will treat the phase gradients $\nabla\xi_I$ as perturbations of their initial conditions $\nabla\hat{\xi}_I$, which are constants. Therefore, employing the approximation

$$
B_\lambda[\Theta_I, \Theta_{\bar{I}}] \approx \theta_I^2 \hat{K}_{\lambda,\mathrm{sym}}^{jl}[\lambda\nabla\xi_I, -\lambda\nabla\xi_I] \approx \theta_I^2 \hat{K}_{\lambda,\mathrm{sym}}^{jl}[\lambda\nabla\hat{\xi}_I, -\lambda\nabla\hat{\xi}_I]
$$

we are motivated by (13) to consider the following linear system

$$
\sum_{I \in \{k\} \times \mathbb{F}} \theta_I^2(x, t)\hat{K}_{\lambda,\mathrm{sym}}^{jl}[\lambda\nabla\hat{\xi}_I, -\lambda\nabla\hat{\xi}_I] = \phi_k^2(t)e(t)M_{[k]}^{jl}.
\tag{23}
$$

In this case, the $\gamma_I(x, t) = \hat{\gamma}_I$ will be constant coefficients and we assume the linear initial conditions $\nabla\hat{\xi}_I$ defined in section 4.1.2. We define a set of constant matrices $M_{[k]}$ as

$$
M_{[k]} := 2^{-1}5^{-3/2}\begin{bmatrix} -8 & 0 \\ 0 & 8 \end{bmatrix} \quad \text{if } k \equiv 0 \mod 2,
$$

$$
M_{[k]} := 2^{-1}5^{-3/2}\begin{bmatrix} 8 & 0 \\ 0 & -8 \end{bmatrix} \quad \text{if } k \equiv 1 \mod 2.
$$

For any index $I = (k, f) \in \mathbb{Z} \times \mathbb{F}$ as defined in section 4.1.1 we can explicitly calculate the left hand tensor of (23) in terms of

$$
\hat{K}^{j\ell}_{\text{sym}}(\nabla \hat{\xi}_{(k,J^k(1,2))}, -\nabla \hat{\xi}_{(k,J^k(1,2))}) =
\begin{cases}
2^{-1}5^{-3/2} \begin{bmatrix} -4 & -3 \\ -3 & 4 \end{bmatrix} & \text{if } [k] = 0 \\[2em]
2^{-1}5^{-3/2} \begin{bmatrix} 4 & 3 \\ 3 & -4 \end{bmatrix} & \text{if } [k] = 1
\end{cases}
$$

$$
\hat{K}^{j\ell}_{\text{sym}}(\nabla \hat{\xi}_{(k,J^k(2,1))}, -\nabla \hat{\xi}_{(k,J^k(2,1))}) =
\begin{cases}
2^{-1}5^{-3/2} \begin{bmatrix} -4 & 3 \\ 3 & 4 \end{bmatrix} & \text{if } [k] = 0 \\[2em]
2^{-1}5^{-3/2} \begin{bmatrix} 4 & -3 \\ -3 & -4 \end{bmatrix} & \text{if } [k] = 1.
\end{cases} \tag{24}
$$

We observe that in either the case $[k] = 0$ or $[k] = 1$ the matrix pair is linearly independent and hence crucially spans the two dimensional space of symmetric, traceless 2×2 matrices. We also note that

$$
\hat{K}^{j\ell}_{\lambda,\text{sym}} \left[\lambda \nabla \hat{\xi}_I, -\lambda \nabla \hat{\xi}_I \right] = \lambda^{-1} \hat{K}^{j\ell}_{\text{sym}} \left[\nabla \hat{\xi}_I, -\nabla \hat{\xi}_I \right].
$$

By applying this identity to (23) and recalling $\theta_I^2 = \lambda \gamma_I^2 e(t) \phi_k(t)$ from section 4.1.6 then for any $k \in \mathbb{Z}$ solving (23) becomes equivalent to solving the following equation of matrices

$$
M_{[k]} = \hat{\gamma}^2_{([k],J^k(1,2))} \hat{K}^{j\ell}_{\text{sym}} \left(\nabla \hat{\xi}_{(k,J^k(1,2))}, -\nabla \hat{\xi}_{(k,J^k(1,2))} \right)
$$
$$
+ \hat{\gamma}^2_{([k],J^k(2,1))} \hat{K}^{j\ell}_{\text{sym}} \left(\nabla \hat{\xi}_{(k,J^k(2,1))}, -\nabla \hat{\xi}_{(k,J^k(2,1))} \right). \tag{25}
$$

The equation of matrices above is solvable by using constant coefficients $\hat{\gamma}^2_{([k],J^k(1,2))} = \hat{\gamma}^2_{([k],J^k(2,1))} = 1$.

4.6.4. Solving the cancellation as a perturbation of a linear system. As typical with convex integration schemes we will perform the error cancellation on a mollified version of the stress tensor R_ϵ as defined in section 4.4. Please refer to [18, section 18] for detailed discussion of the mollified stress tensor. Consider the term $\nabla_j \nabla_j R_S^{jl}$ from section 4.5

$$
\nabla_j \nabla_\ell R_S^{j\ell} = \nabla_j \nabla_l R_\epsilon^{jl} + \nabla_l \sum_{I=(k,f)\in\mathbb{Z}\times F} T^l[\Theta_I]\Theta_{\bar{I}}. \tag{26}
$$

Let $e(t)$ be the lifting function defined in section 4.1.5. Also let $M_{[k]}$ be the constant matrix defined in (24) and let B_λ be the bilinear form defined in (18). Then (26) is equivalent to

$$
(26) = \nabla_j \nabla_\ell \left[\sum_{I=(k,f)\in\mathbb{Z}\times\mathbb{F}} B_\lambda^{jl}[\Theta_I, \Theta_{\bar{I}}] - \sum_{k\in\mathbb{Z}} \phi_k^2(t) \left(e(t) M_{[k]}^{jl} - R_\epsilon^{jl} \right) \right].
$$

If we drop the double divergence from the above expression then we recover the definition of R_S from (13). We can then apply the bilinear microlocal lemma 4.2 to R_S to produce

$$R_S^{j\ell} = \sum_{I=(k,f)\in\mathbb{Z}\times\mathbb{F}} \theta_I^2(x,t)\hat{K}_{\lambda,\text{sym}}^{j\ell}[\lambda\nabla\xi_I, -\lambda\nabla\xi_I]$$

$$- \sum_k e(t)\phi_k^2(t)\left(M_{[k]}^{j\ell} - \frac{R_\epsilon^{j\ell}}{e(t)}\right) + \sum_I \delta B_I^{j\ell}(x,t). \tag{27}$$

We will show that on every time interval, indexed by k, the first two summands add to zero pointwise in (x,t). Without loss of generality consider any single time interval indexed by k. We will prove

$$\sum_{I\in\{k\}\times\mathbb{F}} \theta_I^2(x,t)\hat{K}_{\lambda,\text{sym}}^{j\ell}[\lambda\nabla\xi_I, -\lambda\nabla\xi_I] - e(t)\phi_k^2(t)\left(M_{[k]}^{j\ell} - \frac{R_\epsilon^{j\ell}}{e(t)}\right) = 0. \tag{28}$$

So recall $\theta_I^2 = \lambda\gamma_I^2 e(t)\phi_k^2(t)$ from section 4.1.6 and that $\hat{K}_{\text{sym}}^{j\ell}$ has homogeneity order -1 from (22). By factoring out appropriately we can see that the left-hand side of (28) vanishes if we can solve

$$\sum_{I\in\{k\}\times\mathbb{F}} \gamma_I^2 \hat{K}_{\lambda,\text{sym}}^{jl}[\nabla\xi_I, -\nabla\xi_I] = M_{[k]}^{jl} - \varepsilon^{jl}, \quad \text{where } \varepsilon^{jl} := \frac{R_\epsilon^{jl}}{e(t)}. \tag{29}$$

We view (29) as a perturbation of our solved linear system (23). Let $\mathring{S}_{2\times2}$ be the space symmetric, traceless (2, 0) tensors. For parameters (p_1, p_2) define the linear map $\mathcal{L}_{(p_1,p_2)} : \mathbb{R}\times\mathbb{R} \to \mathring{S}_{2\times2}$

$$\mathcal{L}_{(p_1,p_2)}(x_1, x_2) = \left(x_1\hat{K}_{\lambda,\text{sym}}^{j\ell}[p_1, -p_1] + x_2\hat{K}_{\lambda,\text{sym}}^{j\ell}[p_2, -p_2]\right).$$

If we set

$$p_1 = \nabla\hat{\xi}_{I_1}, \qquad p_2 = \nabla\hat{\xi}_{I_2} \qquad I_1 = [k]\times J^k(1,2), \qquad I_2 = [k]\times J^k(2,1)$$

then by the calculation (25) we know that $\mathcal{L}_{(\nabla\hat{\xi}_{I_1},\nabla\hat{\xi}_{I_2})}(\hat{\gamma}_{I_1}^2, \hat{\gamma}_{I_2}^2) = \mathcal{L}_{(\nabla\hat{\xi}_{I_1},\nabla\hat{\xi}_{I_2})}(1,1) = M_{[k]}$. Moreover for the parameters $\nabla\hat{\xi}_{I_1}$, and $\nabla\hat{\xi}_{I_2}$, we know from the linear independence of the matrices in (24) that $\mathcal{L}_{(\nabla\hat{\xi}_{I_1},\nabla\hat{\xi}_{I_2})}$ maps a basis in $\mathbb{R}^2$ to a basis in $\mathring{S}_{2\times2}$. Therefore the map $\mathcal{L}_{(\nabla\hat{\xi}_{I_1},\nabla\hat{\xi}_{I_2})}$ is an invertible linear map and by (25)

$$\mathcal{L}_{(\nabla\hat{\xi}_{I_1},\nabla\hat{\xi}_{I_2})}^{-1}\left(M_{[k]}\right) = (1,1). \tag{30}$$

Now $\mathcal{L}_{(p_1,p_2)}^{-1}$ is a map that depends smoothly on the parameters (p_1, p_2). Thus we may choose a constant $c_2 > 0$ and sufficiently small in the condition (6) such that

$$|\nabla\xi_I - \nabla\hat{\xi}_I| < c_2 \implies \left\|\mathcal{L}_{(\nabla\xi_{I_1},\nabla\xi_{I_2})}^{-1} - \mathcal{L}_{(\nabla\hat{\xi}_{I_1},\nabla\hat{\xi}_{I_2})}^{-1}\right\| < \frac{1}{4\|M_{[k]}\|}. \tag{31}$$

Furthermore when we define our lifting function $e(t)$ from section 4.1.5 we may choose a sufficiently large constant K such that

$$e(t) > K\|R_\epsilon\|_{C^0} \implies \|\varepsilon\|_{L^\infty} < 8^{-1}\|\mathcal{L}_{(\nabla\hat{\xi}_{I_1},\nabla\hat{\xi}_{I_2})}^{-1}\|_{op}^{-1} \quad \text{and} \quad \|M_{[k]} - \varepsilon\|_{L^\infty} \leqslant 2\|M_{[k]}\|_{L^\infty}. \tag{32}$$

We combine these two conditions with (30) to get

$$
\left\| (1,1) - \mathcal{L}^{-1}_{\left(\nabla \xi_{I_1}, \nabla \xi_{I_2}\right)} \left(M_{[k]} - \varepsilon\right) \right\|_{L^\infty}
$$

$$
\leqslant \left\| \mathcal{L}^{-1}_{\left(\nabla \hat\xi_{I_1}, \nabla \hat\xi_{I_2}\right)} \left(M_{[k]}\right) - \mathcal{L}^{-1}_{\left(\nabla \hat\xi_{I_1}, \nabla \xi_{I_2}\right)} \left(M_{[k]} - \varepsilon\right) \right\|_{L^\infty}
$$

$$
+ \left\| \mathcal{L}^{-1}_{\left(\nabla \hat\xi_{I_1}, \nabla \hat\xi_{I_2}\right)} \left(M_{[k]} - \varepsilon\right) - \mathcal{L}^{-1}_{\left(\nabla \xi_{I_1}, \nabla \xi_{I_2}\right)} \left(M_{[k]} - \varepsilon\right) \right\|_{L^\infty}
$$

$$
\leqslant \left\| \mathcal{L}^{-1}_{\left(\nabla \hat\xi_{I_1}, \nabla \hat\xi_{I_2}\right)} \right\|_{op} \|\varepsilon\|_{L^\infty} + \left\| \mathcal{L}^{-1}_{\left(\nabla \xi_{I_1}, \nabla \xi_{I_2}\right)} - \mathcal{L}^{-1}_{\left(\nabla \hat\xi_{I_1}, \nabla \hat\xi_{I_2}\right)} \right\|_{op} \|M_{[k]} - \varepsilon\|_{L^\infty}
$$

$$
\leqslant 1/2.
$$

Let $(\gamma_{I_1}^2, \gamma_{I_2}^2) = \mathcal{L}_{(\nabla \xi_{I_1}, \nabla \xi_{I_2})}(M_{[k]} - \varepsilon)$. By our previous calculations we can ensure that $1/2 < \gamma_{I_1}^2 \gamma_{I_2}^2 < 2$ on the time support of $\phi_k(t)$. Therefore there exists $1/2 < \gamma_{I_1} \gamma_{I_2} < 2$ that solve (28) pointwise in (x, t). By these calculations we can define the **coefficients**

$$
\gamma_I = \gamma_I(\nabla \xi_I, \varepsilon) \in (2^{-1}, 2) \tag{33}
$$

as the unique, positive functions solving

$$
\sum_{I \in \{k\} \times \mathbb{F}} \theta_I^2(x, t) \hat{K}^{j\ell}_{\lambda, \mathrm{sym}}[\lambda \nabla \xi_I, -\lambda \nabla \xi_I] - e(t) \phi_k^2(t) \left(M_{[k]} - \frac{R_\epsilon^{j\ell}}{e(t)} \right) = 0
$$

from (27). We give C^0-norm estimates for γ_I in section 4.7.1. It follows that (27) reduces to

$$
R_{\mathrm{S}}^{jl} = \sum_{I \in \mathbb{Z} \times \mathbb{F}} \delta B_I^{jl}. \tag{34}
$$

4.7. Estimates

4.7.1. Basic estimates for the construction. We state all of the estimates necessary for controlling the corrections. The following proposition is the main new estimate of this section of our paper.

Proposition 4.1 (Phase gradient estimates). *For the phase gradients $\nabla \xi_I$ defined in section 4.1.2 there exists a positive number, $b_0 \leqslant 1$ such that the conditions (6) on $\nabla \xi_I$ hold for $|t - t(I)| \leqslant \tau$, and such that for all $t \in \mathbb{R}$ with $|t - t(I)| \leqslant \tau$, we have the following estimates*

$$
\|\nabla_{\vec{a}} D_t^r (\nabla \xi_I)\|_{C^0} \lesssim_{\vec{a}} N^{(|\vec{a}| + 1 - L)_+ / L} \Xi^{|\vec{a}|} (\Xi e_u^{1/2})^r, \quad \text{for all } |\vec{a}| \geqslant 0, \ r = 0, 1
$$

$$
\|\nabla_{\vec{a}} D_t^2 (\nabla \xi_I)\|_{C^0} \lesssim_{\vec{a}} N^{(|\vec{a}| + 2 - L)_+ / L} \Xi^{|\vec{a}|} (\Xi e_u^{1/2})^2, \quad \text{for all } |\vec{a}| \geqslant 0, \ r = 2.
$$

Moreover the $r = 1$ bound holds also when using the differential operator $\nabla_{\vec{a}_1} D_t \nabla_{\vec{a}_2}$ applied to $\nabla \xi_I$, where $|\vec{a}_1| + |\vec{a}_2| = |\vec{a}|$, and the $r = 2$ bound holds also when using the differential operator $\nabla_{\vec{a}_1} D_t \nabla_{\vec{a}_2} D_t \nabla_{\vec{a}_3}$ applied to $\nabla \xi_I$, where $|\vec{a}_1| + |\vec{a}_2| + |\vec{a}_3| = |\vec{a}|$.

Proof. See [18, sections 17.1–17.3]. We note that the choice of b_0 depends only on the geometric constant c_2 in the first inequality of (6), which in turn implies the second inequality of (6). $\qquad \square$

Remark 4.1. Due to u being frequency localized by assumption one can identify u with u_ϵ from [18] in this proof. Doing this forgoes the need for any mollification factors like what appears in [18], which allows the above estimates to improve to $\|\nabla_{\vec{a}} D_t^r (\nabla \xi_I)\|_{C^0} \lesssim_{\vec{a}} \Xi^{|\vec{a}|} (\Xi e_u^{1/2})^r$, for all $|\vec{a}| \geqslant 0, \quad r = 0, 1, 2.$

Proposition 4.2 (Mollification estimates). *We have the following bounds for the R_ϵ from section 4.4.*

$$\|\nabla_{\vec{a}} D_t^r R_\epsilon\|_{C^0} \lesssim_{\vec{a}} N^{\frac{3}{2L}(|\vec{a}|+r-L)_+} \Xi^{|\vec{a}|} (\Xi e_u^{1/2})^r \mathcal{D}_R, \quad \text{for all } |\vec{a}| \geqslant 0, \ r = 0, 1$$

$$\|\nabla_{\vec{a}} D_t^2 R_\epsilon\|_{C^0} \lesssim_{\vec{a}} N^{\frac{3}{2L}(|\vec{a}|+1-L)_+} \Xi^{|\vec{a}|} (\Xi e_u^{1/2}) \epsilon_t^{-1} \mathcal{D}_R, \quad \text{for all } |\vec{a}| \geqslant 0$$

$$\|R - R_\epsilon\|_{C^0} \lesssim \left((\Xi e_u^{1/2}) \epsilon_t + \epsilon_x^L \Xi^L \right) \mathcal{D}_R.$$

Proof. We cite the proofs of [18, proposition 18.5] and [18, proposition 18.7]. Here we identify the coarse scale flow u_ϵ from [18] with $u_\epsilon := u = T\theta$ because u is restricted to low frequencies by assumption. Our different choice of mollification factors ϵ_x, ϵ_t in (8) means that the estimates here are the same estimates from [18] but with N replaced by $N^{\frac{3}{2}}$. The bound on $\|R - R_\epsilon\|_{C^0}$ comes from [18, section 18.3], which employs the decomposition $R - R_\epsilon = (R - \eta_{\epsilon_t} *_\Phi R) + \eta_{\epsilon_t} *_\Phi (R - R_{\epsilon_x})$. Here $*_\Phi$ denotes mollification along the flow as defined in section 4.4 and R_{ϵ_x} is the spatial mollification of R in section 4.4. $\qquad \square$

Proposition 4.3 (Coefficients of the stress equation estimates). *For components of the amplitude ε from* (29) *and γ_I from* (33) *the following bounds hold uniformly for $I \in \mathbb{Z} \times F$*

$$\|\nabla_{\vec{a}} D_t^r \varepsilon\|_{C^0} + \|\nabla_{\vec{a}} D_t^r \gamma_I\|_{C^0} \lesssim_{\vec{a}} N^{\frac{3}{2L}(|\vec{a}|+1-L)_+} \Xi^{|\vec{a}|} (\Xi e_u^{1/2})^r, \quad \text{for all } |\vec{a}| \geqslant 0, \ r = 0, 1$$

$$\|\nabla_{\vec{a}} D_t^2 \varepsilon\|_{C^0} + \|\nabla_{\vec{a}} D_t^2 \gamma_I\|_{C^0} \lesssim_{\vec{a}} N^{\frac{3}{2L}(|\vec{a}|+1-L)_+} \Xi^{|\vec{a}|} (\Xi e_u^{1/2}) \epsilon_t^{-1}, \quad \text{for all } |\vec{a}| \geqslant 0.$$

Proof. For the estimates of ε_I we cite the proof of [18, proposition 20.1] but use our estimates R_ϵ from proposition 4.2. It is important to add that our choices of ϵ_t^{-1} from (8) and N from lemma 3.1 are different from [18] but all of the arguments remain the same. For the bounds on γ_I we cite [18, proposition 20.2]. We remark that in the first bound, slightly sharper estimates hold if the terms are controlled separately but we have combined them together here. $\qquad \square$

Proposition 4.4 (Principal amplitude estimates). *For the amplitude θ_I from* (7) *and the induced velocity vector field $u_I = T\theta_I$ from section 4.2 the following bounds hold uniformly for $I \in \mathbb{Z} \times F$*

$$\|\nabla_{\vec{a}} D_t^r \theta_I\|_{C^0} + \|\nabla_{\vec{a}} D_t^r u_I\|_{C^0}$$

$$\lesssim_{\vec{a}} \lambda^{1/2} N^{\frac{3}{2L}(|\vec{a}|+1-L)_+} \Xi^{|\vec{a}|} D_R^{1/2} \tau^{-r}, \quad \text{for all } |\vec{a}| \geqslant 0, \ r = 0, 1$$

$$\|\nabla_{\vec{a}} D_t^2 \theta_I\|_{C^0} + \|\nabla_{\vec{a}} D_t^2 u_I\|_{C^0}$$

$$\lesssim_{\vec{a}} \lambda^{1/2} B_\lambda^{3/2} N^{\frac{3}{2L}(|\vec{a}|+1-L)_+} \Xi^{|\vec{a}|} D_R^{1/2} (\Xi e_u^{1/2}) \epsilon_t^{-1}, \quad \text{for all } |\vec{a}| \geqslant 0.$$

Proof. First we recall from section 4.2 that $u_I^l = \theta_I m^l (\nabla \xi_I)$. The bounds for u_I follow similarly to the bounds on θ_I, so we restrict attention to estimating θ_I. The following calculations are outlined in [18, propositions 21.1–21.4]. They follow the basic scheme that the bounds of

θ_I are roughly a rescaling of the estimates for γ_I. For example we give a calculation of $D_t^2 \theta_I$.

$$
\begin{aligned}
D_t^2 \theta_I &= D_t^2 \left[\lambda^{1/2} \gamma_I(x,t) \phi \left(\frac{t - k\tau}{\tau} \right) e^{1/2}(t) \right] \\
&= \lambda^{1/2} \gamma_I(x,t) \partial_t^2 \left[\phi_k(t) e^{1/2}(t) \right] + \lambda^{1/2} D_t^2 \left[\gamma_I(x,t) \right] \phi_k(t) e^{1/2}(t) \\
&\quad + 2\lambda^{1/2} D_t \left[\gamma_I(x,t) \right] \partial_t \left[\phi_k(t) e^{1/2}(t) \right] \\
&= A_{\mathrm{I}} + A_{\mathrm{II}} + A_{\mathrm{III}}.
\end{aligned}
$$

We begin by estimating the first term using our bounds from sections 4.1.3, 4.1.5 and proposition 4.3. We will also need the observation that $\Xi e_u^{1/2} \leqslant \tau^{-1}$ that is due to our definition of $b^{-1} \geqslant 1$ in section 4.1.3.

$$
\begin{aligned}
\|\nabla_{\vec{a}} A_{\mathrm{I}}\|_{C^0} &\lesssim \lambda^{1/2} \|\nabla_{\vec{a}} \gamma_I\|_{C^0} \tau^{-2} Ð_R^{1/2} \\
&\lesssim_{\vec{a}} \lambda^{1/2} N^{\frac{3}{2L}(|\vec{a}|+1-L)+} \Xi^{|\vec{a}|} \tau^{-2} Ð_R^{1/2}.
\end{aligned}
$$

We can also bound the second term using the estimates from proposition 4.3

$$
\begin{aligned}
\|\nabla_{\vec{a}} A_{\mathrm{II}}\|_{C^0} &\lesssim_{\vec{a}} \lambda^{1/2} \|\nabla_{\vec{a}} D_t^2 \gamma_I\|_{C^0} Ð_R^{1/2} \\
&\lesssim_{\vec{a}} \lambda^{1/2} N^{\frac{3}{2L}(|\vec{a}|+1-L)+} \Xi^{|\vec{a}|} (\Xi e_u^{1/2}) \epsilon_t^{-1} Ð_R^{1/2}.
\end{aligned}
$$

For the third term we use a combination of proposition 4.3 and the definitions from sections 4.1.3 and 4.1.5.

$$
\begin{aligned}
\|\nabla_{\vec{a}} A_{\mathrm{III}}\|_{C^0} &\lesssim_{\vec{a}} \lambda^{1/2} \tau^{-1} Ð_R^{1/2} \|\nabla_{\vec{a}} D_t \gamma_I\|_{C^0} \\
&\lesssim_{\vec{a}} \lambda^{1/2} N^{\frac{3}{2L}(|\vec{a}|+1-L)+} \Xi^{|\vec{a}|} (\Xi e_u^{1/2}) \tau^{-1} Ð_R^{1/2}
\end{aligned}
$$

Finally, we compare these bounds by proving that $\tau^{-1} \lesssim B_\lambda^{3/4} \epsilon_t^{-1}$ and $\tau^{-2} \lesssim B_\lambda^{3/2} (\Xi e_u^{1/2}) \epsilon_t^{-1}$ with implied constants that only depend on c_0 from the definition ϵ_t given in section 4.4. We start by proving $\tau^{-1} \lesssim B_\lambda^{3/4} \epsilon_t^{-1}$

$$
\tau^{-1} \lesssim B_\lambda^{3/4} \epsilon_t^{-1} \iff \frac{Ð_R^{1/4} B_\lambda^{3/4} N^{3/4}}{Ð_u^{1/4}} \Xi e_u^{1/2} \lesssim B_\lambda^{3/4} N^{3/2} \Xi^{3/2} Ð_R^{1/2} \iff
$$

$$
B_\lambda^{3/4} \Xi^{3/2} Ð_R^{1/4} Ð_u^{1/4} N^{3/4} \lesssim B_\lambda^{3/4} N^{3/2} \Xi^{3/2} Ð_R^{1/2} \iff \frac{Ð_u^{1/4}}{Ð_R^{1/4}}
$$

$$
\lesssim N^{3/4}. \tag{35}
$$

The last inequality holds due to the assumption $N \geqslant \frac{Ð_u}{Ð_R} \geqslant 1$ in the main lemma 3.1. Next we verify $\tau^{-2} \lesssim B_\lambda^{3/2} \Xi e_u^{1/2} \epsilon_t^{-1}$.

$$
\tau^{-2} \lesssim B_\lambda^{3/2} \Xi e_u^{1/2} \epsilon_t^{-1} \iff \frac{Ð_R^{1/2} B_\lambda^{3/2} N^{3/2}}{Ð_u^{1/2}} \Xi^2 e_u \lesssim B_\lambda^{3/2} \Xi e_u^{1/2} N^{3/2} \Xi^{3/2} Ð_R^{1/2} \iff
$$

$$
Ð_R^{1/2} N^{3/2} \Xi^3 Ð_u^{1/2} \lesssim \Xi^3 Ð_u^{1/2} N^{3/2} Ð_R^{1/2} \iff 1 \lesssim 1.
$$

The last inequality holds for a sufficiently small constant, c_0, taken in the definition of ϵ_t of line (8). Now that we have shown $\tau^{-1} \lesssim B_\lambda^{3/4}\epsilon_t^{-1}$ and $\tau^{-2} \lesssim B_\lambda^{3/2}(\Xi e_u^{1/2})\epsilon_t^{-1}$ we may apply this to compare our previously found bounds of $\nabla_{\vec{a}}A_I$, $\nabla_{\vec{a}}A_{II}$, and $\nabla_{\vec{a}}A_{III}$. Doing so and taking $B_\lambda \geqslant 1$ so that $B_\lambda^{3/2} \geqslant B_\lambda^{3/4}$ gives the claimed bound on $\|\nabla_{\vec{a}}D_t^2\theta_I\|_{C^0}$. $\qquad\square$

The following Proposition is the main new estimate of this section of this work.

Proposition 4.5 (Bilinear microlocal correction estimates). *Let $L \geqslant 2$ and assume the following bounds for the coarse-scale velocity*

$$\|\nabla_{\vec{a}}u\|_{C^0} \lesssim_{\vec{a}} N^{(|\vec{a}|-L)_+/L}\Xi^{|\vec{a}|}e_u^{1/2}, \quad \text{for all } |\vec{a}| \geqslant 0. \tag{36}$$

For δB_I^{jl} defined in section 4.2 and $D_t = \partial_t + u^i\nabla_i$ the following holds uniformly in $I \in \mathbb{Z} \times \mathbb{F}$

$$\|\nabla_{\vec{a}}D_t^\rho\delta B_I^{jl}\|_{C^0} \lesssim_{\vec{a}} B_\lambda^{-1}N^{-1}N^{\frac{3}{2L}(|\vec{a}|+2-L)_+}\Xi^{|\vec{a}|}\mathcal{D}_R\tau^{-\rho}, \quad \text{for all } |\vec{a}| \geqslant 0, \ 0 \leqslant \rho \leqslant 1. \tag{37}$$

We note that the bound $\overset{*}{\mathcal{D}}_R = \frac{\mathcal{D}_R}{N}$ obtained in the C^0 bound above is consistent with an SQG scheme aimed at the optimal regularity[3].

Proof. We recall the definition of δB_I from lemma 4.2

$$\delta B_I^{j\ell}(x,t) = \int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{i\lambda\nabla\xi_I(x)\cdot h_1}e^{-i\lambda\nabla\xi_I(x)\cdot h_2}K_{\lambda,\text{sym}}^{j\ell}[h_1,h_2]\theta_I(x)Y(x,h_1)\mathrm{d}h_1\,\mathrm{d}h_2$$

$$+ \int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{i\lambda\nabla\xi_I(x)\cdot h_1}e^{-i\lambda\nabla\xi_I(x)\cdot h_2}K_{\lambda,\text{sym}}^{j\ell}[h_1,h_2]\theta_I(x)Y(x,h_2)\mathrm{d}h_1\,\mathrm{d}h_2$$

$$+ \int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{i\lambda\nabla\xi_I(x)\cdot h_1}e^{-i\lambda\nabla\xi_I(x)\cdot h_2}K_{\lambda,\text{sym}}^{j\ell}[h_1,h_2]Y(x,h_1)Y(x,h_2)\mathrm{d}h_1\,\mathrm{d}h_2$$

$$:= A_I + A_{II} + A_{III}$$

$$Y(x,h) := \int_0^1 \frac{\mathrm{d}}{\mathrm{d}r}\left[e^{iZ(r,x,h)}\theta_I(x-rh)\right]\mathrm{d}r$$

$$Z(r,x,h) := r\lambda\int_0^1 h^a h^b\nabla_a\nabla_b\xi(x-sh)(1-s)\mathrm{d}s.$$

We show the bound for just $\|\nabla_{\vec{a}}D_t^\rho A_I\|_{C^0}$, with $0 \leqslant \rho \leqslant 1$, as the corresponding bounds on A_{II} and A_{III} are similar. To aid in the computation, we first decompose $Y(x,h)$ as

$$Y(x,h) = Y_1(x,h) - Y_2(x,h)$$

$$Y_1(x,h) := \int_0^1 e^{iZ(r,x,h)}\left[i\lambda\int_0^1 h^a h^b\nabla_a\nabla_b\xi_I(x-sh)(1-s)\mathrm{d}s\right]\theta_I(x-rh)\mathrm{d}r$$

$$Y_2(x,h) := \int_0^1 e^{iZ(r,x,h)}\nabla_a\theta_I(x-rh)h^a\,\mathrm{d}r.$$

---

[3] In contrast, the approach to the analogous bound in [3, section 5.4] relies on the closeness of ξ_I to its linear initial data, and falls short of the ideal bound.

We return our attention to bounding the integral $\nabla_{\vec{a}} D_t^\rho A_{\mathrm{I}}$ in $\nabla_{\vec{a}} D_t^\rho \delta B_I^{jl}$. Consider

$$A_{\mathrm{I}} = \int_{\mathbb{R}^2 \times \mathbb{R}^2} \mathrm{e}^{\mathrm{i}\lambda\nabla\xi_I(x)\cdot h_1}\, \mathrm{e}^{-\mathrm{i}\lambda\nabla\xi_I(x)\cdot h_2} K_{\lambda,\mathrm{sym}}^{j\ell}[h_1, h_2]\theta_I(x) Y_1(x, h_1)\mathrm{d}h_1\, \mathrm{d}h_2 \qquad (38)$$

$$- \int_{\mathbb{R}^2 \times \mathbb{R}^2} \mathrm{e}^{\mathrm{i}\lambda\nabla\xi_I(x)\cdot h_1}\, \mathrm{e}^{-\mathrm{i}\lambda\nabla\xi_I(x)\cdot h_2} K_{\lambda,\mathrm{sym}}^{j\ell}[h_1, h_2]\theta_I(x) Y_2(x, h_1)\mathrm{d}h_1\, \mathrm{d}h_2. \qquad (39)$$

For the sake of brevity we will just discuss how to show the claimed bounds on $\nabla_{\vec{a}} D_t^\rho (38)$. By similar methods one can prove the same bounds on $\nabla_{\vec{a}} D_t^\rho (39)$ and by linearity we obtain the bounds on $\nabla_{\vec{a}} D_t^\rho A_{\mathrm{I}}$. Hence we consider just the term $\nabla_{\vec{a}} D_t^\rho (38)$. Using the product rule for $\nabla_{\vec{a}} D_t^\rho$ we may write

$$\nabla_{\vec{a}} D_t^\rho (38) = \sum_{\substack{\vec{a}_1 + \cdots + \vec{a}_6 = \vec{a} \\ \rho_1 + \cdots + \rho_6 = \rho}} \int_0^1 \mathrm{d}r \int_{\mathbb{R}^2 \times \mathbb{R}^2} \nabla_{\vec{a}_1} D_t^{\rho_1}\, \mathrm{e}^{\mathrm{i}\lambda\nabla\xi_I(x)\cdot h_1}$$

$$\times \nabla_{\vec{a}_2} D_t^{\rho_2}\, \mathrm{e}^{-\mathrm{i}\lambda\nabla\xi_I(x)\cdot h_2} K_{\lambda,\mathrm{sym}}^{j\ell}[h_1, h_2]$$

$$\cdot \nabla_{\vec{a}_3} D_t^{\rho_3}\, \mathrm{e}^{\mathrm{i}Z(r,x,h_1)} \nabla_{\vec{a}_4} D_t^{\rho_4} \mathrm{i}Z(1, x, h_1) \nabla_{\vec{a}_5} D_t^{\rho_5} \theta_I(x - rh_1)$$

$$\times \nabla_{\vec{a}_6} D_t^{\rho_6} \theta_I(x)\mathrm{d}h_1\, \mathrm{d}h_2. \qquad (40)$$

We begin with estimates for the first factor in (40). We expand this term using the (combinatorial version of the) Faá di Bruno formula as follows

$$|\nabla_{\vec{a}_1} \mathrm{e}^{\mathrm{i}\lambda\nabla\xi_I(x)\cdot h_1}| \lesssim_{\vec{a}_1} \sum_{\pi \in \Pi(|\vec{a}_1|)} |h_1|^{|\pi|}\lambda^{|\pi|} \prod_{B \in \pi} \left\| \nabla^{|B|}\nabla\xi_I \right\|_{C^0}.$$

In the line above $\Pi(|\vec{a}_1|)$ is the set of all partitions of the set of positive integers $\{1, \ldots, |\vec{a}_1|\}$. For a given partition $\pi \in \Pi(|\vec{a}_1|)$, we call one subset, $B \in \pi$, a block and we let $|B|$ denote the size of the subset and $|\pi|$ denote the number of blocks in π. We apply proposition 4.1 and get

$$|\nabla_{\vec{a}_1} \mathrm{e}^{\mathrm{i}\lambda\nabla\xi_I(x)\cdot h_1}| \lesssim_{\vec{a}_1} \sum_{\pi \in \Pi(|\vec{a}_1|)} |h_1|^{|\pi|}\lambda^{|\pi|} \prod_{B \in \pi} N^{(|B|+1-L)_+/L}\Xi^{|B|}.$$

We now recall an elementary *counting inequality*, which states that for any x_i, $M \in \mathbb{R}^+$, $1 \leqslant i \leqslant m$, we have $\sum_{i=1}^m (x_i - M)_+ \leqslant \left(\left(\sum_{i=1}^m x_i \right) - M \right)_+$. A proof can be found in [18, lemma 17.1]. Applying this inequality with $M = L - 1$, and using that $\sum_{B \in \pi} |B| = |\vec{a}_1|$ gives

$$|\nabla_{\vec{a}_1} \mathrm{e}^{\mathrm{i}\lambda\nabla\xi_I(x)\cdot h_1}| \lesssim_{\vec{a}_1} (1 + |h|^{|\vec{a}_1|}\lambda^{|\vec{a}_1|}) N^{(|\vec{a}_1|+1-L)_+/L}\Xi^{|\vec{a}_1|}. \qquad (41)$$

Next we treat the advective derivative of the first factor in (40) using a combination of the Leibniz rule, the Faá di Bruno formula, and our estimates from proposition 4.1

$$|\nabla_{\vec{a}_1} D_t \mathrm{e}^{\mathrm{i}\lambda\nabla\xi_I(x)\cdot h_1}| \lesssim_{\vec{a}_1} \sum_{\vec{b}_1 + \vec{b}_2 = \vec{a}_1} \sum_{\pi \in \Pi(|\vec{b}_1|)} |h_1|^{|\pi|}\lambda^{|\pi|} \left(\prod_{B \in \pi} \left\| \nabla^{|B|}\nabla\xi_I \right\|_{C^0} \right)$$

$$\times \left(|h_1|\lambda \left\| \nabla_{\vec{b}_2} D_t \nabla\xi_I \right\|_{C^0} \right)$$

$$\lesssim_{\vec{a}_1} \sum_{\vec{b}_1 + \vec{b}_2 = \vec{a}_1} \sum_{\pi \in \Pi(|\vec{b}_1|)} |h_1|^{|\pi|+1}\lambda^{|\pi|+1}$$

$$\times \prod_{B \in \pi} \left(N^{(|B|+1-L)_+/L} \Xi^{|B|} \right)$$

$$\times \left(N^{(|b_2|+1-L)_+/L} \Xi^{|b_2|} (\Xi e_u^{1/2}) \right).$$

In order to simplify this expression we can combine the exponents of the N factors. We apply the counting inequality with $M = L - 1$ and use that $\sum_{B \in \pi} |B| = |\vec{a}_1|$ to get

$$|\nabla_{\vec{a}_1} D_t \, \mathrm{e}^{\mathrm{i}\lambda \nabla \xi_I(x) \cdot h_1}| \lesssim_{\vec{a}_1} \left(|h|\lambda + |h|^{|\vec{a}_1|+1} \lambda^{|\vec{a}_1|+1} \right)$$

$$\times N^{(|\vec{a}_1|+1-L)_+/L} \Xi^{|\vec{a}_1|} \left(\Xi e_u^{1/2} \right). \tag{42}$$

Combining estimates (41) and (42) gives

$$|\nabla_{\vec{a}_1} D_t^{\rho_1} \, \mathrm{e}^{\mathrm{i}\lambda \nabla \xi_I(x) \cdot h_1}| \lesssim_{\vec{a}_1} \left(1 + |h|^{|\vec{a}_1|+\rho_1} \lambda^{|\vec{a}_1|+\rho_1} \right)$$

$$\times N^{(|\vec{a}_1|+1-L)_+/L} \Xi^{|\vec{a}_1|} \left(\Xi e_u^{1/2} \right)^{\rho_1}, \tag{43}$$

and the same bound holds for the second factor in (40)

$$|\nabla_{\vec{a}_2} D_t^{\rho_2} \, \mathrm{e}^{-\mathrm{i}\lambda \nabla \xi_I(x) \cdot h_2}| \lesssim_{\vec{a}_2} \left(1 + |h|^{|\vec{a}_2|+\rho_2} \lambda^{|\vec{a}_2|+\rho_2} \right)$$

$$\times N^{(|\vec{a}_2|+1-L)_+/L} \Xi^{|\vec{a}_2|} \left(\Xi e_u^{1/2} \right)^{\rho_2}. \tag{44}$$

Next we prove the bounds on fourth factor of (40), $\nabla_{\vec{a}} D_t^{\rho_4} Z(r, x, h_1)$, by direct calculation. For the bound on the spatial derivative, we use proposition 4.1 and $\Xi = B_\lambda^{-1} N^{-1} \lambda$ to obtain

$$|\nabla_{\vec{a}} Z(r, x, h_1)| \lesssim \lambda |h|^2 \left\| \nabla_{\vec{a}} \nabla^2 \xi_I \right\| \lesssim B_\lambda^{-1} N^{-1} \lambda^2 |h|^2 N^{(|\vec{a}|+2-L)_+/L} \Xi^{|\vec{a}|},$$

where the constant is independent of r. For the advective derivative bound, we need to approximate the value of $u^i(x)$ in D_t by the value of u^i at a nearby point. For example

$$\left(\partial_t + u^i(x) \nabla_i \right) \nabla_a \nabla_b \xi_I(x - sh_1)$$

$$= D_t \nabla_a \nabla_b \xi_I(x - sh_1) + \left(u^i(x) - u^i(x - sh_1) \right) \nabla_i \nabla_a \nabla_b \xi_I(x - sh_1).$$

We can control the second term with Taylor's theorem and the counting inequality with $M = L - 2$

$$\left(u^i(x) - u^i(x - sh_1) \right) \nabla_i \nabla_a \nabla_b \xi_I(x - sh_1)$$

$$= -sh_1^c \int_0^1 \nabla_c u^i(x - \sigma sh_1) \, d\sigma \nabla_i \nabla_a \nabla_b \xi_I(x - sh_1)$$

$$|\nabla_{\vec{a}_4}(45)| \lesssim_{\vec{a}_4} \sum_{|\vec{b}_1|+|\vec{b}_2|=|\vec{a}_4|} |h_1| \left[\left(\Xi e_u^{1/2} \right) \Xi^{|\vec{b}_1|} N^{\left(|\vec{b}_1|+1-L\right)_+/L} \right]$$

$$\times \Xi^{|\vec{b}_2|+2} N^{\left(|\vec{b}_2|+3-L\right)_+/L}$$

$$|\nabla_{\vec{a}_4}(45)| \lesssim_{\vec{a}_4} |h_1| \Xi^{|\vec{a}_4|+2} N^{(|\vec{a}_4|+3-L)_+/L} \left(\Xi e_u^{1/2} \right). \tag{45}$$

We use $N^{(|\vec{a}_4|+3-L)_+/L}\Xi \leqslant N^{(|\vec{a}_4|+2-L)_+/L}\lambda$ and $\Xi^{|\vec{a}_4|+1} = B_\lambda^{-1}N^{-1}\lambda\Xi^{|\vec{a}_4|}$ to obtain

$$
\begin{aligned}
\left|\nabla_{\vec{a}_4}D_t^{\rho_4}Z(r,x,h_1)\right| &\lesssim_{\vec{a}_4} \lambda|h_1|^2\Xi^{|\vec{a}_4|+1}\left(\Xi e_u^{1/2}\right)^{\rho_4} \\
&\quad \cdot \left[N^{(|\vec{a}_4|+2-L)_+} + \mathbf{1}_{\{\rho_4=1\}}|h_1|N^{(|\vec{a}_4|+3-L)_+/L}\Xi\right] \\
&\lesssim_{\vec{a}_4} B_\lambda^{-1}N^{-1}\left(|h_1|^2\lambda^2 + |h_1|^{2+\rho_4}\lambda^{2+\rho_4}\right) \\
&\quad \times N^{(|\vec{a}_4|+2-L)_+/L}\Xi^{|\vec{a}_4|}\left(\Xi e_u^{1/2}\right)^{\rho_4}.
\end{aligned}
\tag{46}
$$

Next we compute a bound on the third factor, $\nabla_{\vec{a}_3}D_t^{\rho_3}\mathrm{e}^{\mathrm{i}Z(r,x,h_1)}$, as follows

$$
\begin{aligned}
\left|\nabla_{\vec{a}_3}\mathrm{e}^{\mathrm{i}Z(r,x,h_1)}\right| &\lesssim_{\vec{a}_3} \sum_{\pi\in\Pi(|\vec{a}_3|)} \lambda^{|\pi|}|h_1|^{2|\pi|}N^{(|\vec{a}_3|+2-L)_+/L}\Xi^{|\vec{a}_3|+|\pi|} \\
&\lesssim_{\vec{a}_3} \left(1 + \lambda^{2|\vec{a}_3|}|h|^{2|\vec{a}_3|}\right)N^{(|\vec{a}_3|+2-L)_+/L}\Xi^{|\vec{a}_3|} \\
\left|\nabla_{\vec{a}_3}D_t\,\mathrm{e}^{\mathrm{i}Z(r,x,h_1)}\right| &\lesssim_{\vec{a}_3} \sum_{\vec{b}_1+\vec{b}_2=\vec{a}_3} \left|\nabla_{\vec{b}_1}\mathrm{e}^{\mathrm{i}Z(r,x,h_1)}\nabla_{\vec{b}_2}D_tZ(r,x,h_1)]\right| \\
&\lesssim_{\vec{a}_3} \sum_{\vec{b}_1+\vec{b}_2=\vec{a}_3}\sum_{\pi\in\Pi(|\vec{b}_1|)} \left(\prod_{B\in\pi}\left|\nabla^{|B|}Z(r,x,h_1)\right|\right) \\
&\quad \times \left|\nabla_{\vec{b}_2}D_tZ(r,x,h_1)\right|.
\end{aligned}
\tag{47}
$$

If we apply this calculation, proposition 4.1, the assumed bounds (36), and the counting inequality with $M = L - 2$ to (47) then we get

$$
\begin{aligned}
\left|\nabla_{\vec{a}_3}D_t\,\mathrm{e}^{\mathrm{i}Z(r,x,h_1)}\right| &\lesssim_{\vec{a}_3} \left(|h|^2\lambda^2 + |h|^{2|\vec{a}_3|+2}\lambda^{2|\vec{a}_3|+2}\right) \\
&\quad \times N^{(|\vec{a}_3|+2-L)_+/L}\Xi^{|\vec{a}_3|}\left(\Xi e_u^{1/2}\right),
\end{aligned}
$$

and then

$$
\begin{aligned}
\left|\nabla_{\vec{a}_3}D_t^{\rho_3}\mathrm{e}^{\mathrm{i}Z(r,x,h_1)}\right| &\lesssim_{\vec{a}_3} \left(1 + |h|^{2|\vec{a}_3|+2}\lambda^{2|\vec{a}_3|+2}\right) \\
&\quad \times N^{(|\vec{a}_3|+2-L)_+/L}\Xi^{|\vec{a}_3|}\left(\Xi e_u^{1/2}\right)^{\rho_3}.
\end{aligned}
\tag{48}
$$

For the fifth factor in (40) we again approximate $u^i(x)$ in D_t by the value $u^i(x - rh)$ at a nearby point. We use propositions 4.1, 4.4, (36), and the counting inequality with $M = L - 2$ to get

$$
\begin{aligned}
\left|\nabla_{\vec{a}_5}\left(\partial_t + u^i(x)\nabla_i\right)^{\rho_5}\theta_I(x-rh_1)\right| & \\
\leqslant \left\|\nabla_{\vec{a}_5}D_t^{\rho_5}\theta_I\right\|_{C^0} &+ \left|\nabla_{\vec{a}_5}\left[\left(\left(u^i(x) - u^i(x-rh_1)\right)\nabla_i\right)^{\rho_5}\theta_I(x-rh_1)\right]\right| \\
\lesssim_{\vec{a}_5} \Xi^{|\vec{a}_5|}N^{\frac{3}{2L}(|\vec{a}_5|+\rho_5+1-L)_+}&\left(1+|h|\lambda\right)\lambda^{1/2}\mathcal{D}_R^{1/2}\tau^{-\rho_5}.
\end{aligned}
\tag{49}
$$

The last factor in (40), $\nabla_{\vec{a}_6}D_t^{\rho_6}\theta_I$, can be controlled by a direct application of proposition 4.4. We can combine this bound with our previously found estimates (43), (44), (48), (46) and (49).

Then applying the counting inequality with $M = L - 2$ with the Kernel scaling estimate from (20) yields

$$\|(40)\|_{C^0} \lesssim_{\vec{a}} \sum_{\substack{\vec{a}_1+\cdots+\vec{a}_6=\vec{a} \\ \rho_1+\cdots+\rho_6=1}} \int_0^1 \mathrm{d}r \int_{\mathbb{R}^2\times\mathbb{R}^2} \mathrm{d}h_1\,\mathrm{d}h_2 |K^{j\ell}_{\lambda,\mathrm{sym}}[h_1,h_2]|$$

$$\cdot \left(1+|h|^{|\vec{a}_1|+\rho_1}\lambda^{|\vec{a}_1|+\rho_1}\right)N^{(|\vec{a}_1|+1-L)_+/L}\Xi^{|\vec{a}_1|}\left(\Xi e_u^{1/2}\right)^{\rho_1}$$

$$\cdot \left(1+|h|^{|\vec{a}_2|+\rho_2}\lambda^{|\vec{a}_2|+\rho_2}\right)N^{(|\vec{a}_2|+1-L)_+/L}\Xi^{|\vec{a}_2|}\left(\Xi e_u^{1/2}\right)^{\rho_2}$$

$$\cdot \left(1+|h|^{2|\vec{a}_3|+2}\lambda^{2|\vec{a}_3|+2}\right)N^{(|\vec{a}_3|+2-L)_+/L}\Xi^{|\vec{a}_3|}\left(\Xi e_u^{1/2}\right)^{\rho_3}$$

$$\cdot \left(|h|^2\lambda^2+|h|^{2+\rho_4}\lambda^{2+\rho_4}\right)N^{(|\vec{a}_4|+2-L)_+/L}\Xi^{|\vec{a}_4|}\left(\Xi e_u^{1/2}\right)^{\rho_4}B_\lambda^{-1}N^{-1}$$

$$\cdot \left(1+|h|\lambda\right)N^{\frac{3}{2L}(|\vec{a}_5|+\rho_5+1-L)_+}\Xi^{|\vec{a}_5|}\lambda^{1/2}\mathrm{D}_R^{1/2}\tau^{-\rho_5}$$

$$\cdot \left(1+|h|\lambda\right)N^{\frac{3}{2L}(|\vec{a}_6|+\rho_6+1-L)_+}\Xi^{|\vec{a}_6|}\lambda^{1/2}\mathrm{D}_R^{1/2}\tau^{-\rho_6}$$

$$\|(40)\|_{C^0} \lesssim B_\lambda^{-1}N^{-1}\Xi^{|\vec{a}|}N^{\frac{3}{2L}(|\vec{a}|+2-L)_+}\mathrm{D}_R\tau^{-\rho},$$

where in the last line we used $(\Xi e_u^{1/2}) \leqslant \tau^{-1}$. This bound completes our work for estimating one summand of $\nabla_{\vec{a}}D_t^\rho A_{\mathrm{I}}$. The other summand (39) may be handled using the same approach, but in this case the factor $B_\lambda^{-1}N^{-1}$ arises from the estimates for $h^a\nabla_a\theta_I(x-rh)$. By linearity and our estimates of (38) and (39), we get the claimed bound (37) for $\nabla_{\vec{a}}D_t^\rho A_{\mathrm{I}}$. The remaining integrals A_{II} and A_{III} may be done by an identical calculation. Hence by linearity we have our claimed bound for $\nabla_{\vec{a}}\delta B_I$. $\qquad\square$

Proposition 4.6 (Microlocal correction estimates). *The following bounds hold uniformly for $I \in \mathbb{Z} \times F$*

$$\|\nabla_{\vec{a}}D_t^r\delta\theta_I\|_{C^0} + \|\nabla_{\vec{a}}D_t^r\delta u_I\|_{C^0}$$
$$\lesssim_{\vec{a}} \lambda^{1/2}B_\lambda^{-1}N^{-1}N^{\frac{3}{2L}(|\vec{a}|+2-L)_+}\Xi^{|\vec{a}|}\mathrm{D}_R^{1/2}\tau^{-r} \quad \textit{for all } |\vec{a}| \geqslant 0, \ r = 0,1$$

$$\|\nabla_{\vec{a}}D_t^2\delta\theta_I\|_{C^0} + \|\nabla_{\vec{a}}D_t^2\delta u_I\|_{C^0}$$
$$\lesssim_{\vec{a}} \lambda^{1/2}B_\lambda^{-1}N^{-1}N^{\frac{3}{2L}(|\vec{a}|+2-L)_+}\Xi^{|\vec{a}|}\mathrm{D}_R^{1/2}(\Xi e_u^{1/2})\epsilon_t^{-1} \quad \textit{for all } |\vec{a}| \geqslant 0.$$

Proof. For the same reasons as those given in proposition 4.4 the calculations for δu_I are comparable to those for $\delta\theta_I$. Hence it will be enough to discuss the estimates for $\delta\theta_I$. These calculations are done via direct calculation (see [23, lemma 7.5], which uses the same microlocal lemma and an analogous scalar correction). Hence all of our estimates follow from our previous estimates on $\nabla\xi_I$ from proposition 4.1 and θ_I from proposition 4.4. $\qquad\square$

Proposition 4.7 (Correction estimates). *Let $U_I = T\Theta_I$. The following bounds hold uniformly for $I \in \mathbb{Z} \times F$*

$$\|\nabla_{\vec{a}}D_t^r\Theta_I\|_{C^0} + \|\nabla_{\vec{a}}D_t^r U_I\|_{C^0}$$
$$\lesssim_{|\vec{a}|} \lambda^{1/2+|\vec{a}|}\mathrm{D}_R^{1/2}\tau^{-r} \quad \textit{for all } |\vec{a}| \geqslant 0, \ r = 0,1$$

$$\|\nabla_{\vec{a}}D_t^2\Theta_I\|_{C^0} + \|\nabla_{\vec{a}}D_t^2U_I\|_{C^0}$$

$$\lesssim_{|\vec{a}|} B_\lambda^{3/2}\lambda^{1/2+|\vec{a}|}\mathcal{D}_R^{1/2}(\Xi e_u^{1/2})\epsilon_t^{-1} \quad \text{for all } |\vec{a}| \geqslant 0.$$

Proof. Consider $\Theta_I = e^{i\lambda\xi_I}(\theta_I + \delta\theta_I) := e^{i\lambda\xi_I}\tilde{\theta}_I$. Observe by propositions 4.4 and 4.6 that $\tilde{\theta}_I$ satisfies the same estimates as θ_I (but with larger constants) since the bounds for $\delta\theta_I$ are lower order. The proposition then follows from the formulas

$$D_t\Theta_I = e^{i\lambda\xi_I}D_t\tilde{\theta}_I, \qquad D_t^2\Theta_I = e^{i\lambda\xi_I}D_t^2\tilde{\theta}_I$$

by appealing to the bounds from propositions 4.1, 4.4 and 4.6. The bounds for U_I follow similarly. $\qquad\square$

4.7.2. Weighted norm. For the purpose of a shorter presentation we introduce the following weighted norm that was used in lemmas 4.1 and 4.2 of [19].

Lemma 4.3 (Weighted norm). *Let $D_t = \partial_t + T^l\theta\nabla_l$ be the advective derivative with respect to θ, $\overset{*}{D}_t = \partial_t + T^l\overset{*}{\theta}\nabla_l$ be the advective derivative with respect to $\overset{*}{\theta} = \theta + \Theta$, $\lambda = B_\lambda N\Xi$ as defined in section 4.1, and define $\overset{\diamond}{e}_u{}^{1/2} := (\lambda\mathcal{D}_R)^{1/2}$. For each advective derivative $\mathring{D}_t \in \{D_t, \overset{*}{D}_t\}$ define the weighted norm $\mathring{H} \in \{H, \overset{*}{H}\}$ by*

$$\mathring{H}[F] := \max_{0\leqslant r\leqslant 1}\max_{0\leqslant|\vec{a}|+r\leqslant L}\frac{\|\nabla_{\vec{a}}\mathring{D}_t^r F\|_{C^0}}{\lambda^{|\vec{a}|}(\lambda\overset{\diamond}{e}_u{}^{1/2})^r}.$$

Furthermore $\mathring{H}[F]$ satisfies the triangle inequality $\mathring{H}[F + G] \leqslant \mathring{H}[F] + \mathring{H}[G]$ and the product rule $\mathring{H}[FG] \leqslant \mathring{H}[F] \cdot \mathring{H}[G]$. The weighted norms are also comparable $\overset{*}{H}[F] \lesssim H[F] \lesssim \overset{*}{H}[F]$.

Proof. The proof is identical to the proof of [19, lemma 4.2] but with $T^l[\Theta] = U^l$ identified with $\widetilde{V}^l$, $T^l[\theta] = u^l$ identified with both v^l and v_ϵ^l, and $\overset{\diamond}{e}_u{}^{1/2}$ identified with $e_\phi^{1/2}$. $\qquad\square$

4.7.3. The second order anti-divergence operator bound. The lemma below is needed for applying the order -2, second order anti-divergence operator from section 4.5.1 to frequency localized scalar functions. This estimate is from [19, lemma 4.3].

Lemma 4.4. *Suppose $Q : \mathbb{R}^2 \to \mathbb{R}$ is Schwartz and satisfies the following bounds*

$$\max_{0\leqslant|\vec{a}|\leqslant|\vec{b}|\leqslant L}\lambda^{-(|\vec{b}|-|\vec{a}|)}\|h^{\vec{a}}\nabla_{\vec{b}}Q]\|_{L^1(\mathbb{R}^2)} \leqslant \Lambda^{-1}$$

for some real number $\Lambda^{-1} \geqslant 0$. Let H be the weighted norm, $\overset{\diamond}{e}_u{}^{1/2}$ be the energy quantity, and D_t be the advective derivative as defined in lemma 4.3. Then for any smooth U on $\mathbb{T}^2$ we have

$$H[Q * U] \lesssim \Lambda^{-1}\left(\|U\|_{C^0} + (\lambda\overset{\diamond}{e}_u{}^{1/2})^{-1}\|D_tU\|_{C^0}\right).$$

Proof. The proof is identical to [19, lemma 4.3]. We present a table where terms used in the proof of the [19, lemma 4.3] are listed in the top row and must be replaced by the corresponding term in the bottom row for doing the analogous proof here.

$$\begin{bmatrix} \bar{H}[\cdot] & \bar{D}_t & e_v & e_\varphi & N \geq (e_v/e_\varphi)^{1/2} \\ \hline H[\cdot] & D_t & e_u & \overset{\diamond}{e}_u & N \geq (e_u/\overset{\diamond}{e}_u)^{1/2} \end{bmatrix}.$$

The last column of the table is the condition $N \geqslant \frac{e_v^{1/2}}{e_\varphi^{1/2}}$ that appears in [19, lemma 4.3] and must replaced by the conditions $N \geqslant \frac{e_u^{1/2}}{\mathring{e}_u^{1/2}}$ for the proof here. The condition $N \geqslant \frac{e_u^{1/2}}{\mathring{e}_u^{1/2}}$ follows from the assumption $N \geqslant Đ_u/Đ_R$ made in the main lemma 3.1 and by taking $B_\lambda \geqslant 1$ in the definition $\lambda = B_\lambda N\Xi$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Corollary 4.1. *Given the second order anti-divergence operator $\mathcal{R}^{jl} : C^0(\mathbb{T}^2) \to C^0(\mathbb{T}^2)$ defined in* (14), *a frequency projection operator P_λ to frequencies $\{\xi \subseteq \hat{\mathbb{R}}^2 : \lambda/10 \leqslant |\xi| \leqslant 10\lambda\}$, and any smooth U on $\mathbb{R}^2$ we have*

$$H[\mathcal{R}^{jl}P_\lambda[U]] \lesssim \lambda^{-2} \left(\|U\|_{C^0} + (\lambda\mathring{e}_u^{1/2})^{-1}\|D_tU\|_{C^0} \right).$$

Proof. The result follows immediately from lemma 4.4 by taking $\Lambda^{-1} = C\lambda^{-2}$. We can define Λ^{-1} in this way due to the frequency support of $\mathcal{R}^{jl}$. $\qquad\qquad\qquad\qquad\qquad\square$

4.7.4. Estimating the stress errors. Recall from (9) that the new stress $\overset{*}{R}$ will be decomposed into four terms. We begin with estimating the term R_T.

4.7.5. Transport stress estimate.

Proposition 4.8 (Transport stress error). *The symmetric trace-free tensor R_T^{jl} defined in* (11) *satisfies the estimate*

$$H[R_T] \lesssim (B_\lambda N\Xi)^{-3/2}Đ_R^{1/2}(\Xi e_u^{1/2})b^{-1}.$$

Proof. Recall from the discussion in section 4.5 that

$$\nabla_j\nabla_l R_T^{jl} = D_t\Theta + T^l[\Theta]\nabla_l\theta. \tag{50}$$

Since the right-hand side of this identity has mean-zero on $\mathbb{T}^2$ there is a well defined stress term

$$R_T := \mathcal{R}^{jl}\left[\partial_t\Theta + T^l\theta\nabla_l\Theta + T^l\Theta\nabla_l\theta\right]$$

as stated in definition (11). The frequency support of θ and u is $\operatorname{supp}\hat{\theta} \cup \operatorname{supp}\hat{u} \subseteq B_\Xi(0)$ whereas Θ has frequency support in an annulus of size λ as explained in section 4.1.4. Therefore the argument of $\mathcal{R}$ above has frequency support in $\{\xi \in \hat{\mathbb{R}}^2 : \lambda/3 \leqslant |\xi| \leqslant 3\lambda\}$ and we may replace $\mathcal{R}$ by $\mathcal{R}P_{\approx\lambda}$ for the appropriate frequency-localization operator $P_{\approx\lambda}$. Corollary 4.1 then gives

$$H[R_T] \lesssim \lambda^{-2} \left(\|(50)\|_{C^0} + (\lambda\mathring{e}_u^{1/2})^{-1}\|D_t(50)\|_{C^0} \right).$$

The term $\|(50)\|_{C^0}$ equals $\|D_t\Theta + T^l[\Theta]\nabla_l\theta\|_{C^0}$. By proposition 4.7 and definition 2.2

$$\|(50)\|_{C^0} \lesssim (\lambda Đ_R)^{1/2}\tau^{-1} + (\lambda Đ_R)^{1/2}(\Xi e_u^{1/2}) \lesssim (\lambda Đ_R)^{1/2}\tau^{-1}. \tag{51}$$

Additionally, we can estimate the term $\|D_t(50)\|_{C^0}$ as

$$\|D_t(50)\|_{C^0} = \|D_t^2\Theta + D_tU^l\nabla_l\theta + U^lD_t\nabla_l\theta\|_{C^0} \tag{52}$$

We can control this quantity using proposition 4.7, definition 2.2, and a bound for $D_t \nabla_l \theta$ that follows from the SQG-Reynold's equation θ. From the SQG-Reynold's equation we have

$$D_t \nabla_l \theta = \nabla_l \nabla_a \nabla_b R^{ab} - (\nabla_l u^k)(\nabla_k \theta)$$

and by definition 2.2

$$\|D_t \nabla_l \theta\|_{C^0} \lesssim \Xi^3 \text{Ð}_R + (\Xi e_u^{1/2})^2 \lesssim (\Xi e_u^{1/2})^2.$$

Using this bound with proposition 4.7 and definition 2.2 we can see that

$$\|(52)\|_{C^0} \lesssim B_\lambda^{3/2} (\lambda \text{Ð}_R)^{1/2} (\Xi e_u^{1/2}) \epsilon_t^{-1} + (\lambda \text{Ð}_R)^{1/2} \tau^{-1} (\Xi e_u^{1/2})$$
$$+ N^{\frac{3}{2L}(1-L)+} (\lambda \text{Ð}_R)^{1/2} (\Xi e_u^{1/2})^2.$$

By our definitions of ϵ_t, τ stated in (8) and section 4.1.3 respectively it follows that

$$(\Xi e_u^{1/2})^2 \leqslant (\Xi e_u^{1/2}) \tau^{-1} \leqslant B_\lambda^{3/2} (\Xi e_u^{1/2}) \epsilon_t^{-1},$$

where $\tau^{-1} \leqslant \epsilon_t^{-1}$ follows from (35). Next we show that $B_\lambda^{3/2} (\Xi e_u^{1/2}) \epsilon_t^{-1} \lesssim B_\lambda^{-3/4} \lambda \overset{\diamond}{e}_u^{1/2} \tau^{-1}$.

$$B_\lambda^{3/2} (\Xi e_u^{1/2}) \epsilon_t^{-1} \lesssim B_\lambda^{-3/4} \lambda \overset{\diamond}{e}_u^{1/2}$$
$$\iff B_\lambda^{3/2} (\Xi e_u^{1/2}) N^{\frac{3}{2L}} \Xi^{3/2} \text{Ð}_R^{1/2} \lesssim B_\lambda^{-3/4} (B_\lambda N \Xi)^{3/2} \text{Ð}_R^{1/2} (\Xi e_u^{1/2}) b^{-1}$$
$$\iff 1 \lesssim B_\lambda^{-3/4} \left(\frac{\text{Ð}_R^{1/2} B_\lambda^{3/2} N^{3/2}}{\text{Ð}_u^{1/2}} \right)^{1/2}$$
$$\iff \frac{\text{Ð}_u^{1/4}}{\text{Ð}_R^{1/4}} \lesssim N^{3/4}.$$

The last inequality holds due to the assumption $N \geqslant \text{Ð}_u / \text{Ð}_R$ in the main lemma 3.1. In summary, we have shown that

$$\|(52)\|_{C^0} \lesssim (\lambda \text{Ð}_R)^{1/2} B_\lambda^{-3/4} \left(\lambda \overset{\diamond}{e}_u^{1/2} \right) \tau^{-1}.$$

By combining our estimates for (51) and (52) we conclude that

$$H[R_T] \lesssim \lambda^{-2} (\lambda \text{Ð}_R)^{1/2} \tau^{-1} + B_\lambda^{-3/4} \lambda^{-2} (\lambda \text{Ð}_R)^{1/2} \tau^{-1} \lesssim \lambda^{-3/2} \text{Ð}_R^{1/2} (\Xi e_u^{1/2}) b^{-1}.$$

$$\square$$

4.7.6. Hi frequency interference stress estimate.

Proposition 4.9. *The symmetric trace-free tensor R_H^{jl} defined in (10) satisfies the estimate*

$$H[R_H] \lesssim b \text{Ð}_R.$$

As is usual in convex integration arguments, the estimate of the high-frequency interference terms relies on extracting a cancellation that is based on the waves behaving to leading order

like steady-state solutions. The key computation (line 60 below) differs from [3] since we do not employ the momentum form of the equation.

Proof. Recall that $R_H^{jl} := \mathcal{R}^{jl} \left[\sum_{J \neq I} U_I^l \nabla_l \Theta_J + U_J^l \nabla_l \Theta_I \right]$ as stated in equation 10. The argument of $\mathcal{R}$ above has frequency support in $\{\xi \subseteq \hat{\mathbb{R}}^2 : \lambda/6 \leqslant |\xi| \leqslant 6\lambda\}$ due to the localizations in frequency and angle of the scalar corrections and velocity corrections defined in section 4.1.4. In particular, $\mathcal{R}$ may be replaced by $\mathcal{R}P_{\approx \lambda}$ so that corollary 4.1 applies.

Since R_H^{jl} is given by a finite sum then by linearity it will be enough to show the claimed bound for a single summand. Let a single summand be $R_{H(I,J)}^{jl} := \mathcal{R}^{jl}[U_I^l \nabla_l \Theta_J + U_J^l \nabla_l \Theta_I]$. By corollary 4.1 this quantity is bonded by

$$H[R_{H(I,J)}^{jl}] \lesssim \lambda^{-2} \left(\|(54)\|_{C^0} + (\lambda \overset{\diamond}{e}_u^{1/2})^{-1} \|D_t(54)\|_{C^0} \right) \tag{53}$$

$$\nabla_j \nabla_l R_{H(I,J)}^{jl} = U_I^l \nabla_l \Theta_J + U_J^l \nabla_l \Theta_I. \tag{54}$$

We begin with estimating $\|(54)\|_{C^0}$. Recall from the definitions given in sections 4.2 and 4.1 respectively $U_I^l = e^{i\lambda \xi_I}(u_I^l + \delta u_I^l)$ where $u_I^l = \theta_I m^l(\lambda \nabla \xi_I)$, and $\Theta_I = e^{i\lambda \xi_I}(\theta_I + \delta \theta_I)$. It follows that

$$U_I^l \nabla_l \Theta_J + U_J^l \nabla_l \Theta_I = \lambda \, e^{i\lambda(\xi_I + \xi_J)} \left[\nabla_l \xi_J u_I^l \theta_J + \nabla_l \xi_I u_J^l \theta_I \right] \tag{55}$$

$$+ \lambda \, e^{i\lambda(\xi_I + \xi_J)} \left[\nabla_l \xi_J u_I^l \delta\theta_J + \nabla_l \xi_I u_J^l \delta\theta_I \right] \tag{56}$$

$$+ \lambda \, e^{i\lambda(\xi_I + \xi_J)} \left[\nabla_l \xi_J \delta u_I^l (\theta_J + \delta\theta_J) + \nabla_l \xi_I \delta u_J^l (\theta_I + \delta\theta_I) \right] \tag{57}$$

$$+ e^{i\lambda(\xi_I + \xi_J)} \left[(u_I + \delta u_I)^l \nabla_l (\theta_J + \delta\theta_J) + (u_J + \delta u_J)^l \nabla_l (\theta_I + \delta\theta_I) \right]. \tag{58}$$

By propositions 4.4 and 4.6 we see that $\|(56)\|_{C^0}, \|(57)\|_{C^0} \lesssim \lambda \Xi \text{Đ}_R$ and $\|(58)\|_{C^0} \lesssim \lambda \Xi \text{Đ}_R$. We will see that these terms are lower order in λ relative to the leading order term in λ, which is $\|(55)\|_{C^0}$. We consider the term (55) and we calculate the amplitude of this wave, $\nabla_l \xi_J u_I^l \theta_J + \nabla_l \xi_I u_J^l \theta_I$, by taking $m^l(\xi) = i\varepsilon^{la}\xi_a/|\xi|$ for $\xi \in \hat{\mathbb{R}}^2$ from definition 2.1.

$$\nabla_l \xi_J u_I^l \theta_J + \nabla_l \xi_I u_J^l \theta_I = i\theta_I \theta_J \epsilon^{la} \left(\nabla_a \xi_I |\nabla \xi_I|^{-1} \nabla_l \xi_J + \nabla_a \xi_J |\nabla \xi_J|^{-1} \nabla_l \xi_I \right). \tag{59}$$

By the anti-symmetry property of ϵ^{li} we see that $\epsilon^{la}(\nabla_a \xi_I \nabla_l \xi_J + \nabla_l \xi_I \nabla_a \xi_J) = 0$ and so we can add a scalar multiple of this identity into the previous line to obtain

$$(59) = i\theta_I \theta_J \epsilon^{la} \left(\nabla_a \xi_I \nabla_l \xi_J (|\nabla \xi_I|^{-1} - |\nabla \hat{\xi}_I|^{-1}) + \nabla_a \xi_J \nabla_l \xi_I (|\nabla \xi_J|^{-1} - |\nabla \hat{\xi}_J|^{-1}) \right). \tag{60}$$

In (60) we have introduced $\hat{\xi}_I, \hat{\xi}_J$, which are the linear initial conditions to ξ_I, ξ_J respectively as defined in section 4.1.2. We have used here that $|\nabla \hat{\xi}_I|^{-1} = |\nabla \hat{\xi}_J|^{-1} = 5^{-1/2}$. In order to control (60) we will use $\left| |\nabla \xi_I|^{-1} - |\nabla \hat{\xi}_I|^{-1} \right|, \left| |\nabla \xi_J|^{-1} - |\nabla \hat{\xi}_J|^{-1} \right| \leqslant b$. This fact comes from a direct calculation of ξ_I that also holds for ξ_J. To start we show that $\|D_t(|\nabla \xi_I|^{-1})\|_{C^0} \lesssim (\Xi e_u^{1/2})$.

$$D_t(|\nabla \xi_I|^{-1}) = -|\nabla \xi_I|^{-3} \nabla^a \xi_I D_t \nabla_a \xi_I \implies \|D_t(|\nabla \xi_I|^{-1})\|_{C^0} \lesssim (\Xi e_u^{1/2}). \tag{61}$$

Where the inequality uses proposition 4.1 and the condition $|\nabla\xi_I| \geqslant 1$ from (6). Then

$$|\nabla\xi_I|^{-1} - |\nabla\hat{\xi}_I|^{-1} \leqslant \int_0^\tau \|D_t(|\nabla\xi_I|^{-1})\|_{C^0}\,ds \lesssim \tau(\Xi e_u^{1/2}) = b. \tag{62}$$

We can apply this bound along with the estimates from propositions 4.1 and 4.4 to (60) to conclude that $\|(60)\|_{C^0} \lesssim \lambda b D_R$, $\|(55)\|_{C^0} \lesssim \lambda^2 b Đ_R$, and finally

$$\lambda^{-2}\|(54)\|_{C^0} \lesssim b Đ_R. \tag{63}$$

We also estimate the term $\|D_t(54)\|_{C^0}$ by direct calculation. For the sake of brevity we will only present calculations for estimating the advective derivative of the first term (55), and the other terms are similar and lower order. We observe that the advective derivative of (55) is equivalent to

$$D_t(55) = \lambda\,e^{i\lambda(\xi_I+\xi_J)}\left[(D_t\nabla_l\xi_J)\,u_I^l\theta_J + (D_t\nabla_l\xi_I)\,u_J^l\theta_I\right] \tag{64}$$

$$+ \lambda\,e^{i\lambda(\xi_I+\xi_J)}\left[\nabla_l\xi_J\left(D_tu_I^l\right)\theta_J + \nabla_l\xi_I\left(D_tu_J^l\right)\theta_I\right] \tag{65}$$

$$+ \lambda\,e^{i\lambda(\xi_I+\xi_J)}\left[\nabla_l\xi_Ju_I^l\left(D_t\theta_J\right) + \nabla_l\xi_Iu_J^l\left(D_t\theta_I\right)\right]. \tag{66}$$

We directly estimate each term in C^0-norm using propositions 4.1 and 4.4 to obtain

$$\|D_t(55)\|_{C^0} \lesssim \lambda^2 Đ_R\left((\Xi e_u^{1/2}) + \tau^{-1}\right) \lesssim \lambda^2 Đ_R\tau^{-1}.$$

All that remains is to verify that $\lambda^{-2}\left(\lambda\mathring{e}_u^{1/2}\right)^{-1}\lambda^2 Đ_R\tau^{-1} \lesssim b Đ_R$. This is equivalent to checking $\tau^{-1}b^{-1} \lesssim \lambda\mathring{e}_u^{1/2}$ via the calculation

$$\tau^{-1}b^{-1} \lesssim \lambda\mathring{e}_u^{1/2} \iff \Xi^{3/2}Đ_u^{1/2}\left(\frac{Đ_R^{1/2}B_\lambda^{3/2}N^{3/2}}{Đ_u^{1/2}}\right)$$

$$\lesssim (B_\lambda N\Xi)^{3/2}Đ_R^{1/2} \iff 1 \lesssim 1.$$

As a result we have

$$\lambda^{-2}\left(\lambda\mathring{e}_u^{1/2}\right)^{-1}\|D_t(55)\|_{C^0} \lesssim b Đ_R.$$

By similar direct calculations of the other terms $D_t(56), D_t(57), D_t(58)$ it can be shown that

$$\lambda^{-2}\left(\lambda\mathring{e}_u^{1/2}\right)^{-1}\|D_t(54)\|_{C^0} \lesssim b Đ_R \tag{67}$$

and by combining (63) and (67) with (53) we conclude the claimed bound on $H[R_{\mathrm{H}}]$. $\qquad\square$

4.7.7. Low frequency stress estimate.

Proposition 4.10. *There exists a symmetric tensor, R_{S}^{jl} as defined in (13) that satisfies the estimate*

$$H[R_{\mathrm{S}}] \lesssim B_\lambda^{-1}N^{-1}Đ_R.$$

Proof. Recall from section 4.5 that the low frequency stress error is defined as

$$R_{\mathrm{S}}^{jl} := \sum_{I=(k,f)\in\mathbb{Z}\times\mathbb{F}} B_\lambda^{jl}[\Theta_I, \Theta_{\bar{I}}] - \phi_k^2(t)\left(e(t)M_{[k]} - R_\epsilon^{jl}\right) = \sum_{I\in\mathbb{Z}\times\mathbb{F}} \delta B_I^{jl},$$

where the last equality holds due to the calculations of section 4.6 and is stated in (34) and where δB_I^{jl} is defined in lemma 4.2. Therefore it is enough to compute $H\left[\sum_{I\in\mathbb{Z}\times\mathbb{F}} \delta B_I^{jl}\right]$ directly. Due to the partition of unity in time $\{\phi_k^2(t)\}_{k\in\mathbb{Z}}$ that appears in the definition of θ_I and in the definition of δB_I^{jl}, R_{S} is a locally finite sum in time. Hence for $0 \leqslant |\vec{a}| \leqslant L$ and $r = 0, 1$

$$\left\| \nabla_{\vec{a}} D_t^r \sum_{I\in\mathbb{Z}\times\mathbb{F}} \delta B_I^{jl} \right\|_{C^0} \lesssim \sup_{I\in\mathbb{Z}\times\mathbb{F}} \|\nabla_{\vec{a}} D_t^r \delta B_I^{jl}\|_{C^0} \lesssim_{\vec{a}} B_\lambda^{-1} N^{-1} N^{\frac{3}{2L}(|\vec{a}|+1-L)+} \Xi^{|\vec{a}|} \mathrm{\Dbar}_R \tau^{-r}.$$

In the last inequality we used proposition 4.5. We remark that the coarse scale velocity assumptions of (36) are satisfied by the flow $u = T[\theta]$ with the assumed bounds by the frequency and energy levels in definition 2.2. Next we show the inequality $\tau^{-1} \lesssim B_\lambda^{-3/4} \lambda \mathring{e}_u^{1/2}$ holds due to the following calculation

$$\tau^{-1} \leqslant B_\lambda^{-3/4} \lambda \mathring{e}_u^{1/2} \iff \Xi^{3/2} \mathrm{\Dbar}_u^{1/2} \left(\frac{\mathrm{\Dbar}_R^{1/2} B_\lambda^{3/2} N^{3/2}}{\mathrm{\Dbar}_u^{1/2}}\right)^{1/2} \lesssim B_\lambda^{-3/4}(B_\lambda N\Xi)^{3/2} \mathrm{\Dbar}_R^{1/2}$$

$$\iff \frac{\mathrm{\Dbar}_u^{1/4}}{\mathrm{\Dbar}_R^{1/4}} \lesssim N^{3/4}.$$

The last inequality holds due to our assumption on $N \geqslant \mathrm{\Dbar}_u/\mathrm{\Dbar}_R$ from the main lemma 3.1. By using the inequality $N^{\frac{3}{2L}(|\vec{a}|+1-L)+}\Xi^{|\vec{a}|} \leqslant \lambda^{|\vec{a}|}$ for $0 \leqslant |\vec{a}| \leqslant L$ and the inequality $\tau^{-1} \lesssim \lambda \mathring{e}_u^{1/2}$, we have that

$$H[R_{\mathrm{S}}] = H\left[\sum_{I\in\mathbb{Z}\times\mathbb{F}} \delta B_I^{jl}\right] \lesssim B_\lambda^{-1} N^{-1} \mathrm{\Dbar}_R.$$

$\square$

4.7.8. Mollification stress estimate.

Proposition 4.11. *There exists a symmetric tensor, R_{M}^{jl} as defined in (12) and there exists a constant $B_\lambda \geqslant 1$ sufficiently large so that the following estimate is satisfied*

$$\overset{*}{H}[R_{\mathrm{M}}] \leqslant \frac{G\mathrm{\Dbar}_R}{1000}, \quad G := \frac{\mathrm{\Dbar}_u^{1/4}}{\mathrm{\Dbar}_R^{1/4} N^{3/4}},$$

where G is the constant defined in the main lemma 3.1.

Proof. We follow the calculations outlined in [18, section 18.3]. Recall from lemma 4.3 that $\overset{*}{H}[R_{\mathrm{M}}] \leqslant C_0 H[R_{\mathrm{M}}]$ for some positive constant C_0 and from (12) and section 4.4, $R_{\mathrm{M}} := R - R_\epsilon$.

We begin with the 0th derivative bound from proposition 4.2:

$$C_0\|R_{\mathrm{M}}\|_{C^0} = C_0\|R - R_\epsilon\|_{C^0} \leqslant C\mathrm{\Dbar}_R\left((\Xi e_u^{1/2})\epsilon_t + \epsilon_x^L \Xi^L\right)$$

for some positive constant C independent of B_λ. By our choice of mollification parameters in (8)

$$\epsilon_x = c_0 N^{-\frac{3}{2L}} \Xi^{-1}, \qquad \epsilon_t = c_0 N^{-3/2} \Xi^{-3/2} Đ_R^{-1/2}$$

and by taking the constant $c_0 > 0$ sufficiently small we obtain the bound $C_0 \|R_M\|_{C^0} \leqslant \frac{Đ_u^{1/2} Đ_R^{1/2}}{1000 N^{3/2}}$. We observe that $\frac{Đ_u^{1/2} Đ_R^{1/2}}{1000 N^{3/2}} \leqslant \frac{GĐ_R}{1000}$ if and only if $\frac{Đ_u^{1/4}}{Đ_R^{1/4}} \leqslant N^{3/4}$. This second condition holds because of the condition $N \geqslant Đ_u/Đ_R$ in the main lemma 3.1 so we are able to conclude that

$$C_0 \|R_M\|_{C^0} \leqslant \frac{GĐ_R}{1000}. \tag{68}$$

Our next step is to show estimates for $C_0 \|\nabla_{\vec{a}} R_M\|_{C^0}$ in the case $1 \leqslant |\vec{a}| \leqslant L$. In this case we use proposition 4.2 and the frequency and energy levels assumed in the main lemma 3.1 to get

$$C_0 \|\nabla_{\vec{a}} R_M\|_{C^0} \leqslant C_0 \left(\|\nabla_{\vec{a}} R\|_{C^0} + \|\nabla_{\vec{a}} R_\epsilon\|_{C^0} \right) \leqslant C \Xi^{|\vec{a}|} Đ_R$$

for some positive constant C that is independent of B_λ. It remains to check $C \Xi^{|\vec{a}|} Đ_R \leqslant \frac{\lambda^{|\vec{a}|} GĐ_R}{1000}$. This is equivalent to checking $C \frac{\Xi^{|\vec{a}|}}{\lambda^{|\vec{a}|}} \leqslant \frac{G}{1000}$. By the definition of $\lambda \geqslant B_\lambda N \Xi$ in section 4.1, the assumption $|\vec{a}| \geqslant 1$, and by taking a constant $B_\lambda \geqslant 1$ sufficiently large we have that

$$C \frac{\Xi^{|\vec{a}|}}{\lambda^{|\vec{a}|}} \leqslant \frac{C}{B_\lambda N} \leqslant \frac{C}{B_\lambda} \frac{1}{N} \leqslant \frac{1}{1000 N} \leqslant \frac{G}{1000}.$$

Where in the last inequality we have used the condition on $N \geqslant 1$ from the main lemma 3.1 and the fact that $\left(Đ_u/Đ_R \right)^{1/4} \leqslant 1$. We have now shown

$$C_0 \frac{\|\nabla_{\vec{a}} R_M\|_{C^0}}{\lambda^{|\vec{a}|}} \leqslant \frac{GĐ_R}{1000}, \quad \text{for } 0 \leqslant |\vec{a}| \leqslant L \tag{69}$$

Lastly, we want to show an estimate for $C_0 \|\nabla_{\vec{a}} D_t R_M\|_{C^0}$ for $0 \leqslant |\vec{a}| \leqslant L - 1$. By proposition 4.2 and definition 2.2

$$C_0 \|\nabla_{\vec{a}} D_t R_M\|_{C^0} \leqslant \|\nabla_{\vec{a}} D_t R\|_{C^0} + \|\nabla_{\vec{a}} D_t R_\epsilon\|_{C^0} \leqslant C \Xi^{|\vec{a}|} Đ_R (\Xi e_u^{1/2})$$

for $0 \leqslant |\vec{a}| \leqslant L - 1$ and some positive constant C independent of B_λ. The $N \geqslant Đ_u/Đ_R$ assumption implies $e_u^{1/2} \leqslant \mathring{e}_u^{1/2}$. Using this bound, the previous bound, and $\lambda \geqslant B_\lambda N \Xi$ shows

$$C_0 \frac{\|\nabla_{\vec{a}} D_t R_M\|_{C^0}}{\lambda^{|\vec{a}|} \left(\lambda \mathring{e}_u^{1/2} \right)} \leqslant C \frac{\Xi^{|\vec{a}|} Đ_R (\Xi e_u^{1/2})}{\lambda^{|\vec{a}|} \left(\lambda \mathring{e}_u^{1/2} \right)} \leqslant C Đ_R \frac{\Xi}{\lambda} = \frac{C}{B_\lambda} \frac{Đ_R}{N}.$$

We observe that if B_λ is a sufficiently large constant then $\frac{C}{B_\lambda} \leqslant \frac{1}{1000}$ and $\frac{Đ_R}{N} \leqslant GĐ_R$ because of the condition $N \geqslant Đ_u/Đ_R$ from the main lemma 3.1. Therefore we have the estimate

$$C_0 \frac{\|\nabla_{\vec{a}} D_t R_M\|_{C^0}}{\lambda^{|\vec{a}|} \left(\lambda \mathring{e}_u^{1/2} \right)} \leqslant \frac{GĐ_R}{1000}, \quad \text{for } 0 \leqslant |\vec{a}| \leqslant L - 1. \tag{70}$$

By our three estimates (68)–(70) we can conclude the claimed bound

$$\overset{*}{H}[R_{\mathrm{M}}] \leqslant C_0 H[R_{\mathrm{M}}] \leqslant \frac{G\mathrm{Đ}_R}{1000}.$$

$\square$

4.8. Verifying the conclusions of the main lemma

In this section we verify the conclusions of the main lemma 3.1. We begin by checking that the new SQG-Reynolds flow $(\overset{*}{\theta}, \overset{*}{u}, \overset{*}{R})$ has frequency and energy levels below $(\overset{*}{\Xi}, \overset{*}{\mathrm{Đ}}_u, \overset{*}{\mathrm{Đ}}_R)$. First consider the new Reynold's Stress $\overset{*}{R}$. By lemma 4.3 $\overset{*}{H}[\overset{*}{R}] \leqslant CH[\overset{*}{R}]$ for some positive constant C that is independent of B_λ. By propositions 4.9, 4.8, 4.10, and 4.11

$$\overset{*}{H}[\overset{*}{R}] \leqslant C \left(H[R_{\mathrm{H}}] + H[R_{\mathrm{T}}] + H[R_{\mathrm{S}}] \right) + \overset{*}{H}[R_{\mathrm{M}}]$$

$$\leqslant C \left(b\mathrm{Đ}_R + (B_\lambda N\Xi)^{-3/2}\mathrm{Đ}_R^{1/2}(\Xi e_u^{1/2})b^{-1} + B_\lambda^{-1}N^{-1}\mathrm{Đ}_R \right) + \frac{G\mathrm{Đ}_R}{1000}.$$

Recalling the definitions from section 4.1.3 and main lemma 3.1

$$b = \left(\frac{\mathrm{Đ}_u^{1/2}}{\mathrm{Đ}_R^{1/2}B_\lambda^{3/2}N^{3/2}} \right)^{1/2} b_0, \qquad G = \frac{\mathrm{Đ}_u^{1/4}}{\mathrm{Đ}_R^{1/4}N^{3/4}}$$

and by taking $B_\lambda \gg 1$ sufficiently large the last sum will be small enough to ensure that

$$\overset{*}{H}[\overset{*}{R}] \leqslant \frac{G\mathrm{Đ}_R}{10}.$$

By the definition of $\overset{*}{H}[\cdot]$ this shows $\overset{*}{R}$ satisfies the bounds of definition 2.2 for the new frequency-energy levels $(\overset{*}{\Xi}, \overset{*}{\mathrm{Đ}}_u, \overset{*}{\mathrm{Đ}}_R)$. Additionally we check that $\overset{*}{\theta}, \overset{*}{u}$ satisfy the desired bounds. We present only the calculations for $\overset{*}{\theta}$ as the $\overset{*}{u}$ calculations are completely identical. For $0 \leqslant |\vec{a}| \leqslant L$ and $r = 0, 1$

$$\|\nabla_{\vec{a}}D_t^r\overset{*}{\theta}\|_{C^0} \leqslant \|\nabla_{\vec{a}}D_t^r\theta\|_{C^0} + \|\nabla_{\vec{a}}D_t^r\Theta\|_{C^0}$$

$$\lesssim \Xi^{|\vec{a}|+1/2}\mathrm{Đ}_u^{1/2}(\Xi e_u^{1/2})^r + \lambda^{|\vec{a}|+1/2}\mathrm{Đ}_R^{1/2}\tau^{-r}.$$

We note that the first term above is less than the second term due to the fact that $\Xi^{|\vec{a}|} \leqslant \lambda^{|\vec{a}|}$ and $\Xi^{1/2}\mathrm{Đ}_u^{1/2} \leqslant \lambda^{1/2}\mathrm{Đ}_R^{1/2} \iff (\mathrm{Đ}_u/\mathrm{Đ}_R)^{1/2} \leqslant (B_\lambda N)^{1/2}$ which follows from our assumption on $N \geqslant \mathrm{Đ}_u/\mathrm{Đ}_R$ in the main lemma 3.1 and by taking $B_\lambda \gg 1$. By these inequalities we have that for $0 \leqslant |\vec{a}| \leqslant L$ and $r = 0, 1$

$$\|\nabla_{\vec{a}}D_t^r\overset{*}{\theta}\|_{C^0} \lesssim \lambda^{|\vec{a}|+1/2}\mathrm{Đ}_R^{1/2}\tau^{-r} \leqslant \overset{*}{\Xi}^{|\vec{a}|}\overset{*}{e}_u^{1/2}\left(\overset{*}{\Xi}\overset{*}{e}_u^{1/2} \right)^r,$$

where again the last inequality holds for a sufficiently large choice of constant $\widehat{C}$. We have now shown the claimed frequency and energy levels $(\overset{*}{\theta}, \overset{*}{u}, \overset{*}{R}) \leqslant (\overset{*}{\Xi}, \overset{*}{\mathrm{Đ}}_u, \overset{*}{\mathrm{Đ}}_R)$. Next we will show the claimed bounds of $\Lambda^{-1/2}\Theta$ and W^i where $\Lambda^{-1/2}\Theta = \nabla_i W^i$. Recall from section 4.1.4

that the scalar correction Θ has a frequency supp in the frequency band of scale λ. The compact frequency supports with standard Littlewood–Paley theory and proposition 4.7 gives the following estimate for $|\vec{a}| \leqslant 1$

$$\|\nabla_{\vec{a}} \Lambda^{-1/2} \Theta\|_{C^0} \lesssim \lambda^{|\vec{a}|-1/2} \|\Theta\|_{C^0} \lesssim \lambda^{|\vec{a}|} D_R^{1/2}.$$

By taking $\widehat{C} \gg B_\lambda$ sufficiently large we get the claimed estimate from the main lemma 3.1.

$$\|\nabla_{\vec{a}} \Lambda^{-1/2} \Theta\|_{C^0} \leqslant \widehat{C}(N\Xi)^{|\vec{a}|} Đ_R^{1/2}.$$

Next let $\mathcal{R}^i := (-\Delta)^{-1} \nabla^i$ be a first order anti-divergence operator and let P_λ be an operator that localizes to frequency $\{|\xi| \sim \lambda\}$ such that $\Theta = P_\lambda \Theta$. Define $W^i := \mathcal{R}^i P_\lambda \Lambda^{-1/2} \Theta$. Then

$$\|W^i\|_{C^0} = \|\mathcal{R}^i P_\lambda \Lambda^{-1/2} \Theta\|_{C^0} \lesssim \lambda^{-1} \|\Lambda^{-1/2} \Theta\|_{C^0} \lesssim \lambda^{-1} D_R^{1/2}.$$

By using the definition of $\lambda \geqslant B_\lambda N\Xi$ from section 4.1 and by taking $\widehat{C} \gg B_\lambda$ sufficiently large this shows the claimed estimate from the main lemma 3.1

$$\|W^i\|_{C^0} \leqslant \widehat{C}(N\Xi)^{-1} D_R^{1/2}.$$

Next we consider $\|\partial_t \Lambda^{-1/2} \Theta\|_{C^0}$. We begin with the term $\partial_t \Lambda^{-1/2} \Theta$

$$\partial_t \Lambda^{-1/2} \Theta = \Lambda^{-1/2} D_t \Theta - \Lambda^{-1/2} \left(u^l \nabla_l \Theta \right). \tag{71}$$

Consider the first term (71). We observe that $D_t \Theta$ is also supported in frequency at scales λ due to the how $u = T\theta$ and $\hat{\theta}$ is supported at scales $\Xi \ll \lambda$ as stated in definition 2.2. By the frequency support and proposition 4.7

$$\|\Lambda^{-1/2} D_t \Theta\|_{C^0} \lesssim \lambda^{-1/2} \|D_t \Theta\|_{C^0} \lesssim \lambda^{-1/2} \lambda^{1/2} Đ_R^{1/2} \tau^{-1} = Đ_R^{1/2} \tau^{-1}.$$

For the second term in (71) we again use the fact that the function $u^l \nabla_l \Theta$ is supported in frequency at scales λ to obtain

$$\|\Lambda^{-1/2} \left(u^l \nabla_l \Theta \right)\|_{C^0} \lesssim \lambda^{-1/2} \|u\|_{C^0} \|\nabla_l \Theta\|_{C^0} \lesssim e_u^{1/2} \lambda Đ_R^{1/2}.$$

In total we have shown

$$\|\partial_t \Lambda^{-1/2} \Theta\|_{C^0} \lesssim Đ_R^{1/2} \left(\tau^{-1} + \lambda e_u^{1/2} \right).$$

We now compare the time scales and we verify that $\tau^{-1} \lesssim B_\lambda^{-1/4} \lambda e_u^{1/2}$ with the calculation below:

$$\tau^{-1} \lesssim B_\lambda^{-1/4} \lambda e_u^{1/2} \iff b^{-1} \lesssim B_\lambda^{3/4} N \iff b_0^{-1} \frac{Đ_R^{1/4} B_\lambda^{3/4} N^{3/4}}{Đ_u^{1/4}}$$

$$\lesssim B_\lambda^{3/4} N \iff \frac{Đ_R^{1/4}}{Đ_u^{1/4}} \lesssim N^{1/4}.$$

Where the last inequality holds because $Đ_R/Đ_u \leqslant 1$ and $N \geqslant 1$. By this calculation we can obtain the claimed estimate by taking a constant $\widehat{C} \gg B_\lambda$ sufficiently large to get

$$\|\partial_t \Lambda^{-1/2} \Theta\|_{C^0} \lesssim Đ_R^{1/2} \lambda e_u^{1/2} \leqslant \widehat{C} Đ_R^{1/2} (N\Xi e_u^{1/2})$$

All that remains to check is the time support of condition (4). This containment follows from the time support of the lifting function $e(t)$ defined in section 4.1.5 and by verifying that $\epsilon_t \ll (\Xi e_u^{1/2})^{-1}$, which in turn follows from the definition of $\epsilon_t = c_0 N^{-3/2} \Xi^{-3/2} Ð_R^{1/2}$ in (8) and the condition $N \geqslant Ð_u / Ð_R$ from the main lemma 3.1.

5. Main lemma implies the main theorem

In this section we prove Theorems 1 and 2 using the main lemma 3.1. Before applying the main lemma 3.1 we fix the parameter $L = 2$. Given $0 < \alpha < 3/10, 0 < \beta < 1/4$, we will produce by iteration of the main lemma a sequence of solutions $(\theta_{(k)}, R_{(k)})$ to the SQG-Reynold's system such that $R_{(k)} \to 0$ uniformly in $C^0(\mathbb{R} \times \mathbb{T}^2)$, the time support of $\theta_{(k)}$ is contained in the interval $J_{(k)}$, and the sequence of functions $\{\Lambda^{-1/2} \theta_{(k)}\}_{k \in \mathbb{N}}$ converges in $C_t^\beta C_x^0 \cap C_t^0 C_x^\alpha (\mathbb{R} \times \mathbb{T}^2)$. We remark that the non-triviality of our constructed solutions will be a consequence of the h-principle proven in section 6.

5.1. The base case $k = 0$

Fix α, β be such that $0 < \alpha < 3/10, 0 < \beta < 1/4$. For stage $k = 0$ we assume that $(\theta_{(0)}, R_{(0)})$ are any given smooth, compactly supported SQG-Reynolds flow on $\mathbb{R} \times \mathbb{T}^2$ (possibly 0). We assume also that the support of the spatial Fourier transform supp $\hat{\theta}$ is bounded in frequency space, uniformly in time.

Let $J_{(0)}$ be a nonempty time interval containing the support of $(\theta_{(0)}, R_{(0)})$. We choose a triple of parameters $(\Xi_{(0)}, Ð_{u(0)}, Ð_{R(0)})$ that are admissible according to definition 2.2 of the frequency and energy levels. Namely, take $Ð_{R(0)}$ to be any number $Ð_{R(0)} \geqslant \|R_{(0)}\|_{C^0}$, and set $Ð_{u(0)}$ to equal $Z Ð_{R(0)}$, where $Z = Z_{\alpha,\beta} \gg 1$ is chosen to satisfy conditions (74) and (76) below. Finally, take $\Xi_{(0)}$ sufficiently large so that the remaining conditions in definition 2.2 are satisfied.

5.2. Choosing the parameters

We iterate the main lemma 3.1 to produce SQG-Reynolds solutions $(\theta_{(k)}, R_{(k)})$ that are bounded above by the corresponding frequency and energy levels $(\Xi_{(k)}, Ð_{u(k)}, Ð_{R(k)})$. We also impose the ansatz

$$Ð_{R(k+1)} = \frac{Ð_{R(k)}}{Z}, \quad \text{for all } k \geqslant 0. \tag{72}$$

The parameter $Z = Z_{\alpha,\beta}$ is chosen sufficiently large to guarantee the conditions (74) and (76).

To achieve this ansatz we must choose the frequency growth parameter N to be

$$N = Z^{5/3}. \tag{73}$$

Note that by the ansatz, after the kth application of the main lemma 3.1 $Z = \frac{Ð_{R(k-1)}}{Ð_{R(k)}} = \frac{Ð_{u(k)}}{Ð_{R(k)}}$ and it follows that after the $k + 1$th application of the main lemma 3.1

$$Ð_{R(k+1)} = G_{(k)} Ð_{R(k)} = \frac{Z^{1/4}}{N^{3/4}} Ð_{R(k)} = \frac{Ð_{R(k)}}{Z}.$$

Hence by our choice of N we can maintain the ansatz for all $k \geqslant 0$.

5.3. Regularity of the solution

In this section we will show the convergence of $\{\Lambda^{-1/2}\theta_{(k)}\}_{k\in\mathbb{N}}$ in $C_t^\beta C_x^0 \cap C_t^0 C_x^\alpha(\mathbb{R}\times\mathbb{T}^2)$ for the given α, β. As a consequence, we have strong convergence of $\theta_{(k)}$ in $L_t^2 H_x^{-1/2}$, which implies weak convergence of the nonlinearity $\nabla_l[\theta_{(k)}T^l[\theta]] \rightharpoonup \nabla_l[\theta T^l[\theta]]$ in $\mathcal{D}'(\mathbb{R}\times\mathbb{T}^2)$. Since $\nabla_j\nabla_l R_{(k)}^{jl} \rightharpoonup 0$, the limiting scalar field θ is therefore a weak solution to the SQG equation.

We first calculate the $C_x^\alpha(\mathbb{T}^2)$ regularity. Define the partial sum

$$\Lambda^{-1/2}\theta_{(K)} := \sum_{k=0}^{K} \Lambda^{-1/2}\Theta_{(k)},$$

where $\Theta_{(k)}$ is the scalar correction from the kth application of the main lemma 3.1. From this definition it follows that

$$\|\Lambda^{-1/2}\theta_{(K)}\|_{C_x^\alpha} \leqslant \sum_{k=0}^{K} \|\Lambda^{-1/2}\Theta_{(k)}\|_{C_x^\alpha}$$

and it will enough to show a bound on $\|\Lambda^{-1/2}\Theta_{(k+1)}\|_{C_x^\alpha}$ that decays exponentially in k. We may interpolate between the estimates for $|\vec{a}| = 0, 1$ derivatives given at the end of the $k+1$th application of the main lemma 3.1 to obtain the following bound for $\alpha \in (0, 1)$

$$\|\Lambda^{-1/2}\Theta_{(k+1)}\|_{C_x^\alpha} \leqslant \widehat{C}(N\Xi_{(k)})^\alpha \mathrm{D}_{R(k)}^{1/2}.$$

Define the upperbound quantity

$$E_{\alpha(k+1)} := (N\Xi_{(k)})^\alpha \mathrm{D}_{R(k)}^{1/2}$$

so that $\|\Lambda^{-1/2}\Theta_{(k+1)}\|_{C_x^\alpha} \leqslant \widehat{C}E_{\alpha(k+1)}$. We will show that this upperbound $E_{\alpha(k+1)}$ decays exponentially in k for an appropriate parameter α. We apply the ansatz (72), our choice of $N = Z^{5/3}$ from (73), and the frequency and energy levels from the main lemma 3.1

$$\begin{aligned} E_{\alpha(k+1)} &= Z^{5\alpha/3}\left(\widehat{C}N\Xi_{(k-1)}\right)^\alpha Z^{-1/2}\mathrm{D}_{R(k-1)}^{1/2} \\ &= \widehat{C}^\alpha Z^{\frac{10\alpha-3}{6}}\left(N\Xi_{(k-1)}\right)^\alpha \mathrm{D}_{R(k-1)}^{1/2} \\ &= \widehat{C}^\alpha Z^{\frac{10\alpha-3}{6}} E_{\alpha(k)}. \end{aligned}$$

Now we observe that $E_{\alpha(k+1)} < E_{\alpha(k)}$ if and only if $\widehat{C}^\alpha Z^{\frac{10\alpha-3}{6}} = \left(\widehat{C}^{\frac{6\alpha}{10\alpha-3}}Z\right)^{\frac{10\alpha-3}{6}} < 1$. For a given $0 < \alpha < 3/10$ we choose the parameter $Z \gg 1$ sufficiently large so that

$$\left(\widehat{C}^{\frac{6\alpha}{10\alpha-3}}Z\right) < 1. \tag{74}$$

It follows that, with Z as above, the upperbound quantity $E_{\alpha(k+1)}$ decays exponentially in k. Hence for the function $\Lambda^{-1/2}\theta := \lim_{K\to\infty} \Lambda^{-1/2}\theta_{(K)}$ and the given $\alpha < 3/10$,

$$\max_t \|\Lambda^{-1/2}\theta\|_{C_x^\alpha} \leqslant \sum_{k=0}^{\infty} \max_t \|\Lambda^{-1/2}\Theta_{(k)}\|_{C_x^\alpha} < \infty.$$

Next we establish the $C_t^\beta C_x^0$ regularity. By interpolating the estimates at the end of $k + 1$th application of the main theorem 3.1 again we obtain the bound

$$\left\| \Lambda^{-1/2} \Theta_{(k+1)} \right\|_{C_t^\beta C_x^0} \leqslant \widehat{C} \mathrm{D}_{R(k)}^{1/2} \left((N\Xi_{(k)}) e_{u(k)}^{1/2} \right)^\beta$$

and we define the upperbound quantity

$$E_{\beta(k+1)} := \mathrm{D}_{R(k)}^{1/2} \left(N\Xi_{(k)} e_{u(k)}^{1/2} \right)^\beta.$$

In order to show that this quantity decays as exponentially in k we will need the identity

$$e_{u(k)}^{1/2} = \widehat{C}^{1/2} Z^{1/3} e_{u(k-1)}^{1/2}. \tag{75}$$

This identity can be directly computed using the main lemma 3.1, our choice of $N = Z^{5/3}$ from (73), ansatz (72), and the identity $\mathrm{D}_{u(k)} = \frac{\mathrm{D}_{u(k-1)}}{Z}$ that comes from combining the ansatz (72) with the main lemma 3.1

$$e_{u(k)}^{1/2} = \Xi_{(k)}^{1/2} \mathrm{D}_{u(k)}^{1/2} = \left(\widehat{C} N\Xi_{(k-1)} \right)^{1/2} \frac{\mathrm{D}_{u(k-1)}^{1/2}}{Z^{1/2}} = \widehat{C}^{1/2} N^{1/2} Z^{-1/2} e_{u(k-1)}^{1/2} = \widehat{C}^{1/2} Z^{1/3} e_{u(k-1)}^{1/2}.$$

We apply (75) and calculate $E_{\beta(k+1)}$ as

$$\begin{aligned}
E_{\beta(k+1)} &= \frac{\mathrm{D}_{R(k-1)}^{1/2}}{Z^{1/2}} \left(Z^{5/3} (\widehat{C} N\Xi_{(k-1)}) \widehat{C}^{1/2} Z^{1/3} e_{u(k-1)}^{1/2} \right)^\beta \\
&= \widehat{C}^{\frac{3\beta}{2}} Z^{2\beta - \frac{1}{2}} \mathrm{D}_{R(k-1)}^{1/2} \left(N\Xi_{(k-1)} e_{u(k-1)}^{1/2} \right)^\beta \\
&= \widehat{C}^{\frac{3\beta}{2}} Z^{\frac{4\beta-1}{2}} E_{\beta(k)}.
\end{aligned}$$

We observe that the upperbound quantity $E_{\beta(k+1)}$ decays exponentially in k if and only if

$$\widehat{C}^{\frac{3\beta}{2}} Z^{\frac{4\beta-1}{2}} = \left(\widehat{C}^{\frac{3\beta}{4\beta-1}} Z \right)^{\frac{4\beta-1}{2}} < 1.$$

For a given $0 < \beta < 1/4$ we can choose a parameter $Z \gg 1$ sufficiently large so that

$$\left(\widehat{C}^{\frac{3\beta}{4\beta-1}} Z \right)^{\frac{4\beta-1}{2}} < 1. \tag{76}$$

From this inequality, it follows that $E_{\beta(k+1)}$ decays as exponentially in k and

$$\left\| \Lambda^{-1/2} \theta \right\|_{C_t^\beta C_x^0} \leqslant \sum_{k=0}^\infty \left\| \Lambda^{-1/2} \Theta_{(k)} \right\|_{C_t^\beta C_x^0} < \infty,$$

for the given $\beta < 1/4$.

Since $C_t^\beta C_x^0$ and $C_t^0 C_x^\alpha$ are Banach spaces, we have shown that $\Lambda^{-1/2} \theta \in C_t^\beta C_x^0 \cap C_t^0 C_x^\alpha$.

Remark. The regularity cannot be improved without a nontrivial modification of this construction. Heuristically, the reason is that the corrections $\Theta_{(k)}$ are orthogonal to each other by the disjointness of their Fourier supports, and by a similar computation to the above, the $\| \Lambda^\gamma \Theta_{(k)}(t) \|_{L_{t,x}^2}$ norms of the corrections, which are of size $\sim \Xi_{(k)}^{\gamma + \frac{1}{2}} \mathrm{D}_{R,(k)}^{1/2}$, diverge geometrically

as the iteration continues whenever $\gamma > -\frac{1}{2} + \frac{3}{10}$. This fact precludes higher regularity in L^2. Furthermore, [22] presents a technique that, when adapted to the present setting, can show that $\Lambda^{-\frac{1}{2}+\frac{3}{10}}\theta(t)$ fails to be in $L^1_x(B)$ for every time slice t and spatial ball B contained in its support. In this sense, the construction implies a complete failure of higher regularity everywhere in the support of the solution.

5.4. Compact support in time of the solution

In this section we will discuss how the limiting SQG solution, $\theta := \lim_{k\to\infty}\theta_{(k)}$, has compact support in time. By (4) it is enough to show that the series $\sum_{k=0}^\infty \left(\Xi_{(k)}e_{u(k)}^{1/2}\right)^{-1}$ converges. By the choice of $N = Z^{5/3}$ from (73) and the identity (75), we have

$$\left(\Xi_{(k)}e_{u(k)}^{1/2}\right)^{-1} = \widehat{C}^{-3/2}Z^{-2}\left(\Xi_{(k-1)}e_{u(k-1)}^{1/2}\right)^{-1}$$

We note that $\widehat{C}^{-3/2}Z^{-2} < 1$, and so $\sum_{k=0}^\infty \left(\Xi_{(k)}e_{u(k)}^{1/2}\right)^{-1} < \infty$.

6. An h-principle

We present the proof to theorem 2 here. The work in this section relies on our organization of the estimates in lemma 3.1, and involves a new step related to the compact frequency support of the iteration.

Let f be a given smooth scalar field with compact support that satisfies $\int_{\mathbb{T}^2} f(x,t)\mathrm{d}x = 0$ for all t, and let $0 < \alpha < 3/10, 0 < \beta < 1/4$ be given. We want to construct a sequence of weak solutions θ_n to the SQG equation of class $\Lambda^{-1/2}\theta_n \in C^0_t C^\alpha_x \cap C^\beta_t C^0_x$ with such that $\Lambda^{-1/2}\theta_n \rightharpoonup \Lambda^{-1/2}f$ in $L^\infty_{t,x}$ weak-$*$.

The first step is to approximate f by scalar fields f_n, $n \in \mathbb{N}$, that have compact frequency support. Let $\eta_n(h) = 2^{2n}\eta(2^n h)$ be a standard mollifying kernel in the spatial variables with compact frequency support $\mathrm{supp}\,\hat{\eta}_n \subseteq \{|\xi| \leqslant 2^n\}$. Set $f_n = \eta_n * f$. With this choice, we have that

$$\sup_n \|\nabla_{\vec{a}}\partial_t^r f_n\|_{C^0} \lesssim \|\nabla_{\vec{a}}\partial_t^r f\|_{C^0}, \text{ for } |\vec{a}| \geqslant 0, \ r \leqslant 1, \quad \text{and} \quad \lim_{n\to\infty}\|\Lambda^{-1/2}(f_n - f)\|_{C^0} = 0.$$

$$(77)$$

The time support of f_n is also contained in the time support of f.

Now observe that the scalar fields f_n can be realized as SQG-Reynolds flows. Namely, consider $\partial_t f_n + T^l f_n \nabla_l f_n$ and observe that this function has mean-0 over $\mathbb{T}^2$ due to the nonlinear term $T^l f_n \nabla_l f_n = \nabla_l [f_n T^l f_n]$ being the divergence of a smooth vector field and the assumption that $\int_{\mathbb{T}^2} f(x,t)\mathrm{d}x = 0$ in theorem 2. Therefore we may apply the second-order anti-divergence operator $\mathcal{R}$ defined in (14). Specifically, we define symmetric and traceless tensors R_n such that

$$R_n := \mathcal{R}\left[\partial_t f_n + T^l f_n \nabla_l f_n\right], \quad \text{which implies } \partial_t f_n + T^l f_n \nabla_l f_n = \nabla_j \nabla_l R_n^{jl}.$$

The pair (f_n, R_n) is thus a smooth SQG-Reynolds flow with frequency support $\mathrm{supp}\,\hat{f}_n \subseteq \{|\xi| \leqslant 2^n\}$. Observe also that by (77), we have a uniform bound on the errors R_n:

$$\sup_n \|R_n\|_{C^0} \leqslant \overset{\diamond}{\mathrm{D}}_R < \infty \quad \text{for some } \overset{\diamond}{\mathrm{D}}_R > 0.$$

Starting with (f_n, R_n), we now construct a weak solution θ_n to SQG close to f_n in the appropriate weak topology. The SQG solution θ_n is constructed by iterating the main lemma 3.1 using an initial choice of frequency and energy levels $\left(\Xi_{n(0)}, Đ_{R,n(0)}, Đ_{u,n(0)}\right)$ chosen as follows. Take $Đ_{R,n(0)} = \overset{\diamond}{Đ}_R > 0$ and $Đ_{u,n(0)} = ZĐ_{R,n(0)}$ where $Z = Z_{\alpha,\beta} \gg 1$ is as in section 5. Take $\Xi_{n(0)}$ sufficiently large such that the SQG-Reynolds flow $(\theta_{n(0)}, R_{n,(0)}) := (f_n, R_n)$ satisfies definition 2.2, and such that

$$\lim_{n\to\infty} \Xi_{n(0)} = \infty. \tag{78}$$

In particular, the frequency support condition will be satisfied for $\Xi_{n(0)} \geqslant 2^n$.

Repeated application of the main lemma using the parameter choices of section 5 yields a sequence of SQG-Reynolds flows $(\theta_{n(k)}, R_{n(k)})$ such that $\theta_{n(k)}$ converges as $k \to \infty$ to an SQG solution θ_n with $\Lambda^{-1/2}\theta_n \in C_t^\beta C_x^0 \cap C_t^0 C_x^\alpha$. We claim that the solutions θ_n satisfy $\Lambda^{-1/2}\theta_n \rightharpoonup \Lambda^{-1/2}f$ in $L_{t,x}^\infty$ weak-*.

Let $\epsilon > 0$ and a test function $\varphi \in L^1(\mathbb{R} \times \mathbb{T}^2)$ be given. We will show that there exists a large index M, depending on φ and ϵ, such that for all $n \geqslant M$

$$\left| \int_{\mathbb{R} \times \mathbb{T}^2} \Lambda^{-1/2}(\theta_n - f)\varphi \, \mathrm{d}x \, \mathrm{d}t \right| < 3\epsilon.$$

First, observe by (77) and Hölder that for all $n \geqslant M$ sufficiently large we have

$$\left| \int_{\mathbb{R} \times \mathbb{T}^2} \Lambda^{-1/2}(f_n - f)\varphi \, \mathrm{d}x \, \mathrm{d}t \right| < \epsilon. \tag{79}$$

Next, take a smooth approximation of compact support $\tilde{\varphi} \in C_c^\infty(\mathbb{R} \times \mathbb{T}^2)$ with

$$\|\varphi - \tilde{\varphi}\|_{L^1(\mathbb{R} \times \mathbb{T}^2)} < \frac{\epsilon}{2\widehat{C}\overset{\diamond}{Đ}_R^{1/2}}, \tag{80}$$

where $\widehat{C}$ is the constant from the main lemma 3.1. It now suffices to estimate the integral

$$\left| \int_{\mathbb{R} \times \mathbb{T}^2} \Lambda^{-1/2}(\theta_n - f_n)\varphi \, \mathrm{d}x \, \mathrm{d}t \right| \leqslant \left| \int_{\mathbb{R} \times \mathbb{T}^2} \Lambda^{-1/2}(\theta_n - f_n)\tilde{\varphi} \, \mathrm{d}x \, \mathrm{d}t \right| \tag{81}$$

$$+ \left| \int_{\mathbb{R} \times \mathbb{T}^2} \Lambda^{-1/2}(\theta_n - f_n)(\varphi - \tilde{\varphi})\mathrm{d}x \, \mathrm{d}t \right|. \tag{82}$$

Let $\Theta_{n(k)}$ be the corrections defined so that $\theta_n - f_n = \sum_{k \geqslant 0} \Theta_{n(k)}$ and the integral (81) is

$$(81) = \left| \int_{\mathbb{R} \times \mathbb{T}^2} \sum_{k \geqslant 0} \Lambda^{-1/2}\Theta_{n(k)}\tilde{\varphi} \, \mathrm{d}x \, \mathrm{d}t \right|.$$

From the main lemma 3.1, $\Lambda^{-1/2}\Theta_{n(k)} = \nabla_i W_{n(k)}^i$, $\|W_{n(k)}\|_{C^0} \leqslant \widehat{C}(N\Xi_{n(k)})^{-1} Ð_{R,n(k)}^{1/2}$, and

$$
\begin{aligned}
(81) &\leqslant \sum_{k\geqslant 0}\left|\int_{\mathbb{R}\times\mathbb{T}^2}\nabla_i W_{n(k)}^i \tilde{\varphi}\, \mathrm{d}x\, \mathrm{d}t\right| \leqslant \sum_{k\geqslant 0}\|W_{n(k)}\|_{C^0}\|\nabla\tilde{\varphi}\|_{L^1}\\
&\leqslant \|\nabla\tilde{\varphi}\|_{L^1}\sum_{k\geqslant 0}\widehat{C}(N\Xi_{n(k)})^{-1}Ð_{R,n(k)}^{1/2}\\
&\leqslant \frac{2\widehat{C}Ð_{R,n(0)}^{1/2}}{N\Xi_{n(0)}}\|\nabla\tilde{\varphi}\|_{L^1} \leqslant \frac{\widehat{C}\overset{\diamond}{Ð}_R^{1/2}}{\Xi_{n(0)}}\|\nabla\tilde{\varphi}\|_{L^1} < \epsilon.
\end{aligned}
$$

Here we used that $Z \gg 1$ and $\Xi_{n(k+1)}^{-1}Ð_{R,n(k+1)}^{1/2} \leqslant Z^{-\frac{5}{3}-\frac{1}{2}}\Xi_{n(k)}^{-1}Ð_{R,n(k)}^{1/2}$ to sum the geometric series, and the last inequality holds by (78) for all $n \geqslant M$ sufficiently large. To bound (82), we now observe

$$
(82) \leqslant \|\Lambda^{-1/2}(\theta_n - f_n)\|_{C^0}\|\varphi - \tilde{\varphi}\|_{L^1}.
$$

Next we show a bound on $\|\Lambda^{-1/2}(\theta_n - f_n)\|_{C^0}$. To start $\|\Lambda^{-1/2}(\theta_n - f_n)\|_{C^0} \leqslant \sum_{k\geqslant 0}\|\Lambda^{-1/2}\Theta_{n(k)}\|_{C^0}$ recall from the main lemma 3.1 that $\|\Lambda^{-1/2}\Theta_{n(k)}\|_{C^0} \leqslant \widehat{C}Ð_{R,n(k)}^{1/2}$ and we use the ansatz from (72) to get $Ð_{R,n(k)} = \frac{Ð_{R,n(0)}}{Z^k}$. From this equation, it follows that

$$
\|\Lambda^{-1/2}(\theta_n - f_n)\|_{C^0} \leqslant \sum_{k\geqslant 0}\widehat{C}Ð_{R,n(0)}^{1/2}Z^{-k/2} \leqslant 2\widehat{C}Ð_{R,n(0)}^{1/2} = 2\widehat{C}\overset{\diamond}{Ð}_R^{1/2},
$$

and therefore

$$
(82) \leqslant 2\widehat{C}\overset{\diamond}{Ð}_R^{1/2}\|\varphi - \tilde{\varphi}\|_{L^1} < \epsilon.
$$

The last inequality is due to our choice of $\tilde{\varphi}$ in (80). By combining the estimates of (79), (81) and (82), we have now shown that there exists an M such that for all $n \geqslant M$

$$
\left|\int_{\mathbb{R}\times\mathbb{T}^2}\Lambda^{-1/2}(\theta_n - f)\varphi\, \mathrm{d}x\, \mathrm{d}t\right| < 3\epsilon,
$$

and we can conclude the claim that $\Lambda^{-1/2}\theta_n \rightharpoonup \Lambda^{-1/2}f$ in the L^∞ weak-* topology.

Acknowledgments

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