Mathematische Zeitschrift (2022) 300:3171–3225
https://doi.org/10.1007/s00209-021-02888-3

Mathematische Zeitschrift

Frobenius–Perron theory of representations of quivers

J. J. Zhang[1] · J.-H. Zhou[2,3]

Received: 3 July 2020 / Accepted: 23 September 2021 / Published online: 15 November 2021

Abstract

The Frobenius–Perron theory of an endofunctor of a category was introduced in recent years (Chen et al. in Algebra Number Theory 13(9):2005–2055, 2019; Chen et al. in Frobenius–Perron theory for projective schemes. Preprint. arXiv:1907.02221, 2019). We apply this theory to monoidal (or tensor) triangulated structures of quiver representations.

Keywords Frobenius–Perron dimension · Derived categories · Quiver representation · Monoidal triangulated category · $\mathbb{ADE}$ Dynkin quiver

Mathematics Subject Classification Primary 16E10 · 16G60 · 16E35

0 Introduction

Throughout let $\Bbbk$ be a base field that is algebraically closed. Algebraic objects are defined over $\Bbbk$.

The Frobenius–Perron dimension of an object in a semisimple finite tensor (or fusion) category was introduced by Etingof–Nikshych–Ostrik in 2005 [19]. Since then it has become an extremely useful invariant in the study of fusion categories and representations of semisimple (weak and/or quasi-)Hopf algebras. By examining the Frobenius–Perron dimension of all objects in a finite tensor category one can determine whether the category is equivalent to the representation category of a finite-dimensional quasi-Hopf algebra [20, Proposition 2.6]. The Frobenius–Perron dimension of a fusion category is also a crucial invariant in the classification of fusion categories as well as that of semisimple Hopf algebras. An important project is to develop the Frobenius–Perron theory for not-necessarily semisimple tensor (or monoidal) categories. A step departing from semisimple categories, or abelian categories of

✉ J. J. Zhang
zhang@math.washington.edu

J.-H. Zhou
zhoujingheng@mail.shufe.edu.cn

1 Department of Mathematics, University of Washington, Box 354350, Seattle, WA 98195, USA

2 School of Mathematics, Shanghai University of Finance and Economics, Shanghai, Shanghai 200433, China

3 Shanghai Center for Mathematical Sciences, Fudan University, Shanghai, Shanghai 200433, China

global dimension 0, is to study the hereditary ones (of global dimension one). Ultimately Frobenius–Perron theory should provide powerful tools and useful invariants for projects like

Project 0.1 Describe and understand weak bialgebras [Definition 1.7] and weak Hopf algebras [11, Definition 2.1] that are hereditary as associative algebras.

Note that an analogous classification project of hereditary prime Hopf algebras was finished in a remarkable paper by Wu–Liu–Ding [63] a few years ago. Some recent efforts pertaining on homological aspects of noetherian weak Hopf algebras were presented in [57]. Recall from [51] that a *monoidal triangulated category* is a monoidal category $\mathcal{T}$ in the sense of [18, Definition 2.2.1] that is triangulated and, for which, the tensor product $\otimes_{\mathcal{T}} : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ is an exact bifunctor. Given a Hopf algebra (respectively, weak and/or quasi- Hopf algebra/bialgebra) H, its comultiplication induces a monoidal structure on the category of representations of H. The corresponding derived category has a canonical monoidal triangulated structure. Monoidal triangulated structures appear naturally in several other subjects.

Algebraic geometry. A classical theorem of Gabriel [21] states that a noetherian scheme $\mathbb{X}$ can be reconstructed from the abelian category of coherent sheaves over $\mathbb{X}$, denoted by $coh(\mathbb{X})$. Hence the abelian category $coh(\mathbb{X})$ captures all information about the space $\mathbb{X}$. Recent development in derived algebraic geometry suggests that the bounded derived category of coherent sheaves over $\mathbb{X}$, denoted by $D^b(coh(\mathbb{X}))$, is sometimes a better category to work with when we are considering many geometric problems such as moduli problems. When $\mathbb{X}$ is smooth, $D^b(coh(\mathbb{X}))$ is equipped with a natural tensor (= symmetric monoidal) triangulated structure in the sense of [5, Definition 1.1].

Tensor triangulated geometry. Tensor triangulated categories have been studied by Balmer [5] and many others, where the study of tensor triangulated categories has been sometimes referred to as *tensor triangulated geometry*. Balmer defined the prime spectrum, denoted by $\mathrm{Spc}(\mathcal{T})$, of a small tensor triangulated category $\mathcal{T}$ by using the thick subcategories which behave like prime ideals under the tensor product. Note that $\mathrm{Spc}(\mathcal{T})$ is a locally ringed space [5, Remark 6.4]. This idea has been shown to be widely applicable to algebraic geometry, homotopy theory and representation theory. Recently Vashaw–Yakimov and Nakano–Vashaw–Yakimov [51,62] developed a noncommutative version of the Balmer spectrum, or *noncommutative tensor triangulated geometry* (in the words of the authors of [51]).

Noncommutative algebraic geometry. Following Grothendieck, *to do geometry you really don't need a space, all you need is a category of sheaves on this would-be space* [48, p. 78]. Following [51,62], we would like to consider or view monoidal triangulated categories as appropriate categories for doing a new kind of noncommutative geometry. For example, if T is a noetherian Koszul Artin–Schelter regular algebra, then the bounded derived category of the noncommutative projective scheme associated to T, denoted by $\mathcal{T} := D^b(\mathrm{proj}\, T)$, has at least two different monoidal triangulated structures [Example 7.9]. In this situation, it would be very interesting to understand how the "geometry" of $\mathrm{proj}\, T$ interacts with "monoidal triangulated structures" on $\mathcal{T}$. Fix a general triangulated category, still denoted by $\mathcal{T}$, it is common that there are many different monoidal triangulated structures on $\mathcal{T}$ (with the same underlying triangulated structure) that reflect on different hidden properties of $\mathcal{T}$. So it is worth distinguishing different types of monoidal triangulated structures on $\mathcal{T}$ and finding a definition of the "size" of these structures.

Quiver representations. A related subject is the representation theory of quivers that has become a popular topic since Gabriel's work in the 1970s [22–24]. For a given quiver, it is naturally equipped with a monoidal structure in the category of its representations, induced by the vertex-wise tensor product of vector spaces (E2.1.1). The monoidal structure of quiver representations has been studied by Strassen [60] in relation with orbit-closure degeneration in 2000, and later by Herschend [29–33] in the relation with the bialgebra structure on the path algebra during 2005–2012. Herschend solved the Clebsch-Gordan problem for quivers of type $\mathbb{A}_n$, $\mathbb{D}_n$ and $\mathbb{E}_{6,7,8}$ in [31,32]. As for tame type, Herschend also gave solutions for type $\tilde{\mathbb{A}}_n$ in [29] and the quivers with relations that correspond to string algebras in [33]. One of our basic objects in this paper is the bounded derived category $D^b(A-\text{mod})$ for a finite dimensional weak bialgebra A. (We usually consider hereditary, but not semisimple, algebras.) Since A is a weak bialgebra, $A-\text{mod}$ has an induced monoidal abelian structure; and hence, $D^b(A-\text{mod})$ is a monoidal triangulated category in the sense of [51]. Note that, even for a finite quiver Q, [30, Proposition 4] and [35, Theorem 3.2] give different weak bialgebra structures on $A := \Bbbk Q$ which produce different monoidal triangulated structures on $D^b(A-\text{mod})$.

Connections between geometry, quiver representations, and weak bialgebras. Going back to classical geometry, let $\mathbb{X}$ be a smooth projective scheme. If $\mathbb{X}$ is equipped with a full strongly exceptional sequence (also called strong full exceptional sequence by some authors) $\{\mathcal{E}_1, \ldots, \mathcal{E}_n\}$ [Definition 7.8], then there is a triangulated equivalence

$$D^b(coh(\mathbb{X})) \cong D^b(A-\text{mod}) \tag{E0.1.1}$$

where A is the finite dimensional algebra $[\text{End}_{D^b(coh(\mathbb{X}))}(\oplus_{i=1}^n \mathcal{E}_i)]^{op}$ (some details can be found at the end of Sect. 7). Since A is finite dimensional (and of finite global dimension), it seems easier to study A than to study $\mathbb{X}$ in some aspects. Equivalence (E0.1.1) induces a monoidal structure on $D^b(A-\text{mod})$ which usually does not come from any weak bialgebra structures of A [Example 7.9]; or in some extreme cases, there is no weak bialgebra structure on A at all. For such an algebra A, it is imperative to understand and even to classify all possible monoidal triangulated structures of $D^b(A-\text{mod})$ (though A may not be a weak bialgebra).

Another well-known example of such a connection is from the study of weighted projective lines, introduced by Geigle-Lenzing [26] in 1985 (see Sect. 6). Since then weighted projective lines have been studied extensively by many researchers. Let $coh(\mathbb{X})$ denote the category of coherent sheaves over a weighted projective line $\mathbb{X}$. When $\mathbb{X}$ is domestic, a version of (E0.1.1) holds and the representation type of A (appeared in the right-hand side of (E0.1.1)) is tame, see Lemma 6.1(2). This is one of the key facts that we will use in this paper.

Recently, a new definition of Frobenius–Perron dimension was introduced in [14,15] where the authors extended its original definition from an object in a semisimple finite tensor category to an endofunctor of any $\Bbbk$-linear category. (We refer to Definition 1.3 for some relevant definitions.) It turns out that new Frobenius–Perron invariants are sensitive to monoidal structures; as a consequence, these are crucial to distinguish different monoidal triangulated structures. One general goal of this paper is to provide evidence that the Frobenius–Perron invariants are effective to study monoidal triangulated structures. Some basic properties and interesting applications of Frobenius–Perron-type invariants can be found in [14,15].

In this paper, we focus on different weak bialgebra structures on the path algebras of finite quivers and the Frobenius–Perron theory for finite dimensional hereditary weak bialgebras. As mentioned above, this is one step beyond the semisimple case.

Definition 0.2 Let $\mathcal{T}$ be a monoidal category and let $\mathcal{P}$ be a function

$$\{\text{endofunctors of } \mathcal{T}\} \longrightarrow \mathbb{R}_{\geq 0}.$$

Note that $\mathcal{P}$ could be Frobenius–Perron dimension or Frobenius–Perron curvature as given in Definition 1.3(6,7). For every object M in $\mathcal{T}$, let $\mathcal{P}(M)$ denote the $\mathcal{P}$-value of the tensor functor $M \otimes_{\mathcal{T}} - : \mathcal{T} \to \mathcal{T}$.

(1) We say $\mathcal{T}$ is $\mathcal{P}$-*finite* if $\mathcal{P}(M) < \infty$ for all objects M. Otherwise, $\mathcal{T}$ is called $\mathcal{P}$-*infinite*.
(2) If $\mathcal{T}$ is $\mathcal{P}$-infinite and if $\mathcal{P}(M) < \infty$ for all indecomposable objects M, then $\mathcal{T}$ is called $\mathcal{P}$-*tame*.
(3) If $\mathcal{T}$ is neither $\mathcal{P}$-finite nor $\mathcal{P}$-tame, then it is called $\mathcal{P}$-*wild*.

Our first main result concerns the trichotomy of fpd-finite/tame/wild property. Let $\text{rep}(Q)$ be the category of finite dimensional representations of a quiver Q.

Theorem 0.3 *Let Q be a finite acyclic quiver and let $\mathcal{T}$ be the triangulated category* $D^b(\text{rep}(Q))$.

(1) *Q is of finite type if and only if $\mathcal{T}$ is* fpd-*finite for every monoidal triangulated structure on $\mathcal{T}$, and if and only if there is one monoidal triangulated structure on $\mathcal{T}$ such that $\mathcal{T}$ is* fpd-*finite.*
(2) *Q is of tame type if and only if there is a monoidal triangulated structure on $\mathcal{T}$ such that $\mathcal{T}$ is* fpd-*tame. In this case, there must be another monoidal triangulated structure on $\mathcal{T}$ such that $\mathcal{T}$ is* fpd-*wild.*
(3) *Q is of wild type if and only if $\mathcal{T}$ is* fpd-*wild for every monoidal triangulated structure on $\mathcal{T}$.*
(4) *If Q is tame or wild, then every monoidal triangulated structure on $\mathcal{T}$ is* fpd-*infinite.*

Note that in part (2) of the above theorem, there are two different monoidal triangulated structures on $\mathcal{T}$, one of which is fpd-tame and the other is not. We refer to Definition 3.1 for the definition of a discrete monoidal structure. By the above theorem, it is rare to have fpd-finite monoidal triangulated structures on $\mathcal{T}$. When it exists, we can say a bit more. The canonical weak bialgebra structure on the path algebra $\Bbbk Q$ is given in Lemma 2.1(1).

Theorem 0.4 *Let A be a finite dimensional hereditary weak bialgebra such that the induced monoidal structure on $A - \text{mod}$ is discrete. Then the following are equivalent:*

(a) *A is of finite representation type,*
(b) $\text{fpd}(M) < \infty$ *for every irreducible representation $M \in A - \text{mod}$,*
(c) $\text{fpd}(M) < \infty$ *for every indecomposable representation $M \in A - \text{mod}$,*
(d) $\text{fpd}(M) < \infty$ *for every representation $M \in A - \text{mod}$,*
(e) $\text{fpd}(X) < \infty$ *for every indecomposable object $X \in D^b(A - \text{mod})$,*
(f) *The induced monoidal triangulated structure on $D^b(A - \text{mod})$ is* fpd-*finite.*

Suppose further that A is the path algebra $\Bbbk Q$ with canonical weak bialgebra structure. It follows from Gabriel's theorem that any of conditions (a) to (e) is equivalent to

(g) Q is a finite union of quivers of type $\mathbb{ADE}$.

Since condition (a) in the above theorem is an algebra property, the fpd-finiteness of $D^b(A - \text{mod})$ only depends on the algebra structure of A, though the definition of $\text{fpd}(X)$ uses the coalgebra structure of A. Note that condition (a) is not equivalent to condition (b) if we remove the hereditary hypothesis in the above theorem [Remark 7.3(3)].

Following BGP-reflection functors [8], Happel showed that, for Dynkin quivers with the same underlying Dynkin diagram, their derived categories are triangulated equivalent [28]. This remarkable theorem is one of most beautiful results in representation theory of finite dimensional algebras. In contrast, the story is very different when we are working with *monoidal triangulated* structures of the derived category of Dynkin quivers, see Theorem 0.5 below. As indicated in [14,15], Frobenius–Perron-type invariants are extremely useful to study derived (or triangulated) categories. Using the Frobenius–Perron curvature, denoted by fpv, of objects in $D^b(A-\text{mod})$ we can prove the following.

Theorem 0.5 *Let A and B be finite dimensional hereditary weak bialgebras. Assume either A is a bialgebra or* $A-\text{mod}$ *is discrete. Suppose that the monoidal triangulated categories* $D^b(A-\text{mod})$ *and* $D^b(B-\text{mod})$ *are equivalent. Then* $A-\text{mod}$ *and* $B-\text{mod}$ *are equivalent as monoidal abelian categories.*

As a consequence, we have

Corollary 0.6 *Suppose that the bounded derived categories of representations of two finite acyclic quivers are equivalent as monoidal triangulated categories. Then the quivers are isomorphic.*

There is also a result concerning an analogue of a t-structure in the monoidal triangulated setting, see Theorem 0.7 below. We introduce the notion of an mtt-structure on a monoidal triangulated category in Sect. 5. Undefined terminologies can be found in Sects. 4 and 5.

Theorem 0.7 *Let A be a finite dimensional weak bialgebra that is hereditary as an algebra. Suppose that the induced monoidal structure on* $A-\text{mod}$ *is discrete. Then there is a unique hereditary* mtt*-structure with deviation zero on the monoidal triangulated category* $D^b(A-\text{mod})$.

It is clear that Theorem 0.7 applies to $D^b(\text{rep}(Q))$ where Q is a finite acyclic quiver. A t-structure on a triangulated category has been studied extensively since it was introduced by Beilinson–Bernstein–Deligne in [7]. It is natural to study all mtt-structures of a monoidal triangulated category. In fact, mtt-structures service as a compelling system of a monoidal triangulated category. It is well-known that (hereditary) t-structures on $D^b(\text{rep}(Q))$ are not unique even for quivers of type $\mathbb{A}_n$, defined below, for $n \geq 3$. Therefore it is surprising that certain mtt-structures (see Theorem 0.7) are unique. This uniqueness result would have other significant consequences than Theorem 0.5 and Corollary 0.6. It is also interesting to search for other classes of monoidal triangulated categories such that the uniqueness property holds for certain mtt-structures.

Though there are more than one tensor structures on the path algebra $\Bbbk Q$ for a quiver Q, one of these structures is from the natural coalgebra structure on $\Bbbk Q$, similar to group algebras [Lemma 2.1(1)]. We will present more results concerning the Frobenius–Perron dimensions of indecomposable representations under such tensor structure. Before that we need to introduce some notation. By definition, a type $\mathbb{A}$ quiver (or more precisely, type $\mathbb{A}_n$ quiver) is a quiver of the following form

$$1 \overset{\alpha_1}{\text{——}} 2 \overset{\alpha_2}{\text{——}} \cdots \overset{\alpha_{i-1}}{\text{——}} i \overset{\alpha_i}{\text{——}} \cdots \overset{\alpha_{n-1}}{\text{——}} n \tag{E0.7.1}$$

where each arrow α_i is either $\longrightarrow$ or $\longleftarrow$. For each quiver of type $\mathbb{A}_n$, the arrows α_i will be specified. It is easy to see that, for each $n \geq 3$, there are more than one quivers of type $\mathbb{A}_n$ up to isomorphisms. Let us fix a quiver of type $\mathbb{A}_n$, say Q, as above. For $1 \leq i \leq j \leq n$, we define a thin representation of Q, denoted by $M\{i, j\}$, by

$$(M\{i, j\})_s = \begin{cases} \Bbbk & i \leq s \leq j, \\ 0 & \text{otherwise} \end{cases} \tag{E0.7.2}$$

and

$$(M\{i,j\})_{\alpha_s} = \begin{cases} Id_{\Bbbk} & i \leq s < j, \\ 0 & \text{otherwise.} \end{cases} \tag{E0.7.3}$$

(This thin representation is sometimes called an interval module by other researchers.) Then by [25, p. 63], all such $M\{i,j\}$ form the complete list of indecomposable representations of Q [Lemma 1.10]. For all $i \leq j$, we say

$$M\{i,j\} \text{ is } \begin{cases} \text{a } sink & \text{if } \alpha_{i-1} = \longrightarrow (\text{ or } i = 1) \text{ and } \alpha_j = \longleftarrow (\text{ or } j = n), \\ \text{a } source & \text{if } \alpha_{i-1} = \longleftarrow (\text{ or } i = 1) \text{ and } \alpha_j = \longrightarrow (\text{ or } j = n), \\ \text{a } flow & \text{if } \alpha_{i-1} = \alpha_j, \text{ and it is either } \longrightarrow \text{ or } \longleftarrow. \end{cases}$$

Since $\mathrm{rep}(Q)$ is hereditary, every indecomposable object in the bounded derived category $D^b(\mathrm{rep}(Q))$ is of the form $M\{i,j\}[m]$ for some $m \in \mathbb{Z}$. We have the following result for type $\mathbb{A}_n$. Some computation in the case of type $\mathbb{D}_n$ quivers is given in [64].

Theorem 0.8 *Let Q be a quiver of type $\mathbb{A}_n$ for some positive integer n. Then the following hold in the bounded derived category $D^b(\mathrm{rep}(Q))$ with tensor defined as in* (E2.1.1).

(1) $\mathrm{fpd}(M\{i,j\}[m]) = 0$ *for all* $m < 0$ *and* $m > 1$.

(2) $$\mathrm{fpd}(M\{i,j\}[0]) = \begin{cases} 1 & \text{if } M\{i,j\} \text{ is a sink,} \\ \min\{i, n-j+1\} & \text{if } M\{i,j\} \text{ is a source,} \\ 1 & \text{if } M\{i,j\} \text{ is a flow.} \end{cases}$$

(3) $$\mathrm{fpd}(M\{i,j\}[1]) = \begin{cases} \min\{i-1, n-j\} & \text{if } M\{i,j\} \text{ is a sink,} \\ 0 & \text{if } M\{i,j\} \text{ is a source,} \\ 0 & \text{if } M\{i,j\} \text{ is a flow.} \end{cases}$$

Related to Project 0.1 we are also very much interested in the following questions.

Question Let A be a finite dimensional weak bialgebra or just an algebra, or $\Bbbk Q$ where Q is a finite acyclic quiver.

(1) How to determine all monoidal abelian structures on the abelian category $A-\mathrm{mod}$?
(2) How to determine all monoidal triangulated structures on the derived category $D^b(A-\mathrm{mod})$?

The paper is organized as follows. Section 1 contains some basic definitions. In particular, we recall the definition of the Frobenius–Perron dimension of an endofunctor. In Sect. 2 we review some preliminaries on quiver representations. The notion of a discrete monoidal abelian category is introduced in Sect. 3. A natural example of a discrete monoidal structure is $\mathrm{rep}(Q)$ which is the main object in this paper. Theorem 0.4 is proved in Sect. 4. In Sect. 5, we introduce the notion of an mtt-structure of a monoidal triangulated category that is a monoidal version of the t-structure of a triangulated category. Theorems 0.5 and 0.7, and Corollary 0.6 are proved near the end of Sect. 5. Section 6 focuses on the proof of Theorem 0.3 which uses some detailed information about weighted projective lines. Section 7 contains various examples which indicate the richness of monoidal triangulated structures from different subjects. Section 8 consists of the proof of Theorem 0.8 with some non-essential details left out.

1 Some basic definitions

This section contains several basic definitions which will be used in later sections.

Recall from [18, Definition 2.1.1] that a *monoidal category* $\mathcal{C}$ consists of the following data:

- (•) a category $\mathcal{C}$,
- (•) a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, called *tensor functor*,
- (•) for each triple (X, Y, Z) in $\mathcal{C}$, a natural isomorphism

$$\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\cong} X \otimes (Y \otimes Z),$$

- (•) an object $\mathbf{1} \in \mathcal{C}$, called *unit object*,
- (•) natural isomorphisms $l_X : \mathbf{1} \otimes X \xrightarrow{\cong} X$ and $r_X : X \otimes \xrightarrow{\cong} X$ for each X in $\mathcal{C}$,

such that the pentagon axiom [18, (2.2)] and the triangle axiom [18, (2.10)] hold. The definitions of a braiding $\{c_{X,Y}\}_{X,Y \in \mathcal{C}}$ on a monoidal category $\mathcal{C}$ and of a braided monoidal category are given in [18, Definitions 8.1.1 and 8.1.2] respectively.

By [18, Definition 8.1.12], a braided monoidal category $\mathcal{C}$ is called *symmetric* if

$$c_{Y,X} \circ c_{X,Y} = id_{X \otimes Y}$$

for all objects $X, Y \in \mathcal{C}$.

We are usually considering $\Bbbk$-linear categories. Now we recall some definitions.

Definition 1.1 Let $\mathcal{C}$ be a monoidal category.

(1) We say $\mathcal{C}$ is *monoidal* $\Bbbk$*-linear* if

- (1a) $\mathcal{C}$ is $\Bbbk$-linear,
- (1b) morphisms and functors involving in the definition of monoidal category are all $\Bbbk$-linear, and
- (1c) the tensor functor preserves direct sums in each argument.

(2) We say $\mathcal{C}$ is *monoidal abelian* if

- (2a) $\mathcal{C}$ is a $\Bbbk$-linear abelian category,
- (2b) $\mathcal{C}$ is monoidal $\Bbbk$-linear in the sense of part (1),
- (2c) the tensor functor preserves exact sequences in each argument.

(3) [51] We say $\mathcal{C}$ is *monoidal triangulated* if

- (3a) $\mathcal{C}$ is $\Bbbk$-linear triangulated category,
- (3b) $\mathcal{C}$ is monoidal $\Bbbk$-linear in the sense of part (1),
- (3c) the tensor functor preserves exact triangles and commutes with the suspension in each argument.
- (3d) the suspension satisfies the anti-commuting diagram given at the end of the definition of a *suspended monoidal* category [61, Definition 1.4].

By the way, we will not be using the axiom (3d) in the above definition in this paper. A tensor triangulated category in the sense of [5, Definition 1.1] is just a symmetric monoidal triangulated category. We refer the reader to [18] for other details.

Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a monoidal category and $\mathcal{A}$ be another category. Following [37, p. 62], by an *action* of $\mathcal{C}$ on $\mathcal{A}$ we mean a strong monoidal functor

$$F = (f, \tilde{f}, f^{\circ}) : \mathcal{C} \longrightarrow [\mathcal{A}, \mathcal{A}],$$

where $[\mathcal{A}, \mathcal{A}]$ is the category of endofunctors of $\mathcal{A}$, provided with a monoidal structure $([\mathcal{A}, \mathcal{A}], \circ, Id_{\mathcal{A}})$ which is strict, wherein $\circ$ denotes composition and $Id_{\mathcal{A}}$ is the identity endofunctor. Here, to give the functor $f : \mathcal{C} \to [\mathcal{A}, \mathcal{A}]$ is equally to give a functor $\odot : \mathcal{C} \times \mathcal{A} \to \mathcal{A}$ where $X \odot A = (fX)A$ for all $X \in \mathcal{C}$ and $A \in \mathcal{A}$; to give the invertible and natural $\tilde{f}_{X,Y} : (fX) \circ (fY) \to f(X \otimes Y)$ (or rather their inverses) is equally to give a natural isomorphism with components

$$\alpha_{X,Y,A} : (X \otimes Y) \odot A \to X \odot (Y \odot A);$$

to give the invertible $f^\circ : Id_{\mathcal{A}} \to f\mathbf{1}$ (or rather its inverse) is equally to give a natural isomorphism with components $\lambda_A : \mathbf{1} \odot A \to A$; and the coherence conditions for F become the commutativity of the three diagrams [37, (1.1), (1.2) and (1.3)] which are the pentagon axiom involving the associator of $\mathcal{C}$ and the triangle axioms for the action of the unit object $\mathbf{1}$ on $\mathcal{A}$ compatible with the left unitor of $\mathcal{C}$ respectively. It is clear that a monoidal category $\mathcal{C}$ acts on itself by defining $\odot = \otimes$. We refer to [37] for more details.

Convention 1.2 *Let $\mathcal{C}$ be a monoidal category acting on another category $\mathcal{A}$.*

(1) *If both $\mathcal{C}$ and $\mathcal{A}$ are $\Bbbk$-linear, we automatically assume that*

  (1a) *morphisms and functors involving in the definition of action are all $\Bbbk$-linear, and*
  (1b) *$\odot$ preserves direct sums in each argument.*

(2) *If both $\mathcal{C}$ and $\mathcal{A}$ are abelian, we automatically assume that*

  (2a) *morphisms and functors involving in the definition of action are all $\Bbbk$-linear,*
  (2b) *$\mathcal{C}$ is monoidal abelian in the sense of Definition 1.1(2),*
  (2c) *$\odot$ preserves exact sequences in each argument.*

(3) *If both $\mathcal{C}$ and $\mathcal{A}$ are triangulated, we automatically assume that*

  (3a) *morphisms and functors involving in the definition of action are all $\Bbbk$-linear,*
  (3b) *$\mathcal{C}$ is monoidal triangulated in the sense of Definition 1.1(3),*
  (3c) *$\odot$ preserves exact triangles and commutes with the suspension in each argument.*

Next we recall some definitions concerning the Frobenius–Perron dimension of an endofunctor. We refer to [14] for other related definitions. Let dim be $\dim_{\Bbbk}$.

Definition 1.3 [14] Let $\mathcal{C}$ be a $\Bbbk$-linear category.

(1) An object X in $\mathcal{C}$ is called a *brick* if

$$\mathrm{Hom}_{\mathcal{C}}(X, X) = \Bbbk.$$

(2) Let $\phi := \{X_1, \ldots, X_n\}$ be a finite subset of nonzero objects in $\mathcal{C}$. We say that ϕ is a *brick set* if each $X_i \in \phi$ is a brick and

$$\mathrm{Hom}_{\mathcal{C}}(X_i, X_j) = 0, \forall\, i \neq j.$$

(3) Let $\phi := \{X_1, \ldots, X_n\}$ and let σ be an endofunctor of $\mathcal{C}$. The *adjacency matrix* of (ϕ, σ) is defined to be

$$A(\phi, \sigma) = (a_{ij})_{n \times n}, \quad \text{where} \quad a_{ij} = \dim \mathrm{Hom}_{\mathcal{C}}(X_i, \sigma(X_j)) \;\; \forall\, i, j.$$

(4) Let Φ_b be the collection of all finite brick sets in $\mathcal{C}$. The *Frobenius–Perron dimension* of an endofunctor σ is defined to be

$$\mathrm{fpd}(\sigma) := \sup_{\phi \in \Phi_b} \{\rho(A(\phi, \sigma))\}$$

where $\rho(A)$ is the spectral radius of a square matrix A [14, Section 1], i.e. the largest absolute value of A.

(5) The *Frobenius–Perron curvature* of σ is defined to be

$$\operatorname{fpv}(\sigma) := \sup_{\phi \in \Phi_b} \left\{ \limsup_{n\to\infty} \left(\rho(A(\phi, \sigma^n))\right)^{1/n} \right\}.$$

(6) If $\mathcal{C}$ is a monoidal $\Bbbk$-linear category acting on a $\Bbbk$-linear category $\mathcal{A}$ and M is an object in $\mathcal{C}$, the *Frobenius–Perron dimension* of M is defined to be

$$\operatorname{fpd}(M) := \operatorname{fpd}(M \odot -)$$

where $M \odot -$ is considered as an endofunctor of $\mathcal{A}$ and $\operatorname{fpd}(M \odot -)$ is defined in part (4). Similarly, the *Frobenius–Perron curvature* of $M \in \mathcal{C}$ is defined to be

$$\operatorname{fpv}(M) := \operatorname{fpv}(M \odot -)$$

where $M \odot -$ is considered as an endofunctor of $\mathcal{A}$ and $\operatorname{fpd}(M \odot -)$ is defined in part (5).

(7) As a special case of (6), if $\mathcal{C}$ is a monoidal $\Bbbk$-linear category and M is an object in $\mathcal{C}$, the *Frobenius–Perron dimension* of M is defined to be

$$\operatorname{fpd}(M) := \operatorname{fpd}(M \otimes -)$$

where $\operatorname{fpd}(M \otimes -)$ is defined in part (4). Similarly, the *Frobenius–Perron curvature* of M is defined to be

$$\operatorname{fpv}(M) := \operatorname{fpv}(M \otimes -)$$

where $\operatorname{fpv}(M \otimes -)$ is defined in part (5).

When $\mathcal{C}$ is $R-\mathrm{mod}$ for an algebra R, a brick set is also called a semibrick [3]. If both "full" and "exceptional" conditions [Definition 7.8(2, 3)] are satisfied, this is also known as a simple-minded collection, see [42, Definition 3.2].

Now we recall the definition of representation types.

Definition 1.4 Let A be a finite dimensional algebra over $\Bbbk$.

(1) We say A is of *finite type* or *finite representation type* if there are only finitely many isomorphism classes of finite dimensional indecomposable left A-modules.
(2) We say A is *tame* or *of tame representation type* if it is not of finite representation type, and for every $n \in \mathbb{N}$, all but finitely many isomorphism classes of n-dimensional indecomposables occur in a finite number of one-parameter families.
(3) We say A is *wild* or *of wild representation type* if, for every finite dimensional $\Bbbk$-algebra B, the representation theory of B can be representation embedded into that of A.

We always assume that the base field $\Bbbk$ is algebraically closed. A famous trichotomy result due to Drozd [17] states that every finite dimensional algebra is either of finite, tame, or wild representation type. By classical theorems of Gabriel [22] and Nazarova [52], the quivers of finite and tame representation types correspond to the $\mathbb{ADE}$ and $\widetilde{\mathbb{A}}\widetilde{\mathbb{D}}\widetilde{\mathbb{E}}$ diagrams respectively. By [14, Theorem 0.3], the representation type of a quiver Q is indicated by the value of the Frobenius–Perron dimension of the suspension functor of the derived category $D^b(\operatorname{rep}(Q))$.

To show some monoidal structure is fpd-infinite [Definition 0.2(1)], we need the following concepts.

Definition 1.5 Let $\mathcal{C}$ be a $\Bbbk$-linear category.

(1) Let ϕ be an infinite set of objects in $\mathcal{C}$. We say ϕ is an *infinite brick set* if

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) = \begin{cases} \Bbbk & \text{if } X = Y \quad \text{in } \phi, \\ 0 & \text{if } X \neq Y \quad \text{in } \phi. \end{cases}$$

(2) Suppose $\mathcal{C}$ is abelian or triangulated. A brick set ϕ (either finite or infinite) is called a *connected brick set* if $\mathrm{Ext}^1_{\mathcal{C}}(X, Y) \neq 0$ for all $X, Y \in \phi$.

The next is about the definition of a weak bialgebra.

Definition 1.6 Let A be an algebra with a $\Bbbk$-linear morphism $\Delta : A \to A \otimes A$. We say Δ is a *prealgebra morphism* if

$$\Delta(ab) = \Delta(a)\Delta(b)$$

for all $a, b \in A$.

A prealgebra morphism is an algebra morphism if and only if $\Delta(1) = 1 \otimes 1$ where 1 is the identity (or unit) element of A.

Definition 1.7 [11, Definition 2.1] A *weak bialgebra* is a vector space B over the base field $\Bbbk$ with the structures of

(a) an associative algebra $(B, m, 1)$ with multiplication $m : B \otimes B \to B$ and unit $1 \in B$, and
(b) a coassociative coalgebra (B, Δ, ε) with comultiplication $\Delta : B \to B \otimes B$ and counit $\varepsilon : B \to \Bbbk$

satisfying the following conditions.

(i) The comultiplication $\Delta : B \to B \otimes B$ is a prealgebra morphism.
(ii) The unit and counit satisfy

$$(\Delta(1) \otimes 1)(1 \otimes \Delta(1)) = (\Delta \otimes Id)\Delta(1) = (1 \otimes \Delta(1))(\Delta(1) \otimes 1) \tag{E1.7.1}$$

and

$$\varepsilon(xyz) = \sum \varepsilon(xy_{(1)})\varepsilon(y_{(2)}z) = \sum \varepsilon(xy_{(2)})\varepsilon(y_{(1)}z), \tag{E1.7.2}$$

where $\Delta(y) = \sum y_{(1)} \otimes y_{(2)}$ is the Sweedler notation.

We refer to [10,11,53,54] for many other basic definitions related to weak bialgebras and weak Hopf algebras. The tensor structure of left modules over a weak bialgebra [53, Proposition 2] is given below.

Definition 1.8 Let A be a weak bialgebra over $\Bbbk$. For two left A-modules M and N, define $M \otimes^l N = \Delta(1)(M \otimes_{\Bbbk} N)$ where $\otimes_{\Bbbk}$ is the tensor product over $\Bbbk$.

The following lemma is clear.

Lemma 1.9 *Let A be a weak bialgebra.*

(1) *With the tensor product $- \otimes^l -$ given in Definition 1.8, both $A - \mathrm{mod}$ and $A - \mathrm{Mod}$ are monoidal abelian categories.*

(2) *Both* $D^b(A-\text{mod})$ *and* $D^b(A-\text{Mod})$ *are monoidal triangulated.*

Finally we mention a fact in quiver representations.

Lemma 1.10 [25, p. 63] *Let* Q *be a quiver of type* $\mathbb{A}_n$*. Then* $M\{i,j\}$*, for* $1 \leq i < j \leq n$*, defined as in* (E0.7.2)–(E0.7.3)*, form the complete list of indecomposable representations of* Q*, up to isomorphisms.*

Convention 1.11 *For the rest of the paper, we will use* A *for an algebra over* $\mathbb{k}$*. It could have a bialgebra or weak bialgebra structure. We will use* $\mathcal{A}$ *for the abelian category of finite dimensional left* A*-modules, also denoted by* $A-\text{mod}$*. Let* $\mathcal{T}$ *be a triangulated category that could have extra monoidal triangulated structure. Sometimes* $\mathcal{T}$ *denotes the bounded derived category* $D^b(\mathcal{A})$*. A general* $\mathbb{k}$*-linear or monoidal category is denoted by* $\mathcal{C}$*.*

2 Preliminaries on quiver representations

We refer to [4] for some basic concepts in quiver representation theory. Here we fix some convention. Let $Q = (Q_0, Q_1, s, t)$ be a quiver where Q_0 is the set of vertices of Q, Q_1 is the set of arrows of Q, and $s, t : Q_1 \to Q_0$ are source and target maps of Q respectively. Let M be a representation of Q. For each vertex $i \in Q_0$, let $(M)_i$ denote the vector space at i. For each arrow $\alpha \in Q_1$ from vertex $i := s(\alpha)$ to vertex $j := t(\alpha)$, let $(M)_\alpha$ denote the $\mathbb{k}$-linear map from $(M)_i$ to $(M)_j$ corresponding to α. Let $\text{Rep}(Q)$ be the category of all representations of Q and $\text{rep}(Q)$ be the full subcategory of $\text{Rep}(Q)$ consisting of finite dimensional representations. By [4, Theorem 1.7 in Chapter VII], every finite dimensional hereditary algebra A is Morita equivalent to the path algebra $\mathbb{k}Q$ of a finite acyclic quiver Q.

The definition of a weak bialgebra is given in Definition 1.7. The path algebra $\mathbb{k}Q$ is naturally equipped with a coalgebra structure that makes it a weak bialgebra, see [54, Example 2.5] and [30, Section 3]. We state this known fact as follows.

Lemma 2.1 *Let* Q *be a finite quiver.*

(1) *Its path algebra* $\mathbb{k}Q$ *is a cocommutative weak bialgebra whose coalgebra structure is determined by*

$$\Delta(p) = p \otimes p \quad \text{and} \quad \varepsilon(p) = 1$$

for any path $p = \alpha_1\alpha_2\cdots\alpha_m$ *of length* $m \geq 0$.

(2) *The weak bialgebra structure in part* (1) *is a bialgebra if and only if* $|Q_0| = 1$.

Since $\mathbb{k}Q$ is a cocommutative weak bialgebra, $\text{rep}(Q)(\cong \mathbb{k}Q - \text{mod})$ is a symmetric monoidal abelian category where the tensor product is given in Definition 1.8. For two representations $M = ((M)_i, (M)_\alpha)$ and $N = ((N)_i, (N)_\alpha)$ of Q where $i \in Q_0$ and $\alpha \in Q_1$, we can define the *vertex-wise tensor product* $M \otimes^v N$ by

$$(M \otimes^v N)_i = (M)_i \otimes_{\mathbb{k}} (N)_i, \quad \text{and} \quad (M \otimes^v N)_\alpha = (M)_\alpha \otimes_{\mathbb{k}} (N)_\alpha, \tag{E2.1.1}$$

for all $i \in Q_0$ and $\alpha \in Q_1$. Then the tensor product $M \otimes^l N$ given in Definition 1.8 is exactly equal to the vertex-wise tensor product $M \otimes^v N$ give in (E2.1.1). Therefore, we do not distinguish these two tensors and denote them by $M \otimes N$. The tensor structure of quiver representations has been studied by many researchers, see, for example, [29–32,40,41]. Note that the bounded derived category $D^b(\text{rep}(Q))$ is a tensor triangulated category in the sense of [5, Definition 1.1]; consequently, it is a monoidal triangulated category.

In this paper we study more than one tensor structures of the quiver representations. But, in this section, we are only working on the tensor structure defined by (E2.1.1). We start with some details about quiver representations.

We have defined the Frobenius–Perron dimension, denoted by fpd, of an object in a monoidal category $\mathcal{C}$ in Definition 1.3(4). A nice property of fpd is a duality property when applied to objects in rep(Q).

Definition 2.2 Let $Q = (Q_0, Q_1, s_Q, t_Q)$ be a quiver and M be a finite-dimensional representation of Q.

(1) Define the *opposite quiver* of Q, denoted by Q^{op}, to be the quiver which reverses all arrows in Q_1, that is

$$Q_0^{op} = Q_0, \, Q_1^{op} = Q_1, s_{Q^{op}} = t_Q, t_{Q^{op}} = s_Q.$$

(2) Define the *dual* of M, denoted by M^*, to be the representation of Q^{op} that is determined by

$$(M^*)_i = ((M)_i)^*, \, (M^*)_\alpha = ((M)_\alpha)^*,$$

for all vertices i and arrows α.

We give an easy example.

Example 2.3 Let Q be $1 \longrightarrow 2$ and M be $\Bbbk \xrightarrow{(1,0)^T} \Bbbk^2$. Then we have

$$Q^{op} : \; 1 \longleftarrow 2 \quad \text{and} \quad M^* = \; \Bbbk \xleftarrow{(1,0)} \Bbbk^2 \; .$$

For two finite dimensional $\Bbbk$-vector spaces U, V, we have

$$(V \otimes U)^* = U^* \otimes V^* \cong V^* \otimes U^*.$$

Furthermore, if we have linear maps between finite dimensional $\Bbbk$-vector spaces, say $f : V \to V'$ and $g : U \to U'$, then we have the commutative diagram

$$\begin{array}{ccc} (V \otimes U)^* & \xleftarrow{(f \otimes g)^*} & (V' \otimes U')^* \\ \downarrow \simeq & & \downarrow \simeq \\ V^* \otimes U^* & \xleftarrow{f^* \otimes g^*} & V'^* \otimes U'^*. \end{array}$$

The above commutative diagram holds for objects in rep(Q) since $\Bbbk Q$ is a commutative weak bialgebra [Lemma 2.1(1)]. It is clear that the $\Bbbk$-linear dual induces a contravariant equivalence between the abelian categories rep(Q) and rep(Q^{op}). Combining these two facts, we have

$$\begin{aligned} \mathrm{Hom}_{(\mathrm{rep}(Q))^{op}}(X, M \otimes N) &\cong \mathrm{Hom}_{\mathrm{rep}(Q)}(M \otimes N, X) \\ &\cong \mathrm{Hom}_{\mathrm{rep}(Q^{op})}(X^*, M^* \otimes N^*) \end{aligned} \tag{E2.3.1}$$

for $M, N, X \in \mathrm{rep}(Q)$. Now the following lemma follows from (E2.3.1).

Lemma 2.4 *Let Q be a finite quiver and M be a finite representation of Q. Then*

$$\mathrm{fpd}(M \otimes_{\mathrm{rep}(Q)^{op}} -) = \mathrm{fpd}(M^* \otimes_{\mathrm{rep}(Q^{op})} -)$$

where M is considered as an object in the tensor category rep$(Q)^{op}$ *and M^* an object in* rep(Q^{op}).

The same statement holds for other Frobenius–Perron invariants such as fpv.

Next we study some brick sets of quiver representations. Let $S(i)$ denote the simple representation (of Q) at vertex i where

$$S(i)_j = \begin{cases} \Bbbk & j = i \\ 0 & j \neq i \end{cases} \quad \text{and} \quad S(i)_\alpha = 0, \quad \forall\ \alpha \in Q_1, \tag{E2.4.1}$$

and e_i denote the trivial path at vertex i. By the tensor structure of rep(Q) (E2.1.1), we have the following.

Lemma 2.5 *Let $S(i)$ be the simple left $\Bbbk Q$-module defined as above and M in* rep(Q). *Then $S(i) \otimes M$ is isomorphic to a direct sum of finitely many copies of $S(i)$.*

In the above lemma, $S(i) \otimes M$ could be 0.

Proposition 2.6 *Let M be in* rep(Q). *Then*

$$\text{fpd}(M) \geq d,$$

where $d = \max_{v \in Q_0}\{\dim((M)_v)\}$.

Proof Let $a = \dim(M)_v$ and let $\phi_0 = \{S(v)\}$ for $v \in Q_0$. Then

$$\text{Hom}_{\text{rep}(Q)}(S(v), M \otimes S(v)) = \text{Hom}_{\text{rep}(Q)}(S(v), S(v)^{\oplus a}) = \Bbbk^{\oplus a}$$

which implies that $A(\phi_0, M \otimes -) = (a)_{1\times 1}$. Therefore fpd$(M) \geq a$ for all a. The assertion follows. □

Note that fpd(M) may be infinite as the next example shows (and as predicted by Lemma 6.4).

Example 2.7 Let Q be the Kronecker quiver $1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2$. Let $S(1)$ be defined as in (E2.4.1). For every $c \in \Bbbk$, we define an object in rep(Q):

$$M_c := \Bbbk \underset{\beta = cId}{\overset{\alpha = Id}{\rightrightarrows}} \Bbbk\ . \tag{E2.7.1}$$

Then M_c is a brick object (and such an object is also called a band module of Q [13, pp. 160–161]). It is easy to see that Hom$(M_c, S(1)) \cong \Bbbk$ and that $\{M_c, M_{c'}\}$ is a brick set if $c \neq c'$. As a consequence, $\{M_c, \mid c \in \Bbbk\}$ is an infinite brick set.

Let T be any finite subset of $\Bbbk$ and let $\phi := \{M_c \mid c \in T\}$. The $A(\phi, S(1) \otimes -)$ is a $|T| \times |T|$ matrix in which all entries are 1. Then $\rho(A(\phi, S(1) \otimes -)) = |T|$. Since $\Bbbk$ is infinite, we obtain that fpd$(S(1)) = \infty$.

Let us consider a slightly more general situation.

Example 2.8 Suppose Q is another quiver and p_1 and p_2 are two paths from vertex i to vertex j that do not intersect except at the two endpoints. Then we can consider a similar brick object M_c so that

$$(M_c)_v = \begin{cases} \Bbbk & \text{if } v \text{ is in either } p_1 \text{ or } p_2, \\ 0 & \text{otherwise,} \end{cases}$$

$$(M_c)_\alpha = \begin{cases} Id & \text{if } \alpha \text{ is in either } p_1 \text{ or } p_2, \text{ but not the first arrow in } p_2, \\ cId & \text{if } \alpha \text{ is the first arrow in } p_2, \\ 0 & \text{otherwise,} \end{cases}$$

or, similar to (E2.7.1), we can write it as

$$M_c := \Bbbk \overset{p_1=Id}{\underset{p_2=cId}{\rightrightarrows}} \Bbbk\ .$$

Then $\{M_c, \mid c \in \Bbbk\}$ is an infinite brick set.

We will use this example later.

3 Discrete categories

In this section we will prove some basic lemmas for monoidal abelian categories that are needed in the proof of Theorem 0.4. We start with a definition.

Definition 3.1 Let $\mathcal{C}$ be a monoidal abelian category. We say $\mathcal{C}$ is *discrete* if

(a) $\mathcal{C}$ is Hom-finite, namely $\mathrm{Hom}_{\mathcal{C}}(M, N)$ is finite dimensional over $\Bbbk$ for objects M, N in $\mathcal{C}$,
(b) every object in $\mathcal{C}$ has finite length,
(c) $\mathcal{C}$ has finitely many simple objects, say $\{S_1, \ldots, S_n\}$, up to isomorphisms, and
(d) for all simple objects S_i and S_j in $\mathcal{C}$,

$$S_i \otimes S_j \cong \begin{cases} S_i & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases} \tag{E3.1.1}$$

Note that an essentially small category $\mathcal{C}$ satisfying condition (b) is called a *length* category [24,43].

Let Q be a finite quiver. Then there is a canonical monoidal abelian structure on $\mathrm{rep}(Q)$ induced by the weak bialgebra structure defined in Lemma 2.1. The following lemma follows immediately from the definition, see (E2.1.1).

Lemma 3.2 *Let Q be a finite acyclic quiver. Then the canonical monoidal abelian structure on* $\mathrm{rep}(Q)$ *is discrete.*

For the rest of this section we assume that $\mathcal{C}$ is discrete. As a consequence, $\mathcal{C}$ is a Krull-Schmidt category. For $M \in \mathcal{C}$, let $\ell(M)$ denote the length of $\mathcal{C}$. Let $IC(M)$ denote the *isomorphism class* of all (possibly repeated) simple subquotients of M. This can be obtained by considering any composition series of M. Even though composition series of M is not unique, $IC(M)$ is unique, so well-defined.

Lemma 3.3 *Let $\mathcal{C}$ be a* Hom*-finite monoidal abelian category with finitely many simple objects. Let $\mathbf{1}$ be the unit object.*

(1) *$\ell(-)$ is additive.*
(2) *For every nonzero object M there is a simple object $S \in IC(\mathbf{1})$ such that $S \otimes M \neq 0$. By symmetry, there is a simple object $T \in IC(\mathbf{1})$ such that $M \otimes T \neq 0$.*
(3) *If $S \in IC(\mathbf{1})$ and T is a simple object in $\mathcal{C}$, then $S \otimes T$ is either 0 or a simple object. For each T, there is only one $S \in IC(\mathbf{1})$ such that $S \otimes T \neq 0$.*
(4) *The multiplicity of any simple object S in $IC(\mathbf{1})$ is 1.*

(5) *If* $S, T \in IC(\mathbf{1})$, *then*

$$S \otimes T \cong \begin{cases} S & \text{if } S = T \\ 0 & \text{if } S \neq T. \end{cases}$$

(6) *$\mathcal{C}$ is discrete if and only if $S \in IC(\mathbf{1})$ for all simple objects S.*

Proof (1) Clear from the definition.
(2) By part (1), we have

$$\ell(M) = \ell(\mathbf{1} \otimes M) = \sum_{S \in IC(\mathbf{1})} \ell(S \otimes M). \tag{E3.3.1}$$

Therefore there is an $S \in IC(\mathbf{1})$ such that $S \otimes M \neq 0$.
(3) If $S \otimes T \neq 0$, then, by (E3.3.1), we have

$$1 = \ell(T) = \ell(\mathbf{1} \otimes T) = \sum_{S' \in IC(\mathbf{1})} \ell(S' \otimes T) \geq \ell(S \otimes T) \geq 1.$$

Therefore $\ell(S \otimes T) = 1$ and $\ell(S' \otimes T) = 0$ for all other $S' \in IC(\mathbf{1})$.
(4) This follows from (E3.3.1) by taking a simple object M with $S \otimes M \neq 0$.
(5) It remains to show that $S \otimes T = 0$ if S and T are distinct elements in $IC(\mathbf{1})$. Suppose on the contrary that $U := S \otimes T \neq 0$. By part (3), it is a simple object. Since $U = S \otimes T$ is a subquotient of $\mathbf{1} \otimes \mathbf{1}$, U is in $IC(\mathbf{1})$. Since $S \neq T$, we have either $U \neq S$ or $U \neq T$. By symmetry, we assume that $U \neq S$. By part (3), there is only one $W \in IC(\mathbf{1})$ such that $W \otimes U \neq 0$. This implies that $W \otimes S \neq 0$ as $U = S \otimes T$. There are two different objects, namely, $S, U \in IC(\mathbf{1})$ such that $W \otimes S \neq 0$ and $W \otimes U \neq 0$. By the left-version of part (3) this is impossible. The assertion follows.
(6) If $IC(\mathbf{1})$ contains all simple objects, then by part (5), $\mathcal{C}$ is discrete. Conversely, suppose $\mathcal{C}$ is discrete. For every simple object T, by part (2), there is an $S \in IC(\mathbf{1})$ such that $S \otimes T \neq 0$. By the definition of discreteness, $T = S$. So $T \in IC(\mathbf{1})$. □

Proposition 3.4 *Let A be a finite dimensional algebra of finite global dimension. Suppose that $(A - \text{mod}, \otimes)$ is a discrete monoidal abelian category. Then, for any simple left A-module S and any $M \in A - \text{mod}$,*

$$M \otimes S \cong S^{\oplus n}$$

where n is the number of copies of S in the the composition series of M.

Proof By the 'no loops conjecture', which was proved by Igusa [36],

$$\mathrm{Ext}^1_A(S, S) = 0. \tag{E3.4.1}$$

By definition, $- \otimes -$ is biexact. Hence $M \otimes S$ has a composition series that is induced by the composition series of M. Let T be a simple subquotient of M. Then $T \otimes S$ is either S when $T \cong S$ or 0 if $T \ncong S$. Thus $M \otimes S$ has a composition series with each simple subquotient being S. The assertion follows from (E3.4.1). □

Recall that $\otimes^v$ is the canonical tensor given in (E2.1.1). We have an immediate consequence.

Corollary 3.5 *Let Q be a finite acyclic quiver. If* $(\mathrm{rep}(Q), \otimes)$ *is another discrete monoidal abelian structure on* $\mathrm{rep}(Q)$*, then for any* $M \in \mathrm{rep}(Q)$ *and any simple representation S over Q,*

$$M \otimes S \cong M \otimes^v S$$

where $\otimes^v$ *is defined as in* (E2.1.1).

There are a lot of monoidal categories that are not discrete. For example, for a finite quiver Q, if $\mathrm{rep}(Q)$ is equipped with other bialgebra structure, it may not be discrete, see Proposition 7.7(a-d). We conclude this section with the definition of a discrete action.

Definition 3.6 Let $\mathcal{C}$ be a monoidal abelian category acting on an abelian category $\mathcal{A}$. Assume that both $\mathcal{C}$ and $\mathcal{A}$ satisfy Definition 3.1(a,b,c). Let $\{T_1, \ldots, T_n\}$ (respectively, $\{S_1, \ldots, S_m\}$) be the complete list of simple objects in $\mathcal{C}$ (respectively, $\mathcal{A}$), where $m \geq n$. The action of $\mathcal{C}$ on $\mathcal{A}$ is called *discrete* if

(d)' there is a permutation $\sigma \in S_n$ such that

$$T_i \odot S_j \cong \begin{cases} S_j & \text{if } j = \sigma(i) \\ 0 & \text{if } j \neq \sigma(i). \end{cases} \tag{E3.6.1}$$

4 Proof of Theorem 0.4

The aim of this section is to prove Theorem 0.4. We need first recall some facts from representation theory of quivers.

Proposition 4.1 [4, Proposition 2.5 in Chapter VII] *Let Q be a finite, connected, and acyclic quiver and M be a brick such that there exists a* $a \in Q_0$ *with* $\dim(M)_a > 1$*. Let Q' be the quiver defined as follows:* $Q' = (Q'_0, Q'_1)$*, where* $Q'_0 = Q_0 \cup \{b\}$*;* $Q'_1 = Q_1 \cup \{\alpha\}$*; and* $\alpha : b \to a$*. Then* $\Bbbk Q'$ *is of infinite representation type.*

By duality and Proposition 4.1, if α is an arrow of the form $a \to b$, then $\Bbbk Q'$ is also of infinite representation type.

Lemma 4.2 [4, Corollary 5.14 in Chapter VII] *If Q is a quiver of type* $\mathbb{ADE}$*, see [4, p. 252], then every indecomposable representation of Q is a brick.*

Recall from Definition 1.4(3) that an algebra A is *wild* or *of wild representation type* if there is a faithful exact embedding of abelian categories

$$Emb : \Bbbk\langle x_1, x_2\rangle - \mathrm{mod} \longrightarrow \mathcal{A} := A - \mathrm{mod} \tag{E4.2.1}$$

that preserves indecomposables and respects isomorphism classes (namely, for all objects M_1, M_2 in $\Bbbk\langle x_1, x_2\rangle - \mathrm{mod}$, $Emb(M_1) \cong Emb(M_2)$ if and only if $M_1 \cong M_2$). A stronger notion of wildness is the following. An algebra A is called *strictly wild*, or *fully wild*, if Emb in (E4.2.1) is a fully faithful embedding [1, Proposition 5]. By definition, strictly wild is wild, but the converse is not true. It is well-known that a wild path algebra $\Bbbk Q$ is always strictly wild, see a comment of Gabriel [23, p. 140] or [1, Proposition 7].

Lemma 4.3 *Let A be a finite dimensional algebra that is strictly wild. Let $\mathcal{C}$ be an abelian category containing $\mathcal{A}$ as a full subcategory. Then $\mathcal{C}$ contains an infinite connected brick set. As a consequence, if Q is a finite acyclic quiver that is wild, then* $\mathrm{rep}(Q)$ *contains an infinite connected brick set.*

Proof The consequence follows from the fact that a wild quiver is strictly wild. So we only prove the main assertion.

Let A be strictly wild. By definition, there is a fully faithful embedding

$$Emb : \Bbbk\langle x_1, x_2\rangle - \text{mod} \longrightarrow \mathcal{A} \longrightarrow \mathcal{C}.$$

For each $c \in \Bbbk$, let $M(c)$ denote the 1-dimensional simple module $\Bbbk\langle x_1, x_2\rangle/(x_1 - c, x_2)$ and let N_c be $Emb(M_c)$. By taking a free resolution $M(c)$, one can check that $\text{Ext}^1_{\Bbbk\langle x_1,x_2\rangle}(M(c), M(c')) \neq 0$ for all c, c'. Hence $\{M(c) \mid c \in \Bbbk\}$ is an infinite connected brick set in $\Bbbk\langle x_1, x_2\rangle - \text{mod}$. Since Emb a fully faithful embedding, $\{N_c \mid c \in \Bbbk\}$ is an infinite connected brick set of $\mathcal{C}$. □

Lemma 4.4 *Let $\mathcal{C}$ be an abelian category of finite global dimension and let $\mathcal{T}$ be the bounded derived category $D^b(\mathcal{C})$. Suppose that*

(1) *$\mathcal{T}$ is triangulated equivalent to $D^b(B - \text{mod})$ for a finite dimensional hereditary algebra B via tilting object X, namely,*

$$\text{RHom}_{\mathcal{T}}(X, -) : \mathcal{T} \to D^b(B - \text{mod})$$

is a triangulated equivalence where $B \cong \text{RHom}_{\mathcal{T}}(X, X)$, and

(2) *$\mathcal{C}$ contains an infinite (respectively, infinite connected) brick set.*

Then $B - \text{mod}$ contains an infinite (respectively, infinite connected) brick set.

Note that, if $\mathcal{C} = A - \text{mod}$ for some finite dimensional algebra A and if $\mathcal{T}$ is triangulated equivalent to $D^b(B - \text{mod})$, then, by tilting theory, the existence of X is automatic.

Proof of Lemma 4.4 We only prove the assertion for "infinite brick set". The proof for "infinite connected brick set" is similar.

Let

$$F := \text{RHom}_{\mathcal{T}}(X, -) : \mathcal{T} \longrightarrow D^b(B - \text{mod})$$

be an equivalence of triangulated categories. Let $\{N(c) \mid c \in U\}$ be an infinite brick set of $\mathcal{C}$ by hypothesis. Then

$$\{F(N(c)) \mid c \in U\}$$

is an infinite brick set of $D^b(B - \text{mod})$. Since X has finite projective dimension, there is an integer n independent of $c \in U$ such that

$$H^i(F(N(c))) = 0 \text{ for all } |i| > n. \tag{E4.4.1}$$

Note that $B - \text{mod}$ is hereditary, which implies that every indecomposable object in $D^b(B - \text{mod})$ is of the form $M[i]$ for some indecomposable object $M \in B - \text{mod}$ and for some i [38, Section 2.5]. By (E4.4.1), $F(N(c)) = M_c[i_c]$ for some indecomposable object $M_c \in B - \text{mod}$ and some integer $|i_c| \leq n$. Since U is infinite, there is an infinite subset $U' \subseteq U$ such that i_c is a constant for all $c \in U'$. Let i_0 denote such i_c. Thus $\{M_c[i_0] \mid c \in U'\}$ is an infinite brick set in $D^b(B - \text{mod})$. Since the suspension [1] is an isomorphism of $D^b(B - \text{mod})$, $\{M_c \mid c \in U'\}$ is an infinite brick set in $D^b(B - \text{mod})$. Finally, using the fact that $B - \text{mod}$ is a full subcategory of $D^b(B - \text{mod})$, we obtain that $\{M_c \mid c \in U'\}$ is an infinite brick set in $B - \text{mod}$. □

Lemma 4.5 *Let A be a finite dimensional hereditary algebra that is not of finite representation type. Then the abelian category $A-\mathrm{mod}$ contains an infinite brick set. As a consequence, if Q is a finite acyclic quiver not of type $\mathbb{ADE}$, then $\mathrm{rep}(Q)$ contains an infinite brick set.*

Proof By [4, Theorem 1.7 in Chapter VII] every such A is Morita equivalent to a path algebra $\Bbbk Q$ for some finite acyclic quiver Q. By Lemma 4.4, we may assume that A is $\Bbbk Q$.

Since A is not of finite type, Q is not of finite type. Lemma 4.3 settles the case where Q is of wild representation type.

Case 1: Q is of type $\widetilde{\mathbb{A}}$. Since Q is acyclic, there exist two different paths p_1 and p_2 from v to u, where $v \neq u \in Q_0$. We can further assume that the length p_1 is smallest among all such choices. In this case, $\mathrm{rep}(Q)$ contains an infinite brick set by Example 2.8.

Case 2: Q is of type $\widetilde{\mathbb{D}}\widetilde{\mathbb{E}}$. We consider a slightly more general situation and then apply the assertion to the special case (see quivers in [4, Corollary 2.7 in Chapter VII]). If there exists a subquiver Q' of Q and an indecomposable representation M of Q' satisfying:

(a) Q' is a quiver of type $\mathbb{D}$ or $\mathbb{E}$,
(b) there exists $x \in Q'_0$, $\dim(M)_x > 1$,
(c) $\{y\} \in Q_0 \backslash Q'_0$,
(d) there exists an arrow $\alpha \in Q_1$ such that $\alpha : y \to x$,

then we construct a new representation $M(\lambda)$ as follows:

$$(M(\lambda))_v = \begin{cases} (M)_v & \text{if } v \in Q'_0 \\ \Bbbk & \text{if } v = y \\ 0 & \text{otherwise,} \end{cases} \qquad (M(\lambda))_\beta = \begin{cases} (M)_\beta & \text{if } \beta \in Q'_1 \\ \lambda & \text{if } \beta = \alpha \\ 0 & \text{otherwise,} \end{cases}$$

where $\lambda : \Bbbk \to (M)_x$ is a $\Bbbk$-linear map.

Then by the proof of [4, Proposition 2.5 in Chapter VII], each $M(\lambda)$ is a brick and there exists infinitely many pairwise non-isomorphic bricks of the form $M(\lambda)$. In fact, the proof of [4, Proposition 2.5 in Chapter VII] shows that there is an infinite set of $U := \{\lambda : \Bbbk \to M_x\}$ such that $\mathrm{Hom}_{\mathrm{rep}(Q)}(M(\lambda), M(\lambda')) = 0$ for all $\lambda, \lambda' \in U$. This means that $\mathrm{rep}(Q)$ contains an infinite brick set. Dually, If we change the condition (d) into (d)':

(d)' there exists an arrow $\alpha \in Q_1$ such that $\alpha : x \to y$,

we can still construct an infinite brick set as above.

Now we go back to a quiver of type $\widetilde{\mathbb{D}}\widetilde{\mathbb{E}}$. By Lemma 4.4, we can assume that

(e) Q' is one of the quivers in [4, Corollary 2.6 in Chapter VII.2], and that
(f) (c) and (d) hold.

Note that (e) implies that (a) holds. By [4, Corollary 2.6 in Chapter VII.2], (b) holds. Therefore we proved that $\mathrm{rep}(Q)$ contains an infinite brick set. □

Lemma 4.6 *Let $\mathcal{C}$ be a monoidal abelian category acting on an abelian category $\mathcal{A}$. Assume that both $\mathcal{C}$ and $\mathcal{A}$ satisfy Definition 3.1(a,b,c). Suppose that*

(a) *$\mathcal{A}$ contains an infinite brick set, and that*
(b) *the action of $\mathcal{C}$ on $\mathcal{A}$ is discrete.*

Then there is a simple $T \in \mathcal{C}$ such that $\mathrm{fpd}(T) = \infty$.

Lemma 4.6 may fail if the action is not discrete. Let Q be the Kronecker quiver in Example 2.7 and A be its path algebra equipped with the cocommutative bialgebra structure in Proposition 7.7(a). Then $S(1)$ is the unit object in $\mathcal{A}$ and $S(2) \otimes M = S(2)^{\oplus \dim(M)}$ for any $M \in \mathcal{A}$. Since all indecomposables in $\mathrm{rep}(Q)$ are well-understood, one can check that $\mathrm{fpd}(S(1)) = \mathrm{fpd}(S(2)) = 1$ (details are omitted).

Proof of Lemma 4.6 Let $\{N(c) \mid c \in U\}$ is an infinite brick set of $\mathcal{A}$ and let $\{S_1, \ldots, S_n\}$ be the complete list of simple objects in $\mathcal{A}$ up to isomorphism. For each $1 \leq i \leq n$, define

$$U_i := \{c \in U \mid \mathrm{Hom}_{\mathcal{A}}(N(c), S_i) \neq 0\}.$$

For each $c \in U$, there is an i such that $\mathrm{Hom}_{\mathcal{A}}(N(c), S_i) \neq 0$. This implies that $U = \bigcup_{i=1}^{n} U_i$. Therefore there is an i such that U_i is infinite. Without loss of generality, we may assume that $U = U_1$ is infinite.

Since the action is discrete, there is a simple object $T \in \mathcal{C}$ such that $T \odot S_1 \cong S_1$. Now $\mathrm{Hom}_{\mathcal{A}}(N(c), S_1) \neq 0$ implies that every simple subquotient of $T \odot N(c)$ is isomorphic to S_1. In particular, $T \odot N(c)$ contains a copy of S_1 for all c.

Let W be any finite subset of U and let $\phi = \{N(c) \mid c \in W\}$. Using the above paragraph,

$$\mathrm{Hom}_{\mathcal{A}}(N(c), T \odot N(c')) \neq 0$$

for all $c, c' \in W$. This implies that $\rho(A(\phi, T \odot -)) \geq |W|$ and $\mathrm{fpd}(T) \geq |W|$. Since $|W|$ can be arbitrarily larger, $\mathrm{fpd}(T) = \infty$. □

The following is a part of Theorem 0.4.

Theorem 4.7 *Let A be a finite dimensional hereditary algebra and let $\mathcal{A} = A - \mathrm{mod}$. Let $\mathcal{C}$ be a monoidal abelian category satisfying Definition 3.1(a,b,c). Suppose that there is an action of $\mathcal{C}$ on $\mathcal{A}$ that is discrete. Then the following are equivalent:*

(a) *A is of finite representation type,*
(b) $\mathrm{fpd}(M) < \infty$ *for every irreducible object $M \in \mathcal{C}$,*
(c) $\mathrm{fpd}(M) < \infty$ *for every indecomposable object $M \in \mathcal{C}$,*
(d) $\mathrm{fpd}(M) < \infty$ *for every object $M \in \mathcal{C}$*

Proof (a) $\Longrightarrow$ (d): If A is of finite representation type, then $\mathcal{A}$ has only finitely many indecomposable objects. This means that there are only finitely many brick sets. Then, by definition, $\mathrm{fpd}(\sigma)$ is finite for every endofunctor σ of $\mathcal{A}$. In particular, $\mathrm{fpd}(M)$ is finite for every representation $M \in \mathcal{C}$.

(d) $\Longrightarrow$ (c) $\Longrightarrow$ (b): Clear.

(b) $\Longrightarrow$ (a): It suffices to show that if A is not of finite representation type, then $\mathrm{fpd}(M) = \infty$ for some irreducible representation $M \in \mathcal{C}$. The assertion follows from Lemmas 4.5 and 4.6 . □

We will use the following lemma concerning a bound of spectral radius of a matrix.

Lemma 4.8 (Gershgorin Circle Theorem [27]) *Let A be a complex $n \times n$ matrix, with entries a_{ij}. For $i \in \{1, \ldots, n\}$, let $R_i = \sum_{j \neq i} |a_{ij}|$ be the sum of the absolute values of the non-diagonal entries in the i-th row. Let $D(a_{ii}, R_i) \subseteq \mathbb{C}$ be a closed disc centered at a_{ii} with radius R_i. Then every eigenvalue of A lies within at least one of the Gershgorin discs $D(a_{ii}, R_i)$. As a consequence, $\rho(A) \leq \max_i\{|a_{ii}| + R_i\}$.*

Proposition 4.9 *Suppose $\mathcal{T}$ is a triangulated category satisfying*

(a) *$\mathcal{T}$ is* Hom*-finite and hence Krull-Schmidt,*
(b) *there are objects $\{X_1, \ldots, X_N\}$ such that every indecomposable object in $\mathcal{T}$ is of the form $X_i[m]$ for some $1 \leq i \leq N$ and $m \in \mathbb{Z}$, and*
(c) *for every two indecomposable objects X, Y in $\mathcal{T}$, $\mathrm{Hom}_{\mathcal{T}}(X, Y[m]) = 0$ for $|m| \gg 0$.*

Then the following hold.

(1) $\text{fpd}(\sigma) < \infty$ *for every endofunctor* σ *of* $\mathcal{T}$.
(2) *If* $\mathcal{C}$ *is a monoidal triangulated category acting on* $\mathcal{T}$, *then* $\text{fpd}(M) < \infty$ *for every object* $M \in \mathcal{C}$.

Proof Let σ be an endofunctor of $\mathcal{T}$. Since there are only finitely many X_i in hypothesis (b), we can assume that every $\sigma(X_i)$ is a direct summand of

$$X = \left(\bigoplus_{i=1}^{N} \bigoplus_{j=-\delta}^{\delta-1} X_i[j] \right)^{\oplus \xi} \tag{E4.9.1}$$

for some fixed δ and ξ.

We make some definitions. Let

$$\alpha = \max\{\dim \text{Hom}_{\mathcal{T}}(X_i[s], X) \mid \forall \ s, i\},$$
$$\gamma = \max\{|s| \mid \ \text{Hom}_{\mathcal{T}}(X_i[s], X) \neq 0 \text{ for some } i\}.$$

For any given finite brick set ϕ, it is always is a subset of

$$\Phi := \bigcup_{j=-D}^{D-1} \{X_1[j], \dots, X_N[j]\}$$

for some large $D \gg 0$. Since ϕ is a subset of Φ, we have

$$\rho(A(\phi, \sigma)) \leq \rho(A(\Phi, \sigma)).$$

By Definition 1.3(4), it is enough to show that $\rho(A(\Phi, \sigma))$ is uniformly bounded on Φ (for each fixed X as given in (E4.9.1)). For the next calculation we make a linear order on the objects in Φ as

$$\begin{aligned} \Phi = \{X_1[-D], \dots, X_N[-D]\} \cup \{X_1[-D+1], \dots, X_N[-D+1]\} \cup \\ \cdots \cup \{X_1[D-2], \dots, X_N[D-2]\} \cup \{X_1[D-1], \dots, X_N[D-1]\} \end{aligned} \tag{E4.9.2}$$

and write is as $\Phi = \{Y_1, \dots, Y_{2ND}\}$. Write the adjacency matrix $A(\Phi, \sigma)$ as (a_{ij}). For each pair (i, j), by definition,

$$a_{ij} = \dim \text{Hom}_{\mathcal{T}}(X_{s_i}[w_i], \sigma(X_{s_j}[w_j])) \leq \dim \text{Hom}_{\mathcal{T}}(X_{s_i}[w_i], X[w_j]) \leq \alpha,$$

for some s_i, s_j, w_i, w_j; and by the ordering in (E4.9.2), we obtain

$$a_{ij} = 0 \quad \text{if } |i-j| > 2N\delta + \gamma + 2.$$

Then each R_i in the Lemma 4.8 is bounded by $(2N\delta + \gamma + 2)\alpha$. By Lemma 4.8 (Gershgorin Circle Theorem), there is a bound of $\rho(A(\Phi, \sigma))$ which is independent of D. Since every finite brick set ϕ is a subset of Φ for some large D, $\rho(A(\phi, \sigma))$ has a bound that is independent of ϕ. Therefore $\text{fpd}(\sigma)$ is finite as desired. □

We will use the following special case. Recall that $\mathcal{A} = A-\text{mod}$ and that $\mathcal{T} = D^b(\mathcal{A})$.

Corollary 4.10 *Let A be a finite dimensional hereditary algebra that is of finite representation type. Then every monoidal triangulated structure on* $\mathcal{T}$ *is* fpd*-finite.*

Proof Since A is of finite type, we can list all indecomposable left A-modules $\{X_1, \dots, X_N\}$. Since A is hereditary, every indecomposable object in $\mathcal{T}$ is of the form $X_i[s]$ for some $1 \leq i \leq N$ and $s \in \mathbb{Z}$ [14, Lemma 3.3]. Finally, since A is hereditary, then $\text{Hom}_{\mathcal{T}}(X_i, X_j[m]) = 0$ for $m \neq 0, 1$. Thus $\mathcal{T}$ satisfies hypotheses (a,b,c) in Proposition 4.9. Then the assertion follows from Proposition 4.9(2) by setting $\mathcal{C} = \mathcal{T}$ and $\odot = \otimes$. □

Lemma 4.11 *Let A be a finite dimensional hereditary algebra. Let $\mathcal{C}$ be a monoidal abelian category satisfying Definition 3.1(a,b,c). Suppose that $\mathcal{C}$ acts on $\mathcal{A}$ via $\odot$. Let $\odot_D$ be the induced action of $D^b(\mathcal{C})$ on $\mathcal{T}$. Let M be an object in $\mathcal{C}$, also viewed as an object in $D^b(\mathcal{C})$.*

(1) *If* $n \neq 0, 1$, *then* $\mathrm{fpd}(M[n] \odot_D -) = 0$.
(2) $\mathrm{fpd}(M \odot_D -) = \mathrm{fpd}(M \odot -)$.

Proof (1) Suppose $n \geq 2$. Let ϕ be a (finite) brick set. Since $\mathcal{A}$ is hereditary, every indecomposable object is of the form $X[m]$. Then we can write $\phi = \bigcup_{\lambda \in \mathbb{Z}} \phi_\lambda$ where ϕ_λ is either empty or $\{X_{\lambda,1}[\lambda], X_{\lambda,2}[\lambda], \ldots, X_{\lambda,t_\lambda}[\lambda]\}$. Since $\mathcal{A}$ is hereditary,

$$\mathrm{Hom}_{\mathcal{T}}(X_{\lambda,s}[\lambda], M[n] \odot_D X_{\delta,s'}[\delta]) = \mathrm{Hom}_{\mathcal{T}}(X_{\lambda,s}[\lambda], (M \odot X_{\delta,s'})[n+\delta]) = 0$$

for all $\lambda \leq \delta$. Then $A(\phi, M[n] \odot_D -)$ is strictly upper triangular. Therefore $\rho(A(\phi, M[n] \odot_D -)) = 0$. As a consequence the assertion follows.

The proof for $n < 0$ is similar.

(2) Let ϕ be a brick set as in part (1). Similar to the proof of part (1), also see [14, Lemma 6.1], we obtain that $A(\phi, M \odot_D -)$ is a block lower triangular matrix. So we only need to consider the case that $\phi = \{X_1[d], X_2[d], \ldots X_t[d]\}$ for the same d. In this case, $A(\phi, M \odot_D -) = A(\phi[-d], M \odot -)$. Therefore the assertion follows. □

Now we are ready to prove Theorem 0.4. We will use the notation introduced in Theorem 4.7 and Lemma 4.11.

Theorem 4.12 *Let A be a finite dimensional hereditary algebra and let $\mathcal{A} = A - \mathrm{mod}$. Let $\mathcal{C}$ be a monoidal abelian category satisfying Definition 3.1(a,b,c). Suppose that there is an action of $\mathcal{C}$ on $\mathcal{A}$ that is discrete. Then the following are equivalent:*

(a) *A is of finite representation type,*
(b) $\mathrm{fpd}(M) < \infty$ *for every irreducible object* $M \in \mathcal{C}$,
(c) $\mathrm{fpd}(M) < \infty$ *for every indecomposable object* $M \in \mathcal{C}$,
(d) $\mathrm{fpd}(M) < \infty$ *for every object* $M \in \mathcal{C}$,
(e) $\mathrm{fpd}(M \odot_D -) < \infty$ *for every indecomposable object* $M \in D^b(\mathcal{C})$,
(f) $\mathrm{fpd}(M \odot_D -) < \infty$ *for every object* $M \in D^b(\mathcal{C})$.

Suppose A is the path algebra $\Bbbk Q$ for some finite quiver Q. Then any of conditions (a) *to* (f) *is equivalent to*

(g) *Q is a finite union of quivers of type $\mathbb{ADE}$.*

Proof By Theorem 4.7, the first four conditions are equivalent.

(a) $\Longrightarrow$ (f): This follows from Proposition 4.9 and the proof of Corollary 4.10.

(f) $\Longrightarrow$ (e): Clear.

(e) $\Longrightarrow$ (c): This follows from Lemma 4.11(2). □

Clearly Theorem 0.4 is a special case of Theorem 4.12.

5 *mtt*-structures of a monoidal triangulated category

First we recall the definition on a t-structure on a triangulated category. The notion of a t-structure was introduced by Beilinson–Bernstein–Deligne in [7]. We make a small change in the definition below.

Definition 5.1 Let $\mathcal{T}$ be a triangulated category.

(1) A *t-structure* on $\mathcal{T}$ is a pair of full subcategories $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ satisfying the following conditions.

(1a) $\mathcal{T}^{\leq 0} \subseteq \mathcal{T}^{\leq 1}$ and $\mathcal{T}^{\geq 0} \supseteq \mathcal{T}^{\geq 1}$ where we use notation $\mathcal{T}^{\leq n} = \mathcal{T}^{\leq 0}[-n]$ and $\mathcal{T}^{\geq n} = \mathcal{T}^{\geq 0}[-n]$.

(1b) If $M \in \mathcal{T}^{\leq 0}$ and $N \in \mathcal{T}^{\geq 1}$, then $\mathrm{Hom}_{\mathcal{T}}(M, N) = 0$.

(1c) For any object $X \in \mathcal{T}$, there is a distinguished (exact) triangle

$$M \to X \to N \to M[1]$$

with $M \in \mathcal{T}^{\leq 0}$ and $N \in \mathcal{T}^{\geq 1}$.

(2) The *heart* of the t-structure is the full subcategory

$$\mathcal{T}^{\geq 0} \cap \mathcal{T}^{\leq 0}$$

which is denoted by $\mathcal{H}$ or $\mathcal{H}(\mathcal{T})$.

(3) [16, p. 1427] A t-structure is called *bounded* if for each $X \in \mathcal{T}$, there exist $m \leq n$ such that $X \in \mathcal{T}^{\leq n} \cap \mathcal{T}^{\geq m}$.

(4) [16, p. 1427] A bounded t-structure is called *hereditary* if $\mathrm{Hom}_{\mathcal{T}}(X, Y[n]) = 0$ for $n \geq 2$ and $X, Y \in \mathcal{H}$.

As a classical example, if $\mathcal{T}$ is the derived category $D^b(A - \mathrm{mod})$, there is a natural t-structure on $\mathcal{T}$ by setting $\mathcal{T}^{\leq 0}$ to be the complexes concentrated in degrees less than or equal to 0 (and similarly for $\mathcal{T}^{\geq 0}$). In this case the heart of this t-structure is $A - \mathrm{mod}$.

Note that hereditary t-structures are very special. Even for the path algebra of a quiver Q of type $\mathbb{A}_3$, there is a t-structure in $D^b(\mathrm{rep}(Q))$ that is not hereditary, see [39] for a classification of t-structures of $D^b(\mathrm{rep}(Q))$ of a quiver of Dynkin type.

We would like to introduce a version of the t-structure in a monoidal triangulated category. We use *mtt* for "monoidal triangulated t" in the next definition.

Definition 5.2 Let $\mathcal{T}$ be a monoidal triangulated category in parts (1,2,3) and a triangulated category in part (4).

(1) A t-structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ on $\mathcal{T}$ is called an *mtt-structure* if the following conditions hold.

(a) $\mathcal{T}^{\leq 0} \otimes \mathcal{T}^{\leq 0} \subseteq \mathcal{T}^{\leq 0}$ and $\mathcal{T}^{\leq 0} \otimes \mathcal{T}^{\leq 0} \not\subseteq \mathcal{T}^{\leq -1}$.

(b) Both $\mathcal{T}^{\leq 0}$ and $\mathcal{T}^{\geq 0}$ are closed under taking direct summands.

(c) There is an integer $D \geq 0$ such that $\mathcal{T}^{\geq D} \otimes \mathcal{T}^{\geq D} \subseteq \mathcal{T}^{\geq D}$.

(2) The minimal integer D in condition (c) is called the *deviation* of the *mtt*-structure of $\mathcal{T}$.

(3) The *deviation* of $(\mathcal{T}, \mathbf{1}, \otimes)$ is defined to be

$$D_{\otimes}(\mathcal{T}) = \inf\{\text{ deviations of all possible } mtt\text{-structures of } (\mathcal{T}, \mathbf{1}, \otimes)\}.$$

(4) Suppose $\mathcal{T}$ is a triangulated category. The *upper deviation* of $\mathcal{T}$ is defined to be

$$UD(\mathcal{T}) = \sup\{D_{\otimes}(\mathcal{T}) \mid \text{all possible monoidal triangulated structures on} \mathcal{T}\}.$$

The *lower deviation* of $\mathcal{T}$ is defined to be

$$LD(\mathcal{T}) = \inf\{D_{\otimes}(\mathcal{T}) \mid \text{all possible monoidal triangulated structures on} \mathcal{T}\}.$$

Example 5.3 We give two classical examples.

(1) If A is a finite dimensional weak Hopf algebra (or a weak bialgebra), then $A - \text{mod}$ has a natural monoidal abelian structure, and consequently, $\mathcal{T} := D^b(A - \text{mod})$ has an induced monoidal triangulated structure. It is clear that $\mathcal{T}$ has a canonical mtt-structure by setting $\mathcal{T}^{\leq 0}$ (respectively, $\mathcal{T}^{\geq 0}$) to be the complexes concentrated in degrees less than or equal to 0 (respectively, greater than or equal to 0). In this case the deviation of the mtt-structure is 0. If A is hereditary as an algebra, then the above t-structure is hereditary.
By definition, $D_{\otimes}(\mathcal{T}) = 0$ when we consider the monoidal triangulated structure given above. As a consequence, $LD(\mathcal{T}) = 0$ when $\mathcal{T}$ is considered as a triangulated category. A special case is $LD(D^b(\text{rep}(Q))) = 0$ for all finite acyclic quivers Q.
(2) If $\mathbb{X}$ is a smooth projective scheme of dimension d, then $\mathcal{T} := D^b(coh(\mathbb{X}))$ has a canonical mtt-structure by setting $\mathcal{T}^{\leq 0}$ (respectively, $\mathcal{T}^{\geq 0}$) to be the complexes concentrated in degrees less than or equal to 0 (respectively, greater than or equal to 0). If $\mathbb{X}$ is of dimension 1, then the above t-structure is hereditary.
Note that the deviation of the canonical mtt-structure of $\mathcal{T}$ is at most d. By definition, $D_{\otimes}(\mathcal{T}) \leq d$ with the natural monoidal triangulated structure. As a consequence, $LD(\mathcal{T}) \leq d$ when $\mathcal{T}$ is considered as a triangulated category.

Lemma 5.4 *Let $\mathcal{T}$ be a monoidal triangulated category with an mtt-structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ of deviation zero. Suppose that $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ is a hereditary t-structure of $\mathcal{T}$. Then the heart of the mtt-structure is a monoidal abelian category.*

Proof By [7, Theorem 1.3.6], the heart $\mathcal{H}$ is an abelian category.

Since $\mathcal{T}$ is a monoidal triangulated category, there is a unit object $\mathbf{1} \in \mathcal{T}$. First we claim that $\mathbf{1} \in \mathcal{H}$. By definition, there is a distinguished triangle

$$M \to \mathbf{1} \to N \to M[1] \tag{E5.4.1}$$

where $M \in \mathcal{T}^{\leq 0}$ and $N \in \mathcal{T}^{\geq 1}$. For any object $X \in \mathcal{H}$, since $X \otimes -$ is an exact functor,

$$X \otimes M \to X \to X \otimes N \to X \otimes M[1]$$

is a distinguished triangle. However, $X \otimes N \in \mathcal{T}^{\geq 1}$ as the deviation is zero. Then $\text{Hom}(X, X \otimes N) = 0$ by the definition of t-structure, and

$$X \otimes M[1] \cong (X \otimes N) \oplus X[1] = X \otimes (N \oplus \mathbf{1}[1]). \tag{E5.4.2}$$

By hypothesis the mtt-structure is hereditary. By [16, Lemma 2.1], (E5.4.2) holds for all $X \in \mathcal{T}$. Take $X = \mathbf{1}$, then $M[1] \cong N \oplus \mathbf{1}[1]$ and in (E5.4.1), the morphism from $\mathbf{1}$ to N is zero. Hence $\mathbf{1}$ is isomorphic to a direct summand of M, which is in $\mathcal{T}^{\leq 0}$.

Similarly, for $Y \in \mathcal{T}^{\leq 0}$ and $f : Y \to \mathbf{1}[-1]$, there is a distinguished triangle:

$$Y \xrightarrow{f} \mathbf{1}[-1] \to Z \to Y[1].$$

Apply the exact functor $X \otimes -$ on the above triangle for all $X \in \mathcal{H}$, and then we obtain $f = 0$, i.e. $\text{Hom}(Y, \mathbf{1}[-1]) = 0$ for all $Y \in \mathcal{T}^{\leq 0}$. Therefore, $\mathbf{1} \in \mathcal{T}^{\geq 0}$. Finally, $\mathbf{1} \in \mathcal{T}^{\geq 0} \cap \mathcal{T}^{\leq 0} = \mathcal{H}$. Thus we proved the claim.

As for the tensor product bifunctor $\otimes$, since the deviation is zero, $\mathcal{H}$ is closed under $\otimes$. Hence $\mathcal{H}$ is a monoidal category with the induced tensor product $\otimes$. The exactness of $\otimes$ in $\mathcal{H}$ follows from the exactness of $\otimes$ in $\mathcal{T}$, see [16, p. 1426]. □

Lemma 5.5 *Let $\mathbb{X}$ be a smooth projective curve and let $\mathcal{T}$ be the monoidal triangulated category $D^b(coh(\mathbb{X}))$.*

(1) *The deviation of every hereditary mtt-structure on $\mathcal{T}$ is positive.*
(2) *For any finite dimensional weak bialgebra A, $D^b(A-\text{mod})$ with canonical monoidal structure is not isomorphic to $\mathcal{T}$ as monoidal triangulated categories.*

Proof (1) Suppose on the contrary that there is a hereditary mtt-structure on $\mathcal{T}$ with deviation zero.

Let $\mathcal{H}$ be its heart. By Lemma 5.4, $\mathcal{H}$ is a monoidal abelian category. Let $\mathcal{O}_x$ be the skyscraper sheaf at a point $x \in \mathbb{X}$. There is an integer n such that $M := \mathcal{O}_x[n]$ is in $\mathcal{H}$. Then $M \otimes M$ is in $\mathcal{H}$. By an easy computation,

$$M \otimes M \cong \mathcal{O}_x[2n] \oplus \mathcal{O}_x[2n-1] \cong M[n] \oplus M[n-1]$$

which cannot be in $\mathcal{H}$ for any n. This yields a contradiction. Therefore the assertion follows.

(2) It is clear that the deviation of the canonical mtt-structure of $D_{\otimes}(D^b(A-\text{mod}))$ is zero [Example 5.3(1)]. This mtt-structure is also hereditary. Now the assertion follows from part (1). □

For the rest of this section, we will use Frobenius–Perron curvature, see Definition 1.3(5), to study the uniqueness of mtt-structures with deviation zero, and then prove Theorems 0.5 and 0.7 .

Definition 5.6 Let $\mathcal{C}$ be a monoidal abelian category and $M \in \mathcal{C}$. The *curvature* of M is defined to be

$$v(M) = \overline{\lim_{n\to\infty}} (\ell(M^{\otimes n}))^{\frac{1}{n}}$$

where $\ell(-)$ denotes the length of an object.

Lemma 5.7 *Let $\mathcal{C}$ be a monoidal abelian category satisfying Definition 3.1(a,b,c). Let A be a finite dimensional weak bialgebra and $\mathcal{A}$ be $A-\text{mod}$. Let M be an object in $\mathcal{C}$ or $\mathcal{A}$.*

(1) *If M is in $\mathcal{C}$, then*

$$\operatorname{fpv}(M) \le v(M) < \infty.$$

(2) *If M is in $\mathcal{A}$, then*

$$\operatorname{fpv}(M) \le v(M) \le \dim M. \tag{E5.7.1}$$

(3) *If $A = \Bbbk Q$ for some finite acyclic quiver Q with the tensor defined as in (E2.1.1), then*

$$\operatorname{fpv}(M) = v(M) = \max_{i\in Q_0}\{\dim(M)_i\}. \tag{E5.7.2}$$

(4) *If $\mathcal{C}$ is discrete, then, for every nonzero object $M \in \mathcal{C}$, $\operatorname{fpv}(M)$ is positive.*
(5) *Suppose that $\mathcal{C}$ acts on a general abelian category $\mathcal{A}$ such that the action is discrete in the sense of Definition 3.6. Then, for every object M in $\mathcal{C}$,*

$$1 \le \operatorname{fpv}(M) < \infty.$$

Proof (1) Let Hom denote $\operatorname{Hom}_{\mathcal{C}}$. Let

$$\alpha := \max\{\ell(X_i \otimes X_j) \mid X_i \text{ and } X_j \text{ are simple}\},$$

and

$$\beta := \max\{\dim \operatorname{End}(X_i) \mid X_i \text{ is simple}\}.$$

Then, for any objects X and Y in $\mathcal{C}$, we have

$$\ell(X \otimes Y) \leq \alpha \ell(X)\ell(Y), \tag{E5.7.3}$$

and

$$\dim \operatorname{Hom}(X, Y) \leq \beta \ell(X)\ell(Y). \tag{E5.7.4}$$

By induction, $\ell(X^{\otimes n}) \leq \alpha^{n-1}\ell(X)^n$ which implies that $v(X) \leq \alpha\ell(X) < \infty$.

Given a brick set $\phi = \{X_1, \ldots, X_r\}$, define $\ell(\phi) := \max_{X \in \phi}\{\ell(X)\}$. By Lemma 4.8,

$$\rho(A(\phi, M^{\otimes n} \otimes_{\mathcal{C}} -)) \leq \max_{i=1,\ldots,r} \left\{ \sum_{j=1}^{r} \dim \operatorname{Hom}(X_i, M^{\otimes n} \otimes X_j) \right\}.$$

By (E5.7.3) and (E5.7.4), we have

$$\begin{aligned} \dim \operatorname{Hom}(X_i, M^{\otimes n} \otimes X_j) &\leq \alpha\beta\ell(X_i)(\ell(M^{\otimes n})\ell(X_j)) \\ &\leq \alpha\beta(\ell(\phi))^2(\ell(M^{\otimes n})) \\ &\leq \alpha\beta(\ell(\phi))^2(v(M)+\varepsilon)^n) \end{aligned}$$

for arbitrary small $\varepsilon > 0$ and for $n \gg 0$. Therefore,

$$\rho(A(\phi, M^{\otimes n} \otimes_{\mathcal{C}} -)) \leq \alpha\beta r(\ell(\phi))^2(v(M)+\varepsilon)^n),$$

which implies that

$$\rho(A(\phi, M^{\otimes n} \otimes_{\mathcal{C}} -))^{\frac{1}{n}} \leq (\alpha\beta r(\dim \phi)^2)^{\frac{1}{n}}(v(M)+\varepsilon), \tag{E5.7.5}$$

for $n \gg 0$. When $n \to \infty$, the limit of right side of inequality (E5.7.5) is $v(M)+\varepsilon$, so $\operatorname{fpv}(M) \leq v(M)+\varepsilon$ for every small ε. The assertion follows.

(2) It follows from Definition 1.8 that

$$\dim M \otimes N \leq (\dim M)(\dim N) \tag{E5.7.6}$$

for all $M, N \in \mathcal{A}$. It is also clear that

$$\dim \operatorname{Hom}_{\mathcal{A}}(M, N) \leq \dim \operatorname{Hom}_{\Bbbk}(M, N) = (\dim M)(\dim N). \tag{E5.7.7}$$

By (E5.7.6), $\dim M^{\otimes n} \leq (\dim M)^n$, which implies that $v(M) \leq \dim M$. Now the assertion follows from part (1).

(3) Let $\phi = \{S(i)\}$ where i is a vertex of Q. Write $\dim(M)_i = d_i$. Then $\rho(A(\phi, M^{\otimes n}))$ is the integer d_i^n and $\lim_{n\to\infty} \rho(A(\phi, M^{\otimes n}))^{\frac{1}{n}} = d_i$. Hence $\operatorname{fpv}(M) \geq d_i$ for all i. It is clear that $v(M) = \max\{d_i \mid i \in Q_0\}$. Therefore part (1) implies that $\operatorname{fpv}(M) = v(M)$.

(4) Suppose $\mathcal{A}$ is discrete. Then there is a simple object S such that $M \otimes S \neq 0$ and $\operatorname{Hom}_{\mathcal{A}}(S, M \otimes S) \neq 0$. By induction, one can show that $\operatorname{Hom}_{\mathcal{A}}(S, M^{\otimes n} \otimes S) \neq 0$ for all n. Therefore $\operatorname{fpv}(M) \geq 1$.

(5) Using a similar proof of part (1), one sees that $\operatorname{fpv}(M) < \infty$. Using the proof of part (4), one can show that $\operatorname{fpv}(M) \geq 1$. Details are omitted. □

Remark 5.8 (1) Let $\mathcal{C}$ be a monoidal abelian category acting on an abelian category $\mathcal{A}$. Assume that $\mathcal{C}$ satisfies Definition 3.1(a,b,c). The action of $\mathcal{C}$ on $\mathcal{A}$ is called fpv-*positive* if

(e) $\operatorname{fpv}(M) > 0$ for every nonzero object M in $\mathcal{C}$. We say $\mathcal{C}$ is fpv-positive if the natural action of $\mathcal{C}$ on itself is fpv-positive.

(2) By Lemma 5.7(5) if an action of $\mathcal{C}$ on $\mathcal{A}$ is discrete, then it is fpv-*positive*.

(3) Suppose an action of $\mathcal{C}$ on $\mathcal{A}$ is fpv-positive. Let $\mathcal{C}'$ be a monoidal abelian subcategory $\mathcal{C}$. Then the induced action of $\mathcal{C}'$ on $\mathcal{A}$ is fpv-positive. In general, such an action is not discrete.

(4) There are other natural examples that the action of $\mathcal{C}$ on $\mathcal{A}$ is not discrete, but fpv-positive, see below.
If A is a finite-dimensional bialgebra and let $\mathcal{C} = A - \text{mod}$, then $\mathcal{C}$ is fpv-positive. Let $0 \neq M \in \mathcal{C}$ and let S_0 be a simple submodule of M. Then $M \otimes S_0 \neq 0$ as $\dim M \otimes S_0 = \dim M \dim S_0$ (when A is a bialgebra). For each $i \geq 1$, we define S_i inductively to be a simple module of $M \otimes S_{i-1}$. So $S_i \subseteq M \otimes S_{i-1}$ for all $i \geq 1$. Continuing this process, we will obtain a set of simple object $\Gamma = \{S_0, S_1, S_2, \ldots\}$. Since A has finite many simples, $|\Gamma| < \infty$. Hence, there exists $m < n \in \mathbb{Z}^+$, such that $S_n \cong S_m$. For all $i \geq n$, we redefine S_i to be $S_{i-k(n-m)}$ where k is an integer such that $m \leq i - k(m-n) < n$.
By the construction, we have, for all $i \geq m$ and all $s \geq 0$,

$$\dim \operatorname{Hom}(S_{i+1}, M \otimes S_i) \geq 1 \text{ and } \dim \operatorname{Hom}(S_{i+s}, M^{\otimes s} \otimes S_i) \geq 1.$$

Therefore, by taking the brick set $\phi = \{S_m, S_{m+1}, \ldots, S_n\}$, one sees that, for each s, $A(\phi, M^{\otimes s} \otimes -)$ is a non-negative matrix that contains a permutation matrix. As a consequence, $\rho(A(\phi, M^{\otimes s} \otimes -)) \geq 1$ for all $s \geq 1$, which implies that $\operatorname{fpv}(M) \geq 1$.

(5) In $\operatorname{rep}(Q)$ where the tensor defined as in (E2.1.1), $\operatorname{fpv}(M)$, unlike $\operatorname{fpd}(M)$, is an invariant only dependent on the the dimension vector of M (which is independent of the orientations of arrows in the quiver).

Next we investigate mtt-structures on $D^b(\mathcal{C})$.

Lemma 5.9 *Suppose that a monoidal abelian category $\mathcal{C}$ acts on an arbitrary abelian category $\mathcal{A}$. Let $\mathcal{T} = D^b(\mathcal{C})$. Assume that*

(a) *the above action is either discrete or* fpv-*positive,*
(b) *$\mathcal{C}$ is hereditary, and*
(c) *$(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ is any hereditary mtt-structure of deviation zero on $\mathcal{T}$.*

Let $\mathcal{H}$ be the heart of the above mtt-structure. Then the following hold.

(1) [16, Lemma 2.1] *If M is an indecomposable object in $\mathcal{T}$, then M is in $\mathcal{T}^{\leq b} \cap \mathcal{T}^{\geq b}$ for some integer b.*
(2) *If $M \in \mathcal{C}$, then M is in the heart $\mathcal{H}$.*
(3) *The mtt-structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ given in (c) is the canonical mtt-structure of $\mathcal{T}$.*

Proof (2,3) We only prove this when the action is discrete. First we claim that $\operatorname{fpv}(M \odot_{\mathcal{T}} -) > 0$ if $0 \neq M \in \mathcal{C}$. It is clear that

$$\operatorname{fpv}(M \odot_{\mathcal{T}} -) \geq \operatorname{fpv}(M \odot_{\mathcal{C}} -).$$

Now the claim follows from Lemma 5.7(5).

Let M be an indecomposable object in $\mathcal{C}$. Then $M \in \mathcal{T}^{\le b} \cap \mathcal{T}^{\ge b}$ for some b. If $b \neq 0$, then $M^{\otimes n} \in \mathcal{T}^{\le nb} \cap \mathcal{T}^{\ge nb}$ by Definition 5.2(a,c). For any fixed brick set ϕ, $A(\phi, M^{\otimes n} \odot_{\mathcal{T}} -)$ is zero for $n \gg 0$ by the hereditary property of Definition 5.1(4). Therefore $\operatorname{fpv}(M \odot_{\mathcal{T}} -) = 0$. By the first paragraph, $\operatorname{fpv}(M \odot_{\mathcal{T}} -) > 0$, yielding a contradiction. Therefore $b = 0$, or equivalently, $M \in \mathcal{T}^{\le 0} \cap \mathcal{T}^{\ge 0} =: \mathcal{H}$. This implies that $\mathcal{C} \subseteq \mathcal{H}$. By [16, Lemma 2.1] and [59, Lemma 3.6], $\mathcal{C} = \mathcal{H}$, and consequently, the mtt-structure $(\mathcal{T}^{\le 0}, \mathcal{T}^{\ge 0})$ in hypothesis (c) must be the canonical mtt-structure of $\mathcal{T}$. □

The following is basically Theorem 0.7.

Theorem 5.10 *Let A be a finite dimensional hereditary weak bialgebra. Suppose that the monoidal abelian category $\mathcal{A}$ is either discrete or* fpv*-positive.*

(1) *There is a unique hereditary mtt-structure with deviation zero on $D^b(\mathcal{A})$.*
(2) *The $\mathcal{A}$ is the heart of any hereditary mtt-structure with deviation zero on $D^b(\mathcal{A})$.*
(3) *The $\mathcal{A}$ is uniquely determined by the monoidal triangulated structure on $D^b(\mathcal{A})$.*

Proof Let $\mathcal{C} = \mathcal{A}$. Then we can easily check all hypotheses in Lemma 5.9. Then part (1) follows from Lemma 5.9(3).

(2, 3) Follow directly from part (1). □

Proof of Theorem 0.5 If A is a bialgebra, by Remark 5.8(4), $\mathcal{A}$ is fpv-positive. Therefore the hypothesis of Theorem 5.10 is satisfied. Now the assertion follows from the uniqueness of hereditary mtt-structure with deviation zero in Theorem 5.10. □

Proof of Corollary 0.6 Let Q and Q' be two quivers such that $D^b(\operatorname{rep}(Q))$ and $D^b(\operatorname{rep}(Q'))$ are equivalent as monoidal triangulated categories. By Theorem 0.5, this equivalent induces an equivalence between $\operatorname{rep}(Q)$ and $\operatorname{rep}(Q')$.

Recall that Q and Q' are acyclic. For each acyclic quiver there are finitely many simple representations, say $\{S_i\}_{i=1}^n$, that are associated to vertices $\{1, \ldots, n\}$ of the quiver. The correspondence between those simple representations gives rise to a bijective map $f : Q_0 \to Q'_0$. By [4, Lemma 2.12, p. 84], $\dim \operatorname{Ext}^1(S_i, S_j)$ is the number of arrows from vertex i to vertex j. Therefore the number of arrows from $f(i)$ to $f(j)$ is the same as that from i to j. Thus $Q \cong Q'$. □

6 Proof of Theorem 0.3

The proof of Theorem 0.3 uses several results about weighted projective lines and takes several pages in total. The final step of the proof is given at the end of this section. First we recall some basic definitions concerning weighted projective lines. Details can be found in [26, Section 1].

For $t \ge 1$, let $\mathbf{p} := (p_0, p_1, \ldots, p_t)$ be a $(t+1)$-tuple of positive integers, called the *weight* or *weight sequence*. Let $\mathbf{D} := (\lambda_0, \lambda_1, \ldots, \lambda_t)$ be a sequence of distinct points of the projective line $\mathbb{P}^1$ over $\Bbbk$. We normalize $\mathbf{D}$ so that $\lambda_0 = \infty$, $\lambda_1 = 0$ and $\lambda_2 = 1$ (if $t \ge 2$). Let R denote the commutative algebra

$$\Bbbk[X_0, X_1, \ldots, X_t]/(X_i^{p_i} - X_1^{p_1} + \lambda_i X_0^{p_0}, i = 2, \ldots, t). \tag{E6.0.1}$$

The image of X_i in R is denoted by x_i for all i. Let $\mathbb{L}$ be the abelian group of rank 1 generated by $\overrightarrow{x_i}$ for $i = 0, 1, \ldots, t$ and subject to the relations

$$p_0 \overrightarrow{x_0} = \cdots = p_i \overrightarrow{x_i} = \cdots = p_t \overrightarrow{x_t} =: \overrightarrow{c}.$$

The algebra R is $\mathbb{L}$-graded by setting $\deg x_i = \overrightarrow{x_i}$. The corresponding *weighted projective line*, denoted by $\mathbb{X}(\mathbf{p}, \mathbf{D})$ or simply $\mathbb{X}$, is a noncommutative space whose category of coherent sheaves is given by the quotient category

$$coh(\mathbb{X}) := \frac{\mathrm{gr}^{\mathbb{L}} - R}{\mathrm{gr}^{\mathbb{L}}_{f.d.} - R},$$

see [45, p. 155].

The weighted projective lines are classified into the following three classes:

$$\mathbb{X} \text{ is } \begin{cases} domestic & \text{if } \mathbf{p} \text{ is } (p, q), (2, 2, n), (2, 3, 3), (2, 3, 4), (2, 3, 5); \\ tubular & \text{if } \mathbf{p} \text{ is } (2, 3, 6), (3, 3, 3), (2, 4, 4), (2, 2, 2, 2); \\ wild & \text{otherwise.} \end{cases} \tag{E6.0.2}$$

In [58, Section 4.4], domestic (respectively, tubular, wild) weighted projective lines are called *parabolic* (respectively, *elliptic, hyperbolic*). Let $\mathbb{X}$ be a weighted projective line. A sheaf $F \in coh(\mathbb{X})$ is called *torsion* if it is of finite length in $coh(\mathbb{X})$. Let $Tor(\mathbb{X})$ denote the full subcategory of $coh(\mathbb{X})$ consisting of all torsion objects. By [58, Lemma 4.16], the category $Tor(\mathbb{X})$ decomposes as a direct product of orthogonal blocks

$$Tor(\mathbb{X}) = \prod_{x \in \mathbb{P}^1 \setminus \{\lambda_0, \lambda_1, \ldots, \lambda_t\}} Tor_x \times \prod_{i=0}^{t} Tor_{\lambda_i} \tag{E6.0.3}$$

where Tor_x is equivalent to the category of nilpotent representations of the Jordan quiver (with one vertex and one arrow) over the residue field $\mathbb{k}_x$ and where Tor_{λ_i} is equivalent to the category of nilpotent representations over $\mathbb{k}$ of the cyclic quiver of length p_i. A simple object in $coh(\mathbb{X})$ is called *ordinary simple* (see [26]) if it is the skyscraper sheaf $\mathcal{O}_x$ of a closed point $x \in \mathbb{P}^1 \setminus \{\lambda_0, \lambda_1, \ldots, \lambda_t\}$.

Let $Vect(\mathbb{X})$ be the full subcategory of $coh(\mathbb{X})$ consisting of all vector bundles. Similar to the elliptic curve case [12, Section 4], one can define the concepts of *degree*, *rank* and *slope* of a vector bundle on a weighted projective line $\mathbb{X}$; details are given in [58, Section 4.7] and [46, Section 2]. For each $\mu \in \mathbb{Q}$, let $Vect_\mu(\mathbb{X})$ be the full subcategory of $Vect(\mathbb{X})$ consisting of all semistable vector bundles of slope μ. By convention, $Vect_\infty(\mathbb{X})$ denotes $Tor(\mathbb{X})$. By [58, Comments after Corollary 4.34], every indecomposable object in $coh(\mathbb{X})$ is in

$$\bigcup_{\mu \in \mathbb{Q} \cup \{\infty\}} Vect_\mu(\mathbb{X}).$$

The *dualizing element* of $\mathbb{X}$ is denoted by

$$\omega_0 := (t-2)\overrightarrow{c} - \sum_{i=1}^{n} \overrightarrow{x}_i \in \mathbb{L}. \tag{E6.0.4}$$

Below we collect some nice properties of weighted projective lines. The definition of a stable tube (or simply tube) was introduced in [56].

Lemma 6.1 [14, Lemma 7.9] *Let $\mathbb{X} = \mathbb{X}(\mathbf{p}, \mathbf{D})$ be a weighted projective line.*

(1) *$coh(\mathbb{X})$ is noetherian and hereditary.*

(2)

$$D^b(coh(\mathbb{X})) \cong \begin{cases} D^b(\text{rep}(\widetilde{\mathbb{A}}_{p,q})) & \text{if } \mathbf{p} = (p, q), \\ D^b(\text{rep}(\widetilde{\mathbb{D}}_n)) & \text{if } \mathbf{p} = (2, 2, n), \\ D^b(\text{rep}(\widetilde{\mathbb{E}}_6)) & \text{if } \mathbf{p} = (2, 3, 3), \\ D^b(\text{rep}(\widetilde{\mathbb{E}}_7)) & \text{if } \mathbf{p} = (2, 3, 4), \\ D^b(\text{rep}(\widetilde{\mathbb{E}}_8)) & \text{if } \mathbf{p} = (2, 3, 5). \end{cases}$$

(3) *Let $\mathcal{S}$ be an ordinary simple object in $coh(\mathbb{X})$. Then* $\text{Ext}^1_{\mathbb{X}}(\mathcal{S}, \mathcal{S}) = \Bbbk$.
(4) *If $\mathbb{X}$ is tubular or domestic, then* $\text{Ext}^1_{\mathbb{X}}(X, Y) = 0$ *for all $X \in Vect_{\mu'}(\mathbb{X})$ and $Y \in Vect_{\mu}(\mathbb{X})$ with $\mu' < \mu$.*
(5) *If $\mathbb{X}$ is domestic, then* $\text{Ext}^1_{\mathbb{X}}(X, Y) = 0$ *for all $X \in Vect_{\mu'}(\mathbb{X})$ and $Y \in Vect_{\mu}(\mathbb{X})$ with $\mu' \leq \mu < \infty$.*
(6) *Suppose $\mathbb{X}$ is tubular or domestic. Then every indecomposable vector bundle $\mathbb{X}$ is semistable.*
(7) *Suppose $\mathbb{X}$ is tubular and let $\mu \in \mathbb{Q}$. Then each $Vect_{\mu}(\mathbb{X})$ is a uniserial category. Accordingly indecomposables in $Vect_{\mu}(\mathbb{X})$ lies in Auslander–Reiten components, which all are stable tubes of finite rank. In fact, for every $\mu \in \mathbb{Q}$,*

$$Vect_{\mu}(\mathbb{X}) \cong Vect_{\infty}(\mathbb{X}) = Tor(\mathbb{X}).$$

Lemma 6.2 *Let $\mathbb{X} = \mathbb{X}(\mathbf{p}, \mathbf{D})$ be a weighted projective line.*

(1) [45, Theorem 2.2(ii)] *Let $\mathcal{T}$ be $D^b(coh(\mathbb{X}))$. Then $\mathcal{T}$ has Serre duality in the form of*

$$\text{Hom}_{\mathcal{T}}(X, Y)^* \cong \text{Hom}_{\mathcal{T}}(Y, S(X)),$$

where the Serre functor S is $-(\omega_0)[1]$ and where the dualizing element ω_0 is in (E6.0.4).
(2) [47, Proposition 1.10] *Each indecomposable vector bundle has a nonzero morphism to Tor_x for every point x in $\mathbb{P}^1$.*

The following linear algebra lemma is needed to estimate the spectral radius of some matrices.

Lemma 6.3 *Let Γ be the $n \times n$-matrix $(a_{ij})_{n\times n}$ where*

$$a_{ij} = \begin{cases} 1 & \textit{if } i = 1, \textit{ or } j = 1, \\ 0 & \textit{otherwise}. \end{cases} \tag{E6.3.1}$$

Then the spectral radius $\rho(\Gamma) \geq \sqrt{n}$.

Proof It is not hard to check that the characteristic polynomial of Γ is

$$f(x) = x^n - x^{n-1} - (n-1)x^{n-2} = x^{n-2}(x^2 - x - (n-1)).$$

Then

$$\rho(\Gamma) = \frac{1 + \sqrt{4n-3}}{2} \geq \sqrt{n}.$$

□

Lemma 6.4 *Suppose $\mathcal{T}$ be a triangulated category satisfying*

(a) *there is an infinite brick set ϕ,*

(b) *there is a brick object* B *in* $\mathcal{T}$ *such that* $\mathrm{Hom}_{\mathcal{T}}(B, X) \neq 0$ *for all* $X \in \phi$,
(c) *there is an integer* m *such that* $\mathrm{Hom}_{\mathcal{T}}(B[s], X) = \mathrm{Hom}_{\mathcal{T}}(X, B[s]) = 0$ *for all* $X \in \Phi$ *and for all* $|s| \geq m$,
(d) $\mathcal{T}$ *has a Serre functor* S, *and*
(e) *there is an integer* m_0 *such that* $\mathrm{Hom}_{\mathcal{T}}(B[m_0], S(X)) \neq 0$ *for all* $X \in \phi$.

Let $\mathcal{C}$ *be a monoidal triangulated category acting on* $\mathcal{T}$. *Then there is an object* $M \in \mathcal{C}$ *such that* $\mathrm{fpd}(M) = \infty$.

Proof In the following proof let $\odot$ denote the action of $\mathcal{C}$ on $\mathcal{T}$ and Hom denote $\mathrm{Hom}_{\mathcal{T}}$.

By condition (d), $\mathcal{T}$ has a Serre functor $S : \mathcal{T} \to \mathcal{T}$ such that

$$\mathrm{Hom}(X, Y)^* \cong \mathrm{Hom}(Y, S(X)) \tag{E6.4.1}$$

for all X, Y in $\mathcal{T}$.

Let $\mathbf{1} \in \mathcal{C}$ be the unit object with respect to the monoidal tensor of $\mathcal{C}$. Let m and m_0 be the integers given in conditions (c) and (e), and let M be the object $\mathbf{1}[m] \oplus \mathbf{1} \oplus \mathbf{1}[m_0 - m]$ in $\mathcal{C}$. It is enough to show that $\mathrm{fpd}(M) = \infty$. Let ϕ_n be a brick set consisting of $(n-1)$ objects in ϕ and one extra special object, namely $B[m]$, where m is in condition (c). Write

$$\phi_n = \{X_1 := B[m], X_2, X_3, \ldots, X_n\}$$

where $X_i \in \phi$ for all $i = 2, 3, \ldots, n$. Let $A := (a_{ij})$ denote the adjacency matrix $A(\phi_n, M \odot -)$. We claim that $a_{1i} \neq 0$ and $a_{j1} \neq 0$ for all i, j.

Case 1:

$$\begin{aligned} a_{11} &= \dim \mathrm{Hom}(B[m], M \odot B[m]) \\ &\geq \dim \mathrm{Hom}(B[m], \mathbf{1} \odot B[m]) \\ &= \dim \mathrm{Hom}(B, B) \\ &= \dim \Bbbk = 1 \qquad \text{by condition (b).} \end{aligned}$$

Case 2: for every $i \geq 2$,

$$\begin{aligned} a_{1i} &= \dim \mathrm{Hom}(B[m], M \odot X_i) \\ &\geq \dim \mathrm{Hom}(B[m], \mathbf{1}[m] \odot X_i) \\ &= \dim \mathrm{Hom}(B[m], X_i[m]) \\ &\geq \dim \Bbbk = 1 \qquad \text{by condition (c).} \end{aligned}$$

Case 3: for every $j \geq 2$,

$$\begin{aligned} a_{j1} &= \dim \mathrm{Hom}(X_j, M \odot B[m]) \\ &\geq \dim \mathrm{Hom}(X_j, \mathbf{1}[m_0 - m] \odot B[m]) \\ &= \dim \mathrm{Hom}(X_j, B[m_0]) \\ &= \dim \mathrm{Hom}(B[m_0], S(X_j)) \qquad \text{by(E6.4.1)} \\ &\geq \dim \Bbbk = 1 \qquad \text{by condition (e).} \end{aligned}$$

Therefore we proved the claim. This means that every entry in A is larger than or equal to the corresponding entry in Γ as given in Lemma 6.3. By linear algebra,

$$\rho(A) \geq \rho(\Gamma) \geq \sqrt{n}$$

where the last inequality is Lemma 6.3. Then, by definition, $\mathrm{fpd}(M) \geq \sqrt{n}$ for all n. Thus $\mathrm{fpd}(M) = \infty$ as desired. □

Now we are ready to show that every monoidal structure on weighted projective line is fpd-infinite.

Proposition 6.5 *Let $\mathbb{X}$ be a weighted projective line and let $\mathcal{T}$ be $D^b(coh(\mathbb{X}))$.*

(1) *Let $\mathcal{C}$ be a monoidal triangulated category acting on $\mathcal{T}$. Then there is an object $M \in \mathcal{C}$ such that* $\operatorname{fpd}(M) = \infty$.
(2) *Every monoidal structure on $\mathcal{T}$ is* fpd-*infinite.*

Proof Since part (2) is a special case of part (1), it suffices to show part (1). We need to verify hypotheses (a)–(e) in Lemma 6.4.

Let ϕ be the set $\{\mathcal{O}_x \mid x \in \mathbb{P}^1 \backslash \{\lambda_0, \ldots, \lambda_t\}\}$ and let B be the trivial bundle $\mathcal{O}_{\mathbb{X}}$. It is clear that ϕ is infinite, so (a) holds. By Lemma 6.2(2), (b) holds. Since $coh(\mathbb{X})$ has global dimension 1, (c) holds. By Lemma 6.2(1), $D^b(coh(\mathbb{X}))$ has a Serre functor S which is $\mathcal{O}_{\mathbb{X}}(\omega_0)[1] \otimes_{\mathbb{X}} -$. Then $S(\mathcal{O}_x) = \mathcal{O}_x[1]$ for all $x \in \mathbb{P}^1 \backslash \{\lambda_0, \ldots, \lambda_t\}$. Therefore (e) holds. Finally the assertion follows from Lemma 6.4. □

It is not hard to check that Proposition 6.5 also holds if $\mathbb{X}$ is an irreducible smooth projective scheme of dimension at least 1.

We still need quite a few lemmas before we can prove Theorem 0.3. Recall that the definition of fpd-wild is given in Definition 0.2(3).

Lemma 6.6 *Let $\mathcal{T}$ be a triangulated category. Suppose that, for each n, there is a connected brick set ϕ with $|\phi| > n$.*

(1) *Let $\mathcal{C}$ be a* Hom-*finite Krull–Schmidt monoidal triangulated category acting on $\mathcal{T}$. Then there is an indecomposable object $M \in \mathcal{C}$ such that* $\operatorname{fpd}(M) = \infty$.
(2) *Suppose further that $\mathcal{T}$ is* Hom-*finite Krull–Schmidt. Then every monoidal triangulated structure on $\mathcal{T}$ is* fpd-*wild.*

Proof Since part (2) is a special case of part (1), it suffices to show part (1).

Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a monoidal triangulated category acting on $\mathcal{T}$ where $\mathbf{1}$ is the unit object of $\mathcal{C}$. Write $\mathbf{1}$ as a direct sum of indecomposable objects

$$\mathbf{1} = \bigoplus_{i=1}^{d} M_i.$$

By hypothesis, for each n, there is a connected brick set ϕ^n with $|\phi^n| > dn$. Define

$$\phi_i^n := \{X \in \phi^n \mid M_i \odot X \neq 0\}.$$

Since $X = \mathbf{1} \odot X = \bigoplus_{i=1}^{d}(M_i \odot X)$ and X is indecomposable, there is exactly one i such that $M_i \odot X \neq 0$, and for that i, we have $M_i \odot X = X$. Hence, for each n, ϕ^n is a disjoint union of ϕ_i^n for $i = 1, \ldots, d$. By the pigeonhole principle, there is at least i such that $|\phi_i^n| > n$. This implies that there is at least one j such that, with this fixed j, there is an infinite sequence n_j such that $|\phi_j^{n_j}| > n_j$. Using this sequence of brick sets, one sees that

$$\operatorname{Hom}_{\mathcal{T}}(M_j[-1] \odot X, Y) = \operatorname{Hom}_{\mathcal{T}}(X, Y[1]) \neq 0$$

for all $X, Y \in \phi_j^{n_j}$. By definition, $\operatorname{fpd}(M_j[-1]) \geq n_j$ as $|\phi_j^{n_j}| \geq n_j$. Since n_j goes to infinity, $\operatorname{fpd}(M_j[-1]) = \infty$ as desired. □

Next we recall more detailed structures concerning weighted projective lines. Let $\mathbf{p}$ be the weight of $\mathbb{X}$ and $B_0 = \gcd(p_i \in \mathbf{p})$. Define ν to be the group homomorphism from $\mathbb{L}$ to $\mathbb{Z}$ such that $\nu(\vec{x}_i) = \prod_{s \neq i} p_s$. It is easy to see that the image of ν is $B_0\mathbb{Z}$. In fact, we can assume that $B_0 = 1$, so $\nu : \mathbb{L} \to \mathbb{Z}$ is a surjective morphism. Since $\operatorname{rank}(\ker(\nu)) = 0$, the kernel of ν is finite.

Lemma 6.7 *Let $\mathbb{X}$ be a weighted projective line and let $\mathcal{T}$ be $D^b(coh(\mathbb{X}))$.*

(1) *There is a positive integer B_1, only dependent on $\mathbb{X}$, such that, if ω_1, ω_2 are in $\mathbb{L}$ satisfying $\nu(\omega_2 - \omega_1) \geq B_1$, then $\operatorname{Hom}_{\mathbb{X}}(\mathcal{O}_{\mathbb{X}}(\omega_1), \mathcal{O}_{\mathbb{X}}(\omega_2)) \neq 0$.*
(2) *For every N, there is a positive integer $B_3(N)$, only dependent on $\mathbb{X}$ and N such that*

$$\dim \operatorname{Hom}_{\mathbb{X}}(\mathcal{O}(\omega_1), \mathcal{O}(\omega_2)) \leq B_3(N)$$

for all ω_1, ω_2 in $\mathbb{L}$ satisfying $0 \leq \nu(\omega_2 - \omega_1) \leq N$.

Proof (1) We may assume that $\omega_1 = 0$. Let $B_1 = (t-1)\prod_{s=0}^{t} p_i$. For $\omega_2 \in \mathbb{L}$ with $\nu(\omega_2) \geq B_1$, write $\omega_2 = \sum_{s=0}^{t-1} a_s \vec{x}_s + a_t \vec{x}_t$ where $0 \leq a_s \leq p_s$ for all $0 \leq s \leq t-1$. Since $\nu(\omega_2) \geq B_1$, $a_t \geq 0$. Then the ω_2-degree component of R (see (E6.0.1)) is not zero and hence $\operatorname{Hom}_{\mathbb{X}}(\mathcal{O}_{\mathbb{X}}, \mathcal{O}_{\mathbb{X}}(\omega_2)) = R_{\omega_2} \neq 0$.

(2) Again we can assume that $\omega_1 = 0$. Since there are only finitely many ω_2 such that $\nu(\omega_2)$ is in between 0 and N. Let $B_3(N)$ be the maximum of all possible

$$\dim \operatorname{Hom}_{\mathbb{X}}(\mathcal{O}, \mathcal{O}(\omega_2))$$

where ω_2 runs over all $\omega_2 \in \mathbb{L}$ such that $0 \leq \nu(\omega_2) \leq N$. Then the assertion follows. □

The next lemma concerns domestic weighted projective lines. Some un-defined terms can be found in [44]. Let ω_0 be the dualizing element defined in (E6.0.4).

Lemma 6.8 *Let $\mathbb{X}$ be a weighted projective line.*

(1) *[44, Proposition 5.1(ii)] Suppose that the weight $\mathbf{p}$ is either $(2, 2, n)$, or $(2, 3, 3)$, or $(2, 3, 4)$ or $(2, 3, 5)$. Let Δ be the attached Dynkin diagram and $\widetilde{\Delta}$ its extended Dynkin diagram. The Auslander-Reiten quiver $\Gamma(Vect(\mathbb{X}))$ of $Vect(\mathbb{X})$ consists of a single standard component having the form $\mathbb{Z}\widetilde{\Delta}$. Moreover, the category of indecomposable vector bundles on $\mathbb{X}$, denoted by $ind(Vect(\mathbb{X}))$, is equivalent to the mesh category of $\Gamma(Vect(\mathbb{X}))$.*
(2) *Under the hypotheses of part (1), there is a finite set of indecomposable vector bundles $\{V_i\}_{i\in I}$ such that every indecomposable vector bundle is of the form $V_i(n\omega_0)$ for some $n \in \mathbb{Z}$ and some $i \in I$.*
(3) [44, Sect. 5.1, page 217] *If the weight $\mathbf{p}$ is of the form (p, q), then each indecomposable vector bundle is a line bundle $\mathcal{O}(\omega)$ for $\omega \in \mathbb{L}$.*
(4) *Under the hypotheses of part (3), there is a finite set of indecomposable vector bundles $\{V_i\}_{i\in I}$ such that every indecomposable vector bundle is of the form $V_i(n\omega_0)$ for some $n \in \mathbb{Z}$.*

Proof (2) There is a ($[-1]$-shifted) Serre functor $F := -(\omega_0)$ which is also a functor from $ind(Vect(\mathbb{X}))$ to itself. It is easy to check that $\nu(\omega_0) < 0$. Then F induces an automorphism of the Auslander–Reiten quiver $\Gamma(Vect(\mathbb{X}))$ by shifting forward a distance $\nu(\omega_0)$. Therefore there is a finite set of indecomposable vector bundles $\{V_i\}_{i\in I}$ such that every indecomposable vector bundle is of the form $V_i(n\omega_0)$ for some $n \in \mathbb{Z}$ and some $i \in I$.

(4) Since the map $\nu : \mathbb{L} \to \mathbb{Z}$ is a group homomorphism with finite kernel, there are only finitely many ω such that $\nu(\omega) = 0$. Similarly, there are only finitely many $\omega \in \mathbb{L}$ such that $\nu(\omega) = 0, 1, \ldots, -\nu(\omega_0) - 1$. Then the set $\{\mathcal{O}(\omega) \mid 0 \leq \nu(\omega) \leq -\nu(\omega_0) - 1\}$ has the desired property. □

We introduce some temporary notation. By Lemma 6.8(2,4), if $\mathbb{X}$ is domestic, then there is a finite set of indecomposable vector bundles, say $\mathbb{K} := \{K_1, \ldots K_{B_4}\}$, such that every indecomposable vector bundle is of the form $K_s(n\omega_0)$ for some $1 \leq s \leq B_4$ and some $n \in \mathbb{Z}$. (Here $\omega_0 \in \mathbb{L}$ is the dualizing element given in (E6.0.4).) For each K_s we fix a sequence of sub-bundles

$$0 =: V_{s,0} \subset V_{s,1} \subset V_{s,2} \subset \cdots \subset V_{s,Y_s} := K_s \tag{E6.8.1}$$

such that each subquotient $V_{s,i}/V_{s,i-1}$ is a line bundle of the form $\mathcal{O}_{\mathbb{X}}(\omega_{s,i})$ for some $\omega_{s,i} \in \mathbb{L}$. Let $\Omega(\mathbb{X})$ be the collection of all such $\omega_{s,i}$'s. Hence $\Omega(\mathbb{X})$ is finite. Let

$$\max(\Omega) = \max\{\nu(\omega) \mid \omega \in \Omega(\mathbb{X})\},$$
$$\min(\Omega) = \min\{\nu(\omega) \mid \omega \in \Omega(\mathbb{X})\}.$$

For every vector bundle V, we write $V = K_s(n\omega_0)$ for some s and n. Then we fix a sequence of sub-bundles of $V := K_s(n\omega_0)$ by applying $-(n\omega_0)$ to (E6.8.1). We have a series of subquotients

$$V_{s,i}(n\omega_0)/V_{s,i-1}(n\omega_0) \cong \mathcal{O}_{\mathbb{X}}(\omega_{s,i} + n\omega_0)$$

induced by (E6.8.1). Let $\nu(V)$ denote the positive different between the largest of all $\nu(\omega_{s,i} + n\omega_0)$ and the smallest of all $\nu(\omega_{s,i} + n\omega_0)$. Then it is clear that $\nu(V) \leq \max(\Omega) - \min(\Omega)$. So we have proved part (1) of the follows proposition.

Proposition 6.9 *Let $\mathbb{X}$ be a domestic weighted projective line.*

(1) *Let V be an indecomposable vector bundle on $\mathbb{X}$. Then the $\nu(V)$ is uniformly bounded by $B_5 := \max(\Omega) - \min(\Omega)$.*
(2) *Let V be an indecomposable vector bundle on $\mathbb{X}$. Then the rank V is uniformly bounded by an integer B_6 (only dependent on $\mathbb{X}$).*
(3) *Suppose ϕ is a brick set consisting of vector bundles on $\mathbb{X}$. Then the size of ϕ is uniformly bounded by B_7 (only dependent on $\mathbb{X}$).*
(4) *Suppose ϕ is a brick set consisting of vector bundles on $\mathbb{X}$. Then, up to a degree shift, ϕ is a subset of $\bigcup_{n=-N}^{N} \mathbb{K}(n\omega_0)$ for some integer N. As a consequence, $\sum_{V \in \phi} \nu(V)$ is uniformly bounded, say, by B_8 (only dependent on $\mathbb{X}$).*
(5) *Fix a vector bundle V on $\mathbb{X}$. For every brick set consisting of vector bundles $\{X_1, \ldots, X_n\}$, $\dim \operatorname{Hom}_{\mathbb{X}}(X_i, V \otimes_{\mathbb{X}} X_j)$ is uniformly bounded by $B_9(V)$ for all i, j (only dependent on V and $\mathbb{X}$).*
(6) *Fix a vector bundle V on $\mathbb{X}$. For every brick set consisting of vector bundles $\{X_1, \ldots, X_n\}$, $\dim \operatorname{Hom}_{\mathbb{X}}(V \otimes_{\mathbb{X}} X_i, X_j)$ is uniformly bounded by $B_{10}(V)$ for all i, j (only dependent on V and $\mathbb{X}$).*

Proof (2) This is part of [47, Theorem 6.1]. It also can be shown directly as follows.

Since every indecomposable vector bundle V is of the form $K_s(\omega)$ for $1 \leq s \leq B_4$, the rank of V is uniformly bounded, say by B_6.

(3) Since $\nu(\omega_0)$ is negative, there is an N_1 such that for all $n \geq N_1$ and for all s_1, s_2,

$$\nu(\omega_{s_2,Y_{s_2}}) - \nu(\omega_{s_1,1} - n\omega_0) \geq B_1$$

where B_1 is the constant given in Lemma 6.7(1). By Lemma 6.7(1), for such n, s_1, s_2,

$$\mathrm{Hom}_{\mathbb{X}}(\mathcal{O}_{\mathbb{X}}(\omega_{s_2,Y_{s_2}})), \mathcal{O}_{\mathbb{X}}(\omega_{s_1,1} - n\omega_0)) \neq 0.$$

By (E6.8.1),

$$\mathrm{Hom}_{\mathbb{X}}(K_{s_2}, K_{s_1}(-n\omega_0)) \neq 0 \tag{E6.9.1}$$

for all s_1, s_2 and all $n \geq N_1$.

Let ϕ be a brick set of vector bundles. We claim that $|\phi| \leq N_1|\mathbb{K}| =: B_7$. If not, by the pigeonhole principle, there is an s such that ϕ contains a subset

$$\{K_s(n_1\omega_0), \dots, K_s(n_q\omega_0)\}$$

for some $q > N_1$ where $n_1 < n_1 < \cdots < n_q$. Then, by (E6.9.1),

$$\mathrm{Hom}_{\mathbb{X}}(K_s(n_q\omega_0), K_s(n_1\omega_0)) = \mathrm{Hom}_{\mathbb{X}}(K_s, K_s((n_1 - n_q)\omega_0)) \neq 0.$$

This contradicts that ϕ is a brick set. Therefore we proved the claim.

(4) Without loss of generality, we may assume that ϕ contains K_1. Let $K_s(n\omega_0)$ be any other object in ϕ. By (E6.9.1), $|n| < N_1$ where N_1 is given in the proof of part (3). Therefore ϕ is a subset of $\bigcup_{n=-N_1}^{N_1} \mathbb{K}(n\omega_0)$. As a consequence, $\sum_{X\in\phi} \nu(X)$ is uniformly bounded, say by B_8.

(5) By part (4), up to a degree shift, we can assume that ϕ is a subset of $\bigcup_{n=-N}^{N} \mathbb{K}(n\omega_0)$ for a fixed integer N. Note that the global degree shift will not change the assertion. Then the assertion follows by the fact that $\bigcup_{n=-N}^{N} \mathbb{K}(n\omega_0)$ is a fixed set.

(6) Similar to the proof of part (5). □

Lemma 6.10 *Let $\mathbb{X}$ be a weighted projective line. Let $\mathcal{T}$ be $D^b(coh(\mathbb{X}))$.*

(1) *Let M be a brick object in $\mathcal{T}$. Then $M \cong N[n]$ where $n \in \mathbb{Z}$ and there $N \in coh(\mathbb{X})$ is either a vector bundle, or an ordinary simple $\mathcal{O}_x$, or an indecomposable object in Tor_{λ_i}.*
(2) *If a brick set ϕ consists of indecomposable objects in Tor_λ for some $\lambda \in \mathbb{P}^1$, then $|\phi|$ is uniformly bounded by B_{11} (only dependent on $\mathbb{X}$).*
(3) *If M is a brick object in $Tor(\mathbb{X})$, then* $\dim M$ *is uniformly bounded by B_{12} (only dependent on $\mathbb{X}$).*

Proof (1) It is well-known that every indecomposable object in $coh(\mathbb{X})$ is either a vector bundle or a torsion sheaf. The assertion follows by (E6.0.3) and the fact that $coh(\mathbb{X})$ is hereditary.

(2) This is trivial if $\lambda \in \mathbb{P}\backslash\{\lambda_0, \dots, \lambda_t\}$. If $\lambda = \lambda_i$ for some i, Tor_{λ_i} is a standard tube of rank p_i with p_i^2 brick objects, see [15, Section 2.2]. So the assertion follows.

(3) By (E6.0.3), $M \in Tor_\lambda$ for some $\lambda \in \mathbb{P}$. It is trivial if $\lambda \in \mathbb{P}\backslash\{\lambda_0, \dots, \lambda_t\}$. Now assume that $\lambda = \lambda_i$. All brick objects in Tor_{λ_i} are given in [15, Corollary 2.8]. As a consequence, $\dim M \leq p_i$. The assertion follows. □

Since R in (E6.0.1) is commutative, there is a natural tensor product on $coh(\mathbb{X})$, denoted by $\otimes_{\mathbb{X}}$. Note that $\otimes_{\mathbb{X}}$ is not (bi)exact. The derived category $\mathcal{T} := D^b(coh(\mathbb{X}))$ has a canonical monoidal structure where the tensor functor is defined by

$$- \otimes_{\mathcal{T}} - := - \otimes^L_{\mathbb{X}} -$$

(the derived tensor product). Note that $\otimes_{\mathcal{T}}$ is biexact so that $\mathcal{T}$ is a monoidal triangulated category. Next we show that this monoidal triangulated structure is fpd-tame when $\mathbb{X}$ is domestic.

Theorem 6.11 *Retain the notation introduced above. If $\mathbb{X}$ is domestic, then the canonical monoidal triangulated structure on $D^b(coh(\mathbb{X}))$ is* fpd*-tame.*

Proof Let $\mathcal{T}$ denote $D^b(coh(\mathbb{X}))$. By Proposition 6.5, $\mathcal{T}$ is fpd-infinite. By definition, it remains to show that $\operatorname{fpd}(M) < \infty$ for every indecomposable object M in $\mathcal{T}$.

Since M is indecomposable and $coh(\mathbb{X})$ is hereditary, by [14, Lemma 3.3], M is of the form $N[n]$ for some $N \in coh(\mathbb{X})$ and $n \in \mathbb{Z}$. By Lemma 6.10(1), N is either a vector bundle or a torsion. So we fix an N and consider the following two cases.

Case 1: N is a vector bundle. In this case $N \otimes_{\mathbb{X}} -$ is exact and $N \otimes_{\mathcal{T}} Y = N \otimes_{\mathbb{X}} Y$ for all $Y \in coh(\mathbb{X})$.

If $n \neq 0, 1$, by the proof of Lemma 4.11, $\operatorname{fpd}(N[n] \otimes_{\mathcal{T}} -) = 0$. Now we deal with the case $n = 0$ or $M = N$. Let ϕ be a brick set. By Lemma 6.10(1), we can write $\phi = \bigcup_{\delta \in \mathbb{Z}} \phi_\delta$, with δ integers ranging from small to large, where ϕ_δ is either empty or of the form

$$\{X_{\delta,1}[\delta], X_{\delta,2}[\delta], \ldots, X_{\delta,t_\delta}[\delta]\}$$

for some $X_{\delta,s} \in coh(\mathbb{X})$. Since

$$\operatorname{Hom}_{\mathcal{T}}(X_{\delta,s}[\delta], N \otimes_{\mathcal{T}} X_{\delta',s'}[\delta']) = 0$$

for all $\delta > \delta'$, the adjacency matrix $A(\phi, N \otimes_{\mathcal{T}} -)$ is a upper triangular block matrix. Now the idea of [14, Lemma 6.1] implies that we only need to consider blocks, namely, we can assume that $\phi = \phi_\delta$ for some δ. For each block associated to ϕ_δ, we can further assume that $\delta = 0$ and $\phi_0 = \{X_1, \ldots, X_t\}$ for some $X_s \in coh(\mathbb{X})$. Without loss of generality, we assume that

$$\phi = \phi_0 = \{X_1, \ldots, X_t\}$$

for some $X_1, \ldots, X_t \in coh(\mathbb{X})$. If ϕ contains an ordinary simple $\mathcal{O}_x$, then, by Lemma 6.2(2), ϕ does not contain any vector bundle. In this case, one can further decompose ϕ according to (E6.0.3) so that $A(\phi, N \otimes_{\mathcal{T}} -)$ is a block diagonal matrix. For each block, ϕ is either $\{\mathcal{O}_x\}$ or consisting of objects in Tor_{λ_i}. So we consider these two subcases. If $\phi = \{\mathcal{O}_x\}$, it is easy to see that $\operatorname{Hom}_{\mathcal{T}}(\mathcal{O}_x, N \otimes_{\mathcal{T}} \mathcal{O}_x)$ has dimension bounded by the rank of N. This is uniformly bounded. If ϕ is a subset of Tor_{λ_i}, then there are only finitely many possibilities [Lemma 6.10(2)]. Hence entries and size of the $A(\phi, N \otimes_{\mathcal{T}} -)$ is uniformly bounded. Therefore $\rho(A(\phi, N \otimes_{\mathcal{T}} -))$ is uniformly bounded. The second case is when ϕ does not contain any ordinary simple $\mathcal{O}_x$. Then the size of ϕ is uniformly bounded by Proposition 6.9(3) and Lemma 6.10(2). We claim that each entry in $A(\phi, N \otimes_{\mathcal{T}} -)$ is uniformly bounded, or $d_{ij} := \dim \operatorname{Hom}_{\mathbb{X}}(X_i, N \otimes_{\mathbb{X}} X_j)$ is uniformly bounded for all X_i, X_j in ϕ. If both X_i and X_j are vector bundles, the assertion follows from Proposition 6.9(5). If X_i is in Tor_{λ_i} and X_j is a vector bundle, then $d_{ij} = 0$. If X_i is a vector bundle and X_j is in Tor_{λ_i}, then d_{ij} is bounded by $\operatorname{rank}(X_i) \operatorname{rank}(N) \dim X_j$, which is uniformly bounded by Proposition 6.9(2) and Lemma 6.10(3). If X_i and X_j are both in Tor_{λ_i}, then d_{ij} is bounded by $(\dim X_i) \operatorname{rank}(N)(\dim X_j)$ which is uniformly bounded. Combining all these cases, one proves that $\operatorname{fpd}(N)$ is finite by Lemma 4.8 (Gershgorin Circle Theorem).

Next we deal with the case $n = 1$ (namely, $M = N[1]$) and re-cycle some notation used in the previous paragraphs. By Lemma 6.10(1), we can write $\phi = \bigcup_{\delta \in \mathbb{Z}} \phi_\delta$, with δ being integers ranging from small to large, where ϕ_δ is either empty or of the form $\{X_{\delta,1}[\delta], X_{\delta,2}[\delta], \ldots, X_{\delta,t_\delta}[\delta]\}$. Since $coh(\mathbb{X})$ is hereditary,

$$\operatorname{Hom}_{\mathcal{T}}(X_{\delta,s}[\delta], N[1] \otimes_{\mathcal{T}} X_{\delta',s'}[\delta']) = 0$$

for all s, s' and all $\delta < \delta'$. Therefore the adjacency matrix $A(\phi, N[1] \otimes_{\mathcal{T}} -)$ is a lower triangular block matrix. For each block we can assume that $\delta = 0$ and $\phi = \{X_1, \ldots, X_t\}$ as in the case $n = 0$. If ϕ contains an ordinary simple $\mathcal{O}_x$, then, by Lemma 6.2(2), ϕ does not contain any vector bundle. In this case, one can further decompose ϕ according to (E6.0.3) so that $A(\phi, N[1] \otimes_{\mathcal{T}} -)$ is a block diagonal matrix. For each block, ϕ is either $\{\mathcal{O}_x\}$ or consisting of objects in Tor_{λ_i}. So we consider these two subcases. If $\phi = \{\mathcal{O}_x\}$, then

$$\mathrm{Hom}_{\mathcal{T}}(\mathcal{O}_x, N[1] \otimes_{\mathcal{T}} \mathcal{O}_x) = \mathrm{Ext}^1_{\mathbb{X}}(\mathcal{O}_x, N \otimes_{\mathcal{T}} \mathcal{O}_x)$$

which is bounded by the rank(N). If ϕ is a subset of Tor_{λ_i}, then there are only finitely many possibilities, see the proof of Lemma 6.10(2). Hence the entries and the size of the $A(\phi, N[1] \otimes_{\mathcal{T}} -)$ are uniformly bounded. Therefore $\rho(A(\phi, N[1] \otimes_{\mathcal{T}} -))$ is uniformly bounded. The second case is when ϕ does not contain any ordinary simple $\mathcal{O}_x$. Then the size of ϕ is uniformly bounded by Proposition 6.9(3) and Lemma 6.10(2). We claim that each entry in $A(\phi, N[1] \otimes_{\mathcal{T}} -)$ is uniformly bounded, or

$$\begin{aligned} d_{ij} &:= \dim \mathrm{Hom}_{\mathbb{X}}(X_i, N[1] \otimes_{\mathbb{X}} X_j) = \dim \mathrm{Ext}^1_{\mathbb{X}}(X_i, N \otimes_{\mathbb{X}} X_j) \\ &= \dim \mathrm{Hom}_{\mathbb{X}}(N \otimes_{\mathbb{X}} X_j, X_i(\omega_0)) = \dim \mathrm{Hom}_{\mathbb{X}}(N(-\omega_0) \otimes_{\mathbb{X}} X_j, X_i) \end{aligned}$$

is uniformly bounded for all X_i, X_j in ϕ. Note that the third equality is Serre duality. If both X_i and X_j are vector bundles, the assertion follows and Proposition 6.9(6). If X_i is in Tor_{λ_i} and X_j is a vector bundle, we obtain that

$$d_{ij} \le \mathrm{rank}(X_j)\, \mathrm{rank}(N(-\omega_0)) \dim X_i,$$

which is uniformly bounded by Proposition 6.9(2) and Lemma 6.10(3). If X_i is a vector bundle and X_j is in Tor_{λ_i}, then $d_{ij} = 0$. If X_i and X_j are both in Tor_{λ_i}, then

$$d_{ij} \le \dim(X_j)\, \mathrm{rank}(N(-\omega_0)) \dim X_i,$$

which is uniformly bounded. Combining all these cases, one proves that fpd$(N[1])$ is finite by Lemma 4.8 (Gershgorin Circle Theorem).

Case 2: N is a torsion. By definition, $N \otimes_{\mathcal{T}} - = N \otimes^L_{\mathbb{X}} -$. If $n \neq -1, 0, 1$, a proof similar to Lemma 4.11(1) shows that fpd$(N[n]) = 0$. We need to analyze the cases $n = -1, 0, 1$. The following proof is independent of n.

Since N is torsion and indecomposable, by (E6.0.3), N is either in Tor_x or Tor_{λ_i}. We will use Gershgorin Circle Theorem [Lemma 4.8]. Let $\phi = \{X_1, \ldots, X_m\}$ be any brick set in $\mathcal{T}$ and let $(d_{ij})_{m \times m}$ denote the adjacency matrix $A(\phi, N[n] \otimes_{\mathcal{T}} -)$ where

$$d_{ij} = \dim \mathrm{Hom}_{\mathcal{T}}(X_i, N[n] \otimes_{\mathcal{T}} X_j).$$

By Lemma 4.8, it suffices to show

(a) each d_{ij} is uniformly bounded (only dependent on $M := N[n]$).
(b) For each j, there are only uniformly-bounded-many i such that $d_{ij} \neq 0$.

Proof of (a): For each j, write $X_j = Y_j[s_j]$ for some $Y_j \in coh(\mathbb{X})$ and $s_j \in \mathbb{Z}$. Since $N \in Tor_{\lambda}$, $H^s_N(X_j) := H^s(N[n] \otimes_{\mathcal{T}} X_j)$ is zero for $s \neq n + s_j - 1, n + s_j$ and $H^s_N(X_j)$ is in Tor_{λ} for $s = n + s_j - 1, n + s_j$. Since $coh(\mathbb{X})$ is hereditary,

$$N[n] \otimes_{\mathcal{T}} X_j = \sum_s H^s(N[n] \otimes_{\mathcal{T}} X_j)[-s],$$

see [16, Lemma 2.1]. If Y_j is a vector bundle, then

$$\dim H^s_N(X_j) \le (\dim N)(\mathrm{rank}(Y_j))$$

for all s. If X_j is torsion, then

$$\dim H_N^s(X_j) \leq (\dim N)(\dim Y_j)$$

for all s. In both cases, $\dim H^s(X_j)$ is uniformly bounded by Proposition 6.9(2) and Lemma 6.10(3). Using the Serre duality and Proposition 6.9(2) and Lemma 6.10(3) again, one sees that

$$\sum_{s,t\in\mathbb{Z}} \dim \mathrm{Hom}_{\mathcal{T}}(X_i[t], H^s(N[n] \otimes_{\mathcal{T}} X_j)[s]) = \sum_{s,t} \dim \mathrm{Hom}_{\mathcal{T}}(X_i[t], H_N^s(X_j)[s])$$

is uniformly bounded. Hence

$$d_{ij} = \mathrm{Hom}_{\mathcal{T}}(X_i, N[n] \otimes_{\mathcal{T}} X_j) = \sum_s \mathrm{Hom}_{\mathcal{T}}(X_i, H^s(N[n] \otimes_{\mathcal{T}} X_j)[s])$$

is uniformly bounded.

Proof of (b): As noted before, $\mathrm{fpd}(N[n]) = 0$ when $n \neq -1, 0, 1$. So, in this proof, we assume that n is -1 or 0 or 1. Without loss of generality, we only prove that there are only uniformly-bounded-many i such that $d_{i1} \neq 0$. By a complex shift, we can assume that $X_1 \in coh(\mathbb{X})$. Since $coh(\mathbb{X})$ is hereditary, one can check that, if $X_i \in coh(\mathbb{X})[m]$ for $|m| \geq 3$, then $d_{i1} = 0$.

For each m with $|m| \leq 2$, let ϕ_m consist of $Y_i \in coh(\mathbb{X})$ such that $X_i = Y_i[m] \in \phi$ and $d_{i1} \neq 0$. If ϕ_m does not contain any ordinary simple $\mathcal{O}_x$, then, by Proposition 6.9(3) and Lemma 6.10(2), $|\phi_m|$ is uniformly bounded. If ϕ_m contain an ordinary simple $\mathcal{O}_x$, then $d_{i1} \neq 0$ implies that x is in the support of $N \otimes_{\mathbb{X}} X_1$. Therefore there are only finitely many possible x. Further, X_1 is either $\mathcal{O}_x$ or a vector bundle, and in the latter case, $d_{i1} \neq 0$ implies that N must be $\mathcal{O}_x$. In both case, $N[n] \otimes_{\mathcal{T}} X_1$ is supported at x. Therefore ϕ_m consists of a single element $\mathcal{O}_x$. Combining above, we obtain that $\sum_{|m|\leq 2} |\phi_m|$ is uniformly bounded. As a consequence, (b) holds.

Now it follows by Lemma 4.8, $\mathrm{fpd}(N[n]) < \infty$. Combining Cases 1 and 2, we finish the proof. □

Now we are ready to prove Theorem 0.3.

Proof of Theorem 0.3 (1) If Q is of finite type, by Corollary 4.10 every monoidal triangulated structure on $D^b(\mathrm{rep}(Q))$ is fpd-finite. The converse follows from Lemmas 6.1(2), 4.3 and 4.5 and Proposition 6.5.

(2) Suppose Q is tame. By Lemma 6.1(2) and Theorem 6.11, there is a fpd-tame monoidal structure on $\mathcal{T}$. Applying Lemma 4.6 to $\mathcal{A} = \mathcal{C} = \mathrm{rep}(Q)$, there is a fpd-wild monoidal structure.

(3) This follows from parts (1,2), Lemmas 4.3 and 6.6.

(4) This follows from part (1). □

Corollary 6.12 *Let Q be a finite acyclic quiver.*

(1) *Q is of finite type if and only if* $\mathrm{rep}(Q)$ *does not contain an infinite brick set.*
(2) *Q is of tame type if and only if* $\mathrm{rep}(Q)$ *contains an infinite brick set and does not contain an infinite connected brick set.*
(3) *Q is of wild type if and only if* $\mathrm{rep}(Q)$ *contains an infinite connected brick set.*

Proof (1) If Q is of finite type, $\mathrm{rep}(Q)$ contains only finitely many indecomposable objects. So $\mathrm{rep}(Q)$ does not contains an infinite brick set.

For the converse, we assume that $\Bbbk Q$ is of tame or wild type. By Lemmas 4.3 and 4.5, rep(Q) contains an infinite brick set. This yields a contradiction. Therefore the assertion follows.

(3) If Q is of wild type, by Lemmas 4.3, rep(Q) contains an infinite connected brick set. Conversely suppose rep(Q) contains an infinite connected brick set. By Lemma 6.6, every monoidal triangulated structure on $D^b(\text{rep}(Q))$ is fpd-wild. By Theorem 0.3(3), Q is of wild type.

(2) Follows from parts (1,3). □

7 Examples

The natural construction of weak bialgebras associated to quivers, given in Lemma 2.1, produces many monoidal triangulated categories by Lemma 1.9(2). The main goal of this section is to construct other examples of (weak) bialgebras most of which are related to finite quivers. We will see that, given a quiver Q, there are different weak bialgebra structures on $\Bbbk Q$ such that the induced tensor products over rep(Q) are different from (E2.1.1). As a consequence, there are several different monoidal abelian structures on rep(Q) generally. We will also see that there are monoidal triangulated structures on derived categories associated to noncommutative projective schemes. The first example comes from [35].

Example 7.1 This example follows some ideas from [35, Theorem 3.2]. Let Q be a quiver with n vertices. We label vertices of Q as $1, 2, \ldots, n$. Suppose that 1 is either a source or a sink, namely, Q satisfies the following condition, either

(1) there is no arrows from 1 to j for every j, or
(2) there is no arrows from j to 1 for every j.

Let e_i be the idempotent corresponding to the vertex i, and we use p for a path of length at least 1.

First we define a bialgebra structure on $\Bbbk Q$ by

$$\begin{aligned} \varepsilon(e_1) &= 1, \quad \Delta(e_1) = e_1 \otimes e_1, \\ \varepsilon(e_i) &= 0, \quad \Delta(e_i) = \sum_{s<i} (e_i \otimes e_s + e_s \otimes e_i) + e_i \otimes e_i, \\ \varepsilon(p) &= 0, \quad \Delta(p) = e_1 \otimes p + p \otimes e_1 \end{aligned}$$

for all $i > 1$ and all paths p of length at least 1. It is routine to check that this defines a cocommutative bialgebra structure on $\Bbbk Q$.

By the above definition, $\Delta(x) = e_1 \otimes x + x \otimes e_1$ for all x in the ideal J generated by arrows of Q (this is also the graded Jacobson radical of $\Bbbk Q$). Let I be any sub-ideal of J. Then it is clear that I is a bialgebra ideal of $\Bbbk Q$. Therefore there is an induced bialgebra structure on $\Bbbk Q/I$.

Let Q is a finite acyclic quiver. Let (Δ, ε) be a coalgebra structure on $\Bbbk Q$. Suppose $|Q_0| = n$, then Δ is called a *partitioning morphism* (cf. [30, p. 460]) if

(1) there are $E_1, \ldots, E_n$ which are subsets of $E = \{(i, j) \mid 1 \le i, j \le n\}$,
(2) $E_i \cap E_j = \emptyset$ if $i \ne j$, and
(3) for every $1 \le k \le n$, $\Delta(e_k) = \sum_{(i,j)\in E_k} e_i \otimes e_j$.

Let $Q(i, j)$ be the set of paths from vertex i to vertex j, then:

Proposition 7.2 *[30, Proposition 4] Let Q be a finite acyclic quiver. Suppose $\Bbbk Q$ has a coalgebra structure $(\Bbbk Q, \Delta, \varepsilon)$. Then $\Bbbk Q_0$ is a subcoalgebra of $\Bbbk Q$ and Δ is a prealgebra map if and only if*

(1) *Δ is a partitioning morphism,*
(2) *$\Delta(\alpha_1 \cdots \alpha_m) = \Delta(\alpha_1) \cdots \Delta(\alpha_m)$ where $\alpha_i \in Q_1$,*
(3) *$\Delta(\alpha) \in \bigoplus_{(i,j)\in E_k,(i',j')\in E_l} \Bbbk Q(i, i') \otimes \Bbbk Q(j, j')$ for any $\alpha : k \to l$.*

Proof Note the fact that if $\Bbbk Q_0$ is a subcoalgebra of $\Bbbk Q$ and Δ is a prealgebra morphism, then Δ is a partitioning morphism. The rest of the proof is similar to [30, Proposition 4], and we omit it here. □

Remark 7.3 (1) Following Proposition 7.2, our first step is to understand all weak bialgebra structures on $\Bbbk^{\oplus n}$. This is already a non-trivial task and we post it as a question.

Can we classify all weak bialgebra structures on $\Bbbk^{\oplus n}$?

When $n = 2$, see Lemma 7.5 below.

(2) There are algebras A which do not admit any weak bialgebra structure. Let A be the algebra $\Bbbk[x]/(x^n)$ for some n. Then A admits a (weak) bialgebra structure if and only if $n = p^t$ where $p = \operatorname{char} \Bbbk > 0$ and $t \geq 1$. We give a sketch proof of one implication. Suppose that $A := \Bbbk[x]/(x^n)$ is a weak bialgebra. Note that A is local which implies that both the target and source counital subalgebras of A are $\Bbbk$. As a consequence, A is a bialgebra. So the augmentation ideal $J := \ker \epsilon$ is the Jacobson radical of A. So the associated graded Hopf algebra $\operatorname{gr}_J A$, which is isomorphic to A as an algebra, is the restricted enveloping algebra of a restricted Lie algebra. Therefore the $\Bbbk$-dimension of A is p^t for some $t \geq 1$. The assertion follows.

(3) Suppose $\operatorname{char} \Bbbk = p > 0$. Let A be the finite dimensional Hopf algebra

$$\Bbbk[x_1, \ldots, x_n]/(x_1^p, \ldots, x_n^p)$$

for some $n \geq 2$. The coalgebra structure of A is determined by

$$\Delta(x_i) = x_i \otimes 1 + 1 \otimes x_i$$

for all i. Since A is local, the only brick object in $\mathcal{A}$ is the trivial module $\Bbbk$. Therefore $\operatorname{fpd}(M) < \infty$ for every object in $\mathcal{A}$. On the other hand, A is wild when $n \geq 2$. Therefore conditions (a) and (b) in Theorem 0.4 are not equivalent if we remove the hereditary hypothesis.

Definition 7.4 Let A be an algebra. Two (weak) bialgebra structures $(\Delta_1, \varepsilon_1)$ and $(\Delta_2, \varepsilon_2)$ on A are called *equivalent* if there is an algebra automorphism σ of A such that $\Delta_1\sigma = (\sigma \otimes \sigma)\Delta_2$ and $\varepsilon_1\sigma = \varepsilon_2$.

Lemma 7.5 *Let $B = \Bbbk^{\oplus 2} = \Bbbk e_1 \oplus \Bbbk e_2$. Then there are five different weak bialgebra structures on B:*

(a) $\Delta(e_1) = e_1 \otimes e_1$, $\Delta(e_2) = e_2 \otimes e_2 + e_1 \otimes e_2 + e_2 \otimes e_1$, $\varepsilon(e_1) = 1$ *and* $\varepsilon(e_2) = 0$.
(b) $\Delta(e_1) = e_1 \otimes e_1 + e_2 \otimes e_2$, $\Delta(e_2) = e_2 \otimes e_1 + e_1 \otimes e_2$, $\varepsilon(e_1) = 1$ *and* $\varepsilon(e_2) = 0$.
(c) $\Delta(e_2) = e_2 \otimes e_2$, $\Delta(e_1) = e_1 \otimes e_1 + e_1 \otimes e_2 + e_2 \otimes e_1$, $\varepsilon(e_1) = 0$ *and* $\varepsilon(e_2) = 1$.
(d) $\Delta(e_2) = e_2 \otimes e_2 + e_1 \otimes e_1$, $\Delta(e_1) = e_1 \otimes e_2 + e_2 \otimes e_1$, $\varepsilon(e_1) = 0$ *and* $\varepsilon(e_2) = 1$.

(e) $\Delta(e_1) = e_1 \otimes e_1$, $\Delta(e_2) = e_2 \otimes e_2$, $\varepsilon(e_1) = 1$ *and* $\varepsilon(e_2) = 1$.

Note that (a) *and* (c) *are equivalent bialgebra structures* (*and so are* (b) *and* (d)). *The fifth one is a weak bialgebra, but not a bialgebra.*

Note that (e) in the above lemma is the direct sum of two copies of trivial Hopf algebra $\Bbbk$. Consequently, it is a weak Hopf algebra. Other bialgebra algebras in the above lemma are not (weak) Hopf algebras.

Proof Fix a (weak) bialgebra structure (Δ, ε) on B. Let B_t and B_s be target and source counital subalgebras of B, see [54, Definition 2.2.3].

Case 1: dim $B_t = 1$, then $B_t = B_s = \Bbbk 1_B$. In this case, B is a bialgebra. As a consequence, $\varepsilon(e_1) + \varepsilon(e_2) = \varepsilon(e_1 + e_2) = \varepsilon(1) = 1$. Since e_i are idempotents, $\varepsilon(e_i)$ is 1 or 0. First we assume that $\varepsilon(e_1) = 1$ and $\varepsilon(e_2) = 0$. Write $\Delta(e_1) = \sum_{i,j} a_{ij} e_i \otimes e_j$. By the counital axiom, we obtain that $\Delta(e_1) = e_1 \otimes e_1$ or $\Delta(e_1) = e_1 \otimes e_1 + e_2 \otimes e_2$. If $\Delta(e_1) = e_1 \otimes e_1$, we obtain case (a); if $\Delta(e_1) = e_1 \otimes e_1 + e_2 \otimes e_2$, we obtain case (b). The other situation is $\varepsilon(e_1) = 0$ and $\varepsilon(e_2) = 1$. By symmetric, we have (c) and (d).

Case 2: dim $B_t = 2$. Then $B_t = B_s = B$. By [6, Lemma 2.7], $\Delta(e_1) = e_1 \otimes e_1$ and $\Delta(e_2) = e_2 \otimes e_2$. Then it is easy to check that we obtain (e). □

Lemma 7.6 *Let A be a bialgebra and J be its Jacobson radical. Suppose that J is nilpotent. If $B := A/J \cong \Bbbk^{\oplus n}$ as an algebra for some positive integer n, then B is a quotient bialgebra of A.*

Proof Let π be the canonical quotient map from A to B. It's clear that π is an algebra map. Consider the composition of algebra maps:

$$A \xrightarrow{\Delta} A \otimes A \xrightarrow{\pi \otimes \pi} B \otimes B.$$

Since $B \otimes B$ doesn't have nilpotent elements and J is nilpotent, the above algebra map from $A \to B \otimes B$ factors through the quotient map π, that is, there exists a unique algebra map Δ_B from $B \to B \otimes B$, such that the following diagram commutes

$$\begin{array}{ccc} A & \xrightarrow{\Delta} & A \otimes A \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi \otimes \pi} \\ B & \xrightarrow{\Delta_B} & B \otimes B. \end{array}$$

Furthermore, $(\Delta_B \otimes Id)\Delta_B$ and $(Id \otimes \Delta_B)\Delta_B$ are the algebra maps induced by algebra maps $(\pi \otimes \pi \otimes \pi)(\Delta \otimes Id)\Delta$ and $(\pi \otimes \pi \otimes \pi)(Id \otimes \Delta)\Delta$ respectively from $A \to B \otimes B \otimes B$. Then Δ_B is coassociative since Δ is coassociative.

Similarly, let $\varepsilon_B : B \to \Bbbk$ be the algebra map induced by $\varepsilon : A \to \Bbbk$. It is not hard to verify that ε_B satisfies the counital axiom. Consequently, B is a quotient bialgebra of A and J is a bi-ideal. □

Now we are ready to classify (weak) bialgebras on a small quiver.

Proposition 7.7 *Suppose Q is the quiver with two vertices* $\{1, 2\}$ *and w arrows from* 1 *to* 2 *with* $w \geq 1$. *Let A be the path algebra $\Bbbk Q$. Then there are 5 types of weak bialgebra structures on A up to equivalences.*

(a) $\Delta(e_1) = e_1 \otimes e_1$, $\Delta(e_2) = e_2 \otimes e_2 + e_1 \otimes e_2 + e_2 \otimes e_1$, $\varepsilon(e_1) = 1$, $\varepsilon(e_2) = 0$, *and for any arrow r from 1 to 2,* $\Delta(r) = e_1 \otimes r + r \otimes e_1$ *and* $\varepsilon(r) = 0$.

(b) $\Delta(e_1) = e_1 \otimes e_1 + e_2 \otimes e_2$, $\Delta(e_2) = e_2 \otimes e_1 + e_1 \otimes e_2$, $\varepsilon(e_1) = 1$, $\varepsilon(e_2) = 0$, *and for any arrow* r *from 1 to 2*, $\Delta(r) = r \otimes e_1 + e_1 \otimes r$ *and* $\varepsilon(r) = 0$.
(c) $\Delta(e_2) = e_2 \otimes e_2$, $\Delta(e_1) = e_1 \otimes e_1 + e_1 \otimes e_2 + e_2 \otimes e_1$, $\varepsilon(e_2) = 1$, $\varepsilon(e_1) = 0$, *and for any arrow* r *from 1 to 2*, $\Delta(r) = e_2 \otimes r + r \otimes e_2$ *and* $\varepsilon(r) = 0$.
(d) $\Delta(e_2) = e_1 \otimes e_1 + e_2 \otimes e_2$, $\Delta(e_1) = e_2 \otimes e_1 + e_1 \otimes e_2$, $\varepsilon(e_2) = 1$, $\varepsilon(e_1) = 0$, *and for any arrow* r *from 1 to 2*, $\Delta(r) = r \otimes e_2 + e_2 \otimes r$ *and* $\varepsilon(r) = 0$.
(e) $\Delta(e_i) = e_i \otimes e_i$, $\varepsilon(e_i) = 1$ *for* $i = 1, 2$, *and the Jacobson radical* J *of* A *is a subcoalgebra of* A.

Proof Let J be the Jacobson radical of A, which is the ideal generated by the arrows from 1 to 2. It is clear that $J^2 = 0$ and $A/J \cong B$ where B is as given in Lemma 7.5.

We first consider bialgebra structures on A.

By Lemma 7.6, A/J is a quotient bialgebra of A and J is a bi-ideal of A. All bialgebra structures on $B \cong A/J$ are classified in Lemma 7.5. We will use this classification to analyze the bialgebra structures on A.

Case 1: Suppose the bialgebra structure on B is as in Lemma 7.5(a). Lifting the bialgebra structure on B to A, we have

$$\Delta(e_1) = e_1 \otimes e_1 + e_1 \otimes t_1 + e_2 \otimes t_2 + t_3 \otimes e_1 + t_4 \otimes e_2 + T,$$
$$\Delta(e_2) = e_1 \otimes e_2 + e_2 \otimes e_1 + e_2 \otimes e_2 - e_1 \otimes t_1 - e_2 \otimes t_2 - t_3 \otimes e_1 - t_4 \otimes e_2 - T,$$

where $T \in J \otimes J$ and $t_i \in J$ for $1 \leq i \leq 4$, and

$$\varepsilon(e_1) = 1, \quad \varepsilon(e_2) = 0, \quad \varepsilon(r) = 0 \text{ for all } r \in J.$$

By counital axiom, we have $t_1 = t_3 = 0$. By using the equation $\Delta(e_1 e_2) = 0$, we have $t_2 = t_4 = 0$. In the bialgebra structure of A, we have, for every arrow r from 1 to 2,

$$\Delta(r) = e_1 \otimes r + r \otimes e_1 + f(r) \otimes e_2 + e_2 \otimes g(r) + w(r)$$

where $f(r), g(r) \in J$ and $w(r) \in J \otimes J$. Using the fact that $r = re_1$, we obtain that $f(r) = g(r) = 0$ for all r.

Pick any $\Bbbk$-basis of J, say $\{r_i\}$, we can write,

$$\Delta(e_1) = e_1 \otimes e_1 + \sum_{i,j} a_{ij} r_i \otimes r_j,$$
$$\Delta(e_2) = e_1 \otimes e_2 + e_2 \otimes e_1 + e_2 \otimes e_2 - \sum_{i,j} a_{ij} r_i \otimes r_j,$$
$$\Delta(r_i) = e_1 \otimes r_i + r_i \otimes e_1 + \sum_{j,k} c_i^{jk} r_j \otimes r_k.$$

Suppose $\deg(e_1) = \deg(e_2) = 0$ and $\deg(r_i) = 1$. Let $\equiv$ denote $=$ modulo higher degree terms. Then the coalgebra structure above can be written as

$$\Delta(e_1) \equiv e_1 \otimes e_1$$
$$\Delta(e_2) \equiv e_1 \otimes e_2 + e_2 \otimes e_1 + e_2 \otimes e_2$$
$$\Delta(r_i) \equiv e_1 \otimes r_i + r_i \otimes e_1.$$

By [35, Lemma 3.1], if two different bialgebra structures on A both satisfy the above equations, then they are isomorphic.

Therefore, in this case, there exists a unique bialgebra structure on A up to isomorphism, that is,

$$\begin{aligned}\Delta(e_1) &= e_1 \otimes e_1,\\ \Delta(e_2) &= e_1 \otimes e_2 + e_2 \otimes e_1 + e_2 \otimes e_2,\\ \Delta(r_i) &= e_1 \otimes r_i + r_i \otimes e_1,\end{aligned}$$

which is exactly (a).

Case 2: Suppose the bialgebra structure on B is as in Lemma 7.5(b). Lifting the bialgebra structure on B to A, we have

$$\Delta(e_1) = e_1 \otimes e_1 + e_2 \otimes e_2 + e_1 \otimes t_1 + e_2 \otimes t_2 + t_3 \otimes e_1 + t_4 \otimes e_2 + T,$$

where $T \in J \otimes J$ and $t_i \in J$ for $1 \le i \le 4$, and

$$\begin{aligned}&\Delta(e_2) = e_1 \otimes e_2 + e_2 \otimes e_1 - e_1 \otimes t_1 - e_2 \otimes t_2 - t_3 \otimes e_1 - t_4 \otimes e_2 - T,\\ &\varepsilon(e_1) = 1, \quad \varepsilon(e_2) = 0, \quad \varepsilon(r) = 0 \text{ for all } r \in J.\end{aligned}$$

By counital axiom, we have $t_1 = t_3 = 0$. By the fact e_i is an idempotent, we have $T = 0$. In the bialgebra structure of A, for every arrow r from 1 to 2, we have

$$\Delta(r) = e_1 \otimes r + r \otimes e_1 + f(r) \otimes e_2 + e_2 \otimes g(r) + w(r)$$

where $f(r), g(r) \in J$ and $w(r) \in J \otimes J$. Using the fact that $e_1 r = 0$, we obtain that $f(r) = g(r) = 0$ for all r and $w(r) + r \otimes t_2 + t_4 \otimes r = 0$. Hence, for all $t \in J$,

$$\Delta(t) = e_1 \otimes t + t \otimes e_1 - t \otimes t_2 - t_4 \otimes t.$$

Moreover, the coassociative axiom, $(Id \otimes \Delta)\Delta(e_2) = (\Delta \otimes Id)\Delta(e_2)$, implies $t_2 = t_4 = 0$. We obtain (b).

Cases 3 and 4: When the bialgebra structure on B is as in Lemma 7.5(c) and (d), it's similarly to case 1 and case 2 respectively, and we obtain (c) and (d).

Next, we consider weak bialgebra, but not bialgebra, structures on A.

Let A_t and A_s be the target and source counital subalgebras. By [11, (2.1) and Proposition 2.4], $\dim A_t = \dim A_s$ and A_s commutes with A_t. If $\dim A_t = \dim A_s = 1$, by [55, Lemma 8.2], A is a bialgebra since $\Delta(1) = 1 \otimes 1$, which is the case we have just finished above. So $\dim A_t = \dim A_s \ge 2$. Since A_s is separable (hence semisimple), $A_s \cap J = \{0\}$. Thus there is an injective map

$$A_s \longrightarrow A \xrightarrow{\pi} B$$

which implies that $\dim A_t = \dim A_s = 2$ and that $\pi(A_s) = B$.

Now we claim that $A_t = A_s \cong B$. Since $\pi(A_s) = B$, we can write $A_s = span\{1, e_1 + p\}$ where $p \in J$. In this case, $A_t = A_s$ since the space of elements which commute with $e_1 + p$ is A_s itself, and they both are weak bialgebras. Let l denote the idempotent $e_1 + p$. Assume that

$$\Delta(1) = a_1 1 \otimes 1 + a_2 l \otimes 1 + a_3 1 \otimes l + a_4 l \otimes l$$

where $a_i \in \Bbbk$. By [11, Equations (2.7a) and (2.7b)], $a_1 + a_2 = 0$ and $\Delta(l) = (a_2 + a_4) l \otimes l$. By $\Delta(1) = \Delta(1^2)$, $a_2 = a_3$ and one of following equalities hold:

(i) $\Delta(1) = l \otimes l$,
(ii) $\Delta(1) = 1 \otimes 1 - l \otimes 1 - 1 \otimes l + l \otimes l$,

(iii) $\Delta(1) = 1 \otimes 1 - l \otimes 1 - 1 \otimes l + 2l \otimes l$.

However, (i) implies that l is a scalar multiple of 1 and (ii) implies that $1 - l$ is a scalar multiple of 1, which both are impossible. So (iii) holds and $\Delta(1) = (1-l) \otimes (1-l) + l \otimes l$, which means $A_t = A_s \cong B$ as weak bialgebra, where the weak bialgebra structure on B as in Lemma 7.5(e).

Re-write $l_1 = l$ and $l_2 = 1 - l$. Then $\Delta(l_1) = l_1 \otimes l_1$ and $\Delta(l_2) = l_2 \otimes l_2$. Note that $A = A_t \oplus J$ as vector space. Then for any arrow r from 1 to 2, we have

$$\Delta(r) = f(r) \otimes l_1 + g(r) \otimes l_2 + l_1 \otimes p(r) + l_2 \otimes q(r) + w(r),$$

where $f(r), g(r), p(r), q(r) \in J$ and $w(r) \in J \otimes J$. By $rl_1 = r$ and $l_2 r = r$, $f(r) = g(r) = p(r) = q(r) = 0$ for all r. That is J is a subcoalgebra of A. It's not hard to check any coalgebras structure over J satisfy conditions in Definition 1.7.

Moreover, let $\sigma : (A, \Delta, \varepsilon) \to (A, \Delta', \varepsilon')$ via $\sigma(l_i) = e_i$ and $\sigma(r) = r$ for $r \in J$, where $(A, \Delta', \varepsilon')$ is the weak bialgebra as in (e). Then σ is an algebra automorphism and (Δ, ε) is equivalent to (Δ', ε'). □

We finish this section with examples related to both commutative projective varieties and noncommutative projective schemes in the sense of [2].

Definition 7.8 [34, p. 1230] Let $\mathbb{X}$ be a smooth projective scheme.

(1) A coherent sheaf $\mathcal{E}$ on $\mathbb{X}$ is called *exceptional* if $\mathrm{Hom}_{\mathbb{X}}(\mathcal{E}, \mathcal{E}) \cong \Bbbk$ and $\mathrm{Ext}^i_{\mathbb{X}}(\mathcal{E}, \mathcal{E}) = 0$ for every $i \geq 0$.
(2) A sequence $\mathcal{E}_1, \ldots, \mathcal{E}_n$ of exceptional sheaves is called an *exceptional sequence* if $\mathrm{Ext}^k_{\mathbb{X}}(\mathcal{E}_i, \mathcal{E}_j) = 0$ for all k and for all $i > j$.
(3) If an exceptional sequence generates $D^b(coh(\mathbb{X}))$, then it is called *full*.
(4) If an exceptional sequence satisfies

$$\mathrm{Ext}^k_{\mathbb{X}}(\mathcal{E}_i, \mathcal{E}_j) = 0$$

for all $k > 0$ and all i, j, then it is called a *strongly exceptional sequence*.

The above concepts are extended to an arbitrary triangulated category in [50, Definition 4.1]. The existence of a full (strongly) exceptional sequence has been proved for many smooth projective schemes. However, on Calabi–Yau varieties there are no exceptional sheaves. When $\mathbb{X}$ has a full exceptional sequence $\mathcal{E}_1, \ldots, \mathcal{E}_n$, then there is a triangulated equivalence

$$\mathrm{RHom}_{\mathbb{X}}(\oplus_{i=1}^n \mathcal{E}_i, -) : \quad D^b(coh(\mathbb{X})) \cong D^b(\mathrm{mod}-A) \tag{E7.8.1}$$

where A is the finite dimensional algebra $\mathrm{End}_{\mathbb{X}}(\oplus_{i=1}^n \mathcal{E}_i)$, see [50, Theorem 4.2] (or [9, Theorem 3.1.7]). By Example 5.3(2), there is a canonical monoidal triangulated structure on $D^b(coh(\mathbb{X}))$ induced by $\otimes_{\mathbb{X}}$. Then we obtain a monoidal triangulated structure on $D^b(\mathrm{Mod}_{f.d}-A)$ via (E7.8.1). By Example 5.3(1), if A is a weak bialgebra, there is a (different) canonical monoidal triangulated structure on $D^b(\mathrm{Mod}_{f.d}-A)$ (or equivalently, on $D^b(coh(\mathbb{X}))$). In short, there are possibly many different monoidal triangulated structures on a given triangulated category.

Next we give an explicit example related to noncommutative projective schemes.

Example 7.9 Let T be a connected graded noetherian Koszul Artin–Schelter regular algebra of global dimension at least 2. If T is commutative, then T is the polynomial ring $\Bbbk[x_0, x_1, \ldots, x_n]$ for some $n \geq 1$. Let $\mathbb{X}$ be the noncommutative projective scheme associated to T in the sense of [2]. In [2] proj T denotes the category of coherent sheaves on $\mathbb{X}$, but here we use $coh(\mathbb{X})$ instead. When T is the commutative polynomial ring $\Bbbk[x_0, x_1, \dots, x_n]$, then $\mathbb{X}$ is the commutative projective n-space $\mathbb{P}^n$. On the other hand, there are many noetherian Koszul Artin–Schelter regular algebras T that are not commutative. Let r be the global dimension of T and $\mathcal{O}$ be the structure sheaf of $\mathbb{X}$. Then

$$\{\mathcal{O}(-(r-1)), \mathcal{O}(-(r-2)), \dots, \mathcal{O}(-1), \mathcal{O}\}$$

is a full strongly exceptional sequence for $\mathbb{X}$ in the sense of [50, Definition 4.1]. By (E7.8.1) or [50, Theorem 4.2],

$$D^b(coh(\mathbb{X})) \cong D^b(A - \text{mod}) \tag{E7.9.1}$$

where A is the opposite ring of $\text{End}_{\mathbb{X}}(\oplus_{i=1}^n \mathcal{O}_i)$. By [50, Definition 4.6 and Theorem 4.7], A is the opposite ring of the Beilinson algebra (which is denoted by R in [50, Definition 4.6]). By the description in [49, Definition 4.7], the Beilinson algebra is an upper triangular matrix with diagonal entries being $\Bbbk$. Then A can be written as $\Bbbk Q/I$ where Q is a quiver with r vertices and the number of arrows from vertex i to vertex j equals the dimension of T_{j-i}. It is clear that Q satisfies condition (2) in Example 7.1. By Example 7.1, there is a cocommutative bialgebra structure on A. Similarly, vertex r in Q satisfies condition (1) in Example 7.1, which implies that there is another cocommutative bialgebra structure on A. Via (E7.9.1), $D^b(coh(\mathbb{X}))$ has at least two different monoidal triangulated structures induced by two different bialgebra structures on A.

Now let T be the polynomial ring $\Bbbk[x_0, x_1]$. Then $\mathbb{X} = \mathbb{P}^1$ and

$$D^b(coh(\mathbb{P}^1)) \cong D^b((B)^{op} - \text{mod})$$

where B is the Beilinson algebra associated to T. By [49, Definition 4.7],

$$B = \begin{pmatrix} \Bbbk & \Bbbk x + \Bbbk y \\ 0 & \Bbbk \end{pmatrix}.$$

It is clear that B is the path algebra of the Kronecker quiver given in Example 2.7. In this case we have two monoidal triangulated structures on $D^b(coh(\mathbb{P}^1))$. One is the monoidal structure induced by $\otimes_{\mathbb{P}^1}$, and the other comes from the canonical weak bialgebra structure of $B = \Bbbk Q$ [Lemma 2.1(1)]. Together with two bialgebra structures on B, see the above paragraph, we obtain four different monoidal triangulated structures on $D^b(coh(\mathbb{P}^1))$. To show these monoidal triangulated structures are not equivalent, one need to use some arguments in the proof of Lemma 5.9 (details are omitted).

8 Proof of Theorems 0.8

It is important and interesting to calculate explicitly $\text{fpd}(M)$ of some objects M in a monoidal abelian (or triangulated) category. Generally this is very difficult task and dependent on complicated combinatorial structures of the brick sets. In this section we will work out one example. Note that some non-essential details are omitted.

A type $\mathbb{A}_n$ quiver is defined to be a quiver of form (E0.7.1):

$$1 \overset{\alpha_1}{\text{——}} 2 \overset{\alpha_2}{\text{——}} \cdots \overset{\alpha_{i-1}}{\text{——}} i \overset{\alpha_i}{\text{——}} \cdots \overset{\alpha_{n-1}}{\text{——}} n$$

where each arrow α_i is either $\longrightarrow$ or $\longleftarrow$. For each $n \geq 3$, there are more than one isomorphism classes of type $\mathbb{A}_n$ quivers with n vertices, though we denote all of them by $\mathbb{A}_n$. In this section we provide fairly detailed computation of $\text{fpd}(M)$ for every indecomposable object in the monoidal abelian category rep($\mathbb{A}_n$). Using Lemma 4.11, we obtain fpd(M) for every indecomposable object M in the monoidal triangulated category $D^b(\text{rep}(\mathbb{A}_n))$. The result is summarized in Theorem 0.8. Throughout this section, the tensor product is defined as in (E2.1.1).

First we try to understand brick sets in rep($\mathbb{A}_n$). Recall that $M\{i, j\}$, for $i \leq j$, denotes the representation of $\mathbb{A}_n$ defined by

$$(M\{i, j\})_s = \begin{cases} \Bbbk & i \leq s \leq j, \\ 0 & \text{otherwise}, \end{cases}$$

$$(M\{i, j\})_{\alpha_s} = \begin{cases} Id_{\Bbbk} & i \leq s < j, \\ 0 & \text{otherwise}. \end{cases}$$

We start with easy observations.

Lemma 8.1 *If $\{M\{1, m\}, M\{k, l\}\}$ is a brick set and $m \geq k \geq 3$, then $\{M\{2, m\}, M\{k, l\}\}$ also is a brick set.*

Proof This is clear since $k \geq 3$. □

Lemma 8.2 *For any $1 \leq i < j \leq n$, $\{M\{1, i\}, M\{1, j\}\}$ is not a brick set.*

Proof There are two cases.

Case 1: $s(\alpha_i) = i$. Let $f : M\{1, j\} \to M\{1, i\}$ be $(f)_k = \begin{cases} Id & k \leq i \\ 0 & k > i \end{cases}$. Then it is clear that $f \in \text{Hom}(M\{1, j\}, M\{1, i\})$ and $\text{Hom}(M\{1, j\}, M\{1, i\}) \neq 0$.

Case 2: $t(\alpha_i) = i$. Let $g : M\{1, i\} \to M\{1, j\}$ be $(g)_k = \begin{cases} Id & k \leq i \\ 0 & k > i \end{cases}$. Then $g \in \text{Hom}(M\{1, i\}, M\{1, j\})$ and $\text{Hom}(M\{1, i\}, M\{1, j\}) \neq 0$.

Combining these two cases, one sees that $\{M\{1, i\}, M\{1, j\}\}$ is not a brick set. □

In the above, we can replace 1 by any positive integer no more that i.

Lemma 8.3 *Suppose $i \leq j \leq k$. Then one of spaces $\text{Hom}(M\{i, j\}, M\{i, k\})$ and $\text{Hom}(M\{i, k\}, M\{i, j\})$ is isomorphic to $\Bbbk$ while the other is zero.*

Proof An idea similar to the proof of Lemma 8.2 shows that one of spaces is nonzero and the other is zero. For the one that is nonzero, it must be $\Bbbk$ by Lemma 4.2. □

Lemma 8.4 *If $f : M\{i, k\} \to M\{i, l\}$ is a non-zero morphism and $k \neq l$, then for any $j \leq i$, $\text{Hom}(M\{i, l\}, M\{j, k\}) = 0$ and $\text{Hom}(M\{j, l\}, M\{i, k\}) = 0$.*

Proof Assume that $g : M\{i, l\} \to M\{j, k\}$ is non-zero morphism, then it can induce a non-zero morphism $\hat{g} : M\{i, l\} \to M\{i, k\}$. By Lemma 8.3, $f = 0$ which contradicts the assumption. Therefore, $\text{Hom}(M\{i, l\}, M\{j, k\}) = 0$. Similarly, $\text{Hom}(M\{j, l\}, M\{i, k\}) = 0$. □

Next we define a binary relation, denoted by $\succ$, that does not necessarily satisfy the usual axioms of an order.

Definition 8.5 For $N, N' \in \text{rep}(\mathbb{A}_n)$, we write $N \succ N'$ if $\text{Hom}(N, N') \cong \Bbbk$. Usually we only consider indecomposable objects N, N'.

Another easy observation, following from Lemma 8.3, is

Lemma 8.6 *Let $I \subset \{1, 2, \ldots, n\}$ and $\mathcal{S}_I = \{X_i \mid X_i = M\{1, i\}, i \in I\}$. Then $(\mathcal{S}_I, \succ)$ is a totally ordered set. Similarly, $\{Y_i \mid Y_i = M\{i, n\}, i \in I\}$ is a totally ordered set.*

Lemma 8.7 *Let $N = M\{i, j\}$, $N' = M\{k, l\}$ and $k \le j < l$.*

(1) *If $s(\alpha_j) = j$ and $i \le k$, then $\mathrm{Hom}(N', N \otimes N') \cong \Bbbk$ where $N \otimes N' = M\{k, j\}$.*
(2) *If $t(\alpha_j) = j$, then for all $m \le j$, $\mathrm{Hom}(N', M\{m, j\}) = 0$.*

Proof (1) In this case, we have $i \le k \le j$. By definition, $N' = M\{k, l\}$ and $N \otimes N' = M\{k, j\}$. Let $f : N' \to N \otimes N$ be defined by $(f)_s = \begin{cases} Id & \text{if } k \le s \le j \\ 0 & \text{otherwise} \end{cases}$. Then it is not hard to check $0 \ne f \in \mathrm{Hom}(N', N \otimes N')$.

If $f' \in \mathrm{Hom}(N', N \otimes N')$, then there is a scalar $c \in \Bbbk$ such that $(f')_s = cId$ for all $k \le s \le j$. Then $f' = cf$ and $\mathrm{Hom}(N', N \otimes N') \cong \Bbbk$.

(2) Since $k \le j < l$, $(N')_{j+1} = \Bbbk$. Let $f \in \mathrm{Hom}(N', M\{m, j\})$. Then, for every $s > j$, $f_s = 0$ as $(M\{m, j\})_s = 0$. So we have

$$(f)_j (N')_{\alpha_j} = (M\{m, j\})_{\alpha_j} (f)_{j+1} = 0.$$

Since $(N')_{\alpha_j} = Id_{\Bbbk}$, we obtain $(f)_j = 0$. Using a similar equation as above and induction, one sees that $f_s = 0$ for all $s < j$. Therefore $f = 0$ as desired. □

For the rest of this section we use ϕ for a brick set in $\mathrm{rep}(\mathbb{A}_n)$. Given a brick set ϕ and an indecomposable representation $M\{i, j\}$, we define three subsets of ϕ according to $\{i, j\}$:

(1) $\phi_i = \{N \in \phi \mid (N)_i \cong \Bbbk, (N)_j = 0\}$,
(2) $\phi_j = \{N \in \phi \mid (N)_i = 0, (N)_j \cong \Bbbk\}$,
(3) $\phi_{ij} = \{N \in \phi \mid (N)_i \cong \Bbbk, (N)_j \cong \Bbbk\}$.

It is clear that ϕ contain the disjoint union of ϕ_i, ϕ_j and ϕ_{ij}. Note that ϕ_l, for l being either i or j, can be divided into the following two parts:

$$\hat{\phi}_l = \{N \in \phi_l \mid M\{i, j\} \otimes N \succ M\{i, j\}\},$$
$$\tilde{\phi}_l = \{N \in \phi_l \mid M\{i, j\} \succ M\{i, j\} \otimes N\}.$$

Lemma 8.8 *Let N be an object in ϕ that satisfies either $M\{i, j\} \otimes N = 0$ or $M\{i, j\} \otimes N = N$. Then*

$$\rho(A(\phi, M\{i, j\} \otimes -)) = \max\{a, \quad \rho(A(\phi \backslash \{N\}, M\{i, j\} \otimes -))\}$$

where $a = \begin{cases} 0 & \text{if } M\{i, j\} \otimes N = 0, \\ 1 & \text{if } M\{i, j\} \otimes N = N. \end{cases}$

Proof Write $\phi = \{N_1, \ldots, N_m\}$ where $N_1 = N$. By the hypothesis on N,

$$\dim \mathrm{Hom}(N_k, M\{i, j\} \otimes N) = 0$$

for $2 \le k \le m$. Hence, in the matrix $A(\phi, M\{i, j\} \otimes -)$, $a_{k1} = 0$ for all $k \ge 2$. As a consequence,

$$\rho(A(\phi, M\{i, j\} \otimes -)) = \max\{a, \quad \rho(A(\phi \backslash \{N_1\}, M\{i, j\} \otimes -))\}$$

where $a := a_{11}$ is the $(1, 1)$-entry in $A(\phi, M\{i, j\} \otimes -)$. Clearly a has the desired property. □

Lemma 8.9 *Let $N \in \phi_i$ and $N' \in \phi_j$. Then $\{N, M\{i, j\} \otimes N'\}$ and $\{M\{i, j\} \otimes N, N'\}$ are brick sets.*

Proof Write N as $M\{i', j'\}$. Then $i' \leq i$ and $j' < j$. Similarly, $N' = M\{k, l\}$ for some $k > i$ and $l \geq j$, and consequently, $M\{i, j\} \otimes N' = M\{k, j\}$. A version of Lemma 8.1 shows that $\{M\{i', j'\}, M\{k, l\}\}$ being a brick set implies that $\{M\{i', j'\}, M\{k, j\}\}$ is a brick set. Therefore $\{N, M\{i, j\} \otimes N'\}$ is a brick set. A similar argument shows that $\{M\{i, j\} \otimes N, N'\}$ is brick set. □

Lemma 8.10 *Let j be a positive integer no more than n. If $j = n$ or $s(\alpha_j) = j$, then $A(\phi_j, M\{i, j\} \otimes -)$ is similar to an upper triangular matrix in which all diagonal entries are 1.*

Proof If $j = n$, then $|\phi_j| = 1$ and $A(\phi_j, M\{i, j\} \otimes -) = (1)_{1\times 1}$ by Lemma 8.2.

If $j < n$ and $s(\alpha_j) = j$, by Lemma 8.6, the set $(\{M\{i, j\} \otimes N \mid N \in \phi_j\}, \succ)$ is a totally ordered set. Let $|\phi_j| = m$ and we can label the objects in ϕ_j so that

$$M\{i, j\} \otimes N_1 \succ \cdots \succ M\{i, j\} \otimes N_m.$$

By Definition 8.5,

$$\operatorname{Hom}(M\{i, j\} \otimes N_k, M\{i, j\} \otimes N_l) \cong \begin{cases} 0 & \text{if } l < k, \\ \Bbbk & \text{if } l \geq k. \end{cases} \tag{E8.10.1}$$

And, by Lemma 8.7(1),

$$\operatorname{Hom}(N_k, M\{i, j\} \otimes N_k) \cong \Bbbk. \tag{E8.10.2}$$

Combine (E8.10.1) and (E8.10.2), then

$$\dim \operatorname{Hom}(N_k, M\{i, j\} \otimes N_l) = \begin{cases} 0 & \text{if } l < k, \\ 1 & \text{if } l \geq k. \end{cases}$$

The assertion follows. □

The next theorem is Theorem 0.8(2).

Theorem 8.11 *Let Q be a quiver of type $\mathbb{A}_n$ given in (E0.7.1) for some $n \geq 2$. Then the following hold in* rep(Q)*:*

$$\operatorname{fpd}(M\{i, j\}) = \begin{cases} 1 & \textit{if } M\{i, j\} \textit{ is a sink}, \\ \min\{i, n-j+1\} & \textit{if } M\{i, j\} \textit{ is a source}, \\ 1 & \textit{if } M\{i, j\} \textit{ is a flow}. \end{cases}$$

Proof First we show that

$$\operatorname{fpd}(M\{i, j\}) \geq \begin{cases} 1 & \text{if } M\{i, j\} \text{ is a sink}, \\ \min\{i, n-j+1\} & \text{if } M\{i, j\} \text{ is a source}, \\ 1 & \text{if } M\{i, j\} \text{ is a flow}. \end{cases} \tag{E8.11.1}$$

Let ϕ be the singleton consisting of $M\{i, j\}$. It is clear that $A(\phi, M\{i, j\} \otimes -)$ is $(1)_{1\times 1}$. Hence $\operatorname{fpd}(M\{i, j\}) \geq 1$. Now suppose that $M\{i, j\}$ is a source. Let $d = \min\{i, n-j+1\}$. We construct a brick set with d elements as follows. By Lemma 8.6, $(\{M\{k, i\} \mid 1 \leq k \leq i\}, \succ)$

and $(\{M\{j, m\} \mid j \leq m \leq n\}, \succ)$ are two totally ordered sets. We list elements in these two sets as

$$M\{k_1, i\} \succ \cdots \succ M\{k_i, i\} \text{ and } M\{j, m_1\} \succ \cdots \succ M\{j, m_{n-j+1}\} \tag{E8.11.2}$$

where $\{k_l\}_{l=1}^{i}$ and $\{m_l\}_{l=1}^{n-j+1}$ are distinct integers from 1 to i and from j to n respectively. Since $d = \min\{i, n-j+1\}$, we have a set of d elements

$$\phi = \{M\{k_1, m_{n-j+1}\}, M\{k_2, m_{n-j}\}, \ldots, M\{k_d, m_{n-j+2-d}\}\}.$$

We claim that ϕ is a brick set. If there is a nonzero map from $M\{k_s, m_{n-j+2-s}\}$ to $M\{k_t, m_{n-j+2-t}\}$ for some $s < t$, then, when restricted to vertices $\{j, j+1, \ldots, n\}$, we obtain a nonzero map from $M\{j, m_{n-j+2-s}\}$ to $M\{j, m_{n-j+2-t}\}$. This contradicts the second half of (E8.11.2). Therefore there is no nonzero morphism from $M\{k_s, m_{n-j+2-s}\}$ to $M\{k_t, m_{n-j+2-t}\}$ for $s < t$. Similarly, there is no nonzero morphism from $M\{k_t, m_{n-j+2-t}\}$ to $M\{k_s, m_{n-j+2-s}\}$ for $s < t$, by using the first half of (E8.11.2). Thus we prove our claim. Using this brick set, one see that every entry in the matrix $A(\phi, M\{i, j\} \otimes -)$ is 1, consequently, $\rho(A(\phi, M\{i, j\} \otimes -)) = d$. Therefore $\operatorname{fpd}(M\{i, j\}) \geq d$ if $M\{i, j\}$ is a source. Combining with the inequality $\operatorname{fpd}(M\{i, j\}) \geq 1$, we obtain (E8.11.1).

It remains to show the opposite inequality of (E8.11.1), or equivalently, to show that

$$\rho(A(\phi, M\{i, j\} \otimes -)) \leq \begin{cases} 1 & \text{if } M\{i, j\} \text{ is a sink,} \\ \min\{i, n-j+1\} & \text{if } M\{i, j\} \text{ is a source,} \\ 1 & \text{if } M\{i, j\} \text{ is a flow,} \end{cases} \tag{E8.11.3}$$

for every brick set ϕ in $\operatorname{rep}(Q)$. We use induction on the integer $|\phi| + n$. If $|\phi| + n$ is 1, nothing needs to be proved. So we assume that $|\phi| + n \geq 2$. If $|\phi| = 1$, then $A(\phi, M\{i, j\} \otimes -)$ is either $(0)_{1\times 1}$ or $(1)_{1\times 1}$. It is clear that the assertion holds. Now we assume that $|\phi| \geq 2$. This forces that $n \geq 3$ (but we will not use this fact directly). If there is an object $N \in \phi$ such that either $M\{i, j\} \otimes N = 0$ or $M\{i, j\} \otimes N = N$, then (E8.11.3) follows from Lemma 8.8 and the induction hypothesis.

For the rest of the proof we can assume that

$$N \ncong M\{i, j\} \otimes N \neq 0$$

for every object $N \in \phi$. Note that the above condition implies that ϕ is the disjoint of ϕ_i, ϕ_j and ϕ_{ij}. Now it suffices to consider ϕ satisfying the following conditions:

(*) $\phi = \phi_i \cup \phi_j \cup \phi_{ij}$,
(**) for every $N \in \phi$, $M\{i, j\} \otimes N \ncong N$.

Let w be the number of objects in ϕ. Suppose that ϕ_j is not empty. If there is an $N \in \phi_j$ such that $N \otimes M\{i, j\} \succ M\{i, j\}$, we let N_w be the object in $\phi_j \cup \phi_{ij}$ such that $N_w \otimes M\{i, j\}$ is largest in the set

$$\{N \otimes M\{i, j\} \mid N \in \phi_j \cup \phi_{ij}\}.$$

Such an object N_w exists by a version of Lemma 8.6. It is easy to see that $N_w \in \phi_j$. By the choice of N_w, one can show that, for every $N_k \in \phi_j \cup \phi_{ij}$ with $k \neq w$,

$$\operatorname{Hom}(N_k, N_w \otimes M\{i, j\}) = 0.$$

If $N_k \in \phi \backslash (\phi_j \cup \phi_{ij})$, then, by Lemma 8.9,

$$\operatorname{Hom}(N_k, N_w \otimes M\{i, j\}) = 0.$$

Therefore $a_{kw} = 0$ for all $k < w$ as an entry in the adjacency matrix $A(\phi, M\{i, j\} \otimes -)$. As a consequence,

$$\rho(A(\phi, M\{i, j\} \otimes -)) = \max\{1, \rho(A(\phi \backslash \{N_w\}, M\{i, j\} \otimes -))\}.$$

Assertion (E8.11.3) follows by induction hypothesis. The other possibility is that for every $N \in \phi_j$ we have $M\{i, j\} \succ N \otimes M\{i, j\}$. Now let N_1 be the object in $\phi_j \cup \phi_{ij}$ such that $N_1 \otimes M\{i, j\}$ is smallest in the set

$$\{N \otimes M\{i, j\} \mid N \in \phi_j \cup \phi_{ij}\}.$$

Such an object N_1 exists by a version of Lemma 8.6. It is easy to see that $N_1 \in \phi_j$. By the choice of N_1, one sees, for every $N_k \in \phi_j \cup \phi_{ij}$ with $k \neq 1$,

$$\mathrm{Hom}(N_1, N_k \otimes M\{i, j\}) = 0.$$

If $N_k \in \phi \backslash (\phi_j \cup \phi_{ij})$, then, by Lemma 8.9

$$\mathrm{Hom}(N_1, N_k \otimes M\{i, j\}) = 0.$$

Therefore $a_{1k} = 0$ for all $k > 1$ as an entry in the adjacency matrix $A(\phi, M\{i, j\} \otimes -)$. As a consequence,

$$\rho(A(\phi, M\{i, j\} \otimes -)) = \max\{1, \rho(A(\phi \backslash \{N_1\}, M\{i, j\} \otimes -))\}.$$

Assertion (E8.11.3) follows by induction hypothesis. Combining these two cases, we show that (E8.11.3) holds by induction when ϕ_j is not empty.

Similarly, (E8.11.3) holds by induction when ϕ_i is not empty. The remaining case is when ϕ_i and ϕ_j are empty, or

(***) $\phi = \phi_{ij}$.

We divide the rest of the proof into 5 small subcases.

Subcase 1: $t(\alpha_{i-1}) = i$. Pick any object in ϕ, say N. Suppose that $(N_1)_{i-1} \neq 0$. Then $\mathrm{Hom}(N_1, M\{i, j\}) = 0$. Note that in this case $M\{i, j\} = M\{i, j\} \otimes N$ for all $N \in \phi$. Therefore $a_{1k} = 0$ for all $k > 1$ as an entry in the adjacency matrix $A(\phi, M\{i, j\} \otimes -)$. As a consequence,

$$\rho(A(\phi, M\{i, j\} \otimes \rho(A(\phi \backslash \{N_1\}, M\{i, j\} \otimes -)).$$

Assertion (E8.11.3) follows by induction hypothesis. Therefore, without loss of generality, we can assume that $(N_1)_{i-1} = 0$ for all $N_1 \in \phi$. Now everything can be computed in the subquiver quiver $Q \backslash \{1\}$. Then we reduce the number of vertices from n to $n - 1$. Again the assertion follows from the induction hypothesis.

Subcase 2: $t(\alpha_j) = j$. This is equivalent to Subcase 1 after one relabels vertices of Q by setting $i' = n + 1 - i$ for all $1 \leq i \leq n$.

Subcase 3: $i = 1$. Since $\phi = \phi_{ij}$, by Lemma 8.6, ϕ consists of single object. As a consequence, $A(\phi, M\{i, j\} \otimes -)$ is either $(0)_{1\times1}$ or $(1)_{1\times1}$. Then $\rho(A(\phi, M\{i, j\} \otimes -)) \leq 1$ and the assertion follows trivially.

Subcase 4: $j = n$. This is equivalent to Subcase 3 after one re-labels vertices of Q by setting $i' = n + 1 - i$ for all $1 \leq i \leq n$.

Subcase 5: Not cases 1-4, namely, $i > 1$, $j < n$, $t(\alpha_{i-1}) = i - 1$ and $t(\alpha_j) = j + 1$. In this case $M\{i, j\}$ is a source. We list all objects in $\phi = \phi_{ij}$ as

$$M\{i_1, j_1\}, \ldots, M\{i_w, j_w\}$$

where $1 \le i_s \le i$ and $j \le j_s \le n$. By Lemma 8.3, all i_s are distinct. The same holds true for j_s. Therefore $|\phi| = w \le d := \min\{i, n-j+1\}$. Since every entry of $A(\phi, M\{i,j\} \otimes -)$ is at most 1, we obtain that $\rho(A(\phi, M\{i,j\} \otimes -)) \le |\phi| \le d$ as desired.

Combining (E8.11.1) with (E8.11.3), we finish the proof. □

Note that, for $M, N \in \text{rep}(Q)$,

$$\text{Hom}_{D^b(\text{rep}(Q))}(M[0], N[1]) \cong \text{Ext}^1_{\text{rep}(Q)}(M, N).$$

For the rest of this section we use $\text{Ext}^1(M, N)$ instead of $\text{Ext}^1_{\text{rep}(Q)}(M, N)$. The *Euler characteristic* of two representations M and N of Q is defined to be

$$\langle \mathbf{dim} M, \mathbf{dim} N \rangle_Q = \sum_{v \in Q_0} x_v y_v - \sum_{\alpha \in Q_1} x_{s(\alpha)} y_{t(\alpha)}$$

where $\mathbf{dim}$ denotes the dimension vector and $x_v = \dim((M)_v)$, $y_v = \dim((N)_v)$ for any $v \in Q_0$. By [25, p.65], we have

$$\dim \text{Hom}(M, N) - \dim \text{Ext}^1(M, N) = \langle \mathbf{dim} M, \mathbf{dim} N \rangle_Q. \tag{E8.11.4}$$

One can verify the following.

Lemma 8.12 *Assume Q is of type $\mathbb{A}_n$. Let $N = M\{i_1, j_1\}$, $N' = M\{i_2, j_2\}$ and $i_1 \le i_2$.*

(1) *If $j_1 \le i_2 - 2$, then* $\text{Ext}^1(N, N') = \text{Ext}^1(N', N) = 0$.
(2) *Suppose that $j_1 = i_2 - 1$.*
   (a) *If $s(\alpha_{j_1}) = j_1$, then* $\text{Ext}^1(N, N') \cong \Bbbk$, $\text{Ext}^1(N', N) = 0$.
   (b) *If $s(\alpha_{j_1}) = i_2$, then* $\text{Ext}^1(N, N') = 0$, $\text{Ext}^1(N', N) \cong \Bbbk$.
(3) *Suppose either $i_1 < i_2 \le j_1 < j_2$ or $i_1 < i_2 \le j_2 < j_1$.*
   (a) *If* $\text{Hom}(N, N') \cong \Bbbk$, *then* $\text{Ext}^1(N, N') = 0$, $\text{Ext}^1(N', N) \cong \Bbbk$.
   (b) *If* $\text{Hom}(N', N) \cong \Bbbk$, *then* $\text{Ext}^1(N, N') \cong \Bbbk$, $\text{Ext}^1(N', N) = 0$.
   (c) *If $\{N, N'\}$ is a brick set, then* $\text{Ext}^1(N, N') = \text{Ext}^1(N', N) = 0$.
(4) *If $i_1 = i_2$ or $j_1 = j_2$, then* $\text{Ext}^1(M, N) = \text{Ext}^1(N, M) = 0$.

Proof When $j_1 \le i_2 - 2$, it is easy to see

$$\dim \text{Hom}(N, N') = \dim \text{Hom}(N', N) = 0$$

and

$$\langle \mathbf{dim} N, \mathbf{dim} N' \rangle_Q = \langle \mathbf{dim} N', \mathbf{dim} N \rangle_Q = 0.$$

Therefore, $\text{Ext}^1(N, N') = \text{Ext}^1(N', N) = 0$.

As for (2), (3) and (4), the proofs are similar and we omit them here. □

A direct corollary of Lemma 8.12 is

Corollary 8.13 *Assume Q is of type $\mathbb{A}_n$. If* $\text{Hom}(M\{i_1, j_1\}, M\{i_2, j_2\}) \cong \Bbbk$, *then*

$$\text{Ext}^1(M\{i_1, j_1\}, M\{i_2, j_2\}) = 0.$$

Any brick set ϕ in $\text{rep}(Q)$ is also a brick set in $D^b(\text{rep}(Q))$. In the next lemma we are working with the category $D^b(\text{rep}(Q))$ and ϕ (respectively, ϕ_i and ϕ_j) still denotes a brick set in $\text{rep}(Q)$.

Lemma 8.14 *Retain the notation above. Then $A(\phi_i, M\{i, j\}[1] \otimes -)$ is similar to a strictly lower triangular matrix.*

Proof By Lemma 8.6, $(\{M\{i, j\} \otimes N \mid N \in \phi_i\}, \succ)$ is a totally ordered set, which can be listed as

$$M\{i, j_1\} \succ M\{i, j_2\} \succ \cdots M\{i, j_{|\phi|_i}\}. \tag{E8.14.1}$$

When we compute the adjacency matrix $A(\phi_i, M\{i, j\}[1] \otimes -)$, we order elements in ϕ_i according to (E8.14.1). For any two objects $M\{i_{s_1}, j_{s_1}\}, M\{i_{s_2}, j_{s_2}\}$ in ϕ_i with $s_1 < s_2$, we have $M\{i, j_{s_1}\} \succ M\{i, j_{s_2}\}$. An easy analysis shows that either $\{M\{i_{s_1}, j_{s_1}\}, M\{i, j_{s_2}\}\}$ is a brick set or $\mathrm{Hom}(M\{i_{s_1}, j_{s_1}\}, M\{i, j_{s_2}\}) \cong \Bbbk$. By Lemma 8.12(3),

$$\mathrm{Ext}^1(M\{i_{s_1}, j_{s_1}\}, M\{i, j\} \otimes M\{i_{s_2}, j_{s_2}\}) = 0.$$

By Lemma 8.12(4), we have

$$\mathrm{Ext}^1(M\{i_{s_1}, j_{s_1}\}, M\{i, j\} \otimes M\{i_{s_1}, j_{s_1}\}) = \mathrm{Ext}^1(M\{i_{s_1}, j_{s_1}\}, M\{i, j_{s_1}\}) = 0.$$

As a consequence, $A(\phi_i, M\{i, j\}[1] \otimes -)$ is a strictly lower triangular matrix. □

Lemma 8.15 *Let $N \in \hat{\phi}_j$, $N' \in \phi_i \cup \phi_{ij}$ and $N'' \in \tilde{\phi}_j$. then*

$$\mathrm{Ext}^1(N, M\{i, j\} \otimes N') = \mathrm{Ext}^1(N', M\{i, j\} \otimes N'') = \mathrm{Ext}^1(N, M\{i, j\} \otimes N'') = 0.$$

Proof Similar to the proof of Lemma 8.12, we only prove the first equation and leave out the proof of the last two equations.

Write $N = M\{i_1, j_1\}$ and $N' = M\{i_2, j_2\}$. By definition, $\hat{\phi}_j$ is nonempty. This implies that $s(\alpha_{i_1-1}) = i_1 - 1$.

First we suppose that $N' \in \phi_i$. If $j_2 \neq i_1 - 1$, then, by Lemmas 8.1 and 8.12 (1,3c), $\mathrm{Ext}^1(N, M\{i, j\} \otimes N') = 0$. If $j_2 = i_1 - 1$, then $\mathrm{Ext}^1(N, M\{i, j\} \otimes N') = 0$ by Lemma 8.12(2). Therefore, $\mathrm{Ext}^1(N, M\{i, j\} \otimes N') = 0$ always holds for $N' \in \phi_i$.

Next we suppose that $N' \in \phi_{ij}$. Then either $\{N, M\{i, j\}\}$ is a brick set or $\mathrm{Hom}(N, M\{i, j\}) \cong \Bbbk$. By Lemma 8.12(3), $\mathrm{Ext}^1(N, M\{i, j\} \otimes N') = 0$ since $M\{i, j\} \otimes N' = M\{i, j\}$.

The assertion follows. □

Now, we prove Theorem 0.8(3).

Theorem 8.16 *Let Q be a quiver of type $\mathbb{A}_n$ given in (E0.7.1) for some $n \geq 2$. Then*

$$\mathrm{fpd}(M\{i, j\}[1]) = \begin{cases} \min\{i-1, n-j\} & \text{if } M\{i, j\} \text{ is a sink,} \\ 1 & \text{if } M\{i, j\} \text{ is a source,} \\ 1 & \text{if } M\{i, j\} \text{ is a flow.} \end{cases}$$

Proof Since this is a statement about the derived category $D^b(\mathrm{rep}(Q))$, we need to consider all brick objects in this derived category. However, by the argument given in the proof of Lemma4.11 (2), we only need to consider brick sets of the form

$$\phi = \{N_1, \ldots, N_m \mid N_s \in \mathrm{rep}(Q)\}$$

which consists of objects in the abelian category $\mathrm{rep}(Q)$.

The rest of the proof is somewhat similar to the proof of Theorem 8.11.

If there exists an object $N_1 = M\{i_0, j_0\} \in \phi$ satisfying $M\{i, j\} \otimes N_1 \cong N_1$, by Lemma 8.2, there exist at most one object $N_2 = M\{i_1, j_1\} \in \phi$ satisfying $j_1 = i_0 - 1$ and at most one object $N_3 = M\{i_2, j_2\} \in \phi$ satisfying $i_2 = j_0 + 1$. Then, by Lemmas 8.1 and 8.12 , in the first column and the first row of $A(\phi, M\{i, j\}[1] \otimes -)$, all entries are zero except for $a_{12}, a_{21}, a_{13}, a_{31}$, and $a_{12}a_{21} = a_{13}a_{31} = 0$. No matter which case is, we always have

$$\rho(A(\phi, M\{i, j\}[1] \otimes -)) = \rho(A(\phi \backslash \{N_1\}, M\{i, j\}[1] \otimes -)).$$

Also, if there is an object $N \in \phi$ satisfying $M\{i, j\} \otimes N = 0$, we also have

$$\rho(A(\phi, M\{i, j\}[1] \otimes -)) = \rho(A(\phi \backslash \{N\}, M\{i, j\}[1] \otimes -)).$$

Similar to the proof of Theorem 8.11, it suffices to consider the brick set ϕ satisfying the following conditions:

(*) $\phi = \phi_i \cup \phi_j \cup \phi_{ij}$,
(**) for every $N \in \phi$, $M\{i, j\} \otimes N \ncong N$.

By Lemma 8.14, if we re-arrange objects in ϕ as $\hat{\phi}_j$, $\hat{\phi}_i$, ϕ_{ij}, $\tilde{\phi}_i$ and $\tilde{\phi}_j$, then $A(\phi, M\{i, j\}[1] \otimes -)$ is a block lower triangular matrix. By Lemma 8.15,

$$\rho(\phi, M\{i, j\}[1] \otimes -) = \rho(\phi_{ij}, M\{i, j\}[1] \otimes -).$$

Therefore, for the rest we consider the brick set ϕ satisfying $\phi_{ij} = \phi$.

We divide the rest of the proof into 3 small cases.

Case 1: $M\{i, j\}$ is a source. In this case, for any $N \in \phi$, $\mathrm{Hom}(N, M\{i, j\}) \cong \Bbbk$. Then by Lemma 8.12(3), $\mathrm{Ext}^1(N, M\{i, j\}) = 0$. As a consequence, the adjacency matrix $A(\phi, M\{i, j\}[1] \otimes -)$ is a zero matrix. Therefore, in this case, $\mathrm{fpd}(M\{i, j\}[1]) = 0$.

Case 2: $M\{i, j\}$ is a flow, without loss of generality, assume that $\alpha_{i-1} = \alpha_j = \longleftarrow$. For any $N = M\{i_1, j_1\} \in \phi$, if $i_1 = i$, by Lemma 8.12(4), $\mathrm{Ext}^1(N, M\{i, j\}) = 0$. If $i_1 < i$, either $\mathrm{Hom}(N, M\{i, j\}) \cong \Bbbk$ or $\{N, M\{i, j\}\}$ is a brick set, then by Lemma 8.12(3), $\mathrm{Ext}^1(N, M\{i, j\}) = 0$. As a consequence, the adjacency matrix $A(\phi, M\{i, j\}[1] \otimes -)$ is a zero matrix. Therefore, in this case, $\mathrm{fpd}(M\{i, j\}[1]) = 0$.

Case 3: $M\{i, j\}$ is a sink. In this case, for any $N = M\{i_1, j_1\} \in \phi$, if $i_1 = i$, by Lemma 8.12(4), $\mathrm{Ext}^1(N, M\{i, j\}) = 0$. If $j_1 = j$, by Lemma 8.12(4), $\mathrm{Ext}^1(N, M\{i, j\}) = 0$. Therefore, since $M\{i, j\} \otimes N = M\{i, j\}$, we can assume that $i_1 < i$ and $j_1 > j$. Now, it's easy to see $\mathrm{Ext}^1(N, M\{i, j\}) \cong \Bbbk$ by Lemma 8.12(3). As a consequence, all entries in the adjacency matrix $A(\phi, M\{i, j\}[1] \otimes -)$ are 1 and $\rho(A(\phi, M\{i, j\}[1] \otimes -)) = |\phi_{ij}| \leq \min\{i-1, n-j\}$.

On the other hand, by Lemma 8.6, $(\{M\{k, i\} \mid 1 \leq k \leq i-1\}, \succ)$ and $(\{M\{j, m\} \mid j+1 \leq m \leq n\}, \succ)$ are two totally ordered sets. We list elements in these two sets as

$$M\{k_1, i\} \succ \cdots \succ M\{k_{i-1}, i\} \text{ and} M\{j, m_1\} \succ \cdots \succ M\{j, m_{n-j}\}$$

where $\{k_l\}_{l=1}^{i-1}$ and $\{m_l\}_{l=1}^{n-j}$ are distinct integers from 1 to $i-1$ and from $j+1$ to n respectively. Let $d = \min\{i-1, n-j\}$, then we have a set of d elements

$$\phi = \{M\{k_1, m_{n-j}\}, M\{k_2, m_{n-j-1}\}, \ldots, M\{k_d, m_{n-j+1-d}\}\}$$

which is a brick set. Using this brick set, one see every entry in the matrix $A(\phi, M\{i, j\}[1] \otimes -)$ is 1 by Lemma 8.12, consequently, $\rho(A(\phi, M\{i, j\}[1] \otimes -)) = d$. Hence, in this case, $\mathrm{fpd}(M\{i, j\}[1]) = \min\{i-1, n-j\}$. □

Proof of Theorem 0.8: (1) This follows from Lemma 4.11(1).
(2) This follows from Lemma 4.11(2) and Theorem 8.11.
(3) This follows from Theorem 8.16. □

Acknowledgements The authors thank the referee for his/her very careful reading and valuable comments and thank Professors Jianmin Chen and Xiao-Wu Chen for many useful conversations on the subject. J. J. Zhang was partially supported by the US National Science Foundation (Grant Nos. DMS-1700825 and DMS-2001015). J.-H. Zhou was partially supported by Fudan University Exchange Program Scholarship for Doctoral Students (Grant No. 2018024).

References

1. Ariki, S.: Hecke algebras of classical type and their representation type. Proc. Lond. Math. Soc. (3) **91**(2), 355–413 (2005)
2. Artin, M., Zhang, J.J.: Noncommutative projective schemes. Adv. Math. **109**, 228–287 (1994)
3. Asai, S.: Semibricks. Int. Math. Res. Not. IMRN **16**, 4993–5054 (2020)
4. Assem, I., Simson, D., Skowroński, A.: Elements of the representation theory of associative algebras, vol. 1, Lond. Math. Soc., p. 65 (2006)
5. Balmer, P.: The spectrum of prime ideals in tensor triangulated categories. J. Reine Angew. Math. **588**, 149–168 (2005)
6. Bao, Y.-H., Xu, X.-W., Ye, Y., Zhang, J.J., Zhao, Z.-B.: Operads associated to weak bialgebras (2021) **(in preparation)**
7. Beilinson, A.A., Bernstein, J., Deligne, P.: Faisceaux pervers. In: Analysis and Topology on Singular Spaces, I (Luminy, 1981), Astérisque, vol. 100, pp. 5–171. Soc. Math. France, Paris (1982)
8. Bernstein, I.N., Gelfand, I.M., Ponomarev, V.A.: Coxeter functors and Gabriel theorem. Uspehi Mat. Nauk **28**(2), 19–33 (1973)
9. Bondal, A., Van den Bergh, M.: Generators and representability of functors in commutative and noncommutative geometry. Mosc. Math. J. **3**(1), 1–36 (2003). (**258**)
10. Böhm, G., Caenepeel, S., Janssen, K.: Weak bialgebras and monoidal categories. Commun. Algebra **39**, 4584–4607 (2011)
11. Böhm, G., Nill, F., Szlachányi, K.: Weak Hopf algebras I. Integral theory and $C^a st$-structures. J. Algebra **221**, 385–438 (1999)
12. Brüning, K., Burban, I.: Coherent sheaves on an elliptic curve (English summary). In: Interactions Between Homotopy Theory and Algebra, vol. 436, pp. 297–315. Contemp. Math. Amer. Math. Soc., Providence (2007)
13. Butler, M.C.R., Ringel, C.M.: Auslander–Reiten sequences with few middle terms and applications to string algebras. Commun. Algebra **15**(1–2), 145–179 (1987)
14. Chen, J.M., Gao, Z.B., Wicks, E., Zhang, J.J., Zhang, X.-H., Zhu, H.: Frobenius–Perron theory of endofunctors. Algebra Number Theory **13**(9), 2005–2055 (2019)
15. Chen, J.M., Gao, Z.B., Wicks, E., Zhang, J. J., Zhang, X-.H., Zhu, H.: Frobenius–Perron theory for projective schemes (2019). arXiv:1907.02221
16. Chen, X.-W., Ringel, C.M.: Hereditary triangulated categories. J. Noncommut. Geom. **12**(4), 1425–1444 (2018)
17. Drozd, J.A., Tame and wild matrix problems, Representation theory, II. In: Proc. Second Internat. Conf., Carleton Univ., Ottawa, Ont., 1979, Lecture Notes in Math., vol. 832, pp. 242–258. Springer, Berlin (1980)
18. Etingof, P., Gelaki, S., Nikshych, D., Ostrik, V.: Tensor Categories, Math. Surveys and Monographs vol. 205. Amer. Math. Soc., Providence (2015)
19. Etingof, P., Nikshych, D., Ostrik, V.: On fusion categories. Ann. Math. (2) **162**(2), 581–642 (2005)
20. Etingof, P., Ostrik, V.: Finite tensor categories. Mosc. Math. J. (3) **4**(627–654), 782–783 (2005)
21. Gabriel, P.: Des catégories abéliennes. Bull. Soc. Math. France **90**, 323–448 (1962)
22. Gabriel, P.: Unzerlegbare Darstellungen. I. Manuscr. Math. **6**, 71–103 (1972)
23. Gabriel, P.: Représentations indécomposables, (French) Séminaire Bourbaki, 26e ann'ee (1973/1974), Exp. No. 444. Lecture Notes in Math., Vol. 431, pp. 143–169. Springer, Berlin (1975)
24. Gabriel, P.: Indecomposable Representations. II. Symposia Mathematica, vol. XI, pp. 81–104. Academic, London (1973)
25. Gabriel, P., Rouiter, A.V.: Representations of finite-dimensional algebras, with a chapter by B. Keller, Encyclopaedia Math. Sci., Algebra, VIII, vol. 73, pp. 1–177. Springer, Berlin (1992)
26. Geigle, W., Lenzing, H.: A class of weighted projective curves arising in representation theory of finite-dimensional algebras, Singularities, representation of algebras, and vector bundles (Lambrecht, 1985), Lecture Notes in Math., vol. 265–297, p. 1273. Springer, Berlin (1987)

27. Geršgorin, S.: Uber die Abgrenzung der Eigenwerte einer Matrix. Bull. Acad. Sc. Leningrad **6**, 749–754 (1931)
28. Happel, D.: On the derived category of a finite-dimensional algebra. Comment. Math. Helv. **62**(3), 339–389 (1987)
29. Herschend, M.: Solution to the Clebsch–Gordan problem for representations of quivers of type $ildemathbbA_n$. J. Algebra Appl. (5) **4**, 481–488 (2005)
30. Herschend, M.: Tensor products on quiver representations. J. Pure Appl. Algebra (2) **212**, 453–469 (2008)
31. Herschend, M.: On the representation rings of quivers of exceptional Dynkin type. Bull. Sci. Math. (5) **132**, 395–418 (2008)
32. Herschend, M.: On the representation ring of a quiver. Algebras Represent. Theory (6) **12**, 513–541 (2009)
33. Herschend, M.: Solution to the Clebsch–Gordan problem for string algebras. J. Pure Appl. Algebra (11) **214**, 1996–2008 (2010)
34. Hille, L., Perling, M.: Exceptional sequences of invertible sheaves on rational surfaces. Compos. Math. **147**(4), 1230–1280 (2011)
35. Huang, H.-L., Torrecillas, B.: Quiver bialgebras and monoidal categories. Colloq. Math. **131**(2), 287–300 (2013)
36. Igusa, K.: Notes on the no loops conjecture. J. Pure Appl. Algebra **69**(2), 161–176 (1990)
37. Janelidze, G., Kelly, G.M.: A note on actions of a monoidal category. In: CT2000 Conference (Como)., Theory Appl. Categ., vol. 9, pp. 61–91 (2001/02)
38. Keller, B.: Derived categories and tilting, Handbook of tilting theory. London Math. Soc. Lecture Note Ser., vol. 332, pp. 49–104. Cambridge Univ. Press, Cambridge (2007)
39. Keller, B., Vossieck, D.: Aisles in derived categories. Deuxiéme Contact Franco-Belge en Algébre (Faulx-les-Tombes, 1987). Bull. Soc. Math. Belg. Sér. A **40**(2), 239–253 (1988)
40. Kinser, R.: Rank functions on rooted tree quivers. Duke Math. J. **152**(1), 27–92 (2010)
41. Kinser, R., Schiffler, R.: Idempotents in representation rings of quivers. Algebra Number Theory **6**(5), 967–994 (2012)
42. Koenig, S., Yang, D.: Silting objects, simple-minded collections, t-structures and co-t-structures for finite-dimensional algebras. Doc. Math. **19**, 403–438 (2014)
43. Krause, H., Vossieck, D.: Length categories of infinite height. In: Geometric and Topological Aspects of the Representation Theory of Finite Groups, Springer Proc. Math. Stat., vol. 242, pp. 213–234. Springer, Cham (2018)
44. Kussin, D., Lenzing, H., Meltzer, H.: Triangle singularities, ADE-chains, and weighted projective lines. Adv. Math. **237**, 194–251 (2013)
45. Lenzing, H.: Weighted projective lines and applications. In: Representations of Algebras and Related Topics, pp. 153–187. EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich (2011)
46. Lenzing, H., Meltzer, H.: Sheaves on a weighted projective line of genus one, and representations of a tubular algebra, Representations of algebras (Ottawa, Canada, 1992). CMS Conf. Proc., vol. 14, pp. 313–337 (1994)
47. Lenzing, H., Reiten, I.: Hereditary Noetherian categories of positive Euler characteristic. Math. Z. **254**(1), 133–171 (2006)
48. Manin, Y.I.: Quantum Groups and Noncommutative Geometry, 2nd edition. With a contribution by Theo Raedschelders and Michel Van den Bergh. CRM Short Courses. Centre de Recherches Mathématiques, [Montreal], QC. Springer, Cham (2018)
49. Minamoto, H., Mori, I.: The structure of AS-Gorenstein algebras. Adv. Math. **226**(5), 4061–4095 (2011)
50. Mori, I.: B-construction and C-construction. Commun. Algebra **41**(6), 2071–2091 (2013)
51. Nakano, D.K., Vashaw, K.B., Yakimov, M.T.: Noncommutative tensor triangular geometry (2019). arXiv:1909.04304
52. Nazarova, L.A.: Representations of quivers of infinite type. Izvestiya Akademii Nauk SSSR. Seriya Matematicheskaya **37**, 752–791 (1973)
53. Nikshych, D., Turaev, V., Vainerman, L.: Invariants of knots and 3-manifolds from quantum groupoids. In: Proceedings of the Pacific Institute for the Mathematical Sciences Workshop “Invariants of Three-Manifolds”, vol. 127, pp. 91–123 (2003)
54. Nikshych, D., Vainerman, L.: Finite quantum groupoids and their applications, New directions in Hopf algebras, Math. Sci. Res. Inst. Publ., vol. 43, pp. 211–262. Cambridge Univ. Press, Cambridge (2002)
55. Nill, F.: Axioms for weak bialgebras. arXiv:math/9805104
56. Ringel, C.M.: Tame Algebras and Integral Quadratic Forms. Lecture Notes in Mathematics, vol. 1099. Springer, Berlin (1984)
57. Rogalski, D., Won, R., Zhang, J.J.: A proof of the Brown–Goodearl conjecture for module-finite weak Hopf algebras. Algebra Number Theory **15**(4), 971–997 (2021)

58. Schiffmann, O.: Lectures on Hall Algebras, Geometric Methods in Representation Theory. Soc. Math. France, Paris (2012)
59. Stanley, D., van Roosmalen, A.C.: Derived equivalences for hereditary Artin algebras. Adv. Math. **303**, 415–463 (2016)
60. Strassen, V.: Asymptotic degeneration of representations of quivers. Comment. Math. Helv. (4) **75**, 594–607 (2000)
61. Suárez-Álvarez, M.: The Hilton–Eckmann argument for the anti-commutativity of cup products. Proc. Am. Math. Soc. **132**(8), 2241–2246 (2004)
62. Vashaw, K., Yakimov, M.: Prime spectra of abelian 2-categories and categorifications of Richardson varieties. In: Representations and Nilpotent Orbits of Lie Algebraic Systems, Progr. Math., vol. 330, pp. 501–553. Birkhäuser/Springer, Cham (2019)
63. Wu, J., Liu, G., Ding, N.: Classification of affine prime regular Hopf algebras of GK-dimension one. Adv. Math. **296**, 1–54 (2016)
64. Zhou, J.H., Wang, Y.H., Ding, J.R.: Frobenius–Perron dimension of representations of a class of D-type quivers (in Chinese). Sci. Sin. Math. **51**, 673–684 (2021). https://doi.org/10.1360/SSM-2020-0093