Degree bounds for Hopf actions on Artin–Schelter regular algebras

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We study semisimple Hopf algebra actions on Artin–Schelter regular algebras and prove several upper bounds on the degrees of the minimal generators of the invariant subring, and on the degrees of syzygies of modules over the invariant subring. These results are analogues of results for group actions on commutative polynomial rings proved by Noether, Fogarty, Fleischmann, Derksen, Sidman, Chardin, and Symonds.

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Syzygy degree bound
Castelnuovo–Mumford regularity

0. Introduction

Throughout, let $\Bbbk$ be a field. The invariant subring T^G of a commutative polynomial ring $T := \Bbbk[x_1, \ldots, x_n]$ under the linear action of a group G has played an important role in commutative algebra and algebraic geometry. Producing a minimal generating set for T^G (as an algebra) is the first step in understanding the invariant subring. In 1916, Noether proved the following fundamental theorem, which is sometimes called Noether's upper bound theorem.

Theorem 0.1 *([44]). If $\Bbbk$ is a field of characteristic zero and G is a finite group of invertible $n \times n$ matrices acting linearly on $T := \Bbbk[x_1, \ldots, x_n]$ then the ring of invariants T^G can be generated as a $\Bbbk$-algebra by polynomials of total degree $\leq |G|$.*

This result is extremely useful in explicitly computing the invariant subring, for in characteristic zero, invariants are linear combinations of elements of the form $\sum_{g \in G} g.m$, the sum of the elements in the G-orbit of a monomial $m \in T$. Knowing an upper bound on the degrees of minimal generators of the invariants affords an algorithm for finding the generators, as one can compute these sums for all monomials m of degree less than or equal to the bound.

Noether's upper bound was extended to the non-modular case (where $\operatorname{char} \Bbbk$ does not divide $|G|$) independently by Fleischmann [23] in 2000, Fogarty [24] in 2001, and Derksen and Sidman [15] in 2002. Derksen and Sidman used a homological invariant, the Castelnuovo–Mumford (CM) regularity of a subspace arrangement associated to the action, to obtain their bound. Hence in the non-modular case there is a bound on the degrees of the minimal generators that is independent of the representation of the group, though the actual degrees of the generators may be quite a bit less than the order of the group (e.g., Domokos and Hegedüs [16] provided a smaller upper bound on the degrees of generators if G is not a cyclic group, and this result was extended to the non-modular case by Sezer [47]). Surveys of results extending Noether's bound that were obtained before 2007 can be found in [43] and [54].

In the modular case, the Noether bound does not hold; for example if $\operatorname{char} \Bbbk = 2$, there is an action of the group of order 2 on $\Bbbk[x_1, x_2, x_3, x_4, x_5, x_6]$ that requires a generator of degree 3 [14, Example 3.5.5(a)]. Using CM regularity (though differently than Derksen and Sidman [15]), Symonds proved a bound that depends on both the order of the group G and the dimension of the representation of G: if G acts on $T = \Bbbk[x_1, \ldots, x_n]$ for $n > 1$ and $|G| \geq 2$ then T^G can be generated by elements of degree $\leq n(|G| - 1)$ [53, Corollary 0.2]. Hence in the modular case the upper bound depends upon both $|G|$ and the degree of the representation, or, equivalently, the global dimension of T.

It is also natural to look for bounds on the maximal degrees of syzygies of T^G as a quotient module over another polynomial ring and the maximal degrees of syzygies of the trivial module $\Bbbk$ over T^G. Castelnuovo–Mumford regularity again has been useful in finding such bounds [13,9,53]. A nice result of Chardin and Symonds [9, Theorem 1.3], generalizing work of Derksen [13], states that, if $\Bbbk$ has characteristic zero, then for all $i \geq 2\,\beta_i(T^G)$, the maximal degree of $\mathrm{Tor}_i^{T^G}(\Bbbk, \Bbbk)$, is bounded by $|G|i+i-2$. There is also an approach using the theory of twisted commutative algebras, which Snowden used to establish bounds on maximal degrees of syzygies [49]. Gandini used similar techniques to extend the Noether bound (and syzygy bound) to a noncommutative ring, the exterior algebra, in the characteristic zero case [25, Theorem V1.13].

Our main goal in this paper is to explore degree bounds on minimal generators, as well as of syzygies, for invariants in noncommutative algebras, where little is known about these bounds. We call an algebra A *connected graded* if it has a $\Bbbk$-vector space decomposition

$$A = \Bbbk \oplus A_1 \oplus A_2 \oplus \cdots$$

with $1 \in A_0$, and $A_i A_j \subseteq A_{i+j}$ for all $i, j \in \mathbb{N}$. Throughout let A be a connected graded noetherian algebra. If M is a graded (left or right) A-module, the shifted A-module $M(n)$ is the graded module defined by $M(n)_i = M_{n+i}$. Denote by $\beta(A) \in \mathbb{N} \cup \{\infty\}$ the largest degree of an element in a minimal generating set of A.

In the noncommutative setting, it is natural to replace the commutative polynomial ring with a noetherian Artin–Schelter regular $\Bbbk$-algebra [1] generated in degree one, as such algebras share many of the homological properties of commutative polynomial rings, and, when commutative, these Artin–Schelter regular algebras are isomorphic to polynomial rings.

Definition 0.2. A connected graded algebra T is called *Artin–Schelter Gorenstein* (or *AS Gorenstein*, for short) if the following conditions hold:

(a) T has injective dimension $d < \infty$ on the left and on the right,
(b) $\mathrm{Ext}_T^i(_T\Bbbk, _TT) = \mathrm{Ext}_T^i(\Bbbk_T, T_T) = 0$ for all $i \neq d$, and
(c) $\mathrm{Ext}_T^d(_T\Bbbk, _TT) \cong \mathrm{Ext}_T^d(\Bbbk_T, T_T) \cong \Bbbk(\mathfrak{l})$ for some integer $\mathfrak{l}$. Here $\mathfrak{l}$ is called the *AS index* of T.

In this case, we say T is of type $(d, \mathfrak{l})$. If in addition,

(d) T has finite global dimension, and
(e) T has finite Gelfand–Kirillov dimension (see (E1.7.1)),

then T is called *Artin–Schelter regular* (or *AS regular*, for short) of dimension d.

Throughout, we will use the letters S and T to denote AS regular (or AS Gorenstein) algebras, while the letters A and B will usually be used for connected graded algebras, more generally.

One reason we focus on AS Gorenstein algebras is that noncommutative CM regularity [Definition 2.9] has been studied for these algebras by Jørgensen [29,30] and Dong and Wu [17]. As in the commutative case, we will see that CM regularity is an important tool for proving bounds on the degrees of generators. Conversely, results on degrees of generators contribute to the further understanding of noncommutative CM regularity.

We will consider groups G that act on AS regular algebras T via graded automorphisms, and, more generally, semisimple Hopf algebras that act homogeneously on T. For a Hopf algebra H, we use the standard notation $\Delta : H \to H \otimes H$ for the coproduct, $\varepsilon : H \to \Bbbk$ for the counit, and $S : H \to H$ for the antipode of H. Further details on Hopf actions on algebras can be found in [42]. In most cases, we will assume the following hypotheses.

Hypothesis 0.3.

(a) H is a semisimple, hence finite-dimensional (by [40]), Hopf algebra,
(b) T is a connected graded noetherian AS Gorenstein algebra of injective dimension at least two, and
(c) T is a left H-module algebra and for each i, T_i is a left H-submodule of T.

Previous work shows that many results concerning group actions on commutative polynomial rings have generalizations to the context of Hopf actions on AS regular algebras (see, e.g., [6–8,10,20–22,32–35,38]).

Unfortunately, there is a serious lack of understanding of degree bounds in the noncommutative context. To illustrate this, note the following two facts that indicate that the noncommutative case is quite different from the commutative case.

(a) In the commutative case, if any finite group G acts nontrivially on the polynomial ring $\Bbbk[x_1, \ldots, x_n]$, then $\beta(\Bbbk[x_1, \ldots, x_n]^G) > 1$. However, when T is noncommutative, $\beta(T^G)$ can be 1 even if T is a Koszul AS regular algebra [Example 1.2(2)].
(b) In the commutative non-modular case, by Theorem 0.1 and [23,24,15], $\beta(\Bbbk[x_1, \ldots, x_n]^G) \leq |G|$. However, if T is noncommutative, then $\beta(T^G)$ can be strictly larger than $|G|$. In [Example 1.2(3)], we provide an example of a $\mathbb{Z}/(2)$ action on $\Bbbk_{-1}[x_1, x_2]$ such that $\beta(\Bbbk_{-1}[x_1, x_2]^{\mathbb{Z}/(2)}) = 3$. Furthermore, Ferraro, Moore, Peng, and the first-named author showed that for n odd there is a cyclic group of order $2n$ acting on $T = \Bbbk_{-1}[x_1, x_2]$ with $\beta(T^G) = 3n$ [22, Theorem 2.5]. Hence, the difference $\beta(T^G) - |G|$ can, in fact, be arbitrarily large.

In our noncommutative context, an upper bound on the degrees of minimal algebra generators for the invariant subring of an AS regular algebra T might depend on both

the algebra T, as well as the group (or Hopf algebra) and its representation, while classical invariant theory is restricted to considering only the single commutative AS regular algebra $\Bbbk[x_1, \ldots, x_n]$. In Example 3.6, we show that for any m, there is a 2-generated noetherian AS regular algebra on which a group of order 2 acts such that the maximal degree of minimal generators of the invariant subring is at least m. Hence, there is no bound on the maximal degree of a generating set of T^G that holds for all noetherian AS regular algebras T that is dependent upon only the number of generators of T and the order of the group G. It would be nice to have a degree bound that does not depend on the action of G (or more generally H) on T. We pose the following question.

Question 0.4. Suppose (T, H) satisfies Hypothesis 0.3. Assume that T is an AS regular domain. Is $\beta(T^H)$ — the maximal degree of a minimal generating set of T^H — bounded by a function of the numerical invariants

$$\dim_{\Bbbk} H, \quad \operatorname{gldim} T, \quad \operatorname{CMreg}(T)?$$

We are able to answer this question in some special cases. First, we generalize a commutative result and show that $\beta(T^H)$ is bounded by $\tau_H(T)$, the τ-saturation degree of the H-action on T (see Definition 1.1(2)).

Theorem 0.5 *(Corollary 3.3). Let A be a connected graded algebra and let H be a semisimple Hopf algebra acting on A homogeneously. Then $\beta(A^H) \leq \tau_H(A)$.*

This result is computationally useful, as $\tau_H(T)$ is often easy to bound. We are able to use this result to answer Question 0.4 in the following two cases. Note that β_i will be defined in (E1.0.6).

Theorem 0.6 *(Theorem 3.5). Suppose that (T, H) satisfies Hypothesis 0.3. Assume further that*

(a) *T is an AS regular domain generated in degree 1 such that $T \# H$ is prime, and*
(b) *T^H has finite global dimension.*

Then $\beta(T^H) \leq \dim H$.

Theorem 0.7. *Suppose $\Bbbk$ is an infinite field, G is a finite group, and $\Bbbk G$ is semisimple. Suppose G acts via graded automorphisms on $\Bbbk_{-1}[x_1, \ldots, x_n]$.*

(1) *[Corollary 3.12] Then*

$$\beta(\Bbbk_{-1}[x_1, \ldots, x_n]^G) \leq 2|G| + n.$$

(2) [Corollary 5.11] *Assume that* $\Bbbk_{-1}[x_1, \ldots, x_n]^G$ *is commutative. Then*

$$\beta_i(\Bbbk_{-1}[x_1, \ldots, x_n]^G) \le i(2|G| + n + 1) - 2$$

for all $i \ge 2$.

An interesting open question is whether there is a similar bound for Hopf algebra actions on $\Bbbk_q[x_1, \ldots, x_n]$ where q is a root of unity (the only interesting group actions occur when $q = \pm 1$ while there are Hopf actions for other values of q).

One difference between the commutative and noncommutative cases is that when a group G acts on $C := \Bbbk[x_1, \ldots, x_n]$ via graded automorphisms, there is a graded surjective map from a polynomial ring onto C^G. On the other hand, if a Hopf algebra H acts on an AS regular algebra T as in Hypothesis 0.3, it is not known if there always exists an analogous AS regular algebra S mapping onto T^H (see Remark 5.8).

Moreover, in the commutative case, by the Noether Normalization Theorem, there exists a polynomial subring B of C^G such that C^G is a finitely generated B-module. In some cases, it is known that an analogous AS regular subalgebra exists (e.g., see Lemma 3.8). The following result provides bounds on the degrees of a minimal generating set of the invariant subring and on the relations among the generators in the case when there is a graded algebra map from an AS regular algebra S to T^H; it is a noncommutative version of [53, Proposition 2.1(2, 3)].

Theorem 0.8 *(Corollary 4.6). Let* (T, H) *be as in Hypothesis 0.3. Suppose there is a graded algebra map* $S \to T^H$ *where* S *is a noetherian AS regular algebra such that* T^H *is finitely generated over* S *on both sides. Let* $\delta(T/S) = \mathrm{CMreg}(T) - \mathrm{CMreg}(S)$. *Then*
(1)

$$\beta(T^H) \le \max\{\beta(S), \delta(T/S)\} \ and$$

(2)

$$\beta_2(T^H) \le \max\left\{2\delta(T/S), \delta(T/S) + \beta(S), \beta_2(S)\right\}.$$

We also provide a noncommutative version of [9, Theorem 1.2 (1)], which gives bounds on the maximal degrees in a projective S-module resolution of T^H when there is a graded surjection from an AS regular algebra S onto T^H. Let J_i denote the annihilator ideal of the finite-dimensional left T-module $\mathrm{Tor}_i^S(T, \Bbbk)$ and let $J_\infty = \cap_{j \ge 0} J_j$. Note that t_i^S will be defined in (E1.0.3)–(E1.0.4).

Theorem 0.9 *(Theorem 5.7). Suppose that* (T, H) *be as in Hypothesis 0.3 and assume that* T *is Koszul. Suppose there exists a graded algebra surjection* $S \to T^H =: R$ *where* S *is a noetherian AS regular algebra such that* $t_j^S(\Bbbk) \le (\deg T/J_\infty + 2)j$ *for all* $j \ge 0$. *Then*

$$t_j^S(R_S) \le (\deg T/J_\infty + 2)j + \deg T/J_\infty$$

for all $j \geq 0$.

In practice, one may bound $\deg T/J_\infty$ by other means: see, for example, Proposition 3.11(2). The following is a noncommutative version of [13, Theorem 2].

Theorem 0.10 *(Theorem 5.12). Let (T, H) be as in Hypothesis 0.3 and suppose that T is AS regular. Suppose further that*

(a) *T is generated in degree 1 and*
(b) *S is a noetherian AS regular algebra such that the minimal generating vector spaces of S and $R := T^H$ have the same dimension and there exists a graded algebra surjection $S \to R$.*

Then the following statements hold.

(1) *We have*

$$\beta_2(R) := t_2^R(\Bbbk) \leq 2 - 2\operatorname{CMreg}(S) + \operatorname{CMreg}(T).$$

(2) *Suppose that $\operatorname{Tor}_1^S(\Bbbk, R) \otimes_R \Bbbk \cong \operatorname{Tor}_1^S(\Bbbk, R)$. Then*

$$t_1^S({}_S R) \leq 2 - 2\operatorname{CMreg}(S) + \operatorname{CMreg}(T).$$

(3) *Suppose the hypothesis of part (2). Let K be the kernel of the algebra map $S \to R$. Then, as a left ideal of A, K is generated in degree at most*

$$2 - 2\operatorname{CMreg}(S) + \operatorname{CMreg}(T).$$

Both $t_1^S({}_S R)$ and $t_2^R(\Bbbk)$ give information on the degrees of the relations between the minimal generators of R (when viewing R as an S-module or as a $\Bbbk$-algebra), and this theorem provides a bound for both $t_1^S({}_S R)$ and $t_2^R(\Bbbk)$ which depends only on the CM regularities of S and T. Other degree bounds for higher syzygies can be found in Corollary 4.8 (bounds on $t_i^S({}_S R)$) and Proposition 5.9 (bounds on $t_i^C(\Bbbk)$ where $C = f(A)$ is the image of a connected graded algebra). We remark that some of the bounds that we obtain for actions on noncommutative AS regular algebras differ from the analogous bounds in the classical case (e.g., Noether's bound does not hold), but many of our bounds reduce to the results known for group actions on commutative polynomial rings.

The paper is organized as follows. Section 1 contains basic definitions and key examples. In Section 2, we review the properties of (noncommutative) local cohomology that are needed in the paper and present Jørgensen's definition of noncommutative Castelnuovo–Mumford regularity. In Section 3, we obtain some bounds on $\beta(A^H)$ using a noncommutative version of τ-saturation degree that was introduced in [24]. In

Sections 4 and 5, assuming the existence of a graded algebra map from an AS regular algebra S to T^H, we prove bounds on the degrees of the projective modules in free resolutions obtaining, in Section 4, generalizations of results of Derksen and Symonds, and, in Section 5, results of Chardin and Symonds and of Derksen that can be used to provide bounds on the higher syzygies of T^H as an S-module. We conclude in Section 6 by listing questions for further study.

1. Hilbert ideals and τ-saturation degree

Let $M = \bigoplus_{d \in \mathbb{Z}} M_d$ be a $\mathbb{Z}$-graded $\mathbb{k}$-vector space. Define the *degree* of M to be the maximum degree of a nonzero homogeneous element in M, namely,

$$\deg(M) = \inf\{d \mid (M)_{\geq d} = 0\} - 1 = \sup\{d \mid (M)_d \neq 0\} \quad \in \quad \mathbb{Z} \cup \{\pm\infty\}. \quad \text{(E1.0.1)}$$

Similarly, we define

$$\gcd(M) = \sup\{d \mid (M)_{\leq d} = 0\} + 1 = \inf\{d \mid (M)_d \neq 0\} \quad \in \quad \mathbb{Z} \cup \{\pm\infty\}. \quad \text{(E1.0.2)}$$

We say that a graded module M is *locally-finite* if $\dim_{\mathbb{k}} M_d < \infty$ for all $d \in \mathbb{Z}$. Let A be a connected graded algebra with trivial A-bimodule denoted by $\mathbb{k}$. For a graded left A-module M, let

$$t_i^A(_A M) = \deg \operatorname{Tor}_i^A(\mathbb{k}, M). \quad \text{(E1.0.3)}$$

If M is a right graded A-module, let

$$t_i^A(M_A) = \deg \operatorname{Tor}_i^A(M, \mathbb{k}). \quad \text{(E1.0.4)}$$

It is clear that $t_i^A(_A\mathbb{k}) = t_i^A(\mathbb{k}_A)$. If the context is clear, we will use $t_i^A(M)$ instead of $t_i^A(_A M)$ (or $t_i^A(M_A)$).

Recall that $\beta(A)$ denotes the largest degree of elements in a minimal generating set of A. This number is independent of the choice of the minimal generating set since it is equal to the largest degree of elements in the graded vector space $\operatorname{Tor}_1^A(\mathbb{k}, \mathbb{k})$, namely,

$$\beta(A) = t_1^A(\mathbb{k}) = \deg(\operatorname{Tor}_1^A(\mathbb{k}, \mathbb{k})). \quad \text{(E1.0.5)}$$

More generally, for any $i \geq 2$, we define

$$\beta_i(A) = t_i^A(\mathbb{k}) = \deg(\operatorname{Tor}_i^A(\mathbb{k}, \mathbb{k})). \quad \text{(E1.0.6)}$$

While $\beta(A)$ gives information about degrees of generators of A, $\beta_2(A)$ yields information about the degrees of the relations of A.

Let H be a semisimple Hopf algebra acting on A homogeneously. Sometimes we will use R to denote the *invariant subring* (or *fixed subring*)

$$R := A^H := \{a \in A \mid h \cdot a = \varepsilon(h)a\}. \tag{E1.0.7}$$

Since the action of H on A is homogeneous, R is a connected graded algebra. We note that $\beta(R) := \beta(A^H)$ generally depends on A, H and the action of H on A.

We now define the Hilbert ideal and the τ-saturation degree, which were used in the non-modular proofs of the Noether bound by Fogarty [24] and Fleischmann [23]. These definitions extend easily to noncommutative algebras. In the noncommutative case, the left Hilbert ideal was introduced in Gandini's thesis [25, Definition II.11].

Definition 1.1. Retain the above notation.

(1) The *left Hilbert ideal* of the H-action on A is the left ideal of A

$$J_H(A) := AR_{\geq 1}.$$

The *right Hilbert ideal* of the H-action on A is the right ideal of A

$$J_H^{\mathrm{op}}(A) := R_{\geq 1}A.$$

(2) The *τ-saturation degree* (or *left τ-saturation degree*) of the H-action on A is defined to be

$$\tau_H(A) = 1 + \deg(A/J_H(A)) = 1 + t_0^R(A_R) \quad \in \quad \mathbb{N} \cup \{\infty\}.$$

The *right τ-saturation degree* of the H-action on A is defined to be

$$\tau_H^{\mathrm{op}}(A) = 1 + \deg(A/J_H^{\mathrm{op}}(A)) = 1 + t_0^R({}_RA) \quad \in \quad \mathbb{N} \cup \{\infty\}.$$

(3) For every integer $i \geq 0$, the *ith annihilator ideal* of the H-action on A is defined to be the (two-sided) ideal of A

$$J_{H,i}(A) := \mathrm{ann}({}_A\mathrm{Tor}_i^R(A_R, \Bbbk))$$

and let

$$J_\infty = \bigcap_{i \geq 0} J_{H,i}.$$

It is easy to see that $J_{H,0}(A) \subseteq J_H(A)$ and $\deg A/J_{H,0}(A) = \deg A/J_H(A)$. It is not known if $\tau_H(A) = \tau_H^{\mathrm{op}}(A)$ in general (see Question 6.1).

Example 1.2.

(1) Suppose that $\Bbbk$ contains a primitive mth root of unity ω and that $A(\neq \Bbbk)$ is generated in degree 1. Define the map $\sigma(a) = \omega a$ for all $a \in A_1$ and extend it to an automorphism of A. Then the invariant subring $A^{\langle \sigma \rangle}$ is the mth Veronese ring of A so the left Hilbert ideal $J_{\langle \sigma \rangle}(A)$ is zero in degrees $< m$ and equal to A in degrees $\geq m$. As a result, $\tau_{\langle \sigma \rangle}(A) = m$. If A is a domain, then $\beta(A^{\langle \sigma \rangle}) = m$. Similarly, $\tau^{\mathrm{op}}_{\langle \sigma \rangle}(A) = m$.

(2) [33, Example 5.4(c)] Let T be the Rees ring of the first Weyl algebra with respect to the standard filtration. So T is generated by x, y, and z subject to the relations

$$xy - yx = z^2, \quad z \text{ is central.}$$

Let σ be the automorphism of T determined by

$$\sigma(x) = x, \quad \sigma(y) = y, \quad \text{and} \quad \sigma(z) = -z.$$

By [33, Example 5.4(c)], $T^{\langle \sigma \rangle}$ is generated in degree 1. In this case

$$\beta(T^{\langle \sigma \rangle}) = 1 < 2 = \tau_{\langle \sigma \rangle}(T).$$

Note that [20, Conjecture 0.3] fails as $T^{\langle \sigma \rangle}$ is AS regular and the product of the degrees of a homogeneous minimal generating set is 1 and $\dim \Bbbk \langle \sigma \rangle = 2$.

(3) [38, Example 3.1] Let T be the (-1)-skew polynomial ring $\Bbbk_{-1}[x_1, x_2]$ and σ be the automorphism of T exchanging x_1 and x_2. Let $G = \langle \sigma \rangle$. By [38, Example 3.1], T^G is generated by $x_1 + x_2$ and $x_1^3 + x_2^3$, so $\beta(T^G) = 3 > 2 = |G|$. One can easily check that $\tau_G(T) = \beta(T^G)$ in this case.

We remark that $\tau_H(A)$ (and $\tau^{\mathrm{op}}_H(A)$) need not be finite when A is not noetherian, as the next example shows.

Example 1.3. Let A be the free algebra generated by two elements, say x and y, in degree 1. Let G be the group $\mathbb{Z}/(2) = \{\mathrm{id}, \sigma\}$ and define an action of G on A by

$$\sigma(x) = -x, \quad \sigma(y) = y.$$

It is clear that the invariant subring A^G is generated by a minimal generating set of homogeneous elements

$$\{y, x^2, xyx, xy^2x, xy^3x, \ldots, xy^n x, \ldots\}.$$

In particular, $\beta(A^G) = \infty$. By a general result in Section 3 (Corollary 3.3), one sees that $\tau_G(A) \geq \beta(A^G) = \infty$, but we can check this directly. Note that A is an infinitely generated right A^G-module with a minimal generating set

$$\Phi := \{1, x, yx, y^2x, y^3x, y^4x, \dots\}.$$

To see this, observe that every homogeneous element can be written as $y^n f$ where f contains an even number of x's or $y^n x f$ where f contains an even number of x's. Thus Φ generates A as a right A^G-module. Further, every $y^n x$ cannot be written as an element in $A^G + \sum_{i=0}^{n-1} y^i x A^G$. Therefore Φ is a minimal generating set of the right A^G-module A. This implies that $A \otimes_{A^G} \Bbbk$ is infinite-dimensional, whence, $\tau_G(A) = \infty$. Similarly, $\tau_G^{\mathrm{op}}(A) = \infty$.

We are mainly interested in noetherian algebras, and in that case the τ-saturation degree is always finite.

Lemma 1.4. *Retain the above notation. Suppose that A is noetherian.*

(1) *Both $t_i^R(A_R)$ and $t_i^R({}_RA)$ are finite for all $i \geq 0$.*
(2) *Both $\tau_H(A)$ and $\tau_H^{op}(A)$ are finite.*

Proof. Part (2) is a special case part (1), so we only need to prove part (1). By symmetry, it suffices to show that $t_i^R(A_R)$ is finite for all $i \geq 0$.

By [42, Corollary 4.3.5 and Theorem 4.4.2], $R := A^H$ is noetherian, and A is a finitely generated module over R on both sides. Since R is noetherian, every term in the minimal free resolution $P^\bullet$ of A_R is a finitely generated free right R-module. Thus, each term in $P^\bullet \otimes_R \Bbbk$ is finite-dimensional. Consequently, $\mathrm{Tor}_i^R(A, \Bbbk)$ is finite-dimensional and the assertion follows. $\square$

Proposition 1.5. *Let A be a noetherian domain and let H be a semisimple Hopf algebra acting on A homogeneously. Suppose that $A\#H$ is prime. Then A is a finitely generated left and right A^H-module of rank equal to $\dim H$.*

If, in addition, A and A^H are AS regular, then A is a free A^H-module.

Proof. By [42, Corollary 4.3.5 and Theorem 4.4.2], $R := A^H$ is noetherian, and A is a finitely generated module over R on both sides.

Let Q be the quotient division ring of A. Since H acts on A, this action extends naturally to an H action on Q. Further, $Q\#H$ is the artinian quotient ring of the prime ring $A\#H$ and so $Q\#H$ is simple. By [11, Corollary 3.10],

$$[Q : Q^H]_r = [Q : Q^H]_l = \dim H.$$

By [48, Theorem 4.3(iii)], $Q^H = Q(A^H)$. Since the (left) rank of A over A^H is equal to the (left) rank of Q over Q^H, the assertion follows.

If A and A^H are AS regular, then by [33, Lemma 1.10(a, b)], A is a finitely generated free module over R on both sides. $\square$

See [3, Lemma 3.10] for a condition when $A\#H$ is prime.

In Corollary 3.3 in Section 3, we will see that the τ-saturation degree of the H-action on A gives a bound on $\beta(A^H)$, so it is important to bound the τ-saturation degree. Next is an easy example.

Example 1.6. Let T be a noetherian AS regular domain generated in degree 1. Suppose that T^H has finite global dimension and that $T\#H$ is prime. By [33, Lemmas 1.10(c) and 1.11(b)], T^H is AS regular. Hence, by Proposition 1.5, T is a free module over $R := T^H$ of rank equal to $d := \dim H$. Then $M := T \otimes_R R/R_{\geq 1}$ has $\Bbbk$-dimension d. As a left T-module, M is generated by the element 1 of degree 0. Since T is generated in degree 1, $M_i \neq 0$ for all i between 0 and $\sup\{j \mid (T/J_H(T))_j \neq 0\}$. So $\sup\{j \mid (T/J_H(T))_j \neq 0\} \leq d-1$ as $\dim M = d$. As a consequence, $\tau_H(T) \leq d$. In summary,

$$\tau_H(T) \leq \dim H.$$

Similarly, we have $\tau_H^{\mathrm{op}}(T) \leq \dim H$.

The Shephard–Todd–Chevalley Theorem describes the invariant subring of a commutative polynomial ring under the action of a reflection group; it shows that the invariant subring has finite global dimension, and the product of the degrees of the minimal generating invariants is equal to the order of the group. A similar phenomenon was observed in examples obtained in the noncommutative setting for group actions [34] and Hopf actions [20,21] where the invariant subrings have finite global dimension. We conclude this section with Proposition 1.8; parts (3) and (4) of this proposition provide a generalization of Proposition 1.5, and describe some conditions under which the product of the degrees of the minimal generators of the invariant subring is equal to the dimension of the Hopf algebra. It provides a partial answer to a variation of [20, Conjecture 0.3].

Definition 1.7. Let A be a locally-finite, connected graded algebra $A := \bigoplus_{i\geq 0} A_i$. The *Hilbert series of A* is defined to be

$$h_A(t) = \sum_{i\in\mathbb{N}} (\dim_\Bbbk A_i) t^i.$$

Similarly, if M is a $\mathbb{Z}$-graded A-module (or $\mathbb{Z}$-graded vector space) $M = \bigoplus_{i\in\mathbb{Z}} M_i$, the *Hilbert series of M* is defined to be

$$h_M(t) = \sum_{i\in\mathbb{Z}} (\dim_\Bbbk M_i) t^i.$$

The *Gelfand–Kirillov dimension* (or *GK dimension*) of a connected graded, locally-finite algebra A is defined to be

$$\mathrm{GKdim}(A) = \limsup_{n\to\infty} \frac{\log(\sum_{i=0}^n \dim_\Bbbk A_i)}{\log(n)}, \tag{E1.7.1}$$

see [41, Chapter 8], [39], or [52, p.1594].

Proposition 1.8. *Let A be a noetherian connected graded domain and H be a semisimple Hopf algebra acting on A homogeneously. Assume that $A\#H$ is prime.*

(1) *Suppose the Hilbert series $p(t) := h_A(t)$ and $q(t) := h_{A^H}(t)$ are rational functions. Then*

$$\big(p(t)/q(t)\big)\,|_{t=1} = \dim H.$$

(2) *If $h_A(t) = \dfrac{1}{\prod_{i=1}^n (1 - t^{a_i})}$ and $h_{A^H}(t) = \dfrac{1}{\prod_{i=1}^n (1 - t^{b_i})}$, then*

$$\prod_{i=1}^n b_i = (\dim H) \prod_{i=1}^n a_i.$$

(3) *If $h_A(t) = \dfrac{1}{(1-t)^n}$, $h_{A^H}(t) = \dfrac{1}{\prod_{i=1}^n (1 - t^{b_i})}$, and there is a minimal generating set $\{x_i\}_{i=1}^n$ of A^H with $\deg x_i = b_i$, then*

$$\prod_{i=1}^n \deg x_i = \dim H.$$

(4) *Suppose that $h_A(t) = \dfrac{1}{(1-t)^n}$ and that A^H is a commutative polynomial ring generated by a minimal homogeneous generating set $\{x_i\}_{i=1}^n$, then*

$$\prod_{i=1}^n \deg x_i = \dim H.$$

Proof. (1) By Proposition 1.5, A is an A^H-module of rank $d := \dim H$ on both sides. Let n be the GK dimension of A (and equal to the GK dimension of A^H). Then, by [52, Corollary 2.2],

$$p(t) = p_1(t)(1-t)^{-n} \quad \text{and} \quad q(t) = q_1(t)(1-t)^{-n}$$

such that $p_1(1)q_1(1) \neq 0$.

Let $D(A)$ denote the graded total quotient ring of A. Then $D(A)$ is free of rank d over $D(A^H)$, as $D(A^H)$ is a graded division ring. Then there is a right graded free A^H-submodule of A of rank d, say M, such that A/M is a torsion graded right A^H-module. Then $\operatorname{GKdim} A/M < \operatorname{GKdim} A^H = n$, and by [52, Corollary 2.2],

$$h_{A/M}(t) = h(t)(1-t)^{-m}$$

with $m < n$ and $h(1) \neq 0$. This implies that

$$\big((1-t)^n h_{A/M}(t)\big)\,|_{t=1} = 0.$$

Since $h_{A/M}(t) = h_A(t) - h_M(t)$, we have

$$\big((1-t)^n h_A(t)\big)\,|_{t=1} = \big((1-t)^n h_M(t)\big)\,|_{t=1}\,. \tag{E1.8.1}$$

The left-hand side of (E1.8.1) is $p_1(1)$. Since M is free of rank d, there are integers $n_1, \ldots, n_d$ such that

$$h_M(t) = (t^{n_1} + \cdots + t^{n_d}) h_{A^H}(t).$$

Then the right-hand side of (E1.8.1) is $dq_1(1)$. As a consequence, $p_1(1) = dq_1(1)$. Now

$$\big(p(t)/q(t)\big)\,|_{t=1} = \big(p_1(t)/q_1(t)\big)\,|_{t=1} = p_1(1)/q_1(1) = d = \dim H.$$

Part (2) is clearly a consequence of part (1). Part (3) is clearly a consequence of part (2). Part (4) is clearly a consequence of part (3). $\quad\square$

We remark that the preceding result is not true if we remove the hypothesis that H is semisimple [12, Observation 4.1(4)].

2. Preliminaries on local cohomology and CM regularity

2.1. Local cohomology

In this subsection we will review some basic ideas related to graded modules, local cohomology, balanced dualizing complexes, and other concepts that will be used in discussing Castelnuovo–Mumford regularity. The definition of the Hilbert series $h_M(t)$ of a module M was given in Definition 1.7. If M is locally-finite bounded above, then $h_M(t)$ is in $\Bbbk((t^{-1}))$. If M is locally-finite bounded below, then $h_M(t)$ is in $\Bbbk((t))$. In both cases, some special $h_M(t)$ can be written as rational functions of t.

Definition 2.1 ([5, Definition 3.6.13]). Let M be nonzero locally-finite and either bounded above or bounded below. Suppose $h_M(t)$ is equal to a rational function, considered as an element in $\Bbbk((t^{-1}))$ or in $\Bbbk((t))$. The *a-invariant* of M, denoted by $a(M)$, is defined to be the t-degree of the rational function $h_M(t)$, namely, $a(M) = \deg_t h_M(t)$.

Example 2.2.

(1) If M is finite-dimensional, then $a(M) = \deg(M)$. A more general case is considered in part (3).

(2) Let A be the commutative polynomial ring $\Bbbk[t_1, \ldots, t_n]$ with $\deg(t_i) = 1$ for all i. Then $h_A(t) = \frac{1}{(1-t)^n}$. Therefore, $a(A) = -n$.

(3) If M is bounded above and $h_M(t)$ is a rational function, then one can check that

$$a(M) = \deg_t h_M(t) = \deg M. \qquad \text{(E2.2.1)}$$

Local cohomology is an important tool in this paper. Let A be a locally-finite $\mathbb{N}$-graded algebra and let $\mathfrak{m}$ denote the graded ideal $A_{\geq 1}$. Let A-Gr denote the category of $\mathbb{Z}$-graded left A-modules. For each graded left A-module M, we define

$$\Gamma_{\mathfrak{m}}(M) = \{x \in M \mid A_{\geq n}x = 0 \text{ for some } n \geq 1 \} = \lim_{n \to \infty} \operatorname{Hom}_A(A/A_{\geq n}, M)$$

and call this the $\mathfrak{m}$-*torsion submodule* of M. It is standard that the functor $\Gamma_{\mathfrak{m}}(-)$ is a left exact functor from A-$\mathsf{Gr} \to A$-Gr. Since this category has enough injectives, the ith right derived functors, denoted by $H^i_{\mathfrak{m}}$ or $R^i\Gamma_{\mathfrak{m}}$, are defined and called the *local cohomology functors*. Explicitly, one has

$$H^i_{\mathfrak{m}}(M) = R^i\Gamma_{\mathfrak{m}}(M) := \lim_{n \to \infty} \operatorname{Ext}^i_A(A/A_{\geq n}, M).$$

See [2,55] for more details.

If M is a left (or right) A-module, let M' denote the graded $\Bbbk$-linear dual of M, where

$$(M')_i = \operatorname{Hom}_{\Bbbk}(M_{-i}, \Bbbk).$$

Definition 2.3. Let A be a locally-finite noetherian $\mathbb{N}$-graded algebra. Let M be a finitely generated graded left A-module. We call M *s-Cohen–Macaulay* or simply *Cohen–Macaulay* if $H^i_{\mathfrak{m}}(M) = 0$ for all $i \neq s$ and $H^s_{\mathfrak{m}}(M) \neq 0$.

The noncommutative version of a dualizing complex was introduced in 1992 by Yekutieli [56]. Roughly speaking, a dualizing complex over a connected graded algebra A is a complex R of graded A-bimodules, such that the two derived functors $\operatorname{RHom}_A(-, R)$ and $\operatorname{RHom}_{A^{\mathrm{op}}}(-, R)$ induce a duality between derived categories $\mathbb{D}^b_{f.g.}(A\text{-}\mathsf{Gr})$ and $\mathbb{D}^b_{f.g.}(A^{\mathrm{op}}\text{-}\mathsf{Gr})$. Let A^e denote the enveloping algebra $A \otimes A^{\mathrm{op}}$.

Definition 2.4 ([56, Definition 3.3]). Let A be a noetherian connected graded algebra. A complex $R \in \mathbb{D}^b(A^e\text{-}\mathsf{Gr})$ is called a *dualizing complex* over A if it satisfies the three conditions below:

(i) R has finite graded injective dimension on both sides.
(ii) R has finitely generated cohomology modules on both sides.
(iii) The canonical morphisms $A \to \operatorname{RHom}_A(R, R)$ and $A \to \operatorname{RHom}_{A^{\mathrm{op}}}(R, R)$ in $\mathbb{D}(A^e\text{-}\mathsf{Gr})$ are both isomorphisms.

Note that local cohomology can be defined for complexes (see e.g. [28]). The following concept will be important in what follows.

Definition 2.5 *([56, Definition 4.1]).* Let A be a noetherian connected graded algebra and let R be a dualizing complex over A. We say that R is a *balanced dualizing complex* if

$$R\Gamma_{\mathfrak{m}}(R) \cong A'$$

in $\mathbb{D}(A^e\text{-}\mathsf{Gr})$.

We recall a result of [35].

Lemma 2.6 *([35, Lemma 3.2(b)]). Assume Hypothesis 0.3. Then*

(1) T^H *admits a balanced dualizing complex.*
(2) T^H *is d-Cohen–Macaulay where* $d = \operatorname{injdim} T$.

Finally, we recall a definition of [31, Definition 1.4]. For each finitely generated graded left A-module M, define

$$B_M(t) = \sum_i (-1)^i h_{H^i_{\mathfrak{m}}(M)}(t),$$

which we can view as an element of $\mathbb{Q}((t^{-1}))$. We say that M is *rational over* $\mathbb{Q}$ if it satisfies the conditions:

(a) $h_M(t)$ and $B_M(t)$ are rational functions over $\mathbb{Q}$ (inside $\mathbb{Q}((t))$ and $\mathbb{Q}((t^{-1}))$ respectively), and
(b) as rational functions over $\mathbb{Q}$, we have

$$h_M(t) = B_M(t). \tag{E2.6.1}$$

Rationality over $\mathbb{Q}$ holds automatically for many graded algebras such as PI algebras and factor rings of AS regular algebras [31, Proposition 5.5]. For simplicity, we assume

Hypothesis 2.7. Every finitely generated graded left and right A-module is rational over $\mathbb{Q}$.

Conjecture 2.8. *Every noetherian connected graded algebra with balanced dualizing complex satisfies Hypothesis 2.7.*

No counterexample to the above conjecture is known.

2.2. Castelnuovo–Mumford regularity

In this subsection we recall the definitions and some basic properties of Castelnuovo–Mumford regularity and Ext-regularity in the noncommutative setting. Noncommutative Castelnuovo–Mumford regularity was first studied by Jørgensen in [29,30] and later by Dong and Wu [17].

Definition 2.9 *([29, Definition 2.1], [17, Definition 4.1]).* Let M be a nonzero graded left A-module. The *Castelnuovo–Mumford regularity* (or *CM regularity*, for short) of M is defined to be

$$\mathrm{CMreg}(M) = \inf\{p \in \mathbb{Z} \mid H_{\mathfrak{m}}^i(M)_{>p-i} = 0, \forall\, i \in \mathbb{Z}\}$$
$$= \sup\{i + \deg(H_{\mathfrak{m}}^i(M)) \mid i \in \mathbb{Z}\}.$$

As noted in [30, Observation 2.3], by [55, Corollary 4.8] $H_{\mathfrak{m}}^i(A) = H_{\mathfrak{m}^{\mathrm{op}}}^i(A)$ and hence $\mathrm{CMreg}({}_A A) = \mathrm{CMreg}(A_A)$, which is simply denoted by $\mathrm{CMreg}(A)$. Also by [2, Theorem 8.3(3)] if B is a noetherian subring of A and A is finitely generated over B on both sides then $\mathrm{CMreg}(A) = \mathrm{CMreg}(A_B) = \mathrm{CMreg}({}_B A)$.

When A has a balanced dualizing complex (for example, if A is commutative or PI), $\mathrm{CMreg}(M)$ is finite for every nonzero finitely generated left graded A-module M [30, Observation 2.3]. But if A does not have a balanced dualizing complex, then $\mathrm{CMreg}(A)$ could be infinite. For example, let R_q be the noetherian connected graded algebra given in [50, Theorem 2.3]. It follows from [50, Theorem 2.3(b)] that $\deg H_{\mathfrak{m}}^1(R_q) = +\infty$ and therefore $\mathrm{CMreg}(R_q) = +\infty$. Next we give some examples where $\mathrm{CMreg}(M)$ is finite.

Example 2.10. The following examples are clear.

(1) If M is a finite-dimensional nonzero graded left A-module, then

$$\mathrm{CMreg}(M) = \deg(M). \qquad (\mathrm{E2.10.1})$$

  A more general case is considered in part (4).
(2) [17, Lemma 4.8] Let T be an AS Gorenstein algebra of type $(d, \mathfrak{l})$. Then $\mathrm{CMreg}(T) = d - \mathfrak{l}$.
(3) Let T be an AS regular algebra of type $(d, \mathfrak{l})$. Then

$$\mathrm{CMreg}(T) = d - \mathfrak{l} = \mathrm{gldim}\, T + \deg_t h_T(t),$$

  which is a non-positive integer, see [52, Proposition 3.1.4]. It is easy to see that T is Koszul if and only if $\mathrm{CMreg}(T) = 0$ [52, Proposition 3.1.5].
(4) If M is s-Cohen–Macaulay, then, by definition,

$$\mathrm{CMreg}(M) = s + \deg(H_{\mathfrak{m}}^s(M)). \qquad (\mathrm{E2.10.2})$$

Definition 2.11 (*[30, Definition 2.2], [17, Definition 4.4]*). Let M be a nonzero graded left A-module. The *Ext-regularity* of M is defined to be

$$\mathrm{Extreg}(M) = \inf\{p \in \mathbb{Z} \mid \mathrm{Ext}_A^i(M, \Bbbk)_{<-p-i} = 0, \forall\, i \in \mathbb{Z}\}$$
$$= -\inf\{i + \mathrm{ged}(\mathrm{Ext}_A^i(M, \Bbbk)) \mid i \in \mathbb{Z}\}.$$

The *Tor-regularity* of M is defined to be

$$\mathrm{Torreg}(M) = \inf\{p \in \mathbb{Z} \mid \mathrm{Tor}_i^A(\Bbbk, M)_{>p+i} = 0, \forall\, i \in \mathbb{Z}\}$$
$$= \sup\{-i + \deg(\mathrm{Tor}_i^A(\Bbbk, M)) \mid i \in \mathbb{Z}\}.$$

If M is a finitely generated graded module over a left noetherian ring A, then $\mathrm{Extreg}(M) = \mathrm{Torreg}(M)$ [17, Remark 4.5], and we will not distinguish between $\mathrm{Extreg}(M)$ and $\mathrm{Torreg}(M)$ in this case.

Example 2.12. The following examples are clear.

(1) If $r = \mathrm{Torreg}(M)$, then

$$t_i^A(_AM) := \deg(\mathrm{Tor}_i^A(\Bbbk, M)) \leq r + i \qquad (\text{E2.12.1})$$

for all i.
(2) [17, Example 4.6] $\mathrm{Extreg}(A) = 0$.

Now we are ready to state some nice results in [29,30,17].

Theorem 2.13. *Let A be a noetherian connected graded algebra with a balanced dualizing complex and let $M \neq 0$ be a finitely generated graded left A-module.*

(1) *[30, Theorems 2.5 and 2.6]*

$$-\mathrm{CMreg}(A) \leq \mathrm{Extreg}(M) - \mathrm{CMreg}(M) \leq \mathrm{Extreg}(\Bbbk).$$

(2) *[30, Corollary 2.8] [17, Theorem 5.4] A is a Koszul AS regular algebra if and only if $\mathrm{Extreg}(M) = \mathrm{CMreg}(M)$ for all M.*

(3) *[17, Proposition 5.6] If M has finite projective dimension, then*

$$-\mathrm{CMreg}(A) = \mathrm{Extreg}(M) - \mathrm{CMreg}(M).$$

In the next lemma, reg can be CMreg, Extreg, or Torreg.

Lemma 2.14 (*[18, Corollary 20.19], [27, Lemma 1.6]*). *Let*

$$0 \to L \to M \to N \to 0$$

be a short exact sequence of finitely generated graded left modules over a left noetherian algebra A.

(1) $\operatorname{reg}(L) \leq \max\{\operatorname{reg}(M), \operatorname{reg}(N) + 1\}$ *with equality if* $\operatorname{reg}(M) \neq \operatorname{reg}(N)$.
(2) $\operatorname{reg}(M) \leq \max\{\operatorname{reg}(L), \operatorname{reg}(N)\}$.
(3) $\operatorname{reg}(N) \leq \max\{\operatorname{reg}(L) - 1, \operatorname{reg}(M)\}$ *with equality if* $\operatorname{reg}(M) \neq \operatorname{reg}(L)$.

Lemma 2.15. *Assume Hypothesis 0.3.*

(1) $\operatorname{CMreg}(T^H) \leq \operatorname{CMreg}(T) \leq 0$.
(2) $\operatorname{CMreg}(T^H) = \operatorname{CMreg}(T)$ *if and only if the H-action on T has trivial homological determinant.*
(3) *If both T and T^H are AS regular and $T^H \neq T$, then*

$$\operatorname{CMreg}(T^H) < \operatorname{CMreg}(T).$$

Proof. (1) By Example 2.10(3), $\operatorname{CMreg}(T) \leq 0$. So it suffices to show the first inequality. By [35, Lemma 2.5(b)], $H_{\mathfrak{m}}^i(T^H)$ is a direct summand of $H_{\mathfrak{m}}^i(T)$ for all i. In particular, if d is the injective dimension of T, then $\deg(H_{\mathfrak{m}}^d(T^H)) \leq \deg(H_{\mathfrak{m}}^d(T))$ and $H_{\mathfrak{m}}^i(T^H) = H_{\mathfrak{m}}^i(T) = 0$ for all $i \neq d$. The assertion now follows from Definition 2.9.

(2) By the proof of part (1), one sees that $\operatorname{CMreg}(T^H) = \operatorname{CMreg}(T)$ if and only if $\deg(H_{\mathfrak{m}}^d(T^H)) = \deg(H_{\mathfrak{m}}^d(T))$, and if and only if $\operatorname{ged}(H_{\mathfrak{m}}^d(T^H)') = \operatorname{ged}(H_{\mathfrak{m}}^d(T)')$. Now the assertion basically follows from [35, Lemma 3.5(f)] (after one matches up the notation).

(3) This follows from [6, Theorem 0.6] and part (2). $\quad\square$

Lemma 2.16. *Let $f : S \to T$ be a graded algebra homomorphism between two noetherian AS regular algebras such that T is finitely generated over S on both sides. Let $\delta(T/S) = \operatorname{CMreg}(T) - \operatorname{CMreg}(S)$.*

(1) $t_i^S(T_S) \leq \delta(T/S) + i$ *for all $i \geq 0$.*
(2) $\deg T/TS_{\geq 1} \leq \delta(T/S)$.
(3) *Assume Hypothesis 0.3 and, in addition, that the image of f is T^H. Then $\tau_H(T) \leq \delta(T/S) + 1$. Similarly, $\tau_H^{op}(T) \leq \delta(T/S) + 1$.*

Proof. (1) By Example 2.12(1) and Theorem 2.13(3), we have

$$t_i^S(T_S) \leq \operatorname{Torreg}(T_S) + i = \operatorname{CMreg}(T_S) - \operatorname{CMreg}(S) + i = \operatorname{CMreg}(T) - \operatorname{CMreg}(S) + i.$$

(2) By definition,

$$\deg T/TS_{\geq 1} = \deg \operatorname{Tor}_0^S(T, \Bbbk) = t_0^S(T_S) \leq \delta(T/S).$$

(3) By definition, $\tau_H(T) = \deg T/TS_{\geq 1} + 1$. The assertion follows from part (2). $\quad\square$

3. τ-saturation degree and $\beta(A)$

In this section, we generalize some arguments in commutative invariant theory (see [13,15,23–25]) to a noncommutative setting. As in Hypothesis 0.3, we usually assume that the Hopf algebra H is semisimple. In the commutative setting, if the group G acts on $\Bbbk[x_1, \ldots, x_n]$ via graded automorphisms, then

$$\beta(\Bbbk[x_1, \ldots, x_n]^G) \leq \tau_G(\Bbbk[x_1, \ldots, x_n]) \leq |G| \qquad \text{(E3.0.1)}$$

[13,24]. Similarly, if T is noncommutative and H is a semisimple Hopf algebra acting homogeneously on T, we relate $\tau_H(T)$ to $\beta(T^H)$. As noted in Question 0.4, in the noncommutative case, we do not have a general upper bound on $\tau_H(T)$; it would be nice to have such a bound even for particular classes of T and H.

3.1. Easy observations

Let A be a connected graded algebra with maximal graded ideal $\mathfrak{m}$. If $\mathfrak{m}$ is generated by a set of homogeneous elements as a left (or, right) ideal, then A is generated by the same set as an algebra. Another way of saying this is that A is generated by $\mathfrak{m}/\mathfrak{m}^2$, as it is well-known that

$$\mathfrak{m}/\mathfrak{m}^2 \cong \operatorname{Tor}_1^A(\Bbbk, \Bbbk) \qquad \text{(E3.0.2)}$$

as graded vector spaces.

If $M = \bigoplus_{i \in \mathbb{Z}} M_i$ is a $\mathbb{Z}$-graded vector space, then let $M_{\leq n}$ denote the subspace $\bigoplus_{i \leq n} M_i$. The following lemma is easy to prove, and the proof is omitted.

Lemma 3.1. *Let A be a connected graded algebra and let H be a semisimple Hopf algebra acting on A homogeneously. Let B be a factor graded ring of A with induced H-action. Let d be a positive integer. Then the following hold.*

(1) $\beta(B) \leq \beta(A)$.
(2) *If $B = A/A_{\geq d}$, then $\beta(B) = \deg(\operatorname{Tor}_1^A(\Bbbk, \Bbbk)_{\leq d-1})$.*
(3) B^H *is a factor ring of A^H.*
(4) $(A/A_{\geq d})^H = A^H/(A^H)_{\geq d}$. *As a consequence,*

$$\beta((A/A_{\geq d})^H) = \deg(\operatorname{Tor}_1^{A^H}(\Bbbk, \Bbbk)_{\leq d-1}).$$

The main result of the next lemma is essentially the same as [25, Lemma VI.7]. Here we prove a slightly more general result.

Lemma 3.2. *Suppose that $f : B \to A$ is a graded algebra homomorphism of connected graded algebras with A generated in degree 1 and that $C := \mathrm{im}(f)$ is a direct summand of A as a right B-module.*

(1) *The left ideal $J = AC_{\geq 1}$ is generated by a set of elements of degree $\leq t_0^B(A_B) + 1$.*
(2) *If J is generated by elements in $C_{\geq 1}$ of degree $\leq r$, then $\beta(C) \leq r$.*
(3) *$\beta(C) \leq t_0^B(A_B) + 1$.*

The analogous results hold for the right ideal $C_{\geq 1}A$ and $t_0^B({}_B A)$.

Proof. (1) Let $g = t_0^B(A_B) + 1$. Since

$$\mathrm{Tor}_0^B(A_B, \Bbbk) = A \otimes_B \Bbbk = A \otimes_B B/B_{\geq 1} = A/AB_{\geq 1} = A/AC_{\geq 1} = A/J$$

is equal to zero in degrees $\geq g$, therefore $J \supseteq A_{\geq g}$. Every element in J of degree $> g$ is generated by A_g since A is generated in degree 1. Therefore J is generated by $\bigoplus_{i=1}^g J_i$ as a left ideal of A. Let $\{f_1, \ldots, f_r\}$ be a basis of $\bigoplus_{i=1}^g J_i$, and write

$$f_i = \sum_{j=1}^n h_{ij} u_j, \quad h_{ij} \in A, 1 \leq i \leq r$$

where $u_1, \ldots, u_n$ are in $C_{\geq 1}$. We may assume that all elements are homogeneous and for each u_j there is at least one i such that $h_{ij} u_j \neq 0$. By removing all terms which can be canceled, we have that $\deg f_i = \deg h_{ij} \deg u_j$ which shows that $\deg u_i \leq \deg f_i \leq g$. It is clear that J is generated by $u_1, \ldots, u_n$ as desired.

(2) Suppose that J is generated as a left ideal of A by $u_1, \ldots, u_n \in C_{\geq 1}$ where $\deg u_i \leq r$ for all i. We claim that $C_{\geq 1}$ is generated as a left ideal of C by $u_1, \ldots, u_n$. Let $f \in C_{\geq 1} \subseteq J$. Then

$$f = g_1 u_1 + \cdots + g_n u_n$$

where each $g_i \in A$. By hypothesis, C is a direct summand of A as a right B-module. Write

$$A_B = C_B \oplus D_B.$$

Then each g_i can be written in the form $g_i = \gamma_i + \gamma_i'$ where $\gamma_i \in C$ and $\gamma_i' \in D$. Hence

$$f = (\gamma_1 u_1 + \cdots + \gamma_n u_n) + (\gamma_1' u_1 + \cdots + \gamma_n' u_n)$$

and since $f \in C$,

$$f = \gamma_1 u_1 + \cdots + \gamma_n u_n.$$

Therefore, $C_{\geq 1}$ is generated by $\{u_i\}$ as a left ideal of C. It follows that C is generated as an algebra by $\{u_i\}$ so $\beta(C) \leq r$.

(3) This is an immediate consequence of parts (1) and (2). $\quad\square$

Corollary 3.3. *Let A be a connected graded algebra and let H be a semisimple Hopf algebra acting on A homogeneously. Then $\beta(A^H) \leq \tau_H(A)$.*

Proof. This follows from Lemma 3.2 by taking $B = C = A^H$ and f to be the natural inclusion into A. $\quad\square$

In many cases $\tau_H(A)$ is easier to compute than $\beta(A^H)$. Furthermore the bound can be sharp.

Example 3.4. Let T be the down-up algebra

$$T := A(0,1) = \Bbbk\langle x,y\rangle/(x^2y = yx^2, xy^2 = y^2x),$$

which is an AS regular algebra of dimension 3. Let σ be the automorphism of T defined by $\sigma(x) = -x$ and $\sigma(y) = y$. Then σ generates a group G of order 2 that acts on T. One can check that a basis for the invariants of degree ≤ 3 is given by:

$$y \,(\deg 1), \quad x^2, y^2 \,(\deg 2), \quad x^2y(= yx^2), xyx, y^3 \,(\deg 3).$$

Taking left (respectively, right) multiples of these invariants and determining the dimension of $J_G(T)_3$, it is not hard to check that $\tau_G(T) = \tau_G^{\mathrm{op}}(T) = 3$, and so by Corollary 3.3, $\beta(T^G) \leq 3$.

One can also check that the invariant xyx of degree 3 is not generated by the lower degree invariants, so that $\beta(T^G) = 3 = \tau_G(T)$.

As a corollary of Lemma 3.2, we obtain the following degree bound.

Theorem 3.5. *Suppose that (T,H) satisfies Hypothesis 0.3. Assume further*

(a) *T is a noetherian AS regular domain generated in degree 1 such that $T\#H$ is prime.*
(b) *T^H has finite global dimension.*

Then the following hold.

(1) *$\beta(T^H) \leq \dim H$.*
(2) *Suppose $\beta(T^H) = \dim H$ and that $\Bbbk$ is algebraically closed of characteristic 0. Then $H = \Bbbk G$ where $G = \mathbb{Z}/(d)$ for $d = \dim H$, the G-action on T is faithful, and G is generated by a quasi-reflection of T in the sense of [33, Definition 2.2].*

Proof. (1) This follows from Example 1.6 and Corollary 3.3.

(2) By Proposition 1.5, T is free over $R := T^H$ of rank $d := \dim H$; moreover, the cyclic graded left T-module $M := T \otimes_R (R/R_{\geq 1})$ has $\Bbbk$-dimension d and $M_i \neq 0$ for $i = 0, \ldots, \tau_H(T) - 1$. But by Corollary 3.3 and Example 1.6 we have $\dim H = \beta(R) \leq \tau_H(T) \leq \dim H$, so $\tau_H(T) = \dim H$. Hence $\deg M = d - 1$ and the Hilbert series of M is $1 + t + t^2 + \cdots + t^{d-1}$, or equivalently, $\dim(TR_{\geq 1})_i = \dim T_i - 1$ for all $0 \leq i \leq d - 1$.

As a result, as an H-representation, $T_1 = (T^H)_1 \oplus N$ where N is a one-dimensional simple H-module, and so T_1 is the direct sum of one-dimensional H-modules. Hence, $[H, H] \cdot T_1 = 0$ (where $[H, H]$ is the commutator ideal of H) and since $[H, H]$ is a Hopf ideal and T is generated in degree 1, we have $[H, H] \cdot T = 0$. Therefore, the quotient Hopf algebra $\overline{H} = H/[H, H]$ acts on T. By part (1),

$$\dim(H) = \beta(T^H) = \beta(T^{\overline{H}}) \leq \dim(\overline{H}),$$

so $\dim(\overline{H}) = \dim(H)$. Therefore, $[H, H] = 0$ and H is commutative.

Since H is semisimple and commutative, it is the dual of a group algebra $H = (\Bbbk K)^*$ for some finite group K. Therefore, $\Bbbk K$ coacts on T and the direct sum decomposition $T_1 = (T^H)_1 \oplus N$ is also a decomposition as $\Bbbk K$-comodules. By [36, Theorem 3.5(3,4)], K is generated by the K-grading of N, so $K = \mathbb{Z}/(d)$ is a cyclic group of order d.

Let x be a basis element of N and let τ be the K-degree of x. By [36, Theorem 3.5(1)], $x = f_\tau$ which is a generator of the T^H-module T_τ (in the notation of [36]). Since $T_1 = (T^H)_1 \oplus N = (T^H)_1 \oplus \Bbbk x$, by [36, Theorem 3.5(1)], x is a normal element in T. Further, for each $0 \leq i \leq d - 1$, x^i equals f_{τ^i} which is a generator of T_{τ^i}. By [36, Theorem 3.5(2)], $TR_{\geq 1}$ is a 2-sided ideal of T and M is isomorphic to the graded algebra $\Bbbk[x]/(x^d)$ with an induced K-comodule structure such that M is a free $\Bbbk K$-comodule of rank one. As a consequence, $T = \bigoplus_{i=0}^{d-1} x^i T^H = \bigoplus_{i=0}^{d-1} T^H x^i$. Write $h_T(t) = \frac{1}{(1-t)^n p(t)}$ and $h_{T^H}(t) = \frac{1}{(1-t)^n q(t)}$ where $n = \text{GKdim } T = \text{GKdim } T^H > 0$ and $p(1), q(1) \neq 0$ (see [33, Definition 2.2]). Then

$$h_T(t) = \frac{1}{(1-t)^n p(t)} = (1 + t + \cdots + t^{d-1}) h_{T^H}(t) \qquad \text{(E3.5.1)}$$

$$= \frac{1 + t + \cdots + t^{d-1}}{(1-t)^n q(t)} = \frac{(1 - t^d)}{(1-t)^{n+1} q(t)}.$$

Since $\Bbbk$ is algebraically closed of characteristic 0, $(\Bbbk K)^*$ is isomorphic to $\Bbbk G$ where G is a group that is also isomorphic to $\mathbb{Z}/(d)$. Since the $\Bbbk K$-coaction on M is free (hence faithful), the $\Bbbk G$-action on M is free (hence faithful). As a consequence, the $\Bbbk G$-action on T is faithful.

Viewing $H = \Bbbk G$, suppose that G is generated by the automorphism σ of T. Examining the action of σ on $T_1 = (T^H)_1 \oplus \Bbbk x$, we have $\sigma(x) = \lambda x$ where λ is a primitive dth root of unity. As a right T^H-module, $T = \bigoplus_{i=0}^{d-1} x^i T^H$ and σ acts on $x^i T^H$ by scaling by λ^i. Then

$$
\begin{aligned}
\mathrm{Tr}_T(\sigma) \quad &= \quad \sum_{i=0}^{d-1} \lambda^i t^i h_{T^H}(t) = \frac{\sum_{i=0}^{d-1} \lambda^i t^i}{(1-t)^n q(t)} \\[2mm]
&= \quad \frac{(1-(\lambda t)^d)}{(1-\lambda t)(1-t)^n q(t)} = \frac{(1-t^d)(1-t)}{(1-\lambda t)(1-t)^{n+1} q(t)} \\[2mm]
&\overset{\text{(E3.5.1)}}{=} \quad \frac{(1-t)}{(1-\lambda t)(1-t)^n p(t)} \\[2mm]
&= \quad \frac{1}{(1-t)^{n-1}(1-\lambda t)p(t)}.
\end{aligned}
$$

By [33, Definition 2.2], σ is a quasi-reflection. $\quad\square$

A bound on $\beta(T^H)$ would be useful in projects such as [20,21], where T^H has finite global dimension and the generators of T^H were determined explicitly.

We now give a family of examples of AS regular algebras T and groups G that show that $\beta(T^G)$ can be arbitrarily larger than $|G| = \dim \Bbbk G$.

Example 3.6. Let m be a fixed positive integer larger than 2. Let $\mathfrak{g}$ be the free Lie algebra generated by x and y. We consider $\mathfrak{g}$ as a graded Lie algebra. The universal enveloping algebra $S := U(\mathfrak{g})$ is isomorphic to the free algebra $\Bbbk\langle x, y\rangle$ as graded algebras by assigning $\deg x = \deg y = 1$.

Consider the graded quotient Lie algebra

$$
\mathfrak{g}_m = \mathfrak{g}/\mathfrak{g}_{\geq m}.
$$

Namely, $\mathfrak{g}_m$ is a graded Lie algebra generated by x, y and subject to relations

$$
[a_1, [a_2, [\cdots, [a_{m-1}, a_m]\cdots]]] = 0
$$

for all a_i being x or y. Then $\mathfrak{g}$ is a finite-dimensional graded Lie algebra. Its universal enveloping algebra $T = U(\mathfrak{g}_m)$ is a noncommutative noetherian AS regular algebra.

There is a natural surjective Lie algebra map $\mathfrak{g} \to \mathfrak{g}_m$ that induces a surjective graded algebra map $\phi : S \to T$. Since every relation in T has degree (at least) m, we have that ϕ is a bijective map for degree less than m. Hence $S/S_{\geq m} = T/T_{\geq m}$.

Now let σ be the automorphism of $\mathfrak{g}$ sending x to $-x$ and y to y. Then σ has order 2, and it induces order 2 graded algebra automorphisms on S, T, $S/S_{\geq m}$ and $T/T_{\geq m}$. Let $G = \langle\sigma\rangle$. We claim that $\beta(T^G) \geq m-1$. To see this, we use Example 1.3 and Lemma 3.1 as below.

$$\beta(T^G) \overset{\text{Lem 3.1(1)}}{\geq} \beta(T^G/(T^G)_{\geq m})$$

$$\overset{\text{Lem 3.1(4)}}{=} \beta((T/T_{\geq m})^G) = \beta((S/S_{\geq m})^G)$$

$$\overset{\text{Lem 3.1(4)}}{=} \beta(S^G/(S^G)_{\geq m})$$

$$\overset{\text{Lem 3.1(4)}}{=} \deg(\mathrm{Tor}_1^{S^G}(\Bbbk, \Bbbk)_{\leq m-1})$$

$$\overset{\text{Exam 1.3}}{=} m - 1.$$

Since m can be arbitrarily large, there is no uniform bound for $\beta(T^G)$ which depends only on $|G|$ when we consider all noetherian AS regular algebras T.

3.2. Using central subrings to bound $\beta(R)$

In this subsection, we obtain bounds on degrees of invariants for certain noncommutative rings by leveraging central subrings. We first note that unlike group actions, Hopf actions need not preserve the center of T.

Example 3.7. Let H be the eight-dimensional Kac–Palyutkin semisimple Hopf algebra. As an algebra, H is generated by x, y, and z subject to the relations

$$x^2 = y^2 = 1, xy = yx, zx = yz, zy = xz, z^2 = \frac{1}{2}(1 + x + y - xy)$$

with coalgebra structure given by

$$\Delta(x) = x \otimes x, \Delta(y) = y \otimes y, \Delta(z) = \frac{1}{2}(1 \otimes 1 + 1 \otimes x + y \otimes 1 - y \otimes x)(z \otimes z)$$

and

$$\varepsilon(x) = \varepsilon(y) = \varepsilon(z) = 1.$$

Let

$$T := \frac{\Bbbk\langle u, v\rangle}{(u^2 - v^2).}$$

This AS regular algebra T is isomorphic to $\Bbbk_{-1}[x_1, x_2]$ and has a basis of elements of the form $u^i(vu)^j v^k$ where $i, j \in \mathbb{N}$ and $k = 0, 1$. By [20, Section 2] H acts on T with the actions of the generators x, y, z of H satisfying

$$x.uv = -uv, \quad x.vu = -vu \quad y.uv = -uv \quad y.vu = -vu$$

$$z.uv = -vu \quad z.vu = uv.$$

This action does not preserve the center of T, as u^2 and $uv + vu$ are central, but $z.(uv + vu) = -vu + uv$ is not.

Our first result uses a noncommutative generalization of Broer's bound. Broer's upper bound on $\beta(A^G)$ when $A := \Bbbk[x_1, \ldots, x_n]$ (proved in [4] when A^G is Cohen–Macaulay and extended to the modular case in [53, Corollary 0.3]) was generalized in [38] to quantum polynomial algebras (namely, noetherian AS regular domains with global dimension n and Hilbert series $1/(1-t)^n$ for some n) and used to obtain bounds on minimal generators of $\Bbbk_{-1}[x_1, \ldots, x_n]^G$ for permutation representations G (providing an analogue of Göbel's theorem [26]). We can use this result to obtain a bound for any group G acting on $\Bbbk_{-1}[x_1, \ldots, x_n]$.

Lemma 3.8 *(Broer's Bound, [38, Lemma 2.2]). Let T be a quantum polynomial algebra of dimension n, H be a semisimple Hopf algebra. Suppose that $C \subset T^H \subset T$, for some graded iterated Ore extension $C = \Bbbk[f_1][f_2 : \tau_2, \delta_2] \cdots [f_n : \tau_n, \delta_n]$ such that T_C is finitely generated, and $\deg f_i > 1$ for at least 2 distinct i's. Then*

$$\beta(T^H) \leq \sum \deg f_i - n.$$

For a more general result, see [38, Lemma 2.1].

Corollary 3.9. *Let $\Bbbk$ be an infinite field and G be a finite group acting as graded automorphisms on $\Bbbk_{-1}[x_1, \ldots, x_n]$. Then*

$$\beta(\Bbbk_{-1}[x_1, \ldots, x_n]^G) \leq n(2|G| - 1).$$

Proof. The group G acts on the commutative polynomial ring $S := \Bbbk[x_1^2, \ldots, x_n^2]$. By Lemma 3.10 below, $\beta(C) \leq 2|G|$ for a subring of primary invariants $C \subseteq S^G$, which is isomorphic to a polynomial ring. The bound follows from Lemma 3.8. $\square$

We will have an improvement to this result in Corollary 3.12. The following lemma is due to Dade, see [51, p.483].

Lemma 3.10. *[51, Proposition 3.4] Let G be a finite group acting linearly on $\Bbbk[x_1, \ldots, x_n]$ with $\deg x_i = 1$ for all i. Assume that $\Bbbk$ is an infinite field. Then there are n primary invariants that have degree $\leq |G|$.*

Note that the proof of [51, Proposition 3.4] uses only the fact that $\Bbbk$ is infinite, not the hypothesis that $\operatorname{char} \Bbbk = 0$.

Our second result in this subsection applies to algebras which are module-finite over their centers. The first step in Fogarty's proof of the Noether bound in the non-modular case [24] shows that the product of any $|G|$ elements of degree 1 in $A := \Bbbk[x_1, \ldots, x_n]$ is contained in the Hilbert ideal $J_G(A)$ (i.e. $\tau_G(A) \leq |G|$). This result is proved by indexing

these G elements as f_α for $\alpha \in G$, and using the identity (sometimes called Benson's Lemma)

$$\sum_{\emptyset \neq S \subseteq G} (-1)^{|G-S|} \sum_{\tau \in G} \prod_{\alpha \in S} (\tau \alpha f_\alpha) \prod_{\alpha \in G-S} f_\alpha \quad \in \quad J_G(A)$$

where the leftmost sum is taken over all nonempty subsets $S \subseteq G$. For an algebra A which is module-finite over its center, we can use this idea of Fogarty to obtain the following result.

Proposition 3.11. *Let A be a connected graded domain and let H be a semisimple Hopf algebra acting on A homogeneously. Suppose that*

(a) *A is a finitely generated module over a central subalgebra Z;*
(b) *Z is stable under the H-action;*
(c) *Z is generated as an algebra by elements of degree $\leq d$;*
(d) *A is generated as a Z-module by elements of degree $\leq m$; and*
(e) *either $\Bbbk$ is an algebraically closed field of characteristic zero or H is a group algebra.*

Then

(1) *$\beta(A^H) \leq \tau_H(A) \leq d \dim H + m$.*
(2) *For the ith annihilator ideal $J_{H,i}$ defined in Definition 1.1(3),*

$$\deg A/J_{H,i} \leq \deg A/J_\infty \leq d \dim H + m - 1.$$

Proof. (1) First we claim that the H-action on A induces a group action on Z. If H is a group algebra, this is clear by Hypothesis (b). If H is not a group algebra, then we assume that $\Bbbk$ is an algebraically closed field of characteristic zero. Since Z is a commutative domain and H acts on Z, by a result of Etingof–Walton [19, Theorem 1.3], the action of H on Z factors through a group action. Let this group algebra be $\Bbbk G$, which is a quotient Hopf algebra of H.

Let $g = |G|$, which is bounded by $\dim H$. Suppose that A is generated as a module over Z by homogeneous elements $x_1, \ldots x_r$ of degree $\leq m$. By hypothesis, G acts on the commutative algebra Z. We remark that since H is semisimple, in the case that $\operatorname{char} \Bbbk \neq 0$, $g = \dim H$ is invertible in $\Bbbk$. Hence, by Fogarty's proof [24], we have that all g-fold products of positive degree elements of Z are contained in the left Hilbert ideal $J_G(Z) = J_H(Z)$ and so $J_H(Z)_i = Z_i$ for all $i \geq gd$ (that is, $\tau_G(Z) \leq gd$). Observe also that $J_H(Z) \subseteq J_H(A)$.

Now since $A = Zx_1 + Zx_2 + \cdots + Zx_r$ and $\deg x_i \leq m$, every homogeneous element a of degree at least $gd + m$ can be written as $a = \sum z_i x_i$, where the z_i are homogeneous of degree at least gd. Since for all i, $z_i \in J_H(A)$, we therefore have $a \in J_H(A)$. Hence,

$A_{\geq gd+m} \subseteq J_H(A)$, and so $\tau_H(A) \leq gd + m$. By Corollary 3.3, $\beta(A^H) \leq d|G| + m \leq d \dim H + m$, as desired.

(2) Since Z^G is a central subalgebra of A^H, the actions of Z^G on the left and the right of $\operatorname{Tor}_i^{A^H}(A, \Bbbk)$ are the same. For every $i \geq gd$ and $z \in Z_i$, by the proof of part (1), we can write $z = \sum y_j f_j$, where $y_j \in (Z^G)_{\geq 1}$ and $f_j \in Z$. For each j, we have $y_j \cdot \Bbbk = 0$, and so $y_j \in J_{H,i}(A)$ for all i. Hence, $z \in J_{H,i}(A)$ so $Z_{\geq gd} \subseteq J_{H,i}(A)$. By the proof of part (1), $A_{\geq gd+m}$ is in the ideal generated by $Z_{\geq gd}$. Therefore $A_{\geq gd+m}$ is in $J_{H,i}(A)$, as desired. $\quad\square$

We remark that $A := \Bbbk_{-1}[x_1, \ldots, x_n]$ is a finite module over the central subalgebra $Z := \Bbbk[x_1^2, \ldots, x_n^2]$ and is generated as a Z-module by elements of degree $\leq n$. It is easy to see that every group action on A induces an action on Z. Hence, we have the following corollary, which is an improvement of the result of Corollary 3.9.

Corollary 3.12. *Let G be a finite group acting as graded automorphisms on $\Bbbk_{-1}[x_1, \ldots, x_n]$ and suppose that $|G|$ is invertible in $\Bbbk$. Then*

$$\beta(\Bbbk_{-1}[x_1, \ldots, x_n]^G) \leq 2|G| + n.$$

Corollary 3.12 suggests the following questions, which are subquestions of Question 0.4 (see also Questions 6.3 and 6.4).

Question 3.13. Suppose G is a finite group and H is a semisimple Hopf algebra.

(1) Is there an upper bound on $\beta(\Bbbk_{-1}[x_1, \ldots, x_n]^G)$ that depends only upon $|G|$ and not on the dimension of the representation of G?
(2) A Hopf action need not preserve the subring $S := \Bbbk[x_1^2, \ldots, x_n^2]$. If H acts homogeneously on $\Bbbk_{-1}[x_1, \ldots, x_n]$, is there an analogous bound for $\beta(\Bbbk_{-1}[x_1, \ldots, x_n]^H)$?

4. Bounds on the degree of $\operatorname{Tor}_i^A(M, \Bbbk)$

In the next two sections we will prove results which provide bounds on $t_i^A(M) = \deg \operatorname{Tor}_i^A(M, \Bbbk)$ for various A-modules M. A special case is when either A is T^H (for T and H satisfying Hypothesis 0.3) or A is a noetherian AS regular algebra mapping to T^H. Recall from (E1.0.6) that we also use the notation

$$\beta_i(A) = t_i^A(\Bbbk)$$

where $\beta_1(A) =: \beta(A)$ is a bound for the degrees of the minimal generators of A and $\beta_2(A)$ provides a bound for the degree of the relations of A.

In general, the connected graded algebra $R := T^H$ need not be AS regular. However, when R is commutative, the Noether Normalization Theorem states that there exists a nonnegative integer d and algebraically independent homogeneous elements

$y_1, y_2, \ldots, y_d$ in R such that R is a finitely generated module over the polynomial subring $S = \Bbbk[y_1, y_2, \ldots, y_d]$. When R is noncommutative, such a result fails, in general, even when allowing S to be a noncommutative AS regular algebra.

Nevertheless, if we suppose the existence of an AS regular subalgebra $S \subseteq R$, then we are able to prove bounds for $\beta(R)$ (as well as $\beta_i(R)$ for $i \geq 2$) by understanding the connection between S and R. Therefore, some results in this section assume the existence of an AS regular version of a Noether normalization (e.g., as in Lemma 3.8) or a map from some AS regular algebra $S \to R$ such that R is a finitely generated S-module. In particular, we generalize some results of Symonds in [53] and of Derksen in [13].

Throughout this section, we fix the following notation.

Notation 4.1. *Let A and B be connected graded algebras, and let $f : A \to B$ be a graded algebra homomorphism making B a finitely generated graded left A-module generated by a set of homogeneous elements, say $\{v_i\}_{i \in S_1}$ including 1, with degree no more than $t_0^A(_A B) := \deg \operatorname{Tor}_0^A(\Bbbk, B)$. Let A be generated as an algebra by a set of homogeneous elements, say $\{x_j\}_{j \in S_2}$, of degrees no more than $\beta(A)$.*

The next lemma is [53, Proposition 2.1(1)].

Lemma 4.2. *Assume Notation 4.1. Then*

$$\beta(B) \leq \max\{\beta(A), t_0^A(_A B)\}.$$

Proof. Write $B = \sum_{i \in S_1} A v_i$. Then, as an algebra, B is generated by $\{v_i\}_{i \in S_1} \cup \{x_j\}_{j \in S_2}$. Hence the assertion follows. $\square$

Next we generalize [53, Proposition 2.1(2,3)] which concerns bounds on the degrees of the relations in A, that is, $\beta_2(A)$. In the noncommutative case, we can obtain only a weaker bound.

Let A be a connected graded algebra and write A as

$$A = \Bbbk\langle \mathcal{G}(A) \rangle / I(A)$$

where $\mathcal{G}(A)$ is a minimal generating set of A and $I(A)$ is the two sided ideal of the relations in A. Here $\Bbbk\langle \mathcal{G}(A) \rangle$ is the free algebra generated by the graded vector space $\mathcal{G}(A)$. Let $N \geq \beta(A)$ be a positive integer. Define

$$\Phi_N(A) = \Bbbk\langle \mathcal{G}(A) \rangle / (I(A)_{\leq N})$$

where $(I(A)_{\leq N})$ is the ideal of the free algebra $\Bbbk\langle \mathcal{G}(A) \rangle$ generated by $I(A)_{\leq N}$. By definition, there is a canonical surjective algebra map $\pi_A : \Phi_N(A) \to A$. It is clear that the degree of the minimal relation set of A is no more than N if and only if $\Phi_N(A) = A$. Namely,

$$N \geq \beta_2(A) \quad \Longleftrightarrow \quad \Phi_N(A) = A. \tag{E4.2.1}$$

Lemma 4.3. *Let $f : A \to B$ be as in Notation 4.1. If $N \geq \max\{\beta(A), \beta_2(A)\}$, then there is a unique lifting of the map f to a map $f' : A \to \Phi_N(B)$.*

Proof. By definition, we have $B_{\leq N} = \Phi_N(B)_{\leq N}$, so we will identify these two graded spaces. If f' exists, then since $N \geq \beta(A)$, we have $f'\mid_{\mathcal{G}(A)} = f\mid_{\mathcal{G}(A)}$. Therefore if f' exists, it is unique.

Next we prove the existence of f'. Let Π be the canonical map $\Bbbk\langle\mathcal{G}(A)\rangle \to A(= \Bbbk\langle\mathcal{G}(A)\rangle/(I(A)))$. Since $\Bbbk\langle\mathcal{G}(A)\rangle$ is a free algebra, we can lift the map $f : A \to B$ to $g : \Bbbk\langle\mathcal{G}(A)\rangle \to B$ by defining

$$g(x) := f(x) = (f \circ \Pi)(x), \quad \text{for all } x \in \mathcal{G}(A).$$

It follows that $g = f \circ \Pi$. Since $\Bbbk\langle\mathcal{G}(A)\rangle$ is a free algebra, there is an algebra map $f'' : \Bbbk\langle\mathcal{G}(A)\rangle \to \Phi_N(B)$ defined by setting $f''(x) := f(x)$ for all $x \in \mathcal{G}(A)$. Since $B_{\leq N} = \Phi_N(B)_{\leq N}$, when restricted to $(\Bbbk\langle\mathcal{G}(A)\rangle)_{\leq N}$, $f'' = \pi_B \circ f'' = g$. Therefore $g = \pi_B \circ f''$. It remains to show that $f''(I(A)) = 0$. By construction, $g(I(A)) = 0$. Then

$$f''((I(A))_{\leq N}) = \pi_B \circ f''((I(A))_{\leq N}) = g((I(A))_{\leq N}) = 0$$

as π_B is the identity when restricted to elements of degree no more than N. Since $I(A)$ is generated by $(I(A))_{\leq N}$, we obtain that $f''(I(A)) = 0$, as desired. We now let f' be the map $A \to \Phi_N(B)$ induced by f''. $\quad\square$

Lemma 4.4. *Let $f : A \to B$ be as in Notation 4.1. If*

$$N \geq \max\{2t_0^A(_AB), t_0^A(_AB) + \beta(A), \beta_2(A)\}$$

then $\Phi_N(B)$ is generated by $\{v_i\}_{i \in S_1}$ as a left A-module.

Proof. We identify $a \in A$ with $f(a) \in B$ and $f'(a) \in \Phi_N(B)$ when there is no confusion. Next we express some products of elements in B. For each $i \in S_1$ and $j \in S_2$ we write

$$v_i x_j = \sum_k y_{ijk} v_k, \tag{E4.4.1}$$

for some $y_{ijk} \in A$; and for $i, j \in S_1$,

$$v_i v_j = \sum_k z_{ijk} v_k, \tag{E4.4.2}$$

for some $z_{ijk} \in A$. The above two equations are in degrees no more than

$$\max\{2t_0^A(_AB), t_0^A(_AB) + \beta(A)\} \leq N.$$

Since we can identify $B_{\leq N}$ with $\Phi_N(B)_{\leq N}$, equations (E4.4.1)–(E4.4.2) hold in $\Phi_N(B)$.

We claim that $\Phi_N(B) = \sum_{i \in S_1} Av_i$. By the proof of Lemma 4.2, B, whence $\Phi_N(B)$, is generated by a subset $W \subset \{v_i\}_{i \in S_1} \cup \{x_j\}_{j \in S_2}$, and it is enough to show that

$$\left(\sum_{i \in S_1} Av_i \right) W \subseteq \left(\sum_{i \in S_1} Av_i \right).$$

But this statement follows from equations (E4.4.1)–(E4.4.2) viewed in $\Phi_N(B)$.   □

Next is a version of [53, Proposition 2.1(2,3)].

Proposition 4.5. *Let $f : A \to B$ be as in Notation 4.1. Then*

$$\beta_2(B) \leq \max \left\{ 2t_0^A({}_AB), t_0^A({}_AB) + \beta(A), \beta_2(A), t_1^A({}_AB) \right\}.$$

Proof. Let

$$N = \max \left\{ 2t_0^A({}_AB), t_0^A({}_AB) + \beta(A), \beta_2(A), t_1^A({}_AB) \right\}.$$

By equation (E4.2.1), it is enough to show that $\Phi_N(B) = B$. By Lemma 4.3, f lifts to an algebra map f' from A to $\Phi_N(B)$. Now we have the following commutative diagram of left A-modules with exact rows, where the vertical arrows can be filled in since the P_i are projective:

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & B & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & \Phi_N(B) & \longrightarrow & B & \longrightarrow & 0.
\end{array}
$$

The map from B to B is the identity. By the definition of $\Phi_N(B)$, $K_{\leq N} = 0$. When we take a minimal resolution of the top row of the above diagram, P_1 is generated in degree at most $t_1^A({}_AB)$, which is $\leq N$. So the map from P_1 to K is zero, thus the composition map $P_1 \to \Phi_N(B)$ is zero. It follows that the bottom row is split as a sequence of left A-modules. By Lemma 4.4, $\Phi_N(B)$ is generated as a left A-module in degree at most N. Hence K is generated as a left A-module in degree at most N. This implies that $K = 0$.   □

Since the CM regularity of an AS regular algebra is easy to compute (see Example 2.10(3)), the following corollary is useful in bounding β or β_2.

Corollary 4.6. *Assume Hypothesis 0.3. Suppose there is a graded algebra map $S \to T^H$ where S is a noetherian AS regular algebra such that T^H is finitely generated over S on both sides. Let $\delta(T/S) = \mathrm{CMreg}(T) - \mathrm{CMreg}(S)$. Then*

(1) $\beta(T^H) \leq \max\{\beta(S), \delta(T/S)\}$, and
(2) $\beta_2(T^H) \leq \max\{2\delta(T/S), \delta(T/S) + \beta(S), \beta_2(S)\}$.

Proof. Let $A = S$ and $B = T^H$. Then by Example 2.12(1), Theorem 2.13(3), and Lemma 2.15(1), we have, for all $i \geq 0$,

$$t_i^A(_AB) \leq \text{Torreg}(_AB) + i = \text{CMreg}(B) - \text{CMreg}(A) + i \qquad \text{(E4.6.1)}$$

$$\leq \text{CMreg}(T) - \text{CMreg}(S) + i = \delta(T/S) + i.$$

(1) The assertion follows from Lemma 4.2 and the inequality (E4.6.1).
(2) The assertion follows from Proposition 4.5, the inequality (E4.6.1), and the fact that $\beta(S) \geq 1$. $\square$

In the remainder of this section, we prove a noncommutative version of [13, Theorem 1], which provides a bound on the degrees of higher syzygies of T^H. Recall from Definition 2.1 that the *a-invariant* of a graded module M, denoted by $a(M)$, is defined to be the t-degree of the Hilbert series $h_M(t)$, viewed as a rational function. Note that if $A = \Bbbk[x_1, \ldots, x_r]$ is the commutative polynomial ring such that $\deg x_i = d_i$ and $d_i \geq d_{i+1}$, then for every $1 \leq k \leq r$, we have

$$t_k^A(\Bbbk) = d_1 + \cdots + d_k. \qquad \text{(E4.6.2)}$$

Hence, the next theorem generalizes [13, Theorem 6].

Theorem 4.7. *Let A be a noetherian connected graded algebra with balanced dualizing complex, and let M be a finitely generated graded left A-module that is s-Cohen–Macaulay.*

(1) *For each i,*

$$t_i^A(_AM) \leq \text{CMreg}(M) - s + t_{i+s}^A(\Bbbk).$$

(2) *Assume Hypothesis 2.7 for A. Then, for each i,*

$$t_i^A(_AM) \leq a(M) + t_{i+s}^A(\Bbbk).$$

Proof. The proof given here is different from the proof of [13, Theorem 6].
(1) Let

$$\cdots \to F_m \to \cdots \to F_1 \to F_0 \to \Bbbk \to 0$$

be a minimal free resolution of the right A-module $\Bbbk$. Then $F_m = \coprod_j A(-\sigma_{m,j})$ with

$$\sigma_{m,j} \leq t_m^A(\Bbbk).$$

Taking graded $\Bbbk$-linear duals, we obtain a minimal injective resolution of the left trivial module

$$0 \to \Bbbk \to F_0' \to F_1' \to \cdots \to F_m' \to \cdots$$

where $F_m' = \coprod_j A'(\sigma_{m,j})$.

Let X be a graded left A-module that is bounded above. Then $\mathrm{Ext}_A^m(X, \Bbbk)$ is a subquotient of

$$\mathrm{Hom}_A(X, F_m') = \bigoplus_j \mathrm{Hom}_A(X, A'(\sigma_{m,j})) = \bigoplus_j X'(\sigma_{m,j}).$$

Hence

$$\mathrm{ged}(\mathrm{Ext}_A^m(X, \Bbbk)) \geq \mathrm{ged}(X') - \max_j \{\sigma_{mj}\} = \mathrm{ged}(X') - t_m^A(\Bbbk). \tag{E4.7.1}$$

Since M is s-Cohen–Macaulay, $H_{\mathfrak{m}}^i(M) = 0$ for all $i \neq s$. Let $X = H_{\mathfrak{m}}^s(M)$. By Example 2.10(4), $\deg(X) = \mathrm{CMreg}(M) - s$, or equivalently,

$$\mathrm{ged}(X') = -\mathrm{CMreg}(M) + s. \tag{E4.7.2}$$

By [30, Proposition 1.1], $\mathrm{RHom}_A(M, \Bbbk)) \cong \mathrm{RHom}_A(\mathrm{R}\Gamma_{\mathfrak{m}}(M), \Bbbk)$. Hence

$$\begin{aligned}
\mathrm{Ext}_A^i(M, \Bbbk) = H^i(\mathrm{RHom}_A(M, \Bbbk)) &\cong H^i(\mathrm{RHom}_A(\mathrm{R}\Gamma_{\mathfrak{m}}(M), \Bbbk)) \\
&\cong H^i(\mathrm{RHom}_A(X[-s], \Bbbk) \cong H^{i+s}(\mathrm{RHom}_A(X, \Bbbk)) \\
&= \mathrm{Ext}_A^{i+s}(X, \Bbbk)
\end{aligned}$$

which implies that

$$\begin{aligned}
\mathrm{ged}(\mathrm{Ext}_A^i(M, \Bbbk)) &= \mathrm{ged}(\mathrm{Ext}^{i+s}(X, \Bbbk)) \\
&\overset{(\text{E4.7.1})}{\geq} \mathrm{ged}(X') - t_{i+s}^A(\Bbbk) \\
&\overset{(\text{E4.7.2})}{=} -\mathrm{CMreg}(M) + s - t_{i+s}^A(\Bbbk).
\end{aligned}$$

This is equivalent to

$$\deg((\mathrm{Ext}^i(M, \Bbbk))') \leq \mathrm{CMreg}(M) - s + t_{i+s}^A(\Bbbk).$$

By [17, Remark 4.5], $(\mathrm{Ext}^i(M, \Bbbk))' \cong \mathrm{Tor}_i^A(\Bbbk, M)$. Thus

$$t_i^A(_A M) := \deg(\mathrm{Tor}_i^A(\Bbbk, M)) \leq \mathrm{CMreg}(M) - s + t_{i+s}^A(\Bbbk).$$

(2) By definition and Hypothesis 2.7, we have

$$a(M) \overset{\text{by def.}}{=} \deg_t h_M(t) \overset{\text{Hyp. 2.7}}{=} \deg_t h_{H^s_{\mathfrak{m}}(M)}(t)$$
$$\overset{\text{(E2.2.1)}}{=} \deg H^s_{\mathfrak{m}}(M) \overset{\text{(E2.10.2)}}{=} \text{CMreg}(M) - s.$$

Now the assertion follows from part (1).   □

As a corollary, we obtain a noncommutative version of [13, Theorem 1]. Let $T = \Bbbk[x_1, \ldots, x_n]$ and suppose we have an action of $H = \Bbbk G$ where G is a finite group. Choose a minimal set of homogeneous generators $y_1, \ldots, y_r$ of the invariant ring T^H where $\deg(y_i) = d_i$ and $d_i \geq d_{i+1}$ for all i. Then there is a graded map from S, the polynomial ring in r variables (with degree given by the d_i's) onto T^H. By (E4.6.2), and the fact that each $d_i \leq \beta(T^H)$, we have that $t^S_{i+n}(\Bbbk) \leq (i+n)\beta(T^H)$ for all i, and hence the corollary below recovers Derksen's result.

Corollary 4.8. *Assume Hypothesis 0.3 and let $R = T^H$. Suppose that S is a noetherian AS regular algebra and there exists a graded algebra homomorphism $S \to R$ such that R is finitely generated over S on both sides. Let n be the global dimension of T. Then*

$$t^S_i(_S R) \leq \text{CMreg}(T) - n + t^S_{i+n}(\Bbbk) \leq t^S_{i+n}(\Bbbk) - n$$

for all $i \geq 0$.

Proof. By Lemma 2.15(1) and Example 2.10(3),

$$\text{CMreg}(R) = \text{CMreg}(T^H) \leq \text{CMreg}(T) \leq 0.$$

By Lemma 2.6(2), R is n-Cohen–Macaulay. The assertion follows from Theorem 4.7(1).   □

We remark that Corollary 4.8 is stronger than (though almost equivalent to) Lemma 2.15(1).

5. Further bounds on the degree of $\text{Tor}^A_i(M, \Bbbk)$

In this section we continue to prove bounds on the degrees of the higher syzygies of T^H, obtaining results that are similar to the main results in [9]. The results in this section require the existence of a graded algebra surjection from an AS regular algebra S onto T^H. We begin with some general lemmas.

Lemma 5.1. *Suppose that $f : A \to B$ is a graded algebra map between two connected graded algebras A and B.*

(1) [46, Theorem 10.59] There is a spectral sequence, called the change of rings spectral sequence,

$$E^2_{p,q} := \operatorname{Tor}^B_p(\Bbbk_B, \operatorname{Tor}^A_q(B, {}_A\Bbbk)) \implies \operatorname{Tor}^A_{p+q}(\Bbbk_A, {}_A\Bbbk).$$

The five-term exact sequence associated to the spectral sequence is

$$\operatorname{Tor}^A_2(\Bbbk, \Bbbk) \to \operatorname{Tor}^B_2(\Bbbk, B \otimes_A \Bbbk) \to \Bbbk \otimes_B \operatorname{Tor}^A_1(B, \Bbbk)$$
$$\to \operatorname{Tor}^A_1(\Bbbk, \Bbbk) \to \operatorname{Tor}^B_1(\Bbbk, B \otimes_A \Bbbk) \to 0.$$

(2) Suppose that f is surjective and that A and B have the same minimal generating set, that is, $\{x_j\}$ is a minimal generating set for A and $\{f(x_j)\}$ is a minimal generating set for B. Then we have an exact sequence of graded vector spaces

$$\operatorname{Tor}^A_2(\Bbbk, \Bbbk) \to \operatorname{Tor}^B_2(\Bbbk, \Bbbk) \to \Bbbk \otimes_B \operatorname{Tor}^A_1(B, \Bbbk) \to 0.$$

(3) Retain the hypotheses in part (2). If $\Bbbk \otimes_B \operatorname{Tor}^A_1(B, \Bbbk) \cong \operatorname{Tor}^A_1(B, \Bbbk)$, then we have the exact sequence.

$$\operatorname{Tor}^A_2(\Bbbk, \Bbbk) \to \operatorname{Tor}^B_2(\Bbbk, \Bbbk) \to \operatorname{Tor}^A_1(B, \Bbbk) \to 0 \qquad (E5.1.1)$$

Proof. (1) This is a special case of [46, Theorem 10.59]. The five term exact sequence is given immediately after [46, Theorem 10.59].

(2) If f is surjective, then $B \otimes_A \Bbbk = \Bbbk$. Hence we have an exact sequence

$$\operatorname{Tor}^A_2(\Bbbk, \Bbbk) \to \operatorname{Tor}^B_2(\Bbbk, \Bbbk) \to \Bbbk \otimes_B \operatorname{Tor}^A_1(B, \Bbbk)$$
$$\to \operatorname{Tor}^A_1(\Bbbk, \Bbbk) \to \operatorname{Tor}^B_1(\Bbbk, \Bbbk) \to 0.$$

Since A and B have the same minimal generating set, $\operatorname{Tor}^A_1(\Bbbk, \Bbbk) \cong \operatorname{Tor}^B_1(\Bbbk, \Bbbk)$. So the assertion follows.

(3) This follows immediately from part (2). □

Part of the E^2 page of the spectral sequence in Lemma 5.1(1) looks like

$$
\begin{array}{llll}
\Bbbk \otimes_B \operatorname{Tor}^A_3(B, \Bbbk) & \operatorname{Tor}^B_1(\Bbbk, \operatorname{Tor}^A_3(B, \Bbbk)) & \operatorname{Tor}^B_2(\Bbbk, \operatorname{Tor}^A_3(B, \Bbbk)) & \operatorname{Tor}^B_3(\Bbbk, \operatorname{Tor}^A_3(B, \Bbbk)) \\
\Bbbk \otimes_B \operatorname{Tor}^A_2(B, \Bbbk) & \operatorname{Tor}^B_1(\Bbbk, \operatorname{Tor}^A_2(B, \Bbbk)) & \operatorname{Tor}^B_2(\Bbbk, \operatorname{Tor}^A_2(B, \Bbbk)) & \operatorname{Tor}^B_3(\Bbbk, \operatorname{Tor}^A_2(B, \Bbbk)) \\
\Bbbk \otimes_B \operatorname{Tor}^A_1(B, \Bbbk) & \operatorname{Tor}^B_1(\Bbbk, \operatorname{Tor}^A_1(B, \Bbbk)) & \operatorname{Tor}^B_2(\Bbbk, \operatorname{Tor}^A_1(B, \Bbbk)) & \operatorname{Tor}^B_3(\Bbbk, \operatorname{Tor}^A_1(B, \Bbbk)) \\
\Bbbk \otimes_B (B \otimes_A \Bbbk) & \operatorname{Tor}^B_1(\Bbbk, B \otimes_A \Bbbk) & \operatorname{Tor}^B_2(\Bbbk, B \otimes_A \Bbbk) & \operatorname{Tor}^B_3(\Bbbk, B \otimes_A \Bbbk)
\end{array}
$$

The differential on the E^r-page has degree $(-r, r-1)$, namely, $d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$. For example, the differential on the E^2-page is

$$d^2 : \operatorname{Tor}^B_p(\Bbbk_B, \operatorname{Tor}^A_q(B, {}_A\Bbbk)) \to \operatorname{Tor}^B_{p-2}(\Bbbk_B, \operatorname{Tor}^A_{q+1}(B, {}_A\Bbbk))$$

for all (p, q).

From now on suppose B_A is finitely generated and A is right noetherian. Then $\operatorname{Tor}_q^A(B, \Bbbk)$ is finite-dimensional. Filtering $\operatorname{Tor}_q^A(B, \Bbbk)$ by degree, we see that

$$
\begin{aligned}
\deg \operatorname{Tor}_p^B(\Bbbk, \operatorname{Tor}_q^A(B, \Bbbk)) &\leq \deg \operatorname{Tor}_q^A(B, \Bbbk) + \deg \operatorname{Tor}_p^B(\Bbbk, \Bbbk) \\
&= t_q^A(B_A) + t_p^B(\Bbbk).
\end{aligned}
\tag{E5.1.2}
$$

The degree of an entry on the E^2 page is bounded by the maximum of the degree of $H_i(\mathrm{Tot})$ corresponding to its diagonal and the degrees of the E^2 entries that are linked to it by a differential on some page. Applying this to the bottom row yields

$$
\deg \operatorname{Tor}_i^B(\Bbbk, B \otimes_A \Bbbk) \leq \max\{\{t_j^B(\Bbbk) + t_{i-j-1}^A(B_A)\}_{0 \leq j \leq i-2}, t_i^A(\Bbbk)\}.
\tag{E5.1.3}
$$

Considering the first column, we obtain

$$
\deg \Bbbk \otimes_B \operatorname{Tor}_i^A(B, \Bbbk) \leq \max\{\{t_j^A(B_A) + t_{i-j+1}^B(\Bbbk)\}_{0 \leq j \leq i-1}, t_i^A(\Bbbk)\}.
\tag{E5.1.4}
$$

Similar to Definition 1.1(3), for each i, let $J_i \subseteq B$ be the annihilator ideal of the finite-dimensional left B-module $\operatorname{Tor}_i^A(B, \Bbbk)$. Let $J_{\leq i}$ denote $\bigcap_{j \leq i} J_j$; when i is clear we will use J for $J_{\leq i}$. Notice that $\operatorname{Tor}_i^A(B, \Bbbk)$ is naturally a graded left B/J-module and is generated as such in degrees at most $\deg(\Bbbk \otimes_B \operatorname{Tor}_i^A(B, \Bbbk))$. Thus

$$
t_i^A(B_A) := \deg \operatorname{Tor}_i^A(B, \Bbbk) \leq \deg(\Bbbk \otimes_B \operatorname{Tor}_i^A(B, \Bbbk)) + \deg B/J_{\leq i}.
\tag{E5.1.5}
$$

For each non-negative integer i, let D_i be a positive number which is greater than or equal to

$$
\max\left\{ \deg B/J_{\leq i} + t_2^B(\Bbbk), \left\{ \frac{t_{j+2}^B(\Bbbk) - t_2^B(\Bbbk)}{j} \right\}_{1 \leq j \leq i-1}, \quad \left\{ \frac{t_j^B(\Bbbk)}{j} \right\}_{1 \leq j \leq i} \right\}.
$$

For every $j \leq i$, set

$$
U_j^i(f) := \max_{i_s > 0, \sum_s i_s = j} \left\{ \sum_s \left(t_{i_s+1}^B(\Bbbk) + D_i - t_2^B(\Bbbk) \right) \right\}
\tag{E5.1.6}
$$

for $j > 0$ and define $U_j^i(f) := -\infty$ for $j \leq 0$. For example, $U_1^i(f) = D_i$ and $U_2^i(f) = \max\left\{ 2D_i, t_3^B(\Bbbk) - t_2^B(\Bbbk) + D_i \right\}$. By definition, for $j + k \leq i$,

$$
U_{j+k}^i(f) \geq U_j^i(f) + t_{k+1}^B(\Bbbk) + D_i - t_2^B(\Bbbk).
\tag{E5.1.7}
$$

Notation 5.2. *For the remainder of the section, we fix notation as in Lemma 5.1. In particular, if $f : A \to B$ is a graded algebra map between connected graded domains,*

and i is a positive integer, then we let D_i be the positive integer defined after (E5.1.5). Further, for every $j \leq i$, we let $U_j^i(f)$ be the value defined in (E5.1.6).

Proposition 5.3. *Fix a positive integer i and retain the above notation. For each $j \leq i$, we have*

$$t_j^A(B_A) \leq \max \left\{ U_j^i(f), \{t_k^A(\Bbbk) + (j-k)D_i\}_{0 \leq k \leq j} \right\} + D_i - t_2^B(\Bbbk).$$

Proof. We prove the assertion by induction on j. First let $j = 0$. By the definition of $J_{\leq 0} = \mathrm{ann}_B(B \otimes_A \Bbbk)$, we have $J_{\leq 0} \cdot (B \otimes_A \Bbbk) = J_{\leq 0} \cdot (B/BA_+) = 0$ so $J_{\leq 0} \subseteq BA_+$. Then

$$t_0^A(B_A) = \deg B/BA_+ \leq \deg B/J_{\leq 0} \leq \deg B/J_{\leq i} \leq D_i - t_2^B(\Bbbk)$$

which is the assertion when $j = 0$.

For the inductive step, we assume that $i > 0$. Fix $j \leq i$. Then for $0 \leq k \leq j-1$,

$$
\begin{aligned}
t_k^A&(B_A) + t_{j-k+1}^B(\Bbbk) \\
&\overset{\text{ind. hyp.}}{\leq} \max \left\{ U_k^i(f), \{t_\ell^A(\Bbbk) + (k-\ell)D_i\}_{0 \leq \ell \leq k} \right\} + D_i - t_2^B(\Bbbk) \\
&\qquad\qquad + t_{j-k+1}^B(\Bbbk) \\
&\leq \max \Big\{ U_k^i(f) + D_i - t_2^B(\Bbbk) + t_{j-k+1}^B(\Bbbk), \\
&\qquad\qquad \{t_\ell^A(\Bbbk) + (k-\ell)D_i + D_i - t_2^B(\Bbbk) + t_{j-k+1}^B(\Bbbk)\}_{0 \leq \ell \leq k} \Big\} \\
&\overset{\text{(E5.1.7)}}{\leq} \max \Big\{ U_j^i(f), \\
&\qquad \{t_\ell^A(\Bbbk) + (j-\ell)D_i + (k+1-j)D_i - t_2^B(\Bbbk) + t_{j-k+1}^B(\Bbbk)\}_{0 \leq \ell \leq k} \Big\} \\
&\overset{\text{def.}D_i}{\leq} \max \left\{ U_j^i(f), \{t_\ell^A(\Bbbk) + (j-\ell)D_i\}_{0 \leq \ell \leq k} \right\} \\
&\leq \max \{ U_j^i(f), \{t_\ell^A(\Bbbk) + (j-\ell)D_i\}_{0 \leq \ell \leq j-1} \}.
\end{aligned}
$$

We use the above inequality to see that

$$
\begin{aligned}
t_j^A(B_A) &\overset{\text{(E5.1.5)}}{\leq} \deg(\Bbbk \otimes_B \mathrm{Tor}_j^A(B, \Bbbk)) + \deg B/J_{\leq j} \\
&\overset{\text{def. } D_i}{\leq} \deg(\Bbbk \otimes_B \mathrm{Tor}_j^A(B, \Bbbk)) + D_i - t_2^B(\Bbbk) \\
&\overset{\text{(E5.1.4)}}{\leq} \max\{\{t_k^A(B_A) + t_{j-k+1}^B(\Bbbk)\}_{0 \leq k \leq j-1}, t_j^A(\Bbbk)\} + D_i - t_2^B(\Bbbk) \\
&\leq \max\{U_j^i(f), \{t_\ell^A(\Bbbk) + (j-\ell)D_i\}_{0 \leq \ell \leq j-1}, t_j^A(\Bbbk)\} + D_i - t_2^B(\Bbbk) \\
&= \max\{U_j^i(f), \{t_\ell^A(\Bbbk) + (j-\ell)D_i\}_{0 \leq \ell \leq j}\} + D_i - t_2^B(\Bbbk),
\end{aligned}
$$

completing the proof. $\square$

Lemma 5.4. *Retain the hypotheses of Proposition 5.3. Then, for every $j \leq i$, $U_j^i(f) \leq jD_i$.*

Proof. By definition,

$$D_i \geq \left\{ \frac{t_{j+1}^B(\Bbbk) - t_2^B(\Bbbk)}{j-1} \right\}_{2 \leq j \leq i}$$

which is equivalent to

$$t_{j+1}^B(\Bbbk) + D_i - t_2^B(\Bbbk) \leq jD_i$$

for all $1 \leq j \leq i$. Now the assertion follows easily from the definition of $U_j^i(f)$ (E5.1.6). □

Lemma 5.5. *Retain the hypotheses of Proposition 5.3 and assume that, for all $0 < k \leq i$,*

$$D_i \geq \frac{t_k^A(\Bbbk)}{k}. \tag{E5.4.1}$$

Then for all $1 \leq j \leq i$,

$$t_j^A(B_A) \leq (j+1)D_i - t_2^B(\Bbbk).$$

Proof. Under the hypothesis, we have that for all $1 \leq j, k \leq i$,

$$t_k^A(\Bbbk) + (j-k)D_i \leq jD_i.$$

By Lemma 5.4, we also have that $U_j^i(f) \leq jD_i$. Hence, by Proposition 5.3,

$$t_j^A(B_A) \leq \max \left\{ U_j^i(f), \{t_k^A(\Bbbk) + (j-k)D_i\}_{0 \leq k \leq j} \right\} + D_i - t_2^B(\Bbbk)$$
$$\leq jD_i + D_i - t_2^B(\Bbbk) = (j+1)D_i - t_2^B(\Bbbk),$$

as desired. □

Now we have an immediate consequence.

Corollary 5.6. *Retain the hypotheses of Proposition 5.3. Suppose that D_i is a number larger than or equal to*

$$\max \left\{ \deg B/J_{\leq i} + t_2^B(\Bbbk), \left\{ \frac{t_{j+2}^B(\Bbbk) - t_2^B(\Bbbk)}{j} \right\}_{1 \leq j \leq i-1}, \left\{ \frac{t_j^A(\Bbbk)}{j} \right\}_{1 \leq j \leq i}, \left\{ \frac{t_j^B(\Bbbk)}{j} \right\}_{1 \leq j \leq i} \right\}.$$

Then for all $1 \leq j \leq i$, $t_j^A(B_A) \leq (j+1)D_i - t_2^B(\Bbbk)$.

This recovers the result in the commutative case [9, Corollary 5.2]. To see this note that in the setting of [9, Corollary 5.2], $B = \Bbbk[x_1, \ldots, x_n]$ with $\deg x_i = 1$ for all i and $A = \Bbbk[y_1, \ldots, y_n]$ with $\deg y_i = d_i$ with $\{d_1, \ldots, d_n\}$ non-increasing. Then

$$t_j^B(\Bbbk) = \begin{cases} j & 0 \leq j \leq n \\ 0 & \text{otherwise} \end{cases}$$

and $t_j^A(\Bbbk)/j \leq d_1 \leq \deg B/BA_{\geq 1} = \deg B/J_{\geq i}$ for all $j \leq i$. Thus we can take $D_i = \deg B/BA_{\geq 1} + 2$ (which is independent of i). By Corollary 5.6, we have

$$t_j^A(B) \leq (j+1)(\deg B/BA_{\geq 1} + 2) - 2$$

which is the second statement of [9, Corollary 5.2]. The first statement of [9, Corollary 5.2] follows from Proposition 5.3 (details are omitted).

Now suppose H is a semisimple Hopf algebra acting on B homogeneously and let $C = B^H$. Suppose that $f : A \to C$ is a surjective map of graded algebras and consider it as a graded algebra map $A \to B$. Recall that $J_i \subseteq B$ denotes the annihilator ideal of the finite-dimensional left B-module $\operatorname{Tor}_i^A(B, \Bbbk)$.

Theorem 5.7. *Suppose that (T, H) satisfies Hypothesis 0.3 and assume that T is Koszul. Let $J_\infty = \cap_{j \geq 0} J_j$. Let S be a noetherian AS regular algebra that maps onto $R := T^H$ surjectively such that $t_j^S(\Bbbk) \leq j(\deg T/J_\infty + 2)$ for all $j \geq 0$. Then*

$$t_i^S(R_S) \leq i(\deg T/J_\infty + 2) + \deg T/J_\infty$$

for all $i \geq 0$.

Proof. Since T is Koszul, $t_j^T(\Bbbk) = j$ for all $0 \leq j \leq \operatorname{gldim} T$. In particular, $t_2^T(\Bbbk) = 2$. Under the hypotheses of this theorem, one can check that $\deg T/J_\infty + 2$ is at least equal to each term in the max-expression in Corollary 5.6 (letting $(A, B) = (S, T)$). Therefore, if we take $D = \deg T/J_\infty + 2$ and apply Corollary 5.6, we obtain that

$$t_i^S(T_S) \leq (i+1)D - t_2^T(\Bbbk) = iD + D - 2.$$

Since R is a direct summand of T as a right S-module, the assertion follows. $\square$

This is a noncommutative version of [9, Theorem 1.2 (part 1)]. To see this we let T be the polynomial ring $\Bbbk[x_1, \ldots, x_n]$ with $\deg x_i = 1$ (that is B in [9, Theorem 1.2]). Let S be another polynomial ring mapping onto $R := T^H$ (where $H = \Bbbk G$ for some finite group in the setting of [9, Theorem 1.2 (part 1)]). In the commutative case

$$D := \deg T/J_\infty + 2 = \deg T/TR_{\geq 1} + 2 = \tau_H(T) + 1 \leq |G| + 1$$

where the last $\leq$ follows from Fogarty's result [24], or equivalently, Proposition 3.11(1) by taking $d = 1$ and $m = 0$. By Theorem 5.7,

$$t_i^S({}_S R) \leq iD + D - 2 = (i+1)D - 2 = (i+1)(\tau_H(T)+1) - 2$$
$$= (i+1)\tau_H(T) + i - 1 \leq (i+1)|G| + i - 1$$

which recovers exactly [9, Theorem 1.2 (part 1)].

Remark 5.8. When T^H is noncommutative, the hypothesis in Theorem 5.7 does not hold automatically. While there are cases where it is known that such an AS regular algebra S exists (see e.g. [37], [7]), it is unknown if this holds in general. For a general connected graded algebra A, we can make the following comments.

(1) If A is finite-dimensional, then there is a noetherian AS regular algebra S and a surjective algebra map $f : S \to A$ [45, p. 34].
(2) Let A be the noetherian connected graded domain given in [50, Theorem 2.3]. Then A has GK dimension 2 and does not satisfy the χ-condition. For each integer $d \geq 2$, let B be the polynomial extension $A[x_1, \ldots, x_{d-2}]$. Then B is a noetherian connected graded domain of GK-dimension d that does not satisfy the χ-condition [2, Theorem 8.3]. We claim that there is no surjective homomorphism from a noetherian AS regular algebra S to B (nor a graded algebra homomorphism from S to B such that B is finitely generated over S on both sides). Suppose to the contrary that there is an AS regular algebra S and a surjective homomorphism from S to B. By [2, Theorem 8.1], S satisfies the χ-condition, and by [2, Theorem 8.3], so does B. This yields a contradiction.
(3) When $\mathrm{GKdim}\, A = 1$, it is still unknown if there exists a surjective homomorphism from a noetherian AS regular algebra S to A.

Next we prove a noncommutative version of [9, Theorem 1.3]. One can recover [9, Theorem 1.3] from Proposition 5.9 by specializing to the commutative situation, but we omit those details here. We return to the setting in Proposition 5.3.

Proposition 5.9. *Let $f : A \to B$ be a graded algebra homomorphism of connected graded algebras and let $C = \mathrm{im}(f)$. Assume the following:*

(a) *B is generated in degree 1,*
(b) *C is a direct summand of B as a right A-module,*
(c) *D_i is the number given in Corollary 5.6, and*
(d) *$D_i \geq \max\left\{ \dfrac{t_j^A(\Bbbk) + t_2^B(\Bbbk)}{j} \right\}_{2 \leq j \leq i}.$*

Then $t_0^C(\Bbbk) = 0$, $t_1^C(\Bbbk) \leq D_i - t_2^B(\Bbbk) + 1$, and, for $2 \leq j \leq i$,

$$t_j^C(\Bbbk) \leq jD_i - t_2^B(\Bbbk).$$

Proof. The assertion for $t_0^C(\Bbbk)$ is obvious. The assertion for $t_1^C(\Bbbk)$ follows from Lemma 3.2(3). Now let $j \geq 2$ and let $F_j = jD_i - t_2^B(\Bbbk)$. It remains to show that $t_j^C(\Bbbk) \leq F_j$. We proceed by induction on j.

Applying (E5.1.3) to the map $f : A \to C$, and using the fact that $C \otimes_A \Bbbk = \Bbbk$, we obtain that

$$t_j^C(\Bbbk) \leq \max \left\{ \{t_k^C(\Bbbk) + t_{j-k-1}^A(C_A)\}_{0 \leq k \leq j-2}, \ t_j^A(\Bbbk) \right\}. \tag{E5.8.1}$$

Note that when $2 \leq j \leq i$, hypothesis (d) on D_i is equivalent to

$$t_j^A(\Bbbk) \leq jD_i - t_2^B(\Bbbk) = F_j.$$

Hence, to show the main assertion it suffices to show that for all $0 \leq k \leq j - 2$,

$$t_k^C(\Bbbk) + t_{j-k-1}^A(C_A) \leq F_j. \tag{E5.8.2}$$

Note that (E5.8.2) holds for $k = 0$ because

$$t_0^C(\Bbbk) + t_{j-1}^A(C_A) = t_{j-1}^A(C_A) \leq t_{j-1}^A(B_A) \leq F_j \tag{E5.8.3}$$

where the first inequality holds because C is a direct summand of B as a right A-module, and the second inequality holds by Corollary 5.6. Further, (E5.8.2) holds for $k = 1$ because

$$t_1^C(\Bbbk) + t_{j-2}^A(C_A) \leq D_i + F_{j-1} = F_j.$$

If $2 \leq k \leq j - 2$, we use the induction hypothesis and the fact $t_{j-k-1}^A(C_A) \leq F_{j-k}$ (as explained in (E5.8.3)) to see that

$$t_k^C(\Bbbk) + t_{j-k-1}^A(C_A) \leq F_k + F_{j-k} < F_j.$$

Therefore (E5.8.2) holds for all $k \leq j - 2$, as desired. $\square$

Theorem 5.10. *Retain the hypotheses of Theorem 5.7. Fix a positive integer i and assume that*

$$\deg T/J_\infty \geq \max \left\{ \frac{t_j^S(\Bbbk) + t_2^T(\Bbbk)}{j} \right\}_{2 \leq j \leq i} - 2.$$

Then $t_0^R(\Bbbk) = 0$, $t_1^R(\Bbbk) \leq \deg T/J_\infty + 1$, and, for $2 \leq j \leq i$,

$$t_j^R(\Bbbk) \leq j(\deg T/J_\infty + 2) - 2.$$

Proof. Note that since T is Koszul, $t_2^T(\Bbbk) = 2$. Letting $(A, B) = (S, T)$, observe that Hypothesis (d) in Proposition 5.9 holds for $D_i = \deg T/J_\infty + 2$.

Under these hypotheses, one can check that $\deg T/J_\infty + 2$ is at least equal to each term in the max-expression in Corollary 5.6 and that the hypotheses in Proposition 5.9 hold. Therefore the assertion follows from Proposition 5.9. $\square$

The commutative result [9, Theorem 1.3(part 1)] is covered by the above theorem. We now give a very special case of Theorem 5.10.

Corollary 5.11. *Let G be a finite group acting as graded automorphisms on $T :=$ $\Bbbk_{-1}[x_1, \ldots, x_n]$, and suppose that $\Bbbk$ is an infinite field and $\Bbbk G$ is semisimple. Assume that $R := T^G$ is commutative. Then $t_0^R(\Bbbk) = 0$, $t_1^R(\Bbbk) \leq 2|G| + n$, and*

$$t_i^R(\Bbbk) \leq i(2|G| + n + 1) - 2$$

for all $i \geq 2$.

Proof. It is clear that $t_0^R(\Bbbk) = 0$. By Corollary 3.12, $t_1^R(R) \leq 2|G| + n$.

When $i \geq 2$, we let S be a commutative polynomial ring generated by elements of degree $\leq \beta(R)$ which maps surjectively onto R.

Since $T := \Bbbk_{-1}[x_1, \ldots, x_n]$ is a finite module over the commutative subalgebra $Z := \Bbbk[x_1^2, \ldots, x_n^2]$ and is generated as a Z-module by elements of degree $\leq n$, we have $d = 2$ and $m = n$ in Proposition 3.11. By Proposition 3.11(2), $\deg T/J_{G,i} \leq 2|G| + n - 1$. Let $D = D_i = 2|G| + n + 1$ (which is independent of i). Then by Proposition 3.11(1), $\beta(R) \leq \tau_G(T) \leq 2|G| + n = D - 1$. Now it is routine to check that $D(= D_i)$ is at least equal to each term in the max-expressions in Corollary 5.6 and Proposition 5.9(d). By Proposition 5.9 with $(B, A, C) = (T, S, R)$, we obtain that

$$t_i^R(\Bbbk) \leq i(2|G| + n + 1) - 2$$

as $t_2^T(\Bbbk) = 2$. $\square$

To conclude this section we prove a version of [13, Theorem 2].

Theorem 5.12. *Let (T, H) be as in Hypothesis 0.3 and suppose that T is AS regular. Let $R = T^H$. Suppose further that*

(a) *T is generated in degree 1.*
(b) *S is a noetherian AS regular algebra such that the minimal generating vector spaces of S and R have the same dimension and there exists a graded algebra surjection $S \to R$.*

Then we can conclude:

(1) *We have*

$$\beta_2(R) := t_2^R(\Bbbk) \le \tau_H(T) + \tau_H^{op}(T) - \mathrm{CMreg}(T)$$
$$\le 2 - 2\,\mathrm{CMreg}(S) + \mathrm{CMreg}(T).$$

(2) *Suppose that* $\mathrm{Tor}_1^S(\Bbbk, R) \otimes_R \Bbbk \cong \mathrm{Tor}_1^S(\Bbbk, R)$. *Then*

$$t_1^S({}_S R) \le \tau_H(T) + \tau_H^{op}(T) - \mathrm{CMreg}(T)$$
$$\le 2 - 2\,\mathrm{CMreg}(S) + \mathrm{CMreg}(T).$$

(3) *Suppose the hypothesis of part* (2). *Let* K *be the kernel of the algebra map* $S \to R$. *Then, as a left ideal of* A, K *is generated in degree at most*

$$\tau_H(T) + \tau_H^{op}(T) - \mathrm{CMreg}(T) \le 2 - 2\,\mathrm{CMreg}(S) + \mathrm{CMreg}(T).$$

Recall that if T is AS regular, then, by Example 2.10(3), $\mathrm{CMreg}(T) \le 0$. The condition that $\mathrm{Tor}_1^S(\Bbbk, R) \otimes_R \Bbbk \cong \mathrm{Tor}_1^S(\Bbbk, R)$ is automatic when R and S are commutative. It holds even for some noncommutative cases; see, for example, Lemma 5.13.

Proof of Theorem 5.12. (1) Let $\{f_1, \ldots, f_r\}$ be a set of homogeneous elements in R that generates R minimally; these are also elements in T. Let $d_i = \deg(f_i)$ for all $1 \le i \le r$.

We consider the left T-module U defined by

$$U := \left\{ (w_1, \cdots, w_r) \in T(-d_1) \oplus \cdots \oplus T(-d_r) \;\middle|\; \sum_{i=1}^{w} w_i f_i = 0 \right\}. \qquad \text{(E5.11.1)}$$

Then U fits into the short exact sequence of graded left T-modules,

$$0 \to U \to \bigoplus_{j=1}^{r} T(-d_j) \to T R_{\ge 1}(=: J_H(T)) \to 0. \qquad \text{(E5.11.2)}$$

By Lemma 3.2(3), for all $1 \le j \le r$, $d_j \le \tau_H(T)$. Applying Lemma 2.14(1) to the exact sequence

$$0 \to J_H(T) \to T \to T/J_H(T) \to 0,$$

and using the fact that $\mathrm{CMreg}(T) \le 0$, we obtain that

$$\mathrm{CMreg}(J_H(T)) \le \max\{\mathrm{CMreg}(T), \mathrm{CMreg}(T/J_H(T)) + 1\}$$
$$= \mathrm{CMreg}(T/J_H(T)) + 1 = \tau_H(T).$$

Applying Lemma 2.14(1) to (E5.11.2) and using the fact that each $d_j \leq \tau_H(T)$ (or equivalently, $\mathrm{CMreg}(T(-d_j)) \leq \tau_H(T)$), we have $\mathrm{CMreg}(U) \leq \tau_H(T)+1$. By Theorem 2.13(3), we have

$$\mathrm{Extreg}(J_H(T)) = \mathrm{CMreg}(J_H(T)) - \mathrm{CMreg}(T) \leq \tau_H(T) - \mathrm{CMreg}(T)$$

and

$$\mathrm{Extreg}(U) = \mathrm{CMreg}(U) - \mathrm{CMreg}(T) \leq \tau_H(T) + 1 - \mathrm{CMreg}(T).$$

Since Ext-regularity is equal to Tor-regularity [Definition 2.11], U is generated in degrees $\leq \tau_H(T) + 1 - \mathrm{CMreg}(T)$ as a left T-module, or

$$U = \sum_{\lambda \leq \tau_H(T)+1-\mathrm{CMreg}(T)} T U_\lambda. \tag{E5.11.3}$$

There is an induced H-action on the left T-module $\bigoplus_{j=1}^{r} T(-d_j)$ that makes (E5.11.2) a short exact sequence of left H-modules. Consider the left R-module M defined by

$$M := \left\{ (w_1, \ldots, w_r) \in R(-d_1) \oplus \cdots \oplus R(-d_r) \ \middle| \ \sum_{i=1}^{w} w_i f_i = 0 \right\}.$$

Since the f_i are H-invariants, (E5.11.2) is an exact sequence of H-equivariant T-modules, so we can apply $(-)^H$. Since H is semisimple, $(-)^H$ is an exact functor. Then the following exact sequence follows from (E5.11.2).

$$0 \to M \to \bigoplus_{j=1}^{r} R(-d_j) \to R_{\geq 1} \to 0. \tag{E5.11.4}$$

Thus M fits into the short exact sequence,

$$0 \to M \to \bigoplus_{j=1}^{r} R(-d_j) \to R \to \Bbbk \to 0.$$

Let $\mathfrak{m} = R_{\geq 1}$. We can identify $M/\mathfrak{m}M$ with $\mathrm{Tor}_2^R(\Bbbk, \Bbbk)$.

Now we consider M as an R-submodule of U. Let J^{op} be $R_{\geq 1}T$. Then $J^{\mathrm{op}} = \sum_j f_j T = \mathfrak{m}T$. Since H is semisimple, applying $(-)^H$ to the exact sequence

$$0 \to J^{\mathrm{op}}U \to U \to U/J^{\mathrm{op}}U \to 0$$

we obtain an exact sequence

$$0 \to (J^{\mathrm{op}}U)^H \to U^H \to (U/J^{\mathrm{op}}U)^H \to 0. \tag{E5.11.5}$$

We have already seen that $U^H = M$. We claim that $(J^{\mathrm{op}}U)^H = \mathfrak{m}M$. Let $\phi \in (J^{\mathrm{op}}U)^H$, which can be written as

$$\phi = \sum_j f_j u_j$$

for some $u_j \in U$. Let e be the integral of H. Then

$$\phi = e \cdot \phi = \sum_j f_j(e \cdot u_j) \in \mathfrak{m}M.$$

Now (E5.11.5) shows that $(U/J^{\mathrm{op}}U)^H \cong M/\mathfrak{m}M$.

Next we claim that $\deg(U/J^{\mathrm{op}}U) \le \tau_H(T) + \tau_H^{\mathrm{op}}(T) - \mathrm{CMreg}(T)$. Each $h \in U$ of degree strictly larger than $\tau_H(T) + \tau_H^{\mathrm{op}}(T) - \mathrm{CMreg}(T)$ can be written as

$$h = \sum_i p_i q_i,$$

where $q_i \in U_\lambda$ with $\lambda \le \tau_H(T) + 1 - \mathrm{CMreg}(T)$ and $p_i \in T_d$ with $d \ge \tau_H^{\mathrm{op}}(T)$. By the definition of $\tau_H^{\mathrm{op}}(T)$, we have $p_i \in J^{\mathrm{op}}$. Hence $h \in J^{\mathrm{op}}U$. Therefore we proved the claim. Since $M/\mathfrak{m}M$ is a subspace of $U/J^{\mathrm{op}}U$, we obtain that $\deg(M/fmM) \le \tau_H(T) + \tau_H^{\mathrm{op}}(T) - \mathrm{CMreg}(T)$ or that $\deg(\mathrm{Tor}_2^R(\Bbbk, \Bbbk)) \le \tau_H(T) + \tau_H^{\mathrm{op}}(T) - \mathrm{CMreg}(T)$. The second inequality follows from Lemma 2.16.

(2) The assertion follows from Lemma 5.1(3) (or (E5.1.1)) and part (1).

(3) By the exact sequence $0 \to K \to S \to R \to 0$, K is generated by elements corresponding to $\mathrm{Tor}_1^S(\Bbbk, R)$. Then the assertion follows from part (2). $\square$

Lemma 5.13. *Suppose that S and R are connected graded algebras and $f : S \to R$ is a graded algebra surjection with kernel K. If K is generated by normal elements in S, then $\mathrm{Tor}_1^S(\Bbbk, R) \otimes_R \Bbbk \cong \mathrm{Tor}_1^S(\Bbbk, R)$. As a consequence, (E5.1.1) holds.*

Proof. Let K be the kernel of the surjective map. Then $\mathrm{Tor}_1^S(\Bbbk, R) \otimes_R \Bbbk \cong \mathrm{Tor}_1^S(\Bbbk, R)$ is equivalent to $S_{\ge 1}K \supseteq KS_{\ge 1}$.

Write $K = \sum_i w_i S = \sum_i S w_i$ for a set of normal elements $\{w_i\} \subseteq S$. Then, for each i, $w_i S_{\ge 1} = S_{\ge 1} w_i$. Therefore

$$KS_{\ge 1} = \sum_i S w_i S_{\ge 1} = \sum_i S S_{\ge 1} w_i = \sum_i S_{\ge 1} S w_w = S_{\ge 1}K$$

as desired. $\square$

6. Further questions

We conclude by posing some further questions on degree bounds. Suppose that A is a noetherian connected graded algebra and H is a semisimple Hopf algebra acting homogeneously on A. Recall the definition of the τ-saturation degree $\tau_H(A)$ [Definition 1.1(2)].

Question 6.1. Under what hypothesis is $\tau_H(A) = \tau_H^{\mathrm{op}}(A)$?

In Corollary 3.3, we showed that an upper bound on $\tau_H(A)$ provides an upper bound for $\beta(A^H)$, the maximum degree of a minimal generating set of A^H. In Example 3.4, Theorem 3.5, and Proposition 3.11, we were able to compute bounds on $\tau_H(A)$.

Question 6.2. For which A and H can we bound $\tau_H(A)$?

If one is able to bound $\beta(A^H)$, there also remains the question of the sharpness of the bound. Noether's bound is sharp in the non-modular case. However, in [16], Domokos and Hegedüs show that if T is a commutative polynomial ring over a field of characteristic zero and G is a finite group that is not cyclic, then there is a strict inequality $\beta(T^G) < |G|$.

Question 6.3. Is there a version of Domokos and Hegedüs's result for skew polynomial rings (or other AS regular algebras) under group (or Hopf) actions?

We saw in Example 3.6 that for a group G, there is no universal bound on $\beta(T^G)$ over all AS regular algebras that depends only upon the order of the group and the degree of the representation, even for a group of order 2. This is in contrast to the commutative case. However, we pose the following question.

Question 6.4. Let $T := \Bbbk[V]$ be a commutative polynomial ring over a field of characteristic zero and V a representation of a finite group G. Define

$$\beta(G, V) := \min\{d : \Bbbk[V]^G \text{ generated by elements of degree } \leq d\}$$
$$\beta(G) = \max\{\beta(G, V) : V \text{ is a finite representation of } G\}.$$

It is a theorem of Weyl that $\beta(G) = \beta(G, V_{\mathrm{reg}})$.

Is there a version of this result, i.e., a particular representation of G or H, which has the highest degrees of minimal generating invariants for particular families of AS regular algebras? For example, one could fix $A = \Bbbk_{-1}[x_1, \ldots, x_n]$. Then if $|G| = n$, the regular representation of G induces an action on A and [38, Theorem 2.5] gives a bound on $\beta(A^G)$. Does this give a bound for all actions of G on (-1)-skew polynomial rings?

In this paper, we have focused on actions by semisimple Hopf algebras. The group algebra $\Bbbk G$ is semisimple precisely in the non-modular case (i.e., when the characteristic of $\Bbbk$ does not divide $|G|$). Hence, as noted in the introduction, when $T = \Bbbk[x_1, \ldots, x_n]$ the bounds on $\beta(T^G)$ depend on whether or not $\Bbbk G$ is semisimple.

Question 6.5. If H is a non-semisimple Hopf algebra acting on a connected graded noetherian AS Gorenstein algebra T, what bounds can be established on $\beta(T^H)$?

We refer the reader to [12] for examples of non-semisimple Hopf algebra actions on AS regular algebras.

Acknowledgments

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