

OPTIMAL REGULARITY OF MIXED DIRICHLET-CONORMAL BOUNDARY VALUE PROBLEMS FOR PARABOLIC OPERATORS

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ABSTRACT. We obtain the regularity of solutions in Sobolev spaces for the mixed Dirichlet-conormal problem for parabolic operators in cylindrical domains with time-dependent separations, which is the first of its kind. Assuming the boundary of the domain to be Reifenberg-flat and the separation to be locally sufficiently close to a Lipschitz function of m variables, where $m = 0, \dots, d-2$, with respect to the Hausdorff distance, we prove the unique solvability for $p \in (2(m+2)/(m+3), 2(m+2)/(m+1)))$. In the case when $m = 0$, the range $p \in (4/3, 4)$ is optimal in view of the known results for Laplace equations.

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1. INTRODUCTION

In this paper, we obtain the maximal regularity for divergence form parabolic equations with mixed boundary conditions:

$$\begin{cases} \mathcal{P}u - \lambda u = D_i g_i + f & \text{in } Q^T, \\ \mathcal{B}u = g_i n_i & \text{on } \mathcal{N}^T, \\ u = 0 & \text{on } \mathcal{D}^T, \end{cases} \quad (1.1)$$

where for some $T \in (-\infty, \infty]$, $Q^T = (-\infty, T) \times \Omega$ is a cylinder with the base $\Omega \subset \mathbb{R}^d$ being either bounded or unbounded. The lateral boundary of Q^T is decomposed into two non-intersecting components \mathcal{D}^T and \mathcal{N}^T on which we impose two different types of boundary conditions. We consider both cases when $\mathcal{D}^T, \mathcal{N}^T$, and their interfacial boundary (separation) Γ^T are cylindrical and non-cylindrical.

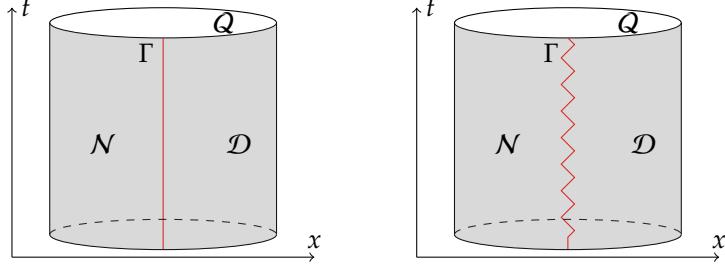


FIGURE 1. Domains with cylindrical and non-cylindrical Γ

We also consider the equations in a finite height cylinder, in which case, we impose the zero initial condition at $t = S$ for some finite $S < T$. For the parabolic operator \mathcal{P} and the conormal derivative operator \mathcal{B}

$$\begin{aligned} \mathcal{P}u &= -u_t + D_i(a^{ij}D_ju + a^i u) + b^i D_i u + cu, \\ \mathcal{B}u &= (a^{ij}D_ju + a^i u)n_i, \end{aligned}$$

where $n = (n_1, \dots, n_d)$ is the outward unit normal to $\partial\Omega$, we always assume that the leading coefficients (a^{ij}) are symmetric, and that there exists $\Lambda \in (0, 1]$ satisfying

$$a^{ij}(t, x)\xi_j\xi_i \geq \Lambda|\xi|^2, \quad |a^{ij}(t, x)| \leq \Lambda^{-1}$$

for any $\xi \in \mathbb{R}^d$ and $(t, x) \in \mathbb{R} \times \mathbb{R}^d$. We also always assume

$$|a^i| + |b^i| + |c| \leq K$$

for some $K > 0$.

Elliptic and parabolic equations with mixed boundary conditions arise naturally in physics and material science. For example, when a block of floating ice is melting into liquid water, the ice-water interface maintains zero temperature (Dirichlet) while the ice-air interface is insulated (Neumann). Such problem also has applications in the combustion theory. See, for example, [27, 28]. We also refer to [21] for an application in modelling exocytosis, which is a form of active transport mechanism. It is worth mentioning that the mixed problems for the heat or Laplace equations are commonly called Zaremba problem in the literature.

In contrast to the purely Dirichlet or conormal boundary value problem, solutions to mixed boundary value problems can be non-smooth near the separation Γ even if the domain, coefficients, and boundary data are all smooth. In the literature, elliptic equations with mixed boundary conditions have been studied quite extensively both from the PDE perspective and harmonic analysis point of view. We refer the reader to [33, 29, 3, 31, 19, 17] and [35, 2, 4, 7, 10] and the references therein. In these papers, regularity of solutions in Hölder, Sobolev, and Besov spaces as well as the non-tangential maximal function estimates were obtained. We also refer the reader to [1, 12] for results about the Kato square root problem for elliptic equations and systems with mixed boundary conditions and their applications.

However, there are relatively few papers dealing with parabolic equations with mixed boundary conditions. In [34], Skubachevskii and Shamin established the existence, uniqueness, and stability of solutions to parabolic equations in a cylindrical domain with time-independent separation Γ under minimal regularity assumptions on initial data, by using a semigroup approach and properties of difference operators in L_2 -based Sobolev spaces. Hieber and Rehberg [20] studied systems of reaction-diffusion equations with mixed Dirichlet-Neumann boundary conditions in 2D and 3D cylindrical Lipschitz domains satisfying Gröger's regular condition. Assuming an implicit topological isomorphism condition on the second-order operator, they proved the maximal L_p estimates by using harmonic analysis and a heat-kernel method. See also [18] for a further result about equations with nonhomogeneous boundary data as well as an earlier result in [16] about the elliptic mixed boundary value problem. We also mention the work [32, 23] for parabolic equations in non-cylindrical domains. In [32] Savaré considered parabolic equations in a non-cylindrical domain with $C^{1,1}$ boundary and separation Γ . Under certain condition on the excess of Γ with respect to t , he introduced an approximation approach to general abstract evolution equations in L_2 -based Sobolev spaces and obtained optimal regularity under quite weak assumptions on the data. Recently in [23], Kim and Cao studied linear and semilinear parabolic equations in non-cylindrical domains with the mixed Dirichlet, Neumann, and Robin conditions and Lipschitz leading coefficients. They considered smooth domains which are C^1 in t and C^2 in x and general \mathcal{D}^T and \mathcal{N}^T , and proved the solvability in L_2 -based Sobolev spaces. In [11], a maximal regularity result was established for parabolic equations with time irregular coefficients and mixed boundary conditions when the exponents are in a neighborhood of 2. See also [14] for an optimal L^p maximal regularity result when $p \leq 2$ and the coefficients are BV in the x variable. We remark that in all these work, either p is assumed to be 2 or an implicit condition is imposed on the operator, so that p needs to be sufficiently close to 2.

In this paper, we consider parabolic equations in a cylindrical domain with rough boundary and separation and with vanishing mean oscillation (VMO) coefficients. Let $m = 0, \dots, d-2$ be an integer. Our main result, Theorem 2.4, reads that if $\partial\Omega$ is Reifenberg-flat and the separation Γ is locally sufficiently close to a Lipschitz function of m variables with respect to the Hausdorff distance, then for any $p \in \left(\frac{2(m+2)}{m+3}, \frac{2(m+2)}{m+1}\right)$, $f, g_i \in L_p(Q^T)$, and sufficiently large λ , there is a unique solution $u \in \mathcal{H}_p^1(Q^T)$ to (1.1). See Section 2.1 for the definitions of function spaces. In the special case when Γ is Reifenberg-flat of co-dimension 2, i.e., $m = 0$, we get the solvability when $p \in (4/3, 4)$, which is an optimal range in view of the known results for the Laplace equation in [33] and for elliptic equations in [7].

Let us give a brief description of the proof. In the first step, we show the L_2 solvability by approximating Γ by piecewise time-independent separations and Ω by extension domains, and solve the approximating equations using the Galerkin method. Note that one cannot directly test the equation with u because the usual Steklov average argument is not applicable here. The second step is to derive an estimate of the form

$$\|Du\|_{L_p^{x'} L_\infty^{t,x''}(Q_{1/2}^+)} \leq C \|Du\|_{L_1(Q_1^+)}, \quad \forall p \in \left[2, \frac{2(m+2)}{m+1}\right),$$

where u is a weak solution to a homogeneous equation with constant coefficients in the half cylinder $Q_1^+ = \{(t, x) : t \in (-1, 0), |x| < 1, x^1 > 0\}$. It satisfies the mixed homogeneous boundary conditions on the flat boundary $\{x^1 = 0\}$ with a time-independent separation Γ defined by a $\{x^2 = \phi(x^3, \dots, x^{m+2})\}$, where ϕ is a Lipschitz function, $x' = (x^1, \dots, x^{m+2})$ and $x'' = (x^{m+3}, \dots, x^d)$. This will be achieved by a boundary Caccioppoli type inequality and the corresponding elliptic estimate obtained in [10], which in turn is a consequence of the Besov type estimate established in [31, 4]. Since $\partial\Omega$ and Γ are not smooth, the usual flattening boundary argument does not work in our case. To approximate the domain by domains with flat boundaries and separations, the third step in the proof is to apply the cutoff and reflection argument first used in [8, 9] for equations with the pure Dirichlet or conormal boundary condition. Such argument was also used recently in [6, 10] for elliptic equations with mixed boundary conditions. Here an additional difficulty is the extra u_t term which appears on the right-hand side after the reflection. For this, we use a delicate multi-step decomposition procedure. Finally, we complete the proof of Theorem 2.4 by utilizing a level set argument introduced by Caffarelli and Peral [5] and the “crawling of ink spots” lemma due to Krylov and Safonov [30, 25].

The rest of the paper is organized as follows. In Section 2, we introduce the notation and function spaces, and then give our main result, Theorem 2.4 for the solvability of (1.1) in Sobolev spaces. In Section 3, we prove several auxiliary results including the L_2 solvability, Poincaré and embedding inequalities suitable to our problem, and a reverse Hölder’s inequality for weak solutions to the mixed boundary value problem. Section 4 is devoted to the boundary estimates for equations with constant coefficients near a curved boundary or a flat boundary. In Section 5, we give the proof of Theorem 2.4.

2. NOTATION AND MAIN RESULTS

2.1. Notation. Throughout the paper, we always assume that Ω is a domain (open and connected, but not necessarily bounded) in \mathbb{R}^d and $Q = (-\infty, \infty) \times \Omega$ is an infinitely long cylinder, which is a subset of

$$\mathbb{R}^{d+1} = \{X = (t, x) : t \in \mathbb{R}, x = (x^1, \dots, x^d) \in \mathbb{R}^d\}.$$

We also assume that the boundary of Q , denoted by $\partial Q = (-\infty, \infty) \times \partial\Omega$, is divided into two disjoint portions \mathcal{D} and \mathcal{N} , separated by Γ . More precisely, let $\mathcal{D} \subset \partial Q$ be an open set (relative to ∂Q) and

$$\mathcal{N} = \partial Q \setminus \mathcal{D}, \quad \Gamma = \overline{\mathcal{D}} \cap \overline{\mathcal{N}}.$$

Note that the separation Γ between \mathcal{D} and \mathcal{N} is time-dependent unless explicitly specified otherwise. In Section 2.2, we will impose certain regularity assumptions on $\partial\Omega$ and Γ .

For $T \in (-\infty, \infty]$, we define

$$\mathcal{Q}^T = \{X \in \mathcal{Q} : t < T\}$$

and similarly define \mathcal{D}^T , \mathcal{N}^T , and Γ^T . For $R > 0$, we set

$$\begin{aligned} B_R(x) &= \{y \in \mathbb{R}^d : |x - y| < R\}, \quad \Omega_R(x) = \Omega \cap B_R(x), \\ Q_R(X) &= (t - R^2, t) \times B_R(x), \quad Q_R(X) = \mathcal{Q} \cap Q_R(X), \\ \mathbb{Q}_R(X) &= (t - R^2, t + R^2) \times B_R(x). \end{aligned}$$

We use the abbreviations $B_R = B_R(0)$ and $\Omega_R = \Omega_R(0)$, etc. For a function f on \mathcal{Q} , we denote

$$(f)_{\mathcal{Q}} = \frac{1}{|\mathcal{Q}|} \int_{\mathcal{Q}} f dX = \int_{\mathcal{Q}} f dX.$$

Now we introduce function spaces and the notion of weak solutions to mixed boundary value problems. To start with, we use the unified notation for cylinders with finite or infinite height: for given S, T with $-\infty \leq S < T \leq \infty$, we set

$$\begin{aligned} \tilde{\mathcal{Q}} &= \{X \in \mathcal{Q} : S < t < T\}, \\ \tilde{\mathcal{D}} &= \{X \in \mathcal{D} : S < t < T\}, \\ \tilde{\mathcal{N}} &= \{X \in \mathcal{N} : S < t < T\}. \end{aligned} \tag{2.1}$$

We write $X = (t, x) = (t, x', x'') \in \mathbb{R}^{d+1}$, where $x' \in \mathbb{R}^{d_1}$ and $x'' \in \mathbb{R}^{d_2}$, $d_1 + d_2 = d$. For $p, q \in [1, \infty)$, we define $L_q^{(t, x'')} L_p^{x'}(\tilde{\mathcal{Q}})$ to be the set of all functions u such that

$$\|u\|_{L_q^{(t, x'')} L_p^{x'}(\tilde{\mathcal{Q}})} := \left(\int_{\mathbb{R}^{d_2+1}} \left(\int_{\mathbb{R}^{d_1}} |u|^p \mathbb{I}_{\tilde{\mathcal{Q}}} dx' \right)^{q/p} dx'' dt \right)^{1/q} < \infty,$$

where $\mathbb{I}_{\tilde{\mathcal{Q}}}$ is the usual characteristic function. Similarly, we define $L_q^{(t, x'')} L_p^{x'}(\tilde{\mathcal{Q}})$ with $p = \infty$ or $q = \infty$, and $L_p^{x'} L_q^{(t, x'')}(\tilde{\mathcal{Q}})$. We abbreviate

$$L_q^t L_p^x(\tilde{\mathcal{Q}}) = L_{q,p}(\tilde{\mathcal{Q}}), \quad L_{p,p}(\tilde{\mathcal{Q}}) = L_p(\tilde{\mathcal{Q}}).$$

Let $C_{\tilde{\mathcal{D}}}^\infty(\tilde{\mathcal{Q}})$ be the set of all infinitely differentiable functions on \mathbb{R}^{d+1} having a compact support in $[S, T] \times \overline{\Omega}$ and vanishing in a neighborhood of $\tilde{\mathcal{D}}$. We denote by $W_{q,p,\tilde{\mathcal{D}}}^{0,1}(\tilde{\mathcal{Q}})$ and $W_{q,p,\tilde{\mathcal{D}}}^{1,1}(\tilde{\mathcal{Q}})$ the closures of $C_{\tilde{\mathcal{D}}}^\infty(\tilde{\mathcal{Q}})$ in $W_{q,p}^{0,1}(\tilde{\mathcal{Q}})$ and $W_{q,p}^{1,1}(\tilde{\mathcal{Q}})$, respectively, where

$$W_{q,p}^{0,1}(\tilde{\mathcal{Q}}) = \{u : u, Du \in L_{q,p}(\tilde{\mathcal{Q}})\}, \quad W_{q,p}^{1,1}(\tilde{\mathcal{Q}}) = \{u : u, Du, u_t \in L_{q,p}(\tilde{\mathcal{Q}})\}.$$

We write $W_{p,p}^{0,1}(\tilde{\mathcal{Q}}) = W_p^{0,1}(\tilde{\mathcal{Q}})$, etc.

By $u \in \mathbb{H}_{p,\tilde{\mathcal{D}}}^{-1}(\tilde{\mathcal{Q}})$ we mean that there exist $g = (g_1, \dots, g_d) \in L_p(\tilde{\mathcal{Q}})^d$ and $f \in L_p(\tilde{\mathcal{Q}})$ such that

$$u = D_i g_i + f \quad \text{in } \tilde{\mathcal{Q}}, \quad g_i n_i = 0 \quad \text{on } \tilde{\mathcal{N}},$$

where $n = (n_1, \dots, n_d)$ is the outward unit normal to $\partial\Omega$, in the distribution sense and the norm

$$\|u\|_{\mathbb{H}_{p,\tilde{\mathcal{D}}}^{-1}(\tilde{\mathcal{Q}})} = \inf \left\{ \|g\|_{L_p(\tilde{\mathcal{Q}})} + \|f\|_{L_p(\tilde{\mathcal{Q}})} : u = D_i g_i + f \text{ in } \tilde{\mathcal{Q}}, g_i n_i = 0 \text{ on } \tilde{\mathcal{N}} \right\}$$

is finite. We set

$$\mathcal{H}_{p,\tilde{\mathcal{D}}}^1(\tilde{Q}) = \{u : u \in W_{p,\tilde{\mathcal{D}}}^{0,1}(\tilde{Q}), u_t \in \mathbb{H}_{p,\tilde{\mathcal{D}}}^{-1}(\tilde{Q})\}$$

equipped with a norm

$$\|u\|_{\mathcal{H}_{p,\tilde{\mathcal{D}}}^1(\tilde{Q})} = \|u\|_{W_{p,\tilde{\mathcal{D}}}^{0,1}(\tilde{Q})} + \|u_t\|_{\mathbb{H}_{p,\tilde{\mathcal{D}}}^{-1}(\tilde{Q})}.$$

When \tilde{Q} has a finite height (i.e., $-\infty < S < T < \infty$), we define $V_2(\tilde{Q})$ as the set of all functions in $W_2^{0,1}(\tilde{Q})$ having the following finite norm

$$\|u\|_{V_2(\tilde{Q})} = \operatorname{ess\,sup}_{S < t < T} \|u(t, \cdot)\|_{L_2(\Omega)} + \|Du\|_{L_2(\tilde{Q})}.$$

When \tilde{Q} has an infinite height, we define $V_2(\tilde{Q})$ as the set of $u \in V_2(\tilde{Q}) \cap \{(t, x) : |t| < T_0\}$ for all $T_0 > 0$ having the finite norm $\|u\|_{V_2(\tilde{Q})}$. We then denote by $V_{2,\tilde{\mathcal{D}}}(\tilde{Q})$ the closure of $W_{2,\tilde{\mathcal{D}}}^{0,1}(\tilde{Q})$ in $V_2(\tilde{Q})$.

We write $u \in L_{q,p,\text{loc}}(\tilde{Q})$ if $\eta u \in L_{q,p}(\tilde{Q})$ for any infinitely differentiable function η on \mathbb{R}^{d+1} having a compact support, and similarly define $\mathcal{H}_{p,\tilde{\mathcal{D}},\text{loc}}^1(\tilde{Q})$, etc.

Definition 2.1. We say that $u \in \mathcal{H}_{p,\tilde{\mathcal{D}},\text{loc}}^1(\tilde{Q})$ satisfies the mixed boundary value problem

$$\begin{cases} \mathcal{P}u = D_i g_i + f & \text{in } \tilde{Q}, \\ \mathcal{B}u = g_i n_i & \text{on } \tilde{\mathcal{N}}, \\ u = 0 & \text{on } \tilde{\mathcal{D}}, \end{cases} \quad (2.2)$$

where $g = (g_1, \dots, g_d) \in L_{p,\text{loc}}(\tilde{Q})^d$ and $f \in L_{p,\text{loc}}(\tilde{Q})$ if

$$\int_{\tilde{Q}} u \varphi_t \, dX + \int_{\tilde{Q}} (-a^{ij} D_j u D_i \varphi - a^i u D_i \varphi + b^i D_i u \varphi + c u \varphi) \, dX = \int_{\tilde{Q}} (-g_i D_i \varphi + f \varphi) \, dX$$

for all $\varphi \in C_{\tilde{\mathcal{D}}}^\infty(\tilde{Q})$ that vanishes for $t = S$ and T . For $S > -\infty$, we also say that $u \in \mathcal{H}_{p,\tilde{\mathcal{D}},\text{loc}}^1(\tilde{Q})$ satisfies (2.2) with the initial condition $u(S, \cdot) = \psi$ on Ω , where $\psi \in L_{p,\text{loc}}(\Omega)$ if

$$\begin{aligned} & \int_{\tilde{Q}} u \varphi_t \, dX + \int_{\tilde{Q}} (-a^{ij} D_j u D_i \varphi - a^i u D_i \varphi + b^i D_i u \varphi + c u \varphi) \, dX \\ &= - \int_{\tilde{Q}} (g_i D_i \varphi - f \varphi) \, dX - \int_{\Omega} \psi \varphi(S, \cdot) \, dx \end{aligned} \quad (2.3)$$

for all $\varphi \in C_{\tilde{\mathcal{D}}}^\infty(\tilde{Q})$ that vanishes for $t = T$.

2.2. Main result. We impose the following regularity assumptions on the domain and the leading coefficients. The first one is the so-called Reifenberg flat conditions on $\partial\Omega$ and Γ .

Assumption 2.2 (γ, m, M). Let $m \in \{0, 1, \dots, d-2\}$ and $M \in (0, \infty)$.

(a) For any $x_0 \in \partial\Omega$ and $R \in (0, R_0]$, there is a coordinate system depending on x_0 and R such that in this coordinate system, we have

$$\{y : y^1 > x_0^1 + \gamma R\} \cap B_R(x_0) \subset \Omega_R(x_0) \subset \{y : y^1 > x_0^1 - \gamma R\} \cap B_R(x_0). \quad (2.4)$$

(b) For any $X_0 = (t_0, x_0) \in \Gamma$ and $R \in (0, R_0]$, there exist a spatial coordinate system and a Lipschitz function ϕ of m variables with Lipschitz constant M , such that in the new coordinate system (called the coordinate system associated with (X_0, R)), we have (2.4),

$$(\partial Q \cap Q_R(X_0) \cap \{(s, y) : y^2 > \phi(y^3, \dots, y^{m+2}) + \gamma R\}) \subset \mathcal{D},$$

$$(\partial Q \cap Q_R(X_0) \cap \{(s, y) : y^2 < \phi(y^3, \dots, y^{m+2}) - \gamma R\}) \subset \mathcal{N},$$

and

$$\phi(x_0^3, \dots, x_0^{m+2}) = x_0^2.$$

Here, if $m = 0$, then the function ϕ is understood as the constant function $\phi \equiv x_0^2$.

The second is the small BMO condition on the leading coefficients.

Assumption 2.3 (θ). For any $X_0 \in \bar{Q}$ and $R \in (0, R_0]$, we have

$$\int_{Q_R(X_0)} |a^{ij}(X) - (a^{ij})_{Q_R(X_0)}| dX \leq \theta.$$

Our theorems deal with cylinders with infinite or finite height. We rewrite (1.1) in the unified form

$$\begin{cases} \mathcal{P}u - \lambda u = D_i g_i + f & \text{in } \tilde{Q}, \\ \mathcal{B}u = g_i n_i & \text{on } \tilde{\mathcal{N}}, \\ u = 0 & \text{on } \tilde{\mathcal{D}}, \end{cases} \quad (2.5)$$

where \tilde{Q} , $\tilde{\mathcal{N}}$, and $\tilde{\mathcal{D}}$ are defined in (2.1), and $-\infty \leq S < T \leq \infty$. The main theorem of the paper is the following solvability result for (2.5) in Sobolev spaces.

Theorem 2.4. Let $R_0 \in (0, 1]$, $m \in \{0, 1, \dots, d-2\}$, $M \in (0, \infty)$, and $p \in \left(\frac{2(m+2)}{m+3}, \frac{2(m+2)}{m+1}\right)$. There exist constants $\gamma, \theta \in (0, 1)$ and $\lambda_0 \in (0, \infty)$ with

$$(\gamma, \theta) = (\gamma, \theta)(d, \Lambda, M, p), \quad \lambda_0 = \lambda_0(d, \Lambda, M, p, K, R_0),$$

such that if Assumptions 2.2 ($\gamma; m, M$) and 2.3 (θ) are satisfied with these γ and θ , then the following assertions hold.

(a) When $S = -\infty$, for any $\lambda \geq \lambda_0$, $g = (g_1, \dots, g_d) \in L_p(\tilde{Q})^d$, and $f \in L_p(\tilde{Q})$, there exists a unique solution $u \in \mathcal{H}_{p, \tilde{\mathcal{D}}}^1(\tilde{Q})$ to (2.5), which satisfies

$$\|Du\|_{L_p(\tilde{Q})} + \lambda^{1/2} \|u\|_{L_p(\tilde{Q})} \leq C \|g\|_{L_p(\tilde{Q})} + C \lambda^{-1/2} \|f\|_{L_p(\tilde{Q})}, \quad (2.6)$$

where $C = C(d, \Lambda, M, p)$.

(b) For the initial boundary value problem on a cylindrical domain of finite height, we can take $\lambda = 0$, i.e., when $S = 0$ and $T \in (0, \infty)$, for any $g = (g_1, \dots, g_d) \in L_p(\tilde{Q})^d$ and $f \in L_p(\tilde{Q})$, there exists a unique solution $u \in \mathcal{H}_{p, \tilde{\mathcal{D}}}^1(\tilde{Q})$ to (2.5) with $\lambda = 0$ and the initial condition $u(0, \cdot) \equiv 0$ on Ω . Moreover, we have

$$\|u\|_{\mathcal{H}_{p, \tilde{\mathcal{D}}}^1(\tilde{Q})} \leq C \|g\|_{L_p(\tilde{Q})} + C \|f\|_{L_p(\tilde{Q})},$$

where $C = C(d, \Lambda, M, p, K, R_0, T)$.

3. AUXILIARY RESULTS

Throughout this paper, we use the following notation.

Notation 3.1. For nonnegative (variable) quantities A and B , we denote $A \lesssim B$ if there exists a generic positive constant C such that $A \leq CB$. We add subscript letters like $A \lesssim_{a,b} B$ to indicate the dependence of the implicit constant C on the parameters a and b .

3.1. L_2 estimate. In this subsection, we prove the solvability of the mixed boundary value problem in $V_{2,\tilde{\mathcal{D}}}(\tilde{Q})$ under the following assumption that the boundary portions \mathcal{D} and \mathcal{N} vary continuously in t , which is weaker than the condition (b) in Assumption 2.2 ($\gamma; m, M$); see Lemma A.1.

Assumption 3.2. For any $\varepsilon > 0$ and $L > 0$, there exist a time partition

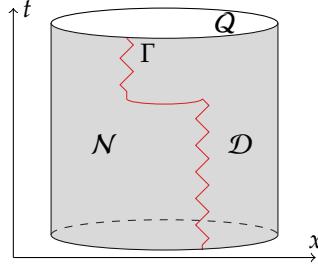
$$\max\{S, -L\} = t_0 < t_1 < \dots < t_n = \min\{T, L\}$$

and decompositions $\partial\Omega = D^{t_k} \cup N^{t_k}$, $k \in \{1, \dots, n\}$, such that

$$\mathcal{D}(t) \subset D^{t_k}, \quad H^d(\mathcal{D}(t), D^{t_k}) < \varepsilon, \quad \forall t \in [t_{k-1}, t_k], \quad (3.1)$$

where $\mathcal{D}(t) = \{x \in \partial\Omega : (t, x) \in \mathcal{D}\}$ and H^d is the usual d -dimensional Hausdorff distance.

Note that Assumption 3.2 excludes the following possibility:



In the proposition below, we do not impose any regularity assumptions on $a^{ij} = a^{ij}(t, x)$. It generalizes the classical solvability result in [26] by allowing a rough Ω and a time-varying Γ .

Proposition 3.3. *Let Ω be a bounded or unbounded domain in \mathbb{R}^d . Under Assumption 3.2, there exists $\lambda_0 \geq 0$ depending only on d, Λ , and K such that the following assertions hold. For any $\lambda \geq \lambda_0$, $g = (g_1, \dots, g_d) \in L_2(\tilde{Q})^d$, $f \in L_2(\tilde{Q})$, and $\psi \in L_2(\Omega)$, there exists a unique $u \in V_{2,\tilde{\mathcal{D}}}(\tilde{Q})$ satisfying*

$$\begin{cases} \mathcal{P}u - \lambda u = D_i g_i + f & \text{in } \tilde{Q}, \\ \mathcal{B}u = g_i n_i & \text{on } \tilde{\mathcal{N}}, \\ u = 0 & \text{on } \tilde{\mathcal{D}}, \\ u = \psi & \text{on } \{S\} \times \Omega \quad \text{if } S > -\infty. \end{cases} \quad (3.2)$$

Moreover, we have

$$\|Du\|_{L_2(\tilde{Q})} + \lambda^{1/2}\|u\|_{L_2(\tilde{Q})} \lesssim_{d,\Lambda} \|g\|_{L_2(\tilde{Q})} + \lambda^{-1/2}\|f\|_{L_2(\tilde{Q})} + \|\psi\|_{L_2(\Omega)} \quad (3.3)$$

and

$$\|u\|_{L_{\infty,2}(\tilde{Q})} \leq \|\psi\|_{L_2(\Omega)} + C(d, \Lambda) \left(\|g\|_{L_2(\tilde{Q})} + \lambda^{-1/2} \|f\|_{L_2(\tilde{Q})} \right). \quad (3.4)$$

Here the initial condition $u = \psi$ and the term $\|\psi\|_{L_2(\Omega)}$ in the estimates only appear when $S > -\infty$.

The rest of this subsection is devoted to the proof of Proposition 3.3. In the following, we call a domain $\Omega \subset \mathbb{R}^d$ an extension domain if it is bounded and admits an bounded extension $W_2^1(\Omega) \rightarrow W_2^1(\mathbb{R}^d)$. In particular, the compact embedding $W_{2,D}^1(\Omega) \hookrightarrow L_2(\Omega)$ holds. The following lemma should be classical, which we state explicitly for completeness.

Lemma 3.4. *Let Ω be an extension domain in \mathbb{R}^d and $D \subset \partial\Omega$. Then there exists an orthogonal basis $\{w_i\}$ for $W_{2,D}^1(\Omega)$ satisfying*

$$\int_{\Omega} w_i w_j \, dx = \delta_{ij},$$

where δ_{ij} is the Kronecker delta symbol.

Proof. The lemma follows from the same argument used in [13, §6.5], by finding Laplacian eigenfunctions with the help of the compact embedding $W_2^1(\Omega) \hookrightarrow L_2(\Omega)$ and the spectral theory for compact operators. We omit the details. ■

Using Lemma 3.4 and the Galerkin method, we obtain the following L_2 solvability of the mixed problem (3.2) with a cylindrical separation.

Lemma 3.5. *Proposition 3.3 holds when $\tilde{\mathcal{D}}$ and $\tilde{\mathcal{N}}$ are time-independent.*

Proof. Approximating g and f by L_2 functions with compact support in time, it suffices to consider the case when $-\infty < S < T < \infty$. When the spatial domain Ω is a bounded and regular (i.e., an extension domain), the lemma follows from the standard Galerkin method together with Lemma 3.4. The energy inequalities (3.3) and (3.4) are standard, with constants independent of the spatial domain. See, for example, [26, §4, Chapter III].

When Ω is unbounded or irregular, we take a sequence of expanding smooth domains $\Omega^k \subset \Omega \cap B_k$ such that for any $x \in (\partial\Omega^k) \cap B_k$, $\text{dist}(x, \partial\Omega) < 1/k$, i.e., $\Omega^k \nearrow \Omega$ as $k \rightarrow \infty$. Set

$$N^k = \{x \in \partial\Omega^k : \text{dist}(x, N) < 1/k\},$$

where $N = \{x \in \partial\Omega : (t, x) \in \tilde{\mathcal{N}}\}$. We also set

$$Q^k = (S, T) \times \Omega^k, \quad N^k = (S, T) \times N^k, \quad D^k = (S, T) \times (\partial\Omega^k \setminus N^k).$$

We may assume that k is sufficiently large so that D^k and N^k are not empty. Now we solve the mixed problem in Q^k with zero Dirichlet boundary condition on D^k , homogeneous conormal boundary condition on N^k , and initial data ψ . Since Ω^k is an extension domain, the solution $u^k \in V_{2,D^k}(Q^k)$ exists and is uniformly bounded. We extend u^k to be zero, still denoted by u^k , on

$$\tilde{Q}^k := (S, T) \times \{x \in \Omega \setminus \Omega^k : \text{dist}(x, N) > 2/k\}.$$

Then u^k is in $V_2(Q^k \cup \tilde{Q}^k)$ and vanishes on

$$\{(t, x) \in \tilde{\mathcal{D}} : \text{dist}(x, N) > 2/k\}.$$

By a diagonal argument, we can pick a subsequence, again denoted by u^k , such that as $k \rightarrow \infty$,

$$\begin{aligned} u^k &\rightharpoonup u, \quad Du^k \rightharpoonup Du \quad \text{weakly in } L_2((S, T) \times K), \\ u^k &\xrightarrow{*} u \quad \text{weakly* in } L_{\infty, 2}((S, T) \times K) \end{aligned}$$

for any compact set $K \subset \Omega$. Due to the estimates (3.3) and (3.4), which are uniform with respect to k , by taking the limit in the weak formulation and using Hölder's inequality, we see that $u \in V_{2, \tilde{\mathcal{D}}}(\tilde{Q})$ satisfies (3.2) as well as the estimates (3.3) and (3.4). The lemma is proved. \blacksquare

We are ready to prove Proposition 3.3.

Proof of Proposition 3.3. Again, by the standard approximation argument, it suffices to consider the case when $-\infty < S < T < \infty$. Let $\varepsilon > 0$ be given. By Assumption 3.2, there exists a time partition

$$S = t_0 < t_1 < \cdots < t_n = T$$

and decompositions $\partial\Omega = D^{t_k} \cup N^{t_k}$, $k \in \{1, 2, \dots, n\}$, such that (3.1) holds. From Lemma 3.5, for $k \in \{1, 2, \dots, n\}$, there exists a unique $u_\varepsilon^k \in V_2^{0,1}((t_{k-1}, t_k) \times \Omega)$ satisfying

$$\begin{cases} \mathcal{P}u_\varepsilon^k - \lambda u_\varepsilon^k = D_i g_i + f & \text{in } (t_{k-1}, t_k) \times \Omega, \\ \mathcal{B}u_\varepsilon^k = g_i n_i & \text{on } (t_{k-1}, t_k) \times N^{t_k}, \\ u_\varepsilon^k = 0 & \text{on } (t_{k-1}, t_k) \times D^{t_k}, \\ u_\varepsilon^k = u_\varepsilon^{k-1} & \text{on } \{t_{k-1}\} \times \Omega, \end{cases} \quad (3.5)$$

where $u_\varepsilon^0 = \psi$, with the estimates

$$\begin{aligned} &\|Du_\varepsilon^k\|_{L_2((t_{k-1}, t_k) \times \Omega)} + \lambda^{1/2} \|u_\varepsilon^k\|_{L_2((t_{k-1}, t_k) \times \Omega)} \\ &\lesssim_{d, \Lambda} \|g\|_{L_2((t_{k-1}, t_k) \times \Omega)} + \lambda^{-1/2} \|f\|_{L_2((t_{k-1}, t_k) \times \Omega)} + \|u_\varepsilon^{k-1}(t_{k-1}, \cdot)\|_{L_2(\Omega)} \end{aligned}$$

and

$$\begin{aligned} &\|u_\varepsilon^k(t, \cdot)\|_{L_{\infty, 2}((t_{k-1}, t_k) \times \Omega)} \\ &\leq \|u_\varepsilon^{k-1}(t_{k-1}, \cdot)\|_{L_2(\Omega)} + C(d, \Lambda) (\|g\|_{L_2((t_{k-1}, t_k) \times \Omega)} + \lambda^{-1/2} \|f\|_{L_2((t_{k-1}, t_k) \times \Omega)}). \end{aligned}$$

We then see that the function u_ε defined by

$$u_\varepsilon = \sum_{k=1}^n u_\varepsilon^k \mathbb{1}_{(t_{k-1}, t_k) \times \Omega} \quad \text{in } \tilde{Q} = (S, T) \times \Omega$$

belongs to $V_{2, \tilde{\mathcal{D}}}(\tilde{Q})$ and that the estimates (3.3) and (3.4) hold with u_ε in place of u . Thus from the weak compactness and Alaoglu's theorems, there exist a subsequence of $\{u_\varepsilon\}$, denoted by $\{u_{\varepsilon_i}\}$, and a function $u \in V_{2, \tilde{\mathcal{D}}}(\tilde{Q})$ such that

$$\begin{aligned} u_{\varepsilon_i} &\rightharpoonup u, \quad Du_{\varepsilon_i} \rightharpoonup Du \quad \text{weakly in } L_2(\tilde{Q}), \\ u_{\varepsilon_i} &\xrightarrow{*} u \quad \text{weakly* in } L_{\infty, 2}(\tilde{Q}). \end{aligned}$$

Moreover, u satisfies (3.3) and (3.4).

To prove that the limit function u satisfies (3.2), we let $\varphi \in C_{\tilde{\mathcal{D}}}^\infty(\tilde{Q})$ vanishing for $t = T$. Since φ vanishes in a neighborhood of $\tilde{\mathcal{D}}$ and has the compact support, we see that

$$\text{dist}(\text{supp } \varphi, \tilde{\mathcal{D}}) > 0.$$

Hence, for sufficiently large i , $\text{dist}(\text{supp } \varphi, \tilde{\mathcal{D}}) > \varepsilon_i$, which in turn implies that φ vanishes in a neighborhood of

$$\bigcup_{k=1}^{n_i} (t_{k-1}^{(i)}, t_k^{(i)}) \times D^{t_k^{(i)}},$$

where $t_k^{(i)}$ and $D^{t_k^{(i)}}$ are given in Assumption 3.2. By testing (3.5) with φ , we have

$$\begin{aligned} & - \int_{\Omega} u_{\varepsilon_i}^k(t_k^{(i)}, \cdot) \varphi(t_k^{(i)}, \cdot) dx + \int_{t_{k-1}^{(i)}}^{t_k^{(i)}} \int_{\Omega} u_{\varepsilon_i}^k \varphi_t dx dt \\ & + \int_{t_{k-1}^{(i)}}^{t_k^{(i)}} \int_{\Omega} (-a^{ij} D_j u_{\varepsilon_i}^k D_i \varphi - a^i u_{\varepsilon_i}^k D_i \varphi + b^i D_i u_{\varepsilon_i}^k \varphi + (c - \lambda) u_{\varepsilon_i}^k \varphi) dx dt \\ & = - \int_{t_{k-1}^{(i)}}^{t_k^{(i)}} \int_{\Omega} (g_i D_i \varphi - f \varphi) dx dt - \int_{\Omega} u_{\varepsilon_i}^{k-1}(t_{k-1}^{(i)}, \cdot) \varphi(t_{k-1}^{(i)}, \cdot) dx. \end{aligned}$$

Thus, by taking summation with respect to k , and then sending $i \rightarrow \infty$, we see that u satisfies (2.3). The proposition is proved. \blacksquare

3.2. Poincaré and embedding inequalities. In this subsection, we present some inequalities suitable to our problem. The first lemma is a Sobolev-Poincaré inequality on small Reifenberg flat domains, proved in [6].

Lemma 3.6 ([6, Theorem 3.5]). *Let $p \in (1, d)$ and Ω be a domain in \mathbb{R}^d satisfying Assumption 2.2 (a) with $\gamma \in (0, 1/48]$, and let $x_0 \in \partial\Omega$ and $R \in (0, R_0/4]$. Then for any $u \in W_p^1(\Omega_{2R}(x_0))$, we have*

$$\|u - (u)_{\Omega_R(x_0)}\|_{L_{dp/(d-p)}(\Omega_R(x_0))} \lesssim_{d,p} \|Du\|_{L_p(\Omega_{2R}(x_0))}.$$

When $u = 0$ on a surface ball, we can remove the average from the left-hand side.

Lemma 3.7. *Let Ω be a domain in \mathbb{R}^d satisfying Assumption 2.2 (a) with $\gamma \in (0, 1/48]$. Let $x_0 \in \partial\Omega$, $R \in (0, R_0/4]$, and $u \in W_p^1(\Omega_{2R}(x_0))$. If there exist some $z_0 \in \partial\Omega \cap B_R(x_0)$ and $\alpha \in (0, 1)$ such that*

$$B_{\alpha R}(z_0) \subset B_R(x_0), \quad u = 0 \text{ on } \partial\Omega \cap B_{\alpha R}(z_0),$$

then

$$\|u\|_{L_{dp/(d-p)}(\Omega_R(x_0))} \lesssim_{d,p,\alpha} \|Du\|_{L_p(\Omega_{2R}(x_0))}, \quad (3.6)$$

provided that $p \in (1, d)$, and

$$\|u\|_{L_p(\Omega_R(x_0))} \lesssim_{d,p,\alpha} R \|Du\|_{L_p(\Omega_{2R}(x_0))}, \quad (3.7)$$

provided that $p \in (1, \infty)$. The same results hold for $x_0 \in \Omega$ and $R \in (0, R_0/8]$ with $\Omega_{5R}(x_0)$ in place of $\Omega_{2R}(x_0)$.

Proof. The estimate (3.7) is a simple consequence of Hölder's inequality and (3.6), the proof of which is the same as that of [7, Corollary 3.2 (a)]. Here the chain of inclusions

$$\Omega_R(x_0) \subset \Omega_{2R}(z_0) \subset \Omega_{4R}(z_0) \subset \Omega_{5R}(x_0)$$

is also used when $x_0 \in \Omega$. \blacksquare

We have the following parabolic Sobolev-Poincaré inequalities with mixed norms, the proof of which is based on the embedding results in [22, Lemmas 5.3 and 5.4] along with Sobolev-Poincaré inequalities for each x and t variables. We present the proof in Appendix A. Note that the usual zero extension technique as in [7, Corollary 3.2] does not work for the parabolic case, since u_t might not be in $\mathbb{H}_{q,p}^{-1}$ after the extension.

Lemma 3.8. *Let Ω be a domain in \mathbb{R}^d satisfying Assumption 2.2 (a) with $\gamma \in (0, 1/48]$, and let $X_0 \in \bar{\Omega}$ and $R \in (0, R_0/4]$ such that either*

$$Q_{2R}(X_0) \subset \Omega \quad \text{or} \quad X_0 \in \partial\Omega.$$

If $p, q \in [1, \infty]$, $p_0 \in [p, \infty]$, $q_0 \in (q, \infty]$, and

$$\frac{d}{p} + \frac{2}{q} < 1 + \frac{d}{p_0} + \frac{2}{q_0},$$

then for $u \in W_{q,p}^{0,1}(Q_{2R}(X_0))$ satisfying

$$u_t = D_i g_i \quad \text{in } Q_{2R}(X_0)$$

in the distribution sense, where $g = (g_1, \dots, g_d) \in L_{q,p}(Q_{2R}(X_0))^d$, we have

$$\begin{aligned} \|u - (u)_{Q_R(X_0)}\|_{L_{q_0,p_0}(Q_R(X_0))} \\ \lesssim_{d,p,q,p_0,q_0} R^{1+d/p_0+2/q_0-d/p-2/q} (\|Du\|_{L_{q,p}(Q_{2R}(X_0))} + \|g\|_{L_{q,p}(Q_{2R}(X_0))}). \end{aligned} \quad (3.8)$$

If we further assume that there exist $Y_0 \in \partial\Omega$ and $\alpha \in (0, 1)$ such that

$$u = 0 \quad \text{on } Q_{\alpha R}(Y_0) \cap \partial\Omega, \quad Q_{\alpha R}(Y_0) \subset Q_R(X_0),$$

then we have

$$\|u\|_{L_{q_0,p_0}(Q_R(X_0))} \lesssim_{d,p,q,p_0,q_0,\alpha} R^{1+d/p_0+2/q_0-d/p-2/q} (\|Du\|_{L_{q,p}(Q_{2R}(X_0))} + \|g\|_{L_{q,p}(Q_{2R}(X_0))}). \quad (3.9)$$

To end this subsection, we prove the following inequality which will be frequently used in our cut-off argument.

Lemma 3.9. *Let $p \in (1, \infty)$ and Assumption 2.2 ($\gamma; m, M$) be satisfied with $\gamma \in \left(0, \frac{1}{160\sqrt{d+3}}\right]$, and let $X_0 \in \Gamma$, $R \in (0, R_0]$, and $\rho \in [R/8, R]$. Then for any $u \in W_p^{0,1}(Q_\rho(X_0))$ vanishing on $\mathcal{D} \cap Q_\rho(X_0)$, we have*

$$\|u\mathbb{I}_A\|_{L_p(Q_{\rho/2}(X_0))} \lesssim_{d,M,p} \gamma \rho \|Du\|_{L_p(Q_\rho(X_0))},$$

where

$$A = \{(s, y) : y^1 < x_0^1 + 2\gamma R, y^2 > \phi - 2\gamma R\}$$

in the coordinate system associated with (X_0, R) .

Proof. By translation we may assume that $X_0 = (0, 0)$. Fix the coordinate system associated with the origin and R , and we denote by \mathcal{D}_{grid} the set of all grid points $z = (\gamma R, k\gamma R)$, where $k = (k_2, \dots, k_d) \in \mathbb{Z}^{d-1}$, such that

$$z \in \Omega_{\rho/2}, \quad \Omega_{\sqrt{d+3}\gamma R}(z) \cap \{x : x^2 > \phi\} \neq \emptyset.$$

By Lemma 3.7 applied to $\Omega_{2\sqrt{d+3}\gamma R}(z)$, we have

$$\|u(t, \cdot)\|_{L_p(\Omega_{2\sqrt{d+3}\gamma R}(z))} \lesssim_{d,M,p} \gamma R \|Du(t, \cdot)\|_{L_p(\Omega_{10\sqrt{d+3}\gamma R}(z))}.$$

Since

$$(\Omega_{\rho/2} \cap A) \subset \bigcup_{z \in \mathcal{D}_{grid}} \Omega_{2\sqrt{d+3}\gamma R}(z) \subset \bigcup_{z \in \mathcal{D}_{grid}} \Omega_{10\sqrt{d+3}\gamma R}(z) \subset \Omega_\rho,$$

we have that

$$\|u(t, \cdot)\|_{L_p(\Omega_{\rho/2})} \lesssim \gamma R \|Du(t, \cdot)\|_{L_p(\Omega_\rho)},$$

from which we get the desired estimate. \blacksquare

3.3. Localization and reverse Hölder's inequality. In this subsection, we assume that the lower-order coefficients of \mathcal{P} are all zero, i.e.,

$$\mathcal{P}u = -u_t + D_i(a^{ij}D_j u).$$

We do not impose any regularity assumptions on a^{ij} . Hereafter in this paper, we always assume that $T \in (-\infty, \infty]$.

We first localize the estimate in Proposition 3.3.

Lemma 3.10. *Let $\Omega \subset \mathbb{R}^d$ and Assumption 3.2 be satisfied. If $u \in \mathcal{H}_{2,\mathcal{D}^T,loc}^1(Q^T)$ satisfies*

$$\begin{cases} \mathcal{P}u = D_i g_i & \text{in } Q^T, \\ \mathcal{B}u = g_i n_i & \text{on } \mathcal{N}^T, \\ u = 0 & \text{on } \mathcal{D}^T, \end{cases} \quad (3.10)$$

where $g = (g_1, \dots, g_d) \in L_{2,loc}(Q^T)^d$, then for any $X_0 \in \overline{Q^T}$ and $R \in (0, 1]$, we have

$$\|Du\|_{L_2(Q_{R/2}(X_0))} \lesssim_{d,\Lambda} R^{-1} \|u\|_{L_2(Q_R(X_0))} + \|g\|_{L_2(Q_R(X_0))}.$$

Proof. We give a proof which also works when the L_2 norms are replaced by the L_p norms provided that the corresponding global estimate is available. By translation we may assume that $X_0 = (0, 0)$. Let

$$R_k = R(1 - 2^{-k}), \quad k \in \{1, 2, \dots\}$$

and η_k be an infinitely differentiable function on \mathbb{R}^{d+1} such that

$$\begin{aligned} 0 \leq \eta_k \leq 1, \quad \eta_k \equiv 1 \text{ on } \mathbb{Q}_{R_k}, \quad \text{supp } \eta_k \subset \mathbb{Q}_{R_{k+1}}, \\ |(\eta_k)_t| + |D\eta_k|^2 \lesssim R^{-2} 2^{2k}. \end{aligned}$$

Then $\eta_k u \in \mathcal{H}_{2,\mathcal{D}^0}^1(Q^0)$ satisfies

$$\begin{cases} \mathcal{P}(\eta_k u) - \lambda_k(\eta_k u) = D_i g_i^k + g^k & \text{in } Q^0, \\ \mathcal{B}(\eta_k u) = g_i^k n_i & \text{on } \mathcal{N}^0, \\ \eta_k u = 0 & \text{on } \mathcal{D}^0, \end{cases} \quad (3.11)$$

where $\lambda_k \geq \lambda_0$, $\lambda_0 = \lambda_0(d, \Lambda) \geq 0$ is from Proposition 3.3, and

$$\begin{aligned} g_i^k &= \eta_k g_i + \sum_{j=1}^d a^{ij} D_j \eta_k u, \\ g^k &= - \sum_{i=1}^d D_i \eta_k g_i - (\eta_k)_t u + \sum_{i,j=1}^d a^{ij} D_j u D_i \eta_k - \lambda_k(\eta_k u). \end{aligned}$$

By (3.3) applied to (3.11), we have

$$\begin{aligned} \|D(\eta_k u)\|_{L_2(Q^0)} + \sqrt{\lambda_k} \|\eta_k u\|_{L_2(Q^0)} &\lesssim \left(\frac{2^k}{R} + \frac{2^{2k}}{R^2 \sqrt{\lambda_k}} + \sqrt{\lambda_k} \right) \|u\|_{L_2(Q_R)} \\ &+ \left(1 + \frac{2^k}{R \sqrt{\lambda_k}} \right) \|g\|_{L_2(Q_R)} + \frac{2^k}{R \sqrt{\lambda_k}} \|D(\eta_{k+1} u)\|_{L_2(Q_0)}. \end{aligned}$$

Set

$$\mathcal{U}_k = \|D(\eta_k u)\|_{L_2(Q^0)}, \quad \mathcal{U} = \|u\|_{L_2(Q_R)}, \quad \mathcal{F} = \|g\|_{L_2(Q_R)}.$$

By multiplying both sides of the above inequality by ε^k and summing the terms respect to k , we obtain

$$\begin{aligned} \sum_{k=1}^{\infty} \varepsilon^k \mathcal{U}_k &\leq C \mathcal{U} \sum_{k=1}^{\infty} \varepsilon^k \left(\frac{2^k}{R} + \frac{2^{2k}}{R^2 \sqrt{\lambda_k}} + \sqrt{\lambda_k} \right) \\ &+ C \mathcal{F} \sum_{k=1}^{\infty} \varepsilon^k \left(1 + \frac{2^k}{R \sqrt{\lambda_k}} \right) + C_0 \sum_{k=1}^{\infty} \frac{\varepsilon^k 2^k}{R \sqrt{\lambda_k}} \mathcal{U}_{k+1}, \end{aligned}$$

where we may assume $C_0 \geq (\lambda_0 + 1)^{1/2}$, and each summation is finite upon choosing

$$\varepsilon = 1/4, \quad \lambda_k = \left(\frac{C_0 2^{k+2}}{R} \right)^2.$$

Indeed we have

$$\begin{aligned} \sum_{k=1}^{\infty} \varepsilon^k \left(\frac{2^k}{R} + \frac{2^{2k}}{R^2 \sqrt{\lambda_k}} + \sqrt{\lambda_k} \right) &\leq \frac{C}{R}, \\ \sum_{k=1}^{\infty} \varepsilon^k \left(1 + \frac{2^k}{R \sqrt{\lambda_k}} \right) &\leq C, \\ C_0 \sum_{k=1}^{\infty} \frac{\varepsilon^k 2^k}{R \sqrt{\lambda_k}} \mathcal{U}_{k+1} &= \sum_{k=1}^{\infty} \varepsilon^{k+1} \mathcal{U}_{k+1} = \sum_{k=2}^{\infty} \varepsilon^k \mathcal{U}_k. \end{aligned}$$

Therefore,

$$\sum_{k=1}^{\infty} \varepsilon^k \mathcal{U}_k \leq \frac{C}{R} \mathcal{U} + C \mathcal{F} + \sum_{k=2}^{\infty} \varepsilon^k \mathcal{U}_k,$$

which implies the desired estimate. \blacksquare

Lemma 3.11. *Let $\alpha \in (0, 1)$, $q_1 \in \left(\frac{2(d+2)}{d+4}, 2 \right)$, and $\gamma \in \left(0, \frac{1}{48} \right]$. If Assumptions 2.2 (a) and 3.2 are satisfied, and if for any $X \in \Gamma$ and $\rho \in (0, R_0/4]$, there exists $Y \in \partial Q$ such that*

$$Q_{\alpha\rho}(Y) \cap \partial Q \subset \mathcal{D}, \quad Q_{\alpha\rho}(Y) \subset Q_\rho(X), \quad (3.12)$$

then the following assertion holds. Let $u \in \mathcal{H}_{2,\mathcal{D}^T,\text{loc}}^1(Q^T)$ satisfy (3.10) with $g = (g_1, \dots, g_d) \in L_{2,\text{loc}}(Q^T)^d$. Then for any $X_0 \in \overline{Q^T}$ and $R \in (0, R_0]$ satisfying either

$$Q_{R/2}(X_0) \subset Q \quad \text{or} \quad X_0 \in \partial Q,$$

we have

$$(|Du|^2)_{Q_{R/16}(X_0)}^{1/2} \lesssim_{d,\Lambda,\alpha,q_1} (|Du|^{q_1})_{Q_R(X_0)}^{1/q_1} + (|g|^2)_{Q_R(X_0)}^{1/2}.$$

Proof. By translation, we may assume that $X_0 = (0, 0)$. In the case when $Q_{R/2} \subset Q$, by Lemma 3.10 applied to $u - (u)_{Q_{R/4}}$, we have

$$(|Du|^2)_{Q_{R/8}}^{1/2} \lesssim R^{-1}(|u - (u)_{Q_{R/4}}|^2)_{Q_{R/4}}^{1/2} + (|g|^2)_{Q_{R/4}}^{1/2},$$

from which together with (3.8), we get the desired estimate. In the case when $X_0 \in \partial Q$, we consider the following two cases:

$$Q_{R/4} \cap \Gamma \neq \emptyset, \quad Q_{R/4} \cap \Gamma = \emptyset.$$

(i) $Q_{R/4} \cap \Gamma \neq \emptyset$. Due to Lemma 3.10, it suffices to show that

$$R^{-1}(|u|^2)_{Q_{R/2}}^{1/2} \lesssim (|Du|_{Q_R}^{q_1})_{Q_R}^{1/q_1} + (|g|^2)_{Q_R}^{1/2}. \quad (3.13)$$

We aim to apply Lemma 3.8. For this, we first fix some $X \in Q_{R/4} \cap \Gamma$. By the assumption, we can find some $Y_0 \in \partial\Omega$ satisfying (3.12) with $\rho = R/4$. Noting u vanishes on $Q_{\alpha R/4}(Y_0) \subset Q_{R/4}(X) \subset Q_{R/2}$, by (3.9) with $R/2$ in place of R , we conclude (3.13).

(ii) $Q_{R/4} \cap \Gamma = \emptyset$. In this case, we have either

$$(Q_{R/4} \cap \partial Q) \subset \mathcal{D} \quad \text{or} \quad (Q_{R/4} \cap \partial Q) \subset \mathcal{N},$$

where the proof of the first case is the same as in (i) and the second case is the same as the interior case.

The lemma is proved. \blacksquare

From Lemma 3.11 along with Gehring's lemma and Agmon's idea, we conclude the following reverse Hölder's inequality. For a given constant $\lambda > 0$ and functions u, f , and $g = (g_1, \dots, g_d)$, we write

$$U = |Du| + \sqrt{\lambda}|u|, \quad F = |g| + \frac{|f|}{\sqrt{\lambda}}.$$

We also denote for a function v defined on a domain Q that

$$\bar{v} = v \mathbb{I}_Q.$$

Lemma 3.12. *Let $\alpha \in (0, 1)$, $p > 2$, and $\gamma \in (0, \frac{1}{48}]$. If Assumptions 2.2 (a) and 3.2 are satisfied, and if for any $X \in \Gamma$ and $\rho \in (0, R_0/4]$, there exists $Y \in \partial Q$ such that (3.12) holds, then we have the following. Let $u \in \mathcal{H}_{2, \mathcal{D}^T, \text{loc}}^1(Q^T)$ satisfy*

$$\begin{cases} \mathcal{P}u - \lambda u = D_i g_i + f & \text{in } Q^T, \\ \mathcal{B}u = g_i n_i & \text{on } \mathcal{N}^T, \\ u = 0 & \text{on } \mathcal{D}^T, \end{cases} \quad (3.14)$$

where $\lambda > 0$, $g = (g_1, \dots, g_d) \in L_{p, \text{loc}}(Q^T)^d$, and $f \in L_{p, \text{loc}}(Q^T)$. There exists a constant $p_0 \in (2, p)$ depending only on d, Λ, α , and p , such that for any $X_0 \in \mathbb{R}^{d+1}$ and $R \in (0, R_0]$, we have

$$(\bar{U}^{p_0})_{Q_{R/2}(X_0)}^{1/p_0} \lesssim_{d, \Lambda, \alpha, p} (\bar{U}^2)_{Q_R(X_0)}^{1/2} + (\bar{F}^{p_0})_{Q_R(X_0)}^{1/p_0}.$$

The same result holds with $|Du|$ and $|g|$ in place of U and F , respectively, provided that $\lambda = 0$ and $f \equiv 0$.

Proof. We first prove the lemma for $\lambda = 0$ and $f \equiv 0$. More precisely, we show that there exists $p_1 \in (2, p]$, depending only on d , Λ , α , and p , such that for any $p_0 \in (2, p_1]$, $X_0 \in \mathbb{R}^{d+1}$, and $R \in (0, R_0]$, we have

$$\left(\overline{|Du|^{p_0}}\right)_{Q_{R/2}(X_0)}^{1/p_0} \lesssim_{d,\Lambda,\alpha,p} \left(\overline{|Du|^2}\right)_{Q_R(X_0)}^{1/2} + \left(\overline{|g|^{p_0}}\right)_{Q_R(X_0)}^{1/p_0}. \quad (3.15)$$

Fix a number $q_1 \in \left(\frac{2(d+2)}{d+4}, 2\right)$. Then by Lemma 3.11 with a covering argument, we have that for $X_0 \in \mathbb{R}^{d+1}$ and $R \in (0, R_0]$,

$$\int_{Q_{R/2}(X_0)} \Phi^{2/q_1} dX \lesssim_{d,\Lambda,\alpha} \left(\int_{Q_R(X_0)} \Phi dX \right)^{2/q_1} + \int_{Q_R(X_0)} \Psi^{2/q_1} dX,$$

where $\Phi = \overline{|Du|^{q_1}}$ and $\Psi = \overline{|g|^{q_1}}$. Thus by Gehring's lemma (see, for instance, [15, Ch.V]), we get the desired result (3.15).

For the case when $\lambda > 0$, we use an idea by S. Agmon. Denote by $(t, z) = (t, x, \tau)$ a point in \mathbb{R}^{d+2} , where $z = (x, \tau) \in \mathbb{R}^{d+1}$. We define

$$\hat{u}(t, z) = u(t, x)\eta(\tau), \quad \eta(\tau) = \cos(\sqrt{\lambda}\tau + \pi/4),$$

$$\hat{Q} = \{(t, z) : t \in \mathbb{R}, z \in \Omega \times \mathbb{R}\}, \quad \hat{Q}^T = \{(t, z) \in \hat{Q} : t < T\},$$

$$\hat{\mathcal{D}} = \{(t, z) : t \in \mathbb{R}, z \in \mathcal{D} \times \mathbb{R}\}, \quad \hat{\mathcal{D}}^T = \{(t, z) \in \hat{\mathcal{D}} : t < T\},$$

$$\hat{\mathcal{N}} = \{(t, z) : t \in \mathbb{R}, z \in \mathcal{N} \times \mathbb{R}\}, \quad \hat{\mathcal{N}}^T = \{(t, z) \in \hat{\mathcal{N}} : t < T\},$$

$$\hat{Q}_r(t_0, z_0) = (t_0 - r^2, t_0) \times \{z \in \mathbb{R}^{d+1} : |z - z_0| < r\}.$$

Since u satisfies (3.14), we see that $\hat{u} \in \mathcal{H}_{2,\hat{\mathcal{D}}^T, \text{loc}}^1(\hat{Q}^T)$ satisfies

$$\begin{cases} \mathcal{P}\hat{u} + D_{\tau\tau}\hat{u} = \sum_{i=1}^d D_i(g_i\eta) + f\eta & \text{in } \hat{Q}^T, \\ \mathcal{B}\hat{u} = \sum_{i=1}^d g_i\eta n_i & \text{on } \hat{\mathcal{N}}^T, \\ \hat{u} = 0 & \text{on } \hat{\mathcal{D}}^T. \end{cases} \quad (3.16)$$

Since $n_\tau = 0$, the operator \mathcal{B} is also the conormal derivative operator associated with $\mathcal{P} + D_{\tau\tau}$. Note that the coefficients of $\mathcal{P} + D_{\tau\tau}$ satisfy the ellipticity condition with the same constant Λ , and that \hat{Q} , $\hat{\mathcal{D}}$, and $\hat{\mathcal{N}}$ satisfy the same conditions of the lemma. Moreover, we can rewrite the source term in the divergence form since

$$f\eta = D_\tau(f\hat{\eta}), \quad \text{where } \hat{\eta}(\tau) = \frac{\sin(\sqrt{\lambda}\tau + \pi/4)}{\sqrt{\lambda}}.$$

Hence, we can apply the above result for $\lambda = 0$ to (3.16) to see that there exists $p_2 \in (2, p]$ determined by $d + 1$, Λ , α , and p , such that for any $p_0 \in (2, p_2]$, $(t_0, z_0) \in \mathbb{R}^{d+2}$, and $R \in (0, R_0]$,

$$\begin{aligned} & \left(\overline{|D_z \hat{u}|^{p_0}}\right)_{\hat{Q}_{R/2}(t_0, z_0)}^{1/p_0} \\ & \lesssim \left(\overline{|D_z \hat{u}|^2}\right)_{\hat{Q}_R(t_0, z_0)}^{1/2} + \left(\overline{|g_i \eta|^{p_0}}\right)_{\hat{Q}_R(t_0, z_0)}^{1/p_0} + \left(\overline{|f \hat{\eta}|^{p_0}}\right)_{\hat{Q}_R(t_0, z_0)}^{1/p_0} \\ & \lesssim_{d,\Lambda,M,p} \left(\overline{|D_z \hat{u}|^2}\right)_{\hat{Q}_R(t_0, z_0)}^{1/2} + \left(\overline{F}^{p_0}\right)_{Q_R(t_0, z_0)}^{1/p_0}. \end{aligned} \quad (3.17)$$

Now we fix p_0 such that $2 < p_0 \leq \min\{p_1, p_2\}$, where p_1 is the constant from the case $\lambda = 0$. Then by (3.17) with $z_0 = (x_0, 0) \in \mathbb{R}^{d+1}$ and the fact that

$$\int_{R/4}^{-R/4} |\eta|^{p_0} d\tau \gtrsim_{p_0} 1, \quad |\overline{Du}(t, x)\eta(\tau)| \leq |\overline{D_z}\hat{u}(t, z)| \leq \overline{U}(t, x),$$

we have

$$(|\overline{Du}|^{p_0})_{Q_{R/4}(t_0, x_0)}^{1/p_0} \lesssim (|\overline{D_z}\hat{u}|^{p_0})_{\hat{Q}_{R/2}(t_0, z_0)}^{1/p_0} \lesssim (\overline{U}^2)_{Q_R(t_0, x_0)}^{1/2} + (\overline{F}^{p_0})_{Q_R(t_0, x_0)}^{1/p_0}.$$

Similarly, using the fact that

$$\int_{R/4}^{-R/4} |\eta'|^{p_0} d\tau \gtrsim_{p_0} \lambda^{p_0/2}, \quad |\overline{u}(t, x)\eta'(\tau)| \leq |\overline{D_z}\hat{u}(t, z)| \leq \overline{U}(t, x),$$

we get

$$(|\sqrt{\lambda}\overline{u}|^{p_0})_{Q_{R/4}(t_0, x_0)}^{1/p_0} \lesssim (|\overline{D_z}\hat{u}|^{p_0})_{\hat{Q}_{R/2}(t_0, z_0)}^{1/p_0} \lesssim (\overline{U}^2)_{Q_R(t_0, x_0)}^{1/2} + (\overline{F}^{p_0})_{Q_R(t_0, x_0)}^{1/p_0}.$$

Combining these together and using a covering argument, we get the desired estimate. The lemma is proved. \blacksquare

4. ESTIMATES WITH TIME-INDEPENDENT SEPARATION AND CONSTANT COEFFICIENTS

In this section, we derive various estimates when the separation is time-independent and the coefficients are all constants. Throughout the section, we consider

$$\mathcal{P}_0 u = -u_t + D_i(a_0^{ij} D_j u),$$

where the coefficients a_0^{ij} are constants. The corresponding conormal derivative of u is denoted by $\mathcal{B}_0 u$.

4.1. Estimates near curved boundary. Recall that

$$U = |Du| + \sqrt{\lambda}|u|,$$

and set

$$U_t = |Du_t| + \sqrt{\lambda}|u_t|.$$

Compared to Lemma 3.12, in the following proposition we further estimate some time derivatives. Here, Γ is assumed to be time-independent, which clearly satisfies Assumption 3.2, and \mathcal{P} is replaced with the constant-coefficient operator \mathcal{P}_0 .

Proposition 4.1. *Let $\alpha \in (0, 1)$ and $\gamma \in (0, \frac{1}{48}]$. Suppose that the spatial domain Ω satisfies Assumption 2.2 (a), the separation Γ between \mathcal{D} and \mathcal{N} is time-independent, and for any $X \in \Gamma$ and $\rho \in (0, R_0/4]$, there exists $Y \in \partial\mathcal{Q}$ such that (3.12) holds. Let $(0, 0) \in \Gamma$, $R \in (0, R_0]$, and $u \in W_2^{0,1}(Q_R)$ satisfy*

$$\begin{cases} \mathcal{P}_0 u - \lambda u = 0 & \text{in } Q_R, \\ \mathcal{B}_0 u = 0 & \text{on } Q_R \cap \mathcal{N}, \\ u = 0 & \text{on } Q_R \cap \mathcal{D}, \end{cases} \quad (4.1)$$

where $\lambda \geq 0$. Then $u_t \in W_2^{0,1}(Q_{R/2})$ and

$$R^{d+2-\frac{d+2}{p_0}} \|U\|_{L_{p_0}(Q_{R/2})} + R^{d+4-\frac{d+2}{p_0}} \|U_t\|_{L_{p_0}(Q_{R/2})} \lesssim_{d, \Lambda, \alpha} \|U\|_{L_1(Q_R)},$$

where $p_0 = p_0(d, \Lambda, \alpha) \in (2, 4)$ is from Lemma 3.12 with $g_i = f = 0$.

Remark 4.2. In the proposition above and throughout the paper, $u \in W_p^{0,1}(Q_R)$ is said to satisfy (4.1) if u vanishes on $Q_R \cap \mathcal{D}$ and

$$\int_{Q_R} u \varphi_t \, dX - \int_{Q_R} a_0^{ij} D_j u D_i \varphi \, dX - \lambda \int_{Q_R} u \varphi \, dX = 0$$

for any $\varphi \in W_{p/(p-1)}^{1,1}(Q_R)$ that vanishes on $\partial Q_R \setminus \mathcal{N}$.

The rest of this subsection is devoted to the proof of Proposition 4.1.

Lemma 4.3. *Under the same conditions in Proposition 4.1 with $\lambda = 0$, we have that for $r \in (0, R)$, $u_t \in W_2^{1,1}(Q_r)$ and*

$$(R-r) \|u_t\|_{L_2(Q_r)} + (R-r)^2 \|Du_t\|_{L_2(Q_r)} + (R-r)^3 \|u_{tt}\|_{L_2(Q_r)} \lesssim_{d,\Lambda} \|Du\|_{L_2(Q_R)}.$$

Proof. By a standard mollification technique, it suffices to prove the desired estimate under the assumption that u and Du are smooth with respect to t . For ρ, τ with $0 < \rho < \tau \leq R$, let $\eta_{\rho,\tau}$ be an infinitely differentiable function in \mathbb{R}^{d+1} such that

$$\begin{aligned} 0 \leq \eta_{\rho,\tau} \leq 1, \quad \eta_{\rho,\tau} \equiv 1 \text{ in } Q_\rho, \quad \text{supp } \eta_{\rho,\tau} \subset Q_\tau, \\ |D\eta_{\rho,\tau}|^2 + |(\eta_{\rho,\tau})_t| \lesssim_d (\tau - \rho)^{-2}. \end{aligned}$$

Since the boundary portions in (4.1) are time-independent, we can apply $\eta_{\rho,\tau}^2 u$ as a test function to get the usual Caccioppoli inequality

$$\|Du\|_{L_2(Q_\rho)} \lesssim_{d,\Lambda} (\tau - \rho)^{-1} \|u\|_{L_2(Q_\tau)}.$$

Since u_t satisfies the same equation and boundary conditions, we also have

$$\|Du_t\|_{L_2(Q_\rho)} \lesssim (\tau - \rho)^{-1} \|u_t\|_{L_2(Q_\tau)}. \quad (4.2)$$

Let

$$\rho \leq \rho_1 < \rho_2 \leq \tau, \quad \rho_0 = \frac{\rho_1 + \rho_2}{2}.$$

We test (4.1) again with $\eta_{\rho_1,\rho_0}^2 u_t$ to get

$$\begin{aligned} \int_{Q_{\rho_0}} |\eta_{\rho_1,\rho_0} u_t|^2 \, dX &= - \int_{Q_{\rho_0}} \eta_{\rho_1,\rho_0}^2 a_0^{ij} D_j u D_i u_t \, dX \\ &\quad - 2 \int_{Q_{\rho_0}} \eta_{\rho_1,\rho_0} D_i \eta_{\rho_1,\rho_0} a_0^{ij} D_j u u_t \, dX. \end{aligned}$$

By Young's inequality and (4.2) with ρ_0 and ρ_2 in place of ρ and τ , respectively, we have

$$\int_{Q_{\rho_1}} |u_t|^2 \, dX \leq \frac{C}{(\rho_2 - \rho_1)^2} \int_{Q_{\rho_2}} |Du|^2 \, dX + \varepsilon \int_{Q_{\rho_2}} |u_t|^2 \, dX$$

for $\varepsilon \in (0, 1)$, where $C = C(d, \varepsilon) > 0$. Since the above inequality holds for any ρ_1, ρ_2 with $\rho \leq \rho_1 < \rho_2 \leq \tau$, by a standard iteration argument, we get

$$\|u_t\|_{L_2(Q_\rho)} \lesssim (\tau - \rho)^{-1} \|Du\|_{L_2(Q_\tau)}. \quad (4.3)$$

From (4.2) with τ replaced by $(\rho + \tau)/2$ and (4.3) with ρ replaced by $(\rho + \tau)/2$, we obtain

$$\|Du_t\|_{L_2(Q_\rho)} \lesssim (\tau - \rho)^{-1} \|u_t\|_{L_2(Q_{(\rho+\tau)/2})} \lesssim (\tau - \rho)^{-2} \|Du\|_{L_2(Q_\tau)}. \quad (4.4)$$

Again, since we can differentiate the problem in t , the u_{tt} estimate follows from (4.3) with u replaced by u_t and (4.4), with parameters chosen properly

$$\|u_{tt}\|_{L_2(Q_\rho)} \lesssim (\tau - \rho)^{-1} \|Du_t\|_{L_2(Q_{(\rho+\tau)/2})} \lesssim (\tau - \rho)^{-3} \|Du\|_{L_2(Q_\tau)}$$

The lemma is proved. \blacksquare

We are ready to prove Proposition 4.1.

Proof of Proposition 4.1. Due to Agmon's idea (cf. the proof of Lemma 3.12), it suffices to prove the proposition for $\lambda = 0$. By a rescaled version of Lemma 3.12 and a covering argument, we see that for $0 < \rho < \tau \leq R$,

$$\|Du\|_{L_{p_0}(Q_\rho)} \lesssim (\tau - \rho)^{\frac{d+2}{p_0} - \frac{d+2}{2}} \|Du\|_{L_2(Q_\tau)}. \quad (4.5)$$

From Hölder's and Young's inequalities, we have

$$\|Du\|_{L_{p_0}(Q_\rho)} \leq \varepsilon \|Du\|_{L_{p_0}(Q_\tau)} + C\varepsilon^{-\frac{p_0}{p_0-2}} (\tau - \rho)^{(d+2)(1/p_0-1)} \|Du\|_{L_1(Q_\tau)}, \quad (4.6)$$

for any $\varepsilon > 0$, where $C = C(d, \Lambda, \alpha, \varepsilon)$. Set

$$\tau_k = \frac{3R}{4} + \frac{R}{4}(1 - 2^{-k}), \quad k \in \{0, 1, 2, \dots\}.$$

Then by (4.6) with τ_k and τ_{k+1} in place of ρ and τ , respectively, we get

$$\|Du\|_{L_{p_0}(Q_{\tau_k})} \leq \varepsilon \|Du\|_{L_{p_0}(Q_{\tau_{k+1}})} + C\varepsilon^{-\frac{p_0}{p_0-2}} R^{(d+2)(1/p_0-1)} 2^{(d+2)(1-1/p_0)k} \|Du\|_{L_1(Q_{\tau_{k+1}})}.$$

Multiplying both sides of the above inequality by ε^k and summing the terms with respect to $k = 0, 1, \dots$, we obtain that

$$\begin{aligned} \sum_{k=0}^{\infty} \varepsilon^k \|Du\|_{L_{p_0}(Q_{\tau_k})} &\leq \sum_{k=1}^{\infty} \varepsilon^k \|Du\|_{L_{p_0}(Q_{\tau_k})} \\ &\quad + C\varepsilon^{-\frac{p_0}{p_0-2}} R^{(d+2)(1/p_0-1)} \sum_{k=0}^{\infty} \left(\varepsilon 2^{(d+2)(1-1/p_0)} \right)^k \|Du\|_{L_1(Q_{\tau_{k+1}})}, \end{aligned}$$

where each summation is finite upon choosing $\varepsilon = 2^{-1-(d+2)(1-1/p_0)}$. This yields

$$\|Du\|_{L_{p_0}(Q_{3R/4})} \lesssim R^{\frac{d+2}{p_0} - (d+2)} \|Du\|_{L_1(Q_R)}. \quad (4.7)$$

Since u_t satisfies the same equation and the boundary conditions, by (4.7), Hölder's inequality, and Lemma 4.3, we have

$$\|Du_t\|_{L_{p_0}(Q_{R/2})} \lesssim R^{\frac{d+2}{p_0} - \frac{d+2}{2}} \|Du_t\|_{L_2(Q_{2R/3})} \lesssim R^{\frac{d+2}{p_0} - \frac{d+2}{2} - 2} \|Du\|_{L_2(Q_{3R/4})} \lesssim R^{\frac{d+2}{p_0} - d - 4} \|Du\|_{L_1(Q_R)}.$$

The proposition is proved. \blacksquare

4.2. Estimates near flat boundary. Now, we prove that the problem that we perturb from – the problem with constant coefficients, cylindrical separation, and flat boundary, can reach the optimal regularity $Du \in L_p^{x'} L_\infty^{(t,x'')}$. In this subsection, we additionally assume that (a_0^{ij}) is symmetric and set

$$Q_R^+ = Q_R \cap \{(t, x) : x_1 > 0\},$$

$$D = B_1 \cap \{x : x_1 = 0, x_2 > \phi\},$$

$$N = B_1 \cap \{x : x_1 = 0, x_2 < \phi\},$$

where

$$\phi = \phi(x^3, \dots, x^{m+2}) : \mathbb{R}^m \rightarrow \mathbb{R}$$

is a Lipschitz function of m variables with Lipschitz constant M and satisfying $\phi(0, \dots, 0) = 0$. Here, $m \in \{0, 1, \dots, d-2\}$ and if $m = 0$, then ϕ is understood as the constant function $\phi \equiv 0$. We write $X = (t, x) = (t, x', x'') \in \mathbb{R}^{d+1}$, where

$$x' = (x^1, \dots, x^{m+2}) \in \mathbb{R}^{m+2}, \quad x'' = (x^{m+3}, \dots, x^d) \in \mathbb{R}^{d-2-m}.$$

Proposition 4.4. *If $u \in W_2^{0,1}(Q_1^+)$ satisfy*

$$\begin{cases} \mathcal{P}_0 u - \lambda u = 0 & \text{in } Q_1^+, \\ \mathcal{B}_0 u = 0 & \text{on } (-1, 0) \times N, \\ u = 0 & \text{on } (-1, 0) \times D, \end{cases}$$

where $\lambda \geq 0$, then for $p \in [2, \frac{2(m+2)}{m+1}]$ we have that

$$U \in L_p^{x'} L_\infty^{(t, x'')}(Q_{1/2}^+)$$

and

$$\|U\|_{L_p^{x'} L_\infty^{(t, x'')}(Q_{1/2}^+)} \lesssim_{d, \Lambda, M, p} \|U\|_{L_1(Q_1^+)}.$$

The rest of this subsection is devoted to the proof of Proposition 4.4.

Lemma 4.5. *Under the same conditions in Proposition 4.4 with $\lambda = 0$, we have that for $0 < r < R \leq 1$,*

$$u \in W_2^{1,1}(Q_r^+), \quad u_t, D_k u \in W_2^{0,1}(Q_r^+), \quad k \in \{m+3, \dots, d\}$$

and

$$\|u_t\|_{L_2(Q_r^+)} + \|Du_t\|_{L_2(Q_r^+)} + \|DD_{x''} u\|_{L_2(Q_r^+)} \lesssim_{d, \Lambda, r, R} \|Du\|_{L_2(Q_R^+)}. \quad (4.8)$$

Generally, for $i \in \{0, 1, 2, \dots\}$, we have

$$\|DD_{(t, x'')}^i u\|_{L_2(Q_r^+)} \lesssim_{d, \Lambda, r, R, i} \|Du\|_{L_2(Q_R^+)}. \quad (4.9)$$

Proof. By the same argument used in the proof of Lemma 4.3, we obtain the bounds of the first two terms in (4.8). The bound of the last term in (4.8) and also the inequality (4.9) follow from the Caccioppoli inequality and the fact that members of $D_{(t, x'')}^i u$ satisfy the same equation and the boundary conditions in Q_R^+ for any $R' < R$. We omit the details. \blacksquare

We are now ready to present the proof of Proposition 4.4.

Proof of Proposition 4.4. Due to Agmon's idea, it suffices to prove the proposition for $\lambda = 0$. Also by a change of variables, we may assume that $a_0^{ij} = \delta_{ij}$.

We first claim that

$$\|Du\|_{L_p^{x'} L_\infty^{(t, x'')}(Q_{1/2}^+)} \lesssim_{d, \Lambda, M, p} \|Du\|_{L_2(Q_1^+)}. \quad (4.10)$$

For almost every (t, x'') with $-1 < t < 0$ and $|x''| < 1$, the function $v = u(t, \cdot, x'')$ satisfies

$$\begin{cases} \Delta_{x'} v = f & \text{in } B_{3/4}'^+, \\ D_1 v = 0 & \text{on } N', \\ v = 0 & \text{on } D', \end{cases}$$

where we set

$$\begin{aligned} f &= u_t(t, \cdot, x'') - \Delta_{x''} u(t, \cdot, x''), \\ B'_{3/4} &= \{x' \in \mathbb{R}^{m+2} : |x'| < 3/4\}, \quad B'^+_{3/4} = B'_{3/4} \cap \{x' \in \mathbb{R}^{m+2} : x_1 > 0\}, \end{aligned}$$

$$D' = B'_{3/4} \cap \{x' \in \mathbb{R}^{m+2} : x_1 = 0, x_2 > \phi\},$$

$$N' = B'_{3/4} \cap \{x' \in \mathbb{R}^{m+2} : x_1 = 0, x_2 < \phi\}.$$

Notice from Lemma 4.5 that $f \in L_2(B'_{3/4})$. Thus by a rescaled version of [10, Lemma 3.2] with a change of variables, we have

$$\|D_{x'} v\|_{L_p(B'_{1/2})} \lesssim_{\Lambda, M, p} \|D_{x'} v\|_{L_2(B'_{3/4})} + \|f\|_{L_2(B'_{3/4})}.$$

Taking L_2 norm in $\{(t, x'') : -1 < t < 0, |x''| < 1/2\}$ and using Lemma 4.5, we obtain

$$\|Du\|_{L_2^{(t,x'')} L_p^{x'}(Q_{1/2}^+)} \lesssim \|Du\|_{L_2(Q_1^+)},$$

and thus by Minkowski's inequality, we have

$$\|Du\|_{L_p^{x'} L_2^{(t,x'')}(Q_{1/2}^+)} \lesssim \|Du\|_{L_2(Q_1^+)}.$$

From the above inequality with scaling and the fact that the members of $D_{(t,x'')}^i u$ satisfy the same equation and the boundary condition in any smaller cylinder $Q_{R'}^+$ with $R' < 1$, we also have

$$\|DD_{(t,x'')}^i u\|_{L_p^{x'} L_2^{(t,x'')}(Q_{1/2}^+)} \lesssim_{d, \Lambda, M, p, i} \|Du\|_{L_2(Q_1^+)}$$

for $i \in \{1, 2, \dots\}$, where we also used Lemma 4.5. Therefore, by the Sobolev embedding in (t, x'') , there exists k such that

$$\|Du\|_{L_p^{x'} L_\infty^{(t,x'')}(Q_{1/2}^+)} \lesssim \sum_{i=0}^k \|DD_{(t,x'')}^i u\|_{L_p^{x'} L_2^{(t,x'')}(Q_{1/2}^+)} \lesssim \|Du\|_{L_2(Q_1^+)},$$

which implies the claim (4.10).

It is well known that (4.10) holds (with $p = \infty$) when u satisfies purely Dirichlet/conormal derivative boundary conditions. The corresponding interior estimate also holds. Hence, by scaling and a covering argument, one can see that for $0 < \rho < r \leq 1$,

$$\|Du\|_{L_p^{x'} L_\infty^{(t,x'')}(Q_\rho^+)} \lesssim (\tau - \rho)^{\frac{m+2}{p} - \frac{d+2}{2}} \|Du\|_{L_2(Q_r^+)},$$

which corresponds to (4.5). Similarly as in the proof of Proposition 4.1, we conclude that

$$\|Du\|_{L_p^{x'} L_\infty^{(t,x'')}(Q_{1/2}^+)} \lesssim \|Du\|_{L_1(Q_1^+)}.$$

This proves the proposition for $\lambda = 0$. The proposition is proved. \blacksquare

5. PROOF OF THEOREM 2.4

This section is devoted to the proof of Theorem 2.4.

5.1. Decomposition. In this subsection, we consider the operator \mathcal{P} without lower-order terms, i.e.,

$$\mathcal{P}u = -u_t + D_i(a^{ij}D_j u).$$

Recall that

$$U = |Du| + \sqrt{\lambda}|u|, \quad F = |g| + \frac{|f|}{\sqrt{\lambda}}.$$

Proposition 5.1. *Let*

$$p > 2, \quad \gamma \in \left(0, \frac{1}{160\sqrt{d+3}}\right], \quad \theta \in (0, 1),$$

and $\lambda_0 = \lambda_0(d, \Lambda)$ be the constant from Proposition 3.3. If Assumptions 2.2 ($\gamma; m, M$) and 2.3 (θ) are satisfied with these γ and θ , then the following assertion holds. Let $u \in \mathcal{H}_{2, \mathcal{D}^T, \text{loc}}^1(\mathcal{Q}^T)$ satisfy

$$\begin{cases} \mathcal{P}u - \lambda u = D_i g_i + f & \text{in } \mathcal{Q}^T, \\ \mathcal{B}u = g_i n_i & \text{on } \mathcal{N}^T, \\ u = 0 & \text{on } \mathcal{D}^T, \end{cases}$$

where $\lambda > 0$, $\lambda \geq \lambda_0$, $g = (g_1, \dots, g_d) \in L_{p, \text{loc}}(\mathcal{Q}^T)^d$, and $f \in L_{p, \text{loc}}(\mathcal{Q}^T)$. Then for any $X_0 \in \overline{\mathcal{Q}^T}$ and $R \in (0, R_0]$, there exist

$$W, V \in L_2(\mathcal{Q}_{\mu R}(X_0)),$$

where $\mu = \frac{1}{16.96}$, such that

$$U \leq W + V \quad \text{in } \mathcal{Q}_{\mu R}(X_0),$$

$$(W^2)_{\mathcal{Q}_{\mu R}(X_0)}^{1/2} \lesssim_{d, \Lambda, M, p} (\gamma + \theta)^{1/2-1/p_0} (U^{p_0})_{\mathcal{Q}_{R/2}(X_0)}^{1/p_0} + (F^2)_{\mathcal{Q}_R(X_0)}^{1/2}, \quad (5.1)$$

and for $q \in \left[1, \frac{2(m+2)}{m+1}\right)$,

$$(V^q)_{\mathcal{Q}_{\mu R}(X_0)}^{1/q} \lesssim_{d, \Lambda, M, p, q} (U^{p_0})_{\mathcal{Q}_{R/2}(X_0)}^{1/p_0} + (F^2)_{\mathcal{Q}_R(X_0)}^{1/2}, \quad (5.2)$$

where $p_0 = p_0(d, \Lambda, M, p) \in (2, p)$ is from Lemma 3.12.

Remark 5.2. In the proposition above, by applying the reverse Hölder's inequality in Lemma 3.12 to (5.1) and (5.2), we have that

$$(W^2)_{\mathcal{Q}_{\mu R}(X_0)}^{1/2} \lesssim (\gamma + \theta)^{1/2-1/p_0} (U^2)_{\mathcal{Q}_R(X_0)}^{1/2} + (F^{p_0})_{\mathcal{Q}_R(X_0)}^{1/p_0}$$

and

$$(V^q)_{\mathcal{Q}_{\mu R}(X_0)}^{1/q} \lesssim (U^2)_{\mathcal{Q}_R(X_0)}^{1/2} + (F^{p_0})_{\mathcal{Q}_R(X_0)}^{1/p_0}.$$

Proof of Proposition 5.1. Based on a covering argument and Lemma A.2, it suffices to consider the following four cases:

- (i) $\mathcal{Q}_R(X_0) \subset \mathcal{Q}$, (ii) $X_0 \in \partial\mathcal{Q}$, $\mathcal{Q}_R(X_0) \cap \partial\mathcal{Q} \subset \mathcal{D}$,
- (iii) $X_0 \in \partial\mathcal{Q}$, $\mathcal{Q}_R(X_0) \cap \partial\mathcal{Q} \subset \mathcal{N}$, (iv) $X_0 \in \Gamma$.

Note that the first three cases can be proved by reducing the proof of Case (iv), in which we need to deal with mixed Dirichlet-conormal boundary conditions. Therefore, we only present here the detailed proof of the proposition with $\mu = 1/96$ for Case (iv).

By translation we may assume that $X_0 = (0, 0)$. Fix the coordinate system associated with the origin and R satisfying the conditions in Assumption 2.2 ($\gamma; m, M$), i.e.,

$$\begin{aligned} & \{x : \gamma R < x^1\} \cap B_R \subset \Omega_R \subset \{x : -\gamma R < x^1\} \cap B_R, \\ & (\partial\mathcal{Q} \cap \mathcal{Q}_R \cap \{(t, x) : x^2 > \phi + \gamma R\}) \subset \mathcal{D}, \\ & (\partial\mathcal{Q} \cap \mathcal{Q}_R \cap \{(t, x) : x^2 < \phi - \gamma R\}) \subset \mathcal{N}. \end{aligned}$$

Let $\chi = \chi(x)$ be an infinitely differentiable function defined on \mathbb{R}^d such that

$$\begin{aligned} 0 \leq \chi \leq 1, \quad |D\chi| \lesssim_d \frac{1+M}{\gamma R}, \\ \chi = 0 \quad \text{in } \{x : x^1 < \gamma R, x^2 > \phi - \gamma R\}, \end{aligned} \quad (5.3)$$

and

$$\chi = 1 \quad \text{in } \mathbb{R}^d \setminus \{x : x^1 < 2\gamma R, x^2 > \phi - 2\gamma R\}. \quad (5.4)$$

We define

$$\mathcal{P}_0 u = -u_t + D_i(a_0^{ij} D_j u),$$

where $a_0^{ij} = (a^{ij})_{Q_R}$ is symmetric, and denote by \mathcal{B}_0 the conormal derivative operator associated with \mathcal{P}_0 . Observe that χu satisfies

$$\begin{cases} \mathcal{P}_0(\chi u) - \lambda(\chi u) = D_i g_i^* + f^* & \text{in } Q_{R/4}, \\ \mathcal{B}_0(\chi u) = g_i^* n_i & \text{on } (-(R/4)^2, 0) \times N_{R/4}, \\ u = 0 & \text{on } (-(R/4)^2, 0) \times D_{R/4}, \end{cases} \quad (5.5)$$

where

$$\begin{aligned} D_{R/4} &= \partial\Omega \cap B_{R/4} \cap \{x : x^2 > \phi - \gamma R\}, \\ N_{R/4} &= \partial\Omega \cap B_{R/4} \cap \{x : x^2 < \phi - \gamma R\}, \\ f^* &= a^{ij} D_j u D_i \chi - g_i D_i \chi + f \chi, \\ g_i^* &= (a_0^{ij} - a^{ij}) D_j(\chi u) + a^{ij} u D_j \chi + g_i \chi. \end{aligned} \quad (5.6) \quad (5.7)$$

Note that the separation between $N_{R/4}$ and $D_{R/4}$ is time-independent. We decompose

$$\chi u = u^{(1)} + u^{(2)} \quad \text{in } Q_{R/4}, \quad (5.8)$$

where $u^{(1)} \in \mathcal{H}_2^1(Q^0)$ is a unique weak solution of the problem

$$\begin{cases} \mathcal{P}_0 u^{(1)} - \lambda u^{(1)} = D_i(g_i^* \mathbb{I}_{Q_{R/4}}) + f^* \mathbb{I}_{Q_{R/4}} & \text{in } Q^0, \\ \mathcal{B}_0 u^{(1)} = (g_i^* \mathbb{I}_{Q_{R/4}}) n_i & \text{on } (-\infty, 0) \times N_{R/4}, \\ u^{(1)} = 0 & \text{on } (-\infty, 0) \times (\partial\Omega \setminus N_{R/4}). \end{cases} \quad (5.9)$$

Note that the existence of $u^{(1)}$ is due to Lemma 3.5, and that $u^{(2)} \in \mathcal{H}_2^1(Q_{R/4})$ satisfies

$$\begin{cases} \mathcal{P}_0 u^{(2)} - \lambda u^{(2)} = 0 & \text{in } Q_{R/4}, \\ \mathcal{B}_0 u^{(2)} = 0 & \text{on } (-(R/4)^2, 0) \times N_{R/4}, \\ u = 0 & \text{on } (-(R/4)^2, 0) \times D_{R/4}. \end{cases} \quad (5.10)$$

We divide the rest of the proof into several steps.

Step 1. In this first step, we estimate

$$U^{(1)} := |Du^{(1)}| + \sqrt{\lambda} |u^{(1)}|$$

and

$$U^{(2)} := |Du^{(2)}| + \sqrt{\lambda} |u^{(2)}|, \quad U_t^{(2)} := |Du_t^{(2)}| + \sqrt{\lambda} |u_t^{(2)}|,$$

where $U_t^{(2)}$ is well defined in $Q_{R/8}$ due to Proposition 4.1. Since the boundary portions in (5.9) are time-independent, we can apply $u^{(1)}$ as a test function to get

$$\int_{\Omega} |u^{(1)}(0, \cdot)|^2 dx + \int_{Q^0} a_0^{ij} D_j u^{(1)} D_i u^{(1)} dX + \lambda \int_{Q^0} |u^{(1)}|^2 dX$$

$$= \int_{Q^0} g_i^* \mathbb{I}_{Q_{R/4}} D_i u^{(1)} dX - \int_{Q'} f^* \mathbb{I}_{Q_{R/4}} u^{(1)} dX.$$

By Young's inequality, the Poincaré inequality, and (5.6)-(5.7), this implies that

$$\|U^{(1)}\|_{L_2(Q^0)} \lesssim_{d,\Lambda} \|F\|_{L_2(Q_{R/4})} + \|Du \mathbb{I}_{\text{supp } D\chi}\|_{L_2(Q_{R/4})} + \|(a_0^{ij} - a^{ij}) D(\chi u)\|_{L_2(Q_{R/4})} + \|uD\chi\|_{L_2(Q_{R/4})}.$$

Notice from Lemma 3.12 that u is in $\mathcal{H}_{p_0, \text{loc}}^1(Q^T)$. Hence, by Hölder's inequality and Lemma 3.9, we have

$$\|Du \mathbb{I}_{\text{supp } D\chi}\|_{L_2(Q_{R/4})} \lesssim (\gamma R^{d+2})^{1/2-1/p_0} \|Du\|_{L_{p_0}(Q_{R/2})}$$

and

$$\begin{aligned} \|uD\chi\|_{L_2(Q_{R/4})} &\lesssim \frac{(\gamma R^{d+2})^{1/2-1/p_0}}{\gamma R} \|u \mathbb{I}_{\text{supp } D\chi}\|_{L_{p_0}(Q_{R/4})} \\ &\lesssim (\gamma R^{d+2})^{1/2-1/p_0} \|Du\|_{L_{p_0}(Q_{R/2})}, \end{aligned} \quad (5.11)$$

which together with Assumption 2.3 (θ) yields

$$\begin{aligned} &\|(a_0^{ij} - a^{ij}) D(\chi u)\|_{L_2(Q_{R/4})} \\ &\lesssim \|a_0^{ij} - a^{ij}\|_{L_{2p_0/(p_0-2)}(Q_{R/4})} \|Du\|_{L_{p_0}(Q_{R/4})} + \|uD\chi\|_{L_2(Q_{R/4})} \\ &\lesssim (\theta + \gamma)^{1/2-1/p_0} R^{(d+2)/2} (|Du|^{p_0})_{Q_{R/2}}^{1/p_0}. \end{aligned}$$

Combining the estimates above, we reach

$$((U^{(1)})^2)_{Q_{R/4}}^{1/2} \lesssim_{d,\Lambda,M,p} (\theta + \gamma)^{1/2-1/p_0} (|Du|^{p_0})_{Q_{R/2}}^{1/p_0} + (F^2)_{Q_R}^{1/2}. \quad (5.12)$$

For the estimates of $U^{(2)}$ and $U_t^{(2)}$, we use (5.8), (5.11), and (5.12) to obtain

$$\begin{aligned} ((U^{(2)})^2)_{Q_{R/4}}^{1/2} &\lesssim ((U^{(1)})^2)_{Q_{R/4}}^{1/2} + ((\chi U)^2)_{Q_{R/4}}^{1/2} + (|uD\chi|^2)_{Q_{R/4}}^{1/2} \\ &\lesssim (U^{p_0})_{Q_{R/2}}^{1/p_0} + (F^2)_{Q_R}^{1/2}. \end{aligned}$$

Therefore, by Proposition 4.1 we get

$$((U^{(2)})^{p_0})_{Q_{R/8}}^{1/p_0} + R^2 ((U_t^{(2)})^{p_0})_{Q_{R/8}}^{1/p_0} \lesssim_{d,\Lambda,M,p} (U^{p_0})_{Q_{R/2}}^{1/p_0} + (F^2)_{Q_R}^{1/2}. \quad (5.13)$$

Step 2. In this step, we decompose $u^{(2)}$. Since $u^{(2)}$ satisfies (5.10), $\chi u^{(2)}$ satisfies

$$\begin{cases} \mathcal{P}_0(\chi u^{(2)}) - \lambda(\chi u^{(2)}) = D_i h_i + h & \text{in } Q_{R/4}^\gamma, \\ \mathcal{B}_0(\chi u^{(2)}) = h_i n_i & \text{on } (-(R/4)^2, 0) \times N_{R/4}^\gamma, \\ \chi u^{(2)} = 0 & \text{on } (-(R/4)^2, 0) \times D_{R/4}^\gamma, \end{cases} \quad (5.14)$$

where

$$\begin{aligned} Q_{R/4}^\gamma &= (-(R/4)^2, 0) \times B_{R/4}^\gamma, \quad B_{R/4}^\gamma = B_{R/4} \cap \{x : x^1 > \gamma R\}, \\ D_{R/4}^\gamma &= B_{R/4} \cap \{x : x^1 = \gamma R, x^2 > \phi - \gamma R\}, \\ N_{R/4}^\gamma &= B_{R/4} \cap \{x : x^1 = \gamma R, x^2 < \phi - \gamma R\}, \\ h(t, x) &= [a_0^{ij} D_j u^{(2)} D_i \chi](t, x) + [u_t^{(2)} \chi + a_0^{ij} D_j u^{(2)} D_i \chi + \lambda u^{(2)} \chi](t, z) \mathbb{I}_{\Omega^*}(x), \\ h_1(t, x) &= [a_0^{1j} D_j \chi u^{(2)}](t, x) + [a_0^{1j} D_j u^{(2)} \chi](t, z) \mathbb{I}_{\Omega^*}(x), \end{aligned}$$

and for $i \in \{2, \dots, d\}$,

$$h_i(t, x) = [a_0^{ij} D_j \chi u^{(2)}](t, x) - [a_0^{ij} D_j u^{(2)} \chi](t, z) \mathbb{I}_{\Omega^*}(x).$$

In the above, we denote

$$z = \mathcal{R}x = (2\gamma R - x^1, x^2, \dots, x^d)$$

to be the reflection x with respect to $\{x^1 = \gamma R\}$ and

$$\Omega^* = \{(x^1, x') : (2\gamma R - x^1, x') \in \Omega_{R/4} \setminus B_{R/4}^\gamma\} \cap B_{R/4}^\gamma.$$

Indeed, one can check that (5.14) holds as follows. Let $\varphi \in C^\infty(\overline{Q_{R/4}^\gamma})$ which vanishes on $(-(R/4)^2, 0) \times (\partial B_{R/4}^\gamma \setminus N_{R/4}^\gamma)$ and $\{t = -(R/4)^2, 0\} \times B_{R/4}^\gamma$. We extend φ to $[-(R/4)^2, 0] \times \{x : x^1 \geq \gamma R\}$ by setting $\varphi \equiv 0$ on $[-(R/4)^2, 0] \times (\{x : x^1 \geq \gamma R\} \setminus B_{R/4}^\gamma)$, and define

$$\tilde{\varphi}(t, x) = \begin{cases} \varphi(t, x) & \text{if } x^1 > \gamma R, \\ \varphi(t, 2\gamma R - x^1, x^2, \dots, x^d) & \text{otherwise.} \end{cases}$$

Since $\chi \tilde{\varphi}$ belongs to $C^\infty(\overline{Q_{R/4}^\gamma})$ and vanishes on $[-(R/4)^2, 0] \times (\partial \Omega_{R/4} \setminus N_{R/4})$, we can test (5.10) with $\chi \tilde{\varphi}$ to get

$$\int_{Q_{R/4}^\gamma} (u_t^{(2)} \chi \tilde{\varphi} + a_0^{ij} D_j u^{(2)} D_i (\chi \tilde{\varphi}) + \lambda u^{(2)} \chi \tilde{\varphi}) dX = 0.$$

From this identity and the definition of $\tilde{\varphi}$, it follows that

$$\begin{aligned} & \int_{Q_{R/4}^\gamma} ((\chi u^{(2)})_t \varphi + a_0^{ij} D_j (\chi u^{(2)}) D_i \varphi + \lambda \chi u^{(2)} \varphi) dX \\ &= \int_{Q_{R/4}^\gamma} a_0^{ij} D_j \chi u^{(2)} D_i \varphi dX - \int_{Q_{R/4} \setminus Q_{R/4}^\gamma} a_0^{ij} D_j u^{(2)} \chi D_i \varphi dX \\ & \quad - \int_{Q_{R/4}^\gamma} a_0^{ij} D_j u^{(2)} D_i \chi \varphi dX \\ & \quad - \int_{Q_{R/4} \setminus Q_{R/4}^\gamma} (u_t^{(2)} \chi + a_0^{ij} D_j u^{(2)} D_i \chi + \lambda u^{(2)} \chi) \tilde{\varphi} dX \\ &= \int_{Q_{R/4}^\gamma} (h_i D_i \varphi - h \varphi) dX, \end{aligned}$$

which is exactly the weak formulation of (5.14).

Now we decompose

$$\chi u^{(2)} = u^{(3)} + u^{(4)} \quad \text{in } Q_{R/4}^\gamma,$$

where $u^{(3)} \in \mathcal{H}_2^1((-\infty, 0) \times B_{R/4}^\gamma)$ is a unique weak solution of the problem

$$\begin{cases} \mathcal{P}_0 u^{(3)} - \lambda u^{(3)} = D_i (h_i \mathbb{I}_{Q_{R/16}^\gamma}) + h \mathbb{I}_{Q_{R/16}^\gamma} & \text{in } Q_{R/4}^\gamma, \\ \mathcal{B}_0 u^{(3)} = (h_i \mathbb{I}_{Q_{R/16}^\gamma})_i & \text{on } (-\infty, 0) \times N_{R/4}^\gamma, \\ u^{(3)} = 0 & \text{on } (-\infty, 0) \times (\partial B_{R/4}^\gamma \setminus N_{R/4}^\gamma), \end{cases} \quad (5.15)$$

and $u^{(4)} \in \mathcal{H}_2^1(Q_{R/4}^\gamma)$ satisfies

$$\begin{cases} \mathcal{P}_0 u^{(4)} - \lambda u^{(4)} = 0 & \text{in } Q_{R/16}^\gamma, \\ \mathcal{B}_0 u^{(4)} = 0 & \text{on } (-(R/16)^2, 0) \times N_{R/16}^\gamma, \\ u^{(4)} = 0 & \text{on } (-(R/16)^2, 0) \times D_{R/16}^\gamma. \end{cases} \quad (5.16)$$

Step 3. In this step, we estimate

$$U^{(3)} := (|Du^{(3)}| + \sqrt{\lambda}|u^{(3)}|) \mathbb{1}_{Q_{R/4}^\gamma}.$$

By applying $u^{(3)}$ as a test function to (5.15), we see that

$$\int_{Q_{R/4}^\gamma} (U^{(3)})^2 dX \lesssim \sum_{i=1}^6 J_i,$$

where

$$\begin{aligned} J_1 &= \int_{Q_{R/16}^\gamma} |u^{(2)} D\chi| |Du^{(3)}| dX, \\ J_2 &= \int_{Q_{R/16}^\gamma} |[Du^{(2)} \chi](t, z) \mathbb{1}_{\Omega^*(x)}| |Du^{(3)}| dX, \\ J_3 &= \int_{Q_{R/16}^\gamma} |Du^{(2)} D\chi| |u^{(3)}| dX, \\ J_4 &= \int_{Q_{R/16}^\gamma} |[u_t^{(2)} \chi](t, z) \mathbb{1}_{\Omega^*(x)}| |u^{(3)}| dX, \\ J_5 &= \int_{Q_{R/16}^\gamma} |[Du^{(2)} D\chi](t, z) \mathbb{1}_{\Omega^*(x)}| |u^{(3)}| dX, \\ J_6 &= \int_{Q_{R/16}^\gamma} |[\lambda u^{(2)} \chi](t, z) \mathbb{1}_{\Omega^*(x)}| |u^{(3)}| dX. \end{aligned}$$

Set

$$\mathcal{K} = \gamma^{1/2-1/p_0} R^{(d+2)/2} \left((U^{p_0})_{Q_{R/2}}^{1/p_0} + (F^2)_{Q_R}^{1/2} \right) \|U^{(3)}\|_{L_2(Q_{R/4}^\gamma)}.$$

Similar to (5.11), we have

$$\|u^{(2)} D\chi\|_{L_2(Q_{R/16}^\gamma)} \lesssim (\gamma R^{d+2})^{1/2-1/p_0} \|Du^{(2)}\|_{L_{p_0}(Q_{R/8})}.$$

Thus by (5.13) we get $J_1 \lesssim \mathcal{K}$. Using Hölder's inequality and (5.13), we obtain

$$\begin{aligned} J_2 &\lesssim \|Du^{(2)}\|_{L_2(Q_{R/16} \setminus Q_{R/16}^\gamma)} \|Du^{(3)}\|_{L_2(Q_{R/16}^\gamma)} \\ &\lesssim (\gamma R^{d+2})^{1/2-1/p_0} \|Du^{(2)}\|_{L_{p_0}(Q_{R/16} \setminus Q_{R/16}^\gamma)} \|Du^{(3)}\|_{L_2(Q_{R/16}^\gamma)} \lesssim \mathcal{K} \end{aligned}$$

and

$$J_6 \lesssim \sqrt{\lambda} \|u^{(2)}\|_{L_2(Q_{R/16} \setminus Q_{R/16}^\gamma)} \cdot \sqrt{\lambda} \|u^{(3)}\|_{L_2(Q_{R/16}^\gamma)} \lesssim \mathcal{K}.$$

Similarly, we obtain

$$\begin{aligned} J_4 &\lesssim \|u_t^{(2)}\|_{L_2(Q_{R/16} \setminus Q_{R/16}^\gamma)} \|u^{(3)}\|_{L_2(Q_{R/16}^\gamma)} \lesssim (\gamma R^{d+2})^{1/2-1/p_0} \|u_t^{(2)}\|_{L_{p_0}(Q_{R/16} \setminus Q_{R/16}^\gamma)} \|u^{(3)}\|_{L_2(Q_{R/16}^\gamma)} \\ &\lesssim (\gamma R^{d+2})^{1/2-1/p_0} R^2 \|Du_t^{(2)}\|_{L_{p_0}(Q_{R/8})} \|Du^{(3)}\|_{L_2(Q_{R/16}^\gamma)} \lesssim \mathcal{K}, \end{aligned}$$

where we also applied Lemma 3.8 to $u_t^{(2)}$ and the boundary Poincaré inequality on half balls to $u^{(3)}$. To estimate J_3 and J_5 , we note that by the same argument as in the proof of Lemma 3.9, we have

$$\|u^{(3)} \mathbb{1}_{\text{supp } D\chi}\|_{L_2(Q_{R/16}^\gamma)} + \|u^{(3)} \mathbb{1}_{\text{supp } (D\chi \circ \mathcal{R})}\|_{L_2(Q_{R/16}^\gamma)} \lesssim \gamma R \|Du^{(3)}\|_{L_2(Q_{R/8})}.$$

where \mathcal{R} is the reflection map: $\mathcal{R}x = (2\gamma R - x^1, x^2, \dots, x^d)$. This together with Hölder's inequality and (5.13) yields that

$$J_3 + J_5 \lesssim \|Du^{(2)}\mathbb{I}_{\text{supp } D\chi}\|_{L_2(Q_{R/16})} \cdot \frac{1}{\gamma R} \left(\|u^{(3)}\mathbb{I}_{\text{supp } D\chi}\|_{L_2(Q_{R/16}^\gamma)} + \|u^{(3)}\mathbb{I}_{\text{supp } (D\chi \circ \mathcal{R})}\|_{L_2(Q_{R/16}^\gamma)} \right) \lesssim \mathcal{K}.$$

Collecting the estimates for J_i and using Young's inequality, we conclude that

$$((U^{(3)})^2)^{1/2}_{Q_{R/4}} \lesssim \gamma^{1/2-1/p_0} \left((U^{p_0})^{1/p_0}_{Q_{R/2}} + (F^2)^{1/2}_{Q_R} \right). \quad (5.17)$$

Step 4. We are ready to complete the proof of the proposition. From the decompositions above, we have

$$u = w + v, \quad U \leq W + V \quad \text{in } Q_{R/4},$$

where

$$w = (1 - \chi)u + u^{(1)} + (1 - \chi)u^{(2)} + \chi u^{(2)}\mathbb{I}_{Q_{R/4} \setminus Q_{R/4}^\gamma} + u^{(3)}\mathbb{I}_{Q_{R/4}^\gamma}, \quad v = u^{(4)}\mathbb{I}_{Q_{R/4}^\gamma},$$

and

$$W = |Dw| + \sqrt{\lambda}|w|, \quad V = |Dv| + \sqrt{\lambda}|v|.$$

Observe that

$$W \leq (1 - \chi)U + |uD\chi| + U^{(1)} + (1 - \chi)U^{(2)} + |u^{(2)}D\chi| + |u^{(2)}\mathbb{I}_{Q_{R/4} \setminus Q_{R/4}^\gamma}| + U^{(3)},$$

where by Hölder's inequality, (5.13), and Lemma 3.9, we have

$$\begin{aligned} ((1 - \chi)^2 U^2)^{1/2}_{Q_{R/4}} &\lesssim \gamma^{1/2-1/p_0} (U^{p_0})^{1/p_0}_{Q_{R/4}}, \\ (|uD\chi|)^{1/2}_{Q_{R/4}} &\lesssim \frac{1}{\gamma R} (|u\mathbb{I}_{\text{supp } (D\chi)}|)^{1/2}_{Q_{R/4}} \lesssim \gamma^{1/2-1/p_0} (U^{p_0})^{1/p_0}_{Q_{R/2}}, \\ ((1 - \chi)^2 (U^{(2)})^2)^{1/2}_{Q_{R/8}} &\lesssim \gamma^{1/2-1/p_0} \left((U^{p_0})^{1/p_0}_{Q_{R/2}} + (F^2)^{1/2}_{Q_R} \right), \end{aligned}$$

and

$$(|u^{(2)}D\chi|^2)^{1/2}_{Q_{R/16}} + (|u^{(2)}\mathbb{I}_{Q_{R/4} \setminus Q_{R/4}^\gamma}|^2)^{1/2}_{Q_{R/16}} \lesssim \gamma^{1/2-1/p_0} \left((U^{p_0})^{1/p_0}_{Q_{R/2}} + (F^2)^{1/2}_{Q_R} \right).$$

These estimates together with (5.12) and (5.17) imply

$$(W^2)^{1/2}_{Q_{R/16}} \lesssim (\gamma + \theta)^{1/2-1/p_0} \left((U^{p_0})^{1/p_0}_{Q_{R/2}} + (F^2)^{1/2}_{Q_R} \right). \quad (5.18)$$

It remains to obtain the estimate of V . Observe that

$$\begin{aligned} B_{R/96}^\gamma &\subset B_{R/48}(y_0) \cap \{x : x^1 > \gamma R\} \\ &\subset B_{R/24}(y_0) \cap \{x : x^1 > \gamma R\} \subset B_{R/16}^\gamma, \end{aligned}$$

where $y_0 = (\gamma R, -\gamma R, 0, \dots, 0) \in \mathbb{R}^d$. Thus by Proposition 4.4 applied to (5.16), $V \leq U + W$, and (5.18), we have

$$(V^q)^{1/q}_{Q_{R/96}} \lesssim (V^2)^{1/2}_{Q_{R/16}} \lesssim (U^2)^{1/2}_{Q_{R/16}} + (W^2)^{1/2}_{Q_{R/16}} \lesssim (U^{p_0})^{1/p_0}_{Q_{R/2}} + (F^2)^{1/2}_{Q_R}.$$

The proposition is proved. ■

5.2. Level set estimates. For a function v on Q^T , we define its maximal function by

$$\mathcal{M}v(X) = \sup_{Q_r(Z) \ni X} \int_{Q_r(Z)} |v(Y)| \mathbb{I}_{Q^T} dY.$$

Let $p > 2$ and $p_0 = p_0(d, \Lambda, M, p) \in (2, p)$ be from Lemma 3.12. For any $s > 0$, denote

$$\begin{aligned} \mathcal{A}(s) &= \{X \in Q^T : (\mathcal{M}U^2(X))^{1/2} > s\}, \\ \mathcal{B}(s) &= \{X \in Q^T : (\mathcal{M}U^2(X))^{1/2} + (\gamma + \theta)^{1/p_0-1/2} (\mathcal{M}F^{p_0}(X))^{1/p_0} > s\}. \end{aligned} \quad (5.19)$$

Lemma 5.3. *Under the same conditions of Proposition 5.1, for any $q \in \left[1, \frac{2(m+2)}{m+1}\right)$, there exists a constant $C_0 > 0$, depending only on d, Λ, M, p , and q , such that the following holds. For any $X_0 = (t_0, x_0) \in \overline{Q^T}$, $R \in (0, R_0]$, $\kappa \geq 4^{(d+2)/2}$, and $s > 0$, if*

$$|Q_{\mu R/2}(X_0) \cap \mathcal{A}(\kappa s)| \geq C_0 \left(\frac{(\gamma + \theta)^{1-2/p_0}}{\kappa^2} + \frac{1}{\kappa^q} \right) \cdot |Q_{\mu R/2}(X_0)|,$$

then

$$Q_{\mu R/2}(X_0) \subset \mathcal{B}(s),$$

where $\mu = \frac{1}{16.96}$.

Proof. By dividing the equation by s and translating the coordinates, we may assume that $s = 1$ and $X_0 = (0, 0)$. We prove the contrapositive of the statement. Suppose that there is a point $Y \in Q_{\mu R/2}$ satisfying

$$Y \notin \mathcal{B}(1).$$

By the definition of $\mathcal{B}(1)$,

$$(\mathcal{M}U^2(Y))^{1/2} + (\gamma + \theta)^{1/p_0-1/2} (\mathcal{M}F^{p_0}(Y))^{1/p_0} \leq 1. \quad (5.20)$$

Set

$$t_0^* = \min\{(\mu R/2)^2, T\}, \quad X_0^* = (t_0^*, 0) \in \overline{Q^T},$$

and observe that

$$Y \in Q_{\mu R/2} \subset Q_{\mu R}(X_0^*). \quad (5.21)$$

From the definition of the maximal function and (5.20), we have

$$\left(U^2 \mathbb{I}_{Q^T} \right)_{Q_{\mu R}(X_0^*)}^{1/2} + (\gamma + \theta)^{1/p_0-1/2} \left(F^{p_0} \mathbb{I}_{Q^T} \right)_{Q_{\mu R}(X_0^*)}^{1/p_0} \leq 1.$$

Hence, by Proposition 5.1 with the estimates in Remark 5.2, there exist functions W, V defined on $Q_{\mu R}(X_0^*)$ such that

$$\begin{aligned} U &\leq W + V \quad \text{in } Q_{\mu R}(X_0^*), \\ (W^2)_{Q_{\mu R}(X_0^*)}^{1/2} &\lesssim_{d, \Lambda, M, p} (\gamma + \theta)^{1/2-1/p_0}, \quad (V^q)_{Q_{\mu R}(X_0^*)}^{1/q} \lesssim_{d, \Lambda, M, p, q} 1. \end{aligned} \quad (5.22)$$

We now claim that for any $X \in Q_{\mu R/2} \cap \mathcal{A}(\kappa)$,

$$(\mathcal{M}(W^2 \mathbb{I}_{Q_{\mu R}(X_0^*)})(X))^{1/2} + (\mathcal{M}(V^2 \mathbb{I}_{Q_{\mu R}(X_0^*)})(X))^{1/2} > \kappa. \quad (5.23)$$

By the definition of \mathcal{A} , we can find $Q_r(Z)$ satisfying

$$X \in Q_r(Z), \quad \left(U^2 \mathbb{I}_{Q^T} \right)_{Q_r(Z)}^{1/2} > \kappa. \quad (5.24)$$

We can always choose the time coordinate of Z to be at most T . Furthermore, we have $r < \mu R/2$ because otherwise, from (5.20), (5.21), and the fact that

$$Q_r(Z) \subset Q_{4r}(X_0^*),$$

we get

$$\begin{aligned} (U^2 \mathbb{I}_{Q^T})_{Q_r(Z)}^{1/2} &\leq 4^{(d+2)/2} (U^2 \mathbb{I}_{Q^T})_{Q_{4r}(X_0^*)}^{1/2} \\ &\leq 4^{(d+2)/2} (\mathcal{M} U^2(Y))^{1/2} \\ &\leq 4^{(d+2)/2} \leq \kappa, \end{aligned}$$

which contradict (5.24). Since $r < \mu R/2$, we have

$$Q_r(Z) \subset Q_{\mu R}(X_0^*),$$

and thus by (5.24),

$$\begin{aligned} \kappa &< (U^2 \mathbb{I}_{Q^T})_{Q_r(Z)}^{1/2} \leq (W^2 \mathbb{I}_{Q^T})_{Q_r(Z)}^{1/2} + (V^2 \mathbb{I}_{Q^T})_{Q_r(Z)}^{1/2} \\ &\leq (W^2 \mathbb{I}_{Q_{\mu R}(X_0^*)})_{Q_r(Z)}^{1/2} + (V^2 \mathbb{I}_{Q_{\mu R}(X_0^*)})_{Q_r(Z)}^{1/2} \\ &\leq (\mathcal{M}(W^2 \mathbb{I}_{Q_{\mu R}(X_0^*)})(X))^{1/2} + (\mathcal{M}(V^2 \mathbb{I}_{Q_{\mu R}(X_0^*)})(X))^{1/2}, \end{aligned}$$

from which we get the claim.

By (5.22), (5.23), and the Hardy-Littlewood theorem, we have

$$\begin{aligned} &|Q_{\mu R/2} \cap \mathcal{A}(\kappa)| \\ &\leq |\{X : (\mathcal{M}(W^2 \mathbb{I}_{Q_{\mu R}(X_0^*)})(X))^{1/2} + (\mathcal{M}(V^2 \mathbb{I}_{Q_{\mu R}(X_0^*)})(X))^{1/2} > \kappa\}| \\ &\leq |\{X : (\mathcal{M}(W^2 \mathbb{I}_{Q_{\mu R}(X_0^*)})(X))^{1/2} > \kappa/2\}| \\ &\quad + |\{X : (\mathcal{M}(V^2 \mathbb{I}_{Q_{\mu R}(X_0^*)})(X))^{1/2} > \kappa/2\}| \\ &\leq \frac{C}{\kappa^2} \|W\|_{L_2(Q_{\mu R}(X_0^*))}^2 + \frac{C}{\kappa^q} \|V\|_{L_q(Q_{\mu R}(X_0^*))}^q \\ &\leq C \left(\frac{(\gamma + \theta)^{1-2/p_0}}{\kappa^2} + \frac{1}{\kappa^q} \right) |Q_{\mu R}(X_0^*)| \\ &\leq C \left(\frac{(\gamma + \theta)^{1-2/p_0}}{\kappa^2} + \frac{1}{\kappa^q} \right) |Q_{\mu R/2}|, \end{aligned}$$

where $C = C(d, \Lambda, M, p, q)$. This completes the proof. \blacksquare

As a consequence of the previous lemma, we have the following regularity result for \mathcal{H}_2^1 weak solutions.

Lemma 5.4. *Let $p \in (2, \frac{2(m+2)}{m+1})$ and $\lambda_0 = \lambda_0(d, \Lambda)$ be from Proposition 3.3. There exist constants $\gamma, \theta \in (0, 1)$ depending only on d, Λ, M , and p , such that if Assumptions 2.2 ($\gamma; m, M$) and 2.3 (θ) are satisfied with these γ and θ , then the following assertions hold. Let $u \in \mathcal{H}_{2,D^T}^1(Q^T)$ satisfy*

$$\begin{cases} \mathcal{P}u - \lambda u = D_i g_i + f & \text{in } Q^T, \\ \mathcal{B}u = g_i n_i & \text{on } \mathcal{N}^T, \\ u = 0 & \text{on } \mathcal{D}^T, \end{cases}$$

where $\lambda > 0$, $\lambda \geq \lambda_0$, and $g_i, f \in L_2(Q^T) \cap L_p(Q^T)$. Then u belongs to $\mathcal{H}_{p, \mathcal{D}^T}^1(Q^T)$ and satisfies

$$\|U\|_{L_p(Q^T)} \lesssim_{d, \Lambda, M, p} R_0^{(d+2)(1/p-1/2)} \|U\|_{L_2(Q^T)} + \|F\|_{L_p(Q^T)}. \quad (5.25)$$

Moreover, if u vanishes outside $Q_{\gamma R_0}(X_0)$ for some $X_0 \in \mathbb{R}^{d+1}$, then

$$\|U\|_{L_p(Q^T)} \lesssim_{d, \Lambda, M, p} \|F\|_{L_p(Q^T)}. \quad (5.26)$$

Proof. We denote

$$q = \frac{1}{2} \left(p + \frac{2(m+2)}{m+1} \right) \in \left(p, \frac{2(m+2)}{m+1} \right).$$

Let γ , θ , and κ be positive constants to be chosen later, such that

$$\gamma \in \left(0, \frac{1}{160\sqrt{d+3}} \right], \quad \theta \in (0, 1), \quad \kappa \geq 4^{(d+2)/2},$$

and that

$$\varepsilon := C_0 \left(\frac{(\gamma + \theta)^{1-2/p_0}}{\kappa^2} + \frac{1}{\kappa^q} \right) < 1,$$

where $C_0 = C_0(d, \Lambda, M, p, q) = C_0(d, \Lambda, M, p)$ is the constant from Lemma 5.3. Recall the notation (5.19). By the Hardy-Littlewood theorem, we have

$$|\mathcal{A}(\kappa s)| \leq \frac{C_1(d)}{(\kappa s)^2} \|U\|_{L_2(Q^T)}^2. \quad (5.27)$$

Using this, Lemma 5.3, and the ‘‘crawling of ink spots’’ lemma in [30, 25], we get

$$|\mathcal{A}(\kappa s)| \leq C_2(d) \varepsilon |\mathcal{B}(s)|$$

for any $s > s_0$, where

$$s_0^2 = \frac{C_1}{\varepsilon \kappa^2 |Q_{\mu R_0/2}|} \|U\|_{L_2(Q^T)}^2.$$

Hence, for a sufficiently large $S > s_0$,

$$\begin{aligned} \int_0^{\kappa S} |\mathcal{A}(s)| s^{p-1} ds &= \kappa^p \int_0^S |\mathcal{A}(\kappa s)| s^{p-1} ds \\ &\leq \kappa^p \int_0^{s_0} |\mathcal{A}(\kappa s)| s^{p-1} ds + C_2 \varepsilon \kappa^p \int_{s_0}^S |\mathcal{B}(s)| s^{p-1} ds \\ &=: I_1 + I_2. \end{aligned}$$

By (5.27) we have

$$I_1 \lesssim_{d, p} \varepsilon^{1-p/2} R_0^{(d+2)(1-p/2)} \|U\|_{L_2(Q^T)}^p,$$

To estimate I_2 , observe that

$$\mathcal{B}(s) \subset \mathcal{A}(s/2) \cup \left\{ X \in Q^T : (\gamma + \theta)^{1/p_0-1/2} (\mathcal{M}F^{p_0}(X))^{1/p_0} > s/2 \right\},$$

from which together with the Hardy-Littlewood maximal function theorem, we obtain

$$\begin{aligned} I_2 &\leq C_2 \varepsilon \kappa^p \int_{s_0}^S |\mathcal{A}(s/2)| s^{p-1} ds + C \varepsilon \kappa^p (\gamma + \theta)^{p(1/p_0-1/2)} \|(\mathcal{M}F^{p_0})^{1/p_0}\|_{L_p(Q^T)}^p \\ &\leq C_* \varepsilon \kappa^p \int_0^{\kappa S} |\mathcal{A}(s)| s^{p-1} ds + C_* \varepsilon \kappa^p (\gamma + \theta)^{p(1/p_0-1/2)} \|F\|_{L_p(Q^T)}^p, \end{aligned}$$

where $C_* = C_*(d, p)$. Note that by (5.27),

$$\int_0^{\kappa S} |\mathcal{A}(s)|s^{p-1} ds < \infty.$$

Thus by taking κ sufficiently large, and then, choosing θ and γ sufficiently small such that

$$C_* \varepsilon \kappa^p = C_* C_0 \left(\kappa^{p-2} (\gamma + \theta)^{1-2/p_0} + \kappa^{p-q} \right) \leq \frac{1}{2},$$

we conclude that for any $S > 0$,

$$\int_0^{\kappa S} |\mathcal{A}(s)|s^{p-1} ds \lesssim_{d, \Lambda, M, p} R_0^{(d+2)(1-p/2)} \|U\|_{L_2(Q^T)}^p + \|F\|_{L_p(Q^T)}^p.$$

This shows (5.25). If we further assume that u vanishes outside $Q_{\gamma R_0}(X_0)$, then by using (5.25), Hölder's inequality, and choosing again γ sufficiently small, we conclude (5.26). The lemma is proved. \blacksquare

5.3. Proof of Theorem 2.4. Thanks to Proposition 3.3 and a duality argument, it suffices to consider the case when $p \in (2, \frac{2(m+2)}{m+1})$.

For the a priori estimate (2.6) in the assertion (a), by moving all the lower-order terms to the right-hand side of the equation, we may assume that the lower-order coefficients of \mathcal{P} are all zero. Then the a priori estimate follows from (5.26) and the standard partition of unity argument. For the solvability in the assertion (a), thanks to the a priori estimate and the method of continuity, we only need to consider the case when the lower-order terms are all zero, which follows from the regularity result in Lemma 5.4 and the a priori estimate along with the standard approximation argument. Finally, the proof of the assertion (b) is by considering the equation of $ue^{-\lambda_0 t}$; cf. [8, Theorem 8.2 (iii)]. \blacksquare

APPENDIX A.

Proof of Lemma 3.8. We may assume that $X_0 = (0, 0)$ and u is smooth with respect to t . By scaling, without loss of generality, we can also assume that $R = 1$.

We first prove (3.8). Take a function $\zeta \in C_0^\infty(\Omega_{3/2})$ such that

$$0 \leq \zeta \leq 1, \quad 1 \lesssim \int_\Omega \zeta dz, \quad |D\zeta| \lesssim 1.$$

Set

$$v(t) = \int_\Omega \zeta u(t, \cdot) dx \Big/ \int_\Omega \zeta dx, \quad c = \int_{-(3/2)^2}^0 v(t) dt.$$

By [22, Lemmas 5.3 and 5.4] with scaling applied to $u - c$, we have

$$\begin{aligned} \|u - (u)_{Q_1}\|_{L_{q_0, p_0}(Q_1)} &\leq 2\|u - c\|_{L_{q_0, p_0}(Q_1)} \lesssim_{d, p, q, p_0, q_0} \|u - c\|_{L_{q, p}(Q_{3/2})} \\ &\quad + \left(\|Du\|_{L_{q, p}(Q_{3/2})} + \|g\|_{L_{q, p}(Q_{3/2})} \right), \end{aligned} \tag{A.1}$$

where, by the triangle inequality, we obtain

$$\|u - c\|_{L_{q, p}(Q_{3/2})} \leq \|u - v\|_{L_{q, p}(Q_{3/2})} + \|v - c\|_{L_{q, p}(Q_{3/2})} =: J_1 + J_2. \tag{A.2}$$

Note that

$$\|u(t, \cdot) - v(t)\|_{L_p(\Omega_{3/2})} \lesssim \|u(t, \cdot) - (u(t, \cdot))_{\Omega_{3/2}}\|_{L_p(\Omega_{3/2})} \lesssim \|Du(t, \cdot)\|_{L_p(\Omega_2)},$$

where we used Lemma 3.6 and the interior Poincaré inequality with a covering argument in the second inequality. This implies that

$$J_1 \lesssim \|Du\|_{L_{q,p}(Q_2)}.$$

By the Sobolev-Poincaré inequality in the t variable, we get

$$J_2 \lesssim \int_{-(3/2)^2}^0 \left| \int_{\Omega} \zeta u_t \, dx \right| dt.$$

Since $u_t = D_i g_i$ and ζ has compact support in $\Omega_{3/2}$, integrating by parts with respect to x and using Hölder's inequality, we see that

$$J_2 \lesssim \|g\|_{L_{q,p}(Q_2)}.$$

Combining (A.1), (A.2), and the estimates of J_i , we conclude (3.8).

We next prove (3.9). Let $Y_0 = (s_0, y_0)$. It is easily seen that the estimates above still holds with

$$c = \int_{s_0-\alpha^2}^{s_0} v(t) dt.$$

Thus, we have

$$\|u\|_{L_{q_0,p_0}(Q_1)} \lesssim_{d,p,q,p_0,q_0} \|Du\|_{L_{q,p}(Q_2)} + \|g\|_{L_{q,p}(Q_2)} + c.$$

Therefore, it suffices to show that

$$c \lesssim \|u\|_{L_1((s_0-\alpha^2, s_0) \times \Omega_1)} \lesssim \|Du\|_{L_{q,p}(Q_2)},$$

which follows from Lemma 3.7 and Hölder's inequality. The lemma is proved. ■

In the lemmas below, we set

$$\Gamma(t) = \{x \in \partial\Omega : (t, x) \in \Gamma\}.$$

Similarly, we define $\mathcal{D}(t)$ and $\mathcal{N}(t)$.

Lemma A.1. *For $\gamma > 0$, the condition (b) in Assumption 2.2 ($\gamma; m, M$) implies Assumption 3.2.*

Proof. Let $R \in (0, R_0]$ and $t_0 \in \mathbb{R}$. It suffices to show that, under the condition (b), there exist decompositions

$$\partial\Omega = D^{t_0} \cup N^{t_0}$$

such that

$$\mathcal{D}(t) \subset D^{t_0}, \quad H^d(\mathcal{D}(t), D^{t_0}) \leq 2\gamma R, \quad \forall t \in [t_0 - R^2, t_0].$$

Indeed, this follows by defining D^{t_0} as the set of all points $x = (x^1, \dots, x^d) \in \partial\Omega$ satisfying either $x \in \mathcal{D}(t_0)$ or

$$\text{dist}(x, \Gamma(t_0)) < R \quad \text{and} \quad x^2 > \phi(x^3, \dots, x^{m+2}) - \gamma R$$

in the coordinate system associated with $((t_0, x_0), R)$, where $x_0 \in \Gamma(t_0)$ satisfies $\text{dist}(x, x_0) = \text{dist}(x, \Gamma(t_0))$. ■

Lemma A.2. *Suppose that the condition (b) in Assumption 2.2 ($\gamma; m, M$) holds with $\gamma \in [0, 1]$.*

(i) *Let $X_0 = (t_0, x_0) \in \Gamma$ and $R \in (0, R_0]$. Then for $t \in (t_0 - R^2, t_0 + R^2)$,*

$$B_R(x_0) \cap \Gamma(t) \neq \emptyset.$$

(ii) Let $X_0 = (t_0, x_0) \in \overline{Q}$ and $R \in (0, R_0]$ with

$$B_R(x_0) \cap \Gamma(t_0) = \emptyset.$$

Then

$$Q_{R/2}(X_0) \cap \Gamma = \emptyset.$$

Proof. We only prove the assertion (i) because (ii) is its easy consequence. Suppose that there exists $t \in (t_0 - R^2, t_0 + R^2)$ such that

$$B_R(x_0) \cap \Gamma(t) = \emptyset.$$

Then either

$$\{t\} \times (B_R(x_0) \cap \partial\Omega) \subset \mathcal{D}(t) \quad \text{or} \quad \{t\} \times (B_R(x_0) \cap \partial\Omega) \subset \mathcal{N}(t),$$

which contradicts with the fact that

$$\{t\} \times (B_R(x_0) \cap \partial\Omega \cap \{x : x^2 > \phi + \gamma R\}) \subset \mathcal{D}(t),$$

$$\{t\} \times (B_R(x_0) \cap \partial\Omega \cap \{x : x^2 < \phi - \gamma R\}) \subset \mathcal{N}(t),$$

and

$$\phi(x_0^3, \dots, x_0^d) = x_0^2.$$

in the coordinate system associated with (X_0, R) . The assertion (a) is proved. ■

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