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CANCELLATION OF MORITA AND SKEW TYPES

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ABSTRACT

We study both Morita cancellative and skew cancellative properties of noncommutative algebras as initiated recently in several papers and explore which classes of noncommutative algebras are Morita cancellative (respectively, skew cancellative). Several new results concerning these two types of cancellations, as well as the classical cancellation, are proved.

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0. Introduction

Let $\Bbbk$ denote a base field. A $\Bbbk$-algebra A is said to be **cancellative** in the category of $\Bbbk$-algebras if any $\Bbbk$-algebra isomorphism $\phi\colon A[x] \cong B[x]$ (where B is another $\Bbbk$-algebra) implies that A is isomorphic to B as a $\Bbbk$-algebra. Geometrically, a $\Bbbk$-variety V is called **cancellative** if any isomorphism $V \times \mathbb{A}^1 \cong W \times \mathbb{A}^1$ for another $\Bbbk$-variety W implies that $V \cong W$. Cancellative properties have been extensively investigated for commutative domains, especially for the commutative polynomial rings, in the literature [AEH, Cr, CM, Fu, Ma1, MS, Ru]. Note that not every commutative domain is cancellative [Da, Fi, Ho]. In the commutative case, we sometimes call this kind of question the "Zariski cancellation problem", as the cancellation problem of fields was first raised by Zariski in 1949 [Se]. In the noncommutative case, the study of cancellative properties dates back to the early 1970s [AEH, As, BR, CE, EH, EK]. Despite great success achieved in the work of Gupta [Gu1, Gu2], the Zariski cancellation problem still remains open for the commutative polynomial ring $\Bbbk[t_1, \ldots, t_n]$ with $n \geq 3$ in the characteristic zero case; see [Kr, Gu3] for a history of this open problem.

Recently, the study of cancellation problems has been revitalized for noncommutative algebras thanks to [BZ1], which mainly employs the famous Makar-Limanov invariants [Ma1, Ma2] and the noncommutative discriminants as investigated in [CPWZ1, CPWZ2]. It is usually very difficult to describe the discriminant for a given algebra; fortunately, many useful results on discriminants have been further established in [BY, CYZ1, CYZ2, GKM, GWY, NTY, WZ]. Ever since [BZ1], there has been much progress made in the study of cancellation problems for noncommutative algebras [BZ2, CYZ1, Ga, LR, LY, LeWZ, LuWZ, LMZ, Ta1, Ta2, TRZ]. In particular, the cancellative property was established in [LeWZ] for many classes of algebras which are not necessarily domains; and the Morita cancellation and derived cancellation were introduced and studied for algebras in [LuWZ]. The cancellative properties for Poisson algebras were most recently examined in [GaW].

The problem of skew cancellations was early considered in [AKP] and revisited in recent papers [Be, BHHV]. One of the main motivations of this paper is to introduce the multi-variable version of the skew cancellative property. In particular, we study the following two closely related topics:

(1) Strong Morita cancellation as initiated in [LuWZ].
(2) Multi-variable version of the skew cancellation as initiated in [AKP, Be, BHHV].

Although we are mainly interested in the Morita cancellation as introduced in [LuWZ], we also make some comments on the derived cancellation in Section 5. Our ideas and methods are inspired by the ones in [AEH, BR, CE, BZ1, CYZ1, CYZ2, LeWZ, LuWZ].

Before we state our results, we need to recall a list of basic definitions about the Morita cancellation from [BZ1, LeWZ, LuWZ]. Later we will recall another list of definitions concerning the skew cancellation. We denote by $A[t_1, \ldots, t_n]$ the polynomial extension of an algebra A with commuting multi-variables $t_1, \ldots, t_n$ and by $M(A)$ the category of all right A-modules. All algebraic objects are defined over the base field $\Bbbk$.

Definition 0.1: Let A be an algebra.

(1) We say A is **strongly cancellative** if any $\Bbbk$-algebra isomorphism
$$A[s_1, \ldots, s_n] \cong B[t_1, \ldots, t_n],$$
for any $n \geq 1$ and any algebra B, implies that A is isomorphic to B as a $\Bbbk$-algebra.

(2) We say A is **universally cancellative** if, for every finitely generated commutative domain R with an ideal $I \subset R$ such that $\Bbbk \longrightarrow R \longrightarrow R/I$ is an isomorphism and every algebra B, any $\Bbbk$-algebra isomorphism
$$A \otimes_{\Bbbk} R \cong B \otimes_{\Bbbk} R$$
implies that $A \cong B$ as $\Bbbk$-algebras.

Definition 0.2: Let A be an algebra.

(1) We say A is **Morita cancellative** if any equivalence of abelian categories
$$M(A[s]) \cong M(B[t]),$$
for any algebra B, implies an equivalence of abelian categories
$$M(A) \cong M(B).$$

(2) We say A is **strongly Morita cancellative** if any equivalence of abelian categories
$$M(A[s_1, \ldots, s_n]) \cong M(B[t_1, \ldots, t_n]),$$
for any $n \geq 1$ and any algebra B, implies an equivalence of abelian categories
$$M(A) \cong M(B).$$

The above (strong) Morita cancellation of noncommutative algebras is a natural generalization of the classical cancellation in the category of commutative algebras. The following universal version of the Morita cancellation is similar to those in Definition 0.1.

Definition 0.3: Let A be an algebra. We say A is **universally Morita cancellative** if, for every finitely generated commutative domain R with an ideal $I \subset R$ such that $\Bbbk \longrightarrow R \longrightarrow R/I$ is an isomorphism and every algebra B, any equivalence of abelian categories

$$M(A \otimes_{\Bbbk} R) \cong M(B \otimes_{\Bbbk} R)$$

implies an equivalence of abelian categories

$$M(A) \cong M(B).$$

We refer the reader to [BZ1, LeWZ, LuWZ] and Section 1 for other basic definitions. Now we can state our results about the Morita cancellation. The first one is a Morita version of [BZ1, Proposition 1.3].

Theorem 0.4: *Let A be an algebra with center being the base field $\Bbbk$. Then A is universally Morita cancellative.*

The next result can be viewed as both a Morita version and a strengthened version of a partial combination of [LeWZ, Theorem 4.1] with [LeWZ, Theorem 4.2]. We denote the nilradical of an algebra A by $N(A)$. The definition of the strongly retractable property is given in Definition 1.1 (see also [LeWZ, Definition 2.1]).

Theorem 0.5: *Let A be an algebra with center Z such that either Z or $Z/N(Z)$ is strongly retractable (respectively, strongly detectable). Then A is strongly cancellative and strongly Morita cancellative.*

This theorem has several consequences. For example, by using Theorem 0.5 (and combining with Lemma 1.2(2)), the hypotheses of being "strongly Hopfian" in [LuWZ, Theorem 0.3, Lemma 3.6, Theorem 4.2(2), Corollary 4.3, Corollary 7.3] and [LeWZ, Theorem 0.2, Theorem 4.2] are superfluous. Next we give an explicit application. Recall that a commutative algebra is called **von Neumann regular** if it is reduced and has Krull dimension zero.

Corollary 0.6: *Let A be an algebra with center Z.*

(1) *If $Z/N(Z)$ is generated by a set of units of $Z/N(Z)$, then Z and A are strongly cancellative and strongly Morita cancellative.*
(2) *If $Z/N(Z)$ is a von Neumann regular algebra, then Z and A are strongly cancellative and strongly Morita cancellative.*
(3) *If Z is a finite direct sum of local algebras, then Z and A are strongly cancellative and strongly Morita cancellative.*

Note that Corollary 0.6(2) answers [LeWZ, Question 0.1] positively. All statements concerning the Morita cancellation in Corollary 0.6 are new. The above corollary also has many applications in practice.

The second part of the paper deals with the skew cancellation which is another natural generalization of the classical cancellation. Here we replace the polynomial extensions by the Ore extensions. Let A be an algebra. Let σ be an algebra automorphism of A and δ be a σ-derivation of A. Then one can form the Ore extension, denoted by $A[t;\sigma,\delta]$, which shares many nice properties with the polynomial extension $A[t]$. The reader is referred to [MR, Chapter 1] for more details. We say σ is **locally algebraic** if every finite dimensional subspace of A is contained in a σ-stable finite-dimensional subspace of A. It is obvious that the identity map is locally algebraic. An iterated Ore extension of A is of the form

$$A[t_1;\sigma_1,\delta_1][t_2;\sigma_2,\delta]\cdots[t_n;\sigma_n,\delta_n]$$

where σ_i is an algebra automorphism of

$$A_{i-1} := A[t_1;\sigma_1,\delta_1]\cdots[t_{i-1};\sigma_{i-1},\delta_{i-1}]$$

and δ_i is a σ_i-derivation of A_{i-1}.

Definition 0.7: Let A be an algebra.

(1) We say A is **skew cancellative** if any isomorphism of algebras
$$A[t;\sigma,\delta] \cong A'[t';\sigma',\delta']$$
for another algebra A', implies an isomorphism of algebras
$$A \cong A'.$$
(2) We say A is **strongly skew cancellative** if any isomorphism of algebras
$$A[t_1;\sigma_1,\delta_1]\cdots[t_n;\sigma_n,\delta_n] \cong A'[t_1';\sigma_1',\delta_1']\cdots[t_n';\sigma_n',\delta_n']$$
for any $n \geq 1$ and any algebra A', implies an isomorphism of algebras
$$A \cong A'.$$

Occasionally, we will restrict our attention to special types of Ore extensions and/or special classes of base algebras. For example, we make the following definition.

Definition 0.8: Let A be an algebra.

(1) We say A is **σ-cancellative** if in Definition 0.7(1), only Ore extensions with $\delta = 0$ and $\delta' = 0$ are considered. We say A is **strongly σ-cancellative** if in Definition 0.7(2), only Ore extensions with $\delta_i = 0$ and $\delta'_i = 0$, for all i, are considered.

(2) We say A is **δ-cancellative** if in Definition 0.7(1), only Ore extensions with $\sigma = \mathrm{Id}_A$ and $\sigma' = \mathrm{Id}_{A'}$ are considered. We say A is **strongly δ-cancellative** if in Definition 0.7(2), only Ore extensions with $\sigma_i = \mathrm{Id}$ and $\sigma'_i = \mathrm{Id}$, for all i, are considered.

(3) We say A is **σ-algebraically cancellative** if in Definition 0.7(1), only Ore extensions with locally algebraic σ and σ' are considered. We say A is **strongly σ-algebraically cancellative** if in Definition 0.7(2), only Ore extensions with locally algebraic σ_i and σ'_i are considered.

A classical cancellation problem is equivalent to a skew cancellation problem with $(\sigma, \delta) = (\mathrm{Id}, 0)$. Therefore, the skew (or σ-, or δ-)cancellation is a natural extension and a strictly stronger version of the classical cancellation. It follows from the definition that the σ-algebraically cancellative property is stronger than the δ-cancellative property. See Figure 1 after Example 5.5. The δ-cancellation was first considered in [AKP], and then in [Be]. In [BHHV, Theorem 1.2], a very nice result concerning both σ- and δ-cancellations was proved, however, the skew cancellative property remains open. As remarked in [BHHV], “[it] would be interesting to give a ‘unification’ of the two results occurring in [BHHV, Theorem 1.2] and prove that skew cancellation holds for general skew polynomial extensions, although this appears to be considerably more subtle than the cases we consider.” One of our main goals in the second half of the paper is to introduce a unified approach to the skew cancellation problem (including both σ- and δ-cancellation).

To state our main results, we need to recall the definition of a divisor subalgebra as introduced in [CYZ1]. Let A be a domain. Let F be a subset of A. Let $Sw(F)$ denote the set of $g \in A$ such that $f = agb$ for some $a, b \in A$ and $0 \neq f \in F$. That is, $Sw(F)$ consists of all the subwords of the elements in F. We set $D_0(F) = F$ and inductively define $D_n(F)$ for $n \geq 1$ as the $\Bbbk$-subalgebra of A generated by $Sw(D_{n-1}(F))$. The subalgebra

$$\mathbb{D}(F) = \bigcup_{n \geq 0} D_n(F)$$

is called the **divisor subalgebra** of A generated by F. If F is the singleton $\{f\}$, we simply write $\mathbb{D}(\{f\})$ as $\mathbb{D}(f)$. See Section 5 for more details.

Theorem 0.9: *Let A be an affine domain of finite GK dimension. Suppose that $\mathbb{D}(1) = A$. Then A is strongly σ-algebraically cancellative. As a consequence, it is strongly δ-cancellative.*

To prove several classes of algebras are skew cancellative, we need to use a structure result of division algebras. Recall from [Sc] that a simple artinian ring S is **stratiform** over $\Bbbk$ if there is a chain of simple artinian rings

$$S = S_n \supseteq S_{n-1} \supseteq \cdots \supseteq S_1 \supseteq S_0 = \Bbbk$$

where, for every i, either

(i) S_{i+1} is finite over S_i on both sides; or
(ii) S_{i+1} is equal to the quotient ring of the Ore extension $S_i[t_i; \sigma_i, \delta_i]$ for an automorphism σ_i of S_i and σ_i-derivation δ_i of S_i.

Such a chain of simple artinian rings is called a stratification of S. The **stratiform length** of S is the number of steps in the chain that are of type (ii). An important fact established in [Sc] is that the stratiform length is an invariant of S. A Goldie prime ring A is called **stratiform** if the quotient division ring of A, denoted by $Q(A)$, is stratiform.

Theorem 0.10: *Let A be a noetherian domain that is stratiform. Suppose that $\mathbb{D}(1) = A$. Then A is strongly skew cancellative in the category of noetherian stratiform domains.*

The following algebras are stratiform with $\mathbb{D}(1) = A$. As a result, they are skew cancellative.

(a) Quantum torus or quantum Laurent polynomial algebras given in Example 4.3(5),
(b) Localized quantum Weyl algebras $B_1^q(\Bbbk)$ in Example 4.3(2).
(c) Affine commutative domain A of GK dimension one satisfying $A^\times \supsetneq \Bbbk^\times$ [Lemma 4.4(9)].

(d) Any noetherian domain that can be written as a finite tensor product (resp. some version of a twisted tensor product) of the algebras in parts (a,b,c).

We further prove a few results concerning the strong cancellation, the strong Morita cancellation, and the skew cancellation of noncommutative algebras; see Theorems 4.6 and 5.4, and Proposition 5.8.

The paper is organized as follows. Section 1 reviews some basic materials. In Section 2 we recall some basic properties about the Gelfand–Kirillov dimension and homological transcendence degree of noncommutative algebras. Section 3 concerns the Morita cancellative property where Theorem 0.5 and Corollary 0.6 are proven. Then we review the definition of a divisor subalgebra and study the skew cancellative property in Section 4. We also prove our main results, namely, Theorems 0.9 and 0.10, in Section 4. The final section contains some comments, examples, remarks and questions.

1. Preliminaries

Throughout $\Bbbk$ denotes a base field. All algebras are $\Bbbk$-algebras and all algebra homomorphisms are $\Bbbk$-linear algebra homomorphisms. As needed, we will continue to use the notation and convention introduced in [BZ1, LeWZ, LuWZ].

We only recall a small selected set of definitions.

Definition 1.1 ([LeWZ, Definition 2.1]): Let A be an algebra.

(1) We say A is **retractable** if, for any algebra B, any algebra isomorphism

$$\phi : A[s] \cong B[t]$$

implies that $\phi(A) = B$.

(2) [AEH, p. 311] We say A is **strongly retractable** if, for any algebra B and integer $n \geq 1$, any algebra isomorphism

$$\phi : A[s_1, \ldots, s_n] \cong B[t_1, \ldots, t_n]$$

implies that $\phi(A) = B$.

The following lemma of Brewer–Rutter [BR] is useful. Suppose A is a subring of a ring B and $f_1, \ldots, f_n$ are elements in B, then the subring generated by A and $f_1, \ldots, f_n$ is denoted by $A\{f_1, \ldots, f_n\}$.

Lemma 1.2 ([BR, Lemma 1]): *Let A be an algebra with center Z.*

(1) *If $f_1, \ldots, f_n$ are Z-generators of the polynomial ring $Z[Y_1, \ldots, Y_n]$, then the A-endomorphism τ of $A[Y_1, \ldots, Y_n]$ defined by $\tau(Y_i) = f_i$ for each $1 \le i \le n$ is an isomorphism.*

(2) *As a special case, if A is commutative, and if $f_1, \ldots, f_n$ are A-generators of the polynomial ring $A[Y_1, \ldots, Y_n]$, then $A\{f_1, \ldots, f_n\} = A[f_1, \ldots, f_n]$.*

For any algebra A, let $Z(A)$ or simply Z denote the center of A and let $N(A)$ denote the nilpotent radical of A. Suppose two algebras R and S are Morita equivalent. Let

$$\omega : Z(R) \to Z(S) \tag{E1.2.1}$$

be the isomorphism of the centers given in [LuWZ, Lemma 1.2(3)]. Note that we can use all facts listed in [LuWZ, Lemma 1.2(3)].

In the following two definitions, we have the following abbreviations.

$$S = \text{strongly},\ M = \text{Morita, and } R = \text{reduced}.$$

Definition 1.3: Let A be an algebra.

(1) We say A is **Morita Z-detectable** or simply **MZ-detectable** if, for any algebra B and any equivalence of abelian categories

$$\mathcal{E}\colon M(A[s]) \longrightarrow M(B[t]),$$

with the induced isomorphism, see (E1.2.1),

$$\omega\colon Z(A[s])(= Z(A)[s]) \longrightarrow Z(B[t])(= Z(B)[t])$$

implies that

$$Z(B)[t] = Z(B)\{\omega(s)\}.$$

By Lemma 1.2, we actually have that

$$Z(B)[t] = Z(B)[\omega(s)].$$

(2) We say A is **strongly Morita Z-detectable** or simply **SMZ-detectable** if, for each $n \ge 1$ and any algebra B, any equivalence of abelian categories

$$\mathcal{E}\colon M(A[s_1, \ldots, s_n]) \longrightarrow M(B[t_1, \ldots, t_n])$$

implies that, with ω given in (E1.2.1) for algebras $R = A[s_1, \dots, s_n]$ and $S = B[t_1, \dots, t_n]$,

$$Z(B)[t_1, \dots, t_n] = Z(B)\{\omega(s_1), \dots, \omega(s_n)\}.$$

Once again, by Lemma 1.2, we actually have that

$$Z(B)[t_1, \dots, t_n] = Z(B)[\omega(s_1), \dots, \omega(s_n)].$$

In the next definition, ω is given as in (E1.2.1) for appropriate R and S and $\overline{\omega}$ is an induced isomorphism in appropriate setting.

Definition 1.4: Let A be an algebra.

(1) We say A is **reduced Morita Z-detectable** or **RMZ-detectable** if, for any algebra B, any equivalence of abelian categories

$$E\colon M(A[s]) \longrightarrow M(B[t])$$

with the induced isomorphism (modulo prime radicals)

$$\overline{\omega}\colon Z(A)/N(Z(A))[\overline{s}] \longrightarrow Z(B)/N(Z(B))[\overline{t}]$$

implies that

$$Z(B)/N(Z(B))[\overline{t}] = Z(B)/N(Z(B))[\overline{\omega}(\overline{s})].$$

(2) We say A is **strongly reduced Morita Z-detectable** or simply **SRMZ-detectable** if, for each $n \geq 1$ and any algebra B, any equivalence of abelian categories

$$\mathcal{E}\colon M(A[s_1, \dots, s_n]) \longrightarrow M(B[t_1, \dots, t_n])$$

implies that

$$Z(B)/N(Z(B))[\overline{t}_1, \dots, \overline{t}_n] = Z(B)/N(Z(B))[\overline{\omega}(\overline{s}_1), \dots, \overline{\omega}(\overline{s}_n)].$$

Several retractabilities are defined in [LeWZ, LuWZ]. It has been observed in [BR, LeWZ, LuWZ] that the cancellative property of an algebra A is controlled by its center $Z(A)$ to a large degree. In the rest of this section, we establish or recall some basic facts. In Section 2, we will show that there is a Morita analogue of [BR, Theorem 1] and [LeWZ, Theorem 4.2] can be strengthened.

The following result is essentially verified in the proof of [BR, Theorem 1], see [BR, pp. 485–486], and in the proof of [EK, Statement #4, pp. 334–335]. For the reader's convenience, we recall it as a lemma and reproduce its proof as follows.

LEMMA 1.5 ([BR, Theorem 1]): *Suppose that A and B are commutative algebras. Let*

$$\sigma\colon A[s_1,\dots,s_n]\longrightarrow B[t_1,\dots,t_n]$$

be an isomorphism of algebras such that the induced isomorphism modulo prime radicals, denoted by

$$\overline{\sigma}\colon A/N(A)[\overline{s}_1,\dots,\overline{s}_n]\longrightarrow B/N(B)[\overline{t}_1,\dots,\overline{t}_n],$$

has the property that

$$B/N(B)[\overline{t}_1,\dots,\overline{t}_n]=B/N(B)\{\overline{f}_1,\dots,\overline{f}_n\}$$

where $f_i=\sigma(s_i)$ for $i=1,\dots,n$. Then

$$B[t_1,\dots,t_n]=B\{f_1,\dots,f_n\}=B[f_1,\dots,f_n]$$

where $f_1,\dots,f_n$ are considered as commutative indeterminates over B.

In essence, Lemma 1.5 implies that a certain detectability lifts from $A/N(A)$ to A.

Proof of Lemma 1.5. Since $B/N(B))[\overline{t}_1,\dots,\overline{t}_n]=B/N(B)\{\overline{f}_1,\dots,\overline{f}_n\}$, there are polynomials in B, say $g_1,\dots,g_n$, such that

$$\overline{t}_i=\overline{g}_i(\overline{f}_1,\dots,\overline{f}_n).$$

As a result, for $i=1,\dots,n$, we have the following:

$$t_i=g_i(f_1,\dots,f_n)+h_i(t_1,\dots,t_n)$$

where $h_i\in N(B[t_1,\dots,t_n])=N(B)[t_1,\dots,t_n]$. Denote by N_0 the ideal of B generated by the coefficients of $h_1,\dots,h_n$. Then, by induction, we have that

$$B[t_1,\dots,t_n]=B\{f_1,\dots,f_n\}+N_0^mB[t_1,\dots,t_n]$$

for each $m\geq 1$. Since N_0 is a finitely generated ideal of B and N_0 is contained in $N(B)$, the prime radical of B, we have that N_0 is nilpotent. As a result, we have that

$$B[t_1,\dots,t_n]=B\{f_1,\dots,f_n\}.$$

Using Lemma 1.2(2), we conclude that

$$B[t_1,\dots,t_n]=B[f_1,\dots,f_n]$$

where the elements $f_1,\dots,f_n$ are regarded as commutative indeterminates over B. ∎

We now state a couple of easy facts about detectability.

Lemma 1.6: *Let Z be the center of an algebra A.*

(1) *If Z is strongly retractable, then A is strongly Morita Z-retractable, and consequently, SMZ-detectable.*
(2) *Suppose that $Z/N(Z)$ is strongly retractable. Then A is strongly reduced Morita Z-retractable. As a consequence, A is SMZ-detectable.*

Proof. (1) The first assertion follows from [LeWZ, Definition 2.6(4)]. For the second assertion, see the proof of [LeWZ, Lemma 3.4].

(2) The first statement is part (1). By part (1), A is strongly reduced Morita detectable. By Lemma 1.5, A is strongly Morita Z-detectable, or SMZ-detectable. ∎

Lemma 1.7: *Let Z be the center of an algebra A.*

(1) *Suppose that A is either strongly Morita Z-retractable or strongly reduced Morita Z-retractable. Then A is SMZ-detectable.*
(2) [As, Theorem 1.2] *If A is reduced, then A is SMZ-detectable if and only if A is strongly Morita Z-retractable.*

Proof. (1) It follows from Lemma 1.6.

(2) If A is strongly Morita Z-retractable, by the proof of [LeWZ, Lemma 3.4], A is SMZ-detectable. The converse statement follows from the proof of [As, Theorem 1.2] which we repeat next.

Suppose B is another algebra such that

$$\mathcal{E}\colon M(A[s_1,\dots,s_n]) \longrightarrow M(B[t_1,\dots,t_n])$$

is an equivalence of abelian categories. Let

$$\omega\colon Z(A)[s_1,\dots,s_n] \longrightarrow Z(B)[t_1,\dots,t_n]$$

be the corresponding induced isomorphism given in (E1.2.1). Denote by f_i the element $\omega(s_i) \in Z(B)[t_1,\dots,t_n]$ for $i = 1,\dots,n$. Since A is SMZ-detectable, by definition,

$$Z(B)[t_1,\dots,t_n] = Z(B)[f_1,\dots,f_n].$$

As a consequence, we have that

$$\omega(Z(A))[f_1,\dots,f_n]=\omega(Z(A)[s_1,\dots,s_n])=Z(B)[t_1,\dots,t_n]=Z(B)[f_1,\dots,f_n].$$

Now we need to show that $\omega(Z(A)) = Z(B)$. To simplify the notation, we will denote $\omega(Z(A))$ by R and $Z(B)$ by S respectively, and f_i by X_i instead. Set

$$S_k = S[X_1, \ldots, X_{k-1}, X_{k+1}, \ldots, X_n]$$

for $k = 1, \ldots, n$. Then $S[X_1, \ldots, X_n] = S_k[X_k]$ is a polynomial algebra in a single indeterminate X_k over S_k. Note that any element α of $R = \omega(Z(A))$ can be written in the following form:

$$\alpha = \beta_0 + \beta_1 X_k + \cdots + \beta_m X_k^m$$

where $\beta_i \in S_k$. Suppose that $f(X_k) = \gamma_0 + \gamma_1 X_k + \cdots \gamma_l X_k^l$ is a polynomial in $S_k[X_k]$ such that $S_k[X_k] = S_k[f(X_k)]$. Then it is true that γ_1 is a unit and $\gamma_2, \ldots, \gamma_l$ are nilpotent elements of S_k. Since A is reduced, its center $Z(A)$ is reduced. Then $R = \omega(Z(A))$ is reduced as well. As a result, $S = Z(B)$ is reduced. Thus, S_k is reduced too. We have that $\gamma_2 = \cdots = \gamma_l = 0$. Note that

$$\begin{aligned} R[X_1, \ldots, X_n] &= R[X_1, \ldots, X_{k-1}, X_k + \alpha, X_{k+1}, \ldots, X_n] \\ &= R[X_1, \ldots, X_{k-1}, X_k + \alpha^2, X_{k+1}, \ldots, X_n]. \end{aligned}$$

As a result, we have that

$$S_k[X_k] = S_k[X_k + \alpha] = S_k[X_k + \alpha^2]$$

which implies that $\beta_1, \ldots, \beta_m$ are nilpotent elements of S_k and thus equal to zero. So we have that $\alpha \in S_k$ for $k=1, \ldots, n$. Note that $\bigcap_{k=1}^n S_k = S$. So we have proved that $\alpha \in S$ as desired. Note that $R \subseteq S$ and $R[X_1, \ldots, X_n] = S[X_1, \ldots, X_n]$ can imply that $R = S$ by [BR, Lemma 2]. ∎

Lemma 1.8: *An algebra A is SMZ-detectable if and only if it is SRMZ-detectable.*

Proof. Suppose that the algebra A is SMZ-detectable and let

$$\mathcal{E} \colon M(A[s_1, \ldots, s_n]) \longrightarrow M(B[t_1, \ldots, t_n])$$

be an equivalence of abelian categories where B is another algebra. Note that the equivalence $\mathcal{E}$ induces an algebra isomorphism

$$\omega \colon Z(A)[s_1, \ldots, s_n] \longrightarrow Z(B)[t_1, \ldots, t_n],$$

as in (E1.2.1). Since A is SMZ-detectable, we have that

$$Z(B)[t_1, \ldots, t_n] = Z(B)[f_1, \ldots, f_n]$$

where $f_i = \omega(s_i)$ for $i = 1, \ldots, n$. Modulo both sides by the nil-radical, we obtain that

$$Z(B)/N(Z(B))[\bar{t}_1, \ldots, \bar{t}_n] = Z(B)/N(Z(B))[\overline{f}_1, \ldots, \overline{f}_n].$$

By definition, A is SRMZ-detectable. The other implication follows from the reversed argument and Lemma 1.5. ∎

However, there exists a commutative algebra which is SRMZ-retractable, but not SMZ-retractable. The following example is borrowed from [As, Example 1]: see also [LeWZ, Example 3.3].

Example 1.9: Let $A = k[x, y]/(x^2, y^2, xy)$. Then A is SRMZ-retractable. Furthermore, it is SMZ-detectable and SRMZ-detectable, but neither strongly retractable nor SMZ-retractable.

2. GK dimension and Homological transcendence degree

2.1. GK dimension. Let A be an algebra over $\Bbbk$. The **Gelfand–Kirillov dimension** (or **GK dimension** for short) of A is defined to be

$$\operatorname{GKdim} A := \sup_V \limsup_{n\to\infty}(\log_n(\dim_{\Bbbk} V^n))$$

where V runs over all finite-dimensional subspaces of A. We refer the reader to [KL] for more details. Next we prove or review some preliminary results concerning the GK dimension of Ore extensions.

Let σ be an automorphism of A. Recall that σ is called **locally algebraic** if every finite-dimensional subspace of A is contained in a σ-stable finite-dimensional subspace of A. If A is affine, then σ is locally algebraic if and only if there is a σ-stable finite-dimensional generating subspace.

Lemma 2.1: *Let A be an affine algebra over $\Bbbk$.*

(1) *Let $B := A[t; \sigma, \delta]$ be an Ore extension of A. If σ is locally algebraic, then* $\operatorname{GKdim} B = \operatorname{GKdim} A + 1$.
(2) *Let B be an iterated Ore extension $A[t_1; \sigma_1, \delta_1] \cdots [t_n; \sigma_n, \delta_n]$ where each σ_i is locally algebraic. Then* $\operatorname{GKdim} B = \operatorname{GKdim} A + n$.
(3) *Let B be an iterated Ore extension $A[t_1; \delta_1] \cdots [t_n; \delta_n]$. Then* $\operatorname{GKdim} B = \operatorname{GKdim} A + n$.

Proof. (1) We may assume that $1 \in V$. Let W be any finite-dimensional generating subspace of A with $1 \in W$. Since V generates A, $W \subseteq V^n$ for some n. Without loss of generality, we can assume that $W = V$. Since V generates A, we have $\delta(V) \subseteq V^m$ for some m. Now the assertion follows from [Zh, Lemma 4.1].

(2) This follows from induction and part (1).

(3) This is a special case of part (2) by setting σ_i to be the identity. ∎

The reader is referred to [KL, p. 64] for the definition of a filtered algebra. The following lemma is similar to [BZ1, Lemma 3.2].

Lemma 2.2: *Let Y be a filtered algebra with an $\mathbb{N}$-filtration $\{F_iY\}_{i\geq 0}$. Assume that the associated graded algebra $\operatorname{gr} Y$ is an $\mathbb{N}$-graded domain. Suppose Z is a subalgebra of Y and let $Z_0 = Z \cap F_0Y$. If*

$$\operatorname{GKdim} Z = \operatorname{GKdim} Z_0 < \infty,$$

then

$$Z = Z_0.$$

Proof. Suppose Z strictly contains the subalgebra Z_0. There is a natural filtration on Z induced from Y by taking $F_iZ := Z \cap F_iY$. As a result, $\operatorname{gr} Z$ is a subalgebra of $\operatorname{gr} Y$. By [KL, Lemma 6.5],

$$\operatorname{GKdim} Z \geq \operatorname{GKdim} \operatorname{gr} Z \geq \operatorname{GKdim} F_0Z = \operatorname{GKdim} Z_0 = \operatorname{GKdim} Z.$$

Since $\operatorname{gr} Z$ is an $\mathbb{N}$-graded subalgebra of $\operatorname{gr} Y$ that strictly contains $Z_0 = F_0Z$, there is an element $a \in \operatorname{gr} Z$ of positive degree. Considering the grading, we see that

$$Z_0 + Z_0a + Z_0a^2 + \cdots$$

is a direct sum contained in $\operatorname{gr} Z$. As a result, we obtain that

$$\operatorname{GKdim} \operatorname{gr} Z \geq \operatorname{GKdim}(\operatorname{gr} Z)_0 + 1 = \operatorname{GKdim} Z_0 + 1,$$

which yields a contradiction. Therefore $Z = Z_0$. ∎

The above lemma has an immediate consequence.

Proposition 2.3: *Let Y be an iterated Ore extension $A[t_1; \sigma_1, \delta_1] \cdots [t_n; \sigma_n, \delta_n]$ of a domain A. Let B be a subalgebra of Y containing A. If*

$$\operatorname{GKdim} A = \operatorname{GKdim} B < \infty,$$

then

$$A = B.$$

Proof. Let $m \leq n$ be the minimal integer such that

$$B \subseteq A[t_1;\sigma_1,\delta_1]\cdots[t_m;\sigma_m,\delta_m].$$

It remains to show that $m = 0$. Suppose on the contrary that $m \geq 1$. Let

$$Y = A[t_1;\sigma_1,\delta_1]\cdots[t_m;\sigma_m,\delta_m]$$

and

$$Y_0 = A[t_1;\sigma_1,\delta_1]\cdots[t_{m-1};\sigma_{m-1},\delta_{m-1}].$$

Define an $\mathbb{N}$-filtration on Y by

$$F_iY = \sum_{s=0}^{i} Y_0 t_m^s.$$

Let $Z = B$ and $Z_0 = Z \cap Y_0$. By the choice of m, we have $Z \neq Z_0$. By the hypothesis on the GK dimension, we have

$$\operatorname{GKdim} Z_0 \geq \operatorname{GKdim} A = \operatorname{GKdim} B = \operatorname{GKdim} Z \geq \operatorname{GKdim} Z_0.$$

By Lemma 2.2, we have $Z = Z_0$, a contradiction. Therefore the assertion follows. ∎

2.2. Homological transcendence degree. Another useful invariant is the Homological transcendence degree introduced in [YZ]. Recall from [YZ, Definition 1.1] that the **Homological transcendence degree** of a division algebra D is defined to be

$$\operatorname{Htr} D := \operatorname{injdim} D \otimes D^{\mathrm{op}}$$

where D^{op} is the opposite algebra of D. One result of [YZ, Proposition 1.8] is that $\operatorname{Htr} D = n$ if D is a stratiform division algebra of stratiform length n. We say A is stratiform if A is Goldie prime and the ring of its fractions, denoted by $Q(A)$, is stratiform. As an immediate consequence, we have

Lemma 2.4: *Let A be a noetherian domain that is stratiform. If B is an n-step iterated Ore extension of A, then*

$$\operatorname{Htr} Q(B) = \operatorname{Htr} Q(A) + n.$$

There is a variety of examples which are stratiform algebras; and the following are some typical examples (details are omitted).

Example 2.5: The following algebras are stratiform.

(1) Affine commutative domains.
(2) PI prime algebras that are finitely generated over their affine centers.
(3) Skew polynomial algebras and their localizations [YZ, Example 1.9(g)].
(4) Quantum Weyl algebras as defined next or their localizations. Let $q \neq 0$ be a scalar in $\Bbbk$. Let $A_1^q(\Bbbk)$ denote the first quantum Weyl algebra, which is a $\Bbbk$-algebra generated by x, y subject to the relation $xy - qyx = 1$.
(5) Prime algebras that can be written as a tensor product of algebras listed as above.

Here is a version of Proposition 2.3 with the GK dimension replaced by the Homological transcendence degree.

PROPOSITION 2.6: *Let A be a noetherian stratiform domain. Let Y be an iterated Ore extension $A[t_1; \sigma_1, \delta_1] \cdots [t_n; \sigma_n, \delta_n]$. Let B be a subalgebra of Y containing A such that it is stratiform. If*

$$\operatorname{Htr} Q(A) = \operatorname{Htr} Q(B),$$

then

$$A = B.$$

Proof. Suppose on the contrary that $A \neq B$. By the proof of Lemma 2.2, there is an element $a \in B$ such that $Q(A) + Q(A)a + Q(A)a^2 + \cdots$ is a direct sum. This implies that $\dim_{Q(A)} Q(B)$ is infinite. Note that $Q(B)$ is a $(Q(A), Q(B))$-bimodule that is finitely generated as a right $Q(B)$-module. Since $Q(A)$ and $Q(B)$ have the same stratiform length by [YZ, Proposition 1.8], $Q(B)$ is finitely generated as left $Q(A)$-module by [Sc, Theorem 24]. This contradicts the fact that $\dim_{Q(A)} Q(B)$ is infinite. Therefore, we have that $A = B$. ∎

3. Morita cancellation

This section concerns Morita cancellative properties. We also prove some of the results stated in the introduction. The first result, namely, Theorem 0.4, is about universally Morita cancellation whose proof is essentially adopted from [BZ1]. Let $\operatorname{GKdim} A$ denote the Gelfand–Kirillov dimension of an algebra A. We refer the reader to [KL] and Section 2 for the basic definitions and properties of Gelfand–Kirillov dimension.

Proof of Theorem 0.4. Let B be another algebra. Let R be an affine commutative domain with an ideal $I \subset R$ such that $R/I = \Bbbk$. Suppose that

$$\mathcal{E}\colon M(A \otimes_{\Bbbk} R) \longrightarrow M(B \otimes_{\Bbbk} R)$$

is an equivalence of abelian categories. By (E1.2.1), $\mathcal{E}$ induces an isomorphism

$$\omega\colon Z(A \otimes_{\Bbbk} R) \cong Z(B \otimes_{\Bbbk} R)$$

between the centers. Since $Z(A) = \Bbbk$, we obtain that

$$R = Z(A) \otimes_{\Bbbk} R = Z(A \otimes_{\Bbbk} R) \cong Z(B \otimes_{\Bbbk} R) = Z(B) \otimes_{\Bbbk} R.$$

As a result, we have that $R \cong Z(B) \otimes_{\Bbbk} R$. In particular, $Z(B)$ is a commutative domain. Due to a consideration of the GK dimension, we see that $\operatorname{GKdim} Z(B) = 0$, regarded as a $\Bbbk$-algebra. Thus $Z(B)$ is indeed a field. We have that $Z(B) = \Bbbk$ due to the fact that there is an ideal $I \subset R$ such that $R/I = \Bbbk$. Consequently, we have that $Z(B \otimes_{\Bbbk} R) = R$. As a result, ω is an isomorphism from $R \longrightarrow R$ which implies that $R/\omega(I) = \Bbbk$. Note that $A \cong (A \otimes_{\Bbbk} R)/I$ is Morita equivalent to $(B \otimes_{\Bbbk} R)/(\omega(I)) \cong B$ [LuWZ, Lemma 2.1(5)]. Thus, we have proved that A is universally Morita cancellative. ∎

The following result is a re-statement of Theorem 0.5. It is an analogue of [BR, Theorem 1] and serves as an improvement of [LeWZ, Theorem 4.2]. Note that our result does not require the strongly Hopfian assumption. We should mention that [BR, Theorem 1] deals with the cancellation problem in the category of rings; but the idea of its proof carries over word for word for $\Bbbk$-algebras.

Theorem 3.1: *Let A be an algebra with center Z. Suppose either*

(1) *Z or $Z/N(Z)$ is strongly retractable, or*

(2) *Z or $Z/N(Z)$ is strongly detectable.*

Then Z and A are strongly cancellative and strongly Morita cancellative.

Proof. For the assertions concern Z, it suffices to take $A = Z$. So it is enough to prove the assertions for A. We only prove that A is strongly Morita cancellative. The proof of strongly cancellative property is similar, and therefore is omitted.

Under the hypothesis of (1), by Lemma 1.6, A is SMZ-detectable. If Z is strongly detectable (part of the hypothesis in (2)), it is clear that A is SMZ-detectable. If $Z/N(Z)$ is strongly detectable, by Lemma 1.5, Z is strongly detectable. Therefore in all cases, we conclude that A is SMZ-detectable.

Let

$$\mathcal{E}\colon M(A[s_1,\dots,s_n]) \longrightarrow M(B[t_1,\dots,t_n])$$

be an equivalence of abelian categories. Then $\mathcal{E}$ induces an algebra isomorphism (E1.2.1) that is, in the current setting,

$$\omega\colon Z(A)[s_1,\dots,s_n] \xrightarrow{\cong} Z(B)[t_1,\dots,t_n].$$

Since A is SMZ-detectable,

$$Z(B)[t_1,\dots,t_n] = Z(B)[f_1,\dots,f_n]$$

where $f_i = \omega(s_i)$ for $i = 1,\dots,n$. Let I be the ideal of $A[s_1,\dots,s_n]$ generated by $s_1,\dots,s_n$. Then $\omega(I) = B[t_1,\dots,t_n](f_1,\dots,f_n) = B[f_1,\dots,f_n](f_1,\dots,f_n)$. As a result, we have that $A \cong A[s_1,\dots,s_n]/(A[s_1,\dots,s_n]I)$ which is Morita equivalent to $B[f_1,\dots,f_n]/(B[f_1,\dots,f_n]\omega(I)) \cong B$. That is, A is Morita equivalent to B. Therefore, B is strongly Morita cancellative. ∎

Next we mention some easy consequences.

Corollary 3.2: *Let A be an algebra with a center Z. Suppose one of the following holds:*

(1) *Either Z or $Z/N(Z)$ is an integral domain of transcendence degree one over a subfield of Z and is not isomorphic to $\Bbbk'[x]$ for any field extension $\Bbbk \subseteq \Bbbk' \subseteq Z$.*
(2) *Z is an integral domain with nonzero Jacobson radical.*

Then Z or $Z/N(Z)$ is strongly retractable. As a consequence, A is strongly cancellative and strongly Morita cancellative.

Proof. The consequence follows from Theorem 3.1. It remains to show that Z or $Z/N(Z)$ is strongly retractable.

(1) This is [LeWZ, Example 2.2].
(2) It follows from [AEH, Statement 1.10 on page 317]. ∎

Now we prove Corollary 0.6 below.

Proof of Corollary 0.6. (1) It follows from the proof of [LeWZ, Lemma 2.3] that Z is strongly retractable. The assertion follows from Theorem 3.1.

(2) By [BR, Theorem 2], a von Neumann regular algebra $Z/N(Z)$ is strongly retractable. The assertion follows from Theorem 3.1.

(3) By [BR, Theorem 3], Z is strongly retractable. The assertion follows from Theorem 3.1. ∎

Remark 3.3: Theorem 3.1 and Corollary 0.6 have many applications. Here is a partial list.

(1) In view of Corollary 0.6(2), if A is von Neumann regular, then the center Z will also be von Neumann regular. By Corollary 0.6(2), A is strongly cancellative and strongly Morita cancellative.

(2) If Z is of Krull dimension zero, then $Z/N(Z)$ is von Neumann regular. By Corollary 0.6(2) again, A is strongly cancellative and strongly Morita cancellative.

(3) If A is a finite direct product of simple algebras, then Z is a finite product of fields. Thus Z has Krull dimension zero. By the above comment, A is strongly cancellative and strongly Morita cancellative.

(4) By the proof of [LeWZ, Lemma 2.3] or [AEH, Statement 2.1, p. 320], the Laurent polynomial algebra $\Bbbk[x_1^{\pm 1}, \ldots, x_m^{\pm}]$ is strongly retractable. If Z or $Z/N(Z)$ is isomorphic to the Laurent polynomial algebra, then A is strongly cancellative and strongly Morita cancellative by Theorem 3.1.

We will explore some skew cancellative properties [Definitions 0.7 and 0.8] when the algebra A has "enough" invertible elements in Section 4.

4. Divisor subalgebras and skew cancellations

Recall from [BZ1, LeWZ, LuWZ] that the (strong) retractability implies the (strong) cancellation. It is clear that the (strong) retractability implies the (strong) Z-retractability, which in turn implies (strong) Morita cancellation, also see Section 3 and Theorem 0.5. In this section we continue our investigation on the (strong) retractability with a twist.

As asked in [LeWZ], one would like to know how localizations affect cancellative properties. Indeed, cancellative and Morita cancellative properties are preserved under localizations for many families of algebras. We add some results along this line; and we will address the problem in a forthcoming paper later on.

Note that even if the discriminant d of an algebra A is dominating or effective, the discriminant of A_d becomes invertible, where A_d is the localization of A with respect to the Ore set $\{d^i \mid i \geq 0\}$. As a result, the discriminant of A_d is neither dominating nor effective. However, since the element d is invertible in A_d, it will be sent to an invertible element by any $\Bbbk$-algebra isomorphism $\phi\colon A_d[s_1, \ldots, s_n] \longrightarrow B[t_1, \ldots, t_n]$. So we may still be able to prove that the algebra A_d is strongly retractable in many situations.

The point of this section is that we can do more. Namely, we can prove a version of the strong retractability even in the Ore extension setting. The main idea is to utilize the notion of divisor subalgebras introduced in [CYZ1].

We first recall the definition of a divisor subalgebra. Let A be a domain. Let F be a subset of A. Let $Sw(F)$ denote the set of $g \in A$ such that $f = agb$ for some $a, b \in A$ and $0 \neq f \in F$. That is, $Sw(F)$ consists of all the subwords of the elements in F. The following definition is quoted from [CYZ1].

Definition 4.1: Let F be a set of elements in a domain A.

(1) We set $D_0(F) = F$ and inductively define $D_n(F)$ for $n \geq 1$ as the $\Bbbk$-subalgebra of A generated by $Sw(D_{n-1}(F))$. The subalgebra

$$\mathbb{D}(F) = \bigcup_{n \geq 0} D_n(F)$$

is called the **divisor subalgebra** of A generated by F. If F is the singleton $\{f\}$, we simply write $\mathbb{D}(\{f\})$ as $\mathbb{D}(f)$. If we need to indicate the ambient algebra A, we write $\mathbb{D}(F)$ as $\mathbb{D}_A(F)$.

(2) If $f = d(A/Z)$ (if the discriminant $d(A/Z)$ indeed exists), we call $\mathbb{D}(f)$ the **discriminant-divisor subalgebra** of A or *DDS* of A, and write it as $\mathbb{D}(A)$.

We now define some elements which play the same role as the dominating or effective elements in the study of cancellative properties.

Definition 4.2: Let F be a set of elements in an algebra A which is a domain.

(1) We say F is a **controlling set** if $\mathbb{D}(F) = A$.
(2) An element $0 \neq f \in A$ is called **controlling** if $\mathbb{D}(f) = A$.

Next we give some examples of controlling elements. For an algebra A, let $A^\times$ denote the set of invertible elements in A.

Example 4.3: Let $q \neq 0, 1$ be a scalar in $\Bbbk$.

(1) Let $A_1^q(\Bbbk)$ be the first quantum Weyl algebra defined as in Example 2.5(4). Set $z = xy - yx$, then

$$xy = \frac{qz - 1}{q - 1} \quad \text{and} \quad yx = \frac{z - 1}{q - 1}.$$

It is obvious that z is controlling, dominating, and effective in $A_1^q(\Bbbk)$.

(2) We can localize $A_1^q(\Bbbk)$ with respect to the Ore set generated by z and denote the localization by $B_1^q(\Bbbk)$. Since z is a controlling element in $A_1^q(\Bbbk)$, we have $\mathbb{D}_{B_1^q(\Bbbk)}(1) = B_1^q(\Bbbk)$. Note that the center of $B_1^q(\Bbbk)$ is $\Bbbk$ if q is not a root of unity. If q is a root of unity of order l, then the center of $B_1^q(\Bbbk)$ is isomorphic to $Z := \Bbbk[x^l, y^l, z^l]/I$ where

$$I = (z^l[1 - (1-q)^l x^l y^l] - 1)$$

by [LY, Proposition 3.2]. It is also clear that $\mathbb{D}_Z(1) = Z$.

(3) The above example can be extended to higher ranks with multiparameters in both root of unity and non-root of unity cases.

(4) Let $D = \Bbbk[h^{\pm 1}]$ with a $\Bbbk$-algebra automorphism σ defined by $\sigma(h) = qh$ for some $q \in \Bbbk^\times$. Let $0 \neq a \in D$ and denote the generalized Weyl algebra by $A = D(a, \sigma)$, which is the $\Bbbk$-algebra generated by $x, y, h^{\pm 1}$ subject to the relations

$$xy = a(qh), \quad yx = a(h), \quad xh = qhx, \quad yh = q^{-1}hy.$$

Then $\mathbb{D}_A(1) = A$.

(5) Fix a positive integer $n \geq 2$, let q be a set of nonzero scalars

$$\{q_{ij} \mid 1 \leq i < j \leq n\}.$$

A quantum torus (or quantum Laurent polynomial algebra) $T_n^q(\Bbbk)$ is generated by generators $\{x_1^{\pm 1}, \ldots, x_n^{\pm 1}\}$ and subject to the relations

$$x_j x_i = q_{ij} x_i x_j$$

for all $i < j$. Note that $T_n^q(\Bbbk)$ is PI if and only if all q_{ij} are roots of unity. It is clear that $\mathbb{D}_{T_n^q(\Bbbk)}(1) = T_n^q(\Bbbk)$. It is well-known that the center of $T_n^q(\Bbbk)$ is isomorphic to a commutative Laurent polynomial algebra of possibly lower rank. Note that the quantum torus is a localization of a skew polynomial ring given in Example 2.5(3).

(6) The tensor products or twisted tensor products A of these examples also satisfy $\mathbb{D}_A(1) = A$.

The proof of the following lemma is easy and omitted.

Lemma 4.4: *Let A be a domain and F be a subset of A. We have the following*

(1) $\mathbb{D}(\mathbb{D}(F)) = \mathbb{D}(F)$.

(2) $\mathbb{D}_{\mathbb{D}(F)}(F) = \mathbb{D}_A(F)$.

(3) *Suppose B is a subalgebra of A containing F. Then $\mathbb{D}_B(F) \subseteq \mathbb{D}_A(F)$.*

(4) *Let C be the Ore extension $A[t;\sigma,\delta]$. Then $\mathbb{D}_C(F)\subseteq A$. As a consequence, $\mathbb{D}_C(F)=\mathbb{D}_A(F)$.*
(5) *Let C be an iterated Ore extension of A. Then $\mathbb{D}_C(F)=\mathbb{D}_A(F)$.*
(6) *Let $\phi: A\to B$ be an injective algebra homomorphism. Then*

$$\mathbb{D}_A(F)\subseteq \mathbb{D}_B(\phi(F)).$$

If ϕ is an isomorphism, then $\mathbb{D}_A(F)=\mathbb{D}_B(\phi(F))$.
(7) *Let $\overline{A}$ be an iterated Ore extension $A[t_1,\sigma_1,\delta_1][t_2,\sigma_2,\delta_2]\cdots[t_n,\sigma_n,\delta_n]$ and $\overline{B}$ be an iterated Ore extension $B[t'_1,\sigma'_1,\delta'_1][t'_2,\sigma'_2,\delta'_2]\cdots[t'_n,\sigma'_n,\delta'_n]$. Suppose*

$$\phi:\overline{A}\to\overline{B}$$

is an isomorphism. Then $\phi(\mathbb{D}_{\overline{A}}(1))=\mathbb{D}_{\overline{B}}(1)\subseteq B$.
(8) *Suppose A and B are algebras such that $A\otimes_{\Bbbk}B$ is a domain. If $\mathbb{D}_A(1)=A$ and $\mathbb{D}_B(1)=B$, then $\mathbb{D}_{A\otimes_{\Bbbk}B}(1)=A\otimes_{\Bbbk}B$.*
(9) *If A is a finitely generated left (or right) module over $\mathbb{D}_A(1)$, then*

$$\mathbb{D}_A(1)=A.$$

Now we are ready to prove Theorems 0.9 and 0.10.

Proof of Theorem 0.9. Let A be an affine domain of finite GK dimension. Let $\overline{A}$ be an iterated Ore extension $A[t_1;\sigma_1,\delta_1][t_2;\sigma_2,\delta_2]\cdots[t_n;\sigma_n,\delta_n]$ where each σ_i is locally algebraic. By Lemma 2.1(2), $\operatorname{GKdim}\overline{A}=\operatorname{GKdim}A+n$. Now let B be an algebra and $\overline{B}$ be an iterated Ore extension $B[t'_1;\sigma'_1,\delta'_1][t'_2;\sigma'_2,\delta'_2]\cdots[t'_n;\sigma'_n,\delta'_n]$. Suppose $\phi:\overline{A}\to\overline{B}$ is an algebra isomorphism. It remains to show that $A\cong B$. By the hypothesis and Lemma 4.4(7),

$$\phi(A)=\phi(\mathbb{D}_{\overline{A}}(1))=\mathbb{D}_{\overline{B}}(1)\subseteq B.$$

Let B' denote $\phi^{-1}(B)$. Then $A\subseteq B'$ and $\operatorname{GKdim}B'=\operatorname{GKdim}B$. By the definition of $\overline{B}$, we have

$$\begin{aligned}\operatorname{GKdim}B'&=\operatorname{GKdim}B\le\operatorname{GKdim}\overline{B}-n\\&=\operatorname{GKdim}\overline{A}-n=\operatorname{GKdim}A\\&\le\operatorname{GKdim}B.\end{aligned}$$

Therefore, $\operatorname{GKdim}A=\operatorname{GKdim}B'<\infty$. By Proposition 2.3, we have $A=B'$. This implies that $\phi: A\to B$ is an isomorphism. ∎

Proof of Theorem 0.10. Let A be a noetherian domain that is stratiform. Let $\overline{A}$ be an iterated Ore extension $A[t_1;\sigma_1,\delta_1][t_2;\sigma_2,\delta_2]\cdots[t_n;\sigma_n,\delta_n]$. By Lemma 2.4, $\operatorname{Htr}\overline{A} = \operatorname{Htr} A + n$. Now let B be another noetherian domain that is stratiform and $\overline{B}$ be an iterated Ore extension $B[t'_1;\sigma'_1,\delta'_1][t'_2;\sigma'_2,\delta'_2]\cdots[t'_n;\sigma'_n,\delta'_n]$. Suppose $\phi:\overline{A}\to\overline{B}$ is an algebra isomorphism. We need to show that $A\cong B$. By the hypothesis and Lemma 4.4(7), $\phi(A)=\phi(\mathbb{D}_{\overline{A}}(1))=\mathbb{D}_{\overline{B}}(1)\subseteq B$. Let B' denote $\phi^{-1}(B)$. Then $A\subseteq B'$ and $\operatorname{Htr} Q(B')=\operatorname{Htr} Q(B)$. By the definition of $\overline{B}$, we have

$$\begin{aligned}\operatorname{Htr} Q(B') &= \operatorname{Htr} Q(B) = \operatorname{Htr} Q(\overline{B}) - n \quad \text{by Lemma 2.4}\\ &= \operatorname{Htr} Q(\overline{A}) - n = \operatorname{Htr} Q(A) \quad \text{by Lemma 2.4}\\ &\le \operatorname{Htr} Q(B).\end{aligned}$$

Therefore, $\operatorname{Htr} Q(A)=\operatorname{Htr} Q(B')<\infty$. According to Proposition 2.6, we have that $A=B'$. That is, $\phi: A\to B$ is indeed an isomorphism. ∎

In the rest of this section, we will prove that a class of simple algebras is strongly σ-cancellative, but might not be δ-cancellative; see [Example 5.5(1)]. First of all, we need a lemma.

Lemma 4.5: *Let $\overline{B}$ be an iterated Ore extension $B[t_1;\sigma_1]\cdots[t_n;\sigma_n]$ of an algebra B. If A is a simple factor ring of $\overline{B}$ such that $A^\times=\Bbbk^\times$ with the quotient map denoted by $\pi\colon \overline{B}\longrightarrow A$, then the image $\pi(t_i)$ of each t_i in A is a scalar and $\pi(B)=A$. Furthermore, if B is a simple algebra, then $A\cong B$.*

Proof. We use an induction argument. If $n=0$, it is trivial. Now we assume that $n>0$. Let π denote the quotient map from $\overline{B}$ onto A. Since t_n is normal in $\overline{B}$, so is $\pi(t_n)$ in A. Since A is simple, $\pi(t_n)$ is either zero or invertible in A. Since $A^\times=\Bbbk^\times$, $\pi(t_n)$ is a scalar in $\Bbbk$. As a result, we have that

$$\pi(B[t_1;\sigma_1]\cdots[t_{n-1};\sigma_{n-1}])=A.$$

By induction, we have that $\pi(t_i)$ is a scalar in A for $1\le i\le n$ and $\pi(B)=A$. If B is simple, then $B\cong A$. ∎

The next result establishes the strongly σ-cancellative property for many simple algebras such as the first Weyl algebra, which is the $\Bbbk$-algebra generated by x,y subject to the relation $xy-yx=1$.

Theorem 4.6: *Let A be a right (resp. left) noetherian simple domain such that $A^{\times} = \Bbbk^{\times}$. Then A is strongly σ-cancellative.*

Proof. Suppose

$$\phi : \overline{A} := A[s_1; \sigma_1] \cdots [s_n; \sigma_n] \to B[t_1; \tau_1] \cdots [t_n; \tau_n] =: \overline{B}$$

is an isomorphism for another algebra B. Here σ_i and τ_i are automorphisms of appropriate algebras. Then $\overline{B}$ is a noetherian domain, and consequently B is a noetherian domain. By [GoW, Theorem 15.19], A and B have the same Krull dimension. Let I be the ideal of A generated by $\{s_i\}_{i=1}^{n}$. Then $A \cong \overline{A}/I$. Let $J = \phi(I)$. Then $\overline{B}/J (\cong A)$ is simple and every invertible element in $\overline{B}/J$ is a scalar by hypothesis. Let ϕ also denote the induced isomorphism

$$A \to \overline{B}/J.$$

Let π be the map from $\overline{B}$ onto $\overline{B}/J$. By Lemma 4.5, $\pi(t_i)$ is a scalar in $\Bbbk$ for each i. Then

$$\pi(B) = \pi(\overline{B}) = \overline{B}/J.$$

Therefore,

$$\phi^{-1} \circ \pi|_B : B \longrightarrow \overline{B}/J \xrightarrow{\phi^{-1}} A$$

is a surjective algebra homomorphism. Since B is a domain with

$$\operatorname{Kdim} B = \operatorname{Kdim} A,$$

we obtain that $\phi^{-1} \circ \pi|_B$ is an isomorphism and that

$$B \cong \overline{B}/J \cong A$$

as desired. ∎

5. Comments, examples, remarks and questions

In this section we give some isolated results, comments, examples, remarks and open questions.

5.1. Cancellative property of some infinitely generated algebras. In most of the results proved in [LeWZ, LuWZ], we have assumed the algebras are either affine or noetherian or having finite GK dimension. In this subsection we make some comments on the cancellation property for some algebras of infinite GK dimension. The following lemma generalizes [BR, Lemma 2] and [CE, Corollary 1].

Lemma 5.1: *Let A and B be $\Bbbk$-algebras and*

$$\phi\colon A[s_1,\ldots,s_n] \longrightarrow B[t_1,\ldots,t_n]$$

be an algebra isomorphism.

(1) *If $\phi(A) \subseteq B$, then $\phi(A) = B$.*
(2) *If $\phi(Z(A)) \subseteq B$, then $A \cong B$.*

Proof. (1) We have the following restriction of the isomorphism ϕ to the respective centers of $A[s_1,\ldots,s_n]$ and $B[t_1,\ldots,t_n]$:

$$\phi\colon Z(A)[s_1,\ldots,s_n] \xrightarrow{\cong} Z(B)[t_1,\ldots,t_n].$$

Note that $Z(B) = B \cap Z(B)[t_1,\ldots,t_n]$. Since ϕ is an isomorphism and, by the hypothesis $\phi(A) \subseteq B$, we obtain that $\phi(Z(A)) \subseteq Z(B)$. By [BR, Lemma 2], we have that $\phi(Z(A)) = Z(B)$. Let $f_i := \phi(s_i)$ for $1 \le i \le n$. We have that

$$Z(B)[t_1,\ldots,t_n] = Z(B)\{f_1,\ldots,f_n\}.$$

According to Lemma 1.2(2), we have $Z(B)[t_1,\ldots,t_n] = Z(B)[f_1,\ldots,f_n]$. Using Lemma 1.2(1), we can further conclude that $B[t_1,\ldots,t_n] = B[f_1,\ldots,f_n]$, where $f_1,\ldots,f_n$ are considered as central indeterminates. Denote by τ the B-automorphism of $B[t_1,\ldots,t_n]$, which is defined by setting $\tau(f_i) = t_i$. Then we have an isomorphism

$$\tau\circ\phi\colon A[s_1,\ldots,s_n] \longrightarrow B[t_1,\ldots,t_n]$$

with $\tau\circ\phi(s_i) = t_i$ for all i and $(\tau\circ\phi)(A) \subseteq B$. As a result, we have that $\phi(A) = B$.

(2) Similar to the proof of part (1), we have the following induced isomorphism:

$$\phi\colon Z(A)[s_1,\ldots,s_n] \longrightarrow Z(B)[t_1,\ldots,t_n].$$

Since $Z(B) = B \cap Z(B)[t_1,\ldots,t_n]$, the hypothesis $\phi(Z(A)) \subseteq B$ implies that $\phi(Z(A)) \subseteq Z(B)$. By [BR, Lemma 2], we have that $\phi(Z(A)) = Z(B)$. Let $f_i := \phi(s_i)$ for $1 \le i \le n$. Similar to the proof of part (1), we have that

$$B[t_1,\ldots,t_n] = B[f_1,\ldots,f_n]$$

where $f_1,\ldots,f_n$ are considered as central indeterminates. Going back to the isomorphism

$$\phi : \overline{A} := A[s_1,\ldots,s_n] \longrightarrow B[t_1,\ldots,t_n] = B[f_1,\ldots,f_n] =: \overline{B},$$

one sees that ϕ maps the ideal of $\overline{A}$ generated by $\{s_i\}_{i=1}^{w}$ to the ideal of $\overline{B}$ generated by $\{f_i\}_{i=1}^{n}$. Therefore, we have that

$$A \cong \overline{A}/(s_i : i = 1, \ldots, n) \cong \overline{B}/(f_i : i = 1, \ldots, n) \cong B. \qquad \blacksquare$$

Combining some ideas in the previous section, we can further have the following result.

Proposition 5.2: *Let A be an algebra such that $\mathbb{D}(1) \supseteq Z(A)$.*

(1) *Then A is strongly cancellative.*
(2) *If A is commutative, then A is strongly retractable.*

Proof. (1) Let $\phi : \overline{A} = A[s_1, \ldots, s_n] \to \overline{B} = B[t_1, \ldots, t_n]$ be an isomorphism. By Lemma 4.4(7), we have that

$$\phi(\mathbb{D}(1)) = \phi(\mathbb{D}_{\overline{A}}(1)) \subseteq \mathbb{D}_{\overline{B}}(1) \subseteq B.$$

Since $Z(A) \subseteq \mathbb{D}(1)$, we have $\phi(Z(A)) \subseteq B$. Now the assertion follows from Lemma 5.1(2).

(2) To prove the second assertion, we repeat the above proof and apply Lemma 5.1(1). $\blacksquare$

By Proposition 5.2(2), any commutative algebra A with $\mathbb{D}(1) = A$ is strongly retractable. For example, the Laurent polynomial algebra with infinitely many variables $\Bbbk[x_i^{\pm 1} : i = 1, 2, 3, \cdots]$ is strongly retractable. As a consequence of Theorem 0.5, any algebra with a center equal to a finite direct sum of (infinite) Laurent polynomial algebras is strongly cancellative and Morita cancellative. It is obvious that the infinite quantum Laurent polynomial algebra $T_{\infty}^{q}(\Bbbk)$ has this kind of property. Below is a similar example.

Example 5.3: Let q be a sequence of scalars $\{q_i\}_{i\geq 1}$ and let $A_{\infty}^{q}(\Bbbk)$ be the $\Bbbk$-algebra generated by an infinite set $\{x_i, y_i\}_{i\geq 1}$ subject to the relations

$$x_i x_j = x_j x_i, \quad y_i y_j = y_j y_i,$$
$$x_i y_j = y_j x_i \, (i \neq j), \quad x_i y_i - q_i y_i x_i = 1$$

for $i, j \in \{1, 2, 3, \ldots\}$. It is clear that $A_{\infty}^{q}(\Bbbk)$ is an infinite tensor product of the quantum Weyl algebras defined in Example 4.3(1). We call this algebra an **infinite quantum Weyl algebra**. It is obvious that $\operatorname{GKdim} A_{\infty}^{q}(\Bbbk) = \infty$. Set $z_i = x_i y_i - y_i x_i$ and let S be the Ore set generated by the products of z_i. Let B be the localization of $A_{\infty}^{q}(\Bbbk)$ with respect to S. It is easy to see that $\mathbb{D}_B(1) = B$ and $\mathbb{D}_{Z(B)}(1) = Z(B)$ (some details are omitted). As a consequence of Proposition 5.2 and Theorem 0.5, B is strongly cancellative and strongly Morita cancellative. However, it is not known to us whether or not it is (strongly) skew cancellative.

5.2. δ-cancellative property of LND-rigid algebras. In this subsection we use some ideas in [BHHV] to show that every LND-rigid algebra is δ-cancellative. To save some space, we refer the reader to [BHHV, Definition 2.1] for the definitions concerning LND-rigidity and Makar-Limanov invariants $ML(A)$.

Theorem 5.4: *Suppose that $\Bbbk$ is a field of characteristic zero. Let A be an affine $\Bbbk$-domain of finite GK dimension. Suppose that $ML(A) = A$. Then A is δ-cancellative.*

Proof. We follow the proof of [BHHV, Proposition 5.6].

Suppose that $\phi : A[s;\delta] \longrightarrow B[t;\delta']$ is an isomorphism for another algebra B. Here δ is a derivation of A and δ' is a derivation of B. By Lemma 2.1(1), we have that

$$\operatorname{GKdim} A[s;\delta] = \operatorname{GKdim} A + 1 < \infty.$$

A similar statement holds true for B. Thus we have that $\operatorname{GKdim} B = \operatorname{GKdim} A$.

Since A is an affine domain of finite GK dimension, it is an Ore domain. Also note that

$$ML(A) = A.$$

By [BHHV, Lemma 5.3], we have that $ML(A[s;\delta]) = ML(A)$, which is further equal to A. As a result, we have the following:

$$A = ML(A) = ML(A[s;\delta]) \xrightarrow[\sim]{\phi} ML(B[t;\delta']) \subseteq B.$$

Equivalently, we have that $\phi(A) \subseteq B$. Set $B' = \phi^{-1}(B)$. By Proposition 2.3, we have that $A = B'$ or equivalently, $A \cong B$. Therefore, A is δ-cancellative. ∎

However, not every algebra satisfying the hypotheses in Theorem 5.4 is σ-cancellative. The following is an example along this line; see Example 5.5(2).

Example 5.5: Here we consider two affine domains of GK dimension two.

(1) Let A be the first Weyl algebra over a field $\Bbbk$ of characteristic zero. Then A is simple with a trivial center $\Bbbk$. By [BZ1, Proposition 1.3] and Theorem 0.4, A is universally cancellative and universally Morita

cancellative. By Theorem 4.6, A is strongly σ-cancellative. Now we claim that A is not δ-cancellative. Note that the first Weyl algebra A can be written as $\Bbbk\langle x, y\rangle/(xy - yx - 1)$. Let $B = \Bbbk[y, z]$ and let δ' be the derivation of B defined by $\delta'(y) = 1$ and $\delta'(z) = 0$. Then

$$A[z; \delta = 0] \cong B[x; \delta'].$$

It is clear that $A \not\cong B$. Therefore, A is not δ-cancellative.

(2) Let A be a different algebra $\Bbbk_{-1}[x, y] = \Bbbk\langle x, y\rangle/(xy+yx)$ which is an affine noetherian PI domain of GK dimension two with center $Z = \Bbbk[x^2, y^2]$. By [BZ1, Theorem 4.7 and Example 4.8], A is strongly LND-rigid and strongly cancellative. By Theorem 5.4, A is δ-cancellative. But A is not σ-cancellative as

$$A[z; \mathrm{Id}_A] \cong \Bbbk[y, z][x; \sigma]$$

for some automorphism σ of the commutative polynomial algebra $\Bbbk[y, z]$.

Figure 1 summarizes the implication relations among several types of skew cancellations, where an arrow (resp. dotted arrow) means "implies" (resp. "does not imply"). All the implications follow directly from the definitions.

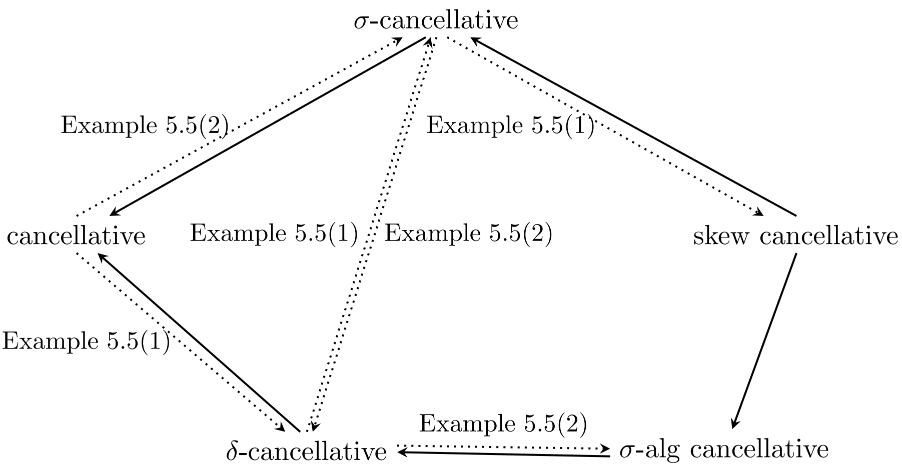

Figure 1. Relations among different types of skew cancellations.

Now a natural question to consider is

Question 5.6: Let A be an algebra as in Theorem 5.4, or specifically, the algebra in Example 5.5(2). Or suppose that A is strongly LND-rigid in the sense of [BZ1, Definition 2.3]. Is then A strongly δ-cancellative?

5.3. Derived cancellative property. First we recall the definition of derived cancellation. Let $M(A)$ denote the category of all right A-modules and $D(A)$ be the corresponding derived category of $M(A)$.

Definition 5.7: Let A be an algebra.

(1) The **derived cancellative** property of A is defined in the same way as in Definition 0.2(1) by replacing the abelian categories $M(-)$ with the triangulated categories $D(-)$.
(2) The **strongly derived cancellative** property of A is defined in the same way as in Definition 0.2(2) by replacing the abelian categories $M(-)$ with the triangulated categories $D(-)$.

The following is a version of [LuWZ, Corollary 7.3] without the strongly Hopfian condition. Its proof is omitted (see the proof of [LuWZ, Corollary 7.3]).

Proposition 5.8: *Let A be an Azumaya algebra over its center Z which has a connected spectrum. Suppose that Z is either (strongly) detectable or (strongly) retractible. Then A is (strongly) cancellative, (strongly) Morita cancellative and (strongly) derived cancellative.*

Corollary 5.9: *Let A be a domain. If A is Azumaya and $\mathbb{D}_{Z(A)}(1) = Z(A)$, then A is strongly cancellative, strongly Morita cancellative and strongly derived cancellative.*

Proof. By Proposition 5.2, Z is strongly retractable. The assertion follows from Proposition 5.8. ∎

The following is also known due to [LuWZ, Corollary 7.3]. Note that all the algebras involved in the example below are Azumaya algebras.

Example 5.10: As an immediate consequence of Corollary 5.9, the following algebras are strongly derived cancellative:

(1) Localized quantum Weyl algebra $B_1^q(\mathbb{k})$ as in Example 4.3(2) where q is a root of unity.

(2) Quantum Laurent polynomial algebras as in Example 4.3(5) where all q_{ij} are roots of unity.

(3) Any finite tensor product of algebras in parts (1) and (2).

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