MULTIPLICITIES AND MIXED MULTIPLICITIES OF ARBITRARY FILTRATIONS

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Dedicated to Jürgen Herzog on the occasion of his 80th birthday

Abstract. We develop a theory of multiplicities and mixed multiplicities of filtrations, extending the theory for filtrations of m-primary ideals to arbitrary (not necessarily Noetherian) filtrations. The mixed multiplicities of r filtrations on an analytically unramified local ring R come from the coefficients of a suitable homogeneous polynomial in r variables of degree equal to the dimension of the ring, analogously to the classical case of the mixed multiplicities of m-primary ideals in a local ring. We prove that the Minkowski inequalities hold for arbitrary filtrations. The characterization of equality in the Minkowski inequality for m-primary ideals in a local ring by Teissier, Rees and Sharp and Katz does not extend to arbitrary filtrations, but we show that they are true in a large and important subcategory of filtrations. We define divisorial and bounded filtrations. The filtration of powers of a fixed ideal is a bounded filtration, as is a divisorial filtration. We show that in an excellent local domain, the characterization of equality in the Minkowski equality is characterized by the condition that the integral closures of suitable Rees like algebras are the same, strictly generalizing the theorem of Teissier, Rees and Sharp and Katz. We also prove that a theorem of Rees characterizing the inclusion of ideals with the same multiplicity generalizes to bounded filtrations in excellent local domains. We give a number of other applications, extending classical theorems for ideals.

1. Introduction

In this paper we extend the theory of multiplicities and mixed multiplicities of filtrations of m_R-primary ideals in a local ring R to arbitrary filtrations. We prove that these multiplicities enjoy good properties and derive applications.

The study of mixed multiplicities of m_R-primary ideals in a local ring R with maximal ideal m_R was initiated by Bhattacharya [1], Rees [32] and Teissier and Risler [40]. In [13] the notion of mixed multiplicities is extended to arbitrary, not necessarily Noetherian, filtrations of R by m_R-primary ideals (m_R-filtrations). It is shown in [13] that many basic theorems for mixed multiplicities of m_R-primary ideals are true for m_R-filtrations.

The development of the subject of mixed multiplicities and its connection to Teissier's work on equisingularity [40] is explained in [16]. A survey of the theory of multiplicities and mixed multiplicities of m_R-primary ideals can be found in [39, Chapter 17], including discussion of the results of the papers [33] of Rees and [38] of Swanson, and the theory of Minkowski inequalities of Teissier [40], [41], Rees and Sharp [35] and Katz [24]. Later, Katz and Verma [25], generalized mixed multiplicities to ideals that are not all m_R-primary. Trung and Verma [43] computed mixed multiplicities of monomial ideals

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from mixed volumes of suitable polytopes. Mixed multiplicities are also used by Huh in the analysis of the coefficients of the chromatic polynomial of graph theory in [23].

A notion of mixed multiplicity for arbitrary ideals is introduced by Bhattacharya in [1]. This notion of mixed multiplicity is extended to arbitrary graded families of ideals by Cid Ruiz and Montaño, [6]. We give an alternate definition of multiplicity and mixed multiplicity for filtrations of ideals in this paper, which more strictly generalizes the definition for m_R-primary ideals.

All local rings will be assumed to be Noetherian. Let R be a d-dimensional local ring with maximal ideal m_R. It is shown in [8, Theorem 1.1] and [11, Theorem 4.2] that in a local ring R, the limit

$$\lim_{n\to\infty}\frac{\ell_R(R/I_n)}{n^d} \tag{1}$$

exists for all filtrations $\mathcal{I}=\{I_n\}$ of m_R-primary ideals if and only if $\dim N(\hat{R})<\dim R$, where $N(\hat{R})$ is the nilradical of R. In local rings R which satisfy $\dim N(\hat{R})<\dim R$, we may then define the multiplicity of a filtration $\mathcal{I}$ of m_R-primary ideals by

$$e_R(\mathcal{I})=\lim_{n\to\infty}\frac{\ell_R(R/I_n)}{n^d/d!}.$$

The problem of existence of such limits (1) has been considered by Ein, Lazarsfeld and Smith [14] and Mustaţă [29]. When the ring R is a domain and is essentially of finite type over an algebraically closed field k with $R/m_R=k$, Lazarsfeld and Mustaţă [27] showed that the limit exists for all m_R-filtrations. In [9], Cutkosky proved it in the complete generality stated above. Lazarsfeld and Mustaţă use in [27] the method of counting asymptotic vector space dimensions of graded families using "Okounkov bodies". This method, which is reminiscent of the geometric methods used by Minkowski in number theory, was developed by Okounkov [30], Kaveh and Khovanskii [26] and Lazarsfeld and Mustaţă [27]. We also use this method. The fact that $\dim N(R)=d$ implies there exists a filtration without a limit was observed by Dao and Smirnov.

It is shown in [13] that if R is a local ring such that $\dim N(\hat{R})<d$ and

$$\mathcal{I}(1)=\{I(1)_m\},\ldots,\mathcal{I}(r)=\{I(r)_m\}$$

are filtrations of m_R-primary ideals, then the limit

$$P(n_1,\ldots,n_r)=\lim_{m\to\infty}\frac{\ell_R(R/I(1)_{mn_1}\cdots I(r)_{mn_r})}{m^d/d!}$$

is a homogeneous real polynomial $P(n_1,\ldots,n_r)$ of degree d for $n_1,\ldots,n_r\in\mathbb{N}$. We may thus define the mixed multiplicities $e_R(\mathcal{I}(1)^{[d_1]},\ldots,\mathcal{I}(r)^{[d_r]})$ of these filtrations from the coefficients of this polynomial, by the expansion

$$P(n_1,\ldots,n_r)=\sum_{d_1+\cdots+d_r=d}\frac{d!}{d_1!\cdots d_r!}e_R(\mathcal{I}(1)^{[d_1]},\ldots,\mathcal{I}(r)^{[d_r]})n_1^{d_1}\cdots n_r^{d_r}.$$

In [13], multiplicities $e_R(\mathcal{I};N)$ and $e_R(\mathcal{I}(1)^{[d_1]},\ldots,\mathcal{I}(r)^{[d_r]};N)$ are defined for a finitely generated R-module N and filtrations of m_R-primary ideals.

We now extend these definitions to arbitrary filtrations. Let R be an analytically unramified local ring. Let $\mathfrak{a}$ be an m_R-primary ideal and $\mathcal{I}=\{I_n\}$ be a filtration of ideals on R. We have that $V(I_n)=V(I_1)$ for all n (Lemma 3.1) where

$$V(I)=\{\mathfrak{p}\in\operatorname{Spec}(R)\mid I\subset\mathfrak{p}\}$$

and we define

$$s(\mathcal{I}) = \dim R/I_1.$$

For $s \in \mathbb{N}$ and a finitely generated R-module N such that $\dim N \le s$ ([37, Section 2 of Chapter V] and [5, Section 4.7])

$$e_s(\mathfrak{a}, N) = \begin{cases} e_{\mathfrak{a}}(N) & \text{if } \dim N = s \\ 0 & \text{if } \dim N < s. \end{cases}$$

In Proposition 4.2, we show that the limit

$$\lim_{m\to\infty} \frac{e_s(\mathfrak{a}, R/I_m)}{m^{d-s}/(d-s)!}$$

exists for $s \ge s(\mathcal{I})$ if R is analytically unramified, and define the multiplicity $e_s(\mathfrak{a},\mathcal{I})$ to be equal to this limit. We have an associativity formula (5),

$$e_s(\mathfrak{a},\mathcal{I}) = \sum_{\mathfrak{p}} e_{R_{\mathfrak{p}}}(\mathcal{I}_{\mathfrak{p}}) e_{\mathfrak{a}}(R/\mathfrak{p}),$$

where $\mathcal{I}_{\mathfrak{p}} = \{(I_n)_{\mathfrak{p}}\}$ and the sum is over all $\mathfrak{p} \in \mathrm{Spec}(R)$ such that $\dim R/\mathfrak{p} = s$ and $\dim R_{\mathfrak{p}} = d - s$.

The condition R analytically unramified is used to ensure that the limits $e_{R_{\mathfrak{p}}}(\mathcal{I}_{\mathfrak{p}})$ exist all for prime ideals $\mathfrak{p}$ in R.

We then define mixed multiplicities of arbitrary filtrations $\mathcal{I}(1) = \{I(1)_m\}, \ldots, \mathcal{I}(r) = \{I(r)_m\}$. We show in Theorem 4.3 that for $s \ge \max\{s(\mathcal{I}(1)), \ldots, s(\mathcal{I}(r))\}$, the limit

$$\lim_{m\to\infty} \frac{e_s(\mathfrak{a}, R/I(1)_{mn_1}\cdots I(r)_{mn_r})}{m^{d-s}/(d-s)!}$$

is a homogeneous real polynomial $H_s(n_1, \ldots, n_r)$ of degree $d-s$ for $n_1, \ldots, n_r \in \mathbb{N}$, which allows us to define the mixed multiplicities

$$e_s(\mathcal{I}(1)^{[d_1]}, \ldots, \mathcal{I}(r)^{[d_r]})$$

from the coefficients of this polynomial in Definition 4.4. We have an associativity formula (15),

$$e_s(\mathfrak{a}, \mathcal{I}(1)^{[d_1]}, \ldots, \mathcal{I}(r)^{[d_r]}) = \sum_{\mathfrak{p}} e_{R_{\mathfrak{p}}}(\mathcal{I}(1)_{\mathfrak{p}}^{[d_1]}, \ldots, \mathcal{I}(r)_{\mathfrak{p}}^{[d_r]}) e_{\mathfrak{a}}(R/\mathfrak{p})$$

where the sum is over all $\mathfrak{p} \in \mathrm{Spec}(R)$ such that $\dim R/\mathfrak{p} = s$ and $\dim R_{\mathfrak{p}} = d - s$.

We define in Proposition 4.2 and Definition 4.4 the multiplicities $e_s(\mathfrak{a}, \mathcal{I}; N)$ and mixed multiplicities $e_s(\mathfrak{a}, \mathcal{I}(1), \ldots, \mathcal{I}(r); N)$ for arbitrary finitely generated R-modules N.

In Section 6, we define divisorial filtrations and s-divisorial filtrations. We prove the converse to the Rees Theorem 1.1 for s-divisorial filtrations in Theorem 6.7. We prove the characterization of equality in the Minkowski inequality (3) for s-divisorial filtrations in Theorem 6.9.

In Section 7, we define bounded filtrations and bounded s-filtrations. We prove the converse to the Rees Theorem 1.1 for bounded s-filtrations in Theorem 7.4. We prove the characterization of equality in the Minkowski inequality (3) for bounded s-filtrations in Theorem 7.6.

In Theorems 6.7, 6.9, 7.4 and 7.6, we have the assumption that R is an excellent local domain. These theorems generalize the corresponding theorems for divisorial and bounded filtrations of m_R-primary ideal of [11, Corollary 7.4], [11, Theorem 12.1], [11, Theorem 13.1] and Theorem [11, Theorem 13.2].

In Examples 7.5 and 7.7 we show that Theorems 7.4 and 7.6 do not extend to arbitrary bounded or divisorial filtrations (when $s > 0$ so that the filtrations do not consist of m_R-primary ideals).

Let R be a local ring and $\mathcal{I} = \{I_n\}$ be a filtration of R. Define the graded R-algebra

$$R[\mathcal{I}] = \sum_{m \geq 0} I_m t^m$$

and let $\overline{R[\mathcal{I}]}$ be the integral closure of $R[\mathcal{I}]$ in the polynomial ring $R[t]$.

In Section 6, we define divisorial filtrations. Suppose that R is a local domain. Let ν be a divisorial valuation of the quotient field of R which is nonnegative on R. We have the valuation ideals

$$I(\nu)_m = \{f \in R \mid \nu(f) \geq m\}$$

for $m \in \mathbb{N}$. The prime ideal $\mathfrak{p} = I(\nu)_1$ is called the center of ν on R. We say that ν is an s-valuation if $\dim R/\mathfrak{p} = s$.

A divisorial filtration of R is a filtration $\mathcal{I} = \{I_m\}$ such that there exist divisorial valuations $\nu_1, \ldots, \nu_r$ and $a_1, \ldots, a_r \in \mathbb{R}_{\geq 0}$ such that for all $m \in \mathbb{N}$,

$$I_m = I(\nu_1)_{\lceil ma_1 \rceil} \cap \cdots \cap I(\nu_r)_{\lceil ma_r \rceil}.$$

An s-divisorial filtration of R is a filtration $\mathcal{I} = \{I_m\}$ such that there exist s-valuations $\nu_1, \ldots, \nu_r$ and $a_1, \ldots, a_r \in \mathbb{R}_{\geq 0}$ such that for all $m \in \mathbb{N}$,

$$I_m = I(\nu_1)_{\lceil ma_1 \rceil} \cap \cdots \cap I(\nu_r)_{\lceil ma_r \rceil}.$$

Observe that the trivial filtration $\mathcal{I} = \{I_m\}$, defined by $I_m = R$ for all m, is a degenerate case of a divisorial filtration and is a degenerate case of an s-divisorial filtration for all s. The nontrivial 0-divisorial filtrations are the divisorial m_R-filtrations of [11].

We will often denote a divisorial filtration $\mathcal{I}$ on a local domain R by $\mathcal{I} = \mathcal{I}(D)$. The motivation for this notation comes from the concept of a representation of a divisorial filtration on an excellent local domain, defined before Theorem 6.7.

In Section 7, we define bounded filtrations and bounded s-filtrations. If $\mathcal{I}(D)$ is a divisorial filtration, then $R[\mathcal{I}(D)]$ is integrally closed in $R[t]$; that is, $\overline{R[\mathcal{I}(D)]} = R[\mathcal{I}(D)]$.

A filtration $\mathcal{I} = \{I_n\}$ on R is said to be a bounded filtration if there exists a divisorial filtration $\mathcal{I}(D)$ on R such that $\overline{R[\mathcal{I}]} = R[\mathcal{I}(D)]$. A filtration $\mathcal{I} = \{I_n\}$ on R is said to be a bounded s-filtration if there exists an s-divisorial filtration $\mathcal{I}(D)$ on R such that $\overline{R[\mathcal{I}]} = R[\mathcal{I}(D)]$.

If I is an ideal, then the I-adic filtration is bounded (Lemma 7.2). Further, the filtration of integral closures of powers of an ideal I, and symbolic filtrations are bounded.

In Section 5, we extend some classical inequalities for multiplicities and mixed multiplicities of m_R-primary ideals to filtrations, generalizing the inequalities of [13] for filtrations of m_R-primary ideals to arbitrary filtrations. We also extend some other classical inequalities for multiplicities of ideals to filtrations. We first prove the following theorem, which is proven in Theorem 5.1. This theorem is proven for m_R-primary filtrations in [13, Theorem 6.9], [10, Theorem 1.4].

Theorem 1.1. *(Theorem 5.1) Let R be an analytically unramified local ring of dimension d, N be a finitely generated R-module, $\mathfrak{a}$ be an m_R-primary ideal and $\mathcal{I} = \{I_n\}$, $\mathcal{J} = \{J_n\}$ be filtrations of ideals of R with $\mathcal{J} \subset \mathcal{I}$. Suppose $R[\mathcal{I}]$ is integral over $R[\mathcal{J}]$. Then*

(i) $s(\mathcal{I}) = s(\mathcal{J})$,

(ii) $e_s(\mathfrak{a}, \mathcal{I}; N) = e_s(\mathfrak{a}, \mathcal{J}; N)$ *for all s such that $s(\mathcal{I}) = s(\mathcal{J}) \leq s \leq d$.*

The converse of Theorem 1.1 fails for arbitrary filtrations, as shown by a simple example of filtrations of m_R-primary ideals in [13]. A famous theorem of Rees [32] (also [39, Theorem 11.3.1]) shows that if R is a formally equidimensional local ring and $J \subset I$ are m_R-primary ideals, then the converse of Theorem 1.1 does hold for the J-adic and I-adic filtrations $\mathcal{J} = \{J^n\} \subset \mathcal{I} = \{I^n\}$. In this situation, the Rees algebra of I, $\bigoplus_{n\geq 0} I^n$ is integral over the Rees algebra of J, $\bigoplus_{n\geq 0} J^n$, if and only if the integral closures of ideals $\overline{I} = \overline{J}$ are equal, which is the condition of Rees's theorem. In Theorem 7.4, we prove that if $\mathcal{J}$ is a bounded s-filtration and $\mathcal{I}$ is an arbitrary filtration with $\mathcal{J} \subset \mathcal{I}$, then the converse of Theorem 1.1 holds. This generalizes the theorem for bounded filtrations of m_R-primary ideals in [11, Theorem 13.1].

The Minkowski inequalities were formulated and proven for m_R-primary ideals in reduced equicharacteristic zero local rings by Teissier [40], [41] and proven in full generality for m_R-primary ideals in arbitrary local rings, by Rees and Sharp [35]. The same inequalities hold for filtrations. They were proven for m_R-filtrations in local rings R such that $\dim N(\hat{R}) < \dim R$ in [13, Theorem 6.3]. We prove them for arbitrary filtrations in an analytically unramified local ring in Theorem 5.3. In the following theorem, we state the fundamental inequality from which all other inequalities follow, (i) of Theorem 1.2, and the most famous inequality (ii) (The Minkowski Inequality).

Theorem 1.2. *(Minkowski Inequalities) Let R be an analytically unramified local ring of dimension d, N be a finitely generated R-module, $\mathfrak{a}$ be an m_R-primary ideal, $\mathcal{I} = \{I_j\}$ and $\mathcal{J} = \{J_j\}$ be filtrations of ideals of R. Let $\max\{s(\mathcal{I}), s(\mathcal{J})\} \leq s < d$ and $k := d - s$.*

(i) Let $k \geq 2$. For $1 \leq i \leq k-1$,

$$e_s(\mathfrak{a}, \mathcal{I}^{[i]}, \mathcal{J}^{[k-i]}; N)^2 \leq e_s(\mathfrak{a}, \mathcal{I}^{[i+1]}, \mathcal{J}^{[k-i-1]}; N) e_s(\mathfrak{a}, \mathcal{I}^{[i-1]}, \mathcal{J}^{[k-i+1]}; N),$$

(ii) $s(\mathcal{I}\mathcal{J}) = \max\{s(\mathcal{I}), s(\mathcal{J})\}$ and $e_s(\mathfrak{a}, \mathcal{I}\mathcal{J}; N)^{\frac{1}{k}} \leq e_s(\mathfrak{a}, \mathcal{I}; N)^{\frac{1}{k}} + e_s(\mathfrak{a}, \mathcal{J}; N)^{\frac{1}{k}}$, where $\mathcal{I}\mathcal{J} = \{I_j J_j\}$.

There is a characterization of when equality holds for m_R-primary ideals in the Minkowski Inequality by Teissier [42] (for Cohen-Macaulay normal two-dimensional complex analytic R), Rees and Sharp [35] (in dimension 2) and Katz [24] (in complete generality).

They have shown that if R is a formally equidimensional local ring and I, J are m_R-primary ideals then the Minkowski inequality

$$e_R(IJ)^{\frac{1}{d}} = e_R(I)^{\frac{1}{d}} + e_R(J)^{\frac{1}{d}}$$

holds if and only if there exists $a, b \in \mathbb{Z}_{>0}$ such that the integral closures $\overline{I^a} = \overline{J^b}$ are equal. This condition is equivalent to the statement that the integral closures of the Rees algebras of I^a and J^b are equal; that is, there exist positive integers a and b such that

$$\overline{\sum_{n\geq 0} I^{an} t^n} = \overline{\sum_{n\geq 0} J^{bn} t^n}. \tag{2}$$

We show in Theorem 5.4 that if $\mathcal{I}$ and $\mathcal{J}$ are filtrations on an analytically unramified local ring R and there exist $a, b \in \mathbb{Z}_{>0}$ such that the integral closures of the R-algebras $\oplus_{n\geq 0} I_{an} t^n$ and $\oplus_{n\geq 0} J_{bn} t^n$ are equal, then the Minkowski equality

$$e_s(\mathfrak{a}, \mathcal{I}\mathcal{J}; N)^{\frac{1}{d-s}} = e_s(\mathfrak{a}, \mathcal{I}; N)^{\frac{1}{d-s}} + e_s(\mathfrak{a}, \mathcal{J}; N)^{\frac{1}{d-s}} \tag{3}$$

holds. However, if $\mathcal{I}$ and $\mathcal{J}$ are filtrations on an analytically unramified local ring R such that the Minkowski Equality (3) holds, then in general, the integral closures of the R-algebras $\oplus_{n\geq 0} I_{an} t^n$ and $\oplus_{n\geq 0} J_{bn} t^n$ are not equal for all $a, b \in \mathbb{Z}_{>0}$, even for filtrations of

m_R-primary ideals in a regular local ring (so that $s = 0$), as is shown in a simple example in [13].

In Theorem 7.6, we show that if $\mathcal{I}(1)$ and $\mathcal{I}(2)$ are two nontrivial bounded s-filtrations in an excellent local domain R, then the Minkowski Equality holds if and only if there exist positive integers a and b such that there is equality of integral closures

$$\overline{\sum_{n\geq 0} I(1)_{an}t^n} = \overline{\sum_{n\geq 0} I(2)_{bn}t^n},$$

giving a complete generalization of the Teissier, Rees and Sharp, Katz Theorem for bounded s-filtrations. This theorem was proven for bounded filtrations of m_R-primary ideals in [11, Theorem 13.2].

In Lemma 5.7, we prove that if $\mathcal{J}(i) \subset \mathcal{I}(i)$ are filtrations for $1 \leq i \leq r$ on an analytically unramified local ring and N is a finitely generated R-module, then we have inequalities of mixed multiplicities

$$e_s(\mathfrak{a}, \mathcal{I}(1)^{[d_1]}, \ldots, \mathcal{I}(r)^{[d_r]}; N) \leq e_s(\mathfrak{a}, \mathcal{J}(1)^{[d_1]}, \ldots, \mathcal{J}(r)^{[d_r]}; N)$$

for $s \geq \max\{s(\mathcal{J}(1)), \ldots, s(\mathcal{J}(r))\}$.

In Proposition 5.9, we generalize a formula on multiplicity of specialization of ideals ([22], [28, formula (2.1)]) to filtrations.

In Theorem 8.5, we extend a theorem of Böger ([3], [39, Corollary 11.3.2]) about equimultiple ideals in a formally equidimensional local ring to a theorem about bounded s-filtrations in an excellent local domain.

An ideal I in a local ring R is equimultiple if $\text{ht}(I) = \ell(I)$ where $\ell(I)$ is the analytic spread of I. In an excellent local domain, our theorem is strictly stronger that Boğer's theorem, even for I-adic filtrations. We show in Corollary 8.3 that the I-adic filtration of an equimultiple ideal is a bounded s-filtration where $s = \dim R - \text{ht}(I)$. In Example 8.4, we show that there are ideals I whose I-adic filtration is a bounded s-filtration, but I is not equimultiple.

2. Notation

We will denote the nonnegative integers by $\mathbb{N}$ and the positive integers by $\mathbb{Z}_{>0}$, the set of nonnegative rational numbers by $\mathbb{Q}_{\geq 0}$ and the positive rational numbers by $\mathbb{Q}_{>0}$. We will denote the set of nonnegative real numbers by $\mathbb{R}_{\geq 0}$ and the positive real numbers by $\mathbb{R}_{>0}$. For a real number x, $\lceil x \rceil$ will denote the smallest integer that is $\geq x$ and $\lfloor x \rfloor$ will denote the largest integer that is $\leq x$. If $E_1, \ldots, E_r$ are prime divisors on a normal scheme X and $a_1, \ldots, a_r \in \mathbb{R}$, then $\lfloor \sum a_i E_i \rfloor$ denotes the integral divisor $\sum \lfloor a_i \rfloor E_i$ and $\lceil \sum a_i E_i \rceil$ denotes the integral divisor $\sum \lceil a_i \rceil E_i$.

A local ring is assumed to be Noetherian. The maximal ideal of a local ring R will be denoted by m_R. The quotient field of a domain R will be denoted by $\text{QF}(R)$. We will denote the length of an R-module M by $\ell_R(M)$. Excellent local rings have many excellent properties which are enumerated in [18, Scholie IV.7.8.3]. We will make use of some of these properties without further reference.

3. Filtrations

A filtration $\mathcal{I} = \{I_n\}_{n\in\mathbb{N}}$ of ideals on a ring R is a descending chain

$$R = I_0 \supset I_1 \supset I_2 \supset \cdots$$

of ideals such that $I_iI_j \subset I_{i+j}$ for all $i,j \in \mathbb{N}$. A filtration $\mathcal{I} = \{I_n\}$ of ideals on a local ring (R, m_R) is a filtration of R by m_R-primary ideals if I_n is m_R-primary for $n \geq 1$. A filtration $\mathcal{I} = \{I_n\}_{n\in\mathbb{N}}$ of ideals on a ring R is called a Noetherian filtration if $\bigoplus_{n\geq 0} I_n$ is a finitely generated R-algebra.

If $I \subset R$ is an ideal, then $V(I) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid I \subset \mathfrak{p}\}$. If $\mathcal{I} = \{I_n\}$ and $\mathcal{J} = \{J_n\}$ are filtrations of R, then we will write $\mathcal{I} \subset \mathcal{J}$ if $I_n \subset J_n$ for all n.

For any filtration $\mathcal{I} = \{I_n\}$ and $\mathfrak{p} \in \operatorname{Spec} R$, let $\mathcal{I}_\mathfrak{p}$ denote the filtration $\mathcal{I}_\mathfrak{p} = \{I_nR_\mathfrak{p}\}$.

Let R be a local ring and $\mathcal{I} = \{I_n\}$ be a filtration of R. As defined in the introduction, the graded R-algebra

$$R[\mathcal{I}] = \sum_{m\geq 0} I_m t^m$$

and $\overline{R[\mathcal{I}]}$ is the integral closure of $R[\mathcal{I}]$ in the polynomial ring $R[t]$.

Lemma 3.1. *Let R be a local ring and $\mathcal{I} = \{I_n\}$ be a filtration of ideals of R. The following hold.*

(*i*) *For all $n \geq 1$, $V(I_1) = V(I_n)$ and $\dim R/I_1 = \dim R/I_n$.*
(*ii*) *If $\mathcal{J} = \{J_n\}$ is a filtration of ideals of R such that $\mathcal{J} \subset \mathcal{I}$ and the R-algebra $R[\mathcal{I}]$ is a finitely generated $R[\mathcal{J}]$-module then $V(I_n) = V(J_m)$ for all $n,m \geq 1$.*

Proof. (*i*) Since $\{I_n\}$ is a filtration, for all integers $n > m \geq 1$, we have $I_m^n \subset I_n \subset I_m$. Therefore $V(I_1) = V(I_n)$ for all $n \geq 1$. Hence $\min \operatorname{Ass} R/I_1 = \min \operatorname{Ass} R/I_n$ for all $n \geq 1$. Thus for all $n \geq 1$, $\dim R/I_1 = \dim R/I_n$.
(*ii*) Let $\{\alpha_1, \ldots, \alpha_r\}$ be a generating set of the R-algebra $R[\mathcal{I}]$ as an $R[\mathcal{J}]$-module. Suppose $\deg \alpha_i = d_i$ for all $i = 1, \ldots, r$ and $d = \max\{d_1, \ldots, d_r\}$. Then for all $n \geq d+1$, we have $J_n \subset I_n \subset J_{n-d}$. Since by (1), $V(J_m) = V(J_1)$ for all $m \geq 1$, we have $V(I_n) = V(J_n)$. Again by using (1), we get $V(I_n) = V(J_m)$ for all $n,m \geq 1$. □

Definition 3.2. *Let R be a local ring and $\mathcal{I} = \{I_n\}$ be a filtration of ideals of R. We define the dimension of the filtration $\mathcal{I}$ to be $s(\mathcal{I}) = \dim R/I_n$ (for any $n \geq 1$).*

The dimension $s(\mathcal{I})$ is well-defined by Lemma 3.1 (*i*). In the case of the trivial filtration $\mathcal{I} = \{I_n\}$, where $I_n = R$ for all n, we have that $s(\mathcal{I}) = -1$.

Definition 3.3. *Let R be a d dimensional local ring and $\mathcal{I} = \{I_n\}$ be a filtration of ideals of R. For $s \in \mathbb{N}$, we define*

$$A(\mathcal{I}) = \min \operatorname{Ass} R/I_1 \bigcap \{\mathfrak{p} \in \operatorname{Spec} R : \dim R/\mathfrak{p} = s\},$$

the set of minimal primes $\mathfrak{p}$ of I_1 such that $\dim R/\mathfrak{p} = s$.

Example 3.4. *The embedded associated primes that appear in the ideals in a filtration $\mathcal{I} = \{I_n\}$ may be infinite in number.*

A simple example is as follows. Let k be an infinite field, and let $\{\alpha_i\}_{i\in\mathbb{Z}_{>0}}$ be a countable set of distinct elements of k. Let $R = k[x,y,z]_{(x,y,z)}$ be the localization of a polynomial ring over k in three variables. Let $I_n = z^{n+1}(z, x - \alpha_n y)$ for $n \in \mathbb{Z}_{>0}$. $\mathcal{I} = \{I_n\}$ is thus a filtration. The associated primes of $I_n = (z^{n+1}) \cap (z^{n+2}, x - \alpha_n y)$ are (z) and $(z, x - \alpha_n y)$. This is in contrast to the fact that the associated primes of the filtration of powers of an ideal I in a local ring is a finite set [4].

Lemma 3.5. *Let R be a local ring and suppose that $\mathcal{I} = \{I_n\}$ is a filtration of R. Then the following are equivalent.*

1) $R[t]$ is integral over $\sum_{n\ge 0} I_n t^n$.
2) $I_1 = R$.
3) $\mathcal{I}$ is the trivial filtration.

Proof. Suppose that t is integral over $\sum_{n\ge 0} I_n t^n$. Then there exist $n \in \mathbb{Z}_{>0}$ and $a_i \in I_i$ for $0 \le i \le n$ such that $t^n + (a_1 t)t^{n-1} + \cdots + (a_n t^n) = 0$. Thus $1 \in (a_1, \ldots, a_n) \subset I_1$. □

Suppose that R is a local ring and $\mathcal{I} = \{I_n\}$ is a filtration of R. The integral closure $\overline{\sum_{n\ge 0} I_n t^n}$ of $\sum_{n\ge 0} I_n t^n$ in $R[t]$ is a graded R-algebra $\overline{\sum_{n\ge 0} I_n t^n} = \sum_{n\ge 0} K_n t^n$, where $\mathcal{K} = \{K_n\}$ is a filtration of R (by [39, Theorem 2.3.2]).

If I is an ideal in a local ring R, let $\overline{I}$ denote its integral closure.

Lemma 3.6. *Let R be a local ring and $\mathcal{I} = \{I_n\}$ be a filtration. Then*

$$\overline{R[\mathcal{I}]} = \sum_{m\ge 0} J_m t^m$$

where $\{J_m\}$ is the filtration

$$J_m = \{f \in R \mid f^r \in \overline{I_{rm}} \text{ for some } r > 0\}.$$

The proof of Lemma 3.6 for m_R-filtrations in [11, Lemma 5.5] extends immediately to arbitrary divisorial filtrations.

Remark 3.7. *If $\mathcal{I} = \{I^n\}$ is the adic-filtration of the powers of a fixed ideal I, then $J_n = \overline{I^n}$ for all n.*

Lemma 3.8. *Suppose that R is a local ring, $\mathcal{I} = \{I_n\}$ is a filtration of R and $p \in Spec(R)$. Let $\overline{R[\mathcal{I}]} = \oplus_{n\ge 0} K_n$. Then the integral closure $\overline{\sum_{n\ge 0} I_n R_p t^n}$ of $\sum_{n\ge 0} I_n R_p t^n$ in $R_p[t]$ is $\sum_{n\ge 0} K_n R_p t^n$.*

Lemma 3.9. *Let R be a local ring, $\mathcal{I} = \{I_n\}$ be a filtration of R and $\overline{R[\mathcal{I}]} = \oplus_{n\ge 0} K_n$. Let $\mathcal{K} = \{K_n\}$. Then*
1) $V(I_1) = V(K_1)$.
2) $s(\mathcal{I}) = s(\mathcal{K})$.

Proof. By Lemmas 3.5 and 3.8, $p \notin V(I_1)$ if and only if $\overline{\sum_{n\ge 0} I_n R_p t^n} = R_p[t]$ which holds if and only if $\sum_{n\ge 0} K_n R_p t^n = R_p[t]$, and this last condition holds if and only if $p \notin V(K_1)$. □

4. Multiplicities of filtrations

Let $\mathfrak{a}$ be an m_R-primary ideal of R and N be a finitely generated R-module with $\dim N = r$. Define

$$e_{\mathfrak{a}}(N) = \lim_{k\to\infty} \frac{l_R(N/\mathfrak{a}^k N)}{k^r/r!}.$$

If $s \ge r = \dim N$, define ([37, V.2], [5, 4.7])

$$e_s(\mathfrak{a}, N) = \begin{cases} e_{\mathfrak{a}}(N) & \text{if } \dim N = s \\ 0 & \text{if } \dim N < s. \end{cases}$$

Example 4.1. *The function $e_{\mathfrak{a}}(N)$ of an m_R-primary ideal $\mathfrak{a}$ does not extend to a function $e_{\mathcal{A}}(N)$ of a filtration $\mathcal{A} = \{a_n\}$ of m_R-primary ideals on finitely generated R-modules N, even on a regular local ring.*

The existence of such an example follows from [8, Example 5.3]. Let k be a field and R be the d dimensional power series ring over k, $R=k[[x_1,\ldots x_{d-1},y]]$. In [8, Example 5.3], a filtration $\mathcal{A}=\{a_n\}$ of m_R-primary ideals is constructed such that if $\mathfrak{p}$ is the prime ideal $\mathfrak{p}=(y)$ of R, then the limit

$$\lim_{k\to\infty}\frac{\ell_R((R/\mathfrak{p}^m)/a_k(R/\mathfrak{p}^m))}{k^{d-1}/(d-1)!}$$

does not exist for any $m\geq 2$. In the example, a function $\sigma:\mathbb{Z}_+\to\mathbb{Q}_+$ is constructed such that letting $N_n=(x_1,\ldots,x_{d-1})^n$, and defining $a_n=(N_n,yN_{n-\sigma(n)},y^2)$, we have that $\mathcal{A}=\{a_n\}$ is a filtration of m_R-primary ideals on R for which the above limits do not exist.

Let R be an analytically unramified local ring of dimension d, N be a finitely generated R-module and $\mathcal{I}=\{I_n\}$ is a filtration of m_R-primary ideals. Then the multiplicity of $\mathcal{I}$ is defined by

$$e_R(\mathcal{I},N):=\lim_{m\to\infty}\frac{\ell_R(N/I_mN)}{m^d/d!}.$$

This limit exists by [8, Theorem 1.1] and [13, Theorem 6.6]. We further define the multiplicity of the trivial filtration $\mathcal{I}=\{I_n\}$, where $I_n=R$ for all n, to be $e_R(\mathcal{I},N)=0$. We write $e_R(\mathcal{I})=e_R(\mathcal{I},R)$.

Suppose that $\mathcal{I}$ is a filtration of R and that $s\geq s(\mathcal{I})$. Let N be a finitely generated R-module. Suppose that $\mathfrak{p}$ is a prime ideal of R such that $\dim R/\mathfrak{p}=s$. Then $\dim R_{\mathfrak{p}}\leq d-s$ with equality if R is equidimensional and catenary. The catenary condition holds, for instance, if R is regular or R is excellent. Define the filtration $\mathcal{I}_{\mathfrak{p}}=\{I_nR_{\mathfrak{p}}\}$. Then we have that

$$e_{R_p}(\mathcal{I}_{\mathfrak{p}},N_{\mathfrak{p}})=\lim_{n\to\infty}\frac{\ell_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}/I_nN_{\mathfrak{p}})}{n^{\dim R_{\mathfrak{p}}}/\dim R_{\mathfrak{p}}!}\tag{4}$$

exists.

We will frequently make use of the fact that if R is a local ring which is analytically unramified and $\mathfrak{p}\in\operatorname{Spec}(R)$ is a prime ideal, then $R_{\mathfrak{p}}$ is analytically unramified ([39, Proposition 9.1.4]).

Proposition 4.2. *Suppose that R is an analytically unramified local ring, N is a finitely generated R-module and $\mathfrak{a}$ is an m_R-primary ideal and $\mathcal{I}$ is a filtration on R. Suppose that $s\in\mathbb{N}$ is such that $s(\mathcal{I})\leq s\leq d$. Then the limit*

$$e_s(\mathfrak{a},\mathcal{I};N):=\lim_{m\to\infty}\frac{e_s(\mathfrak{a},N/I_mN)}{m^{d-s}/(d-s)!}$$

exists. Further,

$$e_s(\mathfrak{a},\mathcal{I};N)=\sum_{\mathfrak{p}}e_{R_{\mathfrak{p}}}(\mathcal{I}_{\mathfrak{p}},N_{\mathfrak{p}})e_{\mathfrak{a}}(R/\mathfrak{p})\tag{5}$$

where the sum is over all $\mathfrak{p}\in\operatorname{Spec}(R)$ such that $\dim R/\mathfrak{p}=s$ and $\dim R_{\mathfrak{p}}=d-s$.

Proof. Let $\mathfrak{p}$ be a prime ideal of R such that $\dim R/\mathfrak{p}=s$. If $\mathfrak{p}\notin A(\mathcal{I})$ then $\ell_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}/I_nN_{\mathfrak{p}})=0$. If $\mathfrak{p}\in A(\mathcal{I})$ then for all $n\geq 1$, $I_nR_{\mathfrak{p}}$ are $\mathfrak{p}R_{\mathfrak{p}}$-primary ideals of $R_{\mathfrak{p}}$. Hence for all $\mathfrak{p}\in A(\mathcal{I})$, the limit

$$\lim_{m\to\infty}\frac{\ell_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}/I_mN_{\mathfrak{p}})}{m^{\dim R_{\mathfrak{p}}}/\dim R_{\mathfrak{p}}!}$$

exists by (4) and since $R_{\mathfrak{p}}$ is analytically unramified. Since $A(\mathcal{I})$ is a finite set, using [5, Corollary 4.7.8], we get that the limit

$$\begin{aligned}
\lim_{m\to\infty}\frac{e_s(\mathfrak{a},N/I_mN)}{m^{d-s}/(d-s)!} &= \lim_{m\to\infty}\frac{(d-s)!}{m^{d-s}}\sum_{\substack{\mathfrak{p}\in\operatorname{Spec}R,\\ \dim R/\mathfrak{p}=s}}\ell_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}/I_mN_{\mathfrak{p}})e_{\mathfrak{a}}(R/\mathfrak{p})\\
&= \sum_{\substack{\mathfrak{p}\in\operatorname{Spec}R,\\ \dim R/\mathfrak{p}=s}}\Big(\lim_{m\to\infty}\frac{\ell_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}/I_mN_{\mathfrak{p}})}{m^{d-s}/(d-s)!}\Big)e_{\mathfrak{a}}(R/\mathfrak{p})\\
&= \sum_{\substack{\mathfrak{p}\in\operatorname{Spec}R,\\ \dim R/\mathfrak{p}=s \text{ and } \dim R_{\mathfrak{p}}=d-s}} e_{R_{\mathfrak{p}}}(\mathcal{I}_{\mathfrak{p}},N_{\mathfrak{p}})e_{\mathfrak{a}}(R/\mathfrak{p})
\end{aligned}$$

exists. ☐

If $\mathcal{I}$ is a filtration of m_R-primary ideals, then $s(\mathcal{I})=0$ and

$$e_0(\mathfrak{a},\mathcal{I};N)=e_R(\mathcal{I};N)e_{\mathfrak{a}}(R/m_R)=e_R(\mathcal{I};N).$$

We will write $e_s(\mathfrak{a},\mathcal{I})=e_s(\mathfrak{a},\mathcal{I};R)$.

4.1. **Mixed multiplicities of filtrations.** Let R be an analytically unramified local ring of dimension d and N be a finitely generated R-module. Suppose that $\mathcal{I}(1)=\{I(1)_i\},\ldots,\mathcal{I}(r)=\{I(r)_i\}$ are filtrations of m_R-primary ideals. It is shown in [13, Theorem 6.6] that the function

$$P(n_1,\ldots,n_r)=\lim_{m\to\infty}\frac{\ell_R(N/I(1)_{mn_1}\cdots I(r)_{mn_r}N)}{m^d/d!} \tag{6}$$

is a homogeneous polynomial of total degree d with real coefficients for all $n_1,\ldots,n_r\in\mathbb{N}$. The mixed multiplicities $e_R(\mathcal{I}(1)^{[d_1]},\ldots,\mathcal{I}(r)^{[d_r]};N)$ of N of type $(d_1,\ldots,d_r)$ with respect to the filtrations $\mathcal{I}(1),\ldots,\mathcal{I}(r)$ are defined from the coefficients of P, generalizing the definition of mixed multiplicities for m_R-primary ideals. Specifically, we write

$$P(n_1,\ldots,n_r)=\sum_{d_1+\cdots+d_r=d}\frac{d!}{d_1!\cdots d_r!}e_R(\mathcal{I}(1)^{[d_1]},\ldots,\mathcal{I}(r)^{[d_r]};N)n_1^{d_1}\cdots n_r^{d_r}. \tag{7}$$

We have need of the formulas (6) and (7) in the slightly more general case that $\mathcal{I}(1)=\{I(1)_i\},\ldots,\mathcal{I}(r)=\{I(r)_i\}$ are filtrations of R such that for each j, $\mathcal{I}(j)$ is either a filtration of m_R-primary ideals or $\mathcal{I}(j)$ is the trivial filtration $I(j)_n=R$ for all $n\in R$. In this case, there exists $\sigma\in\mathbb{N}$ with $0\le\sigma\le r$ such that either $\sigma=0$ and $\mathcal{I}(j)$ are filtrations of m_R-primary ideals for all j or $\sigma>0$ and there exist $1\le i_1<\cdots<i_\sigma\le r$ such that $\mathcal{I}(j)$ is a filtration of m_R-primary ideals if $j\in\{i_1,\ldots,i_\sigma\}$ and $\mathcal{I}(j)$ is the trivial filtration $I(j)_n=R$ for all $n\in\mathbb{N}$ if $j\notin\{i_1,\ldots,i_\sigma\}$. As in (6), we define

$$P(n_1,\ldots,n_r)=\lim_{m\to\infty}\frac{\ell_R(N/I(1)_{mn_1}\cdots I(r)_{mn_r}N)}{m^d/d!} \tag{8}$$

In this case, we have that

$$P(n_1,\ldots,n_r)=P(0,\ldots,0,n_{i_1},0,\ldots,0,n_{i_2},0,\ldots,0,n_{i_\sigma},0,\ldots,0)$$

which is a homogeneous polynomial of degree d by (6) and (7). In this case, we also define the mixed multiplicities $e_R(\mathcal{I}(1)^{[d_1]},\ldots,\mathcal{I}(r)^{[d_r]};N)$ of N of type $(d_1,\ldots,d_r)$ with respect to the filtrations $\mathcal{I}(1),\ldots,\mathcal{I}(r)$ from the coefficients of P, so that P has an expansion

$$(9)\qquad P(n_1,\ldots,n_r)=\sum_{d_1+\cdots+d_r=d}\frac{d!}{d_1!\cdots d_r!}e_R(\mathcal{I}(1)^{[d_1]},\ldots,\mathcal{I}(r)^{[d_r]};N)n_1^{d_1}\cdots n_r^{d_r}.$$

We have that $P(n_1,\ldots,n_r)$ is a homogeneous polynomial of degree d in the variables $n_{i_1},\ldots,n_{i_\sigma}$. Thus

$$(10)\qquad e_R(\mathcal{I}(1)^{[d_1]},\ldots,\mathcal{I}(r)^{[d_r]};N)=0 \text{ if some } d_j>0 \text{ with } j\not\in\{i_1,\ldots,i_\sigma\}.$$

We will write $e_R(\mathcal{I}(1)^{[d_1]},\ldots,\mathcal{I}(r)^{[d_r]})=e_R(\mathcal{I}(1)^{[d_1]},\ldots,\mathcal{I}(r)^{[d_r]};R)$.

Theorem 4.3. *Suppose that R is an analytically unramifed local ring of dimension d, N is a finitely generated R-module, $\mathfrak{a}$ is an m_R-primary ideal and that $\mathcal{I}(1)=\{I(1)_i\},\ldots,\mathcal{I}(r)=\{I(r)_i\}$ are filtrations of ideals. Suppose that $s\in\mathbb{N}$ is such that $\max\{s(\mathcal{I}(1)),\ldots,s(\mathcal{I}(r)\}\le s\le d$ and $n_1,\ldots,n_r\in\mathbb{N}$. Then for $n_1,\ldots,n_r\in\mathbb{N}$,*

$$(11)\qquad H_s(n_1,\ldots,n_r)=\lim_{m\to\infty}\frac{e_s(\mathfrak{a},N/I(1)_{mn_1}\cdots I(r)_{mn_r}N)}{m^{d-s}/(d-s)!}$$

is a homogeneous polynomial of total degree $d-s$.

Proof. Define $A=\bigcup_{j=1}^{r}A(\mathcal{I}(j))$. Then A is a finite set.

For all $(n_1,\ldots,n_r)\in\mathbb{N}^r$, consider the filtrations $\mathcal{J}(n_1,\ldots,n_r)=\{J(n_1,\ldots,n_r)_m=I(1)_{mn_1}\cdots I(r)_{mn_r}\}$ of ideals of R. Note that $V(J(n_1,\ldots,n_r)_m)=V(J(n_1,\ldots,n_r)_{m+1})$ for all $n_1,\ldots,n_r\in\mathbb{N}$ and $m\ge 1$. Let $\mathfrak{p}$ be a prime ideal of R such that $\dim R/\mathfrak{p}=s$. If $\mathfrak{p}\not\in A$ then for all $m\ge 1$, $\ell_{R_\mathfrak{p}}(R_\mathfrak{p}/J(n_1,\ldots,n_r)_mR_\mathfrak{p})=0$. If $\mathfrak{p}\in A$ then for all $m\ge 1$, $J(n_1,\ldots,n_r)_mR_\mathfrak{p}=I(1)_{mn_1}\cdots I(r)_{mn_r}R_\mathfrak{p}$ are either $R_\mathfrak{p}$ or $\mathfrak{p}R_\mathfrak{p}$-primary ideals of $R_\mathfrak{p}$. Therefore by Proposition 4.2, the limit

$$(12)\qquad \begin{aligned}\lim_{m\to\infty}\tfrac{e_s(\mathfrak{a},N/I(1)_{mn_1}\cdots I(r)_{mn_r}N)}{m^{d-s}/(d-s)!} &= e_s(\mathcal{J}(n_1,\ldots,n_r);N)\\ &= \textstyle\sum_\mathfrak{p} e_{R_\mathfrak{p}}(\mathcal{J}(n_1,\ldots,n_r)_\mathfrak{p},N_\mathfrak{p})e_\mathfrak{a}(R/\mathfrak{p}).\end{aligned}$$

where the sum is over $\mathfrak{p}\in\mathrm{Spec}(R)$ such that $\dim R/\mathfrak{p}=s$ and $\dim R_\mathfrak{p}=d-s$. By (8), (9) and (10), we have for each $\mathfrak{p}$ that

(13)

$$e_{R_\mathfrak{p}}(\mathcal{J}(n_1,\ldots,n_r)_\mathfrak{p},N_\mathfrak{p})=\sum_{d_1+\cdots+d_r=d-s}\frac{(d-s)!}{d_1!\cdots d_r!}e_{R_\mathfrak{p}}(\mathcal{I}(1)_\mathfrak{p}^{[d_1]},\ldots,\mathcal{I}(r)_\mathfrak{p}^{[d_r]};N_\mathfrak{p})n_1^{d_1}\cdots n_r^{d_r}$$

is a (possibly zero) homogeneous polynomial of degree $d-s$ in $n_1,\ldots,n_r$. Thus the function (11) is a homogeneous polynomial in $n_1,\ldots,n_r$ of total degree $d-s$. ☐

Definition 4.4. *With the assumptions of Theorem 4.3, the mixed multiplicities*

$$e_s(\mathfrak{a},\mathcal{I}(1)^{[d_1]},\ldots,I(r)^{[d_r]};N)$$

are defined from the expansion

$$(14)\qquad H_s(n_1,\ldots,n_r)=\sum_{d_1+\cdots+d_r=d-s}\frac{(d-s)!}{d_1!\cdots d_r!}e_s(\mathfrak{a},\mathcal{I}(1)^{[d_1]},\ldots,\mathcal{I}(r)^{[d_r]};N)n_1^{d_1}\cdots n_r^{d_r}.$$

Theorem 4.5. *Let assumptions be as in Theorem 4.3. Then we have the formula*

$$(15)\qquad e_s(\mathfrak{a},\mathcal{I}(1)^{[d_1]},\ldots,\mathcal{I}(r)^{[d_r]};N)=\sum_\mathfrak{p} e_{R_\mathfrak{p}}((\mathcal{I}(1)_\mathfrak{p})^{[d_1]},\ldots,(\mathcal{I}(r)_\mathfrak{p})^{[d_r]};N_\mathfrak{p})e_\mathfrak{a}(R/\mathfrak{p})$$

where the sum is over all $\mathfrak{p}\in\mathrm{Spec}(R)$ such that $\dim R/\mathfrak{p}=s$ and $\dim R_\mathfrak{p}=d-s$.

Proof. The theorem follows from equations (12) and (13). □

If $\mathcal{I}(1),\ldots,\mathcal{I}(r)$ are filtrations of m_R-primary ideals (or are trivial filtrations) then

$$e_0(\mathfrak{a},\mathcal{I}(1)^{[d_1]},\ldots,\mathcal{I}(r)^{[d_r]};N)=e_R(\mathcal{I}(1)^{[d_1]},\ldots,\mathcal{I}(r)^{[d_r]};N)e_{\mathfrak{a}}(R/m_R).$$

We will write $e_s(\mathfrak{a},\mathcal{I}(1)^{[d_1]},\ldots,\mathcal{I}(r)^{[d_r]})=e_s(\mathfrak{a},\mathcal{I}(1)^{[d_1]},\ldots,\mathcal{I}(r)^{[d_r]};R)$.

We have that

$$e_s(\mathfrak{a},\mathcal{I}(1)^{[0]},\ldots,\mathcal{I}(i-1)^{[0]},\mathcal{I}(i)^{[d-s]},\mathcal{I}(i+1)^{[0]},\ldots\mathcal{I}(r)^{[0]},N)=e_s(\mathfrak{a},\mathcal{I}(i),N)$$

for all $1\le i\le r$.

5. Inequalities

Suppose that R is a local ring and $I\subset J$ are m_R-primary ideals. Rees showed in [32] that if $R[J]$ is integral over $R[I]$ then the multiplicities $e_I(R)=e_J(R)$ are equal and he proved the converse, that $e_I(R)=e_J(R)$ implies $R[J]$ is integral over $R[I]$ if R is formally equidimensional. We show in [13, Theorem 6.9] and [11, Appendix] that the first statement extends to arbitrary filtrations of m_R-primary ideals, in a local ring such that $\dim N(\hat{R})<\dim R$. However, the converse does not hold for m_R-primary ideals, even if R is a regular local ring. A simple example is given in [13]. We extend this theorem to arbitrary filtrations in the following theorem.

Theorem 5.1. *Let R be an analytically unramified local ring of dimension d, N be a finitely generated R-module, $\mathfrak{a}$ be an m_R-primary ideal and $\mathcal{I}=\{I_n\}$, $\mathcal{J}=\{J_n\}$ be filtrations of ideals of R with $\mathcal{J}\subset\mathcal{I}$. Suppose $R[\mathcal{I}]$ is integral over $R[\mathcal{J}]$. Then*

(*i*) $s(\mathcal{I})=s(\mathcal{J})$,
(*ii*) $e_s(\mathfrak{a},\mathcal{I};N)=e_s(\mathfrak{a},\mathcal{J};N)$ *for all s such that* $s(\mathcal{I})=s(\mathcal{J})\le s\le d$.

Proof. We have that $V(I_1)=V(J_1)$ and $s(\mathcal{I})=s(\mathcal{J})$ by Lemma 3.9. Thus by Proposition 4.2, $e_s(\mathcal{I};N)=e_s(\mathcal{J};N)=0$ for $s>s(\mathcal{I})=s(\mathcal{J})$ and we have that

$$e_s(\mathfrak{a},\mathcal{I};N)=\sum_{\mathfrak{p}}e_{R_\mathfrak{p}}(\mathcal{I}_\mathfrak{p},N_\mathfrak{p})e_\mathfrak{a}(R/\mathfrak{p})\text{ and }e_s(\mathfrak{a},\mathcal{J};N)=\sum_{\mathfrak{p}}e_{R_\mathfrak{p}}(\mathcal{J}_\mathfrak{p},N_\mathfrak{p})e_\mathfrak{a}(R/\mathfrak{p})$$

where the sums are over prime ideals $\mathfrak{p}$ such that $\dim R/\mathfrak{p}=s$ and $\dim R_\mathfrak{p}=d-s$. Suppose that $s=s(\mathcal{I})$. By Lemma 3.1, if $\dim R/\mathfrak{p}=s$, then either $I_1R_\mathfrak{p}$ and $J_1R_\mathfrak{p}$ are both $m_{R_\mathfrak{p}}$-primary ideals or $I_1R_\mathfrak{p}=J_1R_\mathfrak{p}=R_\mathfrak{p}$. In the first case, $\mathcal{J}_\mathfrak{p}\subset\mathcal{I}_\mathfrak{p}$ are filtrations of $m_{R_\mathfrak{p}}$-primary ideals and in the second case, $\mathcal{I}_\mathfrak{p}$ and $\mathcal{J}_\mathfrak{p}$ are both the trivial $R_\mathfrak{p}$-filtration.

Let $\mathfrak{p}_1,\ldots,\mathfrak{p}_r$ be the prime ideals in R such that the first case holds. Then

$$e_s(\mathfrak{a},\mathcal{I};N)=\sum_{i=1}^r e_{R_{\mathfrak{p}_i}}(\mathcal{I}_{\mathfrak{p}_i},N_{\mathfrak{p}_i})e_\mathfrak{a}(R/\mathfrak{p}_i)\text{ and }e_s(\mathfrak{a},\mathcal{J};N)=\sum_{i=1}^r e_{R_{\mathfrak{p}_i}}(\mathcal{J}_{\mathfrak{p}_i},N_{\mathfrak{p}_i})e_\mathfrak{a}(R/\mathfrak{p}_i).$$

We have that $R[\mathcal{J}_{\mathfrak{p}_i}]$ is integral over $R[\mathcal{I}_{\mathfrak{p}_i}]$ for all $1\le i\le r$ and $R_{\mathfrak{p}_i}$ is analytically unramified for all i by [39, Proposition 9.1.4]. Thus $e_{R_{\mathfrak{p}_i}}(\mathcal{I}_{\mathfrak{p}_i},N_{\mathfrak{p}_i})=e_{R_{\mathfrak{p}_i}}(\mathcal{J}_{\mathfrak{p}_i},N_{\mathfrak{p}_i})$ for $1\le i\le r$ by [13, Theorem 6.9] or [10, Appendix], and so $e_s(\mathfrak{a},\mathcal{I};N)=e_s(\mathfrak{a},\mathcal{J};N)$. □

Corollary 5.2. *Suppose that R is an analytically unramified local ring, $\mathfrak{a}$ is an m_R-primary ideal and $\mathcal{I}=\{I_n\}$ is a filtration of R. Let*

$$J_n=\{f\in R\mid f^r\in\overline{I_{rn}}\text{ for some }r>0\}.$$

Then

$$s:=s(\mathcal{I})=s(\{\overline{I_n}\})=s(\{J_n\})$$

and

$$e_s(\mathfrak{a},\mathcal{I}) = e_s(\mathfrak{a},\{\overline{I_n}\}) = e_s(\mathfrak{a},\{J_n\}).$$

Proof. We have that $R[\mathcal{I}] \subset R[\overline{\mathcal{I}}] \subset \overline{R[\mathcal{I}]}$ by Lemma 3.6. Thus $s = s(\mathcal{I}) = s(\{\overline{I_n}\}) = s(\{J_n\})$ and $e_s(\mathfrak{a},\mathcal{I}) = e_s(\mathfrak{a},\{\overline{I_n}\}) = e_s(\mathfrak{a},\{J_n\})$ by Theorem 5.1. □

The Minkowski inequalities were formulated and proven for m_R-primary ideals in reduced equicharacteristic zero local rings by Teissier [40], [41] and proven for m_R-primary ideals in full generality, for local rings, by Rees and Sharp [35]. The same inequalities hold for filtrations. They were proven for m_R-filtrations in local rings R such that $\dim N(\hat{R}) < \dim R$ in [13, Theorem 6.3]. We prove them for arbitrary filtrations in an analytically unramified local ring.

Theorem 5.3. *(Minkowski Inequalities) Let R be an analytically unramified local ring of dimension d, N be a finitely generated R-module, $\mathfrak{a}$ be an m_R-primary ideal, $\mathcal{I}=\{I_j\}$ and $\mathcal{J}=\{J_j\}$ be filtrations of ideals of R. Let $\max\{s(\mathcal{I}),s(\mathcal{J})\} \le s < d$ and $k := d-s$.*

- *(i) Let $k \ge 2$. For $1 \le i \le k-1$,*
$$e_s(\mathfrak{a},\mathcal{I}^{[i]},\mathcal{J}^{[k-i]};N)^2 \le e_s(\mathfrak{a},\mathcal{I}^{[i+1]},\mathcal{J}^{[k-i-1]};N)e_s(\mathfrak{a},\mathcal{I}^{[i-1]},\mathcal{J}^{[k-i+1]};N),$$
- *(ii) For $0 \le i \le k$, $e_s(\mathfrak{a},\mathcal{I}^{[i]},\mathcal{J}^{[k-i]};N)e_s(\mathfrak{a},\mathcal{I}^{[k-i]},\mathcal{J}^{[i]};N) \le e_s(\mathfrak{a},\mathcal{I};N)e_s(\mathfrak{a},\mathcal{J};N)$,*
- *(iii) For $0 \le i \le k$, $e_s(\mathfrak{a},\mathcal{I}^{[k-i]},\mathcal{J}^{[i]};N)^k \le e_s(\mathfrak{a},\mathcal{I};N)^{k-i}e_s(\mathfrak{a},\mathcal{J};N)^i$ and*
- *(iv) $s(\mathcal{I}\mathcal{J}) = \max\{s(\mathcal{I}),s(\mathcal{J})\}$ and $e_s(\mathfrak{a},\mathcal{I}\mathcal{J};N)^{\frac{1}{k}} \le e_s(\mathfrak{a},\mathcal{I};N)^{\frac{1}{k}} + e_s(\mathfrak{a},\mathcal{J};N)^{\frac{1}{k}}$, where $\mathcal{I}\mathcal{J} = \{I_jJ_j\}$.*

Proof. Let $\mathfrak{p} \in \operatorname{Spec} R$ with $\dim R/\mathfrak{p} = s$ and $\dim R_{\mathfrak{p}} = d-s$. If $\mathfrak{p} \in A(\mathcal{I})$ then $I_nR_{\mathfrak{p}}$ are $\mathfrak{p}R_{\mathfrak{p}}$-primary ideals for all $n \ge 1$ (respectively if $\mathfrak{p} \in A(\mathcal{J})$ then $J_nR_{\mathfrak{p}}$ are $\mathfrak{p}R_{\mathfrak{p}}$-primary ideals for all $n \ge 1$). If $\mathfrak{p} \notin A(\mathcal{I})$ then $I_nR_{\mathfrak{p}} = R_{\mathfrak{p}}$ for all $n \ge 1$ (respectively if $\mathfrak{p} \notin A(\mathcal{J})$ then $J_nR_{\mathfrak{p}} = R_{\mathfrak{p}}$ for all $n \ge 1$).
Let $T := (A(\mathcal{I}) \cup A(\mathcal{J})) \cap \{\mathfrak{p} \in \operatorname{Spec}(R) : \dim R_{\mathfrak{p}} = d-s\}$ and $|T| = r$. Suppose $T = \{\mathfrak{p}_1,\ldots,\mathfrak{p}_r\}$.
(i) Suppose that i satisfies $1 \le i \le k-1$. Let $\mathfrak{p} \in \operatorname{Spec} R$ with $\dim R/\mathfrak{p} = s$ and $\dim R_{\mathfrak{p}} = d-s$. We first show that

$$e_{R_{\mathfrak{p}}}(\mathcal{I}_{\mathfrak{p}}^{[i]},\mathcal{J}_{\mathfrak{p}}^{[k-i]};N_{\mathfrak{p}})^2 \le e_{R_{\mathfrak{p}}}(\mathcal{I}_{\mathfrak{p}}^{[i+1]},\mathcal{J}_{\mathfrak{p}}^{[k-i-1]};N_{\mathfrak{p}})e_{R_{\mathfrak{p}}}(\mathcal{I}_{\mathfrak{p}}^{[i-1]},\mathcal{J}_{\mathfrak{p}}^{[k-i+1]};N_{\mathfrak{p}}). \tag{16}$$

If $\mathfrak{p} \in A(\mathcal{I}) \setminus A(\mathcal{J})$ or $\mathfrak{p} \in A(\mathcal{J}) \setminus A(\mathcal{I})$, in both cases we have

$$e_{R_{\mathfrak{p}}}(\mathcal{I}_{\mathfrak{p}}^{[i]},\mathcal{J}_{\mathfrak{p}}^{[k-i]};N_{\mathfrak{p}}) = 0 \text{ for all } 1 \le i \le k-1.$$

Suppose $\mathfrak{p} \in A(\mathcal{I}) \cap A(\mathcal{J})$. Then by [13, Theorem 6.3, 1) , 2)], we have

$$e_{R_{\mathfrak{p}}}(\mathcal{I}_{\mathfrak{p}}^{[i]},\mathcal{J}_{\mathfrak{p}}^{[k-i]};N_{\mathfrak{p}})^2 \le e_{R_{\mathfrak{p}}}(\mathcal{I}_{\mathfrak{p}}^{[i+1]},\mathcal{J}_{\mathfrak{p}}^{[k-i-1]};N_{\mathfrak{p}})e_{R_{\mathfrak{p}}}(\mathcal{I}_{\mathfrak{p}}^{[i-1]},\mathcal{J}_{\mathfrak{p}}^{[k-i+1]};N_{\mathfrak{p}})$$

By (15), we have

$$e_s(\mathfrak{a},\mathcal{I}^{[d_1]},\mathcal{J}^{[d_2]};N) = \sum_{j=1}^{r} e_{R_{\mathfrak{p}_j}}(\mathcal{I}_{\mathfrak{p}_j}^{[d_1]},\mathcal{J}_{\mathfrak{p}_j}^{[d_2]};N_{\mathfrak{p}_j})e_{\mathfrak{a}}(R/\mathfrak{p}_j). \tag{17}$$

For $1 \le j \le r$, set

$$x_1(j) = e_{R_{\mathfrak{p}_j}}(\mathcal{I}_{\mathfrak{p}_j}^{[i+1]},\mathcal{J}_{\mathfrak{p}_j}^{[k-i-1]};N_{\mathfrak{p}_j})^{\frac{1}{2}}e_{\mathfrak{a}}(R/\mathfrak{p}_j)^{\frac{1}{2}} \text{ and}$$
$$x_2(j) = e_{R_{\mathfrak{p}_j}}(\mathcal{I}_{\mathfrak{p}_j}^{[i-1]},\mathcal{J}_{\mathfrak{p}_j}^{[k-i+1]};N_{\mathfrak{p}_j})^{\frac{1}{2}}e_{\mathfrak{a}}(R/\mathfrak{p}_j)^{\frac{1}{2}}.$$

Then, by (17) and (16),

$$
\begin{aligned}
(18)\qquad & e_s(\mathfrak{a},\mathcal{I}^{[i]},\mathcal{J}^{[k-i]};N)^2\\
= \quad & (\sum_{j=1}^{r} e_{R_{\mathfrak{p}_j}}(\mathcal{I}_{\mathfrak{p}_j}^{[i]},\mathcal{J}_{\mathfrak{p}_j}^{[k-i]};N_{\mathfrak{p}_j})e_{\mathfrak{a}}(R/\mathfrak{p}_j))^2\\
\leq \quad & (\sum_{j=1}^{r} x_1(j)x_2(j))^2 \leq (\sum_{j=1}^{r} x_1(j)^2)(\sum_{j=1}^{r} x_2(j)^2)\\
= \quad & (\sum_{j=1}^{r} e_{R_{\mathfrak{p}_j}}(\mathcal{I}_{\mathfrak{p}_j}^{[i+1]},\mathcal{J}_{\mathfrak{p}_j}^{[k-i-1]};N_{\mathfrak{p}_j})e_{\mathfrak{a}}(R/\mathfrak{p}_j))(\sum_{j=1}^{r} e_{R_{\mathfrak{p}_j}}(\mathcal{I}_{\mathfrak{p}_j}^{[i-1]},\mathcal{J}_{\mathfrak{p}_j}^{[k-i+1]};N_{\mathfrak{p}_j})e_{\mathfrak{a}}(R/\mathfrak{p}_j))\\
= \quad & e_s(\mathfrak{a},\mathcal{I}^{[i+1]},\mathcal{J}^{[k-i-1]};N)e_s(\mathfrak{a},\mathcal{I}^{[i-1]},\mathcal{J}^{[k-i+1]};N),
\end{aligned}
$$

which establishes formula (i) of the statement of the theorem. The inequality between the second and third lines of (18) is Hölder's inequality, formula (2.8.3) on page 24 of [19], with $k=2$ (so its conjugate $k'=\frac{k}{k-1}=2$ also).
(ii) and (iii) : Let $e_i =: e_s(\mathfrak{a},\mathcal{I}^{[k-i]},\mathcal{J}^{[i]};N)$ for all $0\leq i\leq k$. If $e_0=0$ or $e_k=0$ then by part (i), we have $e_i=0$ for all $0<i<k$ and we get the inequalities (ii) and (iii). Suppose $e_0>0$ and $e_k>0$. If $e_i=0$ for some $i\in\{1,\ldots,k-1\}$ then using part (i), we get $e_i=0$ for all $0<i<k$ and hence the inequalities (ii) and (iii) hold. If $e_i>0$ for all $0\leq i\leq k$ then the proof follows from the argument given in [39, Corollary 17.7.3, (1) and (2)].
(iv) Since $V(I_1)\cup V(J_1)=V(I_1J_1)$, we have $s(\mathcal{I}\mathcal{J})=\max\{s(\mathcal{I}),s(\mathcal{J})\}$. Let $\mathfrak{p}\in\operatorname{Spec}R$ with $\dim R/\mathfrak{p}=s$ and $\dim R_{\mathfrak{p}}=d-s$. We first show that

$$
(19)\qquad e_{R_{\mathfrak{p}}}(\mathcal{I}_{\mathfrak{p}}\mathcal{J}_{\mathfrak{p}};N_{\mathfrak{p}})^{\frac{1}{k}} \leq e_{R_{\mathfrak{p}}}(\mathcal{I}_{\mathfrak{p}};N_{\mathfrak{p}})^{\frac{1}{k}} + e_{R_{\mathfrak{p}}}(\mathcal{J}_{\mathfrak{p}};N_{\mathfrak{p}})^{\frac{1}{k}}.
$$

If $\mathfrak{p}\in A(\mathcal{I})\setminus A(\mathcal{J})$ (respectively $\mathfrak{p}\in A(\mathcal{J})\setminus A(\mathcal{I})$) then

$$
e_{R_{\mathfrak{p}}}(\mathcal{I}_{\mathfrak{p}}\mathcal{J}_{\mathfrak{p}};N_{\mathfrak{p}})^{\frac{1}{k}} = e_{R_{\mathfrak{p}}}(\mathcal{I}_{\mathfrak{p}};N_{\mathfrak{p}})^{\frac{1}{k}} \leq e_{R_{\mathfrak{p}}}(\mathcal{I}_{\mathfrak{p}};N_{\mathfrak{p}})^{\frac{1}{k}} + e_{R_{\mathfrak{p}}}(\mathcal{J}_{\mathfrak{p}};N_{\mathfrak{p}})^{\frac{1}{k}}
$$
$$
\text{(respectively } e_{R_{\mathfrak{p}}}(\mathcal{I}_{\mathfrak{p}}\mathcal{J}_{\mathfrak{p}};N_{\mathfrak{p}})^{\frac{1}{k}} = e_{R_{\mathfrak{p}}}(\mathcal{J}_{\mathfrak{p}};N_{\mathfrak{p}})^{\frac{1}{k}} \leq e_{R_{\mathfrak{p}}}(\mathcal{I}_{\mathfrak{p}};N_{\mathfrak{p}})^{\frac{1}{k}} + e_{R_{\mathfrak{p}}}(\mathcal{J}_{\mathfrak{p}};N_{\mathfrak{p}})^{\frac{1}{k}}).
$$

Suppose $\mathfrak{p}\in A(\mathcal{I})\cap A(\mathcal{J})$. Then by [13, Theorem 6.3, 4)], we have

$$
e_{R_{\mathfrak{p}}}(\mathcal{I}_{\mathfrak{p}}\mathcal{J}_{\mathfrak{p}};N_{\mathfrak{p}})^{\frac{1}{k}} \leq e_{R_{\mathfrak{p}}}(\mathcal{I}_{\mathfrak{p}};N_{\mathfrak{p}})^{\frac{1}{k}} + e_{R_{\mathfrak{p}}}(\mathcal{J}_{\mathfrak{p}};N_{\mathfrak{p}})^{\frac{1}{k}}
$$

(where $\mathcal{I}_{\mathfrak{p}}\mathcal{J}_{\mathfrak{p}}=\{I_jJ_jR_{\mathfrak{p}}\}$). By (15), we have

$$
(20)\qquad e_s(\mathfrak{a},\mathcal{I}\mathcal{J};N)^{\frac{1}{k}} = \big(\sum_{j=1}^{r} e_{R_{\mathfrak{p}_j}}(\mathcal{I}_{\mathfrak{p}_j}\mathcal{J}_{\mathfrak{p}_j};N_{\mathfrak{p}_j})e_{\mathfrak{a}}(R/\mathfrak{p}_j)\big)^{\frac{1}{k}}.
$$

For $1\leq j\leq r$, set

$$
\begin{aligned}
& x_1(j)=e_{R_{\mathfrak{p}_j}}(\mathcal{I}_{\mathfrak{p}_j};N_{\mathfrak{p}_j})^{\frac{1}{k}}e_{\mathfrak{a}}(R/\mathfrak{p}_j)^{\frac{1}{k}},\ x_2(j)=e_{R_{\mathfrak{p}_j}}(\mathcal{J}_{\mathfrak{p}_j};N_{\mathfrak{p}_j})^{\frac{1}{k}}e_{\mathfrak{a}}(R/\mathfrak{p}_j)^{\frac{1}{k}} \text{ and}\\
& u(j)=e_{R_{\mathfrak{p}_j}}(\mathcal{I}_{\mathfrak{p}_j}\mathcal{J}_{\mathfrak{p}_j};N_{\mathfrak{p}_j})^{\frac{1}{k}}e_{\mathfrak{a}}(R/\mathfrak{p}_j)^{\frac{1}{k}}.
\end{aligned}
$$

Then $u(j) \le x_1(j) + x_2(j)$ for all $j = 1, \ldots, r$ and by (20) and (19),

$$
\begin{aligned}
(21)\ e_s(\mathfrak{a}, \mathcal{I}\mathcal{J}; N)^{\frac{1}{k}} &= \big(\sum_{j=1}^{r} e_{R_{\mathfrak{p}_j}}(\mathcal{I}_{\mathfrak{p}_j}\mathcal{J}_{\mathfrak{p}_j}; N_{\mathfrak{p}_j}) e_{\mathfrak{a}}(R/\mathfrak{p}_j)\big)^{\frac{1}{k}} \\
&= \big(\sum_{j=1}^{r} u(j)^k\big)^{\frac{1}{k}} \le \big(\sum_{j=1}^{r}(x_1(j) + x_2(j))^k\big)^{\frac{1}{k}} \\
&\le \big(\sum_{j=1}^{r}(x_1(j))^k\big)^{\frac{1}{k}} + \big(\sum_{j=1}^{r}(x_2(j))^k\big)^{\frac{1}{k}} \\
&= \big(\sum_{j=1}^{r}(e_{R_{\mathfrak{p}_j}}(\mathcal{I}_{\mathfrak{p}_j}, N_{\mathfrak{p}_j}) e_{\mathfrak{a}}(R/\mathfrak{p}_j)\big)^{\frac{1}{k}} + \big(\sum_{j=1}^{r}(e_{R_{\mathfrak{p}_j}}(\mathcal{J}_{\mathfrak{p}_j}; N_{\mathfrak{p}_j}) e_{\mathfrak{a}}(R/\mathfrak{p}_j)\big)^{\frac{1}{k}} \\
&= e_s(\mathfrak{a}, \mathcal{I}; N)^{\frac{1}{k}} + e_s(\mathfrak{a}, \mathcal{J}; N)^{\frac{1}{k}}
\end{aligned}
$$

which establishes formula (iv) of the statement of the theorem. For $k > 1$, the inequality between the second and third lines of (21) is Minkowski's inequality, Section 2.12 (28) on page 32 of [19]. ☐

Theorem 5.4. *Suppose that R is an analytically unramified local ring of dimension d and that $\mathfrak{a}$ is an m_R-primary ideal. Let $\mathcal{I}(1)$ and $\mathcal{I}(2)$ be two filtrations of ideals on R, $\max\{s(\mathcal{I}(1)), s(\mathcal{I}(1))\} \le s$ and $d - s \ge 1$. Suppose there exist $a, b \in \mathbb{Z}_{>0}$ such that*

$$
\overline{\sum_{n\ge 0} I(1)_{an}} = \overline{\sum_{n\ge 0} I(2)_{bn}}.
$$

Then the Minkowski equality

$$
e_s(\mathfrak{a}, \mathcal{I}(1)\mathcal{I}(2))^{\frac{1}{d-s}} = e_0^{\frac{1}{d-s}} + e_{d-s}^{\frac{1}{d-s}}
$$

holds between $\mathcal{I}(1)$ and $\mathcal{I}(2)$ where $e_0 = e_s(\mathfrak{a}, \mathcal{I}(1)^{[d-s]}, \mathcal{I}(2)^{[0]})$ and $e_{d-s} = e_s(\mathfrak{a}, \mathcal{I}(1)^{[0]}, \mathcal{I}(2)^{[d-s]})$.

Proof. Since $\overline{\sum_{n\ge 0} I(1)_{an}} = \overline{\sum_{n\ge 0} I(2)_{bn}}$, by Lemmas 3.1 and 3.9 we have $V(I(1)_n) = V(I(2)_n)$ for all $n \ge 1$ and $s(\mathcal{I}(1)) = s(\mathcal{I}(2))$.

Let $\mathcal{A}(\mathcal{I}(1)) = \mathcal{A}(\mathcal{I}(2)) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$. For all $n_1, n_2 \in \mathbb{N}$, let

$$
P(n_1, n_2) = \lim_{m\to\infty} \frac{e_s(\mathfrak{a}, R/I(1)_{mn_1} I(2)_{mn_2})}{m^{d-s}/(d-s)!}
$$

and for $1 \le i \le r$, let

$$
P_i(n_1, n_2) = H_i(n_1, n_2) e_{\mathfrak{a}}(R/\mathfrak{p}_i)
$$

where

$$
H_i(n_1, n_2) = \lim_{m\to\infty} \frac{\ell_{R_{\mathfrak{p}_i}}(R_{\mathfrak{p}_i}/I(1)_{mn_1} R_{\mathfrak{p}_i} I(2)_{mn_2} R_{\mathfrak{p}_i})}{m^{d-s}/(d-s)!}.
$$

For all $\mathfrak{p}_i$, we have $\overline{\sum_{n\ge 0} I(1)_{an} R_{\mathfrak{p}_i}} = \overline{\sum_{n\ge 0} I(2)_{bn} R_{\mathfrak{p}_i}}$. Therefore by [11, Theorem 8.4] and its proof, which proves this theorem for 0-filtrations, for all $1 \le i \le r$, there exists $c_i \in \mathbb{R}$, such that for all $n_1, n_2 \in \mathbb{N}$, we have

$$
H_i(n_1, n_2) = c_i(\frac{n_1}{a} + \frac{n_2}{b})^{d-s}
$$

and hence

$$P_i(n_1,n_2)=c_i(\frac{n_1}{a}+\frac{n_2}{b})^{d-s}e_{\mathfrak{a}}(R/\mathfrak{p}_i).$$

Using Theorem 4.5, for all $n_1,n_2\in\mathbb{N}$, we get

$$P(n_1,n_2)=\sum_{i=1}^{r}P_i(n_1,n_2)=\sum_{i=1}^{r}c_i(\frac{n_1}{a}+\frac{n_2}{b})^{d-s}e_{\mathfrak{a}}(R/\mathfrak{p}_i)=c(\frac{n_1}{a}+\frac{n_2}{b})^{d-s} \tag{22}$$

where $c=\sum\limits_{i=1}^{r}c_ie_{\mathfrak{a}}(R/\mathfrak{p}_i)$. Therefore

$$P(1,1)^{\frac{1}{d-s}}=P(1,0)^{\frac{1}{d-s}}+P(0,1)^{\frac{1}{d-s}}.$$

□

The following lemma is well known. We provide a proof for the convenience of the reader. Our proof is an outline of the proof in [15, Lemma 14, page 8].

Lemma 5.5. *Let R be a d-dimensional local ring and let $I_i\subseteq J_i$ be m-primary ideals for $i=1,\ldots,d$. Then*

$$e(I_1^{[1]},\ldots,I_d^{[1]})\geq e(J_1^{[1]},\ldots,J_d^{[1]}).$$

Proof. It is enough to prove the statement when $I_1=J_1,\ldots,I_{d-1}=J_{d-1}$, and $I_d\subseteq J_d$. If $d=1$, then $\ell(R/I_1^n)\geq\ell(R/J_1^n)$, so necessarily $e(I_1)\geq e(J_1)$. Now let $d>1$. We may assume that R has an infinite residue field. Then for a general element $a\in I_1=J_1$, $e(I_1^{[1]},\ldots,I_d^{[1]};R)=e(I_2^{[1]},\ldots,I_d^{[1]};R/(a))$ and $e(J_1^{[1]},\ldots,J_d^{[1]};R)=e(J_2^{[1]},\ldots,J_d^{[1]};R/(a))$. We are done by induction on d. □

Definition 5.6. *Suppose that $\mathcal{I}=\{I_i\}$ is a filtration of ideals on a local ring R. Fix $a\in\mathbb{Z}_+$. The a-th truncated filtration $\mathcal{I}_a=\{I_{a,i}\}$ of $\mathcal{I}$ is defined by*

$$I_{a,n}=\begin{cases} I_n & \text{if } n\leq a\\ \sum\limits_{\substack{i,j>0\\ i+j=n}} I_{a,i}I_{a,j} & \text{if } n>a.\end{cases}$$

Lemma 5.7. *Let R be a d-dimensional local ring, $\mathfrak{a}$ be an m_R-primary ideal, $\mathcal{I}(i)=\{I(i)_n\}$ and $\mathcal{J}(i)=\{J(i)_n\}$ be filtrations of ideals on R for $1\leq i\leq r$ with $J(i)_n\subset I(i)_n$ for all $1\leq i\leq r$ and $n\geq 1$. Let N be a finitely generated R-module.*

(i) *Suppose* $\dim N(\hat{R})<d$ *and $\mathcal{I}(i)$, $\mathcal{J}(i)$ are filtrations of R by m_R-primary ideals. Then*

$$e_R(\mathcal{I}(1)^{[d_1]},\ldots,\mathcal{I}(r)^{[d_r]};N)\leq e_R(\mathcal{J}(1)^{[d_1]},\ldots,\mathcal{J}(r)^{[d_r]};N).$$

(ii) *Suppose R is analytically unramified and* $\max\{s(\mathcal{J}(1)),\ldots,s(\mathcal{J}(r)\}\leq s\leq d$. *Then*

$$e_s(\mathfrak{a},\mathcal{I}(1)^{[d_1]},\ldots,\mathcal{I}(r)^{[d_r]};N)\leq e_s(\mathfrak{a},\mathcal{J}(1)^{[d_1]},\ldots,\mathcal{J}(r)^{[d_r]};N).$$

Proof. (i) For all $a\in\mathbb{Z}_+$, consider the a-th truncated filtrations $\mathcal{I}_a(i)=\{I_a(i)_m\}$ and $\mathcal{J}_a(i)=\{J_a(i)_m\}$ for all $1\leq i\leq r$. Given $a\in\mathbb{Z}_+$, there exists $f_a\in\mathbb{Z}_+$ such that $I_a(i)_{f_am}=(I_a(i)_{f_a})^m$ and $J_a(i)_{f_am}=(J_a(i)_{f_a})^m$ for all $m\geq 0$ and $i=1,\ldots,r$. Define filtrations of R by m_R-primary ideals by $\tilde{\mathcal{I}}_a(i)=\{I_a(i)_{f_am}\}$ and $\tilde{\mathcal{J}}_a(i)=\{J_a(i)_{f_am}\}$ for all $1\le i\le r$. Then by [13, Proposition 6.2], [13, Lemma 3.2] and Lemma 5.5,

$$\begin{aligned}
e_R(\mathcal{I}(1)^{[d_1]},\ldots,\mathcal{I}(r)^{[d_r]};N) &= \lim_{a\to\infty} e_R(\mathcal{I}_a(1)^{[d_1]},\ldots,\mathcal{I}_a(r)^{[d_r]};N)\\
&= \lim_{a\to\infty}\frac{1}{f_a^d}e_R(\tilde{\mathcal{I}}_a(1)_1^{[d_1]},\ldots,\tilde{\mathcal{I}}_a(r)_1^{[d_r]};N)\\
&\le \lim_{a\to\infty}\frac{1}{f_a^d}e_R(\tilde{\mathcal{J}}_a(1)_1^{[d_1]},\ldots,\tilde{\mathcal{J}}_a(r)_1^{[d_r]};N)\\
&= \lim_{a\to\infty} e_R(\mathcal{J}_a(1)^{[d_1]},\ldots,\mathcal{J}_a(r)^{[d_r]};N)\\
&= e_R(\mathcal{J}(1)^{[d_1]},\ldots,\mathcal{J}(r)^{[d_r]};N).
\end{aligned}$$

(ii) This follows from Theorem 4.5 and part (i) of this Lemma. $\square$

In [28, Section 1] and [21, Proposition 3.11], a multiplicity $e(\mathfrak{a},I)$ is defined for ideals $\mathfrak{a}$ and I in a local ring R such that $\mathfrak{a}+I$ is m_R-primary, by

$$e(\mathfrak{a},I)=\sum_{\mathfrak{p}} e_{R_\mathfrak{p}}(I_\mathfrak{p})e_\mathfrak{a}(R/\mathfrak{p}), \tag{23}$$

where the sum is over prime ideals $\mathfrak{p}$ in R which contain I and such that

$$\dim R/\mathfrak{p}=\dim R/I \text{ and } \dim R_\mathfrak{p}=\dim R-\dim R/I.$$

We generalize equation (23) to filtrations. Suppose that R is an analytically unramified local ring of dimension d and $\mathcal{I}$ is a filtration of ideals on R. Let $s=s(\mathcal{I})$. Suppose $\mathfrak{a}$ is an ideal in R such that $\mathfrak{a}+I_1$ is m_R-primary. We define

$$e(\mathfrak{a},\mathcal{I})=\sum_{\mathfrak{p}} e_{R_\mathfrak{p}}(\mathcal{I}_\mathfrak{p})e_\mathfrak{a}(R/\mathfrak{p}), \tag{24}$$

where the sum is over all $\mathfrak{p}\in\operatorname{Spec}(R)$ such that $\dim R/\mathfrak{p}=s$ and $\dim R_\mathfrak{p}=d-s$.

We have the following formula relating equations (23) and (24):

$$e(\mathfrak{a},\mathcal{I})=\lim_{m\to\infty}\frac{e(\mathfrak{a},I_m)}{m^{d-s}}. \tag{25}$$

To prove this formula, we observe that for all $m,n\ge 1$,

$$\begin{aligned}
&\operatorname{Min}(R/I_n)\cap\{p\in\operatorname{Spec}(R):\dim R/\mathfrak{p}=s \text{ and } \dim R_\mathfrak{p}=d-s\}\\
&=\operatorname{Min}(R/I_m)\cap\{p\in\operatorname{Spec}(R):\dim R/\mathfrak{p}=s \text{ and } \dim R_\mathfrak{p}=d-s\},
\end{aligned}$$

so that the limit

$$\begin{aligned}
\lim_{m\to\infty}\frac{e(\mathfrak{a},I_m)}{m^{d-s}} &= \sum_{\mathfrak{p}}\lim_{m\to\infty}\frac{e_{R_\mathfrak{p}}(I_mR_\mathfrak{p})}{m^{d-s}}e_\mathfrak{a}(R/\mathfrak{p})\\
&= \sum_{\mathfrak{p}}\lim_{n\to\infty}\frac{\ell_{R_\mathfrak{p}}(R_\mathfrak{p}/I_nR_\mathfrak{p})}{n^{d-s}/(d-s)!}e_\mathfrak{a}(R/\mathfrak{p})\\
&= \sum_{\mathfrak{p}} e_{R_\mathfrak{p}}(\mathcal{I}_\mathfrak{p})e_\mathfrak{a}(R/\mathfrak{p})
\end{aligned}$$

exists, where the second equality follows from [8, Theorem 6.5] and the sum is over all $\mathfrak{p}\in\operatorname{Spec}(R)$ such that $\dim R/\mathfrak{p}=s$ and $\dim R_\mathfrak{p}=d-s$.

Let $\mathcal{I}$ be a filtration of ideals on a local ring R. We say $x_1, \ldots, x_{s(\mathcal{I})}$ is a system of parameters of $\mathcal{I}$ if $x_1 + I_n, \ldots, x_{s(\mathcal{I})} + I_n$ is a system of parameters of R/I_n for each $n \geq 1$. Note that if $x_1, \ldots, x_t$ is a part of system of parameters of $\mathcal{I}$ then $\{I_n + (x_1, \ldots, x_t)\}$ is a filtration and thus $V(I_m + (x_1, \ldots, x_t)) = V(I_n + (x_1, \ldots, x_t))$ for all $m, n \geq 1$.

Remark 5.8. *Let R be a Regular local ring of dimension $d \geq 1$, $\mathcal{I}$ be a filtration of ideals on R which is not an m_R-filtration. Then using the prime avoidance lemma, we can choose elements $x_i \in m_R \setminus m_R^2 \bigcup \operatorname{Min}(R/I_1 + (x_1, \ldots, x_{i-1}))$ for all $1 \leq i \leq s(\mathcal{I})$ and $x_0 = 0$. Then $x_1, \ldots, x_{s(\mathcal{I})}$ is a system of parameters of $\mathcal{I}$ such that $x_1, \ldots, x_{s(\mathcal{I})}$ is a part of regular system of parameters of R. Define $\underline{x} = x_1, \ldots, x_{s(\mathcal{I})}$. Therefore $R/\underline{x}R$ is a regular local ring and hence $e_{R/\underline{x}R}(\mathcal{J})$ is well-defined where $\mathcal{J} = \{J_n = I_n(R/\underline{x}R)\}$.*

In [22] and [28, formula (2.1), page 118], the following inequality is given. Suppose that I is an ideal in a local ring R. Let $s = \dim R/I$. Suppose that $\underline{x} = x_1, \ldots, x_s$ is part of a system of parameters in R and that the image of $\underline{x}$ in R/I is a system of parameters. Then

$$e(\underline{x}R, I) \leq e_{R/\underline{x}R}(I). \tag{26}$$

In the following proposition, we generalize this inequality to filtrations.

Proposition 5.9. *Let R be an analytically unramified local ring of dimension $d \geq 1$, $\mathcal{I}$ be a filtration of ideals on R which is not an m_R-filtration and $x_1, \ldots, x_{s(\mathcal{I})}$ is a system of parameters of $\mathcal{I}$ such that $\dim(N(\widehat{R/\underline{x}R})) < \dim R/\underline{x}R$ (e.g. R is a Regular local ring and $x_1, \ldots, x_{s(\mathcal{I})}$ is a system of parameters of $\mathcal{I}$ such that $x_1, \ldots, x_{s(\mathcal{I})}$ is a part of regular system of parameters of R) where $N(\widehat{R/\underline{x}R})$ is the nilradical of $R/\underline{x}R$. Let $\underline{x} = x_1, \ldots, x_{s(\mathcal{I})}$. Then for $\mathcal{J} = \{J_n = I_n(R/\underline{x}R)\}$,*

$$e(\underline{x}R, \mathcal{I}) \leq e_{R/\underline{x}R}(\mathcal{J}).$$

Proof. Let $\mathcal{I}_a$ denote the a-th truncated filtration of $\mathcal{I}$ for all $a \geq 1$ and $\mathfrak{p}$ be a prime ideal in R. Then $\mathcal{I}_a(R/\underline{x}R) = \{I_{a,n}(R/\underline{x}R)\}$ and $\mathcal{I}_a R_\mathfrak{p} = \{I_{a,n} R_\mathfrak{p}\}$ are the a-th truncated filtrations of $\mathcal{I}_a(R/\underline{x}R)$ and $\mathcal{I}_a R_\mathfrak{p}$ respectively for all $a \geq 1$. Since for each $a \geq 1$, $\mathcal{I}_a$, $\mathcal{I}_a(R/\underline{x}R)$ and $\mathcal{I}_a R_\mathfrak{p}$ are Noetherian filtrations there exists an integer $f_a \geq 1$ such that $I_{a,nf_a} = I^n_{a,f_a}$, $I_{a,nf_a}(R/\underline{x}R) = I^n_{a,f_a}(R/\underline{x}R)$ and $I_{a,nf_a} R_\mathfrak{p} = I^n_{a,f_a} R_\mathfrak{p}$ for all $n \geq 1$. Also note that $s(\mathcal{I}) = s(\{I^n_{a,f_a}\})$. Then summing over prime ideals $\mathfrak{p}$ with $\dim R/\mathfrak{p} = s(\mathcal{I})$ and $\dim R_\mathfrak{p} = d - s(\mathcal{I})$, we have

$$\begin{aligned}
e(\underline{x}R, \mathcal{I}) = \sum_{\mathfrak{p}} e_{R_\mathfrak{p}}(\mathcal{I}_\mathfrak{p}) e_{\underline{x}R}(R/\mathfrak{p}) &= \sum_{\mathfrak{p}} \lim_{a\to\infty} e_{R_\mathfrak{p}}(\mathcal{I}_a R_\mathfrak{p}) e_{\underline{x}R}(R/\mathfrak{p}) \\
&= \sum_{\mathfrak{p}} \lim_{a\to\infty} \frac{1}{f_a^{d-s(\mathcal{I})}} e_{R_\mathfrak{p}}(I_{a,f_a} R_\mathfrak{p}) e_{\underline{x}R}(R/\mathfrak{p}) \\
&= \lim_{a\to\infty} \frac{1}{f_a^{d-s(\mathcal{I})}} \sum_{\mathfrak{p}} e_{R_\mathfrak{p}}(I_{a,f_a} R_\mathfrak{p}) e_{\underline{x}R}(R/\mathfrak{p}) \\
&= \lim_{a\to\infty} \frac{1}{f_a^{d-s(\mathcal{I})}} e(\underline{x}R, I_{a,f_a}) \\
&\leq \lim_{a\to\infty} \frac{1}{f_a^{d-s(\mathcal{I})}} e(I_{a,f_a}(R/\underline{x}R)) \\
&= \lim_{a\to\infty} e(\mathcal{I}_a(R/\underline{x}R)) = e(\mathcal{I}(R/\underline{x}R))
\end{aligned}$$

where the equalities on the first and last lines are by [13, Proposition 6.2] and the inequality follows from (26). ☐

6. DIVISORIAL FILTRATIONS

Let R be a local domain of dimension d with quotient field K. Let ν be a discrete valuation of K with valuation ring $\mathcal{O}_\nu$ and maximal ideal m_ν. Suppose that $R \subset \mathcal{O}_\nu$. Then for $n \in \mathbb{N}$, define valuation ideals

$$I(\nu)_n = \{f \in R \mid \nu(f) \geq n\} = m_\nu^n \cap R.$$

A divisorial valuation of R ([39, Definition 9.3.1]) is a valuation ν of K such that if $\mathcal{O}_\nu$ is the valuation ring of ν with maximal ideal $\mathfrak{m}_\nu$, then $R \subset V_\nu$ and if $\mathfrak{p} = \mathfrak{m}_\nu \cap R$ then $\mathrm{trdeg}_{\varkappa(\mathfrak{p})}\varkappa(\nu) = \mathrm{ht}(\mathfrak{p}) - 1$, where $\varkappa(\mathfrak{p})$ is the residue field of $R_\mathfrak{p}$ and $\varkappa(\nu)$ is the residue field of V_ν.

By [39, Theorem 9.3.2], the valuation ring of every divisorial valuation ν is Noetherian, hence is a discrete valuation.

Lemma 6.1. *Suppose that R is an excellent local domain. Then a valuation ν of the quotient field K of R which is nonnegative on R is a divisorial valuation of R if and only if the valuation ring $\mathcal{O}_\nu$ is essentially of finite type over R.*

Proof. Since an excellent local domain is analytically unramified, the only if direction follows from [39, Theorem 9.3.2]. Now we establish the if direction. Since $\mathcal{O}_\nu$ is essentially of finite type over R, there exists a finite type R-algebra S and a prime ideal Q in S such that S is a sub R-algebra of $\mathcal{O}_\nu$ and $S_Q = \mathcal{O}_\nu$. Since an excellent local domain is universally catenary, the dimension equality (c.f. [39, Theorem B.3.2.]) holds. Since a Noetherian valuation ring is a discrete valuation ring (c.f. [39, Corollary 6.4.5]) it has dimension 1, so that $\mathrm{ht}(Q) = 1$, from which it follows that ν is a divisorial valuation. ☐

Suppose that $s \in \mathbb{N}$. An s-valuation of R is a divisorial valuation of R such that $\dim R/p = s$ where $p = m_\nu \cap R$.

A divisorial filtration of R is a filtration $\mathcal{I} = \{I_m\}$ such that there exist divisorial valuations $\nu_1, \ldots, \nu_r$ and $a_1, \ldots, a_r \in \mathbb{R}_{\geq 0}$ such that for all $m \in \mathbb{N}$,

$$I_m = I(\nu_1)_{\lceil ma_1 \rceil} \cap \cdots \cap I(\nu_r)_{\lceil ma_r \rceil}.$$

A divisorial filtration is called integral (rational) if $a_i \in \mathbb{Z}_{\geq 0}$ for all i ($a_i \in \mathbb{Q}_{\geq 0}$ for all i).

An s-divisorial filtration of R is a filtration $\mathcal{I} = \{I_m\}$ such that there exist s-valuations $\nu_1, \ldots, \nu_r$ and $a_1, \ldots, a_r \in \mathbb{R}_{\geq 0}$ such that for all $m \in \mathbb{N}$,

$$I_m = I(\nu_1)_{\lceil ma_1 \rceil} \cap \cdots \cap I(\nu_r)_{\lceil ma_r \rceil}. \tag{27}$$

Observe that the trivial filtration $\mathcal{I} = \{I_m\}$, defined by $I_m = R$ for all m, is a degenerate case of a divisorial filtration and is a degenerate case of an s-divisorial filtration for all s. The nontrivial 0-divisorial filtrations are the divisorial m_R-filtrations of [11].

We will often denote a divisorial filtration $\mathcal{I}$ on a local domain R by $\mathcal{I} = \mathcal{I}(D)$, even when R is not excellent and there does not exist a representation of $\mathcal{I}$ as defined before Theorem 6.7.

Even when that a_i are required to be positive for all i, the expression (27) of a divisorial filtration is far from unique. The following example follows from [12, Theorem 4.1].

Example 6.2. *There exist* 0*-valuations* ν_1 *and* ν_2 *on a normal 3-dimensional local ring* R *which is essentially of finite type over an arbitrary algebraically closed field* k*, such that if* $a_1, a_2, b_1, b_2 \in \mathbb{N}$ *and* $a_1 < \left(\frac{3}{9-\sqrt{3}}\right) a_2$*, then*

$$I(\nu_1)_{\lceil ma_1 \rceil} \cap I(\nu_2)_{\lceil ma_2 \rceil} = I(\nu_1)_{\lceil mb_1 \rceil} \cap I(\nu_2)_{\lceil mb_2 \rceil}$$

for all $m \in \mathbb{N}$ *if and only if* $b_2 = a_2$ *and* $b_1 < \left(\frac{3}{9-\sqrt{3}}\right) a_2$*. The filtration has the largest expression as a real divisorial filtration as* $\left\{ I(\nu_1)_{\lceil m\left(\frac{3}{9-\sqrt{3}}\right)a_2 \rceil} \cap I(\nu_2)_{\lceil ma_2 \rceil} \right\}$*. This divisorial filtration is not rational.*

Lemma 6.3. *If* $\mathcal{I}(D)$ *is a divisorial filtration, then* $R[\mathcal{I}(D)] = \overline{R[\mathcal{I}(D)]}$ *is integrally closed.*

The proof of Lemma 6.3 for m_R-filtrations in [11, Lemma 5.7] extends immediately to arbitrary divisorial filtrations.

We will use the following form of the valuative criterion of properness. The proposition is an immediate consequence of [20, Theorem II.4.7].

Proposition 6.4. *Suppose that* R *is a Noetherian domain with quotient field* K *and that* I *is a nonzero ideal of* R*. Let* $\pi : X \to Spec(R)$ *be the blow up of* I*. Suppose that* $\mathcal{O}_\nu$ *is a valuation ring of* K *such that* $R \subset \mathcal{O}_\nu$*. Let* $\mathfrak{p} = m_\nu \cap R$*. Then there exists a unique (not necessarily closed) point* α *of* X *such that* $\mathcal{O}_\nu$ *dominates* $S = \mathcal{O}_{X,\alpha}$*. We have that* S *dominates* $R_\mathfrak{p}$*; that is,* $\pi(\alpha) = \mathfrak{p} \in Spec(R)$*.*

Lemma 6.5. *Let* R *be an excellent local domain and* $\nu_1, \ldots, \nu_t$ *be s-valuations of* R *with associated centers* $\mathfrak{p}_i = m_{\nu_i} \cap R$ *on* R *for all* $1 \le i \le t$*. Then there exists an ideal* K *of* R *such that the associated primes of* R *are the* $\mathfrak{p}_i$ *and if* $\varphi : X \to Spec(R)$ *is the normalization of the blowup of* K*, then there exist prime Weil divisors* $E_1, \ldots, E_t$ *on* X *such that* $\mathcal{O}_{X,E_i} = \mathcal{O}_{\nu_i}$ *for* $1 \le i \le t$*.*

Proof. After reindexing the $\mathfrak{p}_i$ we may suppose that $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ (with $r \le t$) are the distinct primes in the set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_t\}$. Then reindex the ν_i as $\nu_{i,j}$ for $1 \le j \le \beta_i$ so that $m_{\nu_{i,j}} \cap R = \mathfrak{p}_i$ for all i, j.

It follows from Statement (G) of [34] (or [39, Proposition 10.4.4]) that there exists a $(\mathfrak{p}_i)_{\mathfrak{p}_i}$-primary ideal $J_{i,j}$ of $R_{\mathfrak{p}_i}$ such that if $\varphi_{i,j} : X_{i,j} \to \mathrm{Spec}(R_{\mathfrak{p}_i})$ is the normalization of the blowup of $J_{i,j}$, then $\mathcal{O}_{\nu_{i,j}}$ is a local ring of a prime Weil divisor on $X_{i,j}$. Let $J_i = \prod_j J_{i,j}$ and $\varphi_i : X_i \to \mathrm{Spec}(R_{\mathfrak{p}_i})$ be the normalization of the blow up of J_i.

We will now show that $\mathcal{O}_{\nu_{i,j}}$ is a local ring of X_i for all i. Let $Y_{i,j}$ be the blowup of $J_{i,j}$ and Y_i be the blowup of J_i, so that there are natural finite birational projective morphisms $X_{i,j} \to Y_{i,j}$ and $X_i \to Y_i$. Let $J_{i,j} = (a_{i,j,1}, \ldots, a_{i,j,\alpha_{i,j}})$. The ideal sheaves $J_{i,j}\mathcal{O}_{Y_i}$ are locally principal, since Y_i is covered by the open affine sets with the affine coordinate rings $T_{i,k_1,\ldots,k_{\beta_i}} = R_{\mathfrak{p}_i}[J_i/f]$ where $f = a_{i,1,k_1} \cdots a_{i,\beta_i,k_{\beta_i}}$ for some $k_1, \ldots, k_{\beta_i}$. The ideal sheaves $J_{i,j}\mathcal{O}_{X_i}$ are thus locally principal, since X_i is covered by the open affine sets with the affine coordinate rings $S_{i,k_1,\ldots,k_{\beta_i}}$ where $S_{i,k_1,\ldots,k_{\beta_i}}$ is the normalization of $T_{i,k_1,\ldots,k_{\beta_i}}$. Thus we have a birational projective morphism $X_i \to Y_{i,j}$ for all i, j (by the universal property of blowing up, c.f. [20, Proposition II.7.14]). Since X_i is normal, there is a birational projective morphism $X_i \to X_{i,j}$ for all i, j. Now for fixed i, j, by Proposition 6.4, there exists a unique local ring A of X_i such that the valuation ring $\mathcal{O}_{\nu_{i,j}}$ dominates A.

Let B be the local ring of $X_{i,j}$ such that A dominates B. Now B is the unique local ring of $X_{i,j}$ which is dominated by $\nu_{i,j}$ by Proposition 6.4. Since $\mathcal{O}_{\nu_{i,j}}$ is a local ring of $X_{i,j}$ we have that $B = \mathcal{O}_{\nu_{i,j}}$ so that $\mathcal{O}_{\nu_{i,j}} = A$ is a local ring of X_i.

For each i, there exists a $\mathfrak{p}_i$-primary ideal K_i of R such that $(K_i)_{\mathfrak{p}_i} = J_i$. Let $K = \cap_{i=1}^r K_i$. The associated primes of K are thus $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$. Let $\varphi : X \to \mathrm{Spec}(R)$ be the normalization of the blowup of K.

For given i, j, there exists $f_j \in J_i$ such that $\mathcal{O}_{\nu_{i,j}}$ is a local ring of the normalization $T_{i,j}$ of $R_{\mathfrak{p}_i}[J_i/f_j]$. Since $J_i = K_{\mathfrak{p}_i}$, we may assume that $f_j \in K$, so that $R_{\mathfrak{p}_i}[J_i/f_j] = R[K/f_j]_{\mathfrak{p}_i}$. Let $U_{i,j}$ be the normalization of $R[K/f_j]$, so that $U_{i,j}$ is the affine coordinate ring of an open subset of X. Now $T_{i,j}$ is the integral closure of $R[K/f_j]_{\mathfrak{p}_j} = R_{\mathfrak{p}_i}[J_i/f_j]$, so that $(U_{i,j})_{\mathfrak{p}_i} = (T_{i,j})_{\mathfrak{p}_i}$, and so $\mathcal{O}_{\nu_{i,j}}$ is a local ring of $U_{i,j}$, and hence is a local ring of X. $\square$

Remark 6.6. *If R is an excellent local domain and $\nu_1, \ldots, \nu_t$ are divisorial valuations of R, then a slight modification of the above proof gives the weaker statement that there exists an ideal K of R such that if $\varphi : X \to Spec(R)$ is the normalization of the blowup of K, then there exist prime Weil divisors $E_1, \ldots, E_t$ on X such that $\mathcal{O}_{X,E_i} = \mathcal{O}_{\nu_i}$ for $1 \le i \le t$.*

As in [11, Chapter 5], Lemma 6.5 allows us to define a representation of an s-divisorial filtration $\mathcal{I} = \{I_m\}$ on an excllent local domain R, where

$$I_m = I(\nu_1)_{\lceil ma_1 \rceil} \cap \cdots \cap I(\nu_t)_{\lceil ma_t \rceil}.$$

By Lemma 6.5, we may construct a blow up $\varphi : X \to \mathrm{Spec}(R)$ satisfying the conclusions of the lemma for $\nu_1, \ldots, \nu_t$. Let D be the real Weil divisor $D = a_1E_1 + \cdots + a_tE_t$ on X. Define

$$I(mD) = \Gamma(X, \mathcal{O}_X(-\lceil \sum ma_iE_i \rceil) \cap R.$$

giving a filtration $\mathcal{I}(D) = \{I(mD)\}$ on R. We have that

$$I(mD) = \Gamma(X, \mathcal{O}_X(-\lceil \sum ma_iE_i \rceil) \cap R = I(\nu_1)_{\lceil ma_1 \rceil} \cap \cdots \cap I(\nu_t)_{\lceil ma_t \rceil} = I_m$$

so that $I(mD)$ is the s-divisorial filtration $\mathcal{I}$.

Given a representation $\mathcal{I}(mD)$ of an s-divisorial filtration $\mathcal{I}$, it is desirable sometimes to separate the prime components of D which dominate different prime ideals of R, as in the proof of Lemma 6.5.

Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the distinct prime ideals of R which are dominated by a prime component of D. Reindex the E_i as $E_{i,j}$, where for each i, $\{E_{i,j}\}$ are the prime divisors (from the set $\{E_i\}$) which dominate $\mathfrak{p}_i$. Let $a_{ij} \in \mathbb{R}_{>0}$ be defined so that $D = \sum_{i,j} a_{i,j}E_{i,j}$. Define $D(i) = \sum_j a_{i,j}E_{i,j}$ for $1 \le i \le r$. Then the filtrations $\mathcal{I}(D(i)) = \{I(mD(i))\}$ are s-divisorial filtrations on R and $I(mD) = I(mD(1)) \cap \cdots \cap I(mD(r))$ for all $m \ge 0$.

Theorem 6.7. *Suppose that R is an excellent local domain and $\mathfrak{a}$ is an m_R-primary ideal. Let $\mathcal{I}(D)$ be a real s-divisorial filtration and $\mathcal{I}$ be an arbitrary filtration. Suppose that $\mathcal{I}(D) \subset \mathcal{I}$, so that $s(\mathcal{I}) \le s = s(\mathcal{I}(D))$. Then $e_s(\mathfrak{a}, \mathcal{I}(D)) = e_s(\mathfrak{a}, \mathcal{I})$ if and only if $I(mD) = I_m$ for all $m \ge 0$.*

Proof. Let $\mathcal{I} = \{I_m\}$. First suppose that $I(mD) = I_m$ for all $m \ge 0$. Then $e_s(\mathfrak{a}, \mathcal{I}(D)) = e_s(\mathfrak{a}, \mathcal{I})$ by definition of e_s.

Now suppose that $e_s(\mathfrak{a}, \mathcal{I}(D)) = e_s(\mathfrak{a}, \mathcal{I})$. Let $\varphi : X \to \mathrm{Spec}(R)$ be a representation of the filtration $\mathcal{I}(D)$. There are prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ in R such that $\dim R/\mathfrak{p}_i = s$ for all i and X is the normalization of the blowup of an ideal K of R such that $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ are the associated primes of K. Further, $D = \sum_{i=1}^r D(i)$ is an effective Weil divisor on X such that for all i, all prime components E of $D(i)$ satisfy $m_E \cap R = \mathfrak{p}_i$, where m_E is the maximal ideal of $\mathcal{O}_{X,E}$.

Since excellent rings are universally catenary, we have by Proposition 4.2 that

$$e_s(\mathfrak{a},\mathcal{I}(D))=\sum_{i=1}^{r} e_{R_{\mathfrak{p}_i}}(\mathcal{I}(D)_{\mathfrak{p}_i})e_{\mathfrak{a}}(R/\mathfrak{p}_i)$$

and since $V(I_1)\subset V(I(D))$, we have

$$e_s(\mathfrak{a},\mathcal{I})=\sum_{i=1}^{r} e_{R_{\mathfrak{p}_i}}(\mathcal{I}_{\mathfrak{p}_i})e_{\mathfrak{a}}(R/\mathfrak{p}_i).$$

Now for all i, we have that $e_{R_{\mathfrak{p}_i}}(\mathcal{I}(D)_{\mathfrak{p}_i})\ge e_{R_{\mathfrak{p}_i}}(\mathcal{I}_{\mathfrak{p}_i})$ since $\mathcal{I}(D)_{\mathfrak{p}_i}\subset\mathcal{I}_{\mathfrak{p}_i}$. Thus $e_s(\mathfrak{a},\mathcal{I}(D))\ge e_s(\mathfrak{a},\mathcal{I})$ with equality if and only if $e_{R_{\mathfrak{p}_i}}(\mathcal{I}(D)_{\mathfrak{p}_i})=e_{R_{\mathfrak{p}_i}}(\mathcal{I}_{\mathfrak{p}_i})$ for all i.

Suppose that for some i, $\mathcal{I}_{\mathfrak{p}_i}$ is a filtration of $m_{R_{\mathfrak{p}_i}}$-primary ideals. By [10, Theorem 1.4] and [11, Corollary 7.5], $e_{R_{p_i}}(\mathcal{I}(D)_{\mathfrak{p}_i})=e_{R_{\mathfrak{p}_i}}(\mathcal{I}_{\mathfrak{p}_i})$ if and only if $\mathcal{I}(D)_{\mathfrak{p}_i}=\mathcal{I}_{\mathfrak{p}_i}$. Also, if $\mathcal{I}_{\mathfrak{p}_i}$ is the trivial filtration, then since $\mathcal{I}(D)_{p_i}$ is a filtration of $m_{R_{\mathfrak{p}_i}}$-primary ideals, we have that $e_{R_{\mathfrak{p}_i}}(\mathcal{I}(D)_{\mathfrak{p}_i})>0$ by [11, Proposition 5.3] while $e_{R_{\mathfrak{p}_i}}(\mathcal{I}_{\mathfrak{p}_i})=0$, so that $e_{R_{\mathfrak{p}_i}}(\mathcal{I}_{\mathfrak{p}_i})<e_{R_{\mathfrak{p}_i}}(\mathcal{I}(D)_{\mathfrak{p}_i})$. Thus $e_s(\mathfrak{a},\mathcal{I}(D))=e_s(\mathfrak{a},\mathcal{I})$ if and only if $I(mD)_{\mathfrak{p}_i}=(I_m)_{\mathfrak{p}_i}$ for all i and $m\in\mathbb{N}$. In particular, with our assumption that $e_s(\mathfrak{a},\mathcal{I}(D))=e_s(\mathfrak{a},\mathcal{I})$, we have that

$$I(mD)_{\mathfrak{p}_i}=(I_m)_{\mathfrak{p}_i}\text{ for all } i \text{ and } m\in\mathbb{N}. \tag{28}$$

Now each $I(mD)$ is an intersection $Q_1\cap\cdots\cap Q_r$ where each Q_i is a $\mathfrak{p}_i$-primary ideal. Thus $I_m=Q_1\cap\cdots\cap Q_r\cap J$ where J is an ideal of R such that $J\not\subset\mathfrak{p}_i$ for any i. But then $I_m=I(mD)$ since $I(mD)\subset I_m$. ☐

Lemma 6.8. *Suppose that R is a d-dimensional analytically unramified local ring, $\mathfrak{a}$ is an m_R-primary ideal and $\mathcal{I}(1)=\{I(1)_j\}$ and $\mathcal{I}(2)=\{I(2)_j\}$ are filtrations such that $s(\mathcal{I}(1))=s(\mathcal{I}(2))$. Let $s=s(\mathcal{I}(1))=s(\mathcal{I}(2))$ and let $e_i=e_s(\mathfrak{a},\mathcal{I}(1)^{[d-s-i]},\mathcal{I}(2)^{[i]})$ for $0\le i\le d-s$. Then either*

a) $e_0=0$ or $e_{d-s}=0$ and $e_i=0$ for $0<i<d-s$ or
b) $e_0>0$ and $e_{d-s}>0$.

For all $n_1,n_2\in\mathbb{N}$, let

$$P_s(n_1,n_2)=\lim_{m\to\infty}\frac{e_s(\mathfrak{a},R/I(1)_{mn_1}I(2)_{mn_2})}{m^{d-s}/(d-s)!}=\sum_{j=0}^{d-s}\frac{(d-s)!}{(d-s-j)!j!}e_jn_1^{d-s-j}n_2^j.$$

Consider the following conditions.

1) $e_i^2=e_{i-1}e_{i+1}$ for $1\le i\le d-s-1$.
2) $e_ie_{d-s-i}=e_0e_{d-s}$ for $0\le i\le d-s$.
3) $e_i^{d-s}=e_0^{d-s-i}e_{d-s}^i$ for $0\le i\le d-s$.
4) $e_s(\mathfrak{a},\mathcal{I}(1)\mathcal{I}(2)))^{\frac{1}{d-s}}=e_0^{\frac{1}{d-s}}+e_{d-s}^{\frac{1}{d-s}}$, where $\mathcal{I}(1)\mathcal{I}(2)=\{I(1)_jI(2)_j\}$.
5) $P_s(n_1,n_2)=\left(e_0^{\frac{1}{d-s}}n_1+e_{d-s}^{\frac{1}{d-s}}n_2\right)^{d-s}$ for all $n_1,n_2\in\mathbb{N}$.

Then the following hold.

i) Statement 3) holds if and only if statement 4) holds.
ii) If a) holds then the statements 1) - 5) hold.
iii) If the e_i are nonzero for $0\le i\le d-s$, then the statements 1) - 5) are equivalent.

Proof. We have that either a) or b) holds by the Minkowski inequalities for filtrations (Theorem 5.3).

(i) The analysis of [11, Section 9] is valid for arbitrary filtrations, and shows that Statement 3) holds if and only if statement 4) holds.

ii) If a) holds, then all of the equalities 1) - 5) hold.

iii) Suppose that all e_i are nonzero. We show that all of the equalities 1) - 5) hold. The proof of [11, Section 9] applies here to show that conditions 1), 3), 4) and 5) are equivalent.

It remains to show that 2) is equivalent to 3). By the inequality iii) of the Minkowski inequalities for filtrations (Theorem 5.3) we have that

$$e_i^{d-s}e_{d-s-i}^{d-s} \le (e_0^{d-s-i}e_{d-s}^i)(e_0^i e_{d-s}^{d-s-i}) = e_0^{d-s}e_{d-s}^{d-s} \tag{29}$$

for $0 \le i \le d-s$, and equality holds in this equation for all i if and only if equality holds in 2). Taking $(d-s)$-th roots, we have that equality holds in (29) if and only if equality holds in 3) for $0 \le i \le d-s$.

□

Teissier [42] (for Cohen-Macaulay normal two-dimensional complex analytic R), Rees and Sharp [35] (in dimension 2) and Katz [24] (in complete generality) have shown that if R is a formally equidimensional local ring and $I(1), I(2)$ are m_R-primary ideals then the Minkowski inequality is an equality, that is,

$$e_R(I(1)I(2))^{\frac{1}{d}} = e(I(1))^{\frac{1}{d}} + e(I(2))^{\frac{1}{d}}, \tag{30}$$

if and only if there exist positive integers a and b such that

$$\overline{\sum_{n\ge 0} I(1)^{an}t^n} = \overline{\sum_{n\ge 0} I(2)^{bn}t^n}. \tag{31}$$

If $\mathcal{I}(1)$ and $\mathcal{I}(2)$ are filtrations of an analytically unramified local ring R and condition 31) holds then the Minkowski equality holds between $\mathcal{I}(1)$ and $\mathcal{I}(2)$ by Theorem 5.4, but the converse statement, that the Minkowski equality implies condition (31) is not true for filtrations, even for m_R-filtrations in a regular local ring, as is shown in a simple example in [13].

In [10, Theorem 1.6], it is shown that this characterization holds if $\mathcal{I}(1)$ and $\mathcal{I}(2)$ are bounded filtrations of m_R-primary ideals in a d-dimensional analytically irreducible or excellent local domain (s-divisorial filtrations are bounded; bounded filtrations are defined in the following Section 7). We show here that this characterization holds for s-divisorial filtrations.

Theorem 6.9. *Suppose that R is a d-dimensional excellent local domain and that $\mathfrak{a}$ is an m_R-primary ideal. Let $\mathcal{I}(D_1)$ and $\mathcal{I}(D_2)$ be two nontrivial integral s-divisorial filtrations. Then the Minkowski equality*

$$e_s(\mathfrak{a}, \mathcal{I}(D_1)\mathcal{I}(D_2))^{\frac{1}{d-s}} = e_0^{\frac{1}{d-s}} + e_{d-s}^{\frac{1}{d-s}}$$

holds between $\mathcal{I}(D_1)$ and $\mathcal{I}(D_2)$ if and only if there exist $a, b \in \mathbb{Z}_{>0}$ such that $I(amD_1) = I(bmD_2)$ for all $m \in \mathbb{N}$.

Proof. Let $\varphi : X \to \mathrm{Spec}(R)$ be a representation of the filtrations $\mathcal{I}(D_1)$ and $\mathcal{I}(D_2)$; that is, there are prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ in R such that $\dim R/\mathfrak{p}_i = s$ for all i and X is the normalization of the blowup of an ideal K of R such that $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ are the associated primes of K. Further, $D_1 = \sum_{i=1}^t D_1(i)$ and $D_2 = \sum_{i=1}^t D_2(i)$ are effective Weil divisors on

X such that for all i and for $j = 1, 2$, all prime components E of $D_j(i)$ satisfy $m_E \cap R = \mathfrak{p}_i$, where m_E is the maximal ideal of $\mathcal{O}_{X,E}$. Let $\mathcal{I}(D_j(i)) = \{I(mD_j(i))\}$ be the associated divisorial s-filtrations on R.

Let

$$P(n_1, n_2) = \lim_{m\to\infty} \frac{e_s(\mathfrak{a}, R/I(mn_1D_1)I(mn_2D_2))}{m^{d-s}/(d-s)!}$$

and for $1 \le i \le t$, let

$$P_i(n_1, n_2) = \lim_{m\to\infty} \frac{e_s(\mathfrak{a}, R/I(mn_1D_1(i)I(mn_2D_2(i)))}{m^{d-s}/(d-s)!}.$$

Write

$$P(n_1, n_2) = \sum_{j=0}^{d-s} \frac{(d-s)!}{(d-s-j)!j!} e_j n_1^{d-s-j} n_2^j$$

and

$$P_i(n_1, n_2) = \sum_{j=0}^{d-s} \frac{(d-s)!}{(d-s-j)!j!} e(i)_j n_1^{d-s-j} n_2^j.$$

We have that

$$P(n_1, n_2) = \sum_{i=1}^{t} P_i(n_1, n_2) \tag{32}$$

by Theorem 4.5 applied to $P(n_1, n_2)$ and the $P_i(n_2, n_2)$.

First assume that the Minkowski equality holds. By our assumptions, the conclusions of Lemma 6.8 hold for $\mathcal{I}(D(1))$ and $\mathcal{I}(D(2))$. There exists i such that $D_1(i) \ne 0$. We have

$$e_{R_{\mathfrak{p}_i}}(\mathcal{I}(D_1(i))_{\mathfrak{p}_i}^{[d-s]}, \mathcal{I}(D_2(i))_{\mathfrak{p}_i}^{[0]}) = e_{R_{\mathfrak{p}_i}}(\mathcal{I}(D_1(i))_{\mathfrak{p}_i}) \tag{33}$$

by [13, Proposition 6.5].

Let S be the normalization of R, which is dominated by X, and let $\mathfrak{q}_1, \ldots, \mathfrak{q}_u$ be the prime ideals of S which lie over $\mathfrak{p}_i$, so that each $S_{\mathfrak{q}_i}$ is analytically irreducible. Then by Equation (18), Lemma 5.2 and Proposition 5.3 of [11], we have that

$$e_{R_{\mathfrak{p}_i}}(\mathcal{I}(D_1(i))_{\mathfrak{p}_i}) > 0. \tag{34}$$

Now

$$e(i)_0 = e_s(\mathfrak{a}, \mathcal{I}(D_1(i))^{[d-s]}, \mathcal{I}(D_2(i))^{[0]}) = e_{R_{\mathfrak{p}_i}}(\mathcal{I}(D_1(i))_{\mathfrak{p}_i}^{[d-s]}, \mathcal{I}(D_2(i))_{\mathfrak{p}_i}^{[0]}) e_s(\mathfrak{a}, R/\mathfrak{p}_i) \tag{35}$$

by Theorem 4.5. Thus $e(i)_0 > 0$ by (34) and so

$$e_0 = \sum e(i)_0 > 0.$$

Similarly there exists i' such that $D_2(i') \ne 0$. Then using a similar argument to the above, we have $e(i')_{d-s} > 0$ and hence $e_{d-s} > 0$. Now $e_j^{d-s} = e_0^{d-s-j} e_{d-s}^j$ for $0 \le j \le d-s$ by i) of Lemma 6.8, since the Minkowski equality holds. Since $e_0 > 0$ and $e_{d-s} > 0$, we have that $e_j > 0$ for all j.

Since the Minkowski equality holds between $\mathcal{I}(D_1)$ and $\mathcal{I}(D_2)$, and $e_j > 0$ for all j, we have by iii) of Lemma 6.8 that the equalities 1) of Lemma 6.8 hold so there exists $\xi \in \mathbb{R}_{>0}$ such that

$$\xi = \frac{e_1}{e_0} = \cdots = \frac{e_{d-s}}{e_{d-s-1}}. \tag{36}$$

By (32) we have that $e_j = \sum_{i=1}^t e(i)_j$ for all j. By the inequalities $e(i)_j^2 \le e(i)_{j-1}e(i)_{j+1}$ for $1 \le j \le d-s-1$ (Theorem 5.3) and (36) we have that

$$\begin{array}{rcl} 0 & \le & \sum_{i=1}^t (e(i)_{j+1}^{\frac{1}{2}} - \xi e(i)_{j-1}^{\frac{1}{2}})^2 = \sum_{i=1}^t (e(i)_{j+1} - 2\xi e(i)_{j+1}^{\frac{1}{2}} e(i)_{j-1}^{\frac{1}{2}} + \xi^2 e(i)_{j-1}) \\ & \le & \sum_{i=1}^t (e(i)_{j+1} - 2\xi e(i)_j + \xi^2 e(i)_{j-1}) \\ & = & e_{j+1} - 2\xi e_j + \xi^2 e_{j-1} \\ & = & \xi^2 e_{j-1} - 2\xi^2 e_{j-1} + \xi^2 e_{j-1} = 0. \end{array}$$

Thus

$$e(i)_{j+1}^{\frac{1}{2}} = \xi e(i)_{j-1}^{\frac{1}{2}} \text{ and } e(i)_j^2 = e(i)_{j-1}e(i)_{j+1} \tag{37}$$

for all i and $1 \le j \le d-s-1$ respectively. Thus for a particular i, either

$$e(i)_j = 0 \text{ for all } j \tag{38}$$

or

$$e(i)_j > 0 \text{ for all } j. \tag{39}$$

Suppose that (38) holds for a particular i. Then $e(i)_0 = e(i)_{d-s} = 0$ which implies that

$$e_{R_{\mathfrak{p}_i}}(\mathcal{I}(D_1(i))_{\mathfrak{p}_i}) = e_{R_{\mathfrak{p}_i}}(\mathcal{I}(D_2(i))_{\mathfrak{p}_i}) = 0 \tag{40}$$

and so $\mathcal{I}(D_1(i))_{\mathfrak{p}_i}$ is the trivial filtration since we have a contradiction to (40) by (33) and (34) if $\mathcal{I}(D_1(i))_{\mathfrak{p}_i}$ is not trivial. Replacing $D_1(i)$ with $D_2(i)$ in this argument we obtain that $\mathcal{I}(D_2(i))_{\mathfrak{p}_i}$ is also the trivial filtration. In particular, $I(D_1(i))R_{\mathfrak{p}_i} = R_{\mathfrak{p}_i}$ and $I(D_2(i))R_{\mathfrak{p}_i} = R_{\mathfrak{p}_i}$, a contradiction, since $\mathfrak{p}_i$ must be an associated prime of at least one of $I(D_1(i))$ or $I(D_2(i))$. Thus (38) cannot hold, and so (39) holds for all i.

Let us now consider a particular i. Then by (37) and Lemma 6.8, the Minkowski equalities of Lemma 6.8 hold between $\mathcal{I}(D(i)_1)$ and $\mathcal{I}(D(i)_2)$. Thus there exists $\lambda_i \in \mathbb{R}_{>0}$ such that

$$\frac{e(i)_{j+1}}{e(i)_j} = \lambda_i$$

for all j. Thus for $1 \le j \le d-s-1$,

$$\xi^2 = \frac{e(i)_{j+1}}{e(i)_{j-1}} = \frac{e(i)_{j+1}}{e(i)_j}\frac{e(i)_j}{e(i)_{j-1}} = \lambda_i^2$$

so that $\lambda_i = \xi$ and so

$$\frac{e(i)_{d-s}^{\frac{1}{d-s}}}{e(i)_0^{\frac{1}{d-s}}} = \xi.$$

We have that

$$e(i)_j = e_s(\mathfrak{a}, \mathcal{I}(D_1(i))^{[d-s-j]}, \mathcal{I}(D_2(i))^{[j]}) = e_{R_{\mathfrak{p}_i}}(\mathcal{I}(D_1)_{\mathfrak{p}_i}^{[d-s-j]}, \mathcal{I}(D_2)_{\mathfrak{p}_i}^{[j]}) e_s(\mathfrak{a}, R/\mathfrak{p}_i)$$

by Theorem 4.5. Thus for each i, the $e_{R_{\mathfrak{p}_i}}(\mathcal{I}(D_1)_{\mathfrak{p}_i}^{[d-s-j]}, \mathcal{I}(D_2)_{\mathfrak{p}_i}^{[j]})$ satisfy the Minkowski equalities 1) - 3) of Lemma 6.8 with

$$\frac{e_{R_{\mathfrak{p}_i}}(\mathcal{I}(D_2)_{\mathfrak{p}_i})^{\frac{1}{d-s}}}{e_{R_{\mathfrak{p}_i}}(\mathcal{I}(D_1)_{\mathfrak{p}_i})^{\frac{1}{d-s}}} = \xi.$$

Thus $\xi \in \mathbb{Q}$ by [11, Theorem 12.1] and the proof of this theorem.

Write $\xi = \frac{a}{b}$ with $a, b \in \mathbb{Z}_{>0}$. We have that $I(maD_1(i)_{\mathfrak{p}_i}) = I(mbD_2(i)_{\mathfrak{p}_i})$ for all i and $m \in \mathbb{N}$ by [11, Theorem 12.1]. Since the only associated prime of $I(maD_1(i))$ and $I(mbD_2(i))$ is $\mathfrak{p}_i$, we have that $I(maD_1(i)) = I(mbD_2(i))$ for all i and $m \in \mathbb{N}$. Thus

$$I(maD_1) = I(maD_1(1)) \cap \cdots \cap I(maD_1(t)) = I(mbD_2(1)) \cap \cdots \cap I(mbD_2(t)) = I(mbD_2)$$

for all $m \in \mathbb{N}$.

The converse follows from Theorem 5.4 since $\overline{R[\mathcal{I}(D_j)]} = R[\mathcal{I}(D_j)]$ for $j = 1, 2$.

□

The following corollary is proven for m_R-valuations (divisorial valuations which dominate m_R) in [11, Corollary 12.2].

Corollary 6.10. *Suppose that R is an excellent local domain and ν_1 and ν_2 are divisorial valuations of the quotient field of R such that the valuation rings $\mathcal{O}_{\nu_1}$ and $\mathcal{O}_{\nu_2}$ both contain R. Suppose that $s = \dim R/(m_{\nu_1} \cap R) = \dim R/(m_{\nu_2} \cap R)$ and Minkowski's equality holds between the filtrations $\mathcal{I}(\nu_1) = \{I(\nu_1)_m\}$ and $\mathcal{I}(\nu_2) = \{I(\nu_2)_m\}$. Then $\nu_1 = \nu_2$.*

Proof. We have by Theorem 6.9 that $I(\nu_1)_{an} = I(\nu_2)_{bn}$ for all n and some positive integers a and b which we can take to be relatively prime.

Suppose that $0 \neq f \in I(\nu_1)_n$. Then $f^a \in I(\nu_1)_{an} = I(\nu_2)_{bn}$ so that $a\nu_2(f) \geq bn$. If $f^a \in I(\nu_2)_{bn+1}$ then $f^{ab} \in I(\nu_2)_{b(bn+1)} = I(\nu_1)_{a(bn+1)}$ so that $\nu_1(f) > n$. Thus

$$\nu_1(f) = n \text{ if and only if } \nu_2(f) = \frac{b}{a}n. \tag{41}$$

Further, (41) holds for every nonzero $f \in \mathrm{QF}(R)$ since f is a quotient of nonzero elements of R.

Now the maps $\nu_1 : \mathrm{QF}(R) \setminus \{0\} \to \mathbb{Z}$ and $\nu_2 : \mathrm{QF}(R) \setminus \{0\} \to \mathbb{Z}$ are surjective, so there exists $0 \neq f \in \mathrm{QF}(R)$ such that $\nu_1(f) = 1$ and there exists $0 \neq g \in \mathrm{QF}(R)$ such that $\nu_2(g) = 1$ which implies that $a = b = 1$ since a, b are relatively prime. Thus $\nu_1 = \nu_2$.

□

7. Bounded filtrations

Definition 7.1. *Suppose that R is a local domain. A filtration $\mathcal{I} = \{I_n\}$ on R is said to be a bounded filtration if there exists an integral divisorial filtration $\mathcal{I}(D)$ on R such that*

$$\overline{R[\mathcal{I}]} = R[\mathcal{I}(D)].$$

A filtration $\mathcal{I} = \{I_n\}$ on R is said to be a bounded s-filtration if there exists an integral s-divisorial filtration $\mathcal{I}(D)$ on R such that

$$\overline{R[\mathcal{I}]} = R[\mathcal{I}(D)].$$

A filtration $\mathcal{I} = \{I_n\}$ on R is said to be a real bounded s-filtration if there exists a real s-divisorial filtration $\mathcal{I}(D)$ on R such that

$$\overline{R[\mathcal{I}]} = R[\mathcal{I}(D)].$$

Lemma 7.2. *Suppose that R is an excellent local domain and $\mathcal{I} = \{I^n\}$ is the filtration of powers of a fixed ideal I. Then $\mathcal{I}$ is bounded.*

The proof of Lemma 7.2 for m_R-filtrations in [11, Lemma 5.9] extends immediately to arbitrary divisorial filtrations.

Proposition 7.3. *Suppose that R is a local ring with $\dim N(\hat{R}) < d$, $\mathfrak{a}$ is an m_R-primary ideal and*

$$\mathcal{I}(1), \ldots, \mathcal{I}(r), \mathcal{I}'(1), \ldots, \mathcal{I}'(r)$$

are s-filtrations such that $\overline{R(\mathcal{I}'(i))} = \overline{R(\mathcal{I}(i))}$ for $1 \le i \le r$. Then we have equality of all mixed multiplicities

$$e_s(\mathfrak{a}, \mathcal{I}(1)^{[d_1]}, \ldots, \mathcal{I}(r)^{[d_r]}) = e_s(\mathfrak{a}, \mathcal{I}'(1)^{[d_1]}, \ldots, \mathcal{I}'(r)^{[d_r]}). \tag{42}$$

The proof is as the proof for m_R-filtrations, given in [11, Proposition 5.10]. The references to [13, Theorem 6.9] and [10, Appendix] in the proof in [11] must be replaced with a reference to Theorem 5.1 of this paper.

Theorem 7.4. *Suppose that R is an excellent local domain, $\mathfrak{a}$ is an m_R-primary ideal, $\mathcal{I}(1)$ is a real bounded s-filtration and $\mathcal{I}(2)$ is an arbitrary filtration such that $\mathcal{I}(1) \subset \mathcal{I}(2)$, so that $s(\mathcal{I}(2)) \le s = s(\mathcal{I}(1))$. Then $e_s(\mathfrak{a}, \mathcal{I}(1)) = e_s(\mathfrak{a}, \mathcal{I}(2))$ if and only if there is equality of integral closures*

$$\overline{\sum_{m\ge 0} I(1)_m t^m} = \overline{\sum_{m\ge 0} I(2)_m t^m}$$

in $R[t]$.

Proof. First suppose that there is equality of integral closures

$$\overline{\sum_{m\ge 0} I(1)_m t^m} = \overline{\sum_{m\ge 0} I(2)_m t^m}$$

in $R[t]$. Then $e_s(\mathfrak{a}, \mathcal{I}(1)) = e_s(\mathfrak{a}, \mathcal{I}(2))$ by Theorem 5.1.

Now suppose that $e_s(\mathfrak{a}, \mathcal{I}(1)) = e_s(\mathfrak{a}, \mathcal{I}(2))$. Let $\mathcal{I}(D_1)$ be the real divisorial s-filtration such that $\overline{R[\mathcal{I}(1)]} = R[\mathcal{I}(D_1)]$. Let $\mathcal{J}$ be the filtration on R such that $R[\mathcal{J}] = \overline{R[\mathcal{I}(2)]}$. Then

$$R[\mathcal{I}(D_1)] \subset \overline{R[\mathcal{I}(2)]} = R[\mathcal{J}]$$

so that $\mathcal{I}(D_1) \subset \mathcal{J}$. We have that $e_s(\mathcal{I}(1)) = e_s(\mathcal{I}(D_1))$ and $e_s(\mathcal{I}(2)) = e_s(\mathcal{J})$ by Theorem 5.1. Thus $e_s(\mathcal{I}(D_1)) = e_s(\mathcal{J})$ and so $R[\mathcal{I}(D_1)] = R[\mathcal{J}]$ by Theorem 6.7.

Thus the conclusion of the theorem holds for $\mathcal{I}(1)$ and $\mathcal{I}(2)$. ☐

Example 7.5. *Theorem 7.4 does not extend to arbitrary bounded filtrations, or to arbitrary divisorial filtrations.*

There exist bounded filtrations $\mathcal{I}(1) \subset \mathcal{I}(2)$ with $s(\mathcal{I}(1)) = s(\mathcal{I}(2))$ and $e_s(m_R, \mathcal{I}(1)) = e_s(m_R, \mathcal{I}(2))$ but $\overline{R[\mathcal{I}(1)]} \ne \overline{R[\mathcal{I}(2)]}$. Let k be an algebraically closed field and let $R = k[x,y,z]_{(x,y,z)}$, a local ring of the polynomial ring over k in three variables. Let $\mathfrak{p}$ be a height two prime ideal in R such that the symbolic algebra $\oplus_{n\ge 0}\mathfrak{p}^{(n)}$ is not a finitely generated R-algebra. Some examples where this algebra is not finitely generated are given in [36] and [17]. The $\mathfrak{p}$-adic filtration $\mathcal{I}(1) = \{\mathfrak{p}^n\}$ is bounded by Lemma 7.2 and the filtration of symbolic powers of $\mathfrak{p}$, $\mathcal{I}(2) = \{\mathfrak{p}^{(n)}\}$, is a divisorial filtration. We have that $s(\mathcal{I}(1)) = s(\mathcal{I}(2)) = 1$ and since $\mathfrak{p}^n R_{\mathfrak{p}} = \mathfrak{p}^{(n)} R_{\mathfrak{p}}$ for all n, we have by Proposition 4.2 that

$$e_1(m_R, \mathcal{I}(1)) = e_{R_{\mathfrak{p}}}(\mathcal{I}(1)_{\mathfrak{p}}) e_{m_R}(R/\mathfrak{p}) = e_{R_{\mathfrak{p}}}(\mathcal{I}(2)_{\mathfrak{p}}) e_{m_R}(R/\mathfrak{p}) = e_1(m_R, \mathcal{I}(2)).$$

But we cannot have that $R[\mathcal{I}(2)]$ is integral over $R[\mathcal{I}(1)]$ since its integral closure $\overline{R[\mathcal{I}(1)]}$ is a finitely generated R-algebra, and $R[\mathcal{I}(2)] = \sum_{n\ge 0} \mathfrak{p}^{(n)} t^n$ is not. The reason for this example is because of the existence of embedded primes in the filtration $\mathcal{I}(1)$.

We now modify the example to show that Theorem 7.4 does not extend to divisorial filtrations. Let $\mathcal{I}(3)$ be the filtration $\mathcal{I}(3) = \{\overline{\mathfrak{p}^n}\}$. Let X be the normalization of the blowup of $\mathfrak{p}$. Then $\mathfrak{p}\mathcal{O}_X$ is invertible on X and $\overline{\mathfrak{p}^n} = \Gamma(X, \mathfrak{p}^n\mathcal{O}_X) \cap R$ for all n, so that $\mathcal{I}(3)$ is a divisorial filtration on R. Since $R[\mathcal{I}(3)] = \overline{R[\mathcal{I}(1)]}$ by Remark 3.7, we have that $e_1(m_R, \mathcal{I}(3)) = e_1(m_R, \mathcal{I}(1))$ by Theorem 5.1. Thus $e_1(m_R, \mathcal{I}(3)) = e_1(m_R, \mathcal{I}(2))$. Now $R[\mathcal{I}(3)] \subset R[\mathcal{I}(2)]$ since $R[\mathcal{I}(2)]$ is integrally closed in $R[t]$ by Lemma 6.3. Thus $\mathcal{I}(3) \subset \mathcal{I}(2)$.

Theorem 7.6. *Suppose that R is a d-dimensional excellent local domain, $\mathfrak{a}$ is an m_R-primary ideal and $\mathcal{I}(1)$ and $\mathcal{I}(2)$ are two nontrivial bounded s-filtrations. Then the following are equivalent.*

1) *The Minkowski inequality*

$$e_s(\mathfrak{a}, \mathcal{I}(1)\mathcal{I}(2))^{\frac{1}{d-s}} = e_s(\mathfrak{a}, \mathcal{I}(1))^{\frac{1}{d-s}} + e_s(\mathfrak{a}, \mathcal{I}(2))^{\frac{1}{d-s}}$$

holds.

2) *There exist positive integers a, b such that there is equality of integral closures*

$$\overline{\sum_{n\geq 0} I(1)_{an}t^n} = \overline{\sum_{n\geq 0} I(2)_{bn}t^n}$$

in $R[t]$.

Proof. Let $\mathcal{I}(D_1)$ and $\mathcal{I}(D_2)$ be integral divisorial s-filtrations such that $\overline{R(\mathcal{I}(1))} = R(\mathcal{I}(D_1))$ and $\overline{R(\mathcal{I}(2))} = R(\mathcal{I}(D_2))$. By Proposition 7.3, we have equality of functions

$$\lim_{m\to\infty} \frac{e_s(\mathfrak{a}, R/I(1)_{mn_1}I(2)_{mn_2})}{m^{d-s}/(d-s)!} = \lim_{m\to\infty} \frac{e_s(\mathfrak{a}, R/I(mn_1D_1)I(mn_2D_2))}{m^{d-s}/(d-s)!}$$

for all $n_1, n_2 \in \mathbb{N}$. Since 1) and 2) are equivalent for the integral s-divisorial filtrations $\mathcal{I}(D_1)$ and $\mathcal{I}(D_2)$ by Theorem 6.9, they are also equivalent for the bounded s-filtrations $\mathcal{I}(1)$ and $\mathcal{I}(2)$. ☐

Example 7.7. *Theorem 7.6 does not extend to arbitrary bounded filtrations or to arbitrary divisorial filtrations.*

The example constructed in Example 7.5 gives an example. We have that $d = \dim R = 3$, $s = \dim R/p = 1$, $d - s = \dim R_p = 2$ and $\mathfrak{p}^nR_\mathfrak{p} = \mathfrak{p}^{(n)}R_\mathfrak{p}$ for all n, so that $\mathcal{I}(1)_\mathfrak{p} = \mathcal{I}(2)_\mathfrak{p}$ are 0-divisorial filtrations on $R_\mathfrak{p}$. Thus

$$e_0(m_{R_\mathfrak{p}}, \mathcal{I}(1)_\mathfrak{p}\mathcal{I}(2)_\mathfrak{p})^{\frac{1}{\dim R_\mathfrak{p}}} = e_0(m_{R_\mathfrak{p}}, \mathcal{I}(1)_\mathfrak{p})^{\frac{1}{\dim R_\mathfrak{p}}} + e_0(m_{R_\mathfrak{p}}, \mathcal{I}(2)_\mathfrak{p})^{\frac{1}{\dim R_\mathfrak{p}}}$$

by Theorem 7.6. Since

$e_s(m_R, \mathcal{I}(1)) = e_0(m_{R_\mathfrak{p}}, \mathcal{I}(1)_\mathfrak{p})e_{m_R}(R/\mathfrak{p})$, $e_s(m_R, \mathcal{I}(2)) = e_0(m_{R_\mathfrak{p}}, \mathcal{I}(2)_\mathfrak{p})e_{m_R}(R/\mathfrak{p})$ and $e_s(m_R, \mathcal{I}(1)\mathcal{I}(2)) = e_0(m_\mathfrak{p}, \mathcal{I}(1)_\mathfrak{p}\mathcal{I}(2)_\mathfrak{p})e_{m_R}(R/\mathfrak{p})$

we have that the Minkowski equality

$$e_s(m_R, \mathcal{I}(1)\mathcal{I}(2))^{\frac{1}{d-s}} = e_s(m_R, \mathcal{I}(1))^{\frac{1}{d-s}} + e_s(m_R, \mathcal{I}(2))^{\frac{1}{d-s}}$$

holds between $\mathcal{I}(1)$ and $\mathcal{I}(2)$, but as shown in Example 7.5, there do not exist $a, b \in \mathbb{Z}_{>0}$ such that $\overline{\mathfrak{p}^{an}} = \mathfrak{p}^{(bn)}$ for all $n \in \mathbb{N}$.

Similarly, we have that $\mathcal{I}(3) \subset \mathcal{I}(2)$ are divisorial filtrations which satisfy the Minkowski equality, but there do not exist $a, b \in \mathbb{Z}_{>0}$ such that $\overline{\mathfrak{p}^{an}} = \mathfrak{p}^{(bn)}$ for all $n \in \mathbb{N}$.

8. Equimultiple ideals and bounded s-filtrations

The analytic spread of an ideal I in a local ring R is defined to be

$$\ell(I) = \dim R[It]/m_R R[It].$$

We have inequalities

$$\mathrm{ht}(I) \le \ell(I) \le \dim R,$$

proven for instance in [39, Corollary 8.3.9]. An ideal I for which the equality $\mathrm{ht}(I) = \ell(I)$ holds is called equimultiple.

Böger generalized Rees's theorem to equimultiple ideals in a formally equidimensional local ring.

Theorem 8.1. *([3], also [39, Corollary 11.3.2]) Suppose that R is a formally equidimensional local ring and $I \subset J$ are two ideals such that $\ell(I) = ht(I)$ (I is equimultiple). Then $J \subset \overline{I}$ if and only $e_{R_P}(I_P) = e_{R_P}(J_P)$ for every prime P minimal over I.*

Let I be an ideal in an excellent local domain R and $R[I] = \bigoplus_{n\ge 0} I^n$ be the Rees algebra of R, $\pi : X = \mathrm{Proj}(R[I]) \to \mathrm{Spec}(R)$ be the blowup of I. Let B be the normalization of $R[I]$ in its quotient field, which is a finitely generated graded R-algebra. Let $Y = \mathrm{Proj}(B)$, the "normalized blowup" of I. Let $\alpha : Y \to \mathrm{Spec}(R)$ be the natural composition map $Y \to X \stackrel{\pi}{\to} \mathrm{Spec}(R)$.

Let $p \in \mathrm{Spec}(R)$ and $\varkappa(p) := (R/p)_p$. Then (by definition)

$$\pi^{-1}(p) = X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(\varkappa(p)) = \mathrm{Proj}(R[I] \otimes_R \varkappa(p))$$

and

$$\alpha^{-1}(p) = Y \times_{\mathrm{Spec}(R)} \mathrm{Spec}(\varkappa(p)) = \mathrm{Proj}(B \otimes_R \varkappa(p)).$$

Since $Y \to X$ is finite, we have (by [5, Theorems A.6 and A.7]) that if $p \in \mathrm{Spec}(R)$, then

$$\dim \pi^{-1}(p) = \dim \alpha^{-1}(p).$$

We also have that "upper semicontinuity of fiber dimension" holds; that is, if $p \subset p'$ are prime ideals in R, then

$$\dim \pi^{-1}(p) \le \dim \pi^{-1}(p')$$

by [18, (IV.13.1.5)].

Write $I\mathcal{O}_Y = \mathcal{O}_Y(-a_1E_1 - \cdots - a_rE_r)$, where $E_1, \ldots, E_r$ are prime Weil divisors on Y. Then $I\mathcal{O}_Y$ is an ample Cartier divisor on Y. Let $\nu_1, \ldots, \nu_r$ be the corresponding valuations on the quotient field of R. We have that for all $n \ge 0$,

$$\overline{I^n} = \Gamma(Y, I^n\mathcal{O}_Y) \cap R = I(\nu_1)_{a_1 n} \cap \cdots \cap I(\nu_r)_{a_r n}$$

is a primary decomposition of $\overline{I^n}$.

Now we have that for $p \in \mathrm{Spec}(R)$, $\dim \alpha^{-1}(p) \le \dim R_p - 1$ and $\dim \alpha^{-1}(p) = \dim R_p - 1$ if and only if some ν_i dominates p. Further, if p does not contain I, then $\dim \alpha^{-1}(p) = 0$, and if p is a minimal prime of I, then $\dim \alpha^{-1}(p) = \dim R_p - 1$.

Proposition 8.2. *Suppose that I is an ideal of an excellent local domain R and I is equimultiple. Let $s = \dim R - ht(I)$. Then there exist divisorial valuations $\nu_1, \ldots, \nu_r$ such that the center of ν_i on R has dimension s for all i, and $a_1, \ldots, a_r \in \mathbb{Z}_{>0}$ such that*

$$\overline{I^n} = I(\nu_1)_{a_1 n} \cap \cdots \cap I(\nu_r)_{a_r n}$$

for all $n \in \mathbb{N}$.

Proof. By our assumption, $\dim \alpha^{-1}(m_R) = \mathrm{ht}(I) - 1$. Suppose that $p \in \mathrm{Spec}(R)$. If p is dominated by some ν_i, then $I \subset p$ and

$$\mathrm{ht}(I) - 1 \leq \dim R_p - 1 = \dim \alpha^{-1}(p) \leq \dim \alpha^{-1}(m_R) = \mathrm{ht}(I) - 1,$$

so that $\mathrm{ht}(p) = \mathrm{ht}(I)$. □

Corollary 8.3. *Suppose that R is an excellent local domain and I is an equimultiple ideal on R. Then the I-adic filtration $\mathcal{I} = \{I^n\}$ is a bounded s-filtration, where $s = \dim R - ht(I)$.*

A much more difficult to prove form of Proposition 8.2 is true in a locally formally equidimensional Noetherian ring A. If I is an equimultiple ideals in A, then for all $n \geq 0$, every associated prime of $\overline{I^n}$ has height $\ell = \ell(I)$. This statement follows from [31, Theorem 2.12]. By [39, Lemmas 8.42 and B.47], we may assume that R has an infinite residue field. By [39, Proposition 8.3.7], I then has a reduction J generated by ℓ elements. By [31, Theorem 2.12] or [39, Corollary 5.4.2], every associated prime of $\overline{J^n} = \overline{I^n}$ has height ℓ.

Example 8.4. *There exists a height one prime ideal P in a normal, excellent 3 dimensional local ring R and $d > 0$ such that the P-adic filtration $\{P^n\}_{n\in\mathbb{N}}$ of R is a bounded 2-filtration but $2 = \ell(P) > ht(P) = 1$, so that P is not equimultiple.*

Proof. Let k be a field and $k[x, y, z, w]$ be a polynomial ring over k. Let

$$R = (k[x, y, z, w]/(xy - zw))_{(x,y,z,w)}.$$

Let $\overline{x}, \overline{y}, \overline{z}, \overline{w}$ be the respective classes of x, y, z, w in R. Let $P = (\overline{x}, \overline{z})$, which is a height 1 prime ideal in R.

The blowup $\pi : X = \mathrm{Proj}(\oplus_{n\geq 0} P^n) \to \mathrm{Spec}(R)$ of P is such that X is nonsingular, and $\pi^{-1}(m_R) \cong \mathbb{P}^1_k$. This simple and well known calculation is outlined in Exercise 6.16 on page 125 of [7]. Thus $\ell(P) = \dim \pi^{-1}(m_R) + 1 = 2$, and so $1 = \mathrm{ht}(P) < \ell(P) = 2$ and so P is not equimultiple.

Let $E = \mathrm{Spec}(\mathcal{O}_X/P\mathcal{O}_X)$, an integral surface on X, so that $\mathcal{O}_{X,E}$ is a valuation ring, with associated valuation ν_E. Now $P^n\mathcal{O}_X = \mathcal{O}_X(-nE)$ for all n, and

$$\overline{P^n} = \pi_*(P^n\mathcal{O}_X) = \pi_*(\mathcal{O}_X(-nE)) = P^{(n)} = I(\nu_E)_n$$

for all $n \in \mathbb{N}$. Thus $\{P^n\}$ is a bounded 2-filtration. □

We conclude from Corollary 8.3 and Example 8.4, that in an excellent local domain of dimension d, the set of I-adic filtrations of equimultiple height r ideals is a strict subset of the set of bounded $(d - r)$-filtrations.

The following theorem is a generalization in excellent local domains of Böger's theorem from equimultiple ideals to the larger class of s-filtrations. Theorem 8.5 is a consequence of Theorem 7.4.

Theorem 8.5. *Suppose that R is an excellent local domain, $\mathfrak{a}$ is an m_R-primary ideal, $\mathcal{I}(1)$ is a real bounded s-filtration and $\mathcal{I}(2)$ is an arbitrary filtration with $\mathcal{I}(1) \subset \mathcal{I}(2)$. Then $e_{R_\mathfrak{p}}(\mathcal{I}(1)_\mathfrak{p}) = e_{R_\mathfrak{p}}(\mathcal{I}(2)_\mathfrak{p})$ for all $\mathfrak{p} \in \mathrm{Min}(R/I(1)_1)$ if and only if there is equality of integral closures*

$$\overline{\sum_{m\geq 0} I(1)_m t^m} = \overline{\sum_{m\geq 0} I(2)_m t^m} \tag{43}$$

in $R[t]$.

Proof. By Proposition 4.2, we have that

$$e_s(\mathfrak{a},\mathcal{I}(1)) = \sum_{\mathfrak{p}\in \mathrm{Min}(R/I(1)_1)} e_{R_\mathfrak{p}}(\mathcal{I}(1)_\mathfrak{p})e_\mathfrak{a}(R/\mathfrak{p}). \tag{44}$$

Since $I(1)_1 \subset I(2)_1$, we have that $V(I(2)_1) \subset V(I(1)_1)$. Thus $s(\mathcal{I}(2)) \le s(\mathcal{I}(1)) = s$. Since $V(I(1)_1)$ is equidimensional of dimension s, we have by Proposition 4.2 that

$$e_s(\mathfrak{a},\mathcal{I}(2)) = \sum_{\mathfrak{p}\in \mathrm{Min}(R/I(1)_1)} e_{R_\mathfrak{p}}(\mathcal{I}(2)_\mathfrak{p})e_\mathfrak{a}(R/\mathfrak{p}). \tag{45}$$

Now $\mathcal{I}(1) \subset \mathcal{I}(2)$ implies

$$e_{R_\mathfrak{p}}(\mathcal{I}(1)_\mathfrak{p}) \ge e_{R_\mathfrak{p}}(\mathcal{I}(2)_\mathfrak{p}) \tag{46}$$

for all $\mathfrak{p} \in \mathrm{Min}(R/I(1)_1)$. By equations (44), (45) and (46), we have that $e_s(\mathfrak{a},\mathcal{I}(1)) = e_s(\mathfrak{a},\mathcal{I}(2))$ if and only if $e_{R_\mathfrak{p}}(\mathcal{I}(1)_\mathfrak{p}) = e_{R_\mathfrak{p}}(\mathcal{I}(2))$ for all $\mathfrak{p} \in \mathrm{Min}(R/I(1)_1)$. The theorem now follows from Theorem 7.4, which shows that $e_s(\mathfrak{a},\mathcal{I}(1)) = e_s(\mathfrak{a},\mathcal{I}(2))$ if and only if the equality of integral closures of (43) holds. ☐

References

[1] P.B. Bhattacharya, The Hilbert function of two ideals, Proc. Camb. Phil. Soc. 53 (1957), 568 - 575.

[2] N. Bourbaki, Commutative Algebra, Chapters 1-7, Springer Verlag, 1989.

[3] E. Böger, Einige Bemerkungen zur Theorie der ganzalgebraischen Abhandigkeit von Idealen, Math. Ann 185 (1970), 303-308.

[4] M. Brodmann, Asymptotic stability of Ass(M/I^nM) Proc. Amer. Math. Soc. 74 (1979), 16-18.

[5] W. Bruns and J. Herzog, Cohen-Macaulay rings, Cambridge studies in Advanced Mathematics 39, Cambridge University Press, 1993, 13H10 (13-02).

[6] Y. Cid-Ruiz and J. Montaño, Mixed Multiplicities of Graded Families of Ideals, J. Algebra 590 (2022), 394 - 412.

[7] S.D. Cutkosky, Introduction to Algebraic Geometry, Graduate studies in Mathematics 188, American Mathematical Society, 2018.

[8] S.D. Cutkosky, Asymptotic multiplicities of graded families of ideals and linear series, Advances in Mathematics 264 (2014), 55 - 113.

[9] S.D. Cutkosky, Asymptotic Multiplicities, Journal of Algebra 442 (2015), 260 - 298.

[10] S.D. Cutkosky, Mixed Multiplicities of Divisorial Filtrations, Advances in Math. 358 (2019).

[11] S.D. Cutkosky, The Minkowski Equality of Filtrations, Advances in Math. 388 (2021).

[12] S.D. Cutkosky, Examples of multiplicities and mixed multiplicities of filtrations, in Commutative Algebra - 150 years with Roger and Sylvia Wiegand, Contemporary Mathematics 773, Amer. Math. Soc. (2021), 19 - 34.

[13] S.D. Cutkosky, P. Sarkar and H. Srinivasan, Mixed Multiplicities of Filtrations, Trans. Amer. Math. Soc., 372 (2019), 6183 - 6211.

[14] L. Ein, R. Lazarsfeld and K. Smith, Uniform Approximation of Abhyankar valuation ideals in smooth function fields, Amer. J. Math. 125 (2003), 409 - 440.

[15] K. Goel, Mixed Multiplicities of Ideals, Lecture Notes, IIT Bombay.

[16] K. Goel, R.V. Gurjar and J.K. Verma, The Minkowski's equality and inequality for multiplicity of ideals of finite length in Noetherian local rings, Contemporary Math 738 (2019).

[17] S. Goto, K. Nishida and K. Watanabe, Non Cohen-Macaulay symbolic blow-ups for space monomial curves and counterexamples to Cowsik's question, Proc. Amer. Math. Soc. 120 (1994), 383- 392.

[18] A. Grothendieck and J. Dieudonné, Éléments de Géométrie Algébrique IV, Parts 2 and 3, Publ. Math. IHES 24 (1965) and 28 (1966).

[19] G. Hardy, J.E. Littlewood and G. Pólya, Inequalities, second edition, Cambridge University Press, 1952.

[20] R. Hartshorne, Algebraic Geometry, Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977.

[21] M. Herrmann, S. Ikeda, and U. Orbanz, Equimultiplicity and blowing up: an algebraic study, Springer-Verlag, Berlin, 1988.

[22] M. Herrmann, R. Schmidt and W. Vogel, Theorie der normalen flaxhheit, B.G. Teubner, Leipzig, 1977.

[23] J. Huh, Milnor numbers of projective hypersurfaces and the chromatic polynomials of graphs, Journal of the Amer. Math. Soc. 25 (2012), 907 - 927.

[24] D. Katz, Note on multiplicity, Proc. Amer. Math. Soc. 104 (1988), 1021 - 1026.

[25] D. Katz and J. Verma, Extended Rees algebras and mixed multiplicities, Math. Z. 202 (1989), 111-128.

[26] K. Kaveh and G. Khovanskii, Newton-Okounkov bodies, semigroups of integral points, graded algebras and intersection theory, Annals of Math. 176 (2012), 925 - 978.

[27] R. Lazarsfeld and M. Mustaţă, Convex bodies associated to linear series, Ann. Sci. Ec. Norm. Super 42 (2009) 783 - 835.

[28] J. Lipman, Equimultiplicity, reduction and blowing up, R.N. Draper (Ed.), Commutative Algebra, Lect. Notes Pure Appl. Math., vol. 68, Marcel Dekker, New York (1982), pp. 111-147.

[29] M. Mustaţă, On multiplicities of graded sequences of ideals, J. Algebra 256 (2002), 229-249.

[30] A. Okounkov, Why would multiplicities be log-concave?, in The orbit method in geometry and physics, Progr. Math. 213, 2003, 329-347.

[31] J.L. Ratliff Jr., Locally quasi-unmixed Noetherian rings and ideals of the principal class, Pacific J. Math. 52 (1974), 185-205.

[32] D. Rees, $\mathcal{A}$-transforms of local rings and a theorem on multiplicities of ideals, Proc. Cambridge Philos. Soc. 57 (1961), 8 - 17.

[33] D. Rees, Multiplicities, Hilbert functions and degree functions. In Commutative algebra: Durham 1981 (Durham 1981), London Math. Soc. Lecture Note Ser. 72, Cambridge, New York, Cambridge Univ. Press, 1982, 170 - 178.

[34] D. Rees, Izumi's theorem, in Commutative Algebra, C. Huneke and J.D. Sally editors, Springer-Verlag 1989, 407 - 416.

[35] D. Rees and R. Sharp, On a Theorem of B. Teissier on Multiplicities of Ideals in Local Rings, J. London Math. Soc. 18 (1978), 449-463.

[36] P. C. Roberts, A prime ideal in a polynomial ring whose symbolic blow-up is not Noetherian", Proc. AMS 1985, 589-592.

[37] J-P Serre, Algèbre Locale, Multiplicitiés, Springer Verlag, Lecture Notes in Mathematics 11, 1965.

[38] I. Swanson, Mixed multiplicities, joint reductions and a theorem of Rees, J. London Math. Soc. 48 (1993), 1 - 14.

[39] I. Swanson and C. Huneke, Integral Closure of Ideals, Rings and Modules, Cambridge University Press, 2006.

[40] B. Teissier, Cycles évanescents, sections planes et conditions de Whitney, Singularités à Cargèse 1972, Astérisque 7-8 (1973)

[41] B. Teissier, Sur une inégalité à la Minkowski pour les multiplicités (Appendix to a paper by D. Eisenbud and H. Levine), Ann. Math. 106 (1977), 38 - 44.

[42] B. Teissier, On a Minkowski type inequality for multiplicities II, In C.P. Ramanujam - a tribute, Tata Inst. Fund. Res. Studies in Math. 8, Berlin - New York, Springer, 1978.

[43] N.V. Trung and J. Verma, Mixed multiplicities of ideals versus mixed volumes of polytopes, Trans. Amer. Math. Soc. 359 (2007), 4711 - 4727.

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