Characterizing unit spheres in Euclidean spaces via reach and volume

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Abstract

Let M be a smooth, connected, compact submanifold of $\mathbb{R}^n$ without boundary and of dimension $k \geq 2$. Let $\mathbb{S}^k \subset \mathbb{R}^{k+1} \subset \mathbb{R}^n$ denote the k-dimesnional unit sphere. We show if M has reach equal to one, then its volume satisfies $\text{vol}(M) \geq \text{vol}(\mathbb{S}^k)$ with equality holding only if M is congruent to $\mathbb{S}^k$.

1 Introduction

Let M be a smooth, connected, and closed k-dimensional submanifold of $\mathbb{R}^n$. Let $\rho : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ denote the Euclidean metric and $\rho_M : \mathbb{R}^n \to \mathbb{R}$ the distance function to M defined by $\rho_M(x) = \rho(x, M)$. The *reach* of M is the positive real number $\tau(M)$ defined by

$$\tau(M) = \sup\left\{t > 0 \,\middle|\, \text{Each point in } \rho_M^{-1}([0,t)) \text{ has a unique closest point in } M\right\}.$$

The reach of a subset of Euclidean space was introduced by Federer in the influential paper [5]. The reach of a closed submanifold as above equals its normal injectivity radius. The k-dimensional unit sphere

$$\mathbb{S}^k = \left\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \,\middle|\, \|x\| = 1 \text{ and } x_i = 0 \text{ if } k+2 \leq i \leq n\right\}$$

has $\tau(\mathbb{S}^k) = 1$. The scale invariant ratio

$$\text{vol}(M)/\tau(M)^k$$

arises in estimates for the number of metric balls in $\mathbb{R}^n$ needed to cover M when the balls are required to be centered in M and to have equal radii (see,

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e.g., [1, 4, 8, 7]). These estimates have applications in compressive sensing and mathematical data science where they are combined with probabilistic methods to estimate, e.g., the smallest dimension $m < n$ such that M, equipped with the restriction of the metric ρ, admits a bilipshitz map to $\mathbb{R}^m$ with bilipshitz constants close to 1. In [7, Proposition 4.2], Günther's volume comparison theorem and an injectivity radius estimate were applied to establish the inequality

$$\mathrm{vol}(M)/\tau(M)^k \geq \mathrm{vol}(\mathbb{S}^k)/\tau(\mathbb{S}^k)^k = \mathrm{vol}(\mathbb{S}^k).$$

Herein, we show that equality holds only for spheres.

Theorem 1.1. *Let M be a smooth, connected, and closed k-dimensional submanifold of $\mathbb{R}^n$ with $k \geq 2$. If $\tau(M) = 1$, then $\mathrm{vol}(M) \geq \mathrm{vol}(\mathbb{S}^k)$ with equality only if there exists an isometry I of $\mathbb{R}^n$ such that $I(\mathbb{S}^k) = M$.*

The proof consists of two main steps. The first step is to show that M, with the induced Riemannian metric, is isometric to $\mathbb{S}^k$. The second step is to show that M is embedded in $\mathbb{R}^n$ as an isometric image of the standard $\mathbb{S}^k$. In the first step, the hypothesis $\tau(M) = 1$ is used to bound the injectivity radius of M below by π after which Berger's sharp isoembolic inequality [2] is used to show M and $\mathbb{S}^k$ are isometric. Hong's theorem [6] reduces the second step to showing that each geodesic in M, a closed geodesic of length 2π, is the image of the standard $\mathbb{S}^1$ under some isometry of $\mathbb{R}^n$. Finally, the solution of a well known puzzle [10, 11] concerning closed curves in spheres applies to show that the geodesics of M are indeed unit circles in $\mathbb{R}^n$.

2 Preliminaries

2.1 Unit circles in $\mathbb{R}^n$ and closed curves in $\mathbb{S}^{n-1}$

A subset S of $\mathbb{R}^n$ is a *unit circle* if there exists an isometry I of $\mathbb{R}^n$ such that $S = I(\mathbb{S}^1)$. A parametric characterization of unit circles is given in Proposition 2.8 below. It is based on a solution to a puzzle about closed curves in $\mathbb{S}^{n-1}$ appearing in [10, 11].

To state the puzzle, equip $\mathbb{S}^{n-1}$ with the Riemannian metric induced from $\mathbb{R}^n$. The geodesics in $\mathbb{S}^{n-1}$ are unit circles with center of mass the origin. The geodesic distance function $d : \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \to \mathbb{R}$ is given by $d(p, q) = \arccos(\langle p, q\rangle)$. Each $m \in \mathbb{S}^{n-1}$ is the pole of a unique hemisphere

$$H_m = \left\{v \in \mathbb{S}^{n-1} \,|\, \langle v, m\rangle > 0\right\} = \left\{v \in \mathbb{S}^{n-1} \,|\, d(v, m) < \frac{\pi}{2}\right\}$$

and this hemisphere is bounded by an equitorial subsphere

$$E_m = \left\{v \in \mathbb{S}^{n-1} \,|\, \langle v, m\rangle = 0\right\} = \left\{v \in \mathbb{S}^{n-1} \,|\, d(v, m) = \frac{\pi}{2}\right\}.$$

Consider the following puzzle: *Prove if a closed curve on the unit sphere has length less than 2π, then it is contained in some hemisphere.*

We present an elegant solution to this puzzle taken from [10, 11] as Lemma 2.6 below. Understanding the boundary case of the puzzle leads to the desired parametric characterization of unit circles in Proposition 2.8. For our purposes, it is sufficient to work with curves in $\mathbb{R}^n$ admitting smooth parameterizations as described in the following definitions.

Definition 2.1. A *parameterization* is a smooth map $x : \mathbb{R} \to \mathbb{R}^n$. A paramaterization x has *unit speed* if $\|x'(t)\| = 1$ for all $t \in \mathbb{R}$. A *curve* in $\mathbb{R}^n$ is a subset Γ of $\mathbb{R}^n$ for which there exists a parameterization x with $x(\mathbb{R}) = \Gamma$.

Curves in $\mathbb{R}^n$ may not be the image of a unit speed parameterization. For instance, each point in $\mathbb{R}^n$ is a curve as the image of a constant parameterization. Such curves, called *point curves*, do not admit a unit speed parameterization. Given a parameterization x, let

$$\mathrm{Per}(x) = \{T \in \mathbb{R} \,|\, \forall t \in \mathbb{R},\, x(t) = x(t+T)\}.$$

$\mathrm{Per}(x)$ is a closed subgroup of $(\mathbb{R}, +)$. Therefore $\mathrm{Per}(x) = \mathbb{R}$, $\mathrm{Per}(x) = \{0\}$, or there exists $P > 0$ such that $\mathrm{Per}(x) = P \cdot \mathbb{Z}$. $\mathrm{Per}(x) = \mathbb{R}$ if and only if $x(\mathbb{R})$ is a point curve.

Definition 2.2. A paramaterization $x : \mathbb{R} \to \mathbb{R}^n$ is *periodic* with period $P > 0$ if $\mathrm{Per}(x) = P \cdot \mathbb{Z}$. A curve Γ is *closed* if there exists a periodic paramaterization x with $x(\mathbb{R}) = \Gamma$.

The next Lemma is a special case of the possibly intuitive assertion that a parameterized curve $x(t)$ cannot have closed image if there is a one-dimensional subspace L of $\mathbb{R}^n$ such that the velocity vector $x'(t)$ projects to a nonzero vector in L for each $t \in \mathbb{R}$.

Lemma 2.3. *Let $C : \mathbb{R} \to \mathbb{R}^n$ be a unit speed periodic parameterization and let $c : \mathbb{R} \to \mathbb{S}^{n-1}$ be the parametrization defined by $c(t) = C'(t)$ for each $t \in \mathbb{R}$. For each $m \in \mathbb{S}^{n-1}$, there exists $s \in \mathbb{R}$ such that $c(s) \in E_m$.*

Proof. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(t) = \langle C(t), m\rangle$. Let $P > 0$ be the period of C. Then

$$\int_{t=0}^{t=P} \langle c(t), m\rangle \, dt = \int_{t=0}^{t=P} f'(t)\, dt = f(P) - f(0) = \langle C(P) - C(0), m\rangle = 0,$$

from which the Lemma follows. □

If $x : \mathbb{R} \to \mathbb{R}^n$ is a parameterization and if $I \subset \mathbb{R}$ is a bounded interval, then the pair (x, I) has a length defined by the familiar formula

$$L(x, I) = \int_I \|x'(t)\|\, dt.$$

Length is determined by the image $x(I)$. We record the following special case without proof.

Lemma 2.4. *Let Γ be a closed curve in $\mathbb{R}^n$. Suppose that x and y are periodic parameterizations with $x(\mathbb{R}) = \Gamma = y(\mathbb{R})$. If P_x and P_y denote the periods of x and y, then $L(x, [0, P_x]) = L(y, [0, P_y])$.*

Definition 2.5. If Γ is a closed curve, then its *length* is defined as the common value of the lengths appearing in Lemma 2.4.

We now present a solution to the puzzle above. The solution appears in the puzzle books [10, 11]. We include it here since the line of reasoning appears again in the proof of Proposition 2.8 below.

Lemma 2.6. *Let $\Gamma \subset \mathbb{S}^{n-1}$ be a closed curve having the property that for each $m \in \mathbb{S}^{n-1}$, $\Gamma \cap E_m \neq \emptyset$. If L is the length of Γ, then $L \geq 2\pi$.*

Proof. Let $c : \mathbb{R} \to \mathbb{S}^{n-1}$ be a periodic parameterization with $c(\mathbb{R}) = \Gamma$. Let $P_c > 0$ denote the period of c. There exists $t_c \in (0, P_c)$ such that

$$L(c, [0, t_c]) = \frac{L}{2} = L(c, [t_c, P_c]).$$

Let $p = c(0)$ and $q = c(t_c)$ and note that

$$d(p, q) \leq L(c, [0, t_c]) = \frac{L}{2}.$$

Let $\gamma : [0, d(p, q)] \to \mathbb{S}^{n-1}$ be a unit speed minimizing geodesic joining p to q.

We argue by contradiction. If $L < 2\pi$, then

$$d(p, q) \leq \frac{L}{2} < \pi. \tag{2.1}$$

Letting $m = \gamma(\frac{d(p,q)}{2})$ denote the midpoint of γ, it follows

$$d(p, m) = d(q, m) = \frac{d(p, q)}{2} < \frac{\pi}{2}$$

so that $\{p, q\} \subset H_m$. The hypothesis implies there exists $s \in (0, t_c) \cup (t_c, P_c)$ such that $c(s) \in E_m$. After possibly reversing the orientation of c, we may assume that $s \in (0, t_c)$. Let $z = c(s)$. Then

$$d(p, z) \leq L(c, [0, s]) = \int_{t=0}^{t=s} \|c'(t)\| \, dt \tag{2.2}$$

and

$$d(z, q) \leq L(c, [s, t_c]) = \int_{t=s}^{t=t_c} \|c'(t)\| \, dt. \tag{2.3}$$

Summing,

$$d(p, z) + d(z, q) \leq L(c, [0, t_c]) = \frac{L}{2} < \pi. \tag{2.4}$$

Consider the isometric reflection $F : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ about E_m defined by

$$F(x) = x - 2\langle x, m\rangle m$$

for each $x \in \mathbb{S}^{n-1}$. As $z \in E_m$, $F(z) = z$. Use $m = \frac{p+q}{\|p+q\|}$ to evaluate $F(-q) = p$. Now

$$\pi = d(-q, q) \leq d(-q, z) + d(z, q) = d(F(-q), F(z)) + d(z, q) = d(p, z) + d(z, q),$$

contradicting (2.4) and concluding the proof. □

Lemma 2.7. *Let $C : \mathbb{R} \to \mathbb{R}^n$ be a P-periodic unit speed parameterization with $\|C''(t)\| \leq 1$ for each $t \in \mathbb{R}$. Let $c : \mathbb{R} \to \mathbb{S}^{n-1}$ be the parameterization defined by $c(t) = C'(t)$ for each $t \in \mathbb{R}$. Then the length of the closed curve $c(\mathbb{R})$ is less than or equal to P and equal to P if and only if c is a P-periodic unit speed parameterization.*

Proof. Note that c is not constant and that $P \in Per(c)$. Therefore c is a periodic parameterization with period P_c satisfying $P_c \leq P$. The length L of the closed curve $c(\mathbb{R})$ satisfies

$$L = \int_0^{P_c} \|c'(t)\|\, dt = \int_0^{P_c} \|C''(t)\|\, dt \leq \int_0^{P_c} 1\, dt = P_c \leq P.$$

Note that $L = P$ if and only if $P_c = P$ and $\|c'(t)\| = \|C''(t)\| = 1$ for each $t \in \mathbb{R}$, concluding the proof. □

Proposition 2.8. *Let $C : \mathbb{R} \to \mathbb{R}^n$ be a 2π-periodic unit speed parameterization with $\|C''(t)\| \leq 1$ for each $t \in \mathbb{R}$. Then $S = C(\mathbb{R})$ is a unit circle.*

Proof. Let $c : \mathbb{R} \to \mathbb{S}^{n-1}$ be the parameterization defined by $c(t) = C'(t)$ for each $t \in \mathbb{R}$. By Lemmas 2.3, 2.6, and 2.7, c is a 2π-periodic unit speed parameterization of a curve in $\mathbb{S}^{n-1}$ that intersects every equatorial subsphere. Let $p = c(0)$ and $q = c(\pi)$.

We first claim that $d(p, q) = \pi$. If not, then $d(p, q) < \pi = \text{diam}(\mathbb{S}^{n-1})$. As in Lemma 2.6, let m denote the midpoint of the unique minimizing geodesic joining p to q. As above, p and q lie in H_m and so up to changing the orientation of c, there exists $s \in (0, \pi)$ with the property that the point $z = c(s)$ satisfies $z \in E_m$. Note that if we establish $d(p, z) + d(z, q) < \pi$, then the argument presented in Lemma 2.6 applies to obtain a contradiction. First note that $d(p, z) \leq s$ and $d(z, q) \leq \pi - s$ by (2.2) and (2.3). If neither of these inequalities is strict, then the restrictions of c to $[0, s]$ and to $[s, \pi]$ are unit speed geodesics connecting p to z and z to q, respectively. As c is a unit speed parameterization these two geodesics meet smoothly at z. Therefore the restriction of c to $[0, \pi]$ is a geodesic of length π connecting p to q. This contradicts the assumption $d(p, q) < \pi$ since all geodesics in $\mathbb{S}^{n-1}$ of length π are minimizing.

Next, we claim that c parameterizes a geodesic in $\mathbb{S}^{n-1}$. Indeed, the restrictions of c to $[0, \pi]$ and to $[\pi, 2\pi]$ define two curves of length π that connect

the points p and q. As $d(p,q) = \pi$, these two curves are geodesics. These two geodesics meet smoothly at both p and q since c is a unit speed parameterization, concluding the proof that c parameterizes a geodesic in $\mathbb{S}^{n-1}$.

Finally, we argue that C parameterizes a unit circle. Define $x : \mathbb{R} \to \mathbb{R}^n$ by $x(t) = (\cos(t), \sin(t), 0, \ldots, 0)$. Then $x(\mathbb{R}) = \mathbb{S}^1$.. As the isometry group of $\mathbb{S}^{n-1}$ acts transitively on unit tangent vectors, any two unit speed parameterized geodesics in $\mathbb{S}^{n-1}$ differ by an isometry of $\mathbb{S}^{n-1}$. Therefore, there exists an orthogonal matrix $A \in O(n)$ such that for each $t \in \mathbb{R}$,

$$c(t) = x(t) \cdot A.$$

By the fundamental theorem, for each $t \in \mathbb{R}$,

$$C(t) - C(0) = \int_{s=0}^{s=t} c(s)\, ds = \int_{s=0}^{s=t} x(s) \cdot A\, ds.$$

Therefore,

$$C(t) = (\sin(t), -\cos(t), 0, \ldots, 0) \cdot A + (C(0) + (0, 1, 0, \ldots, 0) \cdot A),$$

from which the Proposition follows. □

2.2 Berger's sharp isoembolic inequality

This subsection reviews the definitions of the conjugate and injectivity radii for a closed Riemannian manifold and states Berger's isoembolic inequality, a key tool in the proof of Theorem 1.1. We refer the reader to [3, Chap. 13] and [2] for more details.

Let M be a k-dimensional connected and closed Riemannian manifold. Given $p \in M$, let T_pM and S_pM denote the tangent space and unit tangent sphere of M at the point p. Let

$$\exp_p : T_pM \to M$$

denote the exponential map at p. The *conjugate radius at* p is defined as the supremum of $r > 0$ for which the restriction of $\exp_p$ to the ball $B(0,r)$ is nonsingular. The *conjugate radius of* M, denoted by $\mathrm{conj}(M)$, is defined as the infimum of the conjugate radii of its points.

Given $p \in M$ and $v \in S_pM$, let $\gamma_v(t) = \exp_p(tv)$. Then $\gamma_v : \mathbb{R} \to M$ is a unit speed parameterized geodesic. The *cut time of* (p,v) is the positive real number $c(p,v)$ defined by

$$c(p,v) = \sup\{t > 0 \,|\, d(p, \gamma_v(t)) = t\}.$$

The cut time defines a continuous function on the unit sphere bundle

$$SM = \{(p,v) \,|\, p \in M \text{ and } v \in S_pM\}.$$

The *injectivity radius of* M, denoted $\mathrm{inj}(M)$, is defined by

$$\mathrm{inj}(M) = \min\{c(p,v) \,|\, (p,v) \in SM\}.$$

The following is known as Klingenberg's injectivity radius estimate (see e.g. [3, Chap. 13]).

Lemma 2.9 (Klingenberg)**.** *Let l denote the length of a shortest nonconstant closed geodesic in M. Then* $\mathrm{inj}(M) = \min\{\mathrm{conj}(M), l/2\}$.

Theorem 2.10 (Berger [2])**.** *Let M be a closed k-dimensional Riemannian manifold. Then*

$$\mathrm{vol}(M) \geq \mathrm{vol}(\mathbb{S}^k) \left(\frac{\mathrm{inj}(M)}{\pi} \right)^k$$

with equality holding only for constant curvature spheres.

2.3 Reach one submanifolds of $\mathbb{R}^n$

Lemma 2.11. *Let M be a closed submanifold of $\mathbb{R}^n$ with $\tau(M) = 1$. Then*

1. *If $\gamma : \mathbb{R} \to \mathbb{R}^n$ is a unit speed parameterization of a geodesic in M, then for each $t \in \mathbb{R}$, $\|\gamma''(t)\| \leq 1$.*
2. *The sectional curvatures of M are bounded above by 1.*
3. *The injectivity radius of M satisfies $\mathrm{inj}(M) \geq \pi$.*

Proof. Item (1) follows from the fact that the norm of the second fundamental form of M is bounded above by 1 in all normal directions [9, Proposition 6.1]. Item (2) follows from item (1) and the Gauss equation [3, Chap. 6, Theorem 2.5]. It remains to prove (3). By Lemma 2.9, the injectivity radius of M equals the minimum of its conjugate radius and half the length of a shortest closed geodesic in M. By Lemma 2.11-(2) and the Rauch comparison theorem [3, Chap. 10, Theorem 2.3], $\mathrm{conj}(M) \geq \pi$. It remains to show the shortest closed geodesic in M has length at least 2π. Let $l > 0$ denote the length of a shortest closed geodesic and let $C : \mathbb{R} \to M$ be an l-periodic unit speed parameterization of one such closed geodesic. Let $c = C'$ and let L denote the length of the closed curve $c(\mathbb{R})$. By Lemma 2.11-(1), Lemma 2.7 applies, whence

$$L \leq l.$$

By Lemmas 2.3 and 2.6,

$$2\pi \leq L.$$

Therefore, $2\pi \leq l$, completing the proof. □

3 Proof of Theorem 1.1.

Proof. By Theorem 2.10 and Lemma 2.11-(3),

$$\mathrm{vol}(M) \geq \mathrm{vol}(\mathbb{S}^k) \left(\frac{\mathrm{inj}(M)}{\pi} \right)^k \geq \mathrm{vol}(\mathbb{S}^k).$$

Now suppose that $\mathrm{vol}(M) = \mathrm{vol}(\mathbb{S}^k)$. Then $\mathrm{inj}(M) = \pi$ and M is isometric to the canonical unit sphere $\mathbb{S}^k$. In particular, each of its geodesics is a closed curve of length 2π admitting a unit speed parameterization. By Lemma 2.11-(1) and Proposition 2.8, each geodesic in M is a unit circle in $\mathbb{R}^n$. By [6, Theorem 4], M and $\mathbb{S}^k$ are congruent. □

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