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Functional Analysis / *Analyse fonctionnelle*

Counterexamples for multi-parameter weighted paraproducts

Contre-exemples pour les paraproducts à poids multi-paramétrés

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Abstract. We build the plethora of counterexamples to bi-parameter two weight embedding theorems. Two weight one parameter embedding results (which is the same as results of boundedness of two weight classical paraproducts, or two weight Carleson embedding theorems) are well known since the works of Sawyer in the 80's. Bi-parameter case was considered by S. Y. A. Chang and R. Fefferman but only when underlying measure is Lebesgue measure. The embedding of holomorphic functions on bi-disc requires general input measure. In [9] we classified such embeddings if the output measure has tensor structure. In this note we give examples that without tensor structure requirement all results break down.

Résumé. Dans le présent article, nous construisons une pléthore de contre-exemples aux théorèmes de plongements à deux poids et à deux paramètres. Les résultats de plongement à un paramètre et à deux poids (qui sont la même chose que les résultats de paraproducts bornés classiques à deux poids) sont bien connus depuis les travaux de Sawyer dans les années 80. S. Y. A. Chang et R. Fefferman ont examiné le cas des deux paramètres, mais uniquement lorsque la mesure sous-jacente est la mesure de Lebesgue. Le plongement de fonctions holomorphes sur le bi-disque nécessite une mesure générale en entrée. Dans [9], nous avons classé ces plongements lorsque la mesure obtenue en sortie a une structure tensorielle. Dans cette note, nous donnons des contre-exemples d'après lesquels tous les résultats deviennent faux en l'absence d'hypothèse d'une structure tensorielle.

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Le résultat principal de cet article est la pléthore de contre-exemples qui révèlent que la question de plongement bornée à deux poids et à deux paramètres n'a (peut être) aucune critère qui ressemble le critère pour le plongement bornée à deux poids et à un paramètre (bien connu comme l'imbedding de Carleson à deux poids, voir [14] et aussi [13]). On construit ici les contre-exemple pour quelques conjectures naturelles. C'est fait dans le cas quand une de deux mesure est arbitraire et une autre est une mesure assez simple mais sans une structure tensorielle. Si la structure tensorielle de deuxième mesure est présente, nous avons démontré dans [9] que le critère de Carleson est nécessaire et suffisante pour l'imbedding. En plus nous avons démontré dans [9] que le critère de « boîte » est aussi nécessaire et suffisante pour l'imbedding. C'est un contraste inattendu avec les résultats de S. Y. A. Chang et R. Fefferman où on a le contre-exemple de Carleson qui dit que le critère de « boîte » n'est pas équivalent au critère de Carleson.

1. Hardy inequality on the n -tree and energy of measures

We consider here bi-linear bi-parameter dyadic paraproducts, that is the operators of the type

$$(f, g) \rightarrow \sum_{R=I \times J} (f, \mathbb{1}_R)_\mu (g, h_R) h_R,$$

where I, J are all dyadic sub-intervals of $[0, 1]$, μ is a measure on square $Q_0 := [0, 1]^2$, h_R is a Haar function of rectangle R normalized in $L^2(Q_0, m_2)$. Let T be a dyadic tree. An n -tree T^n , $n > 1$, is a Cartesian product of n identical dyadic trees with order induced by the product structure. In what follows we work only with finite (but very deep) multi-trees. Notice that unlike T , graphs T^n , $n > 1$, have cycles. For instance T^2 is the graph of all dyadic rectangles inside the unit square. For $\mu = m_2 =$ Lebesgue measure on Q_0 , the criterion of boundedness of bi-linear bi-parameter dyadic paraproduct was found by S. Y. A. Chang and used by her and R. Fefferman [3, 4]. L. Carleson [2, 15] constructed a very interesting counterexample in this theory, which we use below. This counterexample was used by J.-L. Journé in answering (in negative) a question of J. Pipher and R. Fefferman on multi-parameter singular integrals. More general bi-linear bi-parameter paraproducts (not necessarily dyadic) were considered in many papers, and their Lebesgue spaces unweighted mapping properties were established, see e.g. [8, 10, 11], see a weighted result in [5], Banach space valued result in [7]. Bi-linear paraproducts are building blocks for bi-linear PDO and, in particular, bi-linear Coifman–Meyer multipliers [6]. They are very important also in PDEs, e.g. in proving fractional Leibniz rules, used by Kato–Ponce. The bi-parameter version of bi-linear paraproducts was used by C. Kenig who used more involved Leibniz rules. There is a large literature on the subject, and many results can be found in [12] together with many references.

On n -tree T^n we have a canonical partial order: $(\gamma'_1, \dots, \gamma'_n) = \gamma' \geq \gamma = (\gamma_1, \dots, \gamma_n)$ iff $\gamma'_i \geq \gamma_i$ for all i . From now on we assume that the *weight* $w: T^n \rightarrow \mathbb{R}_+$ is fixed. The *Hardy operator* associated with w is defined by

$$\mathbf{I}_w \phi(\gamma) := \sum_{\gamma' \geq \gamma} w(\gamma') \phi(\gamma'), \quad \text{and} \quad \mathbf{I}^* \psi(\gamma) = \sum_{\gamma' \leq \gamma} \psi(\gamma').$$

For a measure (non-negative function) μ on T^n we define the $(w-)$ potential to be

$$\mathbf{V}_w^\mu(\alpha) := (\mathbf{I}_w \mathbf{I}^* \mu)(\alpha), \quad \alpha \in T^n,$$

again we usually drop the index w . Let $E \subset T^n$ and μ be a measure on T^n . The E -truncated energy of μ is

$$\mathcal{E}_E[\mu] := \sum_{\alpha \in E} (\mathbf{I}^* \mu)^2(\alpha) w(\alpha).$$

If $E = T^n$, we write $\mathcal{E}[\mu]$ instead, and so $\mathcal{E}[\mu] = \int_{T^n} \mathbf{V}^\mu d\mu := \sum_{\alpha \in T^n} \mathbf{V}^\mu(\alpha) \mu(\alpha)$. If E is a δ -level set of $\mathbf{V}^\mu$ for some $\delta > 0$, i.e. $E = \{\alpha : \mathbf{V}^\mu \leq \delta\}$, then we write $\mathcal{E}_\delta[\mu] := \mathcal{E}_E[\mu]$.

A subset $\mathcal{D}$ of T^n is called a *down-set* if for any $\gamma \in \mathcal{D}$ and $\gamma' \leq \gamma$, one also has $\gamma' \in \mathcal{D}$. We define the *box constant*, *Carleson constant*, *hereditary Carleson constant (HC)*, *Carleson embedding constant* to be the smallest numbers such that

$$\mathcal{E}_{\mathcal{S}(\beta)}[\mu] := \sum_{\alpha \leq \beta} w(\alpha) (\mathbf{I}^* \mu(\alpha))^2 \leq [w, \mu]_{Box} \mu(\mathcal{S}(\beta)), \quad \forall \beta \in T^n; \quad (1)$$

$$\mathcal{E}_{\mathcal{D}}[\mu] \leq [w, \mu]_C \mu(\mathcal{D}), \quad \forall \mathcal{D} \subset T^n \text{ down-set}; \quad (2)$$

$$\mathcal{E}[\mu \mathbb{1}_E] \leq [w, \mu]_{HC} \mu(E), \quad \forall E \subset T^n; \quad (3)$$

$$\mathcal{E}[\psi \mu] \leq [w, \mu]_{CE} \sum_{\omega \in T^n} |\psi(\omega)|^2 \mu(\omega) \quad (4)$$

holds for all functions ψ on T^n . If $[w, \mu]_{CE} < +\infty$, we call (w, μ) the *trace pair* for the weighted Hardy inequality on T^n .

The boundedness of paraproduct operator from $L^2(Q_0, \mu)$ to $L^2(Q_0, m_2)$ is equivalent to boundedness of $\mathbf{I}^*(\cdot \mu)$ from $L^2(\mu)$ to $\ell^2(T^2, w)$, where $w = \{w_R\}$, $w_R = |(g, h_R)|^2$. Measure μ is an input measure, w is an output weight sitting on T^2 .

The inequalities $[w, \mu]_{Box} \leq [w, \mu]_C \leq [w, \mu]_{HC} \leq [w, \mu]_{CE}$ are obvious. The converse inequalities for 1-trees were proved in [13]. Our main result in [9] is the extension to the 2- and 3-trees.

Theorem 1. *Let $\mu : T^n \rightarrow \mathbb{R}_+$, $n = 1, 2, 3$. Let $w : T^n \rightarrow [0, \infty)$ be of tensor product form. Then the reverses of the above inequalities also hold:*

$$[w, \mu]_{CE} \lesssim [w, \mu]_{HC} \lesssim [w, \mu]_C \lesssim [w, \mu]_{Box}.$$

Let $T^2 = T_N^2$ be a finite (but very deep) dyadic bi-tree, and denote the set of minimal elements of this bi-tree, that is, the small squares of size $2^{-N} \times 2^{-N}$, by ∂T^2 , and elements of this set will be denoted by ω . In all examples the measure μ will be supported on ∂T^2 . Thus we are able to write $\mathbb{I}^* \mu(Q)$ as $\mu(Q)$. With this convention the conditions (1)-(4) for the measure μ and weight $w = \{w_Q\}$ become

$$\sum_{Q \in T^2, Q \subset R} \mu^2(Q) w_Q \leq C \mu(R), \quad \text{for any } R \in T^2 \quad (5a)$$

$$\sum_{Q \in T^2, Q \subset E} \mu^2(Q) w_Q \leq C \mu(E), \quad \text{for any } E \subset (\partial T)^2 \quad (5b)$$

$$\sum_{Q \in \mathcal{D}} \mu^2(Q \cap E) w_Q \leq C \mu(E), \quad \text{for any } E \subset (\partial T)^2 \quad (5c)$$

$$\sum_{Q \in \mathcal{D}} \left(\int_Q \varphi d\mu \right)^2 w_Q \leq C \int_{Q_0} \varphi^2 d\mu \quad \text{for any } \varphi \in L^2(Q_0, d\mu). \quad (5d)$$

The implications $(5a) \Leftarrow (5b) \Leftarrow (5c) \Leftarrow (5d)$ hold for arbitrary measures μ and weights w . For product weights w the converse implications hold by Theorem 1, so all these conditions are in fact equivalent. However, if w is not of tensor product form, then one can show that every converse implication can fail. We will show that the converse implications do not hold in general. We will do this by constructing N -coarse measures μ and weights w on finite bi-trees T^2 of depth N such that the discrepancies between box, Carleson, hereditary Carleson, and embedding constants grow with N .

In [2] Carleson constructed families $\mathcal{R}$ of dyadic sub-rectangles of $Q = [0, 1]^2$ having the following two properties:

$$\forall R_0 \in T^2, \quad \sum_{R \subset R_0, R \in \mathcal{R}} m_2(R) \leq C_0 m_2(R_0), \quad (6)$$

but

$$\sum_{R \in \mathcal{R}} m_2(R) > C_1 m_2(\cup_{R \in \mathcal{R}} R), \quad (7)$$

with arbitrarily large ratios $\frac{C_1}{C_0}$, where m_2 is the planar Lebesgue measure. Choosing $\mu = m_2$ and $w_R := \frac{1}{m_2(R)} \cdot \mathbb{1}_{\mathcal{R}}$ we can identify the left-hand sides of (6) and (7) with the left-hand sides of (5a) and (5b), respectively. Hence the box condition (5a) holds with constant C_0 , while the Carleson condition (5b) can only hold with constant $\geq C_1$. Our aim here is to show that for general w, μ the Carleson condition (5b) is no longer sufficient for the Restricted Energy Condition (5c). Namely we prove the following statement.

Theorem 2. *For any $\delta > 0$ there exists a number $N \in \mathbb{N}$, a weight $w : T_N^2 \rightarrow \{0, 1\}$, and a measure μ on ∂T^2 such that μ satisfies the Carleson condition (5b) with the constant $C_\mu = \delta$:*

$$\sum_{Q \subset E} \mu^2(Q) w_Q \leq \delta \mu(E), \quad \text{for any } E \subset (\partial T)^2, \quad (8)$$

but there exists a set $F \subset Q_0$ such that

$$\sum_{Q \in \mathcal{Q}} \mu^2(Q \cap F) w_Q > \mu(F), \quad (9)$$

hence the constant in (5c) is at least 1.

The weight w below has values either 0 or 1, but the support $\mathcal{R}$ of w is an *up-set*, that is, it contains every ancestor of every rectangle in $\mathcal{R}$. The example is based on the fact that potentials on bi-tree may not satisfy the maximal principle. So we start with constructing an N -coarse μ such that we have

$$\mathbb{V}^\mu \lesssim 1 \quad \text{on } \text{supp } \mu, \quad (10)$$

but

$$\max \mathbb{V}^\mu \geq \mathbb{V}^\mu(\omega_0) \gtrsim \log N. \quad (11)$$

We define a collection of rectangles

$$Q_j := [0, 2^{-2^j}] \times [0, 2^{-2^{-j}N}], \quad j = 1, \dots, M \approx \log N. \quad (12)$$

Now we put

$$\begin{aligned} \mathcal{R} &:= \{R : Q_j \subset R \text{ for some } j = 1 \dots M\}, \quad w_Q := \mathbb{1}_{\mathcal{R}}(Q) \\ \mu(\omega) &:= \frac{1}{N} \sum_{j=1}^M \frac{1}{|Q_j^{++}|} \mathbb{1}_{Q_j^{++}}(\omega). \end{aligned} \quad (13)$$

here $|Q|$ denotes the total amount of points $\omega \in (\partial T)^2 \cap Q$, i.e. the amount of the smallest possible rectangles (of size $2^{-N} \times 2^{-N}$) in Q . The non-trivial part of (10) is the estimate

$$\mathbb{V}^\mu(Q_j) \lesssim 1.$$

Let R be a dyadic rectangle containing Q_j , then the set $\{j' : Q_{j'} \subseteq R\}$ is an interval that contains j . For given interval of integers $[m, m+k]$ let

$$C^{[m, m+k]} := \{R \supseteq \omega_0 : \{j' : Q_{j'} \subseteq R\} = [m, m+k]\}.$$

Since each rectangle in $C^{[m,m+k]}$ contains $\left[0, 2^{-2^m}\right] \times \left[0, 2^{-2^{m-k}N}\right]$, we have

$$\#C^{[m,m+k]} \leq (2^m + 1) \left(2^{-m-k}N + 1\right) \lesssim 2^{-k}N. \quad (14)$$

It follows that

$$\mathbb{V}^\mu(Q_j) = \sum_{[m,m+k] \ni j} \left(\#C^{[m,m+k]}\right) (k+1) \frac{1}{N} \lesssim \sum_{k \geq 0} (k+1)^2 2^{-k} N \frac{1}{N} \lesssim 1. \quad (15)$$

This shows that $\mathbb{V}^\mu(Q_j) \lesssim 1$ thus proving (10).

Now we will estimate $\mathbb{V}^\mu(\omega_0)$ from below. To this end we need a more careful *lower* bound on $\#C^{[m,m+k]}$. The set $C^{[j,j]}$ contains all rectangles R that contain Q_j and are contained in $[0, 2^{-2^{j-1}-1}] \times [0, 2^{-2^{j-1}N-1}]$, so

$$\#C^{[j,j]} \geq 2^{j-1} \cdot 2^{-j-1}N \gtrsim N. \quad (16)$$

Hence

$$\mathbb{V}^\mu(\omega_0) \geq \sum_{j=1}^M \left(\#C^{[j,j]}\right) \frac{1}{N} \gtrsim M. \quad (17)$$

This shows (11) as $M \asymp \log N$.

Now to show that the Carleson condition holds, but the hereditary Carleson condition fails we construct a new measure $\nu := \mu + \sigma$, where the pair μ, w is chosen as in (13), and σ is just the $\frac{1}{N}$ mass uniformly distributed over ω_0 . Let us first give a lower bound for the HC constant. Consider $F = \omega_0$. Then by (16) we have

$$\mathcal{E}[\nu|F] \geq \sum_{j=1}^M \left(\#C^{[j,j]}\right) \nu(\omega_0)^2 \gtrsim MN \cdot \nu(\omega_0)^2.$$

This shows that $[w, \nu]_{HC} \gtrsim \nu(\omega_0) \cdot NM = M$. Next we will verify that the Carleson condition (5b) holds with a small constant. We may remove from the sum on the left-hand side of (5b) all rectangles $Q \notin \mathcal{R}$. Then we can replace E by the union of remaining Q 's without changing the left-hand side and decreasing the right-hand side. Hence we may reduce to the case when E is a union of members of $\mathcal{R}$. For each j we have either $Q_j \subseteq E$ or $Q_j^{++} \cap E = \emptyset$. Let $\mathcal{J} := \{j : Q_j \subseteq E\}$. Then by (14) we obtain

$$\begin{aligned} LHS(5b) &\leq \sum_{[m,m+k] \subseteq \mathcal{J}} \sum_{Q \in C^{[m,m+k]}} \left(\frac{(k+1)}{N} + \frac{1}{N}\right)^2 \leq \sum_{[m,m+k] \subseteq \mathcal{J}} 2^{-k} N (k+2)^2 \left(\frac{1}{N}\right)^2 \\ &\lesssim \frac{(\#\mathcal{J})}{N} \leq \mu(E) \leq \nu(E). \end{aligned} \quad (18)$$

so that $[w, \nu]_C \lesssim 1$.

Next we describe a measure-weight pair that satisfies hereditary Carleson condition, but does not provide the embedding. We define $Q_j, \mu, \mathcal{R}, w$ as before. We continue with denoting $Q_{0,j} := Q_j$, $\mu_0 := \mu$ as before and defining a sequence of collections $\mathcal{Q}_k$, $k = 0, \dots, K \approx \log M$ of dyadic rectangles as follows

$$\mathcal{Q}_k := \left\{ Q_{k,j} := \bigcap_{i=j}^{j+2^k-1} Q_{0,i}, j = 1, \dots, M - 2^k \right\}, k = 0, \dots, K. \quad (19)$$

In other words, $\mathcal{Q}_k$ consists of the intersections of 2^k consecutive elements of the basic collection $\mathcal{Q}_0$. The total amount of rectangles in $\mathcal{Q}_k$ is denoted by $M_k = M - 2^k + 1$. For $k = 1, \dots, K$ let

$$\mu_k(\omega) := \frac{2^{-2k}}{N} \sum_{j=1}^{M_k} \frac{1}{|Q_{k,j}^{++}|} \mathbb{1}_{Q_{k,j}^{++}}(\omega), \quad \omega \in (\partial T)^2,$$

and define $\mu := \mu_0 + \sum_{k=1}^K \mu_k$. In [1] it is shown that $[w, \mu]_{HC} \lesssim 1$, but $[w, \mu]_{CE} \gtrsim \log M$.

References

- [1] N. Arcozzi, P. Mozolyako, G. Psaromiligkos, A. Volberg, P. Zorin-Kranich, “Bi-parameter Carleson embeddings with product weights”, <https://arxiv.org/abs/1906.11150>, 2019.
- [2] L. Carleson, *A counter example for measures bounded on H^p for the bi-disc*, Report, vol. 7, Institut Mittag-Leffler, 1974.
- [3] S.-Y. A. Chang, “Carleson measure on the bi-disc”, *Ann. Math.* **109** (1979), no. 3, p. 613-620.
- [4] S.-Y. A. Chang, R. Fefferman, “A continuous version of duality of H^1 with BMO on the bidisc”, *Ann. Math.* **112** (1980), no. 1, p. 179-201.
- [5] J. Chen, G. Lu, “Hörmander type theorems for multi-linear and multi-parameter Fourier multiplier operators with limited smoothness”, *Nonlinear Anal., Theory Methods Appl.* **101** (2014), p. 98-112.
- [6] R. R. Coifman, Y. Meyer, *Au delà des opérateurs pseudo-différentiels*, Astérisque, vol. 57, Société Mathématique de France, 1978.
- [7] F. Di Plinio, Y. Ou, “Banach-valued multilinear singular integrals”, *Indiana Univ. Math. J.* **67** (2018), no. 5, p. 1711-1763.
- [8] M. Lacey, J. Metcalfe, “Paraproducts in one and several parameters”, *Forum Math.* **19** (2007), no. 2, p. 325-351.
- [9] P. Mozolyako, G. Psaromiligkos, A. Volberg, P. Zorin-Kranich, “Combinatorial property of all positive measures in dimensions 2 and 3”, 2020, preprint.
- [10] C. Muscalu, J. Pipher, T. Tao, C. Thiele, “Bi-parameter paraproducts”, *Acta Math.* **193** (2004), p. 269-296.
- [11] ———, “Multi-parameter paraproducts”, *Rev. Mat. Iberoam.* **22** (2006), no. 3, p. 963-976.
- [12] V. Naibo, A. Thomson, “Coifman–Meyer multipliers: Leibniz-type rules and applications to scattering of solutions to PDEs”, *Trans. Am. Math. Soc.* **372** (2019), no. 8, p. 5453-5481.
- [13] F. Nazarov, S. Treil, A. Volberg, “The Bellman functions and two-weight inequalities for Haar multipliers”, *J. Am. Math. Soc.* **12** (1999), p. 909-928.
- [14] E. T. Sawyer, “Weighted inequalities for the two-dimensional Hardy operator”, *Stud. Math.* **82** (1985), no. 1, p. 1-16.
- [15] T. Tao, “Dyadic product H^1 , BMO , and Carleson’s counterexample”, 1999, <http://www.math.ucla.edu/tao/preprints/Expository/product.dvi>.