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The Jacobian, Reflection Arrangement and Discriminant for Reflection Hopf Algebras

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We study finite-dimensional semisimple Hopf algebra actions on noetherian connected graded Artin–Schelter regular algebras and introduce definitions of the Jacobian, the reflection arrangement, and the discriminant in a noncommutative setting.

0 Introduction

The Shephard–Todd–Chevalley theorem states that if G is a finite group acting linearly and faithfully on the commutative polynomial ring $\Bbbk[x_1, \cdots, x_n]$, where the characteristic of the base field $\Bbbk$ is zero, the fixed subring $\Bbbk[x_1, \cdots, x_n]^G$ is isomorphic to $\Bbbk[x_1, \cdots, x_n]$ if and only if G is generated by pseudo-reflections of the space $V := \bigoplus_{i=1}^n \Bbbk x_i^*$. Such a group G is called a *reflection group*. Note that $\Bbbk[x_1, \cdots, x_n]$ is the ring of regular functions on V. This paper is part of a project to extend properties of the action of reflection groups on commutative polynomial algebras to a noncommutative setting.

In the noncommutative setting we consider here, the commutative polynomial ring $\Bbbk[x_1, \cdots, x_n]$ is replaced by an Artin–Schelter regular $\Bbbk$-algebra, denoted by A, and the group G (or the group ring $\Bbbk G$) is replaced by a (finite-dimensional) semisimple Hopf $\Bbbk$-algebra, denoted by H. We say H is a *reflection Hopf algebra* or *reflection quantum*

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group if the fixed subring A^H (E0.1.2) is again Artin–Schelter regular [33, Definition 3.2]. The 1st example of a noncommutative and noncocommutative reflection Hopf algebra (the Kac–Palyutkin algebra [28] acting on $\Bbbk_i[u,v]$ where $i^2 = -1$) was given in [31, Example 7.4]. A systematic study of dual reflection groups (where $H = (\Bbbk G)^*$) was begun in [33]. This noncommutative (and noncocommutative) context for noncommutative invariant theory has proved fruitful, and results include:

(a) The rigidity of (noetherian) Artin–Schelter regular algebras under finite group or semisimple Hopf algebra actions [1, 17, 30, 33].
(b) The homological determinant and Watanabe's theorem. The homological determinant of a group action on Artin–Schelter regular algebras was introduced in [27], and that of a Hopf action in [31].
(c) The Nakayama automorphism and twisted (skew) Calabi–Yau property [19, 33, 43, 44].
(d) The pertinency and radical ideal associated to Hopf actions on Artin–Schelter regular algebras, Auslander's theorem, the McKay correspondence, and noncommutative resolutions [6, 7, 15, 16, 18, 23, 41].

A survey on noncommutative invariant theory in this context is given in [29].

An important topic in classical invariant theory is the arrangements of hyperplanes associated to reflection groups [40]. It is related to combinatorics, algebra, geometry, representation theory, complex analysis, and other fields.

In this paper we investigate the possibility of defining a noncommutative version of a hyperplane arrangement. Some fundamental work of Steinberg [47], Stanley [46], Terao [49], Hartmann–Shepler [24], Orlik–Terao [40], and many others offered an algebraic approach that can be adapted to the noncommutative case. In particular, we will introduce a few concepts that characterize significant structures of the actions of reflection Hopf algebras on Artin–Schelter regular algebras.

Throughout the rest of this paper, let $\Bbbk$ be a base field that is algebraically closed, and all vector spaces, (co)algebras, Hopf algebras, and morphisms are over $\Bbbk$. In general we do not need to assume that the characteristic of $\Bbbk$ is zero. However, in several places where we use results from other papers (e.g., [32, 33]), we add the characteristic zero hypothesis because those results were proved under that extra hypothesis. Let H denote a semisimple (hence finite-dimensional) Hopf algebra, and let K be the $\Bbbk$-linear Hopf dual H^* of H. Throughout we use standard notation (see e.g., [38]) for a Hopf algebra $H(\Delta, \epsilon, S)$. It is well known that a left H-action on an algebra A is equivalent to a right K-coaction on A, and we will use this fact freely.

Let GKdim A denote the Gelfand–Kirillov dimension of the algebra A [34]. Let $\Bbbk^\times$ be the set of invertible elements in $\Bbbk$. If $f, g \in A$ and $f = cg$ for some $c \in \Bbbk^\times$, then we write $f =_{\Bbbk^\times} g$.

Hypothesis 0.1. Assume the following hypotheses:

(a) A is a noetherian connected graded Artin–Schelter regular algebra that is a domain, see Definition 1.1;
(b) H is a semisimple Hopf algebra;
(c) H acts on A inner faithfully [14, Definition 1.5] and homogeneously so that A is a left H-module algebra;
(d) H acts on A as a reflection Hopf algebra in the sense of [33, Definition 3.2], or equivalently, of Definition 1.4.

Let $G(K)$ be the group of grouplike elements in $K := H^*$. For each $g \in G(K)$, define

$$A_g := \{a \in A \mid \rho(a) = a \otimes g\}, \tag{E0.1.1}$$

where $\rho : A \to A \otimes K$ is the corresponding right coaction of K on A. The fixed subring of the H-action on A is defined to be

$$A^H := \{a \in A \mid h \cdot a = \epsilon(h)a \ \forall\, h \in H\}. \tag{E0.1.2}$$

We refer to [31, Section 3] for the definition of the *homological determinant* of the H-action on A. Let hdet : $H \to \Bbbk$ be the homological determinant of the H-action on A. Then hdet, considered as an element in K, is a grouplike element. By [14, Theorem 0.6], hdet is nontrivial (unless $A = A^H$) when H is a reflection Hopf algebra. Since hdet is an element in $G(K)$, both A_{hdet} and $A_{\mathrm{hdet}^{-1}}$ are defined by (E0.1.1).

Theorem 0.2 (Corollary 2.5(1) and Theorem 3.8(1)). Assume Hypotheses 0.1. Let R be the fixed subring A^H.

(1) There is a nonzero element $\mathsf{j}_{A,H} \in A$, unique up to a nonzero scalar, such that $A_{\mathrm{hdet}^{-1}}$ is a free R-module of rank one on both sides generated by $\mathsf{j}_{A,H}$.
(2) There is a nonzero element $\mathsf{a}_{A,H} \in A$, unique up to a nonzero scalar, such that A_{hdet} is a free R-module of rank one on both sides generated by $\mathsf{a}_{A,H}$.

(3) The products $\mathsf{j}_{A,H}\mathsf{a}_{A,H}$ and $\mathsf{a}_{A,H}\mathsf{j}_{A,H}$ in A are elements of R that are either equal, or they differ only by a nonzero scalar in $\Bbbk$, or equivalently, $\mathsf{j}_{A,H}\mathsf{a}_{A,H} =_{\Bbbk^\times} \mathsf{a}_{A,H}\mathsf{j}_{A,H}$.

The above theorem allows us to define the following fundamental concepts.

Definition 0.3. Assume Hypotheses 0.1.

(1) The element $\mathsf{j}_{A,H}$ in Theorem 0.2(1) is called the *Jacobian* of the H-action on A.
(2) The element $\mathsf{a}_{A,H}$ in Theorem 0.2(2) is called the *reflection arrangement* of the H-action on A.
(3) The element $\mathsf{j}_{A,H}\mathsf{a}_{A,H}$, or equivalently, $\mathsf{a}_{A,H}\mathsf{j}_{A,H}$, in Theorem 0.2(3) is called the *discriminant* of the H-action on A, and denoted by $\delta_{A,H}$.

The above concepts are well defined up to a nonzero scalar in $\Bbbk$, and under some hypotheses we show they exist more generally.

In the classical (commutative) setting, when G is a reflection group acting on a vector space V over the field of complex numbers $\mathbb{C}$, the Jacobian (respectively, the reflection arrangement, the discriminant) in Definition 0.3 is essentially equivalent to the classical Jacobian determinant of the basic invariants of G in the commutative polynomial ring $\mathbb{C}[V^*] := \mathcal{O}(V)$ (respectively, the reflection arrangement, the discriminant of the G-action). When we let $A = \mathbb{C}[V^*]$ and $H = \mathbb{C}G$ in Hypotheses 0.1, a well-known result of Steinberg [47] states that

$$\mathsf{j}_{A,H} =_{\mathbb{C}^\times} \prod_{s=1}^{v} f_s^{e_s-1}, \tag{E0.3.1}$$

where $\{f_s\}_{s=1}^{v}$ is the complete list of the linear equations of the reflecting hyperplanes of G, and each e_s is the exponent of the pointwise stabilizer subgroup that consists of pseudo-reflections in G associated to the corresponding reflecting hyperplane. After we identify each hyperplane in V with its linear form in V^*, the set of reflecting hyperplanes is uniquely determined by the following equation [40, Examples 6.39 and 6.40] (where det and $\det^{-1}$ are switched due to different convention used in the book [40])

$$\mathsf{a}_{A,H} =_{\mathbb{C}^\times} \prod_{s=1}^{v} f_s, \tag{E0.3.2}$$

which suggests calling $\mathsf{a}_{A,H}$ in Definition 0.3 the reflection arrangement of the H-action on A. In this paper, we can prove only the following weaker version of Steinberg's theorem [24, 47] in the noncommutative setting [Theorem 0.5].

Hypothesis 0.4. Assume the following hypotheses:

(1) Assume Hypotheses 0.1.
(2) char $\Bbbk = 0$.
(3) H is commutative, or equivalently, $H = (\Bbbk G)^*$ for some finite group G.
(4) A is generated in degree 1.

Theorem 0.5 (Theorem 2.12(2)). Assume Hypotheses 0.4. Then the following hold.

(1) $\mathsf{j}_{A,H}$ is a product of elements of degree 1.
(2) $\mathsf{a}_{A,H}$ is a product of elements of degree 1.

When A is noncommutative, it is usually not a unique factorization domain. Then the decompositions of $\mathsf{j}_{A,H}$ and $\mathsf{a}_{A,H}$ into products of linear forms in Theorem 0.5, formulas like (E0.3.1) and (E0.3.2), are not unique, see Examples 2.2(2) and 4.2. Therefore, it is difficult to imagine and define individual reflecting hyperplane at this point, though, in some special cases, there are natural candidates for such hyperplanes, see (E2.2.2). We have some general results as follows.

Theorem 0.6 (Theorem 3.8(2)). Assume Hypotheses 0.1. Then $\mathsf{a}_{A,H}$ divides $\mathsf{j}_{A,H}$ from the left and the right.

In the classical setting, when $H = \Bbbk G$, for G a reflection group acting on a vector space V, then $\delta_{A,H}$ agrees with the classical definition of discriminant of the G-action [40, Definition 6.44]. When R is central in A and H is a dual reflection group, then $\delta_{A,H}$ is closely related to the noncommutative discriminant $dis(A/R)$ studied in [8, 12, 13].

Theorem 0.7 (Theorem 3.10(2)). Assume Hypotheses 0.4. Suppose that $R := A^H$ (E0.1.2) is central in A. Then $\delta_{A,H}$ and $dis(A/R)$ have the same prime radical.

We refer to [51, 52] for the definition of Auslander regularity and [27, Definition 0.1] for the definition of Artin–Schelter Cohen–Macaulay used in the next theorem and its proof. By Theorem 2.4, the Jacobian $\mathsf{j}_{A,H}$ can be defined in a more general setting, which is used in the next theorem.

Theorem 0.8 (Theorem 3.9). Assume Hypotheses 0.1. Suppose A^H is Auslander regular. Then $A_G := \bigoplus_{g \in G(K)} A_g$ is Artin–Schelter Gorenstein and

$$\mathsf{j}_{A,H} =_{\Bbbk^\times} \mathsf{j}_{A_G,(\Bbbk G)^*}.$$

The theorem above leads to the following question.

Question 0.9. Assume Hypotheses 0.1. Is there a Hopf subalgebra $H_0 \subseteq H$ such that $A^{H_0} = A_G$?

In the classical case, either the Jacobian $\mathsf{j}_{A,H}$ or the reflection arrangement $\mathsf{a}_{A,H}$ completely determines the collection of reflecting hyperplanes via (E0.3.1) or (E0.3.2), respectively. In the noncommutative case, since A is not a unique factorization domain, the decomposition such as (E0.3.1) (or (E0.3.2)) is not unique. Consequently, it is not clear how to define individual reflecting hyperplanes. We propose the following temporary definitions. For any homogeneous element $f \in A$, define the set of left (respectively, right) divisors of degree 1 of f to be

$$\mathfrak{R}^l(f) := \{\Bbbk v \mid v \in A_1, vf_v = f \text{ for some } f_v \in A\}. \tag{E0.9.1}$$

and

$$\mathfrak{R}^r(f) := \{\Bbbk v \mid v \in A_1, f_v v = f \text{ for some } f_v \in A\}. \tag{E0.9.2}$$

Unfortunately, in general (when H is neither commutative nor cocommutative),

$$\mathfrak{R}^l(\mathsf{a}_{A,H}) \neq \mathfrak{R}^r(\mathsf{a}_{A,H}),$$

see (E4.2.6) and (E4.2.7).

Some further results related to other invariants (e.g., the homological determinant, pertinency, and the Nakayama automorphism) are stated as corollaries to Theorem 2.4.

This paper is organized as follows. Section 1 reviews some basic material. We define and study the Jacobian and the reflection arrangement in Section 2. In Section 3 we focus on the discriminant. In Section 4, we give some nontrivial examples with some details.

1 Preliminaries

In this section we recall some basic concepts and fix some notation that will be used throughout.

An algebra A is called *connected graded* if

$$A = \Bbbk \oplus A_1 \oplus A_2 \oplus \cdots$$

and $1 \in A_0$, $A_i A_j \subseteq A_{i+j}$ for all $i, j \in \mathbb{N}$. We say A is *locally finite* if $\dim_{\Bbbk} A_i < \infty$ for all i. The *Hilbert series* of A is defined to be

$$h_A(t) = \sum_{i \in \mathbb{N}} (\dim_{\Bbbk} A_i) t^i.$$

The *Gelfand–Kirillov dimension* (or *GKdimension*) of a connected $\mathbb{N}$-graded, locally finite algebra A is defined to be

$$GKdim(A) = \limsup_{n \to \infty} \frac{\log\left(\sum_{i=0}^{n} \dim_{\Bbbk} A_i\right)}{\log(n)},$$

see [37, Chapter 8], [34], or [48, p.1594].

The algebras that replace the commutative polynomial rings are the so-called Artin–Schelter regular algebras [4]. We recall the definition below.

Definition 1.1. A connected graded algebra A is called *Artin–Schelter Gorenstein* (or *AS Gorenstein*, for short) if the following conditions hold:

- (a) A has injective dimension $d < \infty$ on the left and on the right,
- (b) $Ext^i_A({}_A\Bbbk, {}_A A) = Ext^i_A(\Bbbk_A, A_A) = 0$ for all $i \neq d$, and
- (c) $Ext^d_A({}_A\Bbbk, {}_A A) \cong Ext^d_A(\Bbbk_A, A_A) \cong \Bbbk(\mathfrak{l})$ for some integer $\mathfrak{l}$. Here $\mathfrak{l}$ is called the *AS index* of A.

If in addition,

- (d) A has finite global dimension, and
- (e) A has finite Gelfand–Kirillov dimension,

then A is called *Artin–Schelter regular* (or *AS regular*, for short) of dimension d.

Let M be an A-bimodule, and let μ, ν be algebra automorphisms of A. Then ${}^{\mu}M^{\nu}$ denotes the induced A-bimodule such that ${}^{\mu}M^{\nu} = M$ as a $\Bbbk$-space, and where

$$a * m * b = \mu(a) m \nu(b)$$

for all $a, b \in A$ and $m \in {}^{\mu}M^{\nu} (= M)$. Let 1 denote also the identity map of A. We use ${}^{\mu}M$ (respectively, M^{ν}) for ${}^{\mu}M^{1}$ (respectively, ${}^{1}M^{\nu}$).

Let A be a connected graded finite-dimensional algebra. We say A is a *Frobenius* algebra if there is a nondegenerate associative bilinear form

$$\langle -, - \rangle : A \times A \to \Bbbk,$$

which is graded of degree $-\mathfrak{l}$, or equivalently, there is an isomorphism $A^* \cong A(-\mathfrak{l})$ as graded left (or right) A-modules. There is a (classical) graded Nakayama automorphism $\mu \in Aut(A)$ such that $\langle a, b \rangle = \langle \mu(b), a \rangle$ for all $a, b \in A$. Further, $A^* \cong {}^{\mu}A^{1}(-\mathfrak{l})$ as graded A-bimodules. A connected graded AS Gorenstein algebra of injective dimension 0 is exactly a connected graded Frobenius algebra. The Nakayama automorphism can be defined for certain classes of infinite-dimensional algebras; see the next definition.

Definition 1.2. Let A be an algebra over $\Bbbk$, and let $A^e = A \otimes A^{op}$.

(1) A is called *skew Calabi–Yau* (or *skew CY*, for short) if

(a) A is homologically smooth, that is, A has a projective resolution in the category A^e-Mod that has finite length and such that each term in the projective resolution is finitely generated, and

(b) there is an integer d and an algebra automorphism μ of A such that

$$Ext^i_{A^e}(A, A^e) = \begin{cases} 0 & i \neq d \\ {}^{1}A^{\mu} & i = d, \end{cases} \tag{E1.2.1}$$

as A-bimodules, where 1 denotes the identity map of A.

(2) If (E1.2.1) holds for some algebra automorphism μ of A, then μ is called the *Nakayama automorphism* of A, and is usually denoted by μ_A.

(3) We call A *Calabi–Yau* (or *CY*, for short) if A is skew Calabi–Yau and μ_A is inner (or equivalently, μ_A can be chosen to be the identity map after changing the generator of the bimodule ${}^{1}A^{\mu}$).

If A is connected graded, the above definition should be made in the category of graded modules and (E1.2.1) should be replaced by

$$Ext^i_{A^e}(A, A^e) = \begin{cases} 0 & i \neq d \\ {}^1A^{\mu}(\mathfrak{l}) & i = d, \end{cases} \tag{E1.2.2}$$

where ${}^1A^{\mu}(\mathfrak{l})$ is the shift of the graded A-bimodule ${}^1A^{\mu}$ by degree $\mathfrak{l}$.

We will use local cohomology later. Let A be a locally finite $\mathbb{N}$-graded algebra and $\mathfrak{m}$ be the graded ideal $A_{\geq 1}$. Let A-GrMod denote the category of $\mathbb{Z}$-graded left A-modules. For each graded left A-module M, we define

$$\Gamma_{\mathfrak{m}}(M) = \{x \in M \mid A_{\geq n}x = 0 \text{ for some } n \geq 1\} = \lim_{n\to\infty} \mathrm{Hom}_A(A/A_{\geq n}, M)$$

and call this the $\mathfrak{m}$-*torsion submodule* of M. It is standard that the functor $\Gamma_{\mathfrak{m}}(-)$ is a left exact functor from A-GrMod to itself. Since this category has enough injectives, the ith right derived functors, denoted by $\mathrm{H}^i_{\mathfrak{m}}$ or $R^i\Gamma_{\mathfrak{m}}$, are defined and called the *local cohomology functors*. Explicitly, one has

$$\mathrm{H}^i_{\mathfrak{m}}(M) = R^i\Gamma_{\mathfrak{m}}(M) := \lim_{n\to\infty} Ext^i_A(A/A_{\geq n}, M).$$

See [5, 50] for more details.

The Nakayama automorphism of a noetherian AS Gorenstein algebra can be recovered by using local cohomology [43, Lemma 3.5]:

$$R^d\Gamma_{\mathfrak{m}}(A)^* \cong {}^{\mu}A^1(-\mathfrak{l}), \tag{E1.2.3}$$

where $\mathfrak{l}$ is the AS index of A.

The following notation will be used throughout.

Notation 1.3. ($G, \{p_i\}, \{p_g\}, A_g$). Let H denote a semisimple Hopf algebra. Since $\Bbbk$ is algebraically closed, the Artin–Wedderburn theorem implies that H has a decomposition into a direct sum of matrix algebras

$$H = M_{r_1}(\Bbbk) \oplus M_{r_2}(\Bbbk) \oplus \cdots \oplus M_{r_{N-1}}(\Bbbk) \oplus M_{r_N}(\Bbbk) \tag{E1.3.1}$$

with

$$1 = r_1 = \cdots = r_n < r_{n+1} \leq \cdots \leq r_i \leq r_{i+1} \leq \cdots \leq r_N. \tag{E1.3.2}$$

Each block $M_{r_i}(\Bbbk)$ corresponds to a simple left H-module, denoted by S_i. Then $\{S_i\}_{i=1}^N$ is the complete list of simple left H-modules and $\dim_{\Bbbk} S_i = r_i$ for all i. The center of H is a direct sum of N copies of $\Bbbk$, each of which corresponds to a block $M_{r_i}(\Bbbk)$. Since H is a Hopf algebra, $r_1 = 1$. Further, we can assume that $M_{r_1} = \Bbbk \int$ where $\int$ is the integral of H. Each copy of $M_{r_i}(\Bbbk) = \Bbbk$, for $i = 1, \cdots, n$, gives rise to a central idempotent in H, which is denoted by p_i. Let I_{com} be the ideal of H generated by commutators $[a, b] := ab - ba$ for all $a, b \in H$. Then

$$I_{com} = M_{r_{n+1}}(\Bbbk) \oplus \cdots \oplus M_{r_{N-1}}(\Bbbk) \oplus M_{r_N}(\Bbbk) \quad \text{and} \quad H/I_{com} = \Bbbk^{\oplus n}. \tag{E1.3.3}$$

It is well known that I_{com} is a Hopf ideal, and consequently, $H_{ab} := H/I_{com}$ is a commutative Hopf algebra. Since $\Bbbk$ is algebraically closed, H_{ab} is the dual of a group algebra $\Bbbk G$. By (E1.3.3), the order of G is n. There is another way of interpreting G. Let K be the dual Hopf algebra of H, and let $G(K)$ be the group of grouplike elements in K. Then G is naturally isomorphic to $G(K)$, and we can identify G with $G(K)$. For every grouplike element $g \in G(K)$, the correspondence idempotent in H_{ab} is denoted by p_g. Then the Hopf algebra structure of H_{ab} is given in [33, p.61]. Let e be the unit or identity element of the group G (later, the identity in G is also denoted by 1_G or 1). Lifting the idempotent $p_g \in H_{ab}$ from H_{ab} to the corresponding central idempotent in H, still denoted by p_g, we have, in H,

$$p_g p_h = \begin{cases} p_g & g = h, \\ 0 & g \neq h, \end{cases} \quad \text{and} \quad \sum_{g \in G} p_g \neq 1_H, \quad \text{unless} \ \ n = N, \tag{E1.3.4}$$

and

$$\Delta(p_g) = \sum_{h \in G} p_h \otimes p_{h^{-1}g} + X_g, \quad \text{and} \quad \epsilon(p_g) = \begin{cases} 1_{\Bbbk} & g = e, \\ 0 & g \neq e, \end{cases} \tag{E1.3.5}$$

where X_g is in $I_{com} \otimes H + H \otimes I_{com}$. Since I_{com} is a Hopf ideal, we also have $\Delta(I_{com}) \subseteq I_{com} \otimes H + H \otimes I_{com}$. Note that $\{p_g\}_{g \in G}$ agrees with the idempotents $\{p_i\}_{i=1}^n$. By the duality between H and K, the idempotent in H corresponding to the integral of H is p_1 where $1 \in K$ is the identity element (or the unit element 1_G of the group G). In other words, $p_1 = \int$. Note that p_1 is also the 1st central idempotent corresponding to the decomposition (E1.3.1).

Let A be a connected graded algebra and let H be a semisimple Hopf algebra acting on A homogeneously and inner faithfully [14, Definition 1.5] such that A is an H-module algebra. For each idempotent p_i, where $i = 1, \cdots, n, \cdots, N$, we write $A_{p_i} = p_i \cdot A$.

Then there is a natural decomposition

$$A = \oplus_{i=1}^{N} A_{p_i} \tag{E1.3.6}$$

following from the fact $1_H = \sum_{i=1}^{N} p_i$. Each p_i, for each $i = 1, \cdots, n$, equals p_g, for some $g \in G$, and we write

$$A_g := p_g \cdot A = \{a \in A \mid p_g \cdot a = a\}.$$

We recall a definition.

Definition 1.4. [33, Definition 3.2] Suppose H acts homogeneously and inner faithfully on a noetherian Artin–Schelter regular domain A that is an H-module algebra such that the fixed subring A^H (E0.1.2) is again Artin–Schelter regular. Then we say that H acts on A as a *reflection Hopf algebra* or *reflection quantum groups*. By abuse of language, sometimes we just say that H is a *reflection Hopf algebra* without mentioning A. If, further, $\mathrm{hdet}^{-1} = \mathrm{hdet}$, then H is called a *true reflection Hopf algebra*.

Lemma 1.5. Retain the notation above, and consider A as a K-comodule algebra where $\rho : A \to A \otimes K$ is the right coaction.

(1) For each $g \in G$, $A_g = \{a \in A \mid \rho(a) = a \otimes g\}$.
(2) $A_g A_h \subseteq A_{gh}$ for all $g, h \in G$.
(3) Let A_G be $\bigoplus_{g \in G} A_g$. Then A_G is a subalgebra of A.
(4) If A is a domain, then $G_0 := \{g \in G \mid A_g \neq 0\}$ is a subgroup of G.
(5) Suppose A is a domain and A_g (for some $g \in G$) is a nonzero free module over A^H on the left and the right, then A_g is a rank one free module over A^H on the left and the right.
(6) Assume Hypotheses 0.1. Then each nonzero A_g is a rank one free module over A^H on the left and the right.

Proof. (1) Let $\{h_1, \cdots, h_\alpha\}$ be a $\Bbbk$-linear basis of H and $\{h_1^*, \cdots, h_\alpha^*\}$ be the dual basis of $H^* =: K$. Then the element $\sum_i h_i \otimes h_i^*$ is independent of the choice of $\Bbbk$-linear bases $\{h_i\}_{i=1}^{\alpha}$. Since A is a left H-module, then A is a right K-comodule algebra with coaction given by

$$\rho(a) = \sum_{i=1}^{\alpha} (h_i \cdot a) \otimes h_i^* \tag{E1.5.1}$$

for all $a \in A$.

We pick a nice basis consisting of matrix units that correspond to the matrix decomposition (E1.3.1), making $\{p_g\}_{g\in G}$ a part of the basis for H. Since $\Bbbk G(K)$ is the dual Hopf algebra of $H_{ab} = H/I_{com}$, then $g(I_{com}) = 0$ for each $g \in G(K)$. For every $h \in G = G(K)$, it is easy to see that $g(p_h) = \delta_{gh}$. This implies that $\{g\}_{g\in G}$ is a part of the corresponding dual basis for K. Now the assertion follows from (E1.5.1) and a straightforward calculation.

(2) Let $x \in A_g$ and $y \in A_h$, then $I_{com}A_g = I_{com}A_h = 0$ implies that $I_{com}x = I_{com}y = 0$. By (E1.3.5), $p_{gh}(xy) = p_g(x)p_h(y) = xy$. Thus, $xy \in A_{gh}$.

(3) This follows from part (2).

(4) This follows from part (2) and the fact that A is a domain.

(5) Since A is a domain, $xA_g \subseteq A^H$ for every nonzero $x \in A_{g^{-1}}$. Thus, the rank of A_g over A^H is one.

(6) By [33, Lemma 3.3(2)] (where the hypothesis that the char $\Bbbk$ is zero is not necessary), A_g is free over A^H on both sides. The assertion follows from part (5). ■

Notation 1.6. ($\{f_g\}, \phi_g$). Let R denote the fixed subring A^H. Assume that H is a reflection Hopf algebra acting on a noetherian Artin–Schelter regular domain A. By Lemma 1.5(6), each nonzero A_g is of the form

$$A_g = f_g R = R f_g, \tag{E1.6.1}$$

where $f_g \in A_g$ is a (fixed) nonzero homogeneous element of lowest degree. Note that f_g is unique up to a nonzero scalar in $\Bbbk$. There is a graded automorphism $\phi_g \in Aut(R)$ such that

$$f_g x = \phi_g(x) f_g \tag{E1.6.2}$$

for all $x \in R$ [33, (E3.5.1)]. For every pair (g, h) of elements in G, define $c_{g,h} \in R$ such that

$$f_g f_h = c_{g,h} f_{gh} \tag{E1.6.3}$$

[33, (E3.5.2)]. Then $c_{g,h}$ is a normal element in R and

$$c_{g,h} = f_g f_h f_{fg}^{-1}, \quad \text{and} \quad \phi_{gh}^{-1}(c_{g,h}) = f_{fg}^{-1} f_g f_h. \tag{E1.6.4}$$

Lemma 1.7. Let H be a semisimple Hopf algebra acting on an algebra A.

(1) If M is a simple left H-module and N a one-dimensional left H-module, then both $N \otimes M$ and $M \otimes N$ are simple left H-modules of dimension equal to $\dim_{\Bbbk} M$.

(2) If $M \subseteq A$ is a simple left H-module and $0 \neq b_g \in A_g$ where A_g is defined as in Lemma 1.5(1), then both Mb_g and b_gM (if nonzero) are simple left H-modules of dimension equal to $\dim_{\Bbbk} M$.

Proof. (1) This follows from the fact that $N \otimes -$ and $- \otimes N$ are auto-equivalences of the category of left H-modules.

(2) This follows from the fact that the multiplication map $\mu : A \otimes A \to A$ is a left H-module map. Further, as left H-modules, $b_gM \cong \Bbbk b_g \otimes M$ and $Mb_g \cong M \otimes \Bbbk b_g$ when b_gM and Mb_g are nonzero. ■

Fixed an integer $d > 0$. Let $\{S_{d,i}\}_{i=1}^{w_d}$ be the complete list of simple left H-modules of dimension d. For each $g \in G(K)$, there are permutations in the symmetric group, $\sigma_{g,d}, \tau_{g,d} \in \mathbb{S}_{w_d}$, such that

$$\Bbbk g \otimes S_{d,i} = S_{d,\sigma_{d,g}(i)}, \quad \text{and} \quad S_{d,i} \otimes \Bbbk g \cong S_{d,\tau_{d,g}(i)}. \tag{E1.7.1}$$

Let $\{p_{d,i}\}_{i=1}^{w_d}$ be the complete list of primitive central idempotents of H corresponding to the set $\{S_{d,i}\}_{i=1}^{w_d}$, and let $\mathcal{A}_{d,i} = p_{d,i}A$. By Lemma 1.7(2), we have that

$$b_g\mathcal{A}_{d,i} \subseteq \mathcal{A}_{d,\sigma_{d,g}(i)}, \quad \text{and} \quad \mathcal{A}_{d,i}b_g \subseteq \mathcal{A}_{d,\tau_{d,g}(i)} \tag{E1.7.2}$$

for all g, i.

For every d, define

$$\mathcal{A}_d := \bigoplus_{i=1}^{w_d} \mathcal{A}_{d,i}.$$

Let R be an Ore domain. If M is a left R-module, the rank of M over R is defined to be

$$\operatorname{rk} M := \dim_Q Q \otimes_R M,$$

where Q is the total quotient division ring of R.

Lemma 1.8. Suppose that A is a domain. Let rk denote the rank over A^H. Suppose that $\mathcal{A}_{d,1} \neq 0$ for some d.

(1) $rk\mathcal{A}_d \geq rk\mathcal{A}_1$.

(2) Suppose there are (d', i') such that $\mathcal{A}_{d',i'} \neq 0$ and that $S_{d,1} \otimes S_{d',i'}$ is a direct sum of simple H-modules of dimensions $d_1, \cdots, d_s$. Then

$$\operatorname{rk} \mathcal{A}_{d,1} \leq \sum_{\alpha=1}^{s} \operatorname{rk} \mathcal{A}_{d_\alpha}.$$

(3) Suppose there are (d', i') such that $\mathcal{A}_{d',i'} \neq 0$ and that $(\bigoplus_{i=1}^{w_d} S_{d,j}) \otimes S_{d',i'}$ is a direct sum of simple H-modules of dimensions $d_1, \cdots, d_s$. Then

$$\operatorname{rk} \mathcal{A}_{d} \leq \sum_{\alpha=1}^{s} \operatorname{rk} \mathcal{A}_{d_\alpha}.$$

(4) Suppose that $S_{d,1} \otimes S_{d,i}$ is a direct sum of one-dimensional H-simples for some i such that $\mathcal{A}_{d,i} \neq 0$. Then $\operatorname{rk}\mathcal{A}_{d,1} \leq \operatorname{rk}\mathcal{A}_1$.

(5) If for any $\mathcal{A}_{d,i} \neq 0$, $S_{d,i} \otimes S_{d,1}$ is a direct sum of one-dimensional H-simples, then $\operatorname{rk}\mathcal{A}_d \leq \operatorname{rk}\mathcal{A}_1$.

Proof. (1) Let $0 \neq x \in \mathcal{A}_{d,1}$ such that x is in a simple left H-module M. By Lemma 1.7(2),

$$\mathcal{A}_1 x \subseteq \mathcal{A}_1 M \subseteq \mathcal{A}_d.$$

Therefore,

$$\operatorname{rk} \mathcal{A}_d \geq \operatorname{rk} \mathcal{A}_1 x = \operatorname{rk} \mathcal{A}_1.$$

(2) Let $0 \neq x \in \mathcal{A}_{d',i'}$. By the ideas in the proof of Lemma 1.7(2),

$$\mathcal{A}_{d,1} x \subseteq \bigoplus_{\alpha=1}^{s} \mathcal{A}_{d_\alpha}.$$

Therefore,

$$\operatorname{rk} \mathcal{A}_{d,1} = \operatorname{rk} \mathcal{A}_{d,1} x \leq \sum_{\alpha=1}^{s} \operatorname{rk} \mathcal{A}_{d_\alpha}.$$

(3) The proof is similar to the proof of part (2).

(4,5) These are consequences of parts (2) and (3). ■

The above lemma has some consequences. For example, if H has only one simple S of dimension d larger than 1 and $S \otimes S$ is a direct sum of one-dimensional H-modules, then $\mathcal{A}_1$ and $\mathcal{A}_d$ have the same rank. When $|G| = d^2$ this implies that $\dim_{\Bbbk} H = 2d^2$ [2, 3].

Definition 1.9. Retain the notation as in Lemma 1.5 and let $G = G(K)$.

(1) The subalgebra A_G as defined in Lemma 1.5(3) is called the *G-component* of A.

(2) The $\Bbbk$-vector space $A_{G^c} := \bigoplus_{i=n+1}^{N} p_i \cdot A$ where n and N are defined in (E1.3.2) is called the *G-complement* of A. By Lemma 1.7(2), A_{G^c} is an A_G-bimodule and there is an A_G-bimodule decomposition

$$A = A_G \oplus A_{G^c}.$$

An A-bimodule M is called *H-equivariant* in the sense of [43, Definition 2.2] if

$$h \cdot (amb) = \sum (h_1 \cdot a)(h_2 \cdot m)(h_3 \cdot b)$$

for all $h \in H$, $a, b \in A$ and $m \in M$. The following lemma is more or less proved in [43].

Lemma 1.10. Let Y be an H-equivariant graded A-bimodule that is free of rank one over A on both sides. Then Y is isomorphic to $\mathfrak{e} \otimes A$ such that

(1) $\Bbbk\mathfrak{e}$ is a one-dimensional left H-module and there is an $g \in G(K)$ such that $h \cdot \mathfrak{e} \otimes 1 = g(h)\mathfrak{e} \otimes 1$,

(2) $\mathfrak{e} \otimes 1$ is a generator of the free right A-module Y, namely, $(\mathfrak{e} \otimes 1)a = \mathfrak{e} \otimes a$ for all $a \in A$,

(3) there is a graded algebra automorphism μ of A such that

$$a(\mathfrak{e} \otimes 1) = \mathfrak{e} \otimes \mu(a)$$

for all $a \in A$,

(4) $\mu\big(\Xi^r_g(h) \cdot a\big) = \Xi^l_g(h) \cdot \mu(a)$, where Ξ^r_π is the right winding automorphism of H associated to g, defined to be

$$\Xi^r_g : h \mapsto \sum h_1 g(h_2) \tag{E1.10.1}$$

for all $h \in H$.

In this case, we write

$$Y = (\Bbbk\mathfrak{e}) \otimes {}^{\mu}A^{1}. \tag{E1.10.2}$$

When Y is $R^d\Gamma_{\mathfrak{m}}(A)^*$ for an AS Gorenstein ring A, μ is the Nakayama automorphism of A.

The proof of the above lemma is easy and omitted. If we want to specify the algebra A, (E1.10.2) can be written as

$$Y_A = (\Bbbk\mathfrak{e}_A) \otimes {}^{\mu}A^{1}. \tag{E1.10.3}$$

Definition 1.11. Suppose a Hopf algebra H acts inner faithfully and homogeneously on a connected graded algebra A. Let R be A^H.

(1) The *left covariant module* of the H-action on A is defined to be

$$A^{l,cov\,H} := A/AR_{\geq 1},$$

which is a left A and right R-bimodule.

(2) The *right covariant module* of the H-action on A is defined to be

$$A^{r,cov\,H} := A/R_{\geq 1}A,$$

which is a right A and left R-bimodule.

(3) The *covariant algebra* of the H-action on A is defined to be the factor ring

$$A^{cov\,H} := A/(R_{\geq 1}).$$

(4) We say the H-action on A is *tepid* if $AR_{\geq 1} = R_{\geq 1}A$. In this case we say the covariant ring $A^{cov\,H}$ is *tepid*.

There are reflection Hopf algebras H such that the H-action on A is not tepid and the covariant ring $A^{cov\,H}$ is not Frobenius, see Example 4.2.

2 The Jacobian and the Reflection Arrangement

In this section we will introduce two important concepts for Hopf algebra actions on Artin–Schelter regular algebras: the Jacobian and the reflection arrangement. We also study the connection between the Jacobian and the pertinency ideal.

As in the previous sections, H is a semisimple Hopf algebra. In this section we will use the homological determinant [31, Definition 3.3] in a slightly more general situation. Assume that A is a noetherian connected graded AS Gorenstein algebra (which is not necessarily regular). Let hdet denote both the homological determinant $\mathrm{hdet} : H \to \Bbbk$ and the corresponding grouplike element in K (in [31] it is called co-determinant). As usual, suppose that H acts on A homogeneously and inner faithfully.

To motivate our definition, we first briefly recall some facts in the commutative situation. Let A be the commutative polynomial ring $\Bbbk[V^*] = \Bbbk[x_1, \cdots, x_n]$ and G be a finite subgroup of $\mathrm{GL}(V)$ acting on A naturally. Suppose that G is a reflection group and $R := A^G$ is a polynomial ring, written as $\Bbbk[f_1, \cdots, f_n]$. Then the *Jacobian* J (also called the *Jacobian determinant*) of the basic invariants $\{f_1, \cdots, f_n\}$ is defined to be

$$J := \det \left(\frac{\partial f_i}{\partial x_j} \right)_{i,j=1}^{n},$$

see [24, Introduction]. It is well known that $\deg J = -n + \sum_{i=1}^{n} \deg(f_i)$ and that $g \cdot J = (\det g)^{-1} J$ for all $g \in G$, see [46, p. 139] or [40, p.229]. In the commutative case, we have $\mathrm{hdet} = \det$. It is also well known that $A_{\mathrm{hdet}^{-1}}$ is free over R on both sides and the lowest degree of nonzero elements in $A_{\mathrm{hdet}^{-1}}$ is $-n + \sum_{i=1}^{n} \deg(f_i)$. Hence, $A_{\mathrm{hdet}^{-1}} = JR = RJ$ [46, p.139].

A result of Steinberg [24, 47] says that the Jacobian determinant J in the commutative case is a product of linear forms (with multiplicities) that correspond to the reflecting hyperplanes (E0.3.1). The product of the distinct linear forms, denoted by a, corresponding to the reflecting hyperplanes, namely, the reduced defining equation of the Jacobian determinant (E0.3.2), has the property that $g \cdot \mathsf{a} = \det(g)\mathsf{a}$ for all $g \in G$ and the degree of a is the lowest degree of nonzero elements in A_{hdet}. This means that $A_{\mathrm{hdet}} = \mathsf{a}R = R\mathsf{a}$, see [46, Theorem 2.3] and [40, p. 229].

The following definition attempts to mimic these classical concepts in the noncommutative setting. See Definition 0.3 under Hypotheses 0.1.

Definition 2.1. Let A be AS Gorenstein, $\mathrm{hdet} \in K$ be the homological determinant of the H-action on A and $R = A^H$.

(1) If $A_{\mathrm{hdet}^{-1}}$ is free of rank one over R on both sides, namely, $A_{\mathrm{hdet}^{-1}} = f_{\mathrm{hdet}^{-1}} R = R f_{\mathrm{hdet}^{-1}} \neq 0$, then the *Jacobian* of the H-action on A is defined to be

$$\mathsf{j}_{A,H} :=_{\Bbbk^\times} f_{\mathrm{hdet}^{-1}} \in A.$$

(2) If A_{hdet} is free of rank one over R on both sides and $A_{\text{hdet}} = f_{\text{hdet}} R = R f_{\text{hdet}} \neq 0$, the *reflection arrangement* of the H-action on A is defined to be

$$\mathsf{a}_{A,H} :=_{\Bbbk^\times} f_{\text{hdet}} \in A.$$

In the above definition we do not assume that the fixed subring A^H is Artin–Schelter regular. Next we give some easy examples; in (1) and (3) A^H is not AS regular, but the Jacobian and the reflection arrangement are still defined.

Example 2.2.

(1) If hdet is trivial, then both $\mathsf{j}_{A,H}$ and $\mathsf{a}_{A,H}$ are $1 \in A$.

(2) [33, Example 3.7] In [33] we assume that char $\Bbbk = 0$, but, in fact, it suffices to assume that char $\Bbbk \neq 2$ in this example. Let G be the dihedral group of order 8. It is generated by r of order 2 and ρ of order 4 subject to the relation $r\rho = \rho^3 r$. Let A be generated by x, y, z subject to the relations

$$zx = -xz,$$

$$yx = zy,$$

$$yz = xy.$$

Then A is an AS regular algebra of global dimension 3. Let $H = (\Bbbk G)^*$ and define the G-degree of the generators of A as

$$\deg_G(x) = r, \quad \deg_G(y) = r\rho, \quad \deg_G(z) = r\rho^2.$$

Then $\Bbbk G$ coacts on A. By [33, Example 3.7], the Hopf algebra H acts on A as a (true) reflection Hopf algebra and the fixed subring A^H is isomorphic to the polynomial ring $\Bbbk[t_1, t_2, t_3]$, which is AS regular. (Note that $t_1 = x^2, t_2 = y^2, t_3 = z^2$.) One can check that $\text{hdet} = \text{hdet}^{-1} = r\rho^3$ (so H is a true reflection Hopf algebra) and that

$$\mathsf{j}_{A,H} = \mathsf{a}_{A,H} =_{\Bbbk^\times} zxy =_{\Bbbk^\times} zyz =_{\Bbbk^\times} xyx =_{\Bbbk^\times} xzy =_{\Bbbk^\times} yzx =_{\Bbbk^\times} yxz, \tag{E2.2.1}$$

which is a product of elements of degree 1. By [33, Theorem 3.5(2)], the covariant algebra $A^{cov\, H}$ is always tepid in this setting.

(2) Let us recall the notation introduced in (E0.9.1) and (E0.9.2). For any homogeneous element $f \in A$, define the set of left (respectively, right) divisors of degree 1 of f to be

$$\mathfrak{R}^l(f) := \{\Bbbk v \mid v \in A_1, vf_v = f \text{ for some } f_v \in A\}.$$

and

$$\mathfrak{R}^r(f) := \{\Bbbk v \mid v \in A_1, f_v v = f \text{ for some } f_v \in A\}.$$

It is clear that $\mathfrak{R}^l(\mathsf{j}_{A,H})$ contains $\{\Bbbk x, \Bbbk y, \Bbbk z\}$. By using the fact that y is normal, one can show (with details omitted) that if $yxz =_{\Bbbk^\times} f_1 f_2 f_3$ for three elements f_i of degree 1, then $f_1 f_2 f_3$ must be, up to scalars on f_i, one of the expressions given in (E2.2.1). Therefore,

$$\mathfrak{R}^l(\mathsf{j}_{A,H}) = \mathfrak{R}^l(\mathsf{a}_{A,H}) = \mathfrak{R}^r(\mathsf{j}_{A,H}) = \mathfrak{R}^r(\mathsf{a}_{A,H}) = \{\Bbbk x, \Bbbk y, \Bbbk z\}. \tag{E2.2.2}$$

One might consider the set $\{\Bbbk x, \Bbbk y, \Bbbk z\}$ as (linear forms of) reflecting hyperplanes.

(3) Let A be the down-up algebra

$$\mathbb{D}(0,1) := \frac{\Bbbk\langle u, d\rangle}{(u^2 d - du^2, ud^2 - d^2 u)}.$$

Then A is noetherian, AS regular of global dimension 3. Let H be the Hopf algebra $(\Bbbk G)^*$ where G is the dihedral group of order 8 as in part (2). This is the setting in [17, Example 2.1]. By [17, Example 2.1], we have hdet $=$ hdet^{-1} $= \rho^2$. The fixed subring A^H is not AS regular but is AS Gorenstein. By [17, Lemma 2.2(3)], the Jacobian and the reflection arrangement of the H-action on A are

$$\mathsf{a}_{A,H} = \mathsf{j}_{A,H} =_{\Bbbk^\times} u^2 \in A.$$

One can show directly that the covariant algebra $A^{cov\, H}$ is tepid.

Remark 2.3. The definition of the Jacobian in Definition 2.1(1) agrees with the Jacobian (determinant) when we consider classical reflection groups acting on commutative polynomial rings.

(1) In the commutative case, both $\mathsf{j}_{A,H}$ and $\mathsf{a}_{A,H}$ are products of linear forms (E0.3.1) and (E0.3.2). It is natural to ask if A is generated in degree 1, under what hypotheses, are both $\mathsf{j}_{A,H}$ and $\mathsf{a}_{A,H}$ products of elements of degree 1?

(2) In the commutative case one sees from (E0.3.1) and (E0.3.2) that $\mathsf{a}_{A,H}$ divides $\mathsf{j}_{A,H}$. Is there a generalization of this statement in the noncommutative setting? We will discuss this question in Section 3 (see Theorem 3.8(2)).

(3) More importantly, the definitions of the Jacobian and the reflection arrangement suggest that we should search for a generalization of hyperplane arrangements in the noncommutative setting.

(4) In the classical case, $\mathsf{a}_{A,H}$ is reduced, namely, every factor is squarefree in $\mathsf{a}_{A,H}$. What is the analog of this statement? See Example 2.2(2,3).

Next we have a result concerning the existence of $\mathsf{j}_{A,H}$. Let $\pi : H \to \Bbbk$ be an algebra homomorphism, namely, $\pi \in K$ is a grouplike element. Recall from (E1.10.1) that the right winding automorphism of H associated to π is defined to be

$$\Xi^r_\pi : h \mapsto \sum h_1 \pi(h_2)$$

for all $h \in H$. The left winding automorphism Ξ^l_π of H associated to π is defined similarly, and it is well known that both Ξ^r_π and Ξ^l_π are algebra automorphisms of H. For any element $x \in A$, let η_x denote the "conjugation" map

$$\eta_x : a \to x^{-1} a x$$

whenever $x^{-1}ax$ is defined. In particular, this map could be defined for all a in a subring of A. In the following result we do not assume that the H-action on A is inner faithful.

Recall that

$$\sum \operatorname{hdet}(h_1)\operatorname{hdet}^{-1}(h_2) = \sum \operatorname{hdet}(h_2)\operatorname{hdet}^{-1}(h_1) = \epsilon(h) \tag{E2.3.1}$$

for every $h \in H$.

Theorem 2.4. Let A be a noetherian AS Gorenstein algebra. Let hdet be the homological determinant of the H-action on A.

(1) [43, Lemma 3.10] Let μ be the Nakayama automorphism of A. Then, for every $a \in A$ and $h \in H$,

$$\Xi^l_{\mathrm{hdet}}(h) \cdot \mu(a) = \mu\left(\Xi^r_{\mathrm{hdet}}(h) \cdot a\right). \tag{E2.4.1}$$

As a consequence, $\mu(A^H) = A^H$.

(2) If A^H is AS Gorenstein, then the Jacobian $\mathsf{j}_{A,H}$ is defined and

(a) $\mathfrak{l}_{A^H} = \mathfrak{l} + \deg \mathsf{j}_{A,H}$, where $\mathfrak{l}$ indicates the respective AS indices Definition 1.1(c),

(b) $\mu_{A^H} = \eta_{\mathsf{j}_{A,H}} \circ \mu$.

(3) If $\mathsf{j}_{A,H}$ exists, then A^H is AS Gorenstein.

(4) $\mu(A_{\mathrm{hdet}^{-1}}) = A_{\mathrm{hdet}^{-1}}$. As a consequence, if $\mathsf{j}_{A,H}$ exists, then $\mu(\mathsf{j}_{A,H}) =_{\Bbbk^\times} \mathsf{j}_{A,H}$.

(5) $\mu(A_{\mathrm{hdet}}) = A_{\mathrm{hdet}}$. As a consequence, if $\mathsf{a}_{A,H}$ exists, then $\mu(\mathsf{a}_{A,H}) =_{\Bbbk^\times} \mathsf{a}_{A,H}$.

(6) Let A be a domain. Suppose there is a short exact sequence of Hopf algebras

$$1 \to H_0 \to H \to \overline{H} \to 1$$

such that A^H and A^{H_0} are AS Gorenstein. Then

$$\mathsf{j}_{A,H} =_{\Bbbk^\times} \mathsf{j}_{A,H_0}\mathsf{j}_{A^{H_0},\overline{H}} =_{\Bbbk^\times} \mathsf{j}_{A^{H_0},\overline{H}}\mathsf{j}_{A,H_0}.$$

Proof. (1) Let R denote A^H. The 1st claim is a special case of [43, Lemma 3.10] when the antipode S of H has the property that S^2 is the identity. (Note that since H is semisimple, S^2 is the identity.) For the consequence, we have, for $h \in H$ and $r \in R$,

$$\begin{aligned} h \cdot \mu(r) &= \sum \mathrm{hdet}(h_2)h_1 \cdot \mu(r)\ \mathrm{hdet}^{-1}(h_3) \\ &= \sum \Xi^r_{\mathrm{hdet}}(h_1) \cdot \mu(r)\ \mathrm{hdet}^{-1}(h_2) \\ &= \mu\left(\sum \Xi^l_{\mathrm{hdet}}(h_1) \cdot r\ \mathrm{hdet}^{-1}(h_2)\right) \\ &= \mu\left(\sum \mathrm{hdet}(h_1)h_2 \cdot r\ \mathrm{hdet}^{-1}(h_3)\right) \\ &= \mu\left(\sum \mathrm{hdet}(h_1)\epsilon(h_2)r\ \mathrm{hdet}^{-1}(h_3)\right) \\ &= \epsilon(h)\mu(r). \end{aligned}$$

This implies that $\mu(r) \in R$, and completes the proof of part (1). In the above computation we used (E2.3.1).

We will use the notation introduced in [31]. Let $\mathrm{H}^i_{\mathfrak{m}}(A)$ be the ith local cohomology of A with respect to the graded maximal ideal $\mathfrak{m} := A_{\geq 1}$. Let $(-)^*$ denote the graded $\Bbbk$-linear dual of a graded vector space. Let d be the injective dimension of A. By [31, p.3648] or (E1.2.3),

$$(\mathrm{H}^i_{\mathfrak{m}}(A))^* = \begin{cases} 0 & i \neq d \\ {}^{\mu}A^1(-\mathfrak{l}) =: Y & i = d, \end{cases}$$

and

$$(\mathrm{H}^i_{\mathfrak{m}^R}(R))^* = \begin{cases} 0 & i \neq d \\ Y \cdot \textstyle\int = S(\textstyle\int) \cdot Y & i = d. \end{cases}$$

As a consequence, the injective dimension of R is also d if R is AS Gorenstein. Here μ is the Nakayama automorphism of A, and μ_R is the Nakayama automorphism of R. Note that Y has an A-bimodule structure with compatible H-action, or in other words, Y is an H-equivariant A-bimodule in the sense of [43, Definition 2.2], see [43, Lemma 3.2(a)].

Using the notation in [31, (3.2.1) and (3.2.2)] or in Lemma 1.10, $Y = (\Bbbk\mathfrak{e}) \otimes {}^{\mu}A^1$ as a left H-module (as well as graded A-bimodule) where $\deg(\mathfrak{e}) = \mathfrak{l}$ and the H-action on $\mathfrak{e}$ is given by

$$h \cdot \mathfrak{e} = \mathrm{hdet}(h)\mathfrak{e} \tag{E2.4.2}$$

by [31, Definition 3.3]. (In [31], the authors used the right H-action, one can easily transfer to the left action by composing with the antipode S.) By [31, Lemma 2.4(1)], there is an R-bimodule decomposition

$$A = R \oplus C, \tag{E2.4.3}$$

where $R \subseteq A$ is a graded subalgebra. Further, as a left H-module, R is a direct sum of trivial H-modules, and,

$$R = \{a \in A \mid p_1 \cdot a = a\};$$

and

$$C = \{a \in A \mid (1 - p_1) \cdot a = a\},$$

where p_1 is the idempotent in (E1.3.1) corresponding to the integral of H. The decomposition (E2.4.3) gives rise to a decomposition of Y, as R-bimodules,

$$Y = \left(\mathrm{H}^d_{\mathfrak{m}}(A)\right)^* = \left(\mathrm{H}^d_{\mathfrak{m}^R}(A)\right)^* = \left(\mathrm{H}^d_{\mathfrak{m}^R}(R)\right)^* \oplus \left(\mathrm{H}^d_{\mathfrak{m}^R}(C)\right)^*, \tag{E2.4.4}$$

where $\left(\mathrm{H}^d_{\mathfrak{m}^R}(R)\right)^*$ is preserved by the left action of p_1 and $(\mathrm{H}^d_{\mathfrak{m}^R}(C))^*$ is preserved by the left action of $1-p_1$. Using the fact, $Y = (\Bbbk\mathfrak{e}) \otimes {}^{\mu}A^1$, we can write

$$\left(\mathrm{H}^d_{\mathfrak{m}^R}(R)\right)^* = (\Bbbk\mathfrak{e}) \otimes V, \quad \text{and} \quad \left(\mathrm{H}^d_{\mathfrak{m}^R}(C)\right)^* = (\Bbbk\mathfrak{e}) \otimes W$$

for some graded R-bimodules V, W with ${}^{\mu}A^1 = V \oplus W$.

(2) Assume that R is AS Gorenstein. Then the R-bimodule $(\mathrm{H}^d_{\mathfrak{m}^R}(R))^*$ is isomorphic to ${}^{\mu_R}R^1(-\mathfrak{l}_R)$. In particular, $(\mathrm{H}^d_{\mathfrak{m}^R}(R))^*$ is free of rank one on both sides. This implies that V is a free R-module of rank one on both sides.

Since $(\mathrm{H}^d_{\mathfrak{m}^R}(R))^*$ is preserved by the left action of p_1 and $(\mathrm{H}^d_{\mathfrak{m}^R}(C))^*$ is preserved by the left action of $1-p_1$, by (E2.4.2), V is preserved by the left action of $p_{\mathrm{hdet}^{-1}}$ and W is preserved by the left action of $1-p_{\mathrm{hdet}^{-1}}$. Thus, $V = {}^{\mu}A^1{}_{\mathrm{hdet}^{-1}} = A_{\mathrm{hdet}^{-1}}$ where the last equation follows from the fact that the H-action on ${}^{\mu}A^1$ agrees with the H-action on A. Combining these assertions with ones in the last paragraph, we obtain that $\mathsf{j}_{A,H}$ exists.

For the two sub-statements, note that the right R-module $(\mathrm{H}^d_{\mathfrak{m}^R}(R))^*$ is free with a generator $\mathfrak{e} \otimes \mathsf{j}_{A,H}$. Using the notation introduced in (E1.10.3), we have

$$\mathfrak{e}_R = \mathfrak{e} \otimes \mathsf{j}_{A,H} = \mathfrak{e}_A \otimes \mathsf{j}_{A,H}. \tag{E2.4.5}$$

Then

$$\mathfrak{l}_R = \deg \mathfrak{e}_R = \deg(\mathfrak{e}_A \otimes \mathsf{j}_{A,H}) = \deg \mathfrak{e}_A + \deg \mathsf{j}_{A,H} = \mathfrak{l}_A + \deg \mathsf{j}_{A,H}.$$

Hence, sub-statement (a) follows. Considering elements inside $Y := \mathfrak{e} \otimes A$, for every $r \in R$, using part (1), we have

$$\begin{aligned} r(\mathfrak{e}_R \otimes 1) &= r(\mathfrak{e} \otimes \mathsf{j}_{A,H} 1) \\ &= \mathfrak{e} \otimes \mu(r)\mathsf{j}_{A,H} = \mathfrak{e} \otimes \mathsf{j}_{A,H}\left(\mathsf{j}_{A,H}^{-1}\mu(r)\mathsf{j}_{A,H}\right) \\ &= (\mathfrak{e}_R \otimes 1)\left(\mathsf{j}_{A,H}^{-1}\mu(r)\mathsf{j}_{A,H}\right), \end{aligned}$$

which implies that $\mu_R(r) = \eta_{\mathsf{j}_{A,H}} \circ \mu(r)$; hence, we have verified sub-statement (b).

(3) The proof of the converse is similar. Since $\mathsf{j}_{A,H}$ is defined, $V := A_{\mathrm{hdet}^{-1}}$ is a free R-module of rank one on both sides. Then $(\mathrm{H}^d_{\mathfrak{m}^R}(R))^* = (\Bbbk\mathfrak{e}) \otimes V$ is a free R-module of rank one on both sides. By [43, Lemma 1.7(2)], $(\mathrm{H}^d_{\mathfrak{m}^R}(R))^*$ is isomorphic to ${}^{\mu_R}R^1(-\mathfrak{l}_R)$ for some automorphism μ_R of R and some integer $\mathfrak{l}$. By [31, Lemma 1.6], R is AS Gorenstein.

(4) For $r \in A_{\mathrm{hdet}^{-1}}$ and $h \in H$, we have

$$\begin{aligned}
h \cdot \mu(r) &= \sum \mathrm{hdet}^{-1}(h_1)\ \mathrm{hdet}(h_2) h_3 \cdot \mu(r) \\
&= \sum \mathrm{hdet}^{-1}(h_1) \Xi^l_{\mathrm{hdet}}(h_2) \cdot \mu(r) \\
&= \sum \mathrm{hdet}^{-1}(h_1) \mu(\Xi^r_{\mathrm{hdet}}(h_2) \cdot r) && \text{by (E2.4.1)} \\
&= \sum \mathrm{hdet}^{-1}(h_1) \mu \left(\sum \mathrm{hdet}(h_3) h_2 \cdot r \right) \\
&= \sum \mathrm{hdet}^{-1}(h_1) \mu \left(\sum \mathrm{hdet}(h_3) \mathrm{hdet}^{-1}(h_2) r \right) \\
&= \sum \mathrm{hdet}^{-1}(h_1) \mu \left(\sum \epsilon(h_2) r \right) \\
&= \mathrm{hdet}^{-1}(h) \mu(r).
\end{aligned}$$

Hence, the main assertion follows, and the consequence is clear.

(5) For $r \in A_{\mathrm{hdet}}$ and $h \in H$, we have

$$\begin{aligned}
h \cdot \mu(r) &= \sum \mathrm{hdet}^{-1}(h_1)\ \mathrm{hdet}(h_2) h_3 \cdot \mu(r) \\
&= \sum \mathrm{hdet}^{-1}(h_1) \Xi^l_{\mathrm{hdet}}(h_2) \cdot \mu(r) \\
&= \sum \mathrm{hdet}^{-1}(h_1) \mu \left(\Xi^r_{\mathrm{hdet}}(h_2) \cdot r\right) && \text{by (E2.4.1)} \\
&= \sum \mathrm{hdet}^{-1}(h_1) \mu \left(\sum \mathrm{hdet}(h_3) h_2 \cdot r \right) \\
&= \sum \mathrm{hdet}^{-1}(h_1) \mu \left(\sum \mathrm{hdet}(h_3)\ \mathrm{hdet}(h_2) r \right) \\
&= \sum \mathrm{hdet}^{-1}(h_1)\ \mathrm{hdet}(h_2)\ \mathrm{hdet}(h_3) \mu(r) \\
&= \sum \epsilon(h_1)\ \mathrm{hdet}(h_2) \mu(r) = \mathrm{hdet}(h) \mu(r).
\end{aligned}$$

Hence, the main assertion follows, and the consequence is clear.

(6) Let $r \in H_0$ and $h \in H$. Since H_0 is normal, $\sum S(h_1) r h_2 \in H_0$, and for all $x \in A^{H_0}$, we have

$$\sum S(h_1) r h_2(x) = \epsilon \left(\sum S(h_1) r h_2 \right)(x) = \epsilon(r) \epsilon(h) x.$$

Then

$$rh(x) = \sum h_1 S(h_2) r h_3(x) = \sum h_1 \epsilon(h_2) \epsilon(r)(x) = \epsilon(r)(h(x)),$$

which implies that A^{H_0} is a left H-module algebra. By the definition, H_0-action on A^{H_0} is trivial, so $\overline{H}$ acts on A^{H_0} naturally and

$$A^H = (A^{H_0})^H = (A^{H_0})^{\overline{H}}.$$

By (E2.4.5),

$$\mathfrak{e}_{A^H} = \mathfrak{e}_A \otimes \mathrm{j}_{A,H}, \qquad \text{and}$$

$$\begin{aligned}\mathfrak{e}_{A^H} &= \mathfrak{e}_{A^{H_0}} \otimes \mathrm{j}_{A^{H_0},\overline{H}}\\ &= \left(\mathfrak{e}_A \otimes \mathrm{j}_{A,H_0}\right) \otimes \mathrm{j}_{A^{H_0},\overline{H}}\\ &= \left(\mathfrak{e}_A \otimes \mathrm{j}_{A,H_0}\right)\mathrm{j}_{A^{H_0},\overline{H}}\\ &= \mathfrak{e}_A \otimes \left(\mathrm{j}_{A,H_0}\mathrm{j}_{A^{H_0},\overline{H}}\right)\end{aligned}$$

inside Y. Then

$$\mathrm{j}_{A,H} = \mathrm{j}_{A,H_0}\mathrm{j}_{A^{H_0},\overline{H}}. \tag{E2.4.6}$$

For the 2nd equation, we use part (4). Since both A^H and A^{H_0} are AS Gorenstein, by part (4), we have $\mu(\mathrm{j}_{A,H}) =_{\Bbbk^\times} \mathrm{j}_{A,H}$ and $\mu(\mathrm{j}_{A,H_0}) =_{\Bbbk^\times} \mathrm{j}_{A,H_0}$. Applying μ to the equation $\mathrm{j}_{A,H} = \mathrm{j}_{A,H_0}\mathrm{j}_{A^{H_0},\overline{H}}$, and using the hypothesis that A is a domain, we obtain that

$$\mu(\mathrm{j}_{A^{H_0},\overline{H}}) =_{\Bbbk^\times} \mathrm{j}_{A^{H_0},\overline{H}}. \tag{E2.4.7}$$

Applying $\mu_{A^{H_0}}$ to $\mathrm{j}_{A^{H_0},\overline{H}}$ and using part (4), we have

$$\mu_{A^{H_0}}(\mathrm{j}_{A^{H_0},\overline{H}}) =_{\Bbbk^\times} \mathrm{j}_{A^{H_0},\overline{H}}. \tag{E2.4.8}$$

Combining (E2.4.7), (E2.4.8) with part (2b),

$$\mathrm{j}_{A^{H_0},\overline{H}} =_{\Bbbk^\times} \eta_{\mathrm{j}_{A,H}}(\mathrm{j}_{A^{H_0},\overline{H}})$$

or equivalently,

$$\mathrm{j}_{A^{H_0},\overline{H}}\mathrm{j}_{A,H} =_{\Bbbk^\times} \mathrm{j}_{A,H}\mathrm{j}_{A^{H_0},\overline{H}}. \tag{E2.4.9}$$

Since A is a domain, the combination of (E2.4.6) and (E2.4.9) implies that

$$\mathrm{j}_{A^{H_0},\overline{H}}\mathrm{j}_{A,H_0} =_{\Bbbk^\times} \mathrm{j}_{A,H_0}\mathrm{j}_{A^{H_0},\overline{H}}$$

as desired. ■

Theorem 2.4(6) is useful for the case when H is obtained by an abelian extension of Hopf algebras. We wonder if there is a version of Theorem 2.4(6) for $\mathsf{a}_{A,H}$. Theorem 2.4(2b) is a generalization of [33, Theorem 0.6(1)]. Though the Jacobian exists, it is not clear if the reflection arrangement exists when R is AS Gorenstein. We have three corollaries, including the existence of the reflection arrangement when R is AS regular. The 1st of the corollaries is Theorem 0.2(1,2).

Corollary 2.5. Assume Hypotheses 0.1. Let $R = A^H$ and $\xi(t) = h_A(t)(h_R(t))^{-1}$.

(1) Both $\mathsf{j}_{A,H}$ and $\mathsf{a}_{A,H}$ exist.

(2) $\deg \xi(t) = \deg \mathsf{j}_{A,H}$.

(3) $h_{A^{l,cov\,H}}(t) = h_{A^{r,cov\,H}}(t) = \xi(t)$. As a consequence,

$$\dim A^{l,cov\,H} = \dim A^{r,cov\,H} = \xi(1),$$

where $h_{A^{l,cov\,H}}(t)$ and $h_{A^{r,cov\,H}}(t)$ are the Hilbert series of $A^{l,cov\,H}$ and $A^{r,cov\,H}$.

Proof. (1) By Theorem 2.4, the Jacobian $\mathsf{j}_{A,H}$ exists. In particular, $A_{\mathrm{hdet}^{-1}} \neq 0$. Since K is finite dimensional, hdet is a power of hdet^{-1}. So $A_{\mathrm{hdet}} \neq 0$. By Lemma 1.5(6), A_{hdet} is a free R-module of rank one on both sides, and hence by definition, $\mathsf{a}_{A,H}$ exists.

(2) Let $p_A(t) = (h_A(t))^{-1}$ and $p_R(t) = (h_R(t))^{-1}$. By [48, Proposition 3.1], $\deg p_A(t) = \mathfrak{l}_A$ and $\deg p_R(t) = \mathfrak{l}_R$. By Theorem 2.4(2a),

$$\deg \mathsf{j}_{A,H} = \mathfrak{l}_R - \mathfrak{l}_A = \deg p_R(t) - \deg p_A(t) = \deg \xi(t).$$

(3) Since A_R is a finitely generated free R-module, $h_A(t) = h_{A^{l,cov\,H}}(t) h_R(t)$, and the assertion follows. The consequence is clear. ∎

The next corollary is a rigidity result.

Corollary 2.6. Let A be a noetherian AS Gorenstein algebra with finite GKdimension. Suppose H acts on A such that A^H is AS Gorenstein.

(1) Suppose A is Cohen–Macaulay. If hdet is not trivial, then $\mathsf{p}(A,H) \leq 1$, where $\mathsf{p}(A,H)$ is defined in Definition 2.8(3).

(2) Suppose that there is no graded ideal $I \subseteq A$ such that $GKdimA/I = GKdimA - 1$. Then hdet is trivial.

(3) If A is projectively simple in the sense of [42, Definition 1.1] and if $GKdim A \geq 2$, then there is no graded ideal $I \subseteq A$ such that $GKdim A/I = GKdim A - 1$.

Proof. (1) If hdet is not trivial, then there is an R-bimodule C such that

$$A = A^H \oplus A_{\mathrm{hdet}} \oplus C = R \oplus \mathsf{j}_{A,H} R \oplus C$$

[Theorem 2.4(2)]. Then $End_R(A)$ is not $\mathbb{N}$-graded. Therefore, the natural map $A\#H \to End_R(A)$ cannot be an isomorphism of a graded algebras. By [7, Theorem 3.5], $\mathsf{p}(A,H) \leq 1$.

(2) Suppose to the contrary that hdet is not trivial. By Theorem 2.4, $f := \mathsf{j}_{A,H} \in A_{\geq 1}$ exists. By definition, $fR = Rf$ inside A. Consider the (A,R)-bimodule $M := A/Af$, which is finitely generated on both sides; we have $GKdim(M) = GKdim A - 1$. Let $I = ann_A({}_A M)$. Since M_R is finitely generated, $M = \sum_{i=1}^{s} m_i R$. Then $I = \bigcap ann_A(m_i)$. For each i, $GKdim(A/ann_A(m_i)) \leq GKdim\, M$. Then $GKdim\, A/I \leq GKdim\, M$. Since $I \subseteq Af$, $GKdim\, A/I \geq GKdim\, M$. Therefore, $GKdim\, A/I = GKdim\, M = GKdim\, A - 1$, a contradiction.

(3) This is clear from the definition of a projectively simple ring (also called a just-infinite ring). ■

The 3rd corollary puts some constraints on the homological determinant hdet. Recall from [43, p. 318] that an AS Gorenstein algebra A is called *r-Nakayama*, for some $r \in \Bbbk^{\times}$, the Nakayama automorphism of A is of the form

$$\mu : a \to r^{\deg a} a \tag{E2.6.1}$$

for all homogeneous element $a \in A$. For example, every Calabi–Yau AS regular algebra is 1-Nakayama.

Corollary 2.7. Let A be a noetherian AS Gorenstein algebra that is r-Nakayama for some $r \in \Bbbk^{\times}$. (We need only that $\Xi^{r}_{\mathrm{hdet}}(h)$ is a stable map of μ-isotropy classes.)

(1) Assume that the H-action on A is faithful. Then hdet is a central element in $G(K)$. As a consequence, if the center of $G(K)$ is trivial, then hdet is trivial and A^H is AS Gorenstein.

(2) Suppose that A is an AS regular domain and that $G \neq \{1\}$ is a finite group with trivial center (e.g., G is non-abelian simple). If $H := (\Bbbk G)^*$ acts on A inner faithfully and homogeneously such that A is an H-module algebra, then hdet is trivial and H is not a reflection Hopf algebra in the sense of Definition 1.4.

Proof. (1) Under hypothesis of μ being r-Nakayama and the fact that μ is a graded algebra homomorphism, (E2.4.1) becomes

$$\Xi^r_{\mathrm{hdet}}(h) \cdot a = \Xi^l_{\mathrm{hdet}}(h) \cdot a$$

for all $a \in A$ and $h \in H$. Since the H-action is faithful, we have $\Xi^r_{\mathrm{hdet}}(h) = \Xi^l_{\mathrm{hdet}}(h)$ for all $h \in H$. Applying $g \in G(K)$ to the above equation, we obtain that $(g \circ \mathrm{hdet})(h) = (\mathrm{hdet} \circ g)(h)$. Thus, hdet commutes with all elements $g \in G(K)$. This shows the main assertion, and the consequence is clear.

(2) By Lemma 1.5(4), $G_0 := \{g \in g \mid A_g \neq 0\}$ is a subgroup of G. Since the H-action on A is inner faithful, the K-coaction on A is inner faithful. Thus, $G_0 = G$. This implies that H-action on A is in fact faithful. By part (1), hdet is trivial. By [14, Theorem 0.6], A^H is not AS regular; hence, H is not a reflection Hopf algebra. ■

Definition 2.8. Let H act on A and $\int$ be the integral of H.

(1) The *pertinency ideal* of the H-action on A is defined to be

$$\mathcal{P}_{A,H} := (A\#H)(1\#\textstyle\int)(A\#H) \subseteq A\#H.$$

(2) [25, Definition 1.4] The *radical ideal* of the H-action on A is defined to be

$$\mathfrak{r}_{A,H} := \mathcal{P}_{A,H} \cap A$$

identifying A with $A\#1 \subseteq A\#H$.

(3) [7, Definition 0.1] The *pertinency* of the H-action on A is defined to be

$$\mathsf{p}(A,H) := GKdim(A\#H) - GKdim(A\#H/\mathcal{P}_{A,H}).$$

The *radical ideal* of a group G-action on an algebra A was introduced in [25, Definition 1.4] using pertinence sequences. By the proof of [25, Proposition 2.4], that definition agrees with Definition 2.8(2) when H is a group algebra.

Under some mild hypotheses, we will show that the radical ideal is essentially the Jacobian of the H-action on A when H is a reflection Hopf algebra. For simplicity, let m stand for hdet^{-1} following the notation of [33].

From now on until Theorem 2.12, let $H = (\Bbbk G)^*$ for some finite group G. Assume that $\operatorname{char} \Bbbk = 0$. Then the integral $\int$ of H is p_1 where 1 is the identity of G. Since $H = \bigoplus_{g\in G} \Bbbk p_g$, we have $A = \oplus_{g\in G} A_g$ where $A_g = p_g \cdot A$. By using the comultiplication given in (E1.3.5), one easily checks that the following equations hold.

Lemma 2.9. Let $H = (\Bbbk G)^*$, $g, h \in G$ and $b_h \in A_h$. Then

(1) $(b_h\#1)(1\#p_g) = b_h\#p_g$.
(2) $(1\#p_g)(b_h\#1) = b_h\#p_{h^{-1}g}$.
(3) $(1\#p_{hg})(b_h\#1) = b_h\#p_g$.

Lemma 2.10. Let $\int$ be the integral of $H = (\Bbbk G)^*$. Then

$$(A\#1) \cap (A\#H)(1\#\textstyle\int)(A\#H) = \left(\bigcap_{g\in G} AA_g\right)\#1.$$

As a consequence,

$$\mathfrak{r}_{A,H} = \bigcap_{g\in G} AA_g.$$

Proof. We compute

$$\begin{aligned}(A\#H)(1\#\textstyle\int)(A\#H) &= \left(\displaystyle\sum_h A\#p_h\right)(1\#p_{1_G})\left(\displaystyle\sum_{i,j} A_i\#p_j\right)\\ &= (A\#1)(1\#p_{1_G})\left(\displaystyle\sum_{i,j} A_i\#p_j\right)\\ &= (A\#1)\left(\displaystyle\sum_{i,j} A_i\#p_{i^{-1}}p_j\right)\\ &= \displaystyle\sum_i AA_i\#p_{i^{-1}}.\end{aligned}$$

If $x \in (A\#1) \cap (A\#H)(1\#\textstyle\int)(A\#H)$, then $x = y\#1 = y\#\sum_i p_{i^{-1}}$ for $y \in A$. By the above computation, $y \in AA_i$ for all $i \in G$. Thus, $y \in \bigcap_{g\in G} AA_g$ as required. ■

Lemma 2.11. Assume Hypotheses 0.4. Let $m := \mathrm{hdet}^{-1} \in G$.

(1) For each $g \in G$, there is a nonzero $f_g \in A$ such $A_g = Rf_g = f_gR$.

(2) For each $g \in G$, there is an $h \in G$ such that $f_h f_g =_{\Bbbk^\times} f_m$.

(3)

$$\bigcap_{g\in G} AA_g = \bigcap_{g\in G} Af_g = Af_m = f_mA.$$

Proof. (1) By [33, Theorem 3.5(1)], for each $g \in G$, $A_g = Rf_g = f_gR$ for some homogeneous element $0 \neq f_g \in A$.

(2) By [33, Theorem 3.5(2)], the covariant algebra A^{covH} [Definition 1.11] is the quotient algebra A/I where $I = \oplus_{g\in G}(A^H)_{\geq 1}f_g$, and A^{covH} is Frobenius. Further, A^{covH} has a $\Bbbk$-basis $\{\overline{f_g}\}_{g\in G}$. Since A^{covH} is graded and Frobenius, for every g, there is an $h \in G$ such that $\overline{f_h}\,\overline{f_g} = a\overline{f_m}$ for some $0 \neq a \in \Bbbk$. Then $hg = m$ and $f_h f_g = af_m$.

(3) As a consequence of part (2), $Af_m \subseteq Af_g$ for all g. Therefore, $\bigcap_{g\in G} Af_g = Af_m$. By [33, Theorem 0.5(1)], f_m is a normal element. Then $Af_m = f_mA$. This finishes the proof. ■

Now we prove Theorem 0.5, which is Theorem 2.12(2) below. Following [33], let

$$\mathfrak{G} := \{h \in G \mid \deg f_h = 1\}. \tag{E2.11.1}$$

(In [33], this set is denoted by $\mathfrak{R}$.)

Theorem 2.12. Assume Hypotheses 0.4.

(1) The radical ideal $\mathfrak{r}_{A,H}$ is a principal ideal of A generated by $\mathsf{j}_{A,H}$.

(2) Both $\mathsf{j}_{A,H}$ and $\mathsf{a}_{A,H}$ are products of elements in degree 1 of the form f_h.

(3) $\mathsf{a}_{A,H}$ divides $\mathsf{j}_{A,H}$ from the left and the right.

Proof. (1) The assertion follows from Lemmas 2.10 and 2.11(1).

(2) By [33, Theorem 3.5(5)], the covariant algebra A^{covH} is generated by elements $\{f_h \mid h \in \mathfrak{G}\}$. Using the G-grading and the fact that A^{covH} is a skew Hasse algebra [33, Definition 2.3(2)], every f_g is a product of $f_{h_1}\cdots f_{h_s}$ if $g = h_1\cdots h_s$ where $s = l_{\mathfrak{G}}(g)$ [33, Definition 2.1]. In particular, both $\mathsf{j}_{A,H}$ and $\mathsf{a}_{A,H}$ are products of elements in $\{f_h \mid h \in \mathfrak{G}\}$.

(3) See proof of Lemma 2.11(2). ■

Note that, in general, Theorem 2.12(1) fails, see (E4.2.12) and (E4.2.13). Motivated by the above result, we have the following remarks and questions, which can be viewed as a continuation of Remark 2.3.

Remark 2.13. Assume Hypotheses 0.1.

(1) What is the connection between $\mathfrak{r}_{A,H}$ and $\mathsf{j}_{A,H}$? The relation between them is not obvious, but we believe that $\mathfrak{r}_{A,H}$ is contained in $A\mathsf{j}_{A,H}$. See Lemma 3.13 for a partial result.

(2) As in Remark 2.3(2), we ask: does $\mathsf{a}_{A,H}$ divide $\mathsf{j}_{A,H}$ (from the left and the right)? The answer is yes, see Theorem 3.8(2). As a consequence, $\mathfrak{R}^l(\mathsf{a}_{A,H})$ is a subset of $\mathfrak{R}^l(\mathsf{j}_{A,H})$. This suggests another question: does the equation

$$\mathfrak{R}^l(\mathsf{a}_{A,H}) = \mathfrak{R}^l(\mathsf{j}_{A,H})$$

always hold?

(2) On the other hand, we will give an example where $\mathfrak{R}^l(\mathsf{j}_{A,H}) \neq \mathfrak{R}^r(\mathsf{j}_{A,H})$, see (E4.2.6) and (E4.2.7) in Example 4.2.

(2) One question related to this inequality is: do we have an isomorphism ϕ such that $\phi(\mathfrak{R}^l(\mathsf{j}_{A,H})) = \mathfrak{R}^r(\mathsf{j}_{A,H})$ (respectively, $\phi(\mathfrak{R}^l(\mathsf{a}_{A,H})) = \mathfrak{R}^r(\mathsf{a}_{A,H})$)?

(3) In the classical case, $\deg \mathsf{a}_{A,H}$ is the number of reflecting hyperplanes and $\deg \mathsf{j}_{A,H}$ is the number of pseudo-reflections. What are the meanings of $\deg \mathsf{j}_{A,H}$ and $\deg \mathsf{a}_{A,H}$ in the noncommutative case?

(4) Suppose that A is generated in degree 1. Are $\mathsf{a}_{A,H}$ and $\mathsf{j}_{A,H}$ products of elements of degree 1? If yes, are these products of elements in $\mathfrak{R}^l(\mathsf{a}_{A,H}) \cup \mathfrak{R}^l(\mathsf{a}_{A,H})$?

Further, assume that H is $(\Bbbk G)^*$ and that A is generated in degree 1.

(5) It follows from [33, Theorem 0.4] that $\mathfrak{G}$ can be considered as a subset of both $\mathfrak{R}^l(\mathsf{j}_{A,H})$ and $\mathfrak{R}^r(\mathsf{j}_{A,H})$. As a consequence, $|\mathfrak{G}| \leq |\mathfrak{R}^l(\mathsf{j}_{A,H})|$.

(6) Is the $\deg \mathsf{a}_{A,H} = |\mathfrak{G}|$? For example, in Example 2.2(2) $\mathfrak{G} = \{r, r\rho, r\rho^2\}$ and $\deg \mathsf{a}_{A,H} = 3$.

(7) Does the set $\{f_h \mid h \in \mathfrak{G}\}$ coincide with $\mathfrak{R}^l(\mathsf{a}_{A,H})$? In the ideal situation, we should call $\mathfrak{R}^l(\mathsf{a}_{A,H})$ the collection of "reflecting hyperplanes". In Example 2.2(2) both the "reflecting hyperplanes" and the set $\{f_h \mid h \in \mathfrak{G}\}$ are basically $\{\Bbbk x, \Bbbk y, \Bbbk z\}$. See Lemma 4.1(2) for a case when H is not $(\Bbbk G)^*$.

The Jacobian is defined even when H is not a reflection Hopf algebra and so in Example 2.2(1,3) we note the following.

Example 2.14.

(1) If the H-action on A has trivial homological determinant, then $\mathsf{j}_{A,H} = 1$, but the radical ideal $\mathfrak{r}_{A,H}$ is not the whole algebra A. As a consequence $\mathfrak{r}_{A,H} \subsetneq (\mathsf{j}_{A,H})$.

(2) In Example 2.2(3) it follows from [17, Lemma 2.2] that

$$\mathfrak{r}_{A,H} = u^2(dR + uduR)A \cap (u^3R + dudR)A \subsetneq u^2A = (\mathsf{j}_{A,H}).$$

3 Discriminants

Geometrically the discriminant locus of a reflection group G acting on $\Bbbk[V]$ is the image of reflecting hyperplanes in the corresponding affine quotient space [40, Proposition 6.106]. Algebraically, the discriminant of G is the product of Jacobian and reflection arrangement (as an element in the fixed subring $\Bbbk[V]^G$). In the noncommutative case, we can define the discriminant as the product of the Jacobian and the reflection arrangement. However, the product of two elements in a noncommutative ring is dependent on the order of these elements. Therefore, we make the following definitions.

Definition 3.1. Suppose that both the Jacobian $\mathsf{j}_{A,H}$ and the reflection arrangement $\mathsf{a}_{A,H}$ exist, namely, $A_{\mathrm{hdet}^{-1}} = \mathsf{j}_{A,H}R = R\mathsf{j}_{A,H}$ and that $A_{\mathrm{hdet}} = \mathsf{a}_{A,H}R = R\mathsf{a}_{A,H}$ where $R = A^H$.

(1) The *left discriminant* of the H-action on A, or the left *H-discriminant* of A, is defined to be

$$\delta^l_{A,H} :=_{\Bbbk^\times} \mathsf{a}_{A,H}\, \mathsf{j}_{A,H} \in R.$$

(2) The *right discriminant* of the H-action on A, or the right *H-discriminant* of A, is defined to be

$$\delta^r_{A,H} :=_{\Bbbk^\times} \mathsf{j}_{A,H}\, \mathsf{a}_{A,H} \in R.$$

(3) If $\delta^l_{A,H} =_{\Bbbk^\times} \delta^r_{A,H}$, then $\delta^r_{A,H}$ is called *discriminant* of the H-action on A, or the *H-discriminant* of A, and denoted by $\delta_{A,H}$.

(4) The ideal $\mathfrak{r}_{A,H} \cap R$ of R is called the *H-dis-radical*, and denoted by $\Delta_{A,H}$.

We consider the following list of hypotheses that are weaker than Hypotheses 0.1.

Hypothesis 3.2. Assume the following hypotheses:

(a) A is a noetherian connected graded AS Gorenstein algebra.
(b) Hypotheses 0.1(b,c).
(c) A is a free module over R on both sides.
(d) $G_0 := \{g \in G \mid A_g \neq 0\}$ is a subgroup of $G(K)$ and each A_g, for $g \in G_0$, is a free R-module of rank one on both sides.

Continuing Example 2.2, up to scalars, in Definition 3.1 (1) $\delta_{A,H} = 1$, in (2) $\delta_{A,H} = z^2x^2y^2$, and in (3) $\delta_{A,H} = u^4$. Note that in part (3) $\delta_{A,H}$ exists although Hypothesis 3.2(c) above is not satisfied. It is possible that Hypothesis 3.2(c) can be weakened in part (2) of the following lemma.

Lemma 3.3.

(1) Assume Hypotheses 0.1. Then Hypotheses 3.2 holds.
(2) Assume Hypotheses 3.2. Then R is AS Gorenstein and both $\mathsf{j}_{A,H}$ and $\mathsf{a}_{A,H}$ exist.

Proof. (1) Nothing needs to be proved for Hypotheses 3.2(a,b). Part (c) is [33, Lemma 3.3.(2)]. Part (d) is Lemma 1.5(4,6). (2) By [45, Theorem 11.65], there is a standard spectral sequence for change of rings

$$Ext_A^p\left(Tor_q^R(A,M),A\right) \Rightarrow Ext_R^{p+q}(M,A)$$

for all left R-modules M. Since A is finitely generated and free over R on both sides, the above spectral sequence collapses to

$$Ext_A^p(A \otimes_R M, A) = Ext_R^p(M,A).$$

This implies that R has finite injective dimension and $Ext_R^d(\Bbbk, R)$ is finite dimensional. By [52, Theorem 0.3], R is AS Gorenstein. By Theorem 2.4, $\mathsf{j}_{A,H}$ is defined, or equivalently, $\mathrm{hdet}^{-1} \in G_0$. Since G_0 is a group and $\mathrm{hdet} \in G_0$, A_{hdet} is free of rank one on both sides by Hypothesis 3.2(d). Then $\mathsf{a}_{A,H}$ is defined. ■

The following lemma shows the existence of the discriminant under Hypotheses 3.2.

Lemma 3.4. Assume Hypotheses 3.2. Let $g \in G_0$; then A_g and $A_{g^{-1}}$ are free of rank one over R on both sides. Let f_g and $f_{g^{-1}}$ be the generators of A_g and $A_{g^{-1}}$, respectively, over R; then the following properties hold.

(1) Every $f_{g^{-1}}f_g$ is a normal element in R. In particular, both $\delta^l_{A,H}$ and $\delta^r_{A,H}$ are normal elements in R.

(2) If $g' \in G_0$, then $A_{g'} \cap A_G f_g = Rf_{g'} \cap Rf_{g'g^{-1}}f_g$ and $A_{g'} \cap f_g A_G = f_{g'}R \cap f_g f_{g^{-1}g'}R$. As a consequence, if f_g is a normal element in A_G and $f_{g'}$ divides f_g from the left and the right, then $f_g f_{g^{-1}g'} =_{\Bbbk^\times} f_{g'g^{-1}} f_g$. In particular, if f_g is a normal element in A_G, then $f_g f_{g^{-1}} =_{\Bbbk^\times} f_{g^{-1}} f_g$.

(3) $R\delta^l_{A,H} = R \cap A_G \mathsf{j}_{A,H}$ and $\delta^r_{A,H}R = R \cap \mathsf{j}_{A,H}A_G$.

(4) If $\mathsf{j}_{A,H}$ is a normal element in A_G, then $\delta_{A,H}$ is well defined.

Proof. Since $g \in G_0$, $g \in G(K)$ such that $A_g \neq 0$. Since G_0 is a group [Hypothesis 3.2(d)], $A_{g^{-1}}$ is nonzero. By Hypothesis 3.2(d), A_g and $A_{g^{-1}}$ are free of rank one over R on both sides.

(1) Clearly $f_{g^{-1}}f_g$ is an element in R for every $g \in G_0$. It follows from (E1.6.2) that

$$f_{g^{-1}}f_g x = f_{g^{-1}}\phi_g(x) f_g = \left(\phi_{g^{-1}} \circ \phi_g\right)(x) f_{g^{-1}} f_g.$$

Hence, $f_{g^{-1}}f_g$ is a normal element in R.

(2) We will use Lemma 1.7. For $g, g' \in G_0$, we compute

$$\begin{aligned} A_{g'} \cap A_G f_g &= Rf_{g'} \cap \left(\sum_{d=1,i} \mathcal{A}_{d,i} f_g\right) \\ &= Rf_{g'} \cap \left(\sum_{h \in G} Rf_h f_g\right) \\ &= Rf_{g'} \cap \left(Rf_{g'g^{-1}}f_g \oplus \sum_{h \neq g'g^{-1}} Rf_h f_g\right) \\ &= Rf_{g'} \cap Rf_{g'g^{-1}} f_g. \end{aligned}$$

Similarly, we have $A_{g'} \cap f_g A_G = f_{g'}R \cap f_g A_G = f_{g'}R \cap f_g f_{g^{-1}g'}R$. If f_g is a normal element in A_G, then $A_G f_g = f_g A_G$. Since $f_{g'}R = Rf_{g'}$ and since $f_{g'}$ divides f_g from the left and the right, we have

$$\begin{aligned} f_g f_{g^{-1}g'} R &= f_{g'} R \cap f_g f_{g^{-1}g'} R \\ &= f_{g'} R \cap f_g A_G \\ &= A_{g'} \cap A_G f_g \\ &= A_{g'} \cap f_g A_G \\ &= R f_{g'} \cap R f_{g'g^{-1}} f_g \\ &= R f_{g'g^{-1}} f_g. \end{aligned}$$

Then $f_g f_{g^{-1}g'} =_{\Bbbk^\times} f_{g'g^{-1}} f_g$. Let $g' = 1$; we obtain that $f_g f_{g^{-1}} =_{\Bbbk^\times} f_{g^{-1}} f_g$.

(3) The assertion follows from part (2) by taking $g' = 1$ and $g = \mathrm{hdet}^{-1}$.

(4) The assertion follows from parts (2,3) and the fact that $f_1 = 1$ divides $f_{\mathrm{hdet}^{-1}}$ trivially. ■

The following is Theorem 0.2(3) in a special case.

Theorem 3.5. Assume Hypotheses 0.1. Suppose that char $\Bbbk = 0$ and that H is commutative, namely, $H = (\Bbbk G)^*$.

(1) The discriminant $\delta_{A,H}$ is defined, namely,

$$\delta^l_{A,H} = \delta^r_{A,H} = \delta_{A,H}.$$

(2) $\Delta_{A,H}$ is the principal ideal of R generated by $\delta_{A,H}$.

Proof. (1) By [33, Theorem 0.5(1)], $\mathsf{j}_{A,H} = f_{\mathrm{hdet}^{-1}}$ is a normal element in A. Now the assertion follows from Theorem 2.12(1) and Lemma 3.4(4).

(2) The assertion follows from Theorem 2.12(1) and Lemma 3.4(3). ■

Remark 3.6. Here we make some remarks and ask some questions before we prove one of the main results in this section, namely, Theorem 3.8.

(1) Assuming Hypotheses 0.1 or 3.2, is $\delta_{A,H}$ always defined? The answer is YES, see Theorem 3.8. We might further ask: is $\Delta_{A,H} = (\delta_{A,H})$? This is not true, see Example 4.2.

(2) Note that in the commutative case, $R/(\delta_{A,H})$ is always reduced. So we ask the following questions in the noncommutative case: assuming Hypotheses 0.1. is the factor ring $R/(\delta_{A,H})$ semiprime?

(2) In Example 2.2(2), $\delta_{A,H} =_{\Bbbk^\times} x^2y^2z^2 = t_1t_2t_3$ and $R/(\delta_{A,H})$ is semiprime and reduced.

(3) In the commutative case, $\mathsf{a}_{A,H}$ is reduced in A. When $\mathsf{a}_{A,H}$ is normal in A (which is not always true by Example 4.2), we can ask if $A/(\mathsf{a}_{A,H})$ is semiprime.

(3) In Example 2.2(2), $\mathsf{a}_{A,H} =_{\Bbbk^\times} xyx$ and $A/(\mathsf{a}_{A,H})$ is semiprime, but contains nonzero nilpotent elements.

(4) Suppose that A is generated in degree 1. We ask if

$$\mathfrak{R}^l(\mathsf{a}_{A,H}) = \mathfrak{R}^l(\mathsf{j}_{A,H}) = \mathfrak{R}^l(\delta_{A,H}) = \{\Bbbk f_g \mid g \in \mathfrak{G}\}?$$

A similar question can be asked for $\mathfrak{R}^r$.

To prove the existence of $\delta_{A,H}$, we need to recall some terminology introduced in Section 1. For every left A-module,

$$\mathrm{H}^i_{\mathfrak{m}}(M) = \lim_{n\to\infty} Ext^i_A(A/A_{\geq n}, M).$$

The local cohomology functors are defined similar for right A-modules M. When M is an A-bimodule that is finitely generated on both sides, then $\mathrm{H}^i_{\mathfrak{m}}(M)$ can be computed as a left A-module or a right A-module (the result is the same). If R is a subring of A such that A is finitely generated over R on both sides, then $\mathrm{H}^i_{\mathfrak{m}}(M)$ can be computed by considering M as a module over R. In the next lemma, we might calculate $\mathrm{H}^d_{\mathfrak{m}}(M)$ in the category of graded right R-modules.

Lemma 3.7. Assume Hypotheses 3.2. Let d be the injective dimension of A. Suppose $g \in G_0$.

(1) Then the left action $p_g : A \to A$ is a right R-module map such that it decomposes into

$$p_g : A \xrightarrow{\tilde{p}_g} f_g R \xrightarrow{p_g^{-1}} A. \tag{E3.7.1}$$

(2) Applying $\mathrm{H}^d_{\mathfrak{m}}(-)$ to (E3.7.1), $\mathrm{H}^d_{\mathfrak{m}}(p_g)$ is the left action of $p_{g^{-1}}$ on the module $\mathrm{H}^d_{\mathfrak{m}}(A)$, which decomposes into

$$\mathrm{H}^d_{\mathfrak{m}}(A) \xrightarrow{\mathrm{H}^d_{\mathfrak{m}}(p_g^{-1})} \mathrm{H}^d_{\mathfrak{m}}(f_gR) \xrightarrow{\mathrm{H}^d_{\mathfrak{m}}(\tilde{p}_g)} \mathrm{H}^d_{\mathfrak{m}}(A).$$

(3) Let l_{f_g} be the left multiplication of element f_g on A, then $\mathrm{H}^d_{\mathfrak{m}}(l_{f_g})$ is the right multiplication by f_g on $\mathrm{H}^d_{\mathfrak{m}}(A)$.

(4) The composition

$$\Phi := p_g \circ_{f_g} \circ p_1 : A \xrightarrow{p_1} A \xrightarrow{l_{f_g}} A \xrightarrow{p_g} A \tag{E3.7.2}$$

maps R, as a component of A (E1.3.6), to $f_gR = Rf_g$ and other component of A to zero. The restriction of the map Φ on R with image f_gR is an isomorphism of right R-modules.

(5) After applying $\mathrm{H}^d_{\mathfrak{m}}(-)$ to (E3.7.2),

$$\mathrm{H}^d_{\mathfrak{m}}(\Phi) = \mathrm{H}^d_{\mathfrak{m}}\left(p_g \circ_{f_g} \circ p_1\right) : \mathrm{H}^d_{\mathfrak{m}}(A) \xrightarrow{p_{g^{-1}}} \mathrm{H}^d_{\mathfrak{m}}(A) \xrightarrow{r_{f_g}} \mathrm{H}^d_{\mathfrak{m}}(A) \xrightarrow{p_1} \mathrm{H}^d_{\mathfrak{m}}(A)$$

maps $\mathrm{H}^d_{\mathfrak{m}}(f_gR)$ to $\mathrm{H}^d_{\mathfrak{m}}(R)$ and other component of $\mathrm{H}^d_{\mathfrak{m}}(A)$ to zero where r_{f_g} is the right multiplication by f_g.

(6) $\mathrm{H}^d_{\mathfrak{m}}(f_gR) = \{x \in \mathrm{H}^d_{\mathfrak{m}}(A) \mid h \cdot x = g(h)x\} = \mathfrak{e} \otimes f_{\mathrm{hdet}^{-1}g^{-1}}R$ and $\mathrm{H}^d_{\mathfrak{m}}(R) = \{x \in \mathrm{H}^d_{\mathfrak{m}}(A) \mid h \cdot x = \epsilon(h)x\} = \mathfrak{e} \otimes f_{\mathrm{hdet}^{-1}}R$.

(7) $f_{\mathrm{hdet}^{-1}} =_{\Bbbk^\times} f_{\left(\mathrm{hdet}^{-1}g^{-1}\right)}f_g$.

(8) $f_{\mathrm{hdet}^{-1}} =_{\Bbbk^\times} f_h f_{\left(h^{-1}\mathrm{hdet}^{-1}\right)}$ and $\deg f_{\mathrm{hdet}^{-1}} \geq \deg f_h$ for all $h \in G_0$.

(9) $f_{\mathrm{hdet}^{-1}}$ is a normal element in A_G.

(10) $A_G/(R_{\geq 1})$ is Frobenius.

Proof. (1) In this case $A_g = f_gR$, which is free of rank one over R on both sides. Since the left action of p_g is a right R-module map, we obtain a right R-module decomposition of the map p_g.

(2) Note that $\mathrm{H}^d_{\mathfrak{m}}(A)$ is an H-equivariant A-bimodule where the left H-action comes from the natural right H-action on $\mathrm{H}^d_{\mathfrak{m}}(A)$ [43, Lemma 3.2(a)]. By definition [43, (E2.4.1)] for $i = 0$, $\mathrm{H}^d_{\mathfrak{m}}(p_g)$ is $p_{g^{-1}}$. The decomposition follows from (E3.7.1).

(3) Again this follows from [43, Lemma 3.2(a)] and its proof.

(4) This follows from the decomposition of A and Lemma 1.7(2).

(5) Note that $\mathrm{H}^d_{\mathfrak{m}}(A)$ is an H-equivariant A-bimodule. By the proof of [43, Lemma 3.2(a)], $\mathrm{H}^d_{\mathfrak{m}}(p_1)$, $\mathrm{H}^d_{\mathfrak{m}}(l_{f_g})$, and $\mathrm{H}^d_{\mathfrak{m}}(p_g)$ are p_1, r_{f_g}, and $p_{g^{-1}}$ (by part (2)). The assertion follows.

(6) Note that $g, \mathrm{hdet} \in G_0$, which is a finite group. By part (5), $\mathrm{H}^d_{\mathfrak{m}}(f_g R)$ is the image of the idempotent $p_{g^{-1}}$. Hence,

$$\begin{aligned}
\mathrm{H}^d_{\mathfrak{m}}(f_g R) &= \left\{x \in \mathrm{H}^d_{\mathfrak{m}}(A) \mid p_{g^{-1}} \cdot x = x\right\} \\
&= \left\{x \in \mathrm{H}^d_{\mathfrak{m}}(A) \mid h \cdot x = g^{-1}(g)x;\ \forall\, h \in H\right\} \\
&= (\Bbbk\mathfrak{e}) \otimes R_{\mathrm{hdet}^{-1} g^{-1}} \\
&= (\Bbbk\mathfrak{e}) \otimes f_{\mathrm{hdet}^{-1} g^{-1}} R.
\end{aligned}$$

This proves the 1st equation. The 2nd equation is a consequence by taking $g = 1$.

(7) By part (5), $\mathrm{H}^d_{\mathfrak{m}}(\Phi)$, considered as a map from $\mathrm{H}^d_{\mathfrak{m}}(f_g R)$ to $\mathrm{H}^d_{\mathfrak{m}}(R)$, is the right multiplication by f_g. By part (6), this map agrees with

$$r_{f_g} : (\Bbbk\mathfrak{e}) \otimes f_{\left(\mathrm{hdet}^{-1} g^{-1}\right)} R \to (\Bbbk\mathfrak{e}) \otimes f_{\mathrm{hdet}^{-1}} R.$$

Since $\mathrm{H}^d_{\mathfrak{m}}(\Phi)$ is an isomorphism, we obtain that $f_{(\mathrm{hdet}^{-1} g^{-1})} R f_g = f_{\mathrm{hdet}^{-1}} R$. Hence, the assertion follows.

(8) The 1st assertion follows by taking $g = h^{-1}\mathrm{hdet}^{-1}$. The 2nd assertion is clear.

(9) Every element in A_G is a linear combination of $f_g r$ for some $g \in G_0$ and $r \in R$. Then, by part (8),

$$\begin{aligned}
f_{\mathrm{hdet}^{-1}}(f_g r) &=_{\Bbbk^\times} f_{(\mathrm{hdet}^{-1} g\ \mathrm{hdet})} f_{(\mathrm{hdet}^{-1} g^{-1})} f_g r \\
&=_{\Bbbk^\times} f_{(\mathrm{hdet}^{-1} g\ \mathrm{hdet})} f_{\mathrm{hdet}^{-1}} r \\
&=_{\Bbbk^\times} \left(f_{(\mathrm{hdet}^{-1} g\ \mathrm{hdet})} \phi_{\mathrm{hdet}^{-1}}(r)\right) f_{\mathrm{hdet}^{-1}}.
\end{aligned}$$

The assertion follows.

(10) Let $F := A_G/(R_{\geq 1})$. Then $F = \bigoplus_{g \in G_0} \Bbbk f_g$ with multiplication satisfying part (7) or (8). For every element $x \in F$, write $x = \sum c_g f_g$ with $c_g \neq 0$. Pick g so that $\deg f_g$ is smallest among all g such that $c_g \neq 0$. Then

$$f_{\left(\mathrm{hdet}^{-1} g^{-1}\right)} x = c_g f_{\left(\mathrm{hdet}^{-1} g^{-1}\right)} f_g =_{\Bbbk^\times} f_{\mathrm{hdet}^{-1}},$$

which implies that F is Frobenius. ∎

Now we are ready to prove Theorems 0.2(3) and 0.6. Following (E2.11.1), we define

$$\mathfrak{R}(\mathsf{j}_{A,H}) := \left\{\Bbbk f_g \mid g \in G_0, \deg f_g = 1\right\}. \tag{E3.7.3}$$

Theorem 3.8. Assume Hypotheses 3.2.

(1) $\mathsf{j}_{A,H}\ \mathsf{a}_{A,H} =_{\Bbbk^\times} \mathsf{a}_{A,H}\ \mathsf{j}_{A,H}$. As a consequence, the discriminant $\delta_{A,H}$ of the H-action is defined.

(2) $\mathsf{a}_{A,H}$ divides $\mathsf{j}_{A,H}$.

(3) $\mathfrak{R}(\mathsf{j}_{A,H})$ is a subset of both $\mathfrak{R}^l(\mathsf{j}_{A,H})$ and $\mathfrak{R}^r(\mathsf{j}_{A,H})$.

(4) Assuming the hypotheses of Theorem 0.5, then $\mathfrak{R}(\mathsf{j}_{A,H}) = \{\Bbbk f_g \mid f_g \in \mathfrak{G}\}$ where $\mathfrak{G}$ is defined in (E2.11.1).

Proof. (1) By Lemma 3.7(9), $\mathsf{j}_{A,H} = f_{\mathrm{hdet}^{-1}}$ is a normal element in A_G. The assertions follow from Lemma 3.4(2,4) by setting $g = \mathrm{hdet}^{-1}$.

(2) This is Lemma 3.7(7,8).

(3) This follows from Lemma 3.7(7,8).

(4) This is clear. ■

Now we are to prove Theorems 0.2, 0.6 and 0.8.

Proof. of Theorem 0.2 (1,2) This is Corollary 2.5(1).

(3) This is Theorem 3.8(1). ■

Theorem 0.6 is a consequence of Lemma 3.3(1) and Theorem 3.8(2). The next theorem is Theorem 0.8.

Theorem 3.9. Assume Hypotheses 0.1. Suppose R is Auslander regular. Then A_G is AS Gorenstein and $\mathsf{j}_{A,H} =_{\Bbbk^\times} \mathsf{j}_{A_G,(\Bbbk G)^*}$ in A_G.

Proof. By Hypotheses 0.1, A is a domain, and hence so is A_G.

Since R is AS regular, it is trivially AS Cohen–Macaulay in the sense of [27, Definition 0.1]. Since A_G is a finitely generated free module over R, it also is AS Cohen–Macaulay. Therefore, the hypotheses of [27, Theorem 6.1(1°)] hold, and the hypotheses of [27, Theorem 6.1(3°)] hold because R is AS regular, see [27, Proposition 5.5]. By the proof of [27, Proposition 5.7], using the fact that R is Auslander regular, we see that the hypotheses of [27, Theorem 6.1(2°)] hold. Combining the facts that R is AS regular and

$A_G/(R_{\geq 1})$ is Frobenius [Lemma 3.7(10)], we obtain that the Hilbert series of A_G satisfies

$$h_{A_G}(t) = \pm t^m h_{A_G}(t^{-1}).$$

Now the 1st assertion follows from [27, Theorem 6.1].

For the 2nd assertion, note that $B := A_G$ satisfies Hypotheses 3.2. It is clear that $B^{(\Bbbk G)^*} = A^H = R$. Let f'_g be the generator of B_g as defined in (E1.6.1). Then $f'_g = f_g$ for all $g \in G$. By Lemma 3.7(7), both $f'_{\mathrm{hdet}^{-1}}$ and $f_{\mathrm{hdet}^{-1}}$ (with different meanings of hdet^{-1}) have the highest degree among $\{f'_g \mid g \in G\}$ and $\{f_g \mid g \in G\}$. Thus, $f'_{\mathrm{hdet}^{-1}} = f_{\mathrm{hdet}^{-1}}$. This is equivalent to $\mathsf{j}_{A_G,(\Bbbk G)^*} = \mathsf{j}_{A,H}$ by definition. ■

Next we prove Theorem 0.7. The discriminant has been an important tool in number theory and algebraic geometry for many years. The discriminant of a reflection group is a fundamental invariant of reflection group actions. Next we will compare the H-discriminant in the noncommutative case [Definition 3.1(3)] to the noncommutative discriminant over a central subalgebra, which was used in recent studies of automorphism groups and locally nilpotent derivations [8, 12, 13].

If I is an ideal of a commutative ring, let $\sqrt{I}$ denote the prime radical ideal of I.

Theorem 3.10. Assume Hypotheses 0.1. Further, assume that

(a) char $\Bbbk = 0$,
(b) $H = (\Bbbk G)^*$, and
(c) $R := A^H$ is central in A.

Let $dis(A/R)$ be the discriminant defined in [12, Definition 1.3(3)]. Then

(1) $dis(A/R) =_{\Bbbk^\times} \prod_{g \in G}(f_{g^{-1}} f_g)$.
(2)

$$\sqrt{(dis(A/R))} = \sqrt{(\Delta_{A,H})} = \sqrt{(\delta_{A,H})}$$

as ideals of R.

Proof. (1) Since $A = \oplus_{g \in G} f_g R$, A can be embedded into the matrix algebra $M_r(R)$ by the left multiplication, where $r = |G|$. For each g, the left multiplication by f_g is

$$l_{f_g} : f_h \mapsto f_g f_h = c_{g,h} f_{gh} \text{ for } c_{g,h} \in R \qquad \text{(see ((E1.6.2)))}.$$

If $g \neq e$, $gh \neq g$, then the regular trace of f_g [12, Example 1.2(3)], denoted by $tr(f_g)$, is zero. As a consequence, we have

$$tr(f_g f_h) = tr(c_{g,h} f_{gh}) = \begin{cases} 0 & gh \neq e, \\ c_{h^{-1},h} = f_{h^{-1}} f_h & g = h^{-1}. \end{cases} \tag{E3.10.1}$$

By [12, Definition 1.3(3)], the discriminant $dis(A/R)$ is the determinant of the matrix

$$\Big(tr(f_g f_h)\Big)_{G\times G}.$$

Using (E3.10.1), every row (and every column) contains only one nonzero entry, namely, $f_{h^{-1}} f_h$. Hence, we have

$$dis(A/R) =_{\Bbbk^\times} \prod_{h\in G} f_{h^{-1}} f_h.$$

(As an example, note that in Example 2.2(2)

$$\prod_{h\in G} f_{h^{-1}} f_h = (x^2)(y^2)(z^2)(xyzy)(xzxz)(zyxy)(xzyxzy) =_{\Bbbk^\times} z^8x^8y^8.)$$

(2) By Theorem 3.5, $\Delta_{A,H}$ is the principal ideal of R generated by $\delta_{A,H}$. Hence, $\Delta_{A,H} = (\delta_{A,H})$, and it remains to show that $\sqrt{(dis(A/R))} = \sqrt{(\delta_{A,H})}$.

Since $\delta_{A,H} = f_{m^{-1}} f_m$, by part (1), $\delta_{A,H}$ divides $dis(A/R)$. By the proof of Lemma 2.11, every f_g divides f_m from the left and the right. Hence, there are $a, b \in A$ such that $af_{g^{-1}} f_g b = f_m^2$. Since $f_{g^{-1}} f_g$ is in the central subring R, we have $f_{g^{-1}} f_g ab = f_m^2$. This implies that $f_{g^{-1}} f_g$ divides $f_m^r \in R$. (Note that $f_m^2 \notin R$ in general.) As a consequence, $dis(A/R)$ divides $(f_m^r)^r$. Finally $(f_m^r)^r$ divides $\delta_{A,H}^{r^2}$. Therefore,

$$\sqrt{(\delta_{A,H})} = \sqrt{dis(A/R)} = \sqrt{f_m^r}$$

as desired. ■

Theorem 0.7 is Theorem 3.10(2). Note that there are many examples where R is not central in A, even when R is a commutative polynomial ring [Example 4.2]. Without the hypothesis of $H = (\Bbbk G)^*$, it is easy to construct examples where

$$\sqrt{(\Delta_{A,H})} \neq \sqrt{(\delta_{A,H})},$$

see (E4.2.14).

Remark 3.11. Suppose H is a semisimple Hopf algebra.

(1) Let $G(K)$ be the group of all grouplike elements in K. In general, $\Bbbk G(K)$ is NOT a normal Hopf subalgebra [35].

(2) One could ask if $\Bbbk G(K)$ is a normal Hopf subalgebra under Hypothesis 0.1. This is related to Question 0.9.

(3) If $\Bbbk G(K)$ is a normal Hopf subalgebra, then there is a short exact sequence of Hopf algebras

$$1 \to \Bbbk G(K) \to K \to K_0 \to 1, \tag{E3.11.1}$$

where $K_0 = K/(\Bbbk G(K))_+$. There is a dual short exact sequence

$$1 \to H_0 \to H \to (\Bbbk G(K))^* \to 1, \tag{E3.11.2}$$

where $H_0 = (K_0)^*$. If we further assume Hypotheses 0.1, then $A_G = A^{H_0}$ and Question 0.9 has a positive answer under these extra hypotheses.

We end this section by providing some results that can be used to compute the radical ideal of the H-action, particularly when A has dimension 2.

Definition 3.12. Let H be a semisimple Hopf algebra acting on A.

(1) If

$$\Delta(\textstyle\int) = \Delta(p_1) = \sum_{h \in G(K)} p_h \otimes p_{h^{-1}} + X_1,$$

where $X_1 \in I_{com} \otimes I_{com}$, see (E1.3.5), then H is called *rife*.

(2) Assume Hypotheses 0.1. We say the H-action is *rife* if

(a) H is rife.

(b) $\mathfrak{j}_{A,H}$ is normal in A.

(c) the radical ideal of the H-action $\mathfrak{r}_{A,H}$ is generated by $\mathfrak{j}_{A,H}$.

By Theorem 2.12(1), when H is $(\Bbbk G)^*$, then the H-action is rife. Otherwise, the H-action may not be rife, even when H is rife [Example 4.2].

Lemma 3.13. Assume Hypotheses 0.1. If H is rife, then $\mathfrak{r}_{A,H}$ is a subspace of $A\mathfrak{j}_{A,H}$.

Proof. For each $g \in G(K)$, since H is rife, we have

$$\begin{aligned}(A\#\textstyle\int)(A\#p_g) &= (A\#1)\left(\sum_{h\in G(K)} p_{h^{-1}}A\#p_h p_g + X_1 \cdot (A\#p_g)\right)\\ &= (A\#1)\big(A_{g^{-1}}\#p_g\big)\\ &= AA_{g^{-1}}\#p_g\\ &= Af_{g^{-1}}\#p_g.\end{aligned}$$

If $x \in \mathfrak{r}_{A,H}$, then $x\#1 \in (A\#H)(A\#\int)(A\#H)$. Multiplying $1\#p_g$ from the right, $x\#p_g \in (A\#H)(A\#\int)(A\#p_g)$. By computation,

$$(A\#H)(A\#\textstyle\int)(A\#p_g) = (A\#\int)(A\#p_g) = Af_{g^{-1}}\#p_g,$$

which implies that $x \in Af_{g^{-1}}$ for all $g \in G(K)$. Hence, $x \in \bigcap_{g\in G(K)} Af_g$, which is a subspace of $Af_{\mathrm{hdet}^{-1}} = A\mathfrak{j}_{A,H}$. ■

For every (left) ideal I in a noetherian algebra A, let $\bar{I}$ denote the largest ideal containing I such that $\bar{I}/I$ is finite dimensional. The following lemma is well known.

Lemma 3.14. Let A be AS regular of global dimension 2. (So A is noetherian). Let I be a nonzero graded two-sided ideal. If $\bar{I} = I$, then I is a principal ideal generated by a normal element. In particular, $\bar{I}$ is always a principal ideal generated by a normal element.

Proof. Since $\bar{I} = I$, A/I is $\mathfrak{m}$-torsionfree, so $\mathrm{H}^0_{\mathfrak{m}}(M) = 0$, see definition in Section 1. By Auslander–Buchsbaum formula [26, Theorem 3.2], the left A-module A/I has projective dimension at most 1. Since A/I is not projective, it has projective dimension one. Consequently, the left A-module I is projective. Since A is connected graded, I is free (of rank one). Thus, $I = Ax$ for some homogeneous element $x \in A$. By symmetry, $I = yA$ for some homogeneous element y. Then $Ax = yA$ implies that $x =_{\Bbbk^\times} y$. Thus, x is normal and the assertion follows. ■

We use Lemma 3.14 to make the following definitions in the case that A has global dimension 2.

Definition 3.15. Assume Hypotheses 0.1. Further, assume that A has global dimension 2 and that the radical ideal of the H-action $\mathfrak{r}_{A,H}$ is nonzero.

(1) Any element that generates the principal ideal $\overline{\mathfrak{r}_{A,H}}$ in A is called a *principal radical* of the H-action on A, and is denoted by $\widetilde{\mathfrak{r}}_{A,H}$.

(2) Any element that generates the principal ideal $\overline{\Delta_{A,H}}$ in R is called a *principal dis-radical* of the H-action on A, and is denoted by $\widetilde{\Delta}_{A,H}$.

Note that the principal radical $\widetilde{\mathfrak{r}}_{A,H}$ is always defined for any Hopf algebra H acting on an AS regular algebra A of global dimension 2, while the principal dis-radical $\widetilde{\Delta}_{A,H}$ is defined only when, in addition, H is a reflection Hopf algebra.

4 Examples

When a Hopf algebra H acts on a noetherian AS regular algebra A, there is a list of important invariants that can be studied. Starting from A, we can consider the following data:

(•) the Nakayama automorphism of A, denoted by μ [Definition 1.2].
(•) the AS index of A, denoted by $\mathfrak{l}$ [Definition 1.1].
(•) the twisted superpotential associated to A [20, Definition 1] or [10, p.1502].

For H, since we assume that H is semisimple, it is Calabi–Yau with trivial Nakayama automorphism. When H acts on A, we can consider the following:

(•) the pertinency $\mathsf{p}(A,H)$ [Definition 2.8(3)].
(•) the pertinency ideal $\mathcal{P}(A,H)$ [Definition 2.8(1)].
(•) the radical ideal $\mathfrak{r}_{A,H}$ [Definition 2.8(1)]. In global dimension 2 case, we can ask for the principal radical $\widetilde{\mathfrak{r}}_{A,H}$ [Definition 3.15(1)].
(•) H-dis-radical ideal $\Delta_{A,H}$ [Definition 3.1(4)]. In global dimension 2 case, we can ask for the principal dis-radical $\widetilde{\Delta}_{A,H}$ [Definition 3.15(2)].
(•) the homological determinant hdet of the H-action on A [31, Definition 3.3].
(•) the fusion rules for H, or the McKay quiver for representations of H.

When the fixed subring A^H is AS Gorenstein (or AS regular), we can further consider the following:

(•) the Jacobian $\mathsf{j}_{A,H}$ [Definition 2.1(1)].
(•) the reflection arrangement $\mathsf{a}_{A,H}$ [Definition 2.1(2)].
(•) $\mathfrak{R}^l(\mathsf{j}_{A,H})$ and $\mathfrak{R}^l(\mathsf{a}_{A,H})$, see (E0.9.1).
(•) the discriminant $\delta_{A,H}$ [Definition 3.1(3)].

There are several algebras associated to (A,H): the fixed subring A^H, the covariant ring $A^{cov\, H}$, the G-component A_G, $A/(\mathsf{a}_{A,H})$ if $\mathsf{a}_{A,H}$ is normal, $R/(\delta_{A,H})$ when $\delta_{A,H}$ is defined. If any of the these algebras is AS Gorenstein, we can compute the corresponding data in the first two •s.

First we compute the Jacobian when $A = \Bbbk_{-1}[x, y]$ and H is a group algebra $\Bbbk G$ for some finite group G. Let us recall some facts from [32]. We consider two different kinds of automorphisms of $\Bbbk_{-1}[x, y]$. The 1st is of the form

$$\sigma_a : x \mapsto ax, y \mapsto y, \qquad \text{or} \qquad \tau_b : x \mapsto x, y \mapsto ay. \tag{E4.0.1}$$

and the 2nd one is of the form

$$\tau_{1,2,\lambda} : x \mapsto \lambda y, y \mapsto -\lambda^{-1}x. \tag{E4.0.2}$$

Let α and β be two positive integers such that β is divisible by both 2 and α. Let $M(2, \alpha, \beta)$ be the subgroup of $Aut_{gr}(\Bbbk_{-1}[x, y])$ generated by

$$\{\sigma_a \mid a^\alpha = 1\} \cup \{\tau_{1,2,\lambda} \mid \lambda^\beta = 1\}$$

(see [32] in discussion before [32, Lemma 5.3]). By [32, Lemma 5.3], if G is not generated only by a single σ_a or $\tau_{1,2,a}$ in (E4.0.1) and (E4.0.2) and $\Bbbk_{-1}[x, y]^G$ is AS regular, then $G \cong M(2, \alpha, \beta)$. As one example, the groups $M(2, 1, 2\ell)$ are the binary dihedral groups of order 4ℓ generated by $\tau_{1,2,1}$ and $\tau_{1,2,\lambda}$ for λ a primitive 2ℓth root of unity, that is, the representation generated by the two mystic reflections:

$$g_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{and} g_2 = \begin{pmatrix} 0 & \lambda \\ -\lambda^{-1} & 0 \end{pmatrix}.$$

Lemma 4.1. Suppose that $A = \Bbbk_q[x, y]$ where $1 \neq q \in \Bbbk^\times$ and that $H = \Bbbk G$ for a finite group G. Assume Hypotheses 0.1. Then one of the following holds.

(1) $G = \langle\sigma\rangle \times \langle\tau\rangle \cong C_n \times C_m$ where σ and τ are of the form given in (E4.0.1) and of order n and m, respectively. In this case $\mathsf{j}_{A,H} =_{\Bbbk^\times} x^{n-1}y^{m-1}$, $\mathsf{a}_{A,H} =_{\Bbbk^\times} xy$ and

$$\mathfrak{R}^l(\mathsf{j}_{A,H}) = \mathfrak{R}^l(\mathsf{a}_{A,H}) = \mathfrak{R}^r(\mathsf{j}_{A,H}) = \mathfrak{R}^r(\mathsf{a}_{A,H}) = \{\Bbbk x, \Bbbk y\}.$$

(2) $q = -1$ and $G = M(2, \alpha, \beta)$ for $\alpha \geq 2$. Then

$$\mathsf{a}_{A,H} = xy(x^\beta - y^\beta) \quad \text{and} \quad \mathsf{j}_{A,H} = x^{\alpha-1}y^{\alpha-1}(x^\beta - y^\beta).$$

Further,

$$\mathfrak{R}^l(\mathsf{j}_{A,H}) = \mathfrak{R}^r(\mathsf{j}_{A,H}) = \mathfrak{R}^l(\mathsf{a}_{A,H}) = \mathfrak{R}^r(\mathsf{a}_{A,H}) = \{\Bbbk x, \Bbbk y\} \cup \{\Bbbk(x+\xi y) \mid \xi^\beta = 1\}.$$

(3) $q = -1$ and $G = M(2, 1, \beta)$ (for) $\alpha = 1$. Then

$$\mathsf{a}_{A,H} = \mathsf{j}_{A,H} = (x^\beta - y^\beta).$$

Further,

$$\mathfrak{R}^l(\mathsf{j}_{A,H}) = \mathfrak{R}^r(\mathsf{j}_{A,H}) = \mathfrak{R}^l(\mathsf{a}_{A,H}) = \mathfrak{R}^r(\mathsf{a}_{A,H}) = \{\Bbbk(x+\xi y) \mid \xi^\beta = 1\}.$$

Proof. By [32, Theorem 1.1], G is generated by quasi-reflections in the sense of [32, p. 131]. When $q \neq \pm 1$, $Aut_{gr}(A) = (\Bbbk^\times)^2$ and every quasi-reflection is a reflection in the sense of [32, Definition 2.3(1)], namely, of the form in (E4.0.1). One can check easily from this observation that $G \cong C_n \times C_m$. The statements in part (1) are easy to check now.

When $q = -1$, one extra possibility is that G is generated by mystic reflections in the sense of [32, Definition 2.3(1)]. In this case, by [32, Lemma 5.3], G is the group $M(2, \alpha, \beta)$, and A^G is the commutative polynomial ring $\Bbbk[x^\alpha y^\alpha, x^\beta + y^\beta]$ [32, Proposition 5.4].

(2) When $\alpha \geq 2$ one can check directly that nonzero elements of the minimal degree in A_{hdet} and $A_{\mathrm{hdet}^{-1}}$ are

$$\mathsf{a}_{A,\Bbbk G} = xy(x^\beta - y^\beta) \quad \text{and} \quad \mathsf{j}_{A,\Bbbk G} = x^{\alpha-1}y^{\alpha-1}(x^\beta - y^\beta),$$

respectively. From this we obtain, after an easy calculation, that

$$\mathfrak{R}^l(\mathsf{j}_{A,\Bbbk G}) = \mathfrak{R}^r(\mathsf{j}_{A,\Bbbk G}) = \mathfrak{R}^l(\mathsf{a}_{A,\Bbbk G}) = \mathfrak{R}^r(\mathsf{a}_{A,\Bbbk G}) = \{\Bbbk x, \Bbbk y\} \cup \{\Bbbk(x+\xi y) \mid \xi^\beta = 1\}.$$

The assertion follows.

(3) When $\alpha = 1$, the computation is similar to the one in part (2). ■

By Lemma 4.1(2,3), $\Bbbk M(2, \alpha, \beta)$ is a true reflection Hopf algebra acting on $\Bbbk_{-1}[x, y]$ if and only if $\alpha = 1$ or 2. When $\alpha = 1$, $M(2, 1, \beta)$ is isomorphic to a binary dihedral group. Note in this case that the number of mystic reflections is the degree of $\mathfrak{j}_{A,\Bbbk G}$, also equals to $|\mathfrak{R}^l(\mathfrak{j}_{A,\Bbbk G})|$. Further that the Jacobian (and hence the reflection arrangement) is central, but A^G is not central in A.

For the rest of this section we give an example where H is neither commutative or cocommutative. This example is the smallest possible in terms of dimensions, H having $\Bbbk$-dimension 8 and A having global dimension 2. Even in this "small" example, computations are still quite complicated, unfortunately. To save some space, some non-essential details are omitted, especially toward the end of the example. Some additional information concerning this example is given in [21] and [31, Example 7.4].

Example 4.2. Assume that char $\Bbbk = 0$. Let H be the Kac–Palyutkin Hopf algebra H_8. By [9, p. 341], H is self-dual and it has no nontrivial dual cocycle twist in the sense of [36, 39]. Recall that H is generated by x, y, z and subject to the following relations:

$$x^2 = y^2 = 1, \quad xy = yx, \quad zx = yz,$$

$$zy = xz, \quad z^2 = \frac{1}{2}(1 + x + y - xy).$$

The comultiplication of H is determined by

$$\Delta(x) = x \otimes x,$$

$$\Delta(y) = y \otimes y,$$

$$\Delta(z) = \frac{1}{2}(1 \otimes 1 + 1 \otimes x + y \otimes 1 - y \otimes x)(z \otimes z).$$

The group of grouplike elements in H is $G(H) = \{1, x, y, xy\}$, the Klein four-group. Let $\Bbbk_i[u, v]$ be the skew polynomial algebra generated by u, v and subject to the relation

$$vu = iuv, \tag{E4.2.1}$$

where $i^2 = -1$. By [43, Example 5.5], the Nakayama automorphism of A is determined by

$$\mu : u \mapsto -iu, \quad v \mapsto iv$$

and, in dimension 2, the twisted superpotential is trivially the single relation, namely,

$$\omega = vu - iuv.$$

By [31, Example 7.4], H acts on $A := \Bbbk_i[u, v]$ inner faithfully with commutative (but not central) regular fixed subring $A^H = \Bbbk[u^2 + v^2, u^2v^2]$. Thus, Hypotheses 0.1 (and hence Hypotheses 3.2) holds.

It is easy to check that $AR_{\geq 1} \neq R_{\geq 1}A$. So the H-action on A is not tepid, see Definition 1.11(4). It is routine to check that the covariant algebra $A^{cov\,H} := A/(R_{\geq 1})$ is isomorphic to $A/(\Bbbk(u^2+v^2) \oplus A_{\geq 3})$, which has Hilbert series $1+2t+2t^2$. As a consequence, $A^{cov\,H}$ is not Frobenius, which is different from the classical (commutative) case and the case of the dual reflection groups in [33, Theorem 0.4].

Recall from [31, Example 7.4] that there is a unique 2D H-representation $V = \Bbbk u \oplus \Bbbk v$ given by the assignment:

$$x \to \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad y \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad z \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

which uniquely determines the H-action on $A = \Bbbk_i[u, v]$.

Our 1st goal is to calculate the Jacobian, the reflection arrangement, and the discriminant of this H-action on A. Note that $x + y, xy$, and z^2 are all central in H. Consider the central idempotents in H:

$$f_1 = (1 + x + y + xy)/4, \quad \text{and} \quad f_2 = (1 - x - y + xy)/4.$$

It is easy to check that $f_1z^2 = z^2f_1 = f_1$ and $f_2z^2 = z^2f_2 = -f_2$. In addition we have the following two idempotents in H that are not central:

$$f_3 = (1 - x + y - xy)/4, \quad \text{and} \quad f_4 = (1 + x - y - xy)/4.$$

These idempotents satisfy $f_if_j = 0$ for all $i \neq j$. Using the above information, we define the following central idempotents of H, which correspond to the group of grouplike elements $G(K)(= \{1, g, g', gg'\})$, where K is the dual Hopf algebra of H,

$$\begin{aligned} p_1 = \textstyle\int &= (f_1 + zf_1)/2 = (1 + x + y + xy + z + xz + yz + xyz)/8 \\ p_g &= (f_1 - zf_1)/2 = (1 + x + y + xy - z - xz - yz - xyz)/8 \\ p_{g'} &= (f_2 + izf_2)/2 = (1 - x - y + xy + iz - ixz - iyz + ixyz)/8 \\ p_{gg'} &= (f_2 - izf_2)/2 = (1 - x - y + xy - iz + ixz + iyz - ixyz)/8. \end{aligned}$$

Using the fact that $zu^2 = v^2, zv^2 = u^2, z(uv) = -iuv, z(u^3v) = iuv^3$, etc., we obtain

$$\begin{aligned} p_1 A &= A^H = R = \Bbbk[u^2+v^2, u^2v^2] \\ p_g A &= (u^2 - v^2)R \\ p_{g'} A &= (uv)R \\ p_{gg'} A &= (u^3v + uv^3)R = (uv(u^2 - v^2))R. \end{aligned}$$

As a consequence, we have the decomposition of A_G into graded pieces (as in Lemma 1.5(3))

$$A_G = p_1 A \oplus p_g A \oplus p_{g'} A \oplus p_{gg'} A = A^{(2)},$$

where $A^{(2)}$ is the 2nd Veronese subring of A. It is clear that $(u^2+v^2)u = u(u^2-v^2)$, which is not in $AR_{\geq 1}$. Hence, $R_{\geq 1}A \neq AR_{\geq 1}$, and consequently, the H-action on A is not tepid in the sense of Definition 1.11(4). By an easy calculation,

$$\xi(t) = h_A(t)(h_R(t))^{-1} = (1+t)(1+t+t^2+t^3),$$

which has degree 4. It follows from Corollary 2.5(2) then $\deg \mathsf{j}_{A,H} = 4$. Hence,

$$\mathsf{j}_{A,H} =_{\Bbbk^\times} uv(u^2 - v^2) \tag{E4.2.2}$$

and

$$\mathrm{hdet}^{-1} = gg'. \tag{E4.2.3}$$

Since $\mathrm{hdet}^2 = 1$, we obtain that

$$\mathsf{a}_{A,H} = \mathsf{j}_{A,H} =_{\Bbbk^\times} uv(u^2 - v^2) \tag{E4.2.4}$$

and

$$\delta_{A,H} =_{\Bbbk^\times} u^2v^2(u^2-v^2)^2 = u^2v^2[(u^2+v^2)^2 - 4u^2v^2] \in R. \tag{E4.2.5}$$

As a consequence of (E4.2.4), H is a true reflection Hopf algebra. Using the fact that

$$u^2 - v^2 = \left(u + e^{\frac{3}{8}(2\pi i)}v\right)\left(\left(u + e^{\frac{1}{8}(2\pi i)}v\right) = \left(u + e^{\frac{7}{8}(2\pi i)}v\right)\left(\left(u + e^{\frac{5}{8}(2\pi i)}v\right)$$

and that u and v are normal, we can calculate

$$\mathfrak{R}^l(\mathsf{a}_{A,H}) = \mathfrak{R}^l(\mathsf{j}_{A,H}) = \left\{\Bbbk u, \Bbbk v, \Bbbk(u + e^{\frac{3}{8}(2\pi i)}v), \Bbbk(u + e^{\frac{7}{8}(2\pi i)}v)\right\} \tag{E4.2.6}$$

and

$$\mathfrak{R}^r(\mathsf{a}_{A,H}) = \mathfrak{R}^r(\mathsf{j}_{A,H}) = \left\{\Bbbk u, \Bbbk v, \Bbbk(u + e^{\frac{1}{8}(2\pi i)} v), \Bbbk(u + e^{\frac{5}{8}(2\pi i)} v)\right\}. \tag{E4.2.7}$$

Since $\mathsf{j}_{A,H}$ is not normal (easy to check), $A\mathsf{j}_{A,H}$ is not a 2-sided ideal. So $\mathfrak{r}_{A,H} \neq A\mathsf{j}_{A,H}$. By Lemma 3.13 (after verifying the hypotheses in Lemma 3.13), $\mathfrak{r}_{A,H}$ is a subspace of $A\mathsf{j}_{A,H}$.

Our 2nd goal is to calculate the radical ideal of this H-action. Let

$$E = (1 - xy)/2 = 1 - (p_1 + p_g + p_{g'} + p_{gg'}),$$

a central idempotent of H so

$$H = \Bbbk p_1 \oplus \Bbbk p_g \oplus \Bbbk p_{g'} \oplus \Bbbk p_{gg'} \oplus EH.$$

We have the relations:

$$Ef_3 = f_3, \ \ Ef_4 = f_4, \ \ f_3 f_4 = f_4 f_3 = 0, \ \ zf_3 = f_4 z, \ \ zf_4 = f_3 z.$$

Further,

$$f_3 z^2 = \frac{1}{8}(1 - x + y - xy)(1 + x + y - xy) = \frac{1}{8}(2 - 2x + 2y - 2xy) = f_3$$

and similarly

$$f_4 z^2 = \frac{1}{8}(1 + x - y - xy)(1 + x + y - xy) = \frac{1}{8}(2 + 2x - 2y - 2xy) = f_4.$$

Hence, let $m_{12} = f_3 z f_4 = f_3 z = z f_4$ and $m_{21} = f_4 z f_3 = f_4 z = z f_3$. Then

$$\begin{aligned} m_{12} m_{21} &= (f_3 z)(z f_3) = f_3 z^2 f_3 = f_3 f_3 = f_3, \\ m_{21} m_{12} &= (f_4 z)(z f_4) = f_4, \end{aligned}$$

and

$$\begin{aligned} m_{12}^2 &= (f_3 z f_4)(f_3 z f_4) = 0, \\ m_{21}^2 &= (f_4 z f_3)(f_4 z f_3) = 0. \end{aligned}$$

So the subspace EH is isomorphic to 2×2-matrix, and for convenience, we write

$$EH \cong \begin{pmatrix} f_3 & m_{12} \\ m_{21} & f_4 \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}.$$

Next we find $x_{i,j}, y_{i,j}$ so that

$$\Delta(\textstyle\int) = \Delta(p_1) = p_1 \otimes p_1 + p_g \otimes p_g + p_{g'} \otimes p_{g'} + p_{gg'} \otimes p_{gg'}$$

$$+ \sum_{1 \le i,j \le 2} m_{ij} \otimes x_{ij} + \sum_{1 \le i,j \le 2} y_{ij} \otimes m_{ij}.$$

First compute

$$\begin{aligned}\Delta(\textstyle\int) &= \Delta((1+x+y+xy)(1+z)/8) = \Delta(1+x+y+xy)\Delta(1+z)/8 \\ &= \frac{1}{8}(1 \otimes 1 + x \otimes x + y \otimes y + xy \otimes xy)(1 \otimes 1 \\ &\quad + \frac{z \otimes z + z \otimes xz + yz \otimes z - yz \otimes xz}{2}) \\ &= \frac{(1 \otimes 1 + x \otimes x + y \otimes y + xy \otimes xy)}{8} \\ &\quad + \frac{(z \otimes z + xz \otimes xz + yz \otimes yz + xyz \otimes xyz)}{16} \\ &\quad + \frac{(z \otimes xz + xz \otimes z + yz \otimes xyz + xyz \otimes yz)}{16} \\ &\quad + \frac{(yz \otimes z + xyz \otimes xz + z \otimes yz + xz \otimes xyz)}{16} \\ &\quad - \frac{(yz \otimes xz + xyz \otimes z + z \otimes xyz + xz \otimes yz)}{16}.\end{aligned}$$

After some tedious computation, we obtain that

$$\begin{aligned}\Delta(\textstyle\int) = \Delta(p_1) &= p_1 \otimes p_1 + p_g \otimes p_g + p_{g'} \otimes p_{g'} + p_{gg'} \otimes p_{gg'} \\ &\quad + ((f_3 \otimes f_3) + (f_4 \otimes f_4) + (m_{12} \otimes m_{12}) + (m_{21} \otimes m_{21}))/2.\end{aligned}$$

Let L be the left ideal of A generated by elements w satisfying

$$w = \sum_i b_i(f_3 \cdot a_i) + \sum_j d_j(m_{12} \cdot c_j) \tag{E4.2.8}$$

$$0 = \sum_i b_i(m_{21} \cdot a_i) + \sum_j d_j(f_4 \cdot c_j) \tag{E4.2.9}$$

for some a_i, b_i, c_j, d_j in A. Let L' be the left ideal of A generated by elements w satisfying

$$w = \sum_i b_i(f_4 \cdot a_i) + \sum_j d_j(m_{21} \cdot c_j) \tag{E4.2.10}$$

$$0 = \sum_i b_i(m_{12} \cdot a_i) + \sum_j d_j(f_3 \cdot c_j) \tag{E4.2.11}$$

for some a_i, b_i, c_j, d_j in A.

It follows from the definition of the radical ideal [Definition 2.8(2)] that we have

Lemma 4.3. Retain the above notation. The radical ideal is

$$(\mathfrak{r}_{H,A}) = A\mathsf{j}_{A,H} \cap L \cap L'.$$

Proof. The main idea here is to do finer computations than ones in the proof of Lemma 3.13. To save space, details are omitted. ■

As a consequence, one can calculate the radical ideal in this example:

$$\mathfrak{r}_{A,H} = \overline{\mathfrak{r}_{A,H}} = (\tilde{\mathfrak{r}}_{A,H}), \tag{E4.2.12}$$

where

$$\tilde{\mathfrak{r}}_{A,H} =_{\Bbbk^\times} uv(u^4 - v^4) =_{\Bbbk^\times} \mathsf{j}_{A,H}(u^2 + v^2). \tag{E4.2.13}$$

Further,

$$\Delta_{A,H} = \overline{\Delta_{A,H}} = \left(\mathsf{j}^2_{A,H}\,(u^2 + v^2)\right) = \left(\delta_{A,H}\,(u^2 + v^2)\right). \tag{E4.2.14}$$

This is the end of the example.

Complex reflection groups are important in many areas of current research, for example, in defining rational Cherednik algebras. In this paper we have presented generalizations of the various invariants that are used in studying complex reflection groups, their geometry, and their actions on polynomial rings (see, e.g., [11]). The tools developed here, the Jacobian, the reflection arrangement, and the discriminant, as well as the pertinency ideal, the radical of the H-action, the homological determinant, and the Nakayama automorphism should further the understanding of Hopf actions on AS regular algebras. If there is ever a version of rational Cherednik algebras for Artin–Schelter regular algebras, then one should understand better reflection Hopf algebras, and whence, the invariants introduced in this paper.

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