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KINETIC FOKKER-PLANCK AND LANDAU EQUATIONS WITH SPECULAR REFLECTION BOUNDARY CONDITION

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Abstract. We establish existence of finite energy weak solutions to the kinetic Fokker-Planck equation and the linear Landau equation near Maxwellian, in the presence of specular reflection boundary condition for general domains. Moreover, by using a method of reflection and the S_p estimate of [5], we prove regularity in the kinetic Sobolev spaces S_p and anisotropic Hölder spaces for such weak solutions. Such S_p regularity leads to the uniqueness of weak solutions.

Contents

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1. Introduction and Main results.

1.1. Introduction.

We consider the following generalized kinetic Fokker-Planck equation in a bounded domain Ω with the specular boundary condition:

$$\begin{aligned} &\partial_t f + v \cdot \nabla_x f - \partial_{v_i} (a^{ij}(t, x, v) \partial_{v_j} f) + b(t, x, v) \cdot \nabla_v f = \mathsf{g} \\ &\qquad\qquad f(t, x, v) = f(t, x, R_x v), (x, v) \in \gamma_-. \end{aligned} \tag{1.1}$$

Here

$$R_x v = v - 2(n_x \cdot v) n_x$$

is the specular reflected velocity, and n_x is the outward unit normal vector at $x \in \partial\Omega$, and

$$\gamma_\pm = \{(x, v) : x \in \partial\Omega, \pm n_x \cdot v > 0\}$$

is the outgoing/incoming set. In this paper, we will study two important cases of the generalized Fokker-Planck equations: the celebrated Kolmogorov-Fokker-Planck equation, which is given by Eq. (1.1) with $a^{ij} = \delta_{ij}$, and the linear Landau equation, which we introduce below.

A fundamental model in plasma physics is the nonlinear kinetic Landau equation with the specular reflection boundary condition given by

$$\begin{aligned} &\partial_t F + v \cdot \nabla_x F = \mathcal{Q}\,[F, F] \text{ in } (0, T) \times \Omega \times \mathbb{R}^3, \\ &F(0, x, v) = F_0(x, v), (x, v) \in \Omega \times \mathbb{R}^3, \;\; F(t, x, v) = F(t, x, R_x v), \quad (x, v) \in \gamma_-, \end{aligned}$$

where $\mathcal{Q}$ is the Landau (Fokker-Planck) collision operator with Coulomb interaction defined as

$$\begin{aligned}&\mathcal{Q}\,[F_1, F_2](v) = \nabla_v \cdot \int_{\mathbb{R}^3} \Phi(v-v')[F_1(v')\nabla_v F_2(v) - F_2(v)(\nabla_v F_1)(v')]\,dv', \qquad (1.2)\\ &\Phi(v) = \left(I_3 - \frac{v}{|v|}\otimes\frac{v}{|v|}\right)|v|^{-1}.\end{aligned}$$

Let $\mu(v) := \pi^{-3/2}e^{-|v|^2}$ be the Maxwellian, which is a steady state of Eq. (1.2). We rewrite the equation near the Maxwellian by replacing $\mathcal{Q}\,[F,F]$ with $\mathcal{Q}\,[\mu + \mu^{1/2}f, \mu+\mu^{1/2}f]$:

$$\begin{aligned}&\partial_t f + v\cdot\nabla_x f + \mathsf{L} f - \Gamma[f,f] = 0 \text{ in } (0,T)\times\Omega\times\mathbb{R}^3,\\ &f(0,x,v) = f_0(x,v), (x,v)\in\Omega\times\mathbb{R}^3,\ f(t,x,v) = f(t,x,R_x v), \quad (x,v)\in\gamma_-,\end{aligned} \tag{1.3}$$

where

$$\begin{aligned}&\mathsf{L} = -\mathsf{A} - \mathsf{K}, \quad \mathsf{A} f = \mu^{-1/2}\mathcal{Q}\,[\mu, \mu^{1/2}f], \quad \mathsf{K} f = \mu^{-1/2}\mathcal{Q}\,[\mu^{1/2}f, \mu],\\ &\Gamma[g,f] = \mu^{-1/2}\mathcal{Q}\,[\mu^{1/2}g, \mu^{1/2}f].\end{aligned} \tag{1.4}$$

We consider the linear version of Eq. (1.3) given by

$$\begin{aligned}&\partial_t f + v\cdot\nabla_x f + \mathsf{L} f - \Gamma[g,f] = 0 \text{ in } (0,T)\times\Omega\times\mathbb{R}^3,\\ &f(0,x,v) = f_0(x,v), \quad (x,v)\in\Omega\times\mathbb{R}^3,\ f(t,x,v) = f(t,x,R_x v), \quad (x,v)\in\gamma_-.\end{aligned} \tag{1.5}$$

Such an equation is useful for proving the global in time well-posedness of (1.3) for sufficiently small initial data (see, for example, [4], [10], [15]). Furthermore, we can rewrite (1.5) in the divergence form

$$\begin{aligned}&\partial_t f + v\cdot\nabla_x f = \nabla_v\cdot(\sigma_G\nabla_v f) + a_g\cdot\nabla_v f + \overline{K}_g f \text{ in } (0,T)\times\Omega\times\mathbb{R}^3,\\ &f(0,x,v) = f_0(x,v), (x,v)\in\Omega\times\mathbb{R}^3,\ f(t,x,v) = f(t,x,R_x v), (x,v)\in\gamma_-,\end{aligned} \tag{1.6}$$

where

$$\sigma = \Phi * \mu, \quad \sigma_G = \sigma + \Phi*(\mu^{1/2}g), \quad a_g^i = -\Phi^{ij}*(v_j\mu^{1/2}g + \mu^{1/2}\partial_{v_j}g), \tag{1.7}$$

$$\overline{K}_g = \mathsf{K} + J_g, \tag{1.8}$$

$$\begin{aligned}J_g f &= \partial_{v_i}(\sigma^{ij}v_j)f - \sigma^{ij}v_iv_jf\\ &\quad - \partial_{v_i}\big(\Phi^{ij}*(\mu^{1/2}\partial_{v_j}g)\big)f + \big(\Phi^{ij}*(v_i\mu^{1/2}\partial_{v_j}g)\big)f.\end{aligned} \tag{1.9}$$

See the details in [4], [9], [15]. To establish the existence of the finite energy strong solution to the problem (1.6) (see Definition 1.9), we work with the equation

$$\begin{aligned}&\partial f + v\cdot\nabla_x f = \nabla_v\cdot(\sigma_G\nabla_v f) + \nu\Delta_v f\\ &\qquad\qquad + a_g\cdot\nabla_v f - \lambda f + h \text{ in } (0,T)\times\Omega\times\mathbb{R}^3,\\ &f(0,x,v) = f_0(x,v),\ (x,v)\in\Omega\times\mathbb{R}^3,\\ &f_-(t,x,v) = f_+(t,x,R_x v), \quad (x,v)\in\gamma_-,\end{aligned} \tag{1.10}$$

where $\nu > 0, \lambda \ge 0$. We call this equation *simplified viscous linear Landau equation.*

While such boundary problems play an important role in many applications, there have been very limited study due to possible singularity forming from the grazing set (see [8]). The paper [11] initiated the study of the regularity of the boundary-value problems for generalized kinetic Fokker-Planck equations. For the related works, we refer the reader to [1], [12], [13], [14]. The boundary-value problem for the Landau equation is considered in the articles [2], [4], [10]. Our paper serves as a foundation of the linear theory for the nonlinear Landau equation used in the works [4], [10]. The article is organized as follows. Section 1.2 contains the necessary notation. In Section 1.3, we present the main results of this paper. We explain the key ideas of the proof in Section 2. The proof of the main results is divided into 3 sections: Sections 3, 4, and 5.

1.2. **Notation.** To state precisely our results, we now introduce necessary notation as follows. In this subsection, $G \subset \mathbb{R}^7$ is an open set, $\alpha \in (0,1]$, $p \in (1,\infty]$, $\theta \in [0,\infty)$, and $T \in (0,\infty)$.

- Geometric notation:

$$\begin{aligned}
&z = (t,x,v), t \in \mathbb{R}, x, v \in \mathbb{R}^3, \quad B_r(x_0) = \{\xi \in \mathbb{R}^3 : |\xi - x_0| < r\},\\
&\Omega \subset \mathbb{R}^3 \text{ is a bounded domain}, \quad \Omega_r(x) = \Omega \cap B_r(x),\ \mathbb{R}^3_{\pm} = \{x \in \mathbb{R}^3 : \pm x_3 > 0\},\\
&\mathbb{H}_{\pm} = \{(x,v) \in \mathbb{R}^3_{\pm} \times \mathbb{R}^3\}, \quad \mathbb{H}^T_{\pm} = \{z \in (0,T) \times \mathbb{H}_{\pm}\}, \qquad (1.11)\\
&\mathbb{R}^7_T = \{z \in (0,T) \times \mathbb{R}^6\}, \quad \Sigma^T = (0,T) \times \Omega \times \mathbb{R}^3,\\
&\gamma_{\pm} = \{(x,v) : x \in \partial\Omega, \pm n_x \cdot v > 0\},\ \gamma_0 = \{(x,v) : n_x \cdot v = 0\},\ \Sigma^T_{\pm} = (0,T) \times \gamma_{\pm}.
\end{aligned}$$

- *Matrices.* By I_3, we denote the 3×3 identity matrix and we set $\mathrm{Sym}(\delta), \delta \in (0,1)$ to be the collection of 3×3 symmetric matrices a such that
$$\delta|\xi|^2 \le a^{ij}\xi_i\xi_j \le \delta^{-1}|\xi|^2, \quad \forall \xi \in \mathbb{R}^3.$$
- $$Yf = \partial_t f + v \cdot \nabla_x f.$$
- Function spaces.
  - $C(\overline{G})$ - the set of all bounded continuous functions on $\overline{G}$.
  - $C^1_0(\overline{G})$ - the set of all continuously differentiable functions in $\overline{G}$ that vanish for large z.
  - $C^\infty_0(D)$, $D \subset \mathbb{R}^7$ is the set of all infinitely differentiable functions with support contained in D.
  - *Anisotropic Hölder space.* For an open set $D \subset \mathbb{R}^6$, by $C^{\alpha/3,\alpha}_{x,v}(\overline{D})$, we denote the set of bounded functions $f = f(x,v)$ such that
  $$[f]_{C^{\alpha/3,\alpha}_{x,v}(\overline{D})} := \sup_{(x_i,v_i)\in\overline{D}:(x_1,v_1)\neq(x_2,v_2)} \frac{|f(x_1,v_1) - f(x_2,v_2)|}{(|x_1 - x_2|^{1/3} + |v_1 - v_2|)^\alpha} < \infty.$$
  Furthermore,
  $$\|f\|_{C^{\alpha/3,\alpha}_{x,v}(\overline{D})} := \|f\|_{L_\infty(D)} + [f]_{C^{\alpha/3,\alpha}_{x,v}(\overline{D})}. \qquad (1.12)$$
  - *Weighted Lebesgue space.* For a locally integrable function $w(x,v)$ such that $w > 0$ a.e., by $L_p(G,w)$ we denote the set of all Lebesgue measurable functions u such that
  $$\|u\|_{L_p(G,w)} := \|w^{1/p}u\|_{L_p(G)} < \infty.$$
  In particular, for
  $$\langle v\rangle := (1 + |v|^2)^{1/2},$$
  we set
  $$L_{p,\theta}(G) = L_p(G, \langle v\rangle^\theta).$$

– *Sobolev spaces.* Let $\mathbb{H}_p^1(G) := \{f \in L_p(G) : \nabla_v f \in L_p(G)\}$ be the Banach space equipped with the norm

$$\|f\|_{\mathbb{H}_p^1(G)} := \||f| + |\nabla_v f|\|_{L_p(G)}.$$

Furthermore, $\mathbb{H}_p^{-1}(G)$ is the set of all functions f on G such that there exists $f_i \in L_p(G), i = 0, 1, 2, 3$, such that

$$f = f_0 + \partial_{v_i} f_i.$$

The $\mathbb{H}_p^{-1}(G)$-norm is given by

$$\|f\|_{\mathbb{H}_p^{-1}(G)} = \inf \sum_{i=0}^{3} \|f_i\|_{L_p(G)},$$

where the infimum is taken over all such $f_i, i = 0, 1, 2, 3$.

– *Kinetic (ultraparabolic) Sobolev spaces.*
By $S_{p,\theta}(G) = \{f \in L_{p,\theta}(G) : Yf, \nabla_v f, D_v^2 f \in L_{p,\theta}(G)\}$ we mean the Banach space with the norm

$$\|f\|_{S_{p,\theta}(G)} = \||f| + |\nabla_v f| + |D_v^2 f| + |Yf|\|_{L_{p,\theta}(G)}. \tag{1.13}$$

We also denote $S_p(G) := S_{p,0}(G)$.
Furthermore, we set $\mathbb{S}_p(G)$ to be the space of functions f such that $f, \nabla_v f \in L_p(G)$, and $Yf \in \mathbb{H}_p^{-1}(G)$. The norm is defined as follows:

$$\|f\|_{\mathbb{S}_p(G)} = \|f\|_{\mathbb{H}_p^1(G)} + \|Yf\|_{\mathbb{H}_p^{-1}(G)}. \tag{1.14}$$

- Traces. Let $f \in L_p(\Sigma^T)$ be a function such that $Yf \in L_p(\Sigma^T)$ for some $p \in [1, \infty)$. We set $f_\pm, f(0, \cdot), f(T, \cdot)$ to be the restrictions of the trace of f on $\Sigma_\pm^T$, $\{0\} \times \Omega \times \mathbb{R}^3$, $\{T\} \times \Omega \times \mathbb{R}^3$, respectively (see the definition on p. 393 of [3] or in Remark 3.3).
- *Space of initial values.* For $p \in (1, \infty]$ and $\theta \geq 0$, by $\mathcal{O}_{\theta,p}$ we denote the set of all functions u on $\Omega \times \mathbb{R}^3$ such that

$$u, v \cdot \nabla_x u, \nabla_v u, D_v^2 u \in L_{p,\theta}(\Omega \times \mathbb{R}^3), \ u_\pm \in L_\infty(\gamma_\pm, |v \cdot n_x|),$$

and

$$u_-(x, v) = u_+(x, R_x v) \text{ a.e. } (x, v) \in \gamma_-.$$

The norm is given by

$$\begin{aligned} &\|u\|_{\mathcal{O}_{p,\theta}} = |u|_{\mathcal{O}_{p,\theta}} + \|u_\pm\|_{L_\infty(\gamma_\pm, |v \cdot n_x|)}, \\ &|u|_{\mathcal{O}_{p,\theta}} := \||u| + |v \cdot \nabla_x u| + |\nabla_v u| + |D_v^2 u|\|_{L_{p,\theta}(\Omega \times \mathbb{R}^3)}. \end{aligned} \tag{1.15}$$

For $\theta = 0$, we set $\mathcal{O}_p = \mathcal{O}_{p,0}$.
- *Constants.* By $N = N(\cdots)$ we mean a constant depending only on the parameters inside the parentheses. The constants N might change from line to line. Sometimes, when it is clear what parameters N depends on, we omit them.

1.3. **Main results.**

1.3.1. *Kinetic Fokker-Planck equation.* Let a be a measurable function taking values in the set of symmetric $d \times d$ matrices.

Definition 1.1. Let $\theta \geq 0, T > 0$ be numbers. We say that $(f, f^{\star}_{\pm}, f^{\star}_{T}, f_0)$ is a finite energy weak solution to

$$Yf - \nabla_v \cdot (a \nabla_v f) + b \cdot \nabla_v f + \lambda f = \mathsf{g}, \quad z \in \Sigma^T \tag{1.16}$$

with the specular boundary condition and the initial data f_0 if

i) $f, \nabla_v f \in L_{2,\theta}(\Sigma^T)$, $\mathsf{g} \in L_{2,\theta}(\Sigma^T)$, $f^{\star}_{\pm} \in L_{\infty}(\Sigma^T_{\pm}, |v \cdot n_x|)$, $f^{\star}_T, f_0 \in L_{2,\theta}(\Omega \times \mathbb{R}^3)$,
ii) $f^{\star}_{-}(t, x, v) = f^{\star}_{+}(t, x, R_x v)$ a.e. on Σ^T_{-},
iii) for any $\phi \in C^1_0(\overline{\Sigma^T})$,

$$\begin{aligned}
&- \int_{\Sigma^T} (Y\phi) f \, dz + \int_{\Omega \times \mathbb{R}^3} \big(f^{\star}_T(x,v)\phi(T,x,v) - f_0(x,v)\phi(0,x,v)\big)\, dxdv \\
&+ \int_{\Sigma^T_+} f^{\star}_{+} \phi \, |v \cdot n_x| d\sigma dt - \int_{\Sigma^T_-} f^{\star}_{-} \phi \, |v \cdot n_x| d\sigma dt \\
&+ \lambda \int_{\Sigma^T} f \phi \, dz + \int_{\Sigma^T} (a \nabla_v f) \cdot \nabla_v \phi \, dz + \int_{\Sigma^T} (b \cdot \nabla_v f)\phi \, dz = \int_{\Sigma^T} \mathsf{g}\phi \, dz.
\end{aligned} \tag{1.17}$$

For the sake of brevity, in the sequel, we will say that f is a finite weak energy solution to (1.16), thus, omitting $f^{\star}_{\pm}, f^{\star}_T, f_0$.

Furthermore, we say that f is a finite energy strong solution to Eq. (1.16) if

– f is a finite energy weak solution,
– $f \in S_2(\Sigma^T)$, where S_2 is defined in (1.13),
– the identity
$$Yf = \nabla_v \cdot (a \nabla_v f) - b \cdot \nabla_v f - \lambda f + \mathsf{g} \tag{1.18}$$
is satisfied a.e. in Σ^T.

Remark 1.1. The definition of a finite energy strong solution does not require $\partial_t f$ or $v \cdot \nabla_x f$ to be a measurable function. In fact, these objects are understood in the sense of distributions. However, Yf is required to be an element of $L_2(\Sigma^T)$.

Remark 1.2. We remark that Yf is not known to be in $L_p(\Sigma^T)$ for the natural finite energy weak solutions. This is a basic difficulty in the study of boundary value problems of the kinetic Fokker-Planck type of equations. We need to develop extra $S_p(\Sigma^T)$ estimates to ensure such a trace property.

If f is a finite energy strong solution to (1.16) then, the functions $f^{\star}_{\pm}, f^{\star}_T, f_0$ are traces of f in the sense of [3], i.e.,

$$f, \, f^{\star}_{\pm}, f^{\star}_T, f_0$$

satisfy the Green's identity (2.20) of [3] (see also (3.9)). Since $f, Yf \in L_2(\Sigma^T)$, by Remark 3.2, f has traces $f(0,\cdot), f(T,\cdot), f_{\pm}$. Then, we multiply (1.18) by $\phi \in C^1_0(\overline{\Sigma^T})$, use the Green's formula (3.9), and compare the resulting equality with (1.17). This gives $f(0,\cdot) = f_0, f(T,\cdot) = f^{\star}_T$, $f_{\pm} = f^{\star}_{\pm}$ a.e..

We impose the following assumptions on the coefficients and the domain Ω.

Assumption 1.2. There exists $\delta \in (0,1)$ such that $\forall z$, $a(z) \in \mathrm{Sym}(\delta)$, i.e.,

$$\delta |\xi|^2 \leq a^{ij}(z)\xi_i \xi_j \leq \delta^{-1} |\xi|^2. \tag{1.19}$$

Assumption 1.3. There exists a constant $K > 0$ such that

$$\|b\|_{L_{\infty}(\Sigma^T)} \leq K. \tag{1.20}$$

Assumption 1.4. There exists a constant $K > 0$ such that

$$\|\nabla_v a\|_{L_\infty(\Sigma^T)} \le K, \quad \nabla_v b \in L_\infty(\Sigma^T). \tag{1.21}$$

We present three results on the existence, uniqueness, and the S_p-regularity of the finite energy weak solutions.

Theorem 1.5 (Existence of finite energy weak solutions)**.** *Let Ω be a bounded C^2 domain. Under Assumptions 1.2 - 1.4, for any $T > 0$, $\theta \ge 0$, there exists $\lambda_0 = \lambda_0(\theta, \delta, K) > 0$ such that for any $\lambda \ge \lambda_0$ and $\mathsf{g} \in L_{2,\theta}(\Sigma^T) \cap L_\infty(\Sigma^T)$, $f_0 \in L_{2,\theta}(\Omega \times \mathbb{R}^3) \cap L_\infty(\Omega \times \mathbb{R}^3)$, Eq. (1.16) has a finite energy weak solution f, and, in addition,*

$$\begin{aligned} &\|f_T^\star\|_{L_{2,\theta}(\Omega\times\mathbb{R}^3)} + \lambda^{1/2}\|f\|_{L_{2,\theta}(\Sigma^T)} + \|\nabla_v f\|_{L_{2,\theta}(\Sigma^T)} \\ &\le N(\lambda^{-1/2}\|\mathsf{g}\|_{L_{2,\theta}(\Sigma^T)} + \|f_0\|_{L_{2,\theta}(\Omega\times\mathbb{R}^3)}), \\ &\max\{\|f_T^\star\|_{L_\infty(\Omega\times\mathbb{R}^3)}, \|f\|_{L_\infty(\Sigma^T)}, \|f_\pm^\star\|_{L_\infty(\Sigma_\pm^T, |v\cdot n_x|)}\} \\ &\le \lambda^{-1}\|\mathsf{g}\|_{L_\infty(\Sigma^T)} + \|f_0\|_{L_\infty(\Omega\times\mathbb{R}^3)}, \end{aligned} \tag{1.22}$$

where $N = N(\delta, K, \theta) > 0$.

The following corollary is derived from Theorem 1.5 by using an exponential weight.

Corollary 1.1. *One can take $\lambda_0 = 0$ in Theorem 1.5. However, in that case, the constant N also depends on T, i.e., $N = N(\delta, K, \theta, T)$.*

We remark that the uniqueness of such weak solutions remains an open problem in general. By using the method of reflection (see Section 2), we prove the uniqueness and higher regularity for Eq. (1.16) in the case when $a = I_3$.

Theorem 1.6 (S_2 regularity of a finite energy weak solution)**.** *Let Ω be a bounded C^3 domain, $T > 0, \theta \ge 2$ be numbers, and $\mathsf{g} \in L_{2,\theta-2}(\Sigma^T), f_0 \in \mathcal{O}_{2,\theta-2}$ (see (1.15)), and $a = I_3$. Under Assumptions 1.2 - 1.3, there exists $\lambda_0 = \lambda_0(K, \theta, \Omega) > 0$ such that for any $\lambda \ge \lambda_0$, if f is a finite energy weak solution to Eq. (1.16) with parameter θ, then, $f \in S_{2,\theta-2}(\Sigma^T)$ and $f, \nabla_v f \in L_{7/3,\theta-2}(\Sigma^T)$. Furthermore, we have*

$$\begin{aligned} &\|f\|_{S_{2,\theta-2}(\Sigma^T)} + \||f| + |\nabla_v f|\|_{L_{7/3,\theta-2}(\Sigma^T)} \\ &\le N(\|\mathsf{g}\|_{L_{2,\theta-2}(\Sigma^T)} + |f_0|_{\mathcal{O}_{2,\theta-2}} \\ &\quad + \|\nabla_v f\|_{L_{2,\theta}(\Sigma^T)} + \|f\|_{L_{2,\theta-1}(\Sigma^T)}), \end{aligned} \tag{1.23}$$

where $N = N(K, \Omega, \theta)$. Finally, f satisfies the identity (1.18) a.e., and, hence f is a finite energy strong solution.

Corollary 1.2 (Uniqueness of finite energy solutions)**.** *Under assumptions of Theorem 1.6, any two finite energy weak solutions to Eq. (1.16) must coincide.*

Theorem 1.7 (Higher regularity of the finite energy strong solution)**.** *Let $p > 14$ and invoke the assumptions of Theorem 1.6 and assume, additionally, $\theta \ge 16$, and $\mathsf{g} \in L_{p,\theta-4}(\Sigma^T)$, $f_0 \in \mathcal{O}_{p,\theta-4}(\Sigma^T)$ (see (1.15)). Then, for any $\lambda \ge 0$, one has $f \in S_{p,\theta-16}(\Sigma^T) \cap C(\overline{\Sigma^T})$, and*

$$\begin{aligned} \|f\|_{S_{p,\theta-16}(\Sigma^T)} &+ \|f\|_{L_\infty((0,T), C^{\alpha/3,\alpha}_{x,v}(\overline{\Omega}\times\mathbb{R}^3))} + \|\nabla_v f\|\|_{L_\infty((0,T), C^{\alpha/3,\alpha}_{x,v}(\overline{\Omega}\times\mathbb{R}^3))} \\ &\le N(\|\mathsf{g}\|_{L_{2,\theta-2}(\Sigma^T)} + \|\mathsf{g}\|_{L_{p,\theta-4}(\Sigma^T)} \\ &\quad + |f_0|_{\mathcal{O}_{2,\theta-2}} + |f_0|_{\mathcal{O}_{p,\theta-4}} \\ &\quad + \|\nabla_v f\|_{L_{2,\theta}(\Sigma^T)} + \|f\|_{L_{2,\theta-1}(\Sigma^T)}), \end{aligned}$$

where

- $\alpha = 1 - 14/p$,
- $C^{\alpha/3,\alpha}_{x,v}(\overline{\Omega} \times \mathbb{R}^3)$ *is the anisotropic Hölder space defined in* (1.12),
- $N = N(p, \theta, K, \Omega)$.

1.3.2. *Linear Landau equation.*

Definition 1.8. Let $H_{\sigma,\theta}(\Sigma^T)$ be the Hilbert space of all Lebesgue measurable functions such that the norm

$$\|u\|_{\sigma,\theta} := \bigg(\int_{\Sigma^T} (\sigma^{ij}\partial_{v_i} u\, \partial_{v_j} u + \sigma^{ij} v_i v_j u^2)\langle v\rangle^{\theta}\, dz\bigg)^{1/2} \tag{1.24}$$

is finite. Here σ is defined by (1.7). Since σ is a positive definite matrix (see Remark 1.3), $\|\cdot\|_{\sigma,\theta}$ is, indeed, a norm.

Remark 1.3. Let $\theta \geq 0$ and $u \in L_{2,\theta}(\Sigma^T)$ be a function such that $\nabla_v u \in L_{2,\theta}(\Sigma^T)$. Then,

$$\|u\|_{\sigma,\theta} \leq N\||u| + |\nabla_v u|\|_{L_{2,\theta-1}(\Sigma^T)}. \tag{1.25}$$

This follows from the facts that, for any $v \in \mathbb{R}^3$,

$$\sigma^{ij}(v) v_i v_j \leq N\langle v\rangle^{-1}, \quad \sigma(v) \leq N\langle v\rangle^{-1} I_3.$$

Furthermore, due to the inequality

$$\langle v\rangle^{-3} I_3 \leq C_1 \sigma(v)$$

for some constant $C_1 > 0$, we have

$$\|u\|_{L_{2,\theta-3}(\Sigma^T)} \leq C_1 \|u\|_{\sigma,\theta}.$$

See the details in Lemma 3 of [9].

Definition 1.9. We say that f is a finite energy strong solution to (1.6) if there exists $\theta \geq 0$ such that

1. $f \in S_2(\Sigma^T) \cap L_{2,\theta}(\Sigma^T) \cap H_{\sigma,\theta}(\Sigma^T)$,
2. the traces (see Remark 3.2) have the following regularity: $f(T,\cdot), f(0,\cdot) \in L_{2,\theta}(\Omega \times \mathbb{R}^3)$, $f_{\pm} \in L_{\infty}(\Sigma^T_{\pm}, |v \cdot n_x|)$,
3. for any $\phi \in C^1_0(\overline{\Sigma^T})$,

$$\begin{aligned} &- \int_{\Sigma^T} (Y\phi) f\, dz + \int_{\Omega\times\mathbb{R}^3} \big(f(T,x,v)\phi(T,x,v) - f(0,x,v)\phi(0,x,v)\big)\, dx dv \\ &+ \int_{\Sigma^T_+} f_+ \phi\, |v\cdot n_x| d\sigma dt - \int_{\Sigma^T_-} f_- \phi\, |v \cdot n_x| d\sigma dt \\ &+ \int_{\Sigma^T} [(\sigma_G \nabla_v f)\cdot \nabla_v \phi - (a_g \cdot \nabla_v f)\phi - (\overline{K}_g f)\phi]\, dz = 0, \end{aligned}$$

4. Eq. (1.6) is satisfied almost everywhere, including the initial and the boundary conditions. The latter are understood as conditions on traces of f.

Remark 1.4. Note that the condition (4) implies (3) by the Green's identity (see (3.9)).

Assumption 1.10. Let $\varkappa \in (0, 1]$ and $g \in L_{\infty}((0,T), C^{\varkappa/3,\varkappa}_{x,v}(\overline{\Omega} \times \mathbb{R}^3))$ be the function such that

– $\nabla_v g \in L_\infty((0,T), C^{\varkappa/3,\varkappa}_{x,v}(\overline{\Omega} \times \mathbb{R}^3))$, and

$$\|g\|_{L_\infty((0,T),C^{\varkappa/3,\varkappa}_{x,v}(\overline{\Omega}\times\mathbb{R}^3))} + \|\nabla_v g\|_{L_\infty((0,T),C^{\varkappa/3,\varkappa}_{x,v}(\overline{\Omega}\times\mathbb{R}^3))} \le K \tag{1.26}$$

for some $K > 0$.

–

$$g_-(t,x,v) = g_+(t,x,R_x v) \quad \forall z \in \Sigma^T_-. \tag{1.27}$$

Theorem 1.11. *Let*

- Ω *be a bounded* C^3 *domain,*
- $\varkappa \in (0,1]$, $T > 0$, $p > 14$ *be numbers,*
- *Assumption 1.10 (see* (1.26) - (1.27)*) hold,*
- $f_0 \in \mathcal{O}_{2,\theta} \cap \mathcal{O}_\infty$ *(see* (1.15)*),*
- $\|g\|_{L_\infty(\Sigma^T)} \le \varepsilon$.

Then, there exists a number $\theta_0 = \theta_0(p,\varkappa) > 4$ *such that for any* $\theta \ge \theta_0$, *there exist numbers* $\theta = \theta(p,\varkappa) > 4$, $\varepsilon = \varepsilon(\theta) \in (0,1)$, *and* $\theta' = \theta'(p,\varkappa), \theta'' = \theta''(p,\varkappa) \in (1, \theta - 3)$ *such that Eq.* (1.5) *has a unique finite energy strong solution* f *(see Definition 1.9). Furthermore,* $f \in C(\overline{\Sigma^T})$, *and*

$$\begin{aligned} &\|f\|_{S_{2,\theta'}(\Sigma^T)} + \|f\|_{S_{p,\theta''}(\Sigma^T)} + \|f(T,\cdot)\|_{L_{2,\theta}(\Omega\times\mathbb{R}^3)} + \|f\|_{L_{2,\theta}(\Sigma^T)} + \|f\|_{\sigma,\theta} \\ &+ \|f\|_{L_\infty((0,T),C^{\alpha/3,\alpha}_{x,v}(\overline{\Omega}\times\mathbb{R}^3))} + \|\nabla_v f\|_{L_\infty((0,T),C^{\alpha/3,\alpha}_{x,v}(\overline{\Omega}\times\mathbb{R}^3))} \\ &\le N(\|f_0\|_{\mathcal{O}_{2,\theta}} + \|f_0\|_{\mathcal{O}_\infty}), \end{aligned} \tag{1.28}$$

where $\alpha = 1 - \frac{14}{p}$, *the* $\|\cdot\|_{\sigma,\theta}$*-norm is defined in* (1.24), *and* $N = N(p,\theta,\varkappa,\Omega,K,T)$.

2. Method of the proof.

1. Kinetic Fokker-Planck equation,

Existence. To construct the finite energy weak solutions to (1.16), we discretize the diffusion operator $\nabla_v \cdot (a \nabla_v f)$ and prove uniform bounds in Section 3 by using the results of [3]. The discretized second-order operator A_h (see (3.3)) is inspired by the representation of symmetric matrices in Theorem 3.1 of [17] (see Lemma 3.1). Such representation (see (3.1)) was first stated in the work [18] without smoothness properties of the weights λ_k. The form of A_h enables us to prove an analogue of the L_p-energy inequality for the heat equation (see (3.15)). We then use the weak* compactness argument to conclude the existence of finite energy weak solutions.

Uniqueness and higher regularity. Thanks to the specular reflection boundary condition, we are able to construct a reflection operator (see (2.6)) to extend solutions to $x \in \mathbb{R}^3$. In the present authors' opinion, such a reflection operator is useless for the study of the regularity for the Vlasov type equations in the absence of velocity diffusion. In particular, when the same reflection operator is applied to such an equation, it preserves the form of an equation but yields a drift term with discontinuous coefficients. On the other hand, in the presence of velocity diffusion, one can apply the S_2 regularity theory in $\mathbb{R}^7_T$ (see Section 4) to the extended equation (2.17) and conclude that a finite energy weak solution with $\theta \ge 2$ is of class $S_2(\Sigma^T)$. We then use the S_p regularity results of [5] (see Appendix D) to show that $f \in S_p(\Sigma^T)$. Unfortunately, the mirror extension argument works only for very particular diffusion operators in the velocity variable, for example, Δ_v and $\nabla_v \cdot (\sigma_G \nabla_v)$, where σ_G is defined in (1.7). See the discussion in Section 2.2.

2. **Linear Landau equation.** To prove Theorem 1.11, we use the method of vanishing viscosity. We first consider a simplified version of Eq. (1.6) given by Eq. (1.10). By the existence result for the generalized kinetic Fokker-Planck equation (see Theorem 1.5), we prove the existence of a finite energy weak solution in the sense of Definition 1.1. To prove the unique solvability in the class of the finite energy strong solutions, we use the mirror extension method combined with the S_2 estimate of [5] (see Appendix D).

   Viscous linear Landau equation. By using a perturbation argument, we extend the aforementioned unique solvability result to the viscous linear Landau equation (1.10), which contains the non-local term $\overline{K}_g f$ defined in (1.8). We then prove uniform in ν bounds by combining the standard energy estimate (see Lemma 5.3) with the S_p estimates of [5]. Finally, by using the weak* compactness argument, we prove the existence of the finite energy strong solution (see Definition 1.9) to the linear Landau equation (1.6). The uniqueness in the class of the finite energy strong solutions follows from the aforementioned energy estimate.

Our argument for proving the existence/uniqueness of the finite energy strong solution for Eqs. (1.16) and (1.10) goes as follows:

$$\textit{existence of finite energy weak solution } f \to f \in S_2 \to \textit{uniqueness.}$$

In the rest of this section, we

- define the mirror extension operator and show that it preserves the form of Eq (1.16),
- delineate the proofs of the S_2 bound, S_p estimate, and the Hölder estimate,
- elaborate on the importance of the condition (1.26) in Assumption 1.10.

2.1. **A boundary-flattening diffeomorphism that preserves the specular reflection boundary condition.** We may assume that there exists a sufficiently small number $r_0 > 0$ such that for any $x_0 \in \Omega$, there exists a function $\rho = \rho(x_1, x_2)$ such that

$$\begin{aligned} &\partial\Omega \cap B_{r_0}(x_0) \subset \{x : x_3 = \rho(x_1, x_2)\}, \\ &\Omega_{r_0}(x_0) := \Omega \cap B_{r_0}(x_0) \subset \{x : x_3 < \rho(x_1, x_2)\}, \end{aligned} \tag{2.1}$$

and ρ is a bounded function with bounded continuous partial derivatives up to order 3.

Next, denote $\rho_i = \frac{\partial \rho}{\partial x_i}, i = 1, 2$. Following [10] (see Section 7.1.1 of the reference), we introduce the boundary-flattening diffeomorphism

$$\Psi : \Omega_{r_0}(x_0) \times \mathbb{R}^3 \to \mathbb{H}_- = \mathbb{R}^3_- \times \mathbb{R}^3, \quad (x, v) \to (y, w)$$

given by

$$y = \psi(x), \quad w = D\psi(x)v, \tag{2.2}$$

where ψ is defined as the inverse of the transformation

$$\psi^{-1}(y) = \begin{pmatrix} y_1 \\ y_2 \\ \rho(y_1, y_2) \end{pmatrix} + y_3 \begin{pmatrix} -\rho_1 \\ -\rho_2 \\ 1 \end{pmatrix}. \tag{2.3}$$

It is easy to check that $D\psi^{-1}$ sends the vector $(0,0,1)^T$ to $(-\rho_1,-\rho_2,1)^T$, which is an outward normal vector at $\partial\Omega$. Furthermore, by direct computations (see [10]),

$$R_x v = v - 2(v\cdot n_x)n_x = \begin{pmatrix} w_1+\rho_1 w_3 \\ w_2+\rho_2 w_3 \\ \rho_1 w_1+\rho_2 w_2 - w_3 \end{pmatrix} = \frac{\partial x}{\partial y}|_{y_3=0}(w_1,w_2,-w_3)^T, \quad (2.4)$$

which shows that Ψ preserves the specular reflection boundary condition.

2.1.1. *Mirror extension transformation.* Denote

$$\widehat{z} = (t,y,w), \quad J = \left|\det \frac{\partial x}{\partial y}\right|^2,$$

and, for any function u, supported on $\mathbb{R}\times\Omega_{r_0}(x_0)\times\mathbb{R}^3$, we set

$$\widehat{u}(\widehat{z}) = u(t,x(y),v(y,w)), \quad \widetilde{u}(\widehat{z}) = \widehat{u}(\widehat{z})J(y). \quad (2.5)$$

Then, the mirror extension of u is defined as

$$\overline{u}(t,y,w) := \begin{cases} \widetilde{u}(t,y,w), (t,y,w)\in\mathbb{H}_-^T, \\ \widetilde{u}(t,Ry,Rw), (t,y,w)\in\mathbb{H}_+^T, \end{cases} \qquad R = \operatorname{diag}(1,1,-1). \quad (2.6)$$

Note that, strictly speaking, $\overline{u}$ is not an extension of $\widehat{u}$, however, since the Jacobian $J\approx 1$ near x_0, $\overline{u}$ is close to the function

$$\widehat{z}\to\widehat{u}(t,Ry,Rw)$$

provided that r_0 is sufficiently small.

2.1.2. *The generalized kinetic Fokker-Planck equation under the mirror extension mapping.* We consider Eq. (1.16) and assume that there exists some $\delta\in(0,1)$ such that for any z, $a\in\operatorname{Sym}(\delta)$, i.e., (1.19) holds. Let f be a finite energy solution to Eq. (1.16) (see Definition 1.1) supported on $\mathbb{R}\times\Omega_{r_0/2}(x_0)\times\mathbb{R}^3$. Here we show that the mirror extension $\overline{f}$ (see (2.6)) also solves a generalized kinetic Fokker-Planck type equation. To this end, we change variables in the weak formulation of Eq. (1.16) for some fixed $\phi\in C_0^1(\overline{\Sigma^T})$ (see (1.17) in Definition 1.1).

First, we find an equation satisfied by $\widetilde{f}$ (defined in (2.5)) on $\mathbb{H}_-^T$. By direct computations (see Appendix A), $\widetilde{f}$ satisfies the identity

$$\begin{aligned}
&-\int_{\mathbb{H}_-^T}(Y\widehat{\phi})\,\widetilde{f}d\widehat{z} + \int_{\mathbb{H}_-}\left(\widetilde{f}_T^\star(y,w)\widehat{\phi}(T,y,w) - \widetilde{f}_0(y,w)\widehat{\phi}(0,y,w)\right)dydw \\
&+\int_{\mathbb{H}_-^T}\left((\nabla_w\widetilde{f})^T A\nabla_w\widehat{\phi} - \widetilde{f}X\cdot\nabla_w\widehat{\phi} + \left(B\cdot\nabla_w\widetilde{f}\right)\widehat{\phi} + \lambda\widetilde{f}\,\widehat{\phi}\right)d\widehat{z} \\
&+\int_0^T\int_{\mathbb{R}^2\times\mathbb{R}_+^3}|w_3|\widetilde{f}_+^\star\widehat{\phi}\,dy_1dy_2dwdt \\
&-\int_0^T\int_{\mathbb{R}^2\times\mathbb{R}_-^3}|w_3|\widetilde{f}_-^\star\widehat{\phi}\,dy_1dy_2dwdt = \int_{\mathbb{H}_-^T}\widehat{\phi}\widetilde{\mathrm{g}}\,d\widehat{z},
\end{aligned} \quad (2.7)$$

where $\widehat{\phi}$ is defined by (2.5), and

$$A = \left(\frac{\partial y}{\partial x}\right) \widehat{a} \left(\frac{\partial y}{\partial x}\right)^T, \quad B = \left(\frac{\partial y}{\partial x}\right) b, \tag{2.8}$$

$$X = (X_1, X_2, X_3)^T = \left(\frac{\partial y}{\partial x}\right)\left(\frac{\partial v}{\partial y}\right) w = \left(\frac{\partial y}{\partial x}\right) \frac{\partial \left(\frac{\partial x}{\partial y} w\right)}{\partial y} w. \tag{2.9}$$

Next, we extend the coefficients to $\mathbb{H}^T_-$. First, we set $B(t, \cdot, w), X(\cdot, w)$ to be 0 if $y \in \mathbb{R}^3_- \setminus \psi(\Omega_{r_0}(x_0))$. Second, let $\kappa \in C^\infty_0(\mathbb{R}^3)$ be a function such that $0 \le \kappa \le 1$, and $\kappa = 1$ on $|y - \psi(x_0)| \le 3r_0/4$, $\kappa = 0$ on $|y - \psi(x_0)| \ge 7r_0/8$ and denote

$$\mathcal{A}(t, y, w) = A(t, y, w)\kappa(y) + \delta I_3(1 - \kappa(y)). \tag{2.10}$$

Note that for sufficiently small r_0,

$$\widehat{f} \text{ is supported on } B_{3r_0/4}(\psi(x_0)), \tag{2.11}$$

and then, so is $\widetilde{f}$. Since $\kappa = 1$ on the support of $\widetilde{f}$, Eq. (2.7) holds with A and $\widehat{\phi}$ replaced with $\mathcal{A}$ and any $\eta \in C^1_0(\overline{\mathbb{H}^T_-})$, respectively.

We now extend $\mathcal{A}, B, X$ to $\mathbb{R}^7_T$ by setting

$$\mathbb{A}(t, y, w) = \begin{cases} \mathcal{A}(t, y, w), \quad (t, y, w) \in \mathbb{H}^T_-, \\ R\,\mathcal{A}(t, Ry, Rw)\,R, \ (t, y, w) \in \mathbb{H}^T_+, \end{cases} \tag{2.12}$$

$$\mathbb{X}(y, w) = \begin{cases} X(y, w), \ (y, w) \in \mathbb{H}_-, \\ R\,X(Ry, Rw), \ (y, w) \in \mathbb{H}_+, \end{cases} \tag{2.13}$$

$$\mathbb{B}(t, y, w) = \begin{cases} B(t, y, w), \ (t, y, w) \in \mathbb{H}^T_-, \\ R\,B(t, Ry, Rw), \ (t, y, w) \in \mathbb{H}^T_+. \end{cases} \tag{2.14}$$

Finally, we check that $\overline{f}$ satisfies an equation on the whole space. We fix an arbitrary function $\eta \in C^1_0(\overline{\mathbb{R}^7_T})$ such that $D^2_v \eta \in C(\overline{\mathbb{R}^7_T})$ and denote by $\eta_\pm$ the restriction of η to $\overline{\mathbb{H}^T_\pm}$. Replacing $\widehat{\phi}$ with $\eta_+(t, Ry, Rw)$ in (2.7) and changing variables, we obtain

$$\begin{aligned} &- \int_{\mathbb{H}^T_+} (Y \eta_+) \overline{f}\, d\widehat{z} + \int_{\mathbb{H}_+} [\overline{f^\star_T}(y, w)\eta_+(T, y, w) - \overline{f_0}(y, w)\eta_+(0, y, w)]\, dy dw \\ &+ \int_{\mathbb{H}^T_+} \left((\nabla_w \overline{f})^T \mathbb{A} \nabla_w \eta_+ - \overline{f}\, \mathbb{X} \cdot \nabla_w \eta_+ + (\mathbb{B} \cdot \nabla_w \overline{f})\, \eta_+ + \lambda \overline{f}\, \eta_+\right) d\widehat{z} \\ &+ \int_0^T \int_{\mathbb{R}^2 \times \mathbb{R}^3_-} |w_3| \widetilde{f}^\star_+(t, y_1, y_2, Rw) \eta\, dy_1 dy_2\, dw\, dt \\ &- \int_0^T \int_{\mathbb{R}^2 \times \mathbb{R}^3_+} |w_3| \widetilde{f}^\star_-(t, y_1, y_2, Rw) \eta\, dy_1 dy_2\, dw\, dt \\ &= \int_{\mathbb{H}^T_+} \overline{\mathrm{g}} \eta_+\, d\widehat{z}. \end{aligned} \tag{2.15}$$

Note that by (2.4) and the fact that f satisfies the specular reflection boundary condition, we have

$$\widehat{f}^\star_+(t, y_1, y_2, Rw) = \widehat{f}^\star_-(t, y_1, y_2, w). \tag{2.16}$$

Then, adding (2.15) to (2.7) with $\widehat{\phi} = \eta_-$ and transferring the derivatives in v to the test function η, and using (2.16) give

$$
\begin{aligned}
&- \int_{\mathbb{R}^7_T} (Y\eta)\overline{f}\, d\widehat{z} + \int_{\mathbb{R}^6} [\overline{f^\star_T}(y,w)\eta(T,y,w) - \overline{f_0}(y,w)\eta(0,y,w)]\, dydw \\
&+ \int_{\mathbb{R}^7_T} \big((\nabla_w \overline{f})^T \mathbb{A} \nabla_w \eta - (\mathbb{X} \cdot \nabla_w \eta)\overline{f} + (\mathbb{B} \cdot \nabla_w \overline{f})\eta + \lambda \eta \overline{f}\big)\, d\widehat{z} \\
&= \int_{\mathbb{R}^7_T} \overline{\mathrm{g}} \eta \, d\widehat{z}.
\end{aligned}
$$

Here we used the fact that $\nabla_w \overline{f} \in L_{2,\theta}(\mathbb{R}^7_T)$. Thus, in $\mathbb{R}^7_T$, $\overline{f}$ satisfies the equation

$$
Y\overline{f} - \nabla_w \cdot (\mathbb{A} \nabla_w \overline{f}) + \nabla_w \cdot (\mathbb{X}\overline{f}) + \mathbb{B} \cdot \nabla_w \overline{f} + \lambda \overline{f} = \overline{\mathrm{g}} \tag{2.17}
$$

in the weak sense (cf. Definition 1.1 (1.17)).

2.2. **Strategy for proving the S_2 estimate.**

- **Interior estimate for Eqs. (1.16) and (1.10).** Away from the boundary $\partial\Omega$, by the partition of unity argument, we may reduce these equations to the ones on the whole space $\mathbb{R}^7_T$. To show that $f \in S_2$ away from the boundary, we use the unique solvability result of [5] (see Theorem D.4). As discussed in Appendix D (see Remark D.1), this theorem is applicable if the leading coefficients are of class $L_\infty((0,T), C^{\varkappa/3,\varkappa}_{x,v}(\mathbb{R}^6))$, $\varkappa \in (0,1]$, where $C^{\varkappa/3,\varkappa}_{x,v}(\mathbb{R}^6)$ is defined in (1.12). For Eq. (1.10), the validity of this condition follows from Assumption 1.10 ((1.26) - (1.27)). See Lemma C.1.
- **Boundary estimate.** Near the boundary, we use the boundary flattening diffeomorphism Ψ (see (2.2)) and the mirror extension transformation (2.6) to reduce Eqs. (1.16), (1.10) to the generalized kinetic Fokker-Planck equation (2.17). Let us check if the leading coefficients belong to

$$
L_\infty((0,T), C^{\varkappa/3,\varkappa}_{x,v}(\mathbb{R}^6))
$$

. Due to (2.12), for $(t,y,w) \in \mathbb{H}^T_+$, we have

$$
\mathbb{A}(t,y,w) = \begin{pmatrix} \mathcal{A}^{11} & \mathcal{A}^{12} & -\mathcal{A}^{13} \\ \mathcal{A}^{12} & \mathcal{A}^{22} & -\mathcal{A}^{23} \\ -\mathcal{A}^{13} & -\mathcal{A}^{23} & \mathcal{A}^{33} \end{pmatrix} (t,y,Rw). \tag{2.18}
$$

Note that, unless $\mathcal{A}$ satisfies the condition

$$
\mathcal{A}^{i3}(t,y_1,y_2,0,w) = -\mathcal{A}^{i3}(t,y_1,y_2,0,Rw), i = 1,2, \tag{2.19}
$$

the function $\mathcal{A}(t,\cdot)$ might be discontinuous on the set $\{y_3 = 0\} \times \mathbb{R}^3$. Fortunately, (2.19) holds for both Eqs. (1.16) and (1.10). See the details in Appendix E. By the unique solvability result in the S_2 space (see Theorem D.4) applied to Eq. (2.17), we conclude that $f \in S_2$ near the boundary. Thus, for large λ, finite energy weak solutions to Eqs. (1.16) and (1.10) must be finite energy strong solutions in the sense of Definition 1.1.

2.3. **Strategy for proving the S_p bound and Hölder continuity.** We use a bootstrap argument combined with the localization method described above. In particular, we estimate the S_p norm of a localized solution by using the a priori estimate in Theorem D.5 (see (D.5)). When $p > 14$, by the embedding theorem for the S_p space (see Theorem 2.1 of [19]), our localized solutions and their derivatives in the velocity direction are of class $C^\alpha_{\text{kin}}, \alpha = 1 - 14/p$, where

$$C^\alpha_{\text{kin}} = \{u \in L_\infty(\mathbb{R}^7_T) : [u]_{C^\alpha_{\text{kin}}} < \infty\},$$

and

$$[u]_{C^\alpha_{\text{kin}}} := \sup_{z, z' \in \overline{\mathbb{R}^7_T} : z \neq z'} \frac{|u(z) - u(z')|}{(|t - t'|^{1/2} + |x - x' - (t - t')v'|^{1/3} + |v - v'|)^\alpha} < \infty.$$

Unfortunately, the condition $[u]_{C^\alpha_{\text{kin}}} < \infty$ is not preserved under the *local* diffeomorphism Ψ^{-1}. In other words, if $u(t, y, w) \in C^\alpha_{\text{kin}}$, then, $U(t, x, v) = u(t, \Psi(x, v))$ is not of class C^α_{kin} since $x \to U(t, \cdot, v)$ is not defined globally.

To overcome this issue, we work with a weaker space

$$L_\infty((0, T), C^{\alpha/3, \alpha}_{x,v}(\mathbb{R}^6)) \supset C^\alpha_{\text{kin}}.$$

This space has two important properties which we state below.

- If $u \in S_p(\mathbb{R}^7_T)$, then,
$$u, \nabla_v u \in L_\infty((0, T), C^{\alpha/3, \alpha}_{x,v}(\mathbb{R}^6)) \cap C(\overline{\mathbb{R}^7_T}).$$
This follows from the aforementioned Morrey type embedding theorem for the S_p spaces (see [19]).
- If $\eta = \eta(y)$ is a Lipschitz function with compact support, and
$$u_1(t, y, w) = u(t, y, w)\eta(y) \in L_\infty((0, T), C^{\alpha/3, \alpha}_{x,v}(\mathbb{R}^6)),$$
then
$$U_1(z) = u_1(t, \Psi(x, v)) \in L_\infty((0, T), C^{\alpha/3, \alpha}_{x,v}(\mathbb{R}^6).$$

By the above reasoning, we are able to prove that, under certain assumptions, if f is a finite energy strong solution to Eq. (1.16) or (1.10), then $f \in S_p(\Sigma^T)$, and, in addition, $f, \nabla_v f \in L_\infty((0, T), C^{\alpha/3, \alpha}_{x,v}(\overline{\Omega} \times \mathbb{R}^3)) \cap C(\overline{\Sigma^T})$.

2.4. **The reason we consider the linear Landau equation with rough in time coefficients.** To prove the existence of the local in time solution to the nonlinear Landau equation (Eq. (1.3)), one can consider the Picard iteration sequence $f^{(0)} = f_0$,

$$\begin{aligned}
&Y f^{(n+1)} - \nabla_v \cdot (\sigma_{F^{(n)}} \nabla_v f^{(n+1)}) + a_g \cdot \nabla_v f^{(n+1)} + \overline{K}_{f^{(n)}} f^{(n+1)} = 0 \text{ in } \Sigma^T, \\
&f^{(n+1)}(0, x, v) = f_0(x, v), x \in \Omega, \quad v \in \mathbb{R}^3, \\
&f^{(n+1)}_-(t, x, v) = f^{(n+1)}_+(t, x, R_x v), \quad z \in \Sigma^T_-,
\end{aligned} \tag{2.20}$$

where $\sigma_{F^{(n)}}$ is defined by (1.7) with g replaced with $f^{(n)}$. Let us consider the equation for $f^{(1)}$. Even if f_0 is very regular, by using our method, we can only prove that $f^{(1)}, \nabla_v f^{(1)} \in L_\infty((0, T), C^{\alpha/3, \alpha}_{x,v}(\overline{\Omega} \times \mathbb{R}^3)) \cap C(\overline{\Sigma^T})$, $\alpha \in (0, 1]$. Then, for the equation (2.20) with $n = 0$, Assumption 1.10 (see (1.26) - (1.27)) is satisfied with $g = f^{(1)}$.

3. **Finite energy weak solutions to the generalized kinetic Fokker-Planck equation.** In this section, we prove the existence of finite energy weak solution to Eq. (1.16) (see Theorem 1.5). The conclusion of Theorem 1.5 also implies the existence of finite energy weak solutions to Eq (1.10) which is verified in the proof of Proposition 5.1.

3.1. **Discretization of the second-order operator.** The following lemma, which is due to N.V. Krylov, provides a way to rewrite a second-order operator as a pure second-order derivative operator. By using this result, we construct a finite difference approximation of the operator $\nabla_v \cdot (a\nabla_v)$, which we denote by A_h. Thanks to Lemma 3.1, the stencil depends only on the lower eigenvalue bound of the matrix a, and in addition, the 'diffusion' coefficients of A_h are as regular as a.

Lemma 3.1 (Theorem 3.1 of [17])**.** *Denote $l_i = e_i, i = 1, 2, 3$. Then, there exists a number $d_1 > 3$ and*

- *vectors $l_k \in \mathbb{R}^3, k = 4, \ldots, d_1$,*
- *real-analytic functions $\lambda_k, k = 1, \ldots, d_1$, on Sym($\delta$) (see Section 1.2), and a number $\delta_1 = \delta_1(\delta) > 0$*

such that, for any $a \in$ Sym(δ),

$$a^{ij} = \sum_{k=1}^{d_1} \lambda_k(a) l_k^i l_k^j, \quad \delta_1 < |\lambda_k| < \delta_1^{-1}, \ k = 1, \ldots, d_1. \tag{3.1}$$

Let d_1, $\Lambda = \{l_1, \ldots, l_{d_1}\}$ and $\lambda_k, k = 1, \ldots, d_1$, be the number, stencil, and functions in Lemma 3.1. We set

$$a_k(z) = \lambda_k(a(z)), \ k = 1, \ldots, d_1.$$

Then, by the above lemma, there exist a constant $\delta_1 = \delta_1(\delta) > 0$ such that for any z,

$$\delta_1 \le |a_k(z)| \le \delta_1^{-1}, \quad k = 1, \ldots, d_1. \tag{3.2}$$

For any $h \in \mathbb{R}$ and any function u on $\mathbb{R}^3$, we define the following operators:

$$T_{h,l} u(v) = u(v + hl), \quad \delta_{h,l} u(v) = \frac{u(v + hl) - u(v)}{h},$$
$$A_h u = -\sum_{k=1}^{d_1} \delta_{h,-l_k} (a_k \delta_{h,l_k} u). \tag{3.3}$$

We now check the consistency with $\nabla_v \cdot (a\nabla_v)$. Denote

$$\Delta_{h,\xi} = \frac{T_{h,\xi} - 2I_3 + T_{h,-\xi}}{h^2}, \quad \partial_{(\xi)} = \xi_i \partial_{v_i}, \quad \partial_{(\xi)(\xi)} = \xi_i \xi_j \partial_{v_i} \partial_{v_j}. \tag{3.4}$$

Observe that the following product rule holds:

$$\delta_{h,l}(fg) = g\delta_{h,l} f + (T_{h,l} f)\delta_{h,l} g. \tag{3.5}$$

We fix arbitrary $\phi \in C_0^2(\mathbb{R}^3)$. Then, by (3.5), we have

$$A_h \phi = -a_k(\delta_{h,-l_k} \delta_{h,l_k} \phi) + (-\delta_{h,-l_k} a_k)\, T_{h,-l_k} \delta_{h,l_k} \phi. \tag{3.6}$$

We note that the last term equals

$$(-\delta_{h,-l_k} a_k)\,(-\delta_{h,-l_k} \phi).$$

Then, using the Cauchy-Schwartz inequality, Assumption 1.4 (see (1.21)), and the fact that ϕ has bounded support, we get

$$\lim_{h\to 0} (-\delta_{h,-l_k} a_k)\,(-\delta_{h,-l_k} \phi) = \partial_{(l_k)} a_k \partial_{(l_k)} \phi \quad \text{in } L_2(\Sigma^T).$$

By this, (3.6), and (3.1), we conclude

$$\begin{aligned}\lim_{h\to 0} A_h \phi &= a_k \partial_{(l_k)(l_k)} \phi + \partial_{(l_k)} a_k \partial_{(l_k)} \phi \\ &= \nabla_v \cdot (a\nabla_v \phi) \quad \text{in } L_2(\Sigma^T).\end{aligned} \tag{3.7}$$

3.2. **Discretized equation.** For $\varepsilon, h \in (0,1)$, we consider the equation

$$\begin{aligned}
&Y f_{\varepsilon,h} - A_h f_{\varepsilon,h} + b^i \delta_{h,e_i} f_{\varepsilon,h} + \lambda f_{\varepsilon,h} = \mathsf{g} \quad \text{in } \Sigma^T, \\
&(f_{\varepsilon,h})_-(t,x,v) = (1-\varepsilon)(f_{\varepsilon,h})_+(t,x,R_x v) \quad \text{on } \Sigma^T_-, \\
&f_{\varepsilon,h}(0,x,v) = f_0(x,v), \ (x,v) \in \Omega \times \mathbb{R}^3.
\end{aligned} \tag{3.8}$$

Below we state the definition of the solution space. To do that, we first need to introduce a set of test functions from [3].

Definition 3.2. We say that $\phi \in \Phi$ if

- ϕ is continuously differentiable along the characteristic lines $(t+s, x+vs, v)$,
- ϕ, $Y\phi$ are bounded functions on Σ^T,
- ϕ has bounded support, and there is a positive lower bound of the length of the aforementioned characteristic lines inside Σ^T which intersect the support of ϕ.

Remark 3.1. One can show that

$$C^1_0\big(([0,T] \times \overline{\Omega} \times \mathbb{R}^3) \setminus ((0,T) \times \gamma_0 \cup \{0\} \times \partial\Omega \times \mathbb{R}^3 \cup \{T\} \times \partial\Omega \times \mathbb{R}^3)\big) \subset \Phi.$$

A proof can be found in Lemma 2.1 of [7].

Definition 3.3. For $p \in [1,\infty)$, and $\theta \geq 0$, $E_{p,\theta}(\Sigma^T)$ is the Banach space of functions u with the following properties:

- $u, Yu \in L_{p,\theta}(\Sigma^T)$,
- There exist functions $u_\pm \in L_p(\Sigma^T_\pm, |v \cdot n_x|)$, $u(T,\cdot), u(0,\cdot) \in L_p(\Omega \times \mathbb{R}^3)$ such that for any $\phi \in \Phi$ (see Definition 3.2), the following Green's identity holds:

$$\begin{aligned}
&\int_{\Sigma^T} (Yu)\phi + (Y\phi)u \, dz \\
&= \int_{\Omega \times \mathbb{R}^3} u(T,x,v)\phi(T,x,v)\, dxdv - \int_{\Omega \times \mathbb{R}^3} u(0,x,v)\phi(0,x,v)\, dxdv \\
&\quad + \int_{\Sigma^T_+} u_+ \phi \, |v \cdot n_x| \, d\sigma dt - \int_{\Sigma^T_-} u_- \phi \, |v \cdot n_x| \, d\sigma dt.
\end{aligned} \tag{3.9}$$

Remark 3.2. By Proposition 1 of [3], if $u, Yu \in L_p(\Sigma^T)$, then, there exist unique functions $u_\pm \in L_{p,\mathrm{loc}}(\Sigma^T_\pm)$, $u(T,\cdot), u(0,\cdot) \in L_{p,\mathrm{loc}}(\Omega \times \mathbb{R}^3)$ such that (3.9) holds. See p. 393 of [3] for the definition of $L_{p,\mathrm{loc}}(\Sigma^T_\pm)$.

Remark 3.3. For any $\tau \in (0,T)$, there exist functions $u_\tau, u_{\pm;\tau}$ defined on $\Omega \times \mathbb{R}^3$ and $\Sigma^\tau_\pm$, respectively, such that the identity (3.9) holds with $\tau, u_\tau, u_{\pm;\tau}$ in place of $T, u(T,\cdot)$ and $u_\pm$, respectively. Furthermore, it follows from the proof of Proposition 1 in [3] that for a.e. $\tau \in (0,T)$,

$$u_\tau(\cdot) = u|_{t=\tau}, \quad u_{\pm;\tau} = (uI_{t\in(0,\tau)})|_{\gamma_\pm}.$$

Proposition 3.1. *Under assumptions of Theorem 1.5, for any numbers $p > 1, \theta \geq 0, \varepsilon, h \in (0,1]$, and $\mathsf{g} \in L_{p,\theta}(\Sigma^T)$, Eq. (3.8) has a unique (strong) solution $f_{\varepsilon,h} \in E_{p,\theta}(\Sigma^T)$.*

Proof. By Assumptions 1.2 - Assumptions 1.3 (see (1.19) - (1.20)), A_h is a bounded operator on $L_{p,\theta}(\Sigma^T)$. Now the assertion follows from Theorem 1 of [3]. □

3.3. **Uniform bounds for the discretized equation.** The following energy identity contained Proposition 1 of [3] is crucial in the proof of the uniform L_p bounds for $f_{\varepsilon,h}$.

Lemma 3.4. *For any numbers $p \in [1,\infty), \theta \ge 0$ and $f \in E_{p,\theta}(\Sigma^T)$, one has*

$$\begin{aligned}
&\int_{\Omega\times\mathbb{R}^3} \left(|f(T,x,v)|^p - |f(0,x,v)|^p\right) \langle v\rangle^\theta dxdv \\
&+ \int_{\Sigma_+^T} |f_+|^p \langle v\rangle^\theta\, |v\cdot n_x| d\sigma\, dt - \int_{\Sigma_-^T} |f_-|^p \langle v\rangle^\theta\, |v\cdot n_x| d\sigma\, dt \\
&= p\int_{\Sigma^T} (Yf)|f|^{p-1}(sgn f)\langle v\rangle^\theta\, dz.
\end{aligned} \tag{3.10}$$

Lemma 3.5. *Under the assumptions of Theorem 1.5, for any $\varepsilon, h \in (0,1]$ and $\theta \ge 0$, there exists $\lambda_0 = \lambda_0(\delta,\theta,K) > 0$ such that for any $\lambda \ge \lambda_0$, $\mathsf{g} \in L_{2,\theta}(\Sigma^T)\cap L_\infty(\Sigma^T)$, one has*

$$\begin{aligned}
&\int_{\Omega\times\mathbb{R}^3} |f_{\varepsilon,h}|^2(T,x,v)\langle v\rangle^\theta\, dxdv + \delta_1\sum_{k=1}^{d_1}\int_{\Sigma^T} |\delta_{h,l_k} f_{\varepsilon,h}|^2\langle v\rangle^\theta\, dz \\
&+ (\lambda/2)\int_{\Sigma^T} |f_{\varepsilon,h}|^2\langle v\rangle^\theta\, dz + \varepsilon\int_{\Sigma_+^T} |(f_{\varepsilon,h})_+|^2\langle v\rangle^\theta\, |v\cdot n_x| d\sigma dt \\
&\le \lambda^{-1}\int_{\Sigma^T} |\mathsf{g}|^2\langle v\rangle^\theta\, dz + \int_{\Omega\times\mathbb{R}^3} f_0^2\langle v\rangle^\theta\, dxdv,
\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}
&\max\{\|f(T,\cdot)\|_{L_\infty(\Omega\times\mathbb{R}^3)}, \|f_{\varepsilon,h}\|_{L_\infty(\Sigma^T)}, \|(f_{\varepsilon,h})_\pm\|_{L_\infty(\Sigma_\pm^T,|v\cdot n_x|)}\} \\
&\le \lambda^{-1}\|\mathsf{g}\|_{L_\infty(\Sigma^T)} + \|f_0\|_{L_\infty(\Omega\times\mathbb{R}^3)},
\end{aligned}$$

where $\delta_1 = \delta_1(\delta) > 0$ is defined on page 15.

Proof. For the sake of convenience, we omit the summation with respect to $k \in \{1,\ldots,d_1\}$.

Weighted L_2-estimate. First, by Lemma 3.4 with $p = 2$,

$$\begin{aligned}
&\int_{\Omega\times\mathbb{R}^3} \langle v\rangle^\theta |f_{\varepsilon,h}|^2(T,x,v)\, dxdv + 2\lambda\int_{\Sigma^T}\langle v\rangle^\theta |f_{\varepsilon,h}|^2\, dz \\
&+ \varepsilon\int_{\Sigma_+^T}\langle v\rangle^\theta |(f_{\varepsilon,h})_+|^2\, |v\cdot n_x| d\sigma dt \\
&\underbrace{-2\int_{\Sigma^T}\langle v\rangle^\theta (A_h f_{\varepsilon,h})\, f_{\varepsilon,h}\, dz}_{=I_1} + \underbrace{2\int_{\Sigma^T}\langle v\rangle^\theta b^i(\delta_{h,e_i} f_{\varepsilon,h})\, f_{\varepsilon,h}\, dz}_{=I_2} \\
&\le 2\int_{\Sigma^T}\langle v\rangle^\theta \mathsf{g} f_{\varepsilon,h}\, dz + \int_{\Omega\times\mathbb{R}^3} f_0^2\langle v\rangle^\theta\, dxdv.
\end{aligned}$$

Furthermore, by using the product rule (see (3.5)) and change of variables, we get

$$\begin{aligned} I_1 &= 2\int_{\Sigma^T} \langle v\rangle^\theta f_{\varepsilon,h}\, \delta_{h,-l_k} (a_k \delta_{h,l_k} f_{\varepsilon,h})\, dz \\ &= 2\int_{\Sigma^T} a_k\, (\delta_{h,l_k} f_{\varepsilon,h})\, \delta_{h,l_k} (f_{\varepsilon,h} \langle v\rangle^\theta)\, dz \quad \text{(change of variables)} \\ &= 2\int_{\Sigma^T} \langle v\rangle^\theta a_k |\delta_{h,l_k} f_{\varepsilon,h}|^2\, dz \\ &\quad + 2\int_{\Sigma^T} a_k (\delta_{h,l_k} f_{\varepsilon,h})(T_{h,l_k} f_{\varepsilon,h}) \delta_{h,l_k} \langle v\rangle^\theta\, dz =: I_{1,1} + I_{1,2} \quad \text{(product rule).} \end{aligned}$$

By (3.2),

$$I_{1,1} \geq 2\delta_1 \int_{\mathbb{R}^3} \langle v\rangle^\theta |\delta_{h,l_k} f_{\varepsilon,h}|^2\, dz.$$

By the mean value theorem, for $h \in (0,1]$ and fixed k, one has

$$|\delta_{h,l_k} \langle v\rangle^\theta| \leq N(l_k,\theta) \langle v\rangle^{\theta-1}.$$

Next, separating $\delta_{h,l_k} f_{\varepsilon,h}$ from $T_{h,l_k} f_{\varepsilon,h}$ by the Cauchy-Schwartz inequality, we get

$$\begin{aligned} I_{1,2} \geq &- (\delta_1/2) \int_{\Sigma^T} \langle v\rangle^\theta |\delta_{h,l_k} f_{\varepsilon,h}|^2\, dz \\ &- N(\theta,\Lambda)\delta_1^{-3} \int_{\Sigma^T} \langle v\rangle^\theta |T_{h,l_k} f_{\varepsilon,h}|^2\, dz. \end{aligned}$$

By a change of variables and the triangle inequality,

$$\int_{\Sigma^T} \langle v\rangle^\theta |T_{h,l_k} f_{\varepsilon,h}|^2\, dz = \int_{\Sigma^T} T_{h,-l_k} \langle v\rangle^\theta |f_{\varepsilon,h}|^2\, dz \leq N(\Lambda,\theta) \int_{\Sigma^T} \langle v\rangle^\theta |f_{\varepsilon,h}|^2\, dz.$$

Furthermore,

$$\begin{aligned} &2\int_{\Sigma^T} \langle v\rangle^\theta b^i (\delta_{h,e_i} f_{\varepsilon,h})\, f_{\varepsilon,h}\, dz \\ &\geq -(\delta_1/2) \int_{\Sigma^T} \langle v\rangle^\theta |\delta_{h,e_i} f_{\varepsilon,h}|^2\, dz - N(K,\delta_1) \int_{\Sigma^T} \langle v\rangle^\theta |f_{\varepsilon,h}|^2\, dz. \end{aligned}$$

Thus, by the above and the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned} &\int_{\Omega\times\mathbb{R}^3} \langle v\rangle^\theta |f_{\varepsilon,h}|^2 (T,x,v)\, dxdv + \delta_1 \int_{\Sigma^T} \langle v\rangle^\theta |\delta_{h,l_k} f_{\varepsilon,h}|^2\, dz \\ &+ \varepsilon \int_{\Sigma^T_+} \langle v\rangle^\theta |(f_{\varepsilon,h})_+|^2\, |v\cdot n_x| d\sigma dt \\ &+ (\lambda - N) \int_{\Sigma^T} \langle v\rangle^\theta |f_{\varepsilon,h}|^2\, dz \leq \lambda^{-1} \int_{\Sigma^T} \langle v\rangle^\theta \mathsf{g}^2\, dz + \int_{\Omega\times\mathbb{R}^3} f_0^2 \langle v\rangle^\theta\, dxdv, \end{aligned}$$

where $N = N(d_1,\delta_1,\Lambda,K)$. Thus, for $\lambda_0 > 2N$ and $\lambda \geq \lambda_0$, the weighted energy estimate hold.

L_∞ **estimate.** *Step 1: Higher regularity of* $f_{\varepsilon,h}$. Here we show that

$$f_{\varepsilon,h} \in E_p(\Sigma^T), \quad \forall p \in [2,\infty),$$

which allows us to apply Lemma 3.4 in Step 2.

Fix any $p \in (2,\infty)$. By Proposition 3.1 with $\theta = 0$, Eq. (3.8) has a unique solution $\widetilde{f}_{\varepsilon,h} \in E_p(\Sigma^T)$. Denote

$$\mu_n(v) = e^{-|v|^2/n}, \quad \widetilde{f}^{(n)}_{\varepsilon,h} = \mu_n \widetilde{f}_{\varepsilon,h}.$$

Note that $\widetilde{f}^{(n)}_{\varepsilon,h} \in E_2(\Sigma^T)$, and since μ_n satisfies the specular reflection boundary condition,

$$(\widetilde{f}^{(n)}_{\varepsilon,h})_-(t,x,v) = (1-\varepsilon)(\widetilde{f}^{(n)}_{\varepsilon,h})_+(t,x,R_x v) \text{ on } \Sigma^T_-.$$

Furthermore, the function $F^{(n)} = \widetilde{f}^{(n)}_{\varepsilon,h} - f_{\varepsilon,h}$ satisfies the equation

$$\begin{aligned}
&YF^{(n)} - A_h F^{(n)} + b^i \delta_{h,e_i} F^{(n)} + \lambda F^{(n)} = \mathsf{g}(\mu_n - 1) + \text{Comm}_n,\\
&F^{(n)}_-(t,x,v)) = (1-\varepsilon) F^{(n)}_+(t,x,R_x v),\\
&F^{(n)}(0,\cdot) = f_0(\mu_n - 1),
\end{aligned}$$

where

$$\text{Comm}_n = -A_h[\widetilde{f}_{\varepsilon,h}\mu_n] + A_h[\widetilde{f}_{\varepsilon,h}]\mu_n - b^i[\mu_n(\delta_{h,e_i}\widetilde{f}_{\varepsilon,h}) - \delta_{h,e_i}\widetilde{f}^{(n)}_{\varepsilon,h}].$$

Then, since $F^{(n)} \in E_2(\Sigma^T)$, by the above L_2 estimate (3.11), for $\lambda \ge \lambda_0$, we get

$$\begin{aligned}
&\|F^{(n)}(T,\cdot)\|_{L_2(\Omega\times\mathbb{R}^3)} + \|F^{(n)}_\pm\|_{L_2(\Sigma^T_\pm, |v\cdot n_x|)} + \lambda\|F^{(n)}\|_{L_2(\Sigma^T)}\\
&\le N\|f_0(\mu_n - 1)\|_{L_2(\Omega\times\mathbb{R}^3)} + N\|\mathsf{g}(\mu_n - 1)\|_{L_2(\Sigma^T)} + N\|\text{Comm}_n\|_{L_2(\Sigma^T)},
\end{aligned} \tag{3.12}$$

where N is independent of n and $F^{(n)}$. Note that the first two terms on the right-hand side of (3.12) converge to 0 as $n \to \infty$ by the dominated convergence theorem. Thus, to prove that $\widetilde{f}_{\varepsilon,h}$ and $f_{\varepsilon,h}$ coincide a.e. in Σ^T along with their initial values and traces, it suffices to show that $\text{Comm}_n \to 0$ in $L_2(\Sigma^T)$ as $n \to \infty$.

First, by (3.5), (3.6) and (3.4), for a function u on $\mathbb{R}^3$, we have

$$\begin{aligned}
&\delta_{h,e_i}(u\mu_n) - \mu_n\delta_{h,e_i}u = (\delta_{h,e_i}\mu_n)\, T_h u,\\
&A_h[u\mu_n] = a_k(\Delta_{h,l_k}[u\mu_n]) + (\delta_{h,-l_k}a_k)\,\delta_{h,-l_k}[u\mu_n].
\end{aligned} \tag{3.13}$$

Next, by using (3.6), the product rule for the second-order differences

$$\begin{aligned}
\Delta_{h,l_k}[u\mu_n] = \mu_n\Delta_{h,l_k}u + u\Delta_{h,l_k}\mu_n + (\delta_{h,l_k}u)(\delta_{h,l_k}\mu_n)\\
+ (\delta_{h,-l_k}u)(\delta_{h,-l_k}\mu_n),
\end{aligned}$$

and the product rule for the first-order differences (see (3.5)), we conclude from (3.13) that

$$\begin{aligned}
&A_h[u\mu_n] = \mu_n A_h u\\
&+ a_k\big(u\Delta_{h,l_k}\mu_n + (\delta_{h,l_k}u)(\delta_{h,l_k}\mu_n) + (\delta_{h,-l_k}u)(\delta_{h,-l_k}\mu_n)\big)\\
&+ (\delta_{h,-l_k}a_k)(\delta_{h,-l_k}\mu_n)T_{h,-l_k}u.
\end{aligned}$$

Then, by the above identity and the mean value theorem, for $n \ge 1$,

$$\|\text{Comm}_n\|_{L_2(\Sigma^T)} \le N n^{-1}\big(\|\widetilde{f}_{\varepsilon,h}\|_{L_2(\Sigma^T)} + \sum_{k=1}^{d_1}\|\delta_{h,l_k}\widetilde{f}_{\varepsilon,h}|\|_{L_2(\Sigma^T)}\big),$$

where $N = N(h,\delta,K,\Lambda)$. Hence $\text{Comm}_n \to 0$ as $n \to \infty$ in $L_2(\Sigma^T)$. Thus, the assertion of this step is proved.

Step 2: L_p bound. Fix arbitrary $p \in [2,\infty)$. Let us start with the term containing A_h. By a change of variables and the inequality

$$(a-b)(|a|^q a - |b|^q b) \ge N_0(q)(a-b)^2(|a|^q + |b|^q),\ q \ge 0,\ a,b \in \mathbb{R},\ a \ne b,$$

we have

$$
\begin{aligned}
&p \int_{\mathbb{R}^3} \delta_{h,-l_k} (a_k \delta_{h,l_k} f_{\varepsilon,h}) |f_{\varepsilon,h}|^{p-2} f_{\varepsilon,h} \, dv \\
&= p \int_{\mathbb{R}^3} a_k (\delta_{h,l_k} f_{\varepsilon,h}) \, \delta_{h,l_k} (|f_{\varepsilon,h}|^{p-2} f_{\varepsilon,h}) \, dv \geq N_0 p \delta_1 \int_{\mathbb{R}^3} |f_{\varepsilon,h}|^{p-2} |\delta_{h,l_k} f_{\varepsilon,h}|^2 \, dv.
\end{aligned}
$$

Furthermore, by the Young's inequality,

$$
\begin{aligned}
&p \int_{\mathbb{R}^3} b^i (\delta_{h,e_i} f_{\varepsilon,h}) |f_{\varepsilon,h}|^{p-2} f_{\varepsilon,h} \, dv \\
&\geq -N_0 p (\delta_1/2) \int_{\mathbb{R}^3} |f_{\varepsilon,h}|^{p-2} |\delta_{h,e_i} f_{\varepsilon,h}|^2 \, dv - 2 N_0^{-1} \delta_1^{-1} K^2 p \int_{\mathbb{R}^3} |f_{\varepsilon,h}|^p \, dv.
\end{aligned} \tag{3.14}
$$

Since the stencil Λ contains the standard basis $e_i, i = 1, \ldots, d$, we may replace $|\delta_{h,e_i} f_{\varepsilon,h}|^2$ with the sum $\sum_{k=1}^{d_1} |\delta_{h,l_k} f_{\varepsilon,h}|^2$ in the first integral on the right-hand side of (3.14).

Thus, by the above and the Young's inequality, we obtain

$$
\begin{aligned}
&\int_{\Omega \times \mathbb{R}^3} |f_{\varepsilon,h}|^p (T, x, v) \, dx dv + p N_0 (\delta_1/2) \int_{\Sigma^T} |f_{\varepsilon,h}|^{p-2} |\delta_{h,l_k} f_{\varepsilon,h}|^2 \, dz \\
&+ \varepsilon \int_{\Sigma^T_+} |(f_{\varepsilon,h})_+|^p \, |v \cdot n_x| \, d\sigma dt + p (\lambda/2 - N_1(K, \delta_1)) \int_{\Sigma^T} |f_{\varepsilon,h}|^p \, dz \\
&\leq 2^{p-1} \lambda^{1-p} \int_{\Sigma^T} |\mathsf{g}|^p \, dz + \int_{\Omega \times \mathbb{R}^3} |f_0|^p \, dz.
\end{aligned} \tag{3.15}
$$

Finally, we set $\lambda_0 = 2N_1$, and take the p-th root in the above inequality and pass to the limit as $p \to \infty$. This gives the desired L_∞ bound. □

3.4. **Verification of the weak formulation.** The goal of this section is to prove the following Green's type identity for the solution $f_{\varepsilon,h}$ (see (3.16)). This result is used in the proof of Theorem 1.5 when we show that a solution constructed by the weak* compactness method satisfies the weak formulation (see (1.17) in Definition 1.1).

By $C_0^{1,1,2}(\overline{\Sigma^T})$ we denote the space of all functions ϕ with compact support such that $\partial_t \phi, \nabla_x \phi, \nabla_v \phi, D_v^2 \phi \in C(\overline{\Sigma^T})$.

Lemma 3.6. *Under the assumptions of Proposition 3.1, for any $\phi \in C_0^1(\overline{\Sigma^T})$,*

$$
\begin{aligned}
&- \int_{\Sigma^T} (Y \phi) f_{\varepsilon,h} \, dz + \int_{\Omega \times \mathbb{R}^3} (f_{\varepsilon,h}(T, x, v) \phi(T, x, v) - f_{\varepsilon,h}(0, x, v) \phi(0, x, v)) \, dx dv \\
&+ \int_{\Sigma^T_+} (f_{\varepsilon,h})_+ \phi \, |v \cdot n_x| d\sigma dt - \int_{\Sigma^T_-} (f_{\varepsilon,h})_- \phi \, |v \cdot n_x| d\sigma dt \\
&- \int_{\Sigma^T} f_{\varepsilon,h} A_h \phi \, dz + \int_{\Sigma^T} f_{\varepsilon,h} \delta_{h,-e_i} (b^i \phi) \, dz + \lambda \int_{\Sigma^T} f_{\varepsilon,h} \phi \, dz = \int_{\Sigma^T} \mathsf{g} \phi \, dz.
\end{aligned} \tag{3.16}
$$

It follows from (3.9) that (3.16) holds for any $\phi \in \Phi$ (see Definition 3.2). To prove Lemma 3.6, it suffices to extend the identity (3.9) for $\phi \in C_0^1(\overline{\Sigma^T})$, which is done in the next lemma.

Lemma 3.7. *For any $u \in E_2(\Sigma^T)$, the Green's formula (3.9) holds for any $\phi \in C_0^1(\overline{\Sigma^T})$.*

Proof. Let $\eta_j, j = 1, \dots, m$ be a partition of unity in Ω. Since

$$\sum_j \eta_j (Y\phi) = \sum_j Y(\phi\eta_j) - \sum_j \phi v \cdot \nabla_x \eta_j = \sum_j Y(\phi\eta_j),$$

we may assume that $\operatorname{supp}\phi \subset [0,T] \times \overline{B_r(x_0)} \times \overline{B_r(v_0)}$, where $x_0 \in \partial\Omega$, and $r > 0$ is sufficiently small. Furthermore, we may also assume that (2.1) holds. We will reduce the problem to the case when $\Omega = \mathbb{R}^3_-$ by using the boundary flattening diffeomorphism Π defined below, which is somewhat simpler than Ψ defined in (2.2) - (2.3). The same argument also works if we use Ψ instead of Π but require Ω to be a C^3 domain.

Let $\Pi : \Omega \cap B_{r_0}(x_0) \times \mathbb{R}^3 \to \mathbb{R}^3_- \times \mathbb{R}^3, (x,v) \to (y,w)$ be the diffeomorphism defined as Ψ in Subsection 2.1 but with ψ^{-1} replaced with

$$\pi^{-1}(y) = (y_1, y_2, y_3 + \rho(y_1, y_2)). \tag{3.17}$$

In particular,

$$w = (D\pi)v.$$

We fix any smooth function ξ on $\mathbb{R}$ such that

$$\begin{cases} \xi(t) = 0, & t \le 1, \\ \xi(t) \in (0,1), & t \in (1,2), \\ \xi(t) = 1, & t \ge 2. \end{cases}$$

Furthermore, for $\varepsilon > 0$, denote

$$\xi_\varepsilon(y,w) = \xi\big(\frac{y_3^2 + w_3^2}{\varepsilon^2}\big), \quad \chi_\varepsilon(x,v) = \xi_\varepsilon(\Pi(x,v)),$$
$$\phi_\varepsilon(z) = \phi(z)\chi_\varepsilon(x,v)\xi\left(\frac{t}{\varepsilon}\right)\xi\left(\frac{T-t}{\varepsilon}\right).$$

Note that ϕ_ε vanishes near $t = 0$, $t = T$ and the grazing set $(0,T) \times \gamma_0$ (see (1.11)). By Remark 3.1, $\phi_\varepsilon \in \Phi$ (see Definition 3.2). Therefore, by (3.9),

$$\int_{\Sigma^T} \big[(Yu)\phi_\varepsilon + (Y\phi_\varepsilon)u\big]\, dz - \int_{\Sigma^T_+} u_+ \phi_\varepsilon\, |v \cdot n_x|\, d\sigma dt + \int_{\Sigma^T_-} u_- \phi_\varepsilon\, |v \cdot n_x|\, d\sigma dt = 0.$$

We will pass to the limit as $\varepsilon \to 0$ in this identity. Note that by the dominated convergence theorem we only need to show that

$$\begin{aligned} &\lim_{\varepsilon \to 0} \int_{\Sigma^T} (Y\phi_\varepsilon) u\, dz \\ &\quad = \int_{\Sigma^T} (Y\phi) u\, dz - \int_{\Sigma^T} \big[u(T,x,v)\phi(T,x,v) - u(0,x,v)\phi(0,x,v)\big]\, dx dv. \end{aligned}$$

Invoke the notation of Subsection 2.1 and replace the diffeomorphism Ψ with Π. Then, by (A.3) and the fact that $J = |\det \frac{\partial x}{\partial y}|^2 \equiv 1$, we get

$$\begin{aligned} \int_{\Sigma^T} (Y\phi_\varepsilon) u\, dz &= \int_{\Sigma^T} (\partial_t \phi_\varepsilon) u\, dy dw dt \\ &+ \int_{\mathbb{H}^T_-} (w \cdot \nabla_y \widehat{\phi}_\varepsilon) \widehat{u}\, dy dw dt - \int_{\mathbb{H}^T_-} (X \cdot \nabla_w \widehat{\phi}_\varepsilon) \widehat{u}\, dy dw dt =: I_1 + I_2 + I_3, \end{aligned} \tag{3.18}$$

where $\widehat{u}$ is defined in (2.5).

Convergence of I_1. We split I_1 into 5 integrals given by

$$I_{1,1} = \int_{\Sigma^T} \xi\big(\frac{t}{\varepsilon}\big)\xi\big(\frac{T-t}{\varepsilon}\big) u\partial_t\phi\chi_\varepsilon\, dz,$$

$$I_{1,2} = \varepsilon^{-1}\int_{\Sigma^T} \xi'\big(\frac{t}{\varepsilon}\big)\xi\big(\frac{T-t}{\varepsilon}\big) u(0,x,v)\phi\chi_\varepsilon\, dz,$$

$$I_{1,3} = -\varepsilon^{-1}\int_{\Sigma^T} \xi'\big(\frac{T-t}{\varepsilon}\big)\xi\big(\frac{t}{\varepsilon}\big) u(T,x,v)\phi\chi_\varepsilon\, dz,$$

$$I_{1,4} = \varepsilon^{-1}\int_{\Sigma^T} \xi'\big(\frac{t}{\varepsilon}\big)\xi\big(\frac{T-t}{\varepsilon}\big) \big[u(t,x,v)-u(0,x,v)\big]\phi\chi_\varepsilon\, dz,$$

$$I_{1,5} = -\varepsilon^{-1}\int_{\Sigma^T} \xi\big(\frac{t}{\varepsilon}\big)\xi'\big(\frac{T-t}{\varepsilon}\big) \big[u(t,x,v)-u(T,x,v)\big]\phi\chi_\varepsilon\, dz.$$

The first integral converges to $\int_{\Sigma^T} u\partial_t\phi\, dydwdt$ due to the dominated convergence theorem. By a change of variables and the dominated convergence theorem combined with the fact that $\int \xi'(t)\, dt = 1$, we conclude that

$$I_{1,2} \to \int_{\Sigma^T} u(0,x,v)\phi(0,x,v)\, dxdv, \quad I_{1,3} \to -\int_{\Sigma^T} u(T,x,v)\phi(T,x,v)\, dxdv.$$

Changing variables and using the Cauchy-Schwartz inequality give

$$|I_{1,4}| \le N\varepsilon^{-1}\int_{\varepsilon}^{2\varepsilon} \|u(t,\cdot)-u(0,\cdot)\|_{L_2(\Omega\times\mathbb{R}^3)}\, dt.$$

The expression on the right-hand side of the above inequality converges to 0 as $\varepsilon \to 0$ because $u \in C([0,T], L_2(\Omega\times\mathbb{R}^3))$ (see Lemma B.4). Similarly, the same convergence holds for $I_{1,5}$. Hence, by the above, we conclude

$$I_1 \to \int_{\Sigma^T} [u(0,x,v)\phi(0,x,v) - u(T,x,v)\phi(T,x,v)]\, dxdv. \tag{3.19}$$

Convergence of I_2 and I_3. Note that

$$I_2 = I_{2,1} + I_{2,2} := \int_{\mathbb{H}^T_-} (w\cdot\nabla_y\widehat{\phi})\xi_\varepsilon(y,w)\xi\big(\frac{t}{\varepsilon}\big)\xi\big(\frac{T-t}{\varepsilon}\big)\widehat{u}\, dydwdt$$
$$+ 2\int_{\mathbb{H}^T_-:\varepsilon^2<y_3^2+w_3^2<2\varepsilon^2} \widehat{u}\widehat{\phi}\xi\big(\frac{t}{\varepsilon}\big)\xi\big(\frac{T-t}{\varepsilon}\big)\frac{w_3y_3}{\varepsilon^2}\,\xi'\big(\frac{y_3^2+w_3^2}{\varepsilon^2}\big)\, dydwdt,$$

$$I_3 = I_{3,1} + I_{3,2} := -\int_{\mathbb{H}^T_-} (X\cdot\nabla_w\widehat{\phi})\xi_\varepsilon(y,w)\xi\big(\frac{t}{\varepsilon}\big)\xi\big(\frac{T-t}{\varepsilon}\big)\widehat{u}\, dydwdt$$
$$- 2\varepsilon^{-2}\int_{\mathbb{H}^T_-:\varepsilon^2<y_3^2+w_3^2<2\varepsilon^2} \widehat{u}\widehat{\phi}\xi\big(\frac{t}{\varepsilon}\big)\xi\big(\frac{T-t}{\varepsilon}\big)X_3w_3\,\xi'\big(\frac{y_3^2+w_3^2}{\varepsilon^2}\big)\, dydwdt.$$

By the dominated convergence theorem,

$$\lim_{\varepsilon\to 0}(I_{2,1}+I_{3,1}) = \int_{\mathbb{H}^T_-} (w\cdot\nabla_y\widehat{\phi})\widehat{u}\, dydwdt - \int_{\mathbb{H}^T_-} (X\cdot\nabla_w\widehat{\phi})\widehat{u}\, dydwdt.$$

Then, by the above equality, (3.18) - (3.19), it suffices to show that

$$I_{2,2}, I_{3,2} \to 0. \tag{3.20}$$

By the dominated convergence theorem,

$$|I_{2,2}| \le \int_{\mathbb{H}^T_-:\varepsilon^2<|y_3^2+w_3^2|<2\varepsilon^2} |\widehat{u}\widehat{\phi}|\, dydwdt \to 0 \quad \text{as } \varepsilon\to 0.$$

Furthermore, since Ω is a C^2 bounded domain, it follows from (3.17) and (2.9) that X is bounded on the support of $\widehat{\phi}$. By this, the Cauchy-Schwartz inequality, and a change of variables, we get

$$|I_{3,2}| \le N\varepsilon^{-1}\|\widehat{u}\, I_{y_3^2+w_3^2<2\varepsilon^2}\|_{L_2(\mathbb{H}^T_-)}$$
$$\times \|X_3\widehat{\phi}\|_{L_2^{t,y_1,y_2,w_1,w_2}(\mathbb{R}^5_T)L_\infty^{y_3,w_3}((-\infty,0)\times\mathbb{R})}\|\xi'\big(\frac{y_3^2+w_3^2}{\varepsilon^2}\big)\|_{L_2(\mathbb{R}^2)} \to 0$$

as $\varepsilon \to 0$. Thus, (3.20) holds, and the lemma is proved. □

3.5. **Proof of Theorem 1.5.** By Proposition 3.1, for any $\varepsilon, h \in (0,1)$, there exists a unique (strong) solution $f_{\varepsilon,h} \in E_{2,\theta}(\Sigma^T)$ to Eq. (3.8). Furthermore, by the uniform bounds in Lemma 3.5, the Banach-Alaoglu theorem, and the Eberlein-Smulian theorem, there exists functions $f, f^\star_T$, and $f^\star_\pm$ that satisfy the condition (*i* of Definition 1.1 and

$$\begin{aligned}&f_{\varepsilon,h} \to f \text{ weakly in } L_{2,\theta}(\Sigma^T), \quad f_{\varepsilon,h}(T,\cdot) \to f^\star_T \text{ weakly in } L_{2,\theta}(\Omega\times\mathbb{R}^3),\\ &(f_{\varepsilon,h})_\pm \to f^\star_\pm \text{ in the weak* topology of } L_\infty(\Sigma^T_\pm, |v\cdot n_x|).\end{aligned} \tag{3.21}$$

It remains to show that f, $f^\star_T$, and $f^\star_\pm$ satisfy conditions (*ii* and (*iii* of Definition 1.1.

Fix any $\phi \in C^{1,1,2}_0(\overline{\Sigma^T})$. Then, by Lemma 3.6, the weak formulation (3.16) holds. By (3.21), we pass to the limit as $\varepsilon, h \to 0$ in (3.8) and conclude that the specular reflection boundary condition is satisfied. Next, note that by (1.21) in Assumption 1.4, $\nabla_v b \in L_\infty(\Sigma^T)$, and, therefore,

$$\delta_{h,-e_i}(b^i\phi) \to -\partial_{v_i}(b^i\phi) \quad \text{as } h \to 0 \text{ in } L_\infty(\Sigma^T).$$

Combining this with (3.7) and (3.21), we pass to the limit in (3.16) (with respect to a subsequence). This proves the validity of the condition (*iii* of Definition 1.1 with $\phi \in C^{1,1,2}_0(\overline{\Sigma^T})$. By using a limiting argument, we prove that the weak formulation (1.17) holds with $\phi \in C^1_0(\overline{\Sigma^T})$. Finally, passing to the limit in the bounds in Lemma 3.5, we obtain the estimates (1.22).

4. **Regularity of the finite energy weak solutions to the kinetic Fokker-Planck equation.**

Here we prove Theorem 1.6, Corollary 1.2, and Theorem 1.7.

Proof of Theorem 1.6. Let $\zeta \in C^\infty_0(\mathbb{R})$ such that $\zeta(0) = 1$. Replacing f with $f - f_0\zeta$ and g with

$$\mathsf{g} - f_0\partial_t\zeta - v\cdot\nabla_x f_0\zeta + \Delta_v(f_0\zeta) - b\cdot\nabla_v(f_0\zeta) - \lambda(f_0\zeta) \in L_{2,\theta}(\Sigma^T),$$

we may assume that $f_0 \equiv 0$ in the identity (1.17).

Let $\eta_k, k = 1,\dots,m$, be the standard partition of unity in Ω (see, for example, Section 8.4 of [16]) such that supp $\eta_1 \subset \Omega$, $0 \le \eta_k \le 1$, $k = 1,\dots,m$, and

$$|\nabla_x\eta_k| \le N/r_0, \quad \begin{cases}\eta_k = 1 & \text{in } B_{r_0/4}(\mathsf{x}_k)\\ \eta_k = 0 & \text{in } B^c_{r_0/2}(\mathsf{x}_k)\end{cases}, \quad k = 2,\dots,m, \tag{4.1}$$

where $\mathsf{x}_i \in \partial\Omega$, and r_0 is the number in Subsection 2.1 such that (2.1) and (2.11) hold.

Note that

$$f_k := f\eta_k\langle v\rangle^{\theta-2} \tag{4.2}$$

satisfies the identity

$$\begin{aligned}(4.3)\quad & Yf_k - \Delta_v f_k + b \cdot \nabla_v f_k + \lambda f_k \\ & = \mathsf{g}\eta_k \langle v\rangle^{\theta-2} + f(\langle v\rangle^{\theta-2} v \cdot \nabla_x \eta_k \\ & + \eta_k b \cdot \nabla_v \langle v\rangle^{\theta-2} - \Delta_v \langle v\rangle^{\theta-2}\eta_k) - 2\nabla_v f \cdot \nabla_v \langle v\rangle^{\theta-2}\eta_k =: \mathsf{g}_k\end{aligned}$$

in the weak sense, i.e.,

$$\begin{aligned}(4.4)\quad & -\int_{\Sigma^T} (Y\phi) f_k \, dz + \int_{\Omega\times\mathbb{R}^3} f_k(T,x,v)\phi(T,x,v)\, dxdv \\ & + \int_{\Sigma^T} \nabla_v f_k \cdot \nabla_v \phi \, dz + \int_{\Sigma^T} (b\cdot\nabla_v f_k)\phi\, dz + \lambda \int_{\Sigma^T} f_k \phi\, dz = \int_{\Sigma^T} \mathsf{g}_k \phi \, dz.\end{aligned}$$

Interior estimate. We extend f_1, b, and g_1 for x outside Ω by 0 and for negative t by replacing them with $f_1 1_{t\ge 0}$, $b 1_{t\ge 0}$, and $\mathsf{g}_1 1_{t\ge 0}$, respectively. Taking a test function $\phi \in C_0^\infty((-\infty,T)\times\mathbb{R}^6)$ removes the integral over the hyperplane $t = T$ in (4.4). Furthermore, the identity (4.4) holds with Ω, Σ^T replaced with $\mathbb{R}^3$ and $(-\infty,T)\times\mathbb{R}^6$, respectively. Since $\nabla_v f_1 \in L_2((-\infty,T)\times\mathbb{R}^6)$, we have $Yf_1 \in \mathbb{H}_2^{-1}((-\infty,T)\times\mathbb{R}^6)$, and by this, $f_1 \in \mathbb{S}_2((-\infty,T)\times\mathbb{R}^6)$. Then, f_1 satisfies (4.3) in $\mathbb{H}_2^{-1}((-\infty,T)\times\mathbb{R}^6)$.

Let $\lambda_0 > 0$ be the constant in Theorem D.4 (i) with $p = 2$. Then, due to Theorem D.4 (i), for any $\lambda \ge \lambda_0$, the equation

$$Yu_1 - \Delta_v u_1 + b\cdot\nabla_v u_1 + \lambda u_1 = \mathsf{g}_1$$

has a unique solution $u_1 \in S_2((-\infty,T)\times\mathbb{R}^6)$, and, furthermore, $u_1 = 0$ a.e. on $(-\infty,0)\times\mathbb{R}^6$. Then, $U_1 = f_1 - u_1 \in \mathbb{S}_2((-\infty,T)\times\mathbb{R}^6)$ satisfies the equation

$$YU_1 - \Delta_v U_1 + b\cdot\nabla_v U_1 + \lambda U_1 = 0 \quad \text{in } \mathbb{H}_2^{-1}((-\infty,T)\times\mathbb{R}^6).$$

Therefore, by the energy identity of Lemma B.3, integration by parts, and the Cauchy-Schwartz inequality, for a.e. $s \in (-\infty,T)$, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^6} U_1^2(s,x,v)\, dxdv + \int_{(-\infty,s)\times\mathbb{R}^6} |\nabla_v U_1|^2\, dz \\ &\qquad + (\lambda - N_1(K))\int_{(-\infty,s)\times\mathbb{R}^6} U_1^2\, dz \le 0.\end{aligned}$$

It follows that for $\lambda > N_1(K)$, we have $U_1 = 0$ a.e. in $(-\infty,T)\times\mathbb{R}^6$. Then, by Theorem D.4 (i), for $\lambda \ge \lambda_0$ with, possibly, larger λ_0, one has

$$\begin{aligned}(4.5)\quad & \|f_1\|_{S_2((-\infty,T)\times\mathbb{R}^6)} = \|u_1\|_{S_2((-\infty,T)\times\mathbb{R}^6)} \le N\|\mathsf{g}_1\|_{L_2((-\infty,T)\times\mathbb{R}^6)} \\ & \le N(\|\mathsf{g}\|_{L_{2,\theta-2}(\Sigma^T)} + \|f\|_{L_{2,\theta-1}(\Sigma^T)} + \|\nabla_v f\|_{L_{2,\theta-3}(\Sigma^T)}),\end{aligned}$$

where $N = N(K,\Omega,\theta)$. In addition, by the embedding theorem for S_p spaces (see Theorem 2.1 of [19]),

$$\||f_1| + |\nabla_v f_1|\|_{L_{7/3}((-\infty,T)\times\mathbb{R}^6)}$$

is bounded by the right-hand side of (4.5).

Boundary estimate. We fix some $k \in \{2,\dots,m\}$. We redo the construction of the mirror extension mapping in Section 2.1.1 with x_0 replaced with x_k. As above, we extend $\overline{f_k}$ and $\overline{\mathsf{g}_k}$ by 0 for $t \le 0$. Then, by the argument of Section 2.1.2, for any $\phi \in C_0^\infty((-\infty,T)\times\mathbb{R}^6)$, $\overline{f_k}$ satisfies the equation

$$\int_{(-\infty,T)\times\mathbb{R}^6} \big(- Y\phi + (\nabla_w \overline{f_k})^T \mathbb{A} \nabla_w \phi - (\mathbb{X}\cdot\nabla_w\phi)\overline{f_k} \\ + (\mathbb{B}\cdot\nabla_w \overline{f_k})\phi + \lambda \overline{f_k}\phi - \overline{\mathsf{g}_k}\phi\big)\, d\widehat{z} = 0. \tag{4.6}$$

Here

– $\mathbb{A} = \mathbb{A}(y)$ is defined by (2.8), (2.10), (2.12) with $a = I_3$,
– $\mathbb{B}$ is defined by formulas (2.8), (2.14) for $t \geq 0$,
– $\mathbb{X}$ is given by (2.9), (2.13) for $t \geq 0$,

and for $t < 0$, $\mathbb{B}, \mathbb{X}$ are equal to 0. As above, due to $\nabla_w \overline{f_k} \in L_2((-\infty,T)\times\mathbb{R}^6)$ and (4.6), we have $\overline{f_k} \in \mathbb{S}_2((-\infty,T)\times\mathbb{R}^6)$. Below, we will show that $\overline{f} \in S_2((-\infty,T)\times\mathbb{R}^6)$.

First, observe that since ψ is a local C^3 diffeomorphism,

$$|\mathbb{X}| \leq N(\Omega)|w|^2, \quad |\nabla_w \mathbb{X}| \leq N(\Omega)|w|, \tag{4.7}$$

and, therefore by (4.7) and (4.3),

$$\overline{\mathsf{g}_k} - \nabla_w\cdot(\mathbb{X}\overline{f_k}) \in \mathbb{H}_2^{-1}((-\infty,T)\times\mathbb{R}^6).$$

Next, note that $\mathbb{A}$ satisfies the nondegeneracy condition (see (1.19) in Assumption 1.2) with $\delta = \delta(\Omega) > 0$ because ψ defined by (2.3) is a local diffeomorphism. Furthermore, by the conclusion of Appendix E, $\mathbb{A}$ is a Lipschitz function on $\mathbb{R}^3$, and, thus, by Remark D.1, Theorem D.4 (i) is applicable. By this theorem, there exist $\lambda_0 = \lambda_0(K,\Omega) > 0$ such that for any $\lambda \geq \lambda_0$, the equation

$$\partial_t u_k + w\cdot\nabla_y u_k - \nabla_w\cdot(\mathbb{A}\nabla_w u_k) + \mathbb{B}\cdot\nabla_w u_k \\ + \lambda u_k = \overline{\mathsf{g}_k} - \nabla_w\cdot(\mathbb{X}\overline{f_k}) \tag{4.8}$$

has a unique solution $u_k \in S_2((-\infty,T)\times\mathbb{R}^6)$ (see (1.14)). Then, repeating the argument used for the interior estimate, we conclude that $\overline{f_k} \equiv u_k$ for $\lambda \geq \lambda_0(\Omega,K,\theta)$ with, perhaps, different λ_0. Hence, by Theorem D.4 (i),

$$\|\overline{f_k}\|_{S_2((-\infty,T)\times\mathbb{R}^6)} \leq N\bigg(\|\overline{\mathsf{g}_k}\|_{L_2((-\infty,T)\times\mathbb{R}^6)} \\ + \|\nabla_w\overline{f_k}\|_{L_{2,2}((-\infty,T)\times\mathbb{R}^6)} + \|\overline{f_k}\|_{L_{2,1}((-\infty,T)\times\mathbb{R}^6)}\bigg), \tag{4.9}$$

where $N = N(K,\Omega,\theta)$. Again, by the embedding theorem for S_p spaces,

$$\||\overline{f_k}| + |\nabla_w\overline{f_k}|\|_{L_{7/3}((-\infty,T)\times\mathbb{R}^6)}$$

is controlled by the right-hand side of (4.9). Then, by Lemma B.1, we have

$$\|f_k\|_{S_2(\Sigma^T)} + \||f_k| + |\nabla_v f_k|\|_{L_{7/3}(\Sigma^T)} \\ \leq N\bigg(\|\mathsf{g}\|_{L_{2,\theta-2}(\Sigma^T)} + \|\nabla_v f\|_{L_{2,\theta}(\Sigma^T)} + \|f\|_{L_{2,\theta-1}(\Sigma^T)}\bigg).$$

Finally, combining the above estimate with (4.5), we prove the theorem. □

Proof of Corollary 1.2. Let f_1, f_2 be finite energy weak solutions to Eq. (1.16). Then, by Theorem 1.6, $f_1, f_2 \in S_2(\Sigma^T)$.

Next, let $\lambda' > 0$ be a number, which we will choose later, and denote $F = (f_1 - f_2)e^{-\lambda' t}$. Note that $F \in S_2(\Sigma^T)$ satisfies Eq. (1.16) with $\mathsf{g} \equiv 0$, $f_0 \equiv 0$, and

$\lambda + \lambda'$ in place of λ. Then, by a variant of the energy identity (see Lemma B.2), we get

$$\int_{\Omega \times \mathbb{R}^3} |F(T, x, v)|^2 \, dx dv + 2(\lambda + \lambda') \int_{\Sigma^T} |F|^2 \, dz$$
$$- 2 \int_{\Sigma^T} (\Delta_v F) F \, dz + 2 \int_{\Sigma^T} (b \cdot \nabla_v F) F \, dz = 0.$$

Integrating by parts and using the Cauchy-Schwartz inequality, we obtain

$$\int_{\Omega \times \mathbb{R}^3} |F(T, x, v)|^2 \, dx dv + \int_{\Sigma^T} |\nabla_v F|^2 \, dz + 2(\lambda + \lambda' - N(K)) \int_{\Sigma^T} |F|^2 \, dz \le 0.$$

Thus, taking $\lambda' > N$, we conclude $F \equiv 0$. The corollary is proved. □

To prove Theorem 1.7, we need the following result.

Lemma 4.1. *Let*

- $p \ge 2, T > 0, \lambda \ge 0, \theta \ge 2$ *be numbers, and* $r > 1$ *be determined by the relation*
$$\frac{1}{r} = \frac{1}{p} - \frac{1}{14}, \tag{4.10}$$
- *Assumptions 1.2 - 1.3 be satisfied,*
- $f \in S_{p,\theta'}(\Sigma^T)$ *with* $\theta' \ge \theta - 2$, *and* $f, \nabla_v f \in L_{r,\theta}(\Sigma^T)$,
- $f_{\pm} \in L_{\infty}(\Sigma^T_{\pm}, |v \cdot n_x|)$, $f(T, \cdot) \in L_2(\Omega \times \mathbb{R}^3)$, $f(\cdot, 0) \equiv 0$,
- $\mathsf{g} \in L_{r,\theta-2}(\Sigma^T)$,
- f *satisfy the specular boundary condition and the identity (1.18) a.e. and, in addition,* $f(0, \cdot) \equiv 0$.

Then, the following assertions hold.

(i) One has $f \in S_{r,\theta-2}(\Sigma^T)$, *and*

$$\|f\|_{S_{r,\theta-2}(\Sigma^T)} \le N \Big(\|\mathsf{g}\|_{L_{r,\theta-2}(\Sigma^T)} + \|\nabla_v f\|_{L_{r,\theta}(\Sigma^T)} + \|f\|_{L_{r,\theta-1}(\Sigma^T)} \Big), \tag{4.11}$$

where $N = N(p, K, \theta, \Omega)$.

(ii) If $p < 14$, *then,* $f, \nabla_v f \in L_{r,\theta-2}(\Sigma^T)$, *and, furthermore,*

$$\| |f| + |\nabla f| \|_{L_{r,\theta-2}(\Sigma^T)}$$

is less than the right-hand side of (4.11).

(iii) If $p > 14$, *then,* $f, \nabla_v f \in L_{\infty}((0, T), C^{\alpha/3,\alpha}_{x,v}(\overline{\Omega} \times \mathbb{R}^3)) \cap C(\overline{\Sigma^T})$, *where* $\alpha = 1 - 14/p$, *and* $C^{\alpha/3,\alpha}_{x,v}(\overline{\Omega} \times \mathbb{R}^3)$ *is defined in (1.12). In addition, the norms*

$$\|f\|_{L_{\infty}((0,T), C^{\alpha/3,\alpha}_{x,v}(\overline{\Omega} \times \mathbb{R}^3))}, \quad \|\nabla f\|_{L_{\infty}((0,T), C^{\alpha/3,\alpha}_{x,v}(\overline{\Omega} \times \mathbb{R}^3))}$$

are bounded above by the right-hand side of (4.11).

Proof of Lemma 4.1. Denote

$$B^p = \begin{cases} L_r(\mathbb{R}^7_T), \text{ if } p \in [2, 14), \text{where } r \text{ is determined by (4.10)}, \\ L_{\infty}((0, T), C^{\alpha/3,\alpha}_{x,v}(\mathbb{R}^6)), \text{ if } p > 14, \text{ where } \alpha = 1 - 14/p. \end{cases} \tag{4.12}$$

We follow the argument of Theorem 1.6 closely.

Interior estimate. Recall that

$$f_1 = f \eta_1 \langle v \rangle^{\theta - 2} \in S_p(\mathbb{R}^7_T)$$

solves Eq. (4.3) with the zero initial-value condition and note that due to the assumptions of this lemma, for any $k = 1, \dots, m$, we have

$$\mathsf{g}_k = \mathsf{g}\eta_k \langle v\rangle^{\theta-2} + f(\langle v\rangle^{\theta-2} v\cdot\nabla_x\eta_k \\ + \eta_k b\cdot\nabla_v\langle v\rangle^{\theta-2} - \Delta_v\langle v\rangle^{\theta-2}\eta_k) - 2\nabla_v f_k\cdot\nabla_v\langle v\rangle^{\theta-2}\eta_k \in L_r(\mathbb{R}^7_T).$$

Then, due to Lemma D.6, $f_1 \in S_r(\mathbb{R}^7_T)$. Using Theorem D.5 (see Remark D.2), we get

$$\|f_1\|_{S_r(\Sigma^T)} \le N(p, K, \theta, \Omega)\bigg(\|\mathsf{g}\|_{L_{r,\theta-2}(\Sigma^T)} \\ + \|f\|_{L_{r,\theta-1}(\Sigma^T)} + \|\nabla_v f\|_{L_{r,\theta-3}(\Sigma^T)}\bigg). \tag{4.13}$$

Boundary estimate. Recall that f_k is given by (4.2), and $\overline{f_k}$ is its "mirror extension" defined as in (2.6). The function $\overline{f_k} \in S_p(\mathbb{R}^7_T)$ satisfies Eq. (4.8). In addition, due to (4.7),

$$\overline{\mathsf{g}_k} - \nabla_w\cdot(\mathbb{X}\overline{f_k}) \in L_r(\mathbb{R}^7_T).$$

Then, again, by Lemma D.6, we conclude that $\overline{f_k} \in S_r(\mathbb{R}^7_T)$.

Next, by the a priori estimate of Theorem D.5 applied to Eq. (4.8) and using (4.7), we get

$$\|\overline{f_k}\|_{S_r(\mathbb{R}^7_T)} \le N\big(\|\overline{\mathsf{g}_k}\|_{L_r(\mathbb{R}^7_T)} + \|\nabla_w\overline{f_k}\|_{L_{r,2}(\mathbb{R}^7_T)} + \|\overline{f_k}\|_{L_{r,1}(\mathbb{R}^7_T)}\big),$$

where $N = N(p, K, \theta, \Omega) > 0$. Then, by this and Lemma B.1,

$$\|f_k\|_{S_r(\Sigma^T)} \le N\big(\|\mathsf{g}\|_{L_{r,\theta-2}(\Sigma^T)} + \|\nabla_v f\|_{L_{r,\theta}(\Sigma^T)} + \|f\|_{L_{r,\theta-1}(\Sigma^T)}\big). \tag{4.14}$$

Thus, the desired estimate (4.11) follows from (4.13) combined with (4.14). As in the proof of Theorem 1.6, by using the embedding theorem for the S_p space (Theorem 2.1 of [19]), we prove the assertions (ii) and (iii). □

Proof of Theorem 1.7. We will proof the theorem by using a bootstrap argument. As in the proof of Theorem 1.6, we may assume that $f_0 \equiv 0$.

Let $r_k, k \ge 1$ be the numbers defined as follows:

$$r_1 = 2, \quad \frac{1}{r_k} = \frac{1}{r_{k-1}} - \frac{1}{14},\ k = 2, \dots, 6, \\ \frac{1}{r_7} = \frac{1}{p} + \frac{1}{14},\ r_8 = p. \tag{4.15}$$

Note that $r_6 = 7$, $r_7 \in \big(7, 14)$ since $p > 14$. First, by Theorem 1.6, the inequality (1.23) holds. By this estimate, Lemma 4.1, and an induction argument for $k = 1, \dots, 6$, we conclude

$$\|f\|_{S_{7,\theta-12}(\Sigma^T)} + \||f| + |\nabla_v f|\|_{L_{14,\theta-12}(\Sigma^T)} \\ \le N\sum_{k=1}^{6}\|\mathsf{g}\|_{L_{r_k,\theta-2k}(\Sigma^T)} + N\|\nabla_v f\|_{L_{2,\theta}(\Sigma^T)} + N\|f\|_{L_{2,\theta-1}(\Sigma^T)}. \tag{4.16}$$

Then, by (4.16) and the interpolation inequality, $f, \nabla_v f \in L_{r_7,\theta-12}(\Sigma^T)$. Again, by Lemma 4.1, (4.16), and the fact that $r_8 = p$, we get

$$\|f\|_{S_{r_7,\theta-14}(\Sigma^T)} + \||f| + |\nabla_v f|\|_{L_{p,\theta-14}(\Sigma^T)} \\ \le N\big(\|\mathsf{g}\|_{L_{2,\theta-2}(\Sigma^T)} + \|\mathsf{g}\|_{L_{p,\theta-4}(\Sigma^T)} \\ + \|\nabla_v f\|_{L_{2,\theta}(\Sigma^T)} + \|f\|_{L_{2,\theta-1}(\Sigma^T)}\big).$$

Finally, applying Lemma 4.1 once more, we prove the theorem. □

5. Proof of the main result for the linear Landau equation.

5.1. Unique solvability result for the simplified viscous linear Landau equation.

Here, we prove the existence/uniqueness of the finite energy strong solutions to Eq. (1.10) in the sense of Definition 1.1. We follow the scheme that we used to show the existence and uniqueness for the kinetic Fokker-Planck equation (1.16).

Proposition 5.1 (Existence of finite energy weak solution)**.** *Let*

- *Ω be a bounded C^2 domain,*
- *$T > 0$, $\nu \in (0, 1]$, $\varkappa \in (0, 1]$, $\theta \ge 0$ be numbers,*
- *Assumption 1.10 (see (1.26) - (1.27)) be satisfied,*
- *$f_0 \in L_{2,\theta}(\Omega \times \mathbb{R}^3) \cap L_\infty(\Omega \times \mathbb{R}^3)$, $h \in L_{2,\theta}(\Sigma^T) \cap L_\infty(\Sigma^T)$.*

Then, there exists a number $\varepsilon \in (0, 1)$ (independent of $\Omega, T, \nu, \varkappa, \theta, K$) such that, if, additionally,

$$\|g\|_{L_\infty(\Sigma^T)} \le \varepsilon,$$

then, there exists $\lambda_0 = \lambda_0(\nu, K, \theta) > 0$, such that for any $\lambda \ge \lambda_0$, Eq. (1.10) has a finite energy weak solution f in the sense of Definition 1.1, and, furthermore,

$$\begin{aligned}
&\|f^{\star}_T\|_{L_{2,\theta}(\Omega\times\mathbb{R}^3)} + \|\nabla_v f\|_{L_{2,\theta}(\Sigma^T)} + \lambda^{1/2}\|f\|_{L_{2,\theta}(\Sigma^T)}\\
&\le N(\lambda^{-1/2}\|h\|_{L_{2,\theta}(\Sigma^T)} + \|f_0\|_{L_{2,\theta}(\Omega\times\mathbb{R}^3)}),\\
&\max\{\|f^{\star}_T\|_{L_\infty(\Omega\times\mathbb{R}^3)}, \|f\|_{L_\infty(\Sigma^T)}, \|f^{\star}_{\pm}\|_{L_\infty(\Sigma^T_{\pm}, |v\cdot n_x|)}\}\\
&\le \lambda^{-1}\|h\|_{L_\infty(\Sigma^T)} + \|f_0\|_{L_\infty(\Omega\times\mathbb{R}^3)},
\end{aligned}$$

where $N = N(\theta, K, \nu) > 0$.

Proof. We use Theorem 1.5. We need to check that $\sigma_G + \nu I_3$ and a_g satisfy Assumptions 1.2 - 1.4 (see (1.19) - (1.21)).

Assumption 1.2. By Lemma Lemma 2.4 of [15], for sufficiently small ε, there exist constants $C_1, C_2 > 0$ such that

$$C_1\langle v\rangle^{-3} I_3 \le \sigma_G(z) \le C_2\langle v\rangle^{-1} I_3, \ \forall z \in \mathbb{R}^7_T. \tag{5.1}$$

Then, (1.19) in Assumption 1.2 holds with $\delta = \nu$.

Assumption 1.3 and 1.4. By using the argument of Lemma 2 in [9] (see the details in the proof of Corollary 3.2.1 in [4]) combined with Assumption 1.10 ((1.26) - (1.27)), one can show that

$$\||a_g| + |\nabla_v a_g| + |\nabla_v \sigma_G|\|_{L_\infty(\Sigma^T)} \le N(K), \tag{5.2}$$

and, hence, (1.20) and (1.21) hold.

Now all the assertions of this proposition follow directly from Theorem 1.5. □

The following proposition is analogous to Theorems 1.6 and 1.7.

Proposition 5.2 (S_p regularity of the simplified linear viscous Landau equation)**.** *Let*

- *Ω be a bounded C^3 domain,*
- *$T > 0$, $\nu \in (0, 1]$, $\varkappa \in (0, 1]$, $\theta \ge 16$, $p > 14$ be numbers,*
- *$f_0 \in \mathcal{O}_{2,\theta-2} \cap \mathcal{O}_{p,\theta-4}$, $h \in L_{2,\theta-2}(\Sigma^T) \cap L_{p,\theta-4}(\Sigma^T)$,*
- *Assumption 1.10 ((1.26) - (1.27)) be satisfied,*
- *$\|g\|_{L_\infty(\Sigma^T)} \le \varepsilon$,*

– *f be a finite energy weak solution to Eq.* (1.10) *with parameter θ in the sense of Definition 1.1, which exists due to Proposition 5.1.*

Then, there exists a number $\varepsilon \in (0,1)$ (independent of $\Omega, T, \nu, \varkappa, \theta, K, p$) and $\lambda_0 = \lambda_0(\nu, K, \theta, \varkappa, \Omega) \geq 1$ such that $f \in S_{2,\theta-2}(\Sigma^T) \cap S_{p,\theta-16}(\Sigma^T) \cap C(\overline{\Sigma^T})$, and, furthermore,

$$\begin{aligned} \|f\|_{S_{2,\theta-2}(\Sigma^T)} + \|f\|_{S_{p,\theta-16}(\Sigma^T)} \leq N(&\|\nabla_v f\|_{L_{2,\theta}(\Sigma^T)} + \|f\|_{L_{2,\theta-1}(\Sigma^T)} \\ &+ \|h\|_{L_{2,\theta-2}(\Sigma^T)} + \|h\|_{L_{p,\theta-4}(\Sigma^T)} \\ &+ |f_0|_{\mathcal{O}_{2,\theta-2}} + |f_0|_{\mathcal{O}_{p,\theta-4}}), \end{aligned} \tag{5.3}$$

where $N = N(\nu, K, \theta, \varkappa, p, \Omega) > 0$. Finally, f is a finite energy strong solution to Eq. (1.10).

Proof. We fix $\varepsilon > 0$ sufficiently small such that (5.1) holds.

Step 1: S_2-regularity. We inspect the argument of Theorem 1.6. As in the aforementioned theorem, since the right-hand side of (5.3) contains the terms $|f_0|_{\mathcal{O}_{2,\theta-2}}, |f_0|_{\mathcal{O}_{p,\theta-4}}$, we may assume that $f_0 \equiv 0$. We use a partition of unity argument combined with the mirror extension method and the S_p estimate of Theorem D.4. Let $\eta_k, k = 1, \dots, m$ be the partition of unity defined in the proof of Theorem 1.6 (see (4.1)). Note that $f_k = f \eta_k \langle v \rangle^{\theta-2}$ satisfies the equation

$$Y f_k - \nabla_v \cdot \big((\sigma_G + \nu I_3) \nabla_v f_k\big) - a_g \cdot \nabla_v f_k + \lambda f_k =: h_k, \tag{5.4}$$

where

$$\begin{aligned} h_k = & h \eta_k \langle v \rangle^{\theta-2} + f(\langle v \rangle^{\theta-2} v \cdot \nabla_x \eta_k) \\ & - (\sigma_G^{ij} + \nu \delta_{ij})(\partial_{v_i v_j} \langle v \rangle^{\theta-2}) f \eta_k \\ & + (-\partial_{v_i} \sigma_G^{ij} - a_g^j)(\partial_{v_j} \langle v \rangle^{\theta-2}) f \eta_k \\ & - 2(\sigma_G^{ij} + \nu \delta_{ij})(\partial_{v_i} \langle v \rangle^{\theta-2})(\partial_{v_j} f) \eta_k. \end{aligned}$$

Interior estimate. We conclude that $f_1 \in S_2(\mathbb{R}^7_T)$ by using the same argument as in the proof of Theorem 1.6. One minor difference is that one needs to apply Theorem D.4 of Appendix D with $a = \sigma_G + \nu I_3$, $b^i = -a_g^i - \partial_{v_j} \sigma^{ji}, i = 1,2,3,$ and $c = 0$. Let us check the conditions of this theorem. Note that Assumptions 1.2 (see (1.19)) and D.3 (see (D.3)) hold due to (5.1) and (5.2). Furthermore, by Lemma C.1, $\sigma_G \in L_\infty((0,T), C^{\varkappa/3,\varkappa}_{x,v}(\overline{\Omega} \times \mathbb{R}^3))$. We extend σ_G to $\mathbb{R}^7_T$ so that $L_\infty(0,T), C^{\varkappa/3,\varkappa}_{x,v}(\mathbb{R}^6))$ norm is bounded by $N(K, \varkappa, \Omega)$. Then, due to Remark D.1, Assumption D.1 (γ) (see (D.1)) is satisfied for any $\gamma \in (0,1)$. Then, by the aforementioned theorem, for sufficiently large $\lambda_0 = \lambda_0(\nu, K, \varkappa, \theta, \Omega) \geq 1$ and any $\lambda \geq \lambda_0$, one has $f_1 \in S_2(\mathbb{R}^7_T)$, and

$$\begin{aligned} \|f_1\|_{S_2(\mathbb{R}^7_T)} &\leq N \|h_1\|_{L_2(\mathbb{R}^7_T)} \\ &\leq N(\|h\|_{L_{2,\theta}(\Sigma^T)} + \|f\|_{L_{2,\theta-1}(\Sigma^T)} + \|\nabla_v f\|_{L_{2,\theta-3}(\Sigma^T)}), \end{aligned} \tag{5.5}$$

where $N = N(\nu, K, \varkappa, \theta, \Omega) > 0$.

Boundary estimate. As in the proof of Theorem 1.6, for $k \in \{2, \dots, m\}$, the function $\overline{f_k} 1_{t \geq 0} \in \mathbb{S}_2((-\infty, T) \times \mathbb{R}^6)$ satisfies the identity

$$\begin{aligned} &\int_{(-\infty,T)\times\mathbb{R}^6} \big(-Y\phi + (\nabla_w \overline{f_k})^T \mathbb{A} \nabla_w \phi + (\mathbb{B} \cdot \nabla_w \overline{f_k})\phi + \lambda \overline{f_k} \phi\big)\, dy dw dt \\ &= \int_{(-\infty,T)\times\mathbb{R}^6} \big(\overline{h_k} \phi - (\mathbb{X} \cdot \nabla_w \phi) \overline{f_k}\big)\, dy dw dt \end{aligned} \tag{5.6}$$

for any $\phi \in C_0^\infty((-\infty, T) \times \mathbb{R}^6)$, where

- $\mathbb{A}$ is defined by (2.8) (2.10), and (2.12) with a replaced with $\sigma_G + \nu I_3$,
- $\mathbb{B}$ is defined by (2.8) and (2.14) with b replaced with $-a_g$,
- $\mathbb{X}$ is given by (2.9) and (2.13)

for $t \geq 0$, and $\mathbb{A} = \nu I_3$ and $\mathbb{B}, \mathbb{X}$ are both zero for $t < 0$. Recall the argument used in the proof of Theorem 1.6. This time, one needs to apply Theorem D.4 to Eq. (5.6). Let us check their assumptions. Note that by (5.1) - (5.2), $\mathbb{A}$ and $\mathbb{B}$ are bounded functions, and by the discussion in Appendix E, and by Lemma C.1,

$$\|\mathbb{A}\|_{L_\infty((0,T), C_{x,v}^{\varkappa/3,\varkappa}(\mathbb{R}^6))} \leq N(K, \varkappa, \Omega).$$

Then, by Theorem D.4 and the estimates (4.7), we get

$$\begin{aligned} \|\overline{f_k}\|_{S_2(\mathbb{R}^7_T)} &\leq N(\|\overline{h_k}\|_{L_2(\mathbb{R}^7_T)} + \|\nabla_w \cdot (\mathbb{X}\overline{f_k})\|_{L_2(\mathbb{R}^7_T)}) \\ &\leq N(\|h\|_{L_{2,\theta-2}(\Sigma^T)} + \|f\|_{L_{2,\theta-1}(\Sigma^T)} + \|\nabla_v f\|_{L_{2,\theta}(\Sigma^T)}), \end{aligned} \tag{5.7}$$

where $N = N(\nu, K, \theta, \varkappa, \Omega) > 0$. Finally combining (5.5) with (5.7) and using Lemma B.1, we conclude that $f \in S_{2,\theta-2}(\Sigma^T)$ and that the estimate (5.3) holds. As in the proof of Theorem 1.6, the $L_{7/3,\theta-2}(\Sigma^T)$ norm estimate of $f, \nabla_v f$ is obtained by combining the above estimates with the embedding theorem for S_p spaces in [19].

Step 2: S_p **regularity.** To finish the proof, we repeat word-for-word the argument of Theorem 1.7 and Lemma 4.1. □

Corollary 5.1. *Any two finite energy solutions to Eq.* (1.10) *coincide.*

Proof. Since due to Proposition 5.2, any finite energy weak solution to Eq. (1.10) must be of class $S_2(\Sigma^T)$, we may use the energy identity of Lemma B.2. The rest of the argument is similar to that of Corollary 1.2 (see page 25). □

5.2. **Unique solvability result for the viscous linear Landau equation.** We now prove the unique solvability for large $\lambda > 0$ for the equation

$$\begin{aligned} &Yf - \nabla_v \cdot (\sigma_G \nabla_v f) - \nu \Delta_v f - a_g \cdot \nabla_v f + Cf + \lambda f = h \text{ in } \Sigma^T, \\ &f(0, \cdot) = f_0(\cdot) \text{ in } \Omega \times \mathbb{R}^3, \quad f_-(t, x, v) = f_+(t, x, R_x v), \quad z \in \Sigma^T_-, \end{aligned} \tag{5.8}$$

where C is some linear operator, for example $-\overline{K}_g$ defined in (1.8).

To implement a perturbation argument, we will work with the following weighted kinetic Sobolev spaces.

Definition 5.1. Let $T > 0$, $\theta \geq 16$, $p > 14$, $\lambda > 0$ be numbers. We say that $f \in \mathcal{K}_{\theta,\lambda,p}(\Sigma^T)$ if the following hold:

1. $f \in S_{2,\theta-2}(\Sigma^T) \cap S_{p,\theta-16}(\Sigma^T) \cap L_{2,\theta}(\Sigma^T) \cap L_\infty(\Sigma^T)$,
2. $\nabla_v f \in L_{2,\theta}(\Sigma^T)$,
3. $f(0, \cdot) \in L_{2,\theta}(\Omega \times \mathbb{R}^3) \cap L_\infty(\Omega \times \mathbb{R}^3) \cap \mathcal{O}_{2,\theta-2} \cap \mathcal{O}_{p,\theta-16}$, $f(T, \cdot) \in L_{2,\theta}(\Omega \times \mathbb{R}^3)$, $f_\pm \in L_\infty(\Sigma^T_\pm, |v \cdot n_x|)$,
4. $f_-(t, x, v) = f_+(t, x, R_x v)$ a.e. on Σ^T_-.

The $\mathcal{K}_{\theta,\lambda,p}(\Sigma^T)$-norm is defined as

$$\begin{aligned}
&\|f\|_{\mathcal{K}_{\theta,\lambda,p}(\Sigma^T)} = \lambda\|f\|_{L_{2,\theta}(\Sigma^T)} + \lambda\|f\|_{L_\infty(\Sigma^T)} \\
&+\|\nabla_v f\|_{L_{2,\theta}(\Sigma^T)} + \|f\|_{S_{2,\theta-2}(\Sigma^T)} + \|f\|_{S_{p,\theta-16}(\Sigma^T)} \\
&+\|f_\pm\|_{L_\infty(\Sigma^T_\pm,|v\cdot n_x|)} + \|f(T,\cdot)\|_{L_{2,\theta}(\Omega\times\mathbb{R}^3)} \\
&+\|f(0,\cdot)\|_{L_{2,\theta}(\Omega\times\mathbb{R}^3)} + \|f(0,\cdot)\|_{L_\infty(\Omega\times\mathbb{R}^3)} \\
&+\|f(0,\cdot)\|_{\mathcal{O}_{2,\theta-2}} + \|f(0,\cdot)\|_{\mathcal{O}_{p,\theta-16}}.
\end{aligned} \tag{5.9}$$

Assumption 5.2. There exists a linear operator C on $\mathcal{K}_{\theta,\lambda,p}(\Sigma^T)$ and a constant $\kappa > 0$ such for any $u \in \mathcal{K}_{\theta,\lambda,p}(\Sigma^T)$,

$$\|Cu\|_{L_{2,\theta}(\Sigma^T)} + \|Cu\|_{L_\infty(\Sigma^T)} \le \kappa(\|u\|_{L_{2,\theta}(\Sigma^T)} + \|u\|_{L_\infty(\Sigma^T)}). \tag{5.10}$$

Remark 5.1. An example of an operator C that satisfies (5.10) is the operator $-\overline{K}_g$ given by (1.8). If Assumption 1.10 (see (1.26) - (1.27)) holds and $\|g\|_{L_\infty(\Sigma^T)}$ is sufficiently small, then, by Lemmas 2.9 and formula (7.5) of [15], for any $u \in L_{2,\theta}(\Sigma^T) \cap L_\infty(\Sigma^T)$ such that $\nabla_v u \in L_2(\Sigma^T)$, one has

$$\begin{aligned}
&\|\overline{K}_g u\|_{L_{2,\theta}(\Sigma^T)} \le N(\theta)(\|u\|_{L_{2,\theta}(\Sigma^T)} + \|u\|_{L_\infty(\Sigma^T)}), \\
&\|\overline{K}_g u\|_{L_\infty(\Sigma^T)} \le N(\theta)\|u\|_{L_\infty(\Sigma^T)}.
\end{aligned}$$

Proposition 5.3. *Let*

- *Ω be a bounded C^3 domain,*
- *$T > 0$, $\nu \in (0,1]$, $\varkappa \in (0,1]$, $\theta \ge 16$, $p > 14$ be numbers,*
- *$f_0 \in L_{2,\theta}(\Omega\times\mathbb{R}^3) \cap L_\infty(\Omega\times\mathbb{R}^3) \cap \mathcal{O}_{2,\theta-2} \cap \mathcal{O}_{p,\theta-4}$, $h \in L_{2,\theta}(\Sigma^T) \cap L_\infty(\Sigma^T)$,*
- *Assumption 1.10 (see (1.26) - (1.27)) be satisfied,*
- *$\|g\|_{L_\infty(\sigma^T)} \le \varepsilon$.*

Then, for sufficiently small $\varepsilon > 0$ (independent of $\Omega, T, \nu, \varkappa, \theta, K, p$), there exists some $\lambda_0 = \lambda_0(\theta, K, \varkappa, p, \Omega, \nu) > 0$ such that Eq. (5.8) has a unique finite energy strong solution in the sense of Definition 1.1, and

$$\begin{aligned}
\|f\|_{\mathcal{K}_{\theta,\lambda,p}(\Sigma^T)} \le N(&\|h\|_{L_{2,\theta}(\Sigma^T)} + \|h\|_{L_\infty(\Sigma^T)} \\
&+ \|f_0\|_{L_{2,\theta}(\Omega\times\mathbb{R}^3)} + \|f_0\|_{L_\infty(\Omega\times\mathbb{R}^3)} \\
&+ |f_0|_{\mathcal{O}_{2,\theta-2}} + |f_0|_{\mathcal{O}_{p,\theta-4}}),
\end{aligned} \tag{5.11}$$

where $N = N(\theta, K, \Omega, p, \varkappa, \nu) > 0$.

Proof. Let $\varepsilon > 0$ be a number such that (5.1) holds. For an arbitrary function $\xi \in \mathcal{K}_{\theta,\lambda,p}(\Sigma^T)$, let us consider the equation

$$\begin{aligned}
&Yu - \nabla_v\cdot(\sigma_G\nabla_v u) - \nu\Delta_v u - a_g\cdot\nabla_v u + \lambda u = h - C\xi \text{ in } \Sigma^T, \\
&u(0,\cdot) = f_0(\cdot) \text{ in } \Omega\times\mathbb{R}^3, \quad u_-(t,x,v) = u_+(t,x,R_x v), \quad z \in \Sigma^T_-.
\end{aligned} \tag{5.12}$$

Note that by Assumption 5.2 (see (5.10)), the right-hand side of Eq. (5.12) is of class $L_{2,\theta}(\Sigma^T) \cap L_\infty(\Sigma^T)$. Then, by this, Propositions 5.1 - 5.2 and Corollary 5.1, there exists $\lambda_0 = \lambda_0(\theta, K, \Omega, \varkappa, \nu) \ge 1$, such that for any $\lambda \ge \lambda_0$, Eq. (5.12), has a unique finite energy strong solution u in the sense of Definition 1.1, and, in addition,

$$
\begin{aligned}
&\|u(T,\cdot)\|_{L_{2,\theta}(\Omega\times\mathbb{R}^3)} + \lambda^{1/2}\|u\|_{L_{2,\theta}(\Sigma^T)} + \|\nabla_v u\|_{L_{2,\theta}(\Sigma^T)}\\
&\le N\big(\|f_0\|_{L_{2,\theta}(\Omega\times\mathbb{R}^3)} + \lambda^{-1/2}\|h\|_{L_{2,\theta}(\Sigma^T)}\\
&\quad + \lambda^{-1/2}\|\xi\|_{L_{2,\theta}(\Sigma^T)} + \lambda^{-1/2}\|\xi\|_{L_\infty(\Sigma^T)}\big),\\
&\max\{\|u(T,\cdot)\|_{L_\infty(\Omega\times\mathbb{R}^3)}, \|u\|_{L_\infty(\Sigma^T)}, \|u_\pm\|_{L_\infty(\Sigma^T_\pm,|v\cdot n_x|)}\}\\
&\le N\big(\|f_0\|_{L_\infty(\Omega\times\mathbb{R}^3)} + \lambda^{-1}\|h\|_{L_\infty(\Sigma^T)}\\
&\quad + \lambda^{-1}\|\xi\|_{L_{2,\theta}(\Sigma^T)} + \lambda^{-1}\|\xi\|_{L_\infty(\Sigma^T)}\big),\\
&\|u\|_{S_{2,\theta-2}(\Sigma^T)} + \|u\|_{S_{p,\theta-16}(\Sigma^T)}\\
&\le N\big(\||u| + |\nabla_v u|\|_{L_{2,\theta}(\Sigma^T)} + |f_0|_{\mathcal{O}_{2,\theta-2}} + |f_0|_{\mathcal{O}_{p,\theta-4}}\\
&\quad + \|h\|_{L_{2,\theta-2}(\Sigma^T)} + \|h\|_{L_{p,\theta-4}(\Sigma^T)} + \|\xi\|_{L_{2,\theta}(\Sigma^T)} + \|\xi\|_{L_\infty(\Sigma^T)}\big),
\end{aligned}
\tag{5.13}
$$

where $N = N(\theta, p, K, \Omega, \kappa, \varkappa, \nu) > 0$. Furthermore, for any $\xi \in \mathcal{K}_{\theta,\lambda,p}(\Sigma^T)$, we denote $R_\lambda \xi = u$, where u is the finite energy strong solution to (5.12). Thus, by (5.13), R_λ is a bounded linear operator on $\mathcal{K}_{\theta,\lambda,p}(\Sigma^T)$.

Next, note that, since a solution to (5.8) is a fixed point of R_λ, it suffices to show that R_λ is a contraction on $\mathcal{K}_{\theta,\lambda,p}(\Sigma^T)$. For any $\xi_1, \xi_2 \in \mathcal{K}_{\theta,\lambda,p}(\Sigma^T)$, by (5.9), (5.13), and the fact that $\lambda_0 \ge 1$, we get

$$\|R_\lambda\xi_1 - R_\lambda\xi_2\|_{\mathcal{K}_{\theta,\lambda,p}(\Sigma^T)} \le N\big(\|\xi_1-\xi_2\|_{L_\infty(\Sigma^T)} + \|\xi_1-\xi_2\|_{L_{2,\theta}(\Sigma^T)}\big),$$

where N is independent of λ. Furthermore, by the definition of the $\mathcal{K}_{\theta,\lambda,p}(\Sigma^T)$ norm, for $\lambda \ge \lambda_0$,

$$\|\xi_1-\xi_2\|_{L_{2,\theta}(\Sigma^T)} + \|\xi_1-\xi_2\|_{L_\infty(\Sigma^T)} \le \lambda_0^{-1}\|\xi_1-\xi_2\|_{\mathcal{K}_{\theta,\lambda,p}(\Sigma^T)}.$$

Thus, for $\lambda_0 > N+1$, R_λ is a contraction mapping. Thus, Eq. (5.8) has a unique solution f of class $\mathcal{K}_{\theta,\lambda,p}(\Sigma^T)$. To prove that f satisfies the weak formulation (1.17) for any $\phi \in C^1_0(\overline{\Sigma^T})$, we define a Picard iteration sequence $f^{(0)} \equiv f_0$, $f^{(n+1)} = R_\lambda f^{(n)}$, which converges to f in $\mathcal{K}_{\theta,\lambda,p}(\Sigma^T)$. We pass to the limit in the weak formulation for $f^{(n)}$ and use (5.10) in Assumption 5.2. □

5.3. **Proof of Theorem 1.11.** To prove the existence part, we work with the viscous linear Landau equation

$$
\begin{aligned}
&Yf + \mathsf{L}f - \nu\Delta_v f + \Gamma[g,f] + \lambda f = h,\\
&f(0,\cdot,\cdot) = f_0, \quad f_-(t,x,v) = f_+(t,x,R_x v), \quad z \in \Sigma^T_-.
\end{aligned}
\tag{5.14}
$$

An equivalent form of this equation is

$$
\begin{aligned}
&Yf - \nabla_v\cdot(\sigma_G\nabla_v f) - \nu\Delta_v f - a_g\cdot\nabla_v f - \overline{K}_g f + \lambda f = h \text{ in } \Sigma^T,\\
&f(0,\cdot) = f_0(\cdot) \text{ in } \Omega\times\mathbb{R}^3, \quad f_-(t,x,v) = f_+(t,x,R_x v), \quad z \in \Sigma^T_-.
\end{aligned}
\tag{5.15}
$$

Thanks to Proposition 5.3, Eq. (5.15) has a unique strong solution (in the sense of Definition 1.9) of class $\mathcal{K}_{\theta,\lambda,p}(\Sigma^T)$. However, the a priori bound in (5.11) has a constant depending on ν, and, hence, such estimate cannot be used in the method of vanishing viscosity. To establish a priori estimates that are uniform in ν, we use the following ingredients: the weighted L_2 bound (see Lemma 5.3) and the S_p estimate in Proposition 5.4. Once we prove the former, we bootstrap by using the latter. We point out that in contrast to the kinetic Fokker-Planck equation, we do not derive a priori bounds of $\|f_\pm\|_{L_\infty(\Sigma^T_\pm,|v\cdot n_x|)}$ by applying the L_p energy identity for the operator Y (c.f. Lemma 3.5). The present authors are not aware of such estimates for Eq. (5.14). Instead, we use the S_p estimates to bound $\|f_\pm\|_{L_\infty(\Sigma^T_\pm, |v\cdot n_x|)}$.

Lemma 5.3. *Let $\nu \in [0,1], \lambda, \theta \ge 0$, $T > 0$ be numbers, f be a finite energy strong solution to Eq. (5.14) in the sense of Definition 1.9. Assume that $h \in L_{2,\theta}(\Sigma^T)$ and $f_0 \in L_{2,\theta}(\Omega \times \mathbb{R}^3)$. Then, there exists a constant $\varepsilon = \varepsilon(\theta) > 0$ such that if*

$$\|g\|_{L_\infty(\Sigma^T)} \le \varepsilon,$$

then, one has

$$\begin{aligned}&\|f(T,\cdot)\|_{L_{2,\theta}(\Omega\times\mathbb{R}^3)} + \|f\|_{\sigma,\theta} + (\lambda^{1/2}+1)\|f\|_{L_{2,\theta}(\Sigma^T)}\\ &\le N(\|h\|_{L_{2,\theta}(\Sigma^T)} + \|f_0\|_{L_{2,\theta}(\Omega\times\mathbb{R}^3)}),\end{aligned}$$

where $\|\cdot\|_{\sigma,\theta}$ is defined in (1.24) and $N = N(\theta, T) > 0$.

Proof. We follow the argument of Lemma 8.2 in [15].

Let $\lambda' > 0$ be a number which we will determine later. Multiplying Eq. (5.14) by $e^{-2\lambda' t}$, and using the energy identity in Lemma B.2 with $\langle v\rangle^\theta$ give

$$\begin{aligned}&\|f(T,\cdot)e^{-\lambda' T}\|^2_{L_{2,\theta}(\Omega\times\mathbb{R}^3)} + 2\int_{\Sigma^T} (\mathsf{L} f) f \langle v\rangle^\theta e^{-2\lambda' t}\, dz\\ &+ 2\int_{\Sigma^T} \Gamma[g,f] f\langle v\rangle^\theta e^{-2\lambda' t}\, dz + 2(\lambda+\lambda')\|f e^{-\lambda' t}\|^2_{L_{2,\theta}(\Sigma^T)}\\ &\le 2\int_{\Sigma^T} f h \langle v\rangle^\theta e^{-2\lambda' t}\, dz + \|f_0\|^2_{L_{2,\theta}(\Omega\times\mathbb{R}^3)}.\end{aligned} \tag{5.16}$$

By Lemmas 2.7 and 2.8 of [15] and (1.25), there exist $N_0, N_1 > 0$, depending only on θ, such that

$$\begin{aligned}&\int_{\Sigma^T} (\mathsf{L} f) f\langle v\rangle^\theta e^{-2\lambda' t}\, dz \ge (1/2)\|f e^{-\lambda' t}\|^2_{\sigma,\theta} - N_0 \|f e^{-\lambda' t}\|^2_{L_{2,\theta}(\Sigma^T)},\\ &\left|\int_{\Sigma^T} \Gamma[g,f] f\langle v\rangle^\theta e^{-2\lambda' t}\, dz\right| \le N_1 \|g\|_{L_\infty(\mathbb{R}^7_T)} \|f e^{-\lambda' t}\|^2_{\sigma,\theta} \le N_1 \varepsilon \|f e^{-\lambda' t}\|^2_{\sigma,\theta}.\end{aligned}$$

Combining this with (5.16) and the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}&\|f(T,\cdot)e^{-\lambda' T}\|^2_{L_{2,\theta}(\Omega\times\mathbb{R}^3)} + (1-2N_1\varepsilon)\|f e^{-\lambda' t}\|^2_{\sigma,\theta}\\ &+ (2\lambda + \lambda' - N_0(\theta))\|f e^{-\lambda' t}\|^2_{L_{2,\theta}(\Sigma^T)}\\ &\le \|f_0\|^2_{L_{2,\theta}(\Omega\times\mathbb{R}^3)} + (\lambda')^{-1}\|h e^{-\lambda' t}\|^2_{L_{2,\theta}(\Sigma^T)}.\end{aligned}$$

Taking $\varepsilon < (4N_1)^{-1}$ and $\lambda' > 2N_0$, we prove the desired estimate. □

Remark 5.2. Note that by Remark 1.3, one can extract the estimate of $\|f\|_{L_{2,\theta-3}(\Sigma^T)}$ from the above result.

We will use the next proposition with θ replaced with $\theta - 3$.

Proposition 5.4 (S_p bound)**.** *Let*

- *Ω be a bounded C^3 domain,*
- *$T > 0$, $\nu \in [0,1]$, $\lambda \ge 0$, $\varkappa \in (0,1]$, $p > 14$ be numbers,*
- *Assumption 1.10 (see (1.26) - (1.27)) hold,*
- *$\|g\|_{L_\infty(\Sigma^T)} \le \varepsilon$,*
- $$\|f\|_{L_{2,\theta}(\Sigma^T)} + \|\nabla_v f\|_{L_{2,\theta}(\Sigma^T)} \le M$$
  for some $M > 0$,

– *f is a finite energy strong solution to Eq.* (5.15) *(see Definition 1.9) with $f_0 \equiv 0$ and $h \in L_{2,\theta}(\Sigma^T) \cap L_{p,\theta}(\Sigma^T)$,*
– *$f \in \mathcal{K}_{\theta,\lambda,p}(\Sigma^T)$ (see Definition 5.1).*

Then, there exist numbers $\varepsilon \in (0,1)$, $\theta = \theta(\varkappa, p) > 1$, and $\theta' = \theta'(\varkappa, p), \theta'' = \theta''(\varkappa, p) \in (1, \theta)$ such that

$$\begin{aligned} &\|f\|_{S_{2,\theta'}(\Sigma^T)} + \|f\|_{S_{p,\theta''}(\Sigma^T)} \\ &+ \|f_{\pm}\|_{L_\infty(\Sigma^T_{\pm}, |v\cdot n_x|)} + \|f\|_{L_\infty((0,T), C^{\alpha/3,\alpha}_{x,v}(\overline{\Omega}\times\mathbb{R}^3))} \\ &+ \|\nabla_v f\|_{L_\infty((0,T), C^{\alpha/3,\alpha}_{x,v}(\overline{\Omega}\times\mathbb{R}^3))} + \|f\|_{L_\infty(\Sigma^T_{\pm}, |v\cdot n_x|)} \\ &\le N(M + \|h\|_{L_{2,\theta}(\Sigma^T)} + \|h\|_{L_{p,\theta}(\Sigma^T)}), \end{aligned} \tag{5.17}$$

where $N = N(\theta, p, \varkappa, K, \Omega) > 0$, and $\alpha = 1 - 14/p$. Furthermore, $f \in C(\overline{\Sigma^T})$.

To prove this estimate, we need a technical result, which is similar to Lemma 4.1. We will state the lemma after we introduce some notation.

Let $\zeta_0 \in C_0^\infty(B_2)$, $\zeta \in C_0^\infty(\{2^{-1} < |v| < 2^{3/2}\})$ be radially symmetric functions such that

– $\zeta_0 = 1$ on $\overline{B}_{2^{1/2}}$, and $0 \le \zeta_0 < 1$ on the complement of this set,
– $\zeta = 1$ if $\{2^{-1/2} \le |v| \le 2\}$, and $0 \le \zeta < 1$ otherwise.

For $n \in \{1, 2, \dots\}$ and a bounded measurable function $b = (b^1, b^2, b^3)^T$, we set

$$\zeta_n(v) = \zeta(v 2^{-n}), \tag{5.18}$$

$$\sigma_{G,n} = (\sigma_G + \nu I_3)\zeta_n + (1 - \zeta_n) I_3. \tag{5.19}$$

Lemma 5.4. *Let*

– *Ω be a bounded C^3 domain,*
– *$T > 0$, $\nu \in [0,1]$, $\theta \ge 2$, $\varkappa \in (0,1]$, $p \ge 2$, $\lambda \ge 0$ be numbers,*
– *Assumption 1.10 (see (1.26) - (1.27)) be satisfied,*
– *$\|b\|_{L_\infty(\Sigma^T)} \le K$,*
– *$\|g\|_{L_\infty(\Sigma^T)} \le \varepsilon$ for some $\varepsilon > 0$,*
– *$f \in S_{p,\widetilde{\theta}(\Sigma^T)} \cap L_{p,\theta}(\Sigma^T)$ for some $\widetilde{\theta} \ge \theta - 2$, $\nabla_v f \in L_{p,\theta}(\Sigma^T)$, $f_{\pm} \in L_\infty(\Sigma^T_{\pm}, |v \cdot n_x|)$,*
– *f satisfies the equation*

$$Yf = \nabla_v \cdot (\sigma_{G,n} \nabla_v f) - b \cdot \nabla_v f - \lambda f + h \quad \text{a.e. in } \Sigma^T \tag{5.20}$$

with $h \in L_{p,\theta-2}(\Sigma^T)$ and $f(0,\cdot) \equiv 0$, and the specular reflection boundary condition

$$f_-(t,x,v) = f_+(t,x,R_x v) \quad \text{a.e. in } \Sigma^T_-.$$

Then, there exists a number $\varepsilon \in (0,1)$ (independent of $\Omega, T, \nu, \varkappa, \theta, K, p$) and $\beta = \beta(p, \varkappa) > 0$ such that

$$\begin{aligned} &\|f\|_{S_{p,\theta-2}(\Sigma^T)} \\ &\le N 2^{\beta n}(\|h\|_{L_{p,\theta-2}(\Sigma^T)} + \|f\|_{L_{p,\theta-1}(\Sigma^T)} + \|\nabla_v f\|_{L_{p,\theta}(\Sigma^T)}), \end{aligned} \tag{5.21}$$

where $N = N(K, \varkappa, p, \Omega, \theta)$.

Furthermore,

- *if $p < 14$, the norm $\||f| + |\nabla_v f|\|_{L_{r,\theta-2}(\Sigma^T)}$ is bounded by the right-hand side of (5.21), where r is given by (4.10).*

- *if $p > 14$, then, for $\alpha = 1 - 14/p$, the norms*
  $$\|f\|_{L_\infty((0,T),C^{\alpha/3,\alpha}_{x,v}(\overline{\Omega}\times\mathbb{R}^3))}, \|\nabla_v f\|_{L_\infty((0,T),C^{\alpha/3,\alpha}_{x,v}(\overline{\Omega}\times\mathbb{R}^3))}, \|f_\pm\|_{L_\infty(\Sigma^T_\pm,\langle v\rangle^{\theta-2})}$$
  are bounded by the right-hand side of (5.21).

Proof. We follow the argument of Lemma 4.1. Let $\varepsilon > 0$ be a number such that (5.1) is true and $f_k, k = 1, \dots, m$ be the functions defined by (4.2).

Interior estimate. We prove the lemma by applying the a priori estimate in Theorem D.5 to Eq. (5.4). Let us check its assumptions.

Assumption 1.2 (see (1.19)*).* Due to (5.1) and (5.19), for sufficiently small $\varepsilon > 0$,
$$N_0 2^{-3n} \le \sigma_{G,n} \le N_0^{-1} I_3$$
for some constant $N_0 > 0$ independent of n.

Assumption D.3 (see (D.3)*).* By (5.1) - (5.2) and (5.19), one has
$$\|\nabla_v \sigma_{G,n}\|_{L_\infty(\Sigma^T)} \le N(K),$$
and hence, (D.3) holds with with b^i replaced with $b^i - \partial_{v_j}\sigma^{ji}_{G,n}, i = 1, 2, 3$, and $c = 0$.

Assumption D.1 (see (D.1)*).* By Assumption 1.10 (see (1.26) - (1.27)) and Lemma C.1, we have
$$\|\sigma_{G,n}\|_{L_\infty((0,T),C^{\varkappa/3,\varkappa}_{x,v}(\overline{\Omega}\times\mathbb{R}^3))} \le N(K,\varkappa).$$
Then we extend $\sigma_{G,n}$ to $\mathbb{R}^7_T$ so that the above inequality holds with Ω replaced with $\mathbb{R}^3$ and with $N = N(K,\varkappa,\Omega)$. Then, by Remark D.1, for any $r \in (0,1)$ and $Q_r(z_0)$,
$$\text{osc}_{x,v}(\sigma_{G,n}, Q_r(z_0)) \le N(K,\varkappa,\Omega) r^{\varkappa},$$
where $\text{osc}_{x,v}(\sigma_{G,n}, Q_r(z_0))$ is defined by (D.2). Hence, for any $\gamma \in (0,1)$, (D.1) in Assumption D.1 (γ) holds with
$$R_0 = N(K,\varkappa,\Omega)\gamma^{1/\varkappa}. \tag{5.22}$$
Furthermore, let
$$\beta = \beta(p) > 0, \quad \kappa = \kappa(p) > 0, \quad \gamma_\star = \delta^\kappa \widetilde{\gamma}_\star(p) > 0$$
be the numbers in Theorem D.5 with $\delta = 2^{-3n}$. Then, by the above, (D.1) in Assumption D.1 ($\gamma_\star$) holds with
$$R_0 = N_1(K,\varkappa,\Omega,p) 2^{-3\kappa n/\varkappa}.$$

Next, by Theorem D.5 (see Remark D.2) and Eq. (5.20),
$$\begin{aligned}\|f_1\|_{S_p(\mathbb{R}^7_T)} &\le N 2^{3\beta n}\|h_1\|_{L_p(\mathbb{R}^7_T)} + N 2^{6\kappa n/\varkappa}\|f_1\|_{L_p(\mathbb{R}^7_T)} \\ &\le N 2^{3\beta n + 6\kappa n/\varkappa}(\|h\|_{L_{p,\theta-2}(\mathbb{R}^7_T)} \\ &\quad + \|f\|_{L_{p,\theta-1}(\mathbb{R}^7_T)} + \|\nabla_v f\|_{L_{p,\theta}(\mathbb{R}^7_T)}),\end{aligned} \tag{5.23}$$
where $N = N(p,K,\varkappa,\theta,\Omega) > 0$. Next, recall the notation B^p (see (4.12)). By the embedding theorem for the S_p space (see [19]), $\|f_1\|_{B^p}$ is bounded above by the right-hand side of (5.23).

Boundary estimate. Recall that $\overline{f_k}, k = 2, \dots, n$, defined as the mirror extension of f_k, satisfies Eq. (5.6) where

- $\mathbb{A}$ is defined by (2.8), (2.10), and (2.12) with a replaced with $\sigma_{G,n}$, where the latter is given by (5.19),
- $\mathbb{B}$ is defined by (2.8) and (2.14) with b,
- $\mathbb{X}$ is given by (2.9) and (2.13).

By the conclusion in Appendix E (see Lemma E.1), since ζ_n is a radially symmetric cutoff function, we have

$$\mathbb{A} \in L_\infty((0,T), C_{x,v}^{\varkappa/3,\varkappa}(\mathbb{R}^6)),$$

and, furthermore, by Lemma C.1,

$$\|\mathbb{A}\|_{L_\infty((0,T),C_{x,v}^{\varkappa/3,\varkappa}(\mathbb{R}^6))} \le N(K,\varkappa,\Omega)$$

because $|\zeta_n| + |\nabla_v \zeta_n| \le N$ with N independent of n. Hence, as above, for any $\gamma_\star > 0$, the function $\mathbb{A}$ satisfies (D.1) in Assumption D.1 ($\gamma_\star$) with R_0 given by (5.22). Next, by Theorem D.5 and Eq. (5.20) combined with the estimates of $\mathbb{X}, \nabla_w \mathbb{X}$ in (4.7), we obtain

$$\begin{aligned}
&\|\overline{f_k}\|_{S_p(\mathbb{R}^7_T)}\\
&\le N 2^{3\beta n} (\||\overline{h_k}| + |\nabla_w \cdot (\mathbb{X}\overline{f_k})|\|_{L_p(\mathbb{R}^7_T)}) + N 2^{6\kappa n/\varkappa} \|\overline{f_k}\|_{L_p(\mathbb{R}^7_T)}\\
&\le N 2^{3\beta n + 6\kappa n/\varkappa} (\|h\|_{L_{p,\theta-2}(\mathbb{R}^7_T)} + \|f\|_{L_{p,\theta-1}(\mathbb{R}^7_T)} + \|\nabla_v f\|_{L_{p,\theta}(\Sigma^T)}),
\end{aligned}$$

where $N = N(p, K, \varkappa, \theta, \Omega)$. Combining the above inequality with (5.23), we prove the desired estimate of $\|f\|_{S_{p,\theta-2}(\Sigma^T)}$. Again, by using the embedding theorem for the S_p spaces, we bound the norms of $\|\overline{f_k}\|_{B^p}, k \ge 2$, where B^p is defined by (4.12). This and the bound of $\|f_1\|_{B^p}$ yield the estimates of

$$\|f_k\|_{L_\infty((0,T),C_{x,v}^{\alpha/3,\alpha}(\overline{\Omega}\times\mathbb{R}^3))}, \|\nabla_v f_k\|_{L_\infty((0,T),C_{x,v}^{\alpha/3,\alpha}(\overline{\Omega}\times\mathbb{R}^3))}$$

with $\alpha = 1 - 14/p$, when $p > 14$. Since $f_k = f\eta_j \langle v\rangle^{\theta-2}$, we also obtain the bound of $\|f_\pm\|_{L_\infty(\Sigma^T_\pm, \langle v\rangle^{\theta-2})}$. The lemma is proved. □

Proof of Proposition 5.4. We follow the argument of Theorem 1.7 and Lemma 4.1 with minor modifications. The central part of the argument is the following assertion.

Claim. Let $\varepsilon > 0$ be a sufficiently small number such that (5.1) is satisfied. Let $r \in [2,\infty) \setminus \{14\}$ and $\beta = \beta(r,\varkappa)$ be the number in the statement of Lemma 5.4, and $\theta > 2 + \beta$. Assume that $f \in \mathcal{K}_{\theta,\lambda,p}(\Sigma^T)$ is a function that satisfies Eq. (5.15) with $f_0 \equiv 0$ weakly and in the almost everywhere sense (see conditions (3) and (4) of Definition 1.9), and, in addition, one has

$$\||f| + |\nabla_v f|\|_{L_{r,\theta_r}(\Sigma^T)} \le M_r$$

for some $2 + \beta < \theta_r \le \theta$ and $M_r > 0$. Then, the following assertions hold:

1. $$\|f\|_{S_{r,\theta_r-2-\beta}(\Sigma^T)} \le N\big(\|h\|_{L_{r,\theta_r}(\Sigma^T)} + M_r\big), \tag{5.24}$$
   where $N = N(r, K, \theta, \Omega, \varkappa) > 0$.
2. If $r \in [2, 14)$,
   $$\||f| + |\nabla_v f|\|_{L_{r',\theta_r-2-\beta}(\Sigma^T)}$$
   is bounded by the right-hand side of (5.24) where r' is determined by the relation
   $$\frac{1}{r'} = \frac{1}{r} - \frac{1}{14}.$$
3. If $r > 14$, then for $\alpha = 1 - 14/r$, the norms
   $$\|f\|_{L_\infty((0,T),C_{x,v}^{\alpha/3,\alpha}(\overline{\Omega}\times\mathbb{R}^3))}, \ \|\nabla_v f\|_{L_\infty((0,T),C_{x,v}^{\alpha/3,\alpha}(\overline{\Omega}\times\mathbb{R}^3))}, \ \|f_\pm\|_{L_\infty(\Sigma^T_\pm,\langle v\rangle^{\theta_r-2-\beta})}$$
   are bounded by the right-hand side of (5.24).

If this claim is valid, then, repeating the argument of Step 2 in the proof of Theorem 1.7 (see p. 27), we prove the desired estimate (5.17). In particular, we use the same powers $r_k, k = 1, \dots, 8$ defined in (4.15), and similar weight parameters

$$\theta_1 = \theta, \quad \theta_{k+1} = \theta_k - 2 - \beta, \quad k = 1, \dots, 7.$$

Proof of the claim. Let $\zeta_n, n \in \{0, 1, 2, \dots\}$ be the functions defined above Lemma 5.4 (see p. 34), and $\xi_0 \in C_0^\infty(B_{2^{1/2}})$, $\xi \in C_0^\infty(\{2^{-1/2} < |v| < 2\})$ be functions such that

– $\xi_0 = 1$ on $\overline{B}_1$, $0 \le \xi_0 < 1$ on $\{|v| > 1\}$,
– $\xi = 1$ on $\{1 \le |v| \le 2^{1/2}\}$, otherwise $0 \le \xi < 1$.

For $n \in \{1, 2, \dots\}$, we set $\xi_n(v) = \xi(v 2^{-n})$ and observe that

$$\zeta_n = 1 \quad \text{on } \operatorname{supp} \xi_n.$$

In this proof, we assume that N is a constant depending only on $K, r, \theta, \Omega, \varkappa$.

By direct calculations, the function $f^{(n)} := f \xi_n$ satisfies

$$\mathfrak{L}_n f^{(n)} + \lambda f^{(n)} = \eta^{(n)} \text{ in } \Sigma^T, \quad f^{(n)}(0, \cdot) \equiv 0,$$

and the specular reflection boundary condition, where

$$\begin{aligned}
&\mathfrak{L}_n = Y - \nabla_v \cdot (\sigma_{G,n} \cdot \nabla_v) - a_g \cdot \nabla_v + \lambda, \\
&\sigma_{G,n} = (\sigma_G + \nu I_3)\zeta_n + (1 - \zeta_n) I_3, \\
&\eta^{(n)} = \xi_n h + \xi_n \overline{K}_g f - a_g \cdot (\nabla_v \xi_n) f - (\partial_{v_i} \sigma^{ij}_{G,n})(\partial_{v_j} \xi_n) f \\
&\qquad - \sigma^{ij}_{G,n} (\partial_{v_i v_j} \xi_n) f - 2\sigma^{ij}_{G,n} (\partial_{v_j} \xi_n)(\partial_{v_i} f).
\end{aligned}$$

By Lemma 5.4 with $\theta_r - \beta$ in place of θ, we get

$$\begin{aligned}
\|f^{(n)}\|_{S_{p,\theta_r - \beta - 2}(\Sigma^T)} \le N 2^{\beta n} \big(& \| |f^{(n)}| + |\nabla_v f^{(n)}| \|_{L_{r,\theta_r - \beta}(\Sigma^T)} \\
& + \|\eta^{(n)}\|_{L_{r,\theta_r - \beta - 2}(\Sigma^T)} \big).
\end{aligned}$$

By Lemma C.2 with θ_r in place of θ, we get

$$\| \langle v \rangle^{\theta_r - \beta - 2} \xi_n \overline{K}_g f \|_{L_r(\Sigma^T)} \le N 2^{-\beta n - 2n} \| |f| + |\nabla_v f| \|_{L_{r,\theta_r}(\Sigma^T)},$$

and, therefore,

$$\begin{aligned}
\|f^{(n)}\|_{S_{p,\theta_r - \beta - 2}(\Sigma^T)} \le\ & N \| (|f| + |\nabla_v f| + |h|) |\zeta_n| \|_{L_{r,\theta_r}(\Sigma^T)} \\
& + N 2^{-2n} \| |f| + |\nabla_v f| \|_{L_{r,\theta_r}(\Sigma^T)}.
\end{aligned} \tag{5.25}$$

Raising (5.25) to the power p and summing up, we prove the validity of the claim (1). Finally, the assertions (2) and (3) of the claim follow from Lemma 5.4 and the estimate (5.25). The proposition is proved. □

Proof of Theorem 1.11. **Uniqueness.** Let $\varepsilon = \varepsilon(\theta)$ be a number in Lemma 5.3. Let f_1 and f_2 be any two finite energy strong solutions to Eq. (1.5). Note that $u = f_1 - f_2$ satisfies Eq. (5.14) with $\nu = 0, \lambda = 0$, $h \equiv 0$ and $f_0 \equiv 0$. The uniqueness now follows from Lemma 5.3.

Existence. We assume, additionally, that $\varepsilon > 0$ is small enough so that (5.1) holds. Replacing f with $f - f_0 \phi$ where $\phi = \phi(t)$ is a cutoff function such that $\phi(0) = 1$, we reduce (1.6) to the forced Landau equation

$$\begin{aligned}
& Y f - \nabla_v \cdot (\sigma_G \nabla_v f) - a_g \cdot \nabla_v f - \overline{K}_g f = h \quad \text{in } \Sigma^T, \\
& f(0, x, v) = 0, \ (x, v) \in \Omega \times \mathbb{R}^3, \quad f_-(t, x, v) = f_+(t, x, R_x v), \quad z \in \Sigma^T_-,
\end{aligned} \tag{5.26}$$

where

$$h = -\partial_t \phi f_0 - v \cdot \nabla_x (f_0)\phi + \nabla_v \cdot (\sigma_G \nabla_v f_0)\phi + a_g \cdot (\nabla_v f_0)\phi + \phi \overline{K}_g f_0.$$

Note that due to the L_∞ estimates of $\sigma_G, \nabla_v \sigma_G, a_g$ (see (5.1) - (5.2)) and Remark 5.1, one has

$$\|h\|_{L_{2,\theta}(\Sigma^T)} + \|h\|_{L_\infty(\Sigma^T)} \le N(K, \theta, \Omega)(|f_0|_{\mathcal{O}_{2,\theta}} + |f_0|_{\mathcal{O}_\infty}). \tag{5.27}$$

Step 1: well-posedness of a viscous approximation scheme. We consider the equation

$$\begin{aligned} & Y f_\lambda^{(\nu)} - \nabla_v \cdot (\sigma_G \nabla_v f_\lambda^{(\nu)}) - \nu \Delta_v f_\lambda^{(\nu)} \\ & - a_g \cdot \nabla_v f_\lambda^{(\nu)} - \overline{K}_g f_\lambda^{(\nu)} + \lambda f_\lambda^{(\nu)} = h \text{ in } \Sigma^T, \\ & f_\lambda^{(\nu)}(0, \cdot) = 0 \text{ in } \Omega \times \mathbb{R}^3, \ (f_\lambda^{(\nu)})_-(t, x, v) = (f_\lambda^{(\nu)})_+(t, x, R_x v), \ z \in \Sigma^T_-. \end{aligned} \tag{5.28}$$

By Proposition 5.3, for any $\nu \in (0, 1]$ and $\theta \ge 16$, there exists $\lambda = \lambda(\theta, K, \nu, \varkappa, \Omega) > 0$ such that Eq. (5.28) has a unique finite energy strong solution $f_\lambda^{(\nu)} \in \mathcal{K}_{\theta,\lambda,p}(\Sigma^T)$ in the sense of Definition 1.9. Then, the function

$$f^{(\nu)} := e^{\lambda t} f_\lambda^{(\nu)} \in \mathcal{K}_{\theta,\lambda,p}(\Sigma^T)$$

is a finite energy strong solution to Eq. (5.26) in the sense of Definition 1.9.

Step 2: uniform bounds for $f^{(\nu)}$.

Weighted energy bound. By Lemma 5.3 and Remark 1.3, for sufficiently small $\varepsilon = \varepsilon(\theta) > 0$, we have

$$\begin{aligned} & \|f^{(\nu)}(T, \cdot)\|_{L_{2,\theta}(\Omega \times \mathbb{R}^3)} + \|f^{(\nu)}\|_{\sigma,\theta} + \|\nabla_v f^{(\nu)}\|_{L_{2,\theta-3}(\Sigma^T)} \\ & + \|f^{(\nu)}\|_{L_{2,\theta}(\Sigma^T)} \le N(\theta, T)\|h\|_{L_{2,\theta}(\Sigma^T)}. \end{aligned} \tag{5.29}$$

S_p bound. By Proposition 5.4 with $\theta - 3$ in place of θ, for sufficiently small $\varepsilon > 0$ and sufficiently large $\theta = \theta(\varkappa, p) > 4$, and $\theta' = \theta'(\varkappa, p), \theta'' = \theta''(\varkappa, p) \in (1, \theta - 3)$, one has

$$\begin{aligned} & \|f^{(\nu)}\|_{S_{2,\theta'}(\Sigma^T)} + \|f^{(\nu)}\|_{S_{p,\theta''}(\Sigma^T)} + \|f_\pm^{(\nu)}\|_{L_\infty(\Sigma^T_\pm, |v \cdot n_x|)} \\ & \le N(\|h\|_{L_{2,\theta-3}(\Sigma^T)} + \|h\|_{L_{p,\theta-3}(\Sigma^T)} \\ & + \|f^{(\nu)}\|_{L_{2,\theta-3}(\Sigma^T)} + \|\nabla_v f^{(\nu)}|\|_{L_{2,\theta-3}(\Sigma^T)}), \end{aligned} \tag{5.30}$$

where $N = N(\theta, p, \varkappa, K, \Omega)$. Combining (5.29) with (5.30) gives

$$\begin{aligned} & \|f^{(\nu)}\|_{S_{2,\theta'}(\Sigma^T)} + \|f^{(\nu)}\|_{S_{p,\theta''}(\Sigma^T)} + \|f_\pm^{(\nu)}\|_{L_\infty(\Sigma^T_\pm, |v \cdot n_x|)} \\ & \quad \le N(\|h\|_{L_{2,\theta}(\Sigma^T)} + \|h\|_{L_\infty(\Sigma^T)}), \end{aligned} \tag{5.31}$$

where $N = N(\theta, p, \varkappa, K, \Omega, T)$.

Step 3: limiting argument. By (5.30), the Banach-Alaoglu theorem, and Eberlein-Smulian theorem, there exists a subsequence ν' and functions

$$f \in S_{2,\theta'}(\Sigma^T), \quad f_\pm^\star \in L_\infty(\Sigma^T_\pm, |v \cdot n_x|), \quad f_T^\star \in L_{2,\theta}(\Omega \times \mathbb{R}^3)$$

such that

- $f^{(\nu')} \to f$ weakly in $S_{2,\theta'}(\Sigma^T)$;
- $f_\pm^{(\nu')} \to f_\pm^\star$ in the weak* topology of $L_\infty(\Sigma^T_\pm, |v \cdot n_x|)$, respectively;
- $f^{(\nu')}(T, \cdot) \to f_T^\star(\cdot)$ weakly in $L_{2,\theta}(\Omega \times \mathbb{R}^3)$;
- $f^{(\nu')} \to f$ weakly in $H_{\sigma,\theta}(\Sigma^T)$.

Passing to the limit in the Green's identity (3.9) (see Remark 3.2), we conclude that $f^\star_\pm, f^\star_T$ are, indeed, traces of the function f so that $f_\pm \equiv f^\star_\pm$, $f^\star_T(\cdot) \equiv f(T, \cdot)$. Using the weak and the weak* convergence, we pass to the limit in (5.29) and (5.31) and prove

$$\begin{aligned} &\|f(T,\cdot)\|_{L_{2,\theta}(\Omega\times\mathbb{R}^3)} + \|f\|_{\sigma,\theta} + \|\nabla_v f\|_{L_{2,\theta-3}(\Sigma^T)} \\ &+ \|f\|_{S_{2,\theta'}(\Sigma^T)} + \|f\|_{S_{p,\theta''}(\Sigma^T)} + \|f_\pm\|_{L_\infty(\Sigma^T_\pm, |v\cdot n_x|)} \\ &\le N(\|h\|_{L_{2,\theta}(\Sigma^T)} + \|h\|_{L_\infty(\Sigma^T)}). \end{aligned} \tag{5.32}$$

Next, we show that f satisfies the weak formulation of Eq. (5.26) (cf. Definition 1.9). Recall that due to Lemma 5.3, $f^{(\nu)}$ satisfies the weak formulation of (5.28) with any $\phi \in C^1_0(\overline{\Sigma^T})$. Testing Eq. (5.28) with $\phi \in C^1_0(\overline{\Sigma^T})$ gives

$$\begin{aligned} &-\int_{\Sigma^T} (Y\phi) f^{(\nu)}\, dz + \int_{\Omega\times\mathbb{R}^3} f^{(\nu)}(T,\cdot)\phi(T,x,v)\, dxdv \\ &+ \int_{\Sigma^T_+} f^{(\nu)}_+ \phi\, |v\cdot n_x|\, d\sigma dt - \int_{\Sigma^T_-} f^{(\nu)}_- \phi |v\cdot n_x|\, d\sigma dt \\ &+ \int_{\Sigma^T} [(\sigma_G + \nu I_3)\nabla_v f^{(\nu)}]\cdot \nabla_v \phi\, dz = \int_{\Sigma^T} [a_g \cdot \nabla_v f^{(\nu)} + \overline{K}_g f^{(\nu)} + h]\phi\, dz. \end{aligned} \tag{5.33}$$

It follows from the definition of K (see (1.4)) that

$$\int_{\Sigma^T} (\mathsf{K} f^{(\nu)})\phi\, dz = \int_{\Sigma^T} (\mathsf{K}\phi) f^{(\nu)}\, dz,$$

and, hence, by the explicit expression of the term $J_g f^{(\nu)}$ (see (1.9)) and the fact that $\overline{K}_g = \mathsf{K} + J_g$, we obtain

$$\int_{\Sigma^T} (\overline{K}_g f^{(\nu)})\phi\, dz = \int_{\Sigma^T} (\overline{K}_g \phi) f^{(\nu)}\, dz.$$

Then, by the weak convergence $f^{(\nu)} \to f$ in $L_2(\Sigma^T)$ and Lemma C.2, we conclude

$$\lim_{\nu\to 0} \int_{\Sigma^T} (\overline{K}_g f^{(\nu)})\phi\, dz = \int_{\Sigma^T} (\overline{K}_g f)\phi\, dz.$$

Finally, by what just said and the aforementioned weak convergence, we pass to the limit in (5.33) and we conclude that f satisfies the weak formulation of (1.6) (see Definition 1.9). By using the weak* convergence of $f^\nu_\pm$ in $L_\infty(\Sigma^T_\pm, |v\cdot n_x|)$, we pass to the limit in the specular boundary condition for $f^{(\nu)}$ and prove that this boundary condition holds for the function f as well. Using the Green's identity again (see Remark 3.2), we prove that f satisfies Eq. (1.6) a.e. and, thus, f is a finite energy strong solution in the sense of Definition 1.9.

Step 4: Hölder estimate. To bound the

$$L_\infty((0,T), C^{\alpha/3,\alpha}_{x,v}(\overline{\Omega}\times\mathbb{R}^3))$$

norm of f and $\nabla_v f$, we repeat the above L_2 to S_p bootstrap argument. Again, by Proposition 5.4 with θ replaced with θ'', for sufficiently small $\varepsilon > 0$ and sufficiently large $\theta = \theta(\varkappa, p)$, we get

$$\begin{aligned} &\|f\|_{L_\infty((0,T), C^{\alpha/3,\alpha}_{x,v}(\overline{\Omega}\times\mathbb{R}^3))} + \|\nabla_v f\|_{L_\infty((0,T), C^{\alpha/3,\alpha}_{x,v}(\overline{\Omega}\times\mathbb{R}^3))} \\ &\quad \le N(\|h\|_{L_{2,\theta''}(\Sigma^T)} + \|h\|_{L_{p,\theta''}(\Sigma^T)} + \|f\|_{L_{2,\theta''}(\Sigma^T)} + \|\nabla_v f\|_{L_{2,\theta''}(\Sigma^T)}). \end{aligned}$$

In addition, by the same proposition, $f \in C(\overline{\Sigma^T})$. Combining this with (5.32), we obtain

$$\begin{aligned} &\|f\|_{L_\infty((0,T),C^{\alpha/3,\alpha}_{x,v}(\overline{\Omega}\times\mathbb{R}^3))} + \|\nabla_v f\|_{L_\infty((0,T),C^{\alpha/3,\alpha}_{x,v}(\overline{\Omega}\times\mathbb{R}^3))} \\ &\quad \le N(\|h\|_{L_{2,\theta''}(\Sigma^T)} + \|h\|_{L_\infty(\Sigma^T)}). \end{aligned} \tag{5.34}$$

Finally, recall that at the beginning of the proof, we replaced f with $f - f_0\phi$. By this, the definition of the $\mathcal{O}_{p,\theta}$ norm (see (1.15)), and (5.32), and (5.34) , we prove the desired estimate (1.28) □

Appendix A. **Verification of the identity (2.7).** Invoke the assumptions and the notation of Section 2.1 and denote

$$M = (\frac{\partial x}{\partial y}).$$

Let f be a finite energy weak solution to Eq. (1.16) (see Definition 1.1) supported on $\mathbb{R}\times\Omega_{r_0/2}\times\mathbb{R}^3$ and $\phi \in C^1_0(\overline{\Sigma^T})$. The goal of this section is to justify the identity (2.7).

Drift term. By using a change of variables and the identity

$$\big(\nabla_v f\big)(t, x(y), v(y,w)) = \Big((\frac{\partial x}{\partial y})^T \Big)^{-1} \nabla_w \widehat{f}(t,y,w) = (M^{-1})^T \nabla_w \widehat{f}(t,y,w), \tag{A.1}$$

we get

$$\int_{\Sigma^T} (b\cdot\nabla_v f)\phi\, dz = \int_{\mathbb{H}^T_-} \Big[\underbrace{(M^{-1}b)}_{=B}\cdot\nabla_w(\widehat{f}J)\Big]\widehat{\phi}\, d\widehat{z}, \tag{A.2}$$

where $\mathbb{H}^T_-$ is defined in (1.11).

Transport term. First, changing variables gives

$$\int_{\Sigma^T} (v\cdot\nabla_x\phi) f\, dz = \int_{\mathbb{H}^T_-} (\widehat{f}J)\,(Mw)^T\big(\nabla_x\phi\big)(t,x(y),v(y,w))\, d\widehat{z}.$$

Next, by the chain rule and (A.1),

$$\begin{aligned} &\big(\nabla_x\phi\big)(t,x(y),v(y,w)) \\ &= \Big((\frac{\partial x}{\partial y})^T \Big)^{-1} \big[\nabla_y\widehat{\phi}(t,y,w) - (\frac{\partial v}{\partial y})^T(\nabla_v\phi)(t,x(y),v(y,w))\big] \\ &= (M^{-1})^T\nabla_y\widehat{\phi}(t,y,w) - (M^{-1})^T(\frac{\partial v}{\partial y})^T(M^{-1})^T\nabla_w\widehat{\phi}(t,y,w), \end{aligned}$$

and then,

$$\begin{aligned} &v^T(y,w)\big(\nabla_x\phi\big)(t,x(y),v(y,w)) \\ &= w^T\nabla_y\widehat{\phi}(t,y,w) - w^T(\frac{\partial v}{\partial y})^T(M^{-1})^T\nabla_w\widehat{\phi}(t,y,w) \\ &= w\cdot\nabla_y\widehat{\phi}(t,y,w) - X\cdot\nabla_w\widehat{\phi}(t,y,w), \end{aligned}$$

where

$$X = (X_1,X_2,X_3)^T = M^{-1}(\frac{\partial v}{\partial y})w = M^{-1}\frac{\partial(Mw)}{\partial y}w.$$

Thus, by the above computations,

$$\int_{\Sigma^T} (Y\phi) f\, dz = \int_{\mathbb{H}^T_-} \big(Y(\widehat{\phi}) - X\cdot\nabla_w\widehat{\phi}\big)(\widehat{f}J) d\widehat{z}. \tag{A.3}$$

Diffusion term. Applying formula (A.1) gives

$$\int_{\Sigma^T} (a \nabla_v \phi)^T \nabla_v f \, dz = \int_{\mathbb{H}^T_-} (A \nabla_w \widehat{\phi})^T \nabla_w (\widehat{f} J) \, d\widehat{z}, \tag{A.4}$$

where

$$A = M^{-1} \widehat{a} (M^{-1})^T.$$

Finally, combining (A.1) - (A.4) and recalling $\widetilde{f} = \widehat{f} J$, we arrive to (2.7).

Appendix B. **.**

Lemma B.1. *Let $p > 1$ be a number, $T \in (0, \infty]$, and Ψ be the diffeomorphism defined in Subsection 2.1 (see (2.2) - (2.3)). Then, for any function u on Σ^T such that $u(z) = 0$ for $x \in (B_r(x_0))^c$, one has*

$$\|u\|_{S_p(\Sigma^T)} \le N \|\widehat{u}\|_{S_p(\mathbb{H}^T_-)} + N \|\nabla_v \widehat{u}\|_{L_{p,2}(\mathbb{H}^T_-)} + N \|\widehat{u}\|_{L_{p,1}(\mathbb{H}^T_-)},$$

where $N = N(\Omega, p)$, and $\widehat{u}$ is defined in (2.5).

Proof. It follows from the compuations in Appendix A and (4.7) that

$$\| |u| + |\nabla_v u| + |D^2_v u| \|_{L_p(\Sigma^T)} \le N \| |\widehat{u}| + |\nabla_w \widehat{u}| + |D^2_w \widehat{u}| \|_{L_p(\mathbb{H}^T_-)},$$

$$\begin{aligned} \|Y u\|_{L_p(\Sigma^T)} &\le N \|(\partial_t + w \cdot \nabla_y) \widehat{u}\|_{L_p(\mathbb{H}^T_-)} + N \|\nabla_w \cdot (X \widehat{u})\|_{L_p(\mathbb{H}^T_-)} \\ &\le N \|(\partial_t + w \cdot \nabla_y) \widehat{u}\|_{L_p(\mathbb{H}^T_-)} + \|\widehat{u}\|_{L_{p,1}(\mathbb{H}^T_-)} + \|\nabla_w \widehat{u}\|_{L_{p,2}(\mathbb{H}^T_-)}. \end{aligned}$$

The desired estimate follows from the above inequalities. □

Lemma B.2. *Let $\theta \ge 0$ be a number and u be a function on Σ^T such that $u, Y u \in L_{2,\theta}(\Sigma^T)$, $u(T, \cdot), u(0, \cdot) \in L_{2,\theta}(\Omega \times \mathbb{R}^3)$, $u_\pm \in L_\infty(\Sigma^T_\pm, |v \cdot n_x|)$, and the specular reflection boundary condition is satisfied. Then, the following variant of the energy identity holds:*

$$\int_{\Omega \times \mathbb{R}^3} \big(u^2(T, x, v) - u^2(0, x, v)\big) \langle v \rangle^\theta dx dv = 2 \int_{\Sigma^T} u (Y u) \langle v \rangle^\theta dz.$$

Proof. For $\varepsilon > 0$, denote

$$\mu_\varepsilon(v) = e^{-|v|^2/\varepsilon}.$$

Note that $u_\varepsilon := u \mu_\varepsilon \in E_{2,\theta}(\Sigma^T)$ (see Definition 3.3 and (3.9)) and it satisfies the specular reflection boundary condition. Then, by the energy identity (see (3.10) in Lemma 3.4),

$$\int_{\Omega \times \mathbb{R}^3} [u^2_\varepsilon(T, x, v) - u^2_\varepsilon(0, x, v)] \langle v \rangle^\theta dx dv = 2 \int_{\Sigma^T} u_\varepsilon (Y u_\varepsilon) \langle v \rangle^\theta dz.$$

Passing to the limit in the above equality and using the dominated convergence theorem, we prove the lemma. □

For $T \in \mathbb{R}$, let $\langle \cdot, \cdot \rangle_T$ be the duality pairing between $\mathbb{H}^{-1}_2(\mathbb{R}^7_T)$ and $\mathbb{H}^1_2(\mathbb{R}^7_T)$ given by

$$\langle f, g \rangle_T = \int_{-\infty}^T \int_{\mathbb{R}^3} [f(t, x, \cdot), g(t, x, \cdot)] \, dx dt,$$

where

$$[f, g] = \int ((1 - \Delta_v)^{-1/2} f) \, ((1 - \Delta_v)^{1/2} g) \, dv.$$

The proof of the following variant of the energy identity can be found in [6].

Lemma B.3. *Let $T \in \mathbb{R}$ be a number and $u \in \mathbb{H}^1_2((-\infty, T) \times \mathbb{R}^6)$, and $Yu \in \mathbb{H}^{-1}_2((-\infty, T) \times \mathbb{R}^6)$. Then, for a.e. $s \in (-\infty, T]$,*

$$\langle Yu, u\rangle_s = (1/2)\|u\|^2_{L_2(\mathbb{R}^6)}(s).$$

Lemma B.4. *Let $T > 0$ be a number and $u \in E_2(\Sigma^T)$. Then, u is of class $C([0, T], L_2(\Sigma^T))$.*

Proof. We only prove the continuity at $t = 0$ because the argument for other points is similar. First, note that by the energy identity (see (3.10)),

$$\lim_{t\to 0+} \|u(t, \cdot)\|_{L_2(\Omega\times\mathbb{R}^3)} = \|u(0, \cdot)\|_{L_2(\Omega\times\mathbb{R}^3)}.$$

Hence, we only need to show that $u(t, \cdot)$ is weakly continuous in $L_2(\Omega \times \mathbb{R}^3)$ at $t = 0$.

We fix an arbitrary test function

$$\phi \in C^1_0\big(([0, T] \times \overline{\Omega} \times \mathbb{R}^3) \setminus ((0, T) \times \gamma_0 \cup \{0\} \times \partial\Omega \times \mathbb{R}^3 \cup \{T\} \times \partial\Omega \times \mathbb{R}^3)\big),$$

which belongs to the set Φ (see Definition 3.2) by Lemma 2.1 of [7]. Note that by the Green's identity (see (3.9)), we have

$$\lim_{t\to 0+} \int_{\Omega\times\mathbb{R}^3} u(t, x, v)\phi(t, x, v)\, dxdv = \int_{\Omega\times\mathbb{R}^3} u(0, x, v)\phi(0, x, v)\, dxdv.$$

We claim that the above convergence also holds if we replace $\phi(t, x, v)$ with $\phi(0, x, v)$. To prove this, we first note that by Remark 3.3 and the energy identity (3.10) in Lemma 3.4, we have

$$u \in L_\infty((0, T), L_2(\Omega \times \mathbb{R}^3)). \tag{B.1}$$

By this and the Cauchy-Schwartz inequality,

$$\begin{aligned}&\Big|\int_{\Omega\times\mathbb{R}^3} u(t, x, v)(\phi(0, x, v) - \phi(t, x, v))\, dxdv\Big|\\ &\le \|u\|_{L_\infty((0,T),L_2(\Omega\times\mathbb{R}^3))}\|\phi(0, \cdot) - \phi(t, \cdot)\|_{L_2(\Omega\times\mathbb{R}^3)}.\end{aligned}$$

The right-hand side of the above inequality converges to 0 as $t \to 0+$ due our choice of ϕ, and this proves the above claim. Hence, for any continuously differentiable function ξ with compact support in $\Omega \times \mathbb{R}^3$,

$$\lim_{t\to 0+} \int_{\Omega\times\mathbb{R}^3} u(t, x, v)\xi(x, v)\, dxdv = \int_{\Omega\times\mathbb{R}^3} u(0, x, v)\xi(x, v)\, dxdv.$$

By using a standard approximation argument combined with (B.1), we conclude that $u(t, \cdot) \to u(0, \cdot)$ weakly in $L_2(\Omega \times \mathbb{R}^3)$ as $t \to 0+$. The lemma is proved. □

Appendix C. .

Lemma C.1. *Let $\varkappa \in (0, 1]$, $g \in L_\infty((0, T), C^{\varkappa/3,\varkappa}_{x,v}(\overline{\Omega} \times \mathbb{R}^3))$, and σ_G be a function defined in (1.7). Then, we have*

$$\|\sigma_G\|_{L_\infty((0,T),C^{\varkappa/3,\varkappa}_{x,v}(\overline{\Omega}\times\mathbb{R}^3))} \le N(1 + \|g\|_{L_\infty((0,T),C^{\varkappa/3,\varkappa}_{x,v}(\overline{\Omega}\times\mathbb{R}^3))}),$$

where $N = N(\varkappa) > 0$.

Proof. Fix almost any $t \in (0, T)$ and arbitrary $x_1, x_2 \in \overline{\Omega}$ recall that by Lemma 3 of [9], for any $v \in \mathbb{R}^3$,

$$|\sigma_G(t, x_1, v)| + |\nabla_v\sigma_G(t, x_1, v)| \le N(1 + \|g(t, x, \cdot)\|_{L_\infty(\mathbb{R}^3)}), \tag{C.1}$$

where N is independent of $\varkappa$. Next, observe that

$$\sigma_G(t,x_1,v)-\sigma_G(t,x_2,v)=\big(\Phi*[\mu^{1/2}(g(t,x_1,\cdot)-g(t,x_2,\cdot))]\big)(v).$$

Replacing $g(t,x_1,v)$ with $g(t,x_1,v)-g(t,x_2,v)$ and using (C.1), we get

$$|\sigma_G(t,x_1,v)-\sigma_G(t,x_2,v)|\le N|x_1-x_2|^{\varkappa/3}\sup_{t\in(0,T),v\in\mathbb{R}^3}\|g(t,\cdot,v)\|_{C^{\varkappa/3}(\overline{\Omega})},$$

where $C^{\varkappa/3}(\overline{\Omega})$ is the usual Hölder space on $\overline{\Omega}$. Combining the above inequalities and using the interpolation inequality for the Hölder norms, we prove the lemma. □

Lemma C.2. *Let $p>3/2$ be a number and $u\in L_p(\Sigma^T)$ be a function such that $\nabla_v u\in L_p(\Sigma^T)$. For any $m=\{0,1,2,\dots\}$, we denote*

$$\{v\sim m\}=\begin{cases}|v|<1,\ m=0,\\ cm<|v|<c^{-1}m,\ m\ge 1,\end{cases}$$

where $c\in(0,1)$ is some number. If the condition (1.26) holds, then for any $\theta\ge 0$,

$$\|\overline{K}_g u\|_{L_p((0,T)\times\Omega\times\{v\sim m\})}\le Nm^{-\theta}\||u|+|\nabla_v u|\|_{L_{p,\theta}(\Sigma^T)},$$

where $\overline{K}_g$ is defined in (1.8) and $N=N(\varkappa,K,\theta,p,c)>0$.

Proof. Recall that $\overline{K}_g=\mathsf{K}+J_g$, where K and J_g are defined in (1.4) and (1.9). Furthermore, by the definition of the collision kernel $\mathcal{Q}$ (see (1.2)),

$$\begin{aligned}\mathsf{K}u&=-\mu^{-1/2}\partial_{v_i}\bigg(\mu\big(\Phi^{ij}*(\mu^{1/2}[\partial_{v_j}u+v_ju])\big)\bigg)\\&=2v_i\mu^{1/2}\bigg(\Phi^{ij}*(\mu^{1/2}[\partial_{v_j}u+v_ju])\bigg)\\&\quad-\mu^{1/2}\bigg(\Phi^{ij}*\partial_{v_i}(\mu^{1/2}v_ju)\bigg)-\mu^{1/2}\bigg(\partial_{v_i}\Phi^{ij}*(\mu^{1/2}\partial_{v_j}u)\bigg)\\&=:\mathsf{K}_1u+\mathsf{K}_2u+\mathsf{K}_3u.\end{aligned}$$

Estimate of J_g. By using Lemmas 2 and 3 of [9] and the condition (1.26), one can show that there exist a constant $N=N(K)>0$ such that

$$\||\sigma|+|\nabla_v\sigma|+|\partial_{v_i}\Phi^{ij}*(\mu^{1/2}\partial_{v_j}g)|+|\Phi^{ij}*(\mu^{1/2}v_i\partial_{v_j}g)|\|_{L_\infty(\mathbb{R}^7_T)}\le N(K).$$

See the details in Lemma 3.6 of [4]. Furthermore, by Lemma 3 of [9],

$$|\sigma^{ij}v_iv_j|\le N\langle v\rangle^{-1}.$$

By the above inequalities,

$$\|J_gu\|_{L_p((0,T)\times\mathbb{R}^3\times\{v\sim m\})}\le N(K)\|u\|_{L_p((0,T)\times\mathbb{R}^3\times\{v\sim m\})}.$$

Estimate of K_1 and K_2. Note that for any $\theta\ge 0$,

$$\begin{aligned}&\|\mathsf{K}_1u\|_{L_p(\{v\sim m\})}\\&\le N(\theta,c)m^{-\theta}\|\mu^{1/4}\big(|v|^{-1}*[\mu^{1/4}(|u|+|\nabla_vu|)]\big)\|_{L_p(\{v\sim m\})}.\end{aligned}\tag{C.2}$$

Furthermore, using the Hölder's inequality with p and $q = p/(p-1) \in (1,3)$, for any $(t,x,v) \in \mathbb{R}^7_T$, we have

$$\int |v-v'|^{-1}\mu^{1/4}(v')|u|(t,x,v')\,dv'$$
$$\le \Big(\int |v-v'|^{-q}\mu^{q/4}(v')\,dv'\Big)^{1/q}\|u(t,x,\cdot)\|_{L_p(\mathbb{R}^3)}.$$

By using the fact that for $q \in (1,3)$ and $v \in \mathbb{R}^3$,

$$\int_{\mathbb{R}^3} |v-v'|^{-q}\mu^{q/4}(v')\,dv' \le N(q)$$

combined with (C.2), we obtain

$$\|\mathsf{K}_1 u\|_{L_p((0,T)\times\mathbb{R}^3\times\{v\sim m\})} \le N(\theta,c)m^{-\theta}\||u|+|\nabla_v u|\|_{L_p(\Sigma^T)}. \tag{C.3}$$

By the same argument, we show that K_2 is bounded by the right-hand side of (C.3).

Estimate of K_3. By direct calculations,

$$\partial_{v_i v_j}\Phi^{ij} = -8\pi\delta(x),$$

and since $u \in L_p(\Sigma^T)$, we get

$$\partial_{v_i v_j}\Phi^{ij} * u = -8\pi u.$$

Hence, integrating by parts in $\mathsf{K}_3 u$ and using the above identity, we prove that

$$\mathsf{K}_3 u = 8\pi\mu u - \mu^{1/2}\Phi^{ij} * \partial_{v_i}(v_j\mu^{1/2}u).$$

Then, repeating the above argument we used to estimate $\mathsf{K}_1 u$, we get

$$\|\mathsf{K}_3 u\|_{L_p((0,T)\times\mathbb{R}^3\times\{v\sim m\})} \le N(\theta,c)m^{-\theta}\||u|+|\nabla_v u|\|_{L_p(\Sigma^T)}.$$

The assertion of the lemma follows from the above estimates. □

Appendix D. S_p **regularity theory for kinetic Fokker-Planck equations with rough coefficients.** In this section, we present the main results of [5]. Throughout the section, $T \in (-\infty, \infty]$. Denote

$$Q_r(z_0) := \{z : t_0 - r^2 < t < t_0, |x - x_0 - (t-t_0)v_0|^{1/3} < r, |v - v_0| < r\},$$

which we call a kinetic cylinder.

Assumption D.1. $(\gamma_\star)$ There exists $R_0 > 0$ such that for any z_0 such that $t < T$ and $r \in (0, R_0]$,

$$\mathrm{osc}_{x,v}(a, Q_r(z_0)) \le \gamma_\star, \tag{D.1}$$

where

$$\mathrm{osc}_{x,v}(a, Q_r(z_0)) \tag{D.2}$$
$$= r^{-14}\int_{t_0-r^2}^{t_0}\int_{D_r(z_0,t)\times D_r(z_0,t)} |a(t,x_1,v_1) - a(t,x_2,v_2)|\,dx_1dv_1dx_2dv_2\,dt,$$

and

$$D_r(z_0,t) = \{(x,v) : |x - x_0 - (t-t_0)v_0|^{1/3} < r, |v - v_0| < r\}.$$

Remark D.1. Note that the following assumption is stronger than Assumption D.1, but somewhat easier to verify in practice.

Assumption D.2. There exists an increasing function $\omega : [0,\infty) \to [0,\infty)$ such that $\omega(0+) = 0$ and

$$\sup_{t<T,x,v} r^{-24} \int_{x_1,x_2 \in B_{r^3}(x)} \int_{v_1,v_2 \in B_r(v)} |a(t,x_1,v_1) - a(t,x_2,v_2)| \, dx_1 dx_2 \, dv_1 dv_2 \le \omega(r).$$

Furthermore, note that if $a \in L_\infty((-\infty,T), C^{\alpha/3,\alpha}_{x,v}(\mathbb{R}^6)), \alpha \in (0,1]$, then, Assumption D.2 holds with $\omega(r) = N r^\alpha$ for some constant $N > 0$.

Assumption D.3. Let $b = (b^1, b^2, b^3)^T$ and c be functions such that

$$\| |b| + |c| \|_{L_\infty((-\infty,T)\times\mathbb{R}^6)} \le K. \tag{D.3}$$

The following theorem is a simplified version of Theorem 2.4 of [5].

Theorem D.4. *Let $p > 1$, $K > 0$, be numbers. Let Assumptions 1.2 (see (1.19)), D.3 hold. There exists a constant*

$$\gamma_\star = \gamma_\star(\delta, p) > 0$$

such that if Assumption D.1 $(\gamma_\star)$ holds, then, the following assertions are valid.

(i) There exists a constant

$$\lambda_0 = \lambda_0(p, \delta, K, R_0) \ge 0$$

such that for any $\lambda \ge \lambda_0$ and any $u \in S_p((-\infty,T)\times\mathbb{R}^6)$, one has

$$\begin{aligned} &\| \lambda |u| + \lambda^{1/2} |\nabla_v u| + |D_v^2 u| + |(-\Delta_x)^{1/3} u| \\ &+ |D_v (-\Delta_x)^{1/6} u| + |Y u| \|_{L_p((-\infty,T)\times\mathbb{R}^6)} \\ &\le N \| Y u - a^{ij} \partial_{v_i v_j} u + b \cdot \nabla_v u + c u + \lambda u \|_{L_p((-\infty,T)\times\mathbb{R}^6)}, \end{aligned} \tag{D.4}$$

where $R_0 \in (0,1)$ is the constant in Assumption D.1 $(\gamma_\star)$, and $N = N(p,\delta,K)$. In addition, for any $f \in L_p((-\infty,T)\times\mathbb{R}^6)$, the equation

$$Y u - a^{ij} \partial_{v_i v_j} u + b \cdot \nabla_v u + c u + \lambda u = f$$

has a unique solution $u \in S_p((-\infty,T)\times\mathbb{R}^6)$.

(ii) For any numbers $-\infty < S < T < \infty$, $\lambda \ge 0$, and $f \in L_p((S,T)\times\mathbb{R}^6)$, the equation

$$Y u - a^{ij} \partial_{v_i v_j} u + b \cdot \nabla_v u + c u + \lambda u = f, \quad u(0,\cdot) = 0$$

has a unique solution $u \in S_p((S,T)\times\mathbb{R}^6)$. In addition,

$$\begin{aligned} &\|u\| + \|D_v u\| + \|D_v^2 u\| + \|(-\Delta_x)^{1/3} u\| \\ &+ \|D_v (-\Delta_x)^{1/6} u\| + \|Y u\| \le N \|f\|, \end{aligned}$$

where $\| \cdot \| = \| \cdot \|_{L_p((S,T)\times\mathbb{R}^6)}$ and $N = N(\delta, p, K, T - S)$.

Theorem D.5 (Corollary 2.6 of [5]). *Under Assumptions 1.2 (see (1.19)), D.3 (see (D.3)), there exist constants*

$$\kappa = \kappa(p) > 0, \quad \beta = \beta(p) > 0, \quad \gamma_\star = \delta^\kappa \widetilde{\gamma}_\star(p) > 0,$$

such that if (D.1) in Assumption D.1 $(\gamma_\star)$ holds, then for any $u \in S_p((-\infty,T)\times\mathbb{R}^6)$ and $\lambda \ge 0$,

$$\begin{aligned} &\|u\|_{S_p((-\infty,T)\times\mathbb{R}^6)} \\ &\le N \delta^{-\beta} (\| Y u - a^{ij} \partial_{v_i v_j} u + b \cdot \nabla_v u + c u + \lambda u \|_{L_p((-\infty,T)\times\mathbb{R}^6)} \\ &\quad + R_0^{-2} \|u\|_{L_p((-\infty,T)\times\mathbb{R}^6)}), \end{aligned} \tag{D.5}$$

where $N = N(p,K)$, and $R_0 \in (0,1)$ is the constant in Assumption D.1 $(\gamma_\star)$.

Remark D.2. The a priori estimates (D.4) and (D.5) hold for $u \in S_p(\mathbb{R}^7_T)$ such that $u(0,\cdot) \equiv 0$. To prove this, we apply these estimates to $u 1_{t \geq 0} \in S_p((-\infty, T) \times \mathbb{R}^6)$.

Lemma D.6. *Let $T > 0$, $\lambda \geq 0$, $1 < q < p$ be numbers, functions a, b, c satisfy Assumptions 1.2 (see (1.19)), D.2 and D.3, and $u \in S_q(\mathbb{R}^7_T)$ be a function such that $u(0,\cdot) \equiv 0$, and*

$$h := Yu - a^{ij}\partial_{v_i v_j} u + b \cdot \nabla_v u + (c+\lambda)u \in L_p(\mathbb{R}^7_T).$$

Then, $u \in S_p(\mathbb{R}^7_T)$.

Proof. We follow the proof of Theorem 4.3.12 (i) of [16]. By Theorem D.4 (ii), the equation

$$YU - a^{ij}\partial_{v_i v_j} U + b \cdot \nabla_v U + (c+\lambda)U = h, \quad U(0,\cdot) \equiv 0$$

has a unique solution $U \in S_p(\mathbb{R}^7_T)$. We will prove that $u = U$ a.e. By using an induction argument, we may assume that

$$\frac{1}{q} - \frac{1}{p} < \frac{1}{12}. \tag{D.6}$$

Let $\phi, \xi \in C_0^\infty(\mathbb{R}^6)$ be functions such that $\phi(0) = 1$, $\xi(0) = 1$, and $\operatorname{supp}\phi \subset \operatorname{supp}\xi$. For $n \geq 1$, denote

$$\phi_n(x,v) = \phi(x/n^3, v/n), \quad \xi_n(x,v) = \xi(x/n^3, v/n).$$

Then, $U_n := U\phi_n \in S_q(\mathbb{R}^7_T)$ satisfies the equation

$$\begin{aligned}
&YU_n - a^{ij}\partial_{v_i v_j} U_n + b \cdot \nabla_v U_n + (c+\lambda)U_n \\
&= h\phi_n + U(v \cdot \nabla_x \phi_n - a^{ij}\partial_{v_i v_j}\phi_n + b \cdot \nabla_v \phi_n) \\
&- 2(a\nabla_v \phi_n) \cdot \nabla_v U, \quad U_n(0,\cdot) \equiv 0,
\end{aligned}$$

and, in addition, by Theorem D.4 (ii),

$$\|U_n\|_{S_q(\mathbb{R}^7_T)} \leq N\|h\phi_n\|_{L_q(\mathbb{R}^7_T)} + Nn^{-1}\|(|U| + |\nabla_v U|)|\xi_n|\|_{L_q(\mathbb{R}^7_T)},$$

where $N = N(q, \delta, K, T)$. By using the Hölder's inequality with p/q and $p/(p-q)$ and changing variables, we get

$$\begin{aligned}
&n^{-1}\|(|U| + |\nabla_v U|)|\xi_n|\|_{L_q(\mathbb{R}^7_T)} \\
&\leq N(p,q) n^{12(p-q)/(pq)-1}\||U| + |\nabla_v U|\|_{L_p(\mathbb{R}^7_T)}.
\end{aligned} \tag{D.7}$$

Note that the $12(p-q)/(pq) - 1 < 0$ due to (D.6), and, hence, passing to the limit in (D.7) as $n \to \infty$, we prove that $U \in S_q(\mathbb{R}^7_T)$. Then, by the uniqueness part of Theorem D.4 (ii), we conclude that $U \equiv u$. The lemma is proved. □

Appendix E. **Hölder continuity of the extended leading coefficients $\mathbb{A}$ on the whole space.** Invoke the notation of Section 2.1.2.

E.1. **Kinetic Fokker-Planck equation.** Here we show that the coefficients $\mathbb{A} = \mathbb{A}(y)$ defined by (2.8) (2.10), (2.12), with $a = I_3$, are of Lipschitz class on $\mathbb{R}^3$. Since $\mathbb{A} \in C^1(\overline{\mathbb{R}_\pm})$, we only need to check the continuity on the set $\{y_3 = 0\} \times \mathbb{R}^3$.

Note that for $a = I_3$, by (2.8)

$$A = \left(\frac{\partial y}{\partial x}\right)\left(\frac{\partial y}{\partial x}\right)^T.$$

By (2.18), (2.19) and the fact that A is independent of w, it suffices to show that

$$A^{i3}(t, y_1, y_2, 0) = 0, \quad i = 1, 2, \tag{E.1}$$

By using (2.3), we compute

$$\frac{\partial x}{\partial y} = \begin{pmatrix} 1 - y_3\rho_{11} & -y_3\rho_{12} & -\rho_1 \\ -y_3\rho_{12} & 1 - y_3\rho_{22} & -\rho_2 \\ \rho_1 & \rho_2 & 1 \end{pmatrix},$$

where ρ_{ij} is the second-order partial derivatives with respect to $y_i y_j$ variables. Hence,

$$A^{-1}(y_1, y_2, 0) = (\frac{\partial x}{\partial y}|_{y_3=0})^T \frac{\partial x}{\partial y}|_{y_3=0} \tag{E.2}$$
$$= \begin{pmatrix} 1 & 0 & \rho_1 \\ 0 & 1 & \rho_2 \\ -\rho_1 & -\rho_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -\rho_1 \\ 0 & 1 & -\rho_2 \\ \rho_1 & \rho_2 & 1 \end{pmatrix} = \begin{pmatrix} 1+\rho_1^2 & \rho_1\rho_2 & 0 \\ \rho_1\rho_2 & 1+\rho_2^2 & 0 \\ 0 & 0 & 1+\rho_1^2+\rho_2^2 \end{pmatrix}.$$

It follows from the formula for the matrix inverse that the condition (E.1) is satisfied. Thus, the desired regularity holds.

E.2. **Linear Landau equation.** Let $A, \mathcal{A}, \mathbb{A}$ be the functions defined by (2.8), (2.10), (2.12) with $a = \sigma_G$.

In this subsection, will show that, under Assumption 1.10 (see (1.26) - (1.27)),

– $\mathbb{A} \in L_\infty((0,T), C^{\varkappa/3,\varkappa}_{x,v}(\mathbb{R}^6))$.
– for any radially symmetric $\xi \in C_0^\infty(\mathbb{R}^3)$, $(t,y,w) \to \xi(\frac{\partial x}{\partial y} w)\mathbb{A}(t,y,w) \in L_\infty((0,T), C^{\varkappa/3,\varkappa}_{x,v}(\mathbb{R}^6))$.

Due to Lemma C.1, $\sigma_G \in L_\infty((0,T), C^{\varkappa/3,\varkappa}_{x,v}(\overline{\Omega} \times \mathbb{R}^3))$, and, hence,

$$\mathbb{A} \in L_\infty((0,T), C^{\varkappa/3,\varkappa}_{x,v}(\overline{\mathbb{H}_\pm})).$$

Then, we only need to show that, for any $t \in (0,T)$, the functions $(y,w) \to \mathbb{A}(t,y,w)$ and $(y,w) \to \xi(\frac{\partial x}{\partial y} w)\mathbb{A}(t,y,w)$ are continuous on the set $\{y_3 = 0\} \times \mathbb{R}^3$. These assertions will follow directly from the next lemma and corollary.

Lemma E.1. *Let u be a function on Σ^T satisfying the specular reflection boundary condition and such that the convolution U defined below makes sense. Denote*

$$U^{ij}(z) = \Phi^{ij} * u(z),$$
$$\mathfrak{U}(t,y,w) = \left(\frac{\partial y}{\partial x}\right)\widehat{U}(t,y,w)\left(\frac{\partial y}{\partial x}\right)^T,$$

where $\widehat{U}$ is defined by (2.5) with U in place of u. Then,

$$\mathfrak{U}^{i3}(t, y_1, y_2, 0, w) = -\mathfrak{U}^{i3}(t, y_1, y_2, 0, Rw), i = 1, 2. \tag{E.3}$$

Proof. Denote

$$\Xi(y,w) = \left(\frac{\partial y}{\partial x}\right)\widehat{\Phi}(y,w)\left(\frac{\partial y}{\partial x}\right)^T,$$

where Φ is the Landau kernel (see (1.2)). Then, by the change of variables $v' = \frac{\partial x}{\partial y} w'$,

$$\mathfrak{U}(t,y,w) = \left|\det \frac{\partial x}{\partial y}\right| \int_{\mathbb{R}^3} \Xi(y,w')\widehat{u}(t,y,w-w')\,dw',$$

and then, by changing variables $w' \to Rw'$, we obtain

$$\mathfrak{U}(t,y,Rw) = \left|\det \frac{\partial x}{\partial y}\right| \int_{\mathbb{R}^3} \Xi(y,Rw')\widehat{u}(t,y,R(w-w'))\,dw'.$$

Since u satisfies the specular reflection boundary condition, we have, for any $t \in (0,T)$, $y_1, y_2 \in \mathbb{R}$, $w \in \mathbb{R}^3$,

$$\widehat{u}(t, y_1, y_2, 0, w) = \widehat{u}(t, y_1, y_2, 0, Rw).$$

Thus, it suffices to show that (E.3) holds with $\mathfrak{U}$ replaced with Ξ.

By direct computations,

$$\widehat{\Phi}(y, w) = \left|\frac{\partial x}{\partial y} w\right|^{-1} I_3 - \left|\frac{\partial x}{\partial y} w\right|^{-3} \frac{\partial x}{\partial y} w w^T \left(\frac{\partial x}{\partial y}\right)^T,$$

and, therefore,

$$\Xi(y, w) = \left|\frac{\partial x}{\partial y} w\right|^{-1} \frac{\partial y}{\partial x} \left(\frac{\partial y}{\partial x}\right)^T - \left|\frac{\partial x}{\partial y} w\right|^{-3} w w^T.$$

Observe that (E.3) trivially holds with $\mathfrak{U}$ replaced with ww^T, and the matrix $\frac{\partial y}{\partial x}(\frac{\partial y}{\partial x})^T$ satisfies the condition (E.1). Hence, we only need to show that the function

$$V(y, w) := \left|\frac{\partial x}{\partial y} w\right|$$

satisfies the specular reflection boundary condition

$$V(y_1, y_2, 0, w) = V(y_1, y_2, 0, Rw). \tag{E.4}$$

Note that

$$|V(y, Rw)|^2 = C^{ij}(y)(Rw)_i (Rw)_j, \quad C(y) = \left(\frac{\partial x}{\partial y}\right)^T \frac{\partial x}{\partial y}.$$

Note that when $y_3 = 0$, the matrix $C(y)$ is given by the right-hand side of (E.2). Hence, since (E.1) holds with A replaced with C, the condition (E.4) is valid. The lemma is proved. □

The above lemma holds if we cutoff U by a radially symmetric function depending only on v.

Corollary E.1. *Let $\xi = \xi(v) \in C_0^\infty(\mathbb{R}^3)$ be a radially symmetric function. Then, the condition (E.3) holds with $\mathfrak{U}$ is replaced with the matrix-valued function*

$$(t, y, w) \to \xi(\frac{\partial x}{\partial y} w) \mathfrak{U}(t, y, w).$$

Proof. The assertion follows from Lemma E.1 and the fact that due to the radial symmetry and (E.4), the function $(y, w) \to \xi(\frac{\partial x}{\partial y} w)$ satisfies the specular reflection boundary condition on $\mathbb{H}^T_-$. □

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REFERENCES

[1] Dallas Albritton, Scott Armstrong, Jean-Christophe Mourrat, Matthew Novack, Variational methods for the kinetic Fokker-Planck equation, arXiv:1902.04037.

[2] Renjun Duan, Shuangqian Liu, Shota Sakamoto, Robert M Strain, Global mild solutions of the Landau and non-cutoff Boltzmann equations, *Comm. Pure Appl. Math.* **74** (2021), no. 5, 932–1020.

[3] Richard W. Beals, Vladimir Protopopescu, Abstract time-dependent transport equations, J. Math. Anal. Appl. **121** (1987), no. 2, 370–405.

[4] Hongjie Dong, Yan Guo, Zhimeng Ouyang, The Vlasov-Poisson-Landau System with the Specular Reflection Boundary Condition, arXiv:2010.05314.

[5] Hongjie Dong, Timur Yastrzhembskiy, Global L_p estimates for kinetic Kolmogorov-Fokker-Planck equations in nondivergence form, arXiv:2107.08568, to appear in Arch. Ration. Mech. Anal.

[6] Hongjie Dong, Timur Yastrzhembskiy, Global L_p estimates for kinetic Kolmogorov-Fokker-Planck equations in divergence form, in preparation.

[7] Yan Guo, Global weak solutions of the Vlasov-Maxwell system with boundary conditions, *Comm. Math. Phys.* **154** (1993), no. 2, 245–263.

[8] Yan Guo, Singular Solutions of the Vlasov-Maxwell System on a Half Line, *Arch. Rational Mech. Anal.* **131** (1995), 241–304.

[9] Yan Guo, The Landau equation in a periodic box, *Comm. Math. Phys.* **231** (2002), no. 3, 391–434.

[10] Yan Guo, Hyung Ju Hwang, Jin Woo Jang, Zhimeng Ouyang, The Landau equation with the specular reflection boundary condition, *Arch. Ration. Mech. Anal.* **236** (2020), no. 3, 1389–1454.

[11] Hyung Ju Hwang, Juhi Jang, Juan J. L. Velázquez, The Fokker-Planck Equation with Absorbing Boundary Conditions, *Arch. Ration. Mech.* **214** (2014), 183–233.

[12] Hyung Ju Hwang, Juhi Jang, Jaewoo Jung, The Fokker-Planck equation with absorbing boundary conditions in bounded domains, *SIAM J. Math. Anal.* **50** (2018), no. 2, 2194–2232.

[13] Hyung Ju Hwang, Juhi Jang, Juan J. L. Velázquez, Nonuniqueness for the kinetic Fokker-Planck equation with inelastic boundary conditions, *Arch. Ration. Mech. Anal.* **231** (2019), no. 3, 1309–1400.

[14] Malte Litsgård, Kaj Nyström, The Dirichlet problem for Kolmogorov-Fokker-Planck type equations with rough coefficients, *J. Funct. Anal.* **281** (2021), no. 10, Paper No. 109226.

[15] Jinoh Kim, Yan Guo, Hyung Ju Hwang, An L^2 to L^∞ framework for the Landau equation, *Peking Math. J.* **3** (2020), no. 2, 131–202.

[16] Nicolai V. Krylov, Lectures on elliptic and parabolic equations in Sobolev spaces, Graduate Studies in Mathematics, 96. American Mathematical Society, Providence, RI, 2008.

[17] Nicolai V. Krylov, On a representation of fully nonlinear elliptic operators in terms of pure second order derivatives and its applications, *Problems in mathematical analysis*, No. **59**., J. Math. Sci. (N.Y.) 177 (2011), no. 1, 1–26.

[18] Theodore S. Motzkin, Wolfgang R. Wasow, On the approximation of linear elliptic differential equations by difference equations with positive coefficients, *J. Math. Physics* **31**, (1953), 253–259.

[19] Sergio Polidoro, Maria A. Ragusa, Sobolev-Morrey spaces related to an ultraparabolic equation, *Manuscripta Math.* **96** (1998), no. 3, 371–392.

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