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## CANCELLATION PROBLEM FOR AS-REGULAR ALGEBRAS OF DIMENSION THREE

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**We study a noncommutative version of the Zariski cancellation problem for some classes of connected graded Artin–Schelter regular algebras of global dimension three.**

### Introduction

The classical Zariski cancellation problem for commutative polynomial rings has a long history; see the very nice survey paper [Gupta 2015]. A noncommutative version of the Zariski cancellation problem was investigated as early as 1970s, see [Coleman and Enochs 1971; Brewer and Rutter 1972], and was reintroduced in [Bell and Zhang 2017a]. During the past few years several research groups have been making significant contributions to this topic; see, for example, [Bell et al. 2021; Bell and Zhang 2017a; 2017b; Brown and Yakimov 2018; Ceken et al. 2015; 2016; Chan et al. 2016; 2018; Gaddis 2017; Gaddis et al. 2017; 2019b; Levitt and Yakimov 2018; Lezama et al. 2019; Lu et al. 2020; Lü et al. 2017; Nguyen et al. 2017; Tang 2017; 2018; Tang et al. 2020; Wang and Zhang 2018]. Very recently, the Zariski cancellation problem was introduced for commutative Poisson algebras by Gaddis and Wang [2020].

We are following the terminology introduced in [Gupta 2015; Bell and Zhang 2017a]. Recall that an algebra  $A$  is called *cancellative* if any algebra isomorphism

$$A[t] \cong B[t]$$

of polynomial extensions for some algebra  $B$  implies that

$$A \cong B.$$

The famous Zariski cancellation problem (abbreviated as ZCP) asks if

*the commutative polynomial ring  $\mathbb{k}[x_1, \dots, x_n]$  over a field  $\mathbb{k}$  is cancellative*

for  $n \geq 1$ ; see [Kraft 1996; Gupta 2015; Bell and Zhang 2017a]. It is well-known that  $\mathbb{k}[x_1]$  is cancellative by a result of Abhyankar, Eakin and Heinzer [Abhyankar

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et al. 1972]. For  $n = 2$ ,  $\mathbb{k}[x_1, x_2]$  is cancellative by a result of Fujita [1979] and Miyanishi and Sugie [1980] in characteristic zero and by a result of Russell [1981] in positive characteristic. The ZCP for  $n \geq 3$  has been open for many years. A major breakthrough in this research area is a remarkable result of Gupta [2014a; 2014b], which settled the ZCP negatively in positive characteristic for  $n \geq 3$ . *The ZCP in characteristic zero remains open for  $n \geq 3$ .* Examples of noncancellative algebras were given by Hochster [1972], Danielewski [1989] and Gupta [2014a; 2014b], and can be found in [Lu et al. 2020, Example 2.5].

Our main goal is to study the ZCP for noncommutative noetherian connected graded Artin–Schelter regular algebras [Definition 0.1] of global dimension three.

**Definition 0.1** [Artin and Schelter 1987, p. 171]. A connected graded algebra  $A$  is called *Artin–Schelter Gorenstein* (or *AS Gorenstein*, for short) if the following conditions hold:

- (a)  $A$  has injective dimension  $d < \infty$  on the left and on the right,
- (b)  $\text{Ext}_A^i(A\mathbb{k}, {}_A A) = \text{Ext}_A^i(\mathbb{k}_A, A_A) = 0$  for all  $i \neq d$ , and
- (c)  $\text{Ext}_A^d(A\mathbb{k}, {}_A A) \cong \text{Ext}_A^d(\mathbb{k}_A, A_A) \cong \mathbb{k}(I)$  for some integer  $I$ .

If in addition,

- (d)  $A$  has finite global dimension, and
- (e)  $A$  has finite Gelfand–Kirillov dimension,

then  $A$  is called *Artin–Schelter regular* (or *AS regular*, for short) of dimension  $d$ .

AS-regular algebras are considered as a noncommutative analogue of the commutative polynomial rings. We refer to [Levasseur 1992; Lü et al. 2017; Stafford 1994] for related concepts such as the Auslander regularity and the Cohen–Macaulay property which will be used later in Proposition 0.6. It is well-known that the only AS-regular algebra of global dimension one is the polynomial ring  $\mathbb{k}[x_1]$ . Combining classical results in [Fujita 1979; Miyanishi and Sugie 1980] with [Bell and Zhang 2017a, Theorem 0.5], every AS-regular algebra of global dimension two (over a base field of characteristic zero) is cancellative. On the other hand, by the results in [Gupta 2014a; 2014b], not every AS-regular algebra of global dimension three (or higher) is cancellative. Therefore it is natural and sensible to ask which AS-regular algebras of global dimension three (or higher) are cancellative. In [Lü et al. 2017, Corollary 0.9], the authors showed that several classes of AS-regular algebras of global dimension three are cancellative. We say  $A$  is *PI* if it satisfies a polynomial identity. Our first result is:

**Theorem 0.2.** *Suppose  $\text{char } \mathbb{k} = 0$ . Let  $A$  be a noetherian connected graded AS-regular algebra of global dimension three that is generated in degree 1. If  $A$  is not PI, then it is cancellative.*

**Theorem 0.2** covers [Lü et al. 2017, Corollary 0.9]. For an algebra  $A$ , let  $Z(A)$  denote the center of  $A$ . Let  $\text{GKdim } A$  (respectively,  $\text{gldim } A$ ) be the Gelfand–Kirillov dimension (respectively, the global dimension) of  $A$ . For AS-regular algebras of higher global dimension, we have the following.

**Theorem 0.3.** *Suppose  $\text{char } \mathbb{k} = 0$ . Let  $A$  be a noetherian connected graded domain of finite global dimension that is generated in degree 1. Suppose that*

- (a)  $\text{GKdim } Z(A) \leq 1$ , and
- (b)  $\text{gldim } A/(t) = \infty$  for every homogeneous central element  $t$  in  $Z(A)$  of positive degree.

*Then  $A$  is cancellative.*

We will also show a result similar to **Theorem 0.3** for graded isolated singularities which have infinite global dimension; see **Theorem 4.3**. For a general noncommutative algebra we have the following conjecture, which extends both **Theorems 0.2** and **0.3**.

**Conjecture 0.4.** *Suppose  $\text{char } \mathbb{k} = 0$ . Let  $A$  be a noetherian finitely generated prime algebra.*

- (1) *If  $\text{GKdim } Z(A) \leq 1$ , then  $A$  is cancellative.*
- (2) *If  $\text{GKdim } A = 3$  and  $A$  is not PI, then  $A$  is cancellative.*

Originally this conjecture was made without the hypothesis that  $\text{char } \mathbb{k} = 0$ . When  $\text{char } \mathbb{k} > 0$ , a counterexample to part (1) was given in [Bell et al. 2021, Theorem 1.1(b)]. See [Bell et al. 2021, Question 4.1] for a modified and weaker version of **Conjecture 0.4**(1) in positive characteristic.

The cancellation property of Veronese subrings of skew polynomial rings was considered in [Chan et al. 2018]. We have the following improvement of **Theorem 0.2** concerning the Veronese subrings which provides some evidence for **Conjecture 0.4**.

**Corollary 0.5.** *Suppose  $\text{char } \mathbb{k} = 0$ . Let  $A$  be a noetherian connected graded AS-regular algebra of global dimension three that is generated in degree 1. If  $A$  is not PI, then the  $d$ -th Veronese subring  $A^{(d)}$  of  $A$  is cancellative for every  $d \geq 1$ .*

The proofs of **Theorems 0.2** and **0.3** are related to the following result that establishes that the center of the algebras in these two theorems is either  $\mathbb{k}$  or  $\mathbb{k}[t]$ .

**Proposition 0.6.** *Let  $A$  be a noetherian connected graded Auslander regular Cohen–Macaulay algebra. If  $\text{GKdim } Z(A) \leq 1$ , then  $Z(A)$  is either  $\mathbb{k}$  or  $\mathbb{k}[t]$ .*

When  $Z(A) = \mathbb{k}[t]$ , we can use **Theorem 1.5** which was proved in [Lu et al. 2020]. And we have a question along this line.

**Question 0.7.** Let  $A$  be a noetherian connected graded Auslander regular algebra. If  $\text{GKdim } Z(A) = 2$ , what can we say about the center  $Z(A)$ ? For example, is  $Z(A)$  always noetherian in this case?

As in the commutative case, it is usually difficult to determine whether or not an AS-regular algebra is cancellative. For example, we are unable to answer the following question.

**Question 0.8.** Let  $q \in \mathbb{k} \setminus \{0, 1\}$  be a root of unity. Is the skew polynomial ring of three variables  $\mathbb{k}_q[x_1, x_2, x_3]$  (or of odd number variables  $\mathbb{k}_q[x_1, x_2, \dots, x_{2n+3}]$ ) cancellative?

When  $q = 1$  and  $\text{char } \mathbb{k} = 0$ , the above question is the classical ZCP which has been open for many years [Gupta 2015]. Note that if  $q = 1$  and  $\text{char } \mathbb{k} > 0$ , then  $\mathbb{k}[x_1, \dots, x_n]$  for  $n \geq 3$  is not cancellative by [Gupta 2014a; 2014b]. Question 0.8 is a special case of [Chan et al. 2018, Question 1.5] which was stated for a larger class of rings, namely, for Veronese subrings of the skew polynomial rings. Surprisingly, if  $q \in \mathbb{k} \setminus \{0, 1\}$ , then the skew polynomial ring of even number variables  $\mathbb{k}_q[x_1, \dots, x_{2n}]$  is cancellative by [Bell and Zhang 2017a, Theorem 0.8(a)].

Several new methods were introduced to deal with the noncommutative version of the ZCP. For example, methods of discriminants and Makar-Limanov invariants were introduced and used in [Bell and Zhang 2017a]. In [Lezama et al. 2019], the retractability and detectability were introduced to relate the cancellation property. In [Lü et al. 2017], Nakayama automorphisms were used to show some classes of algebras are cancellative. In [Lu et al. 2020], Azumaya locus and  $\mathcal{P}$ -discriminant methods were introduced to study the cancellation property. One should continue to look for new invariants and methods to handle the algebras given in Question 0.8.

The paper is organized as follows. Section 1 contains definitions and preliminaries. In Section 2, we prove some preliminary results necessary for the last section. In Section 3, we present some lemmas related to cancellation problem. As an application we establish that the universal enveloping algebra of any 3-dimensional nonabelian Lie algebra is cancellative (Example 3.10). Theorems 0.2, 0.3 and Corollary 0.5 are proven in Section 4.

## 1. Definitions and Preliminaries

We recall some definitions from [Bell and Zhang 2017a; Lezama et al. 2019; Lu et al. 2020]. Let  $\mathbb{k}$  be a base field that is algebraically closed. Objects in this paper are  $\mathbb{k}$ -linear.

**Definition 1.1** [Bell and Zhang 2017a, Definition 1.1]. Let  $A$  be an algebra.

- (1) We call  $A$  *cancellative* if any algebra isomorphism  $A[t] \cong B[s]$  for some algebra  $B$  implies that  $A \cong B$ .

(2) We call  $A$  *strongly cancellative* if, for each  $n \geq 1$ , any algebra isomorphism

$$A[t_1, \dots, t_n] \cong B[s_1, \dots, s_n]$$

for some algebra  $B$  implies that  $A \cong B$ .

For any algebra  $A$ , let  $M(A)$  denote the category of right  $A$ -modules.

**Definition 1.2** [Lu et al. 2020, Definition 2.2]. Let  $A$  be an algebra.

(1) We call  $A$  *m-cancellative* if any equivalence of abelian categories  $M(A[t]) \cong M(B[s])$  for some algebra  $B$  implies that  $M(A) \cong M(B)$ .

(2) We call  $A$  *strongly m-cancellative* if, for each  $n \geq 1$ , any equivalence of abelian categories

$$M(A[t_1, \dots, t_n]) \cong M(B[s_1, \dots, s_n])$$

for some algebra  $B$  implies that  $M(A) \cong M(B)$ .

The letter  $m$  here stands for the word “Morita”.

This Morita version of the cancellation property is a natural generalization of the original Zariski cancellation property when we study noncommutative algebras.

Let  $Z$  be a commutative ring over the base field  $\mathbb{k}$ , which is usually the center of a noncommutative algebra. We now recall the definition of  $\mathcal{P}$ -discriminant for a property  $\mathcal{P}$ . Let  $\text{Spec } Z$  denote the prime spectrum of  $Z$  and let

$$\text{MaxSpec}(Z) := \{\mathfrak{m} \mid \mathfrak{m} \text{ is a maximal ideal of } Z\}$$

denote the maximal spectrum of  $Z$ . For any  $S \subseteq \text{Spec } Z$ ,  $I(S)$  is the ideal of  $Z$  vanishing on  $S$ , namely,

$$I(S) = \bigcap_{\mathfrak{p} \in S} \mathfrak{p}.$$

For any algebra  $A$ ,  $A^\times$  denotes the set of invertible elements in  $A$ . A property  $\mathcal{P}$  considered in the following means a property defined on a class of algebras that is an invariant under algebra isomorphisms.

**Definition 1.3** [Lu et al. 2020, Definition 3.3]. Let  $A$  be an algebra,  $Z := Z(A)$  be the center of  $A$ . Let  $\mathcal{P}$  be a property defined for  $\mathbb{k}$ -algebras (not necessarily a Morita invariant).

(1) The  $\mathcal{P}$ -locus of  $A$  is defined to be

$$L_{\mathcal{P}}(A) := \{\mathfrak{m} \in \text{MaxSpec}(Z) \mid A/\mathfrak{m}A \text{ has the property } \mathcal{P}\}.$$

(2) The  $\mathcal{P}$ -discriminant set of  $A$  is defined to be

$$D_{\mathcal{P}}(A) := \text{MaxSpec}(Z) \setminus L_{\mathcal{P}}(A).$$

(3) The  $\mathcal{P}$ -discriminant ideal of  $A$  is defined to be

$$I_{\mathcal{P}}(A) := I(D_{\mathcal{P}}(A)) \subseteq Z.$$

- (4) If  $I_{\mathcal{P}}(A)$  is a principal ideal of  $Z$  generated by  $d \in Z$ , then  $d$  is called the  $\mathcal{P}$ -discriminant of  $A$ , denoted by  $d_{\mathcal{P}}(A)$ . In this case  $d_{\mathcal{P}}(A)$  is unique up to an element in  $Z^{\times}$ .
- (5) Let  $\mathcal{C}$  be a class of algebras over  $\mathbb{k}$ . We say that  $\mathcal{P}$  is  $\mathcal{C}$ -stable if for every algebra  $A$  in  $\mathcal{C}$  and every  $n \geq 1$ ,

$$I_{\mathcal{P}}(A \otimes \mathbb{k}[t_1, \dots, t_n]) = I_{\mathcal{P}}(A) \otimes \mathbb{k}[t_1, \dots, t_n]$$

as an ideal of  $Z \otimes \mathbb{k}[t_1, \dots, t_n]$ . If  $\mathcal{C}$  is a singleton  $\{A\}$ , we simply call  $\mathcal{P}$   $A$ -stable. If  $\mathcal{C}$  is the whole collection of  $\mathbb{k}$ -algebras with the center affine over  $\mathbb{k}$ , we simply call  $\mathcal{P}$  stable.

In general, neither  $L_{\mathcal{P}}(A)$  nor  $D_{\mathcal{P}}(A)$  is a subscheme of  $\text{Spec } Z(A)$ .

In this paper we will use another property that is closely related to the m-cancellative property.

Recall from the Morita theory that if  $A' := A[t_1, \dots, t_n]$  is Morita equivalent to  $B' := B[s_1, \dots, s_n]$ , then there is an  $(A', B')$ -bimodule  $\Omega$  that is invertible and induces naturally algebra isomorphisms  $A' \cong \text{End}(\Omega_{B'})$  and  $(B')^{op} \cong \text{End}_{(A')}\Omega$  such that

$$Z(A') \cong \text{Hom}_{(A', B')}(\Omega, \Omega) \cong Z(B').$$

The above isomorphism is denoted by

$$(1.3.1) \quad \omega : Z(A') \rightarrow Z(B').$$

The retractable property was introduced in [Lezama et al. 2019, Definitions 2.1 and 2.5] and the Morita  $Z$ -retractability in [Lu et al. 2020, Definition 3.6].

**Definition 1.4.** Let  $A$  be an algebra.

- (1) We call  $A$   $Z$ -retractable if, for any algebra  $B$ , any algebra isomorphism  $\phi : A[t] \cong B[s]$  implies that  $\phi(Z(A)) = Z(B)$ . If further  $\phi(A) = B$ , we just say  $A$  is *retractable*.
- (2) We call  $A$  *strongly  $Z$ -retractable* if, for any algebra  $B$  and integer  $n \geq 1$ , any algebra isomorphism  $\phi : A[t_1, \dots, t_n] \cong B[s_1, \dots, s_n]$  implies that  $\phi(Z(A)) = Z(B)$ . If further  $\phi(A) = B$ , we just say  $A$  is *strongly retractable*.
- (3) We call  $A$  *m- $Z$ -retractable* if, for any algebra  $B$ , any equivalence of categories  $M(A[t]) \cong M(B[s])$  implies that  $\omega(Z(A)) = Z(B)$ , where

$$\omega : Z(A)[t] \rightarrow Z(B)[s]$$

is given as in (1.3.1).

- (4) We call  $A$  *strongly  $m$ - $Z$ -retractable* if, for any algebra  $B$  and any  $n \geq 1$ , any equivalence of categories  $M(A[t_1, \dots, t_n]) \cong M(B[s_1, \dots, s_n])$  implies that  $\omega(Z(A)) = Z(B)$ , where

$$\omega : Z(A)[t_1, \dots, t_n] \rightarrow Z(B)[s_1, \dots, s_n]$$

is given as in (1.3.1).

The following theorem was proved in [Lu et al. 2020, Corollary 3.11 and Lemma 4.4] and will be used several times in later sections.

**Theorem 1.5.** *Let  $A$  be a noetherian algebra such that its center  $Z(A)$  is  $\mathbb{k}[x]$ . Let  $\mathcal{P}$  be a Morita invariant property (respectively, stable property) such that the  $\mathcal{P}$ -discriminant of  $A$ , denoted by  $d$ , is a nonzero noninvertible element in  $Z(A)$ . Then  $A$  is strongly  $m$ - $Z$ -retractable (respectively, strongly  $Z$ -retractable) and strongly  $m$ -cancellative (respectively, strongly cancellative).*

*Proof.* By [Lu et al. 2020, Lemma 6.1],  $\mathcal{P}$  is stable (when  $\mathbb{k}$  is algebraically closed). The assertion follows from [Lu et al. 2020, Corollary 3.11 and Lemma 4.4]  $\square$

## 2. Results not involving cancellation properties

In this section we collect some results that do not directly involve cancellation properties, but are needed in later sections. In the next section, we collect some lemmas that are directly related to cancellation properties.

**Lemma 2.1.** *Let  $\mathbb{k}$  be a field of characteristic zero and  $q \neq 1$  be a nonzero scalar in  $\mathbb{k}$ . The following hold.*

- (1) *Algebras  $\mathbb{k}\langle x, y \rangle / (xy - yx)$ ,  $\mathbb{k}\langle x, y \rangle / (xy - yx - 1)$  and  $\mathbb{k}\langle x, y \rangle / (xy - yx - x)$  are pairwise not Morita equivalent.*
- (2) *If  $q$  is not a root of unity,  $\mathbb{k}\langle x, y \rangle / (xy - qyx)$  is not Morita equivalent to  $\mathbb{k}\langle x, y \rangle / (xy - yx - 1)$ .*
- (3) *The Jordan plane (algebra)  $\mathbb{k}\langle x, y \rangle / (xy - yx + x^2)$  is not Morita equivalent to  $\mathbb{k}\langle x, y \rangle / (xy - yx + x^2 - 1)$ .*
- (4) *The Jordan plane (algebra)  $\mathbb{k}\langle x, y \rangle / (xy - yx + x^2)$  is not Morita equivalent to  $\mathbb{k}\langle x, y \rangle / (xy - yx + x^2 - x)$ .*
- (5) *The Jordan plane (algebra)  $\mathbb{k}\langle x, y \rangle / (xy - yx + x^2)$  is not Morita equivalent to  $\mathbb{k}\langle x, y \rangle / (xy - yx + x^2 - y)$ .*

*Proof.* (1) First of all  $\mathbb{k}\langle x, y \rangle / (xy - yx)$  and  $\mathbb{k}\langle x, y \rangle / (xy - yx - x)$  have global dimension two while  $\mathbb{k}\langle x, y \rangle / (xy - yx - 1)$  has global dimension one. So either the algebra  $\mathbb{k}\langle x, y \rangle / (xy - yx)$  or the algebra  $\mathbb{k}\langle x, y \rangle / (xy - yx - x)$  is not Morita equivalent to  $\mathbb{k}\langle x, y \rangle / (xy - yx - 1)$ . Second, the centers of  $\mathbb{k}\langle x, y \rangle / (xy - yx)$



and  $\mathbb{k}\langle x, y \rangle / (xy - yx - x)$  are nonisomorphic, so these algebras are not Morita equivalent.

(2) Let  $A := \mathbb{k}\langle x, y \rangle / (xy - qyx)$  and  $B := \mathbb{k}\langle x, y \rangle / (xy - qyx - 1)$ . Suppose on the contrary that  $A$  is Morita equivalent to  $B$ . Let  $J$  be the height one prime ideal of  $B$  generated by  $(1 - q)xy - 1$  such that  $B/J = \mathbb{k}[x^{\pm 1}]$  (with the image of  $y$  being  $(1 - q)^{-1}x^{-1}$ ). Since  $A$  is Morita equivalent to  $B$ , there is an ideal  $I$  of  $A$  such that  $A/I$  is Morita equivalent to  $\mathbb{k}[x^{\pm 1}]$ . Since every projective module over  $\mathbb{k}[x^{\pm 1}]$  is free,  $A/I$  is a matrix algebra over  $\mathbb{k}[x^{\pm 1}]$ . When  $q$  is not a root of unity, the only height one prime ideals  $I$  of  $A$  are  $(x)$  or  $(y)$  [Brown and Goodearl 2002, Example II.1.2]. In both cases,  $A/I$  is isomorphic to  $\mathbb{k}[t]$ , which is not a matrix algebra over  $\mathbb{k}[x^{\pm 1}]$ . This yields a contradiction and therefore  $A$  is not Morita equivalent to  $B$ .

(3) Let  $A := \mathbb{k}\langle x, y \rangle / (xy - yx + x^2)$  and  $B := \mathbb{k}\langle x, y \rangle / (xy - yx + x^2 - 1)$  by recycling notation from the proof of part (2) and suppose on the contrary that  $A$  is Morita equivalent to  $B$ . Let  $J_{\pm}$  be the height one prime ideals of  $B$  generated by  $(xy - yx, x \pm 1)$ . Since  $A$  is Morita equivalent to  $B$ , there is an ideal  $I_{\pm}$  of  $A$  such that  $A/I_{\pm}$  is Morita equivalent to  $B/J_{\pm}$ . Since  $\text{char } \mathbb{k} = 0$ ,  $A$  has only a single height one prime that is  $(x)$  [Shirikov 2005, Theorem 2.4]. This yields a contradiction. Therefore  $A$  is not Morita equivalent to  $B$ .

(4) The assertion follows from part (3) because  $\mathbb{k}\langle x, y \rangle / (xy - yx + x^2 - 1) \cong \mathbb{k}\langle x, y \rangle / (xy - yx + x^2 - x)$ .

(5) Let  $A := \mathbb{k}\langle x, y \rangle / (xy - yx + x^2)$  and  $B := \mathbb{k}\langle x, y \rangle / (xy - yx + x^2 - y)$  by recycling notation from the proof of part (2) and suppose on the contrary that  $A$  is Morita equivalent to  $B$ . Let  $y' = y - x^2$ . Then the relation in  $B$  becomes  $xy' - y'x - y' = 0$ . Exchanging  $x$  and  $y'$ , one sees that  $B$  is isomorphic to  $\mathbb{k}\langle x, y \rangle / (xy - yx + x)$ . Let  $I$  be the unique height one prime ideal of  $B$  generated by  $x$ . Then  $B/I \cong \mathbb{k}[y]$ . Since  $A$  and  $B$  are Morita equivalent, there is a height one prime  $J$  of  $A$ . Since the only height one prime of  $A$  is  $(x)$ . Let  $J = (x)$ , then  $I^2$  corresponds to  $J^2$ . This implies that  $B/I^2 \cong \mathbb{k}\langle x, y \rangle / (xy - yx + x, x^2)$  is Morita equivalent to  $A/J^2 \cong \mathbb{k}[x, y]/(x^2)$ . Since the center is preserved by Morita equivalence,

$$\mathbb{k}[x, y]/(x^2) \cong Z(\mathbb{k}\langle x, y \rangle / (xy - yx + x, x^2)) \cong \mathbb{k},$$

yielding a contradiction. Therefore  $A$  and  $B$  are not Morita equivalent.  $\square$

We thank one of the referees for the following remark.

**Remark 2.2** (Referee). With only a little more work, one can show the following.

Claim: Let  $A$  and  $B$  be two filtered AS-regular algebras of global dimension 2 that are not PI. Then  $A$  and  $B$  are Morita equivalent if and only if they are isomorphic.

*Sketch proof of the claim.* Such algebras all have the form  $\mathbb{k}\langle x, y \rangle / (f)$ , where  $f$  is a (not necessarily homogeneous) polynomial of degree 2. By [Gaddis 2016, Corollary 2.12], up to isomorphism,  $f$  is one of

- (a)  $xy - yx - 1$ ,
- (b)  $xy - yx - x$ ,
- (c)  $xy - yx + x^2$ ,
- (d)  $xy - yx + x^2 + 1$ ,
- (e)  $xy - qyx$  for  $q \neq 1$  a nonroot of unity (the algebra is denoted by  $\mathcal{O}_q(\mathbb{k}^2)$ ), or
- (f)  $xy - qyx - 1$  for  $q \neq 1$  a nonroot of unity.

By [Richard and Solotar 2006, Lemma 3.1.1], if two noetherian domains  $A$  and  $B$  are Morita equivalent, then the rings of fractions  $Q(A)$  and  $Q(B)$  are isomorphic.

Note that the algebra corresponding to (a) is simple, consequently, it is not Morita equivalent to any other algebras that are not simple (in cases (b)–(f)). In cases (b)–(d) the corresponding algebras all have  $Q(A) \cong D_1(\mathbb{k})$ , the Weyl division ring. By Lemma 2.1, for the most part, these algebras are pairwise non-Morita equivalent. The proof of (b) not being Morita equivalent to (d) is similar to the proof of Lemma 2.1(3).

On the other hand, for cases (e) and (f),  $Q(A) \cong D_1^q(\mathbb{k})$ , the skew Weyl division ring. By [Alev and Dumas 1994, Corollaire 3.11(c)],  $D_1^q(\mathbb{k}) \cong D_1^{q'}(\mathbb{k})$  if and only if  $q' = q^{\pm 1}$ . By Lemma 2.1(2), algebras in cases (e) and (f) are Morita distinct from each other. Since  $\mathcal{O}_q(\mathbb{k}^2) \cong \mathcal{O}_{q^{-1}}(\mathbb{k}^2)$  in the case (e) (and similarly for the quantum Weyl algebras in case (f)), this completes the proof.  $\square$

Next we prove Proposition 0.6. To save some space, we refer the reader to [Levasseur 1992; Stafford 1994] for the definitions of Auslander regularity and Cohen–Macaulay property. A nice result of [Levasseur 1992, Corollary 6.2] is that every AS-regular algebra of global dimension three is Auslander regular and Cohen–Macaulay. A ring  $A$  is called *stably free* if, for every finitely generated projective  $A$ -module  $P$ , there exist integers  $n$  and  $m$  such that  $P \oplus A^{\oplus n} \cong A^{\oplus m}$ . Connected graded algebras are automatically stably free [Stafford 1994]. An Ore domain  $A$  is called a *maximal order* if  $A \subseteq B$  inside the quotient division ring  $Q(A)$  of  $A$  for some ring  $B$  with the property that  $aBb \subseteq A$ , for some  $a, b \in A \setminus \{0\}$ , then  $A = B$ . The main result of [Stafford 1994] is:

**Theorem 2.3** [Stafford 1994, Theorem]. *Let  $A$  be a noetherian algebra that is Auslander regular, Cohen–Macaulay and stably free. Then,  $A$  is a domain and a maximal order in its quotient division ring  $Q(A)$ .*

**Lemma 2.4.** *Let  $Z$  be a connected graded domain of GK dimension one.*

- (1) *It is noetherian and finitely generated over  $\mathbb{k}$ .*
- (2) *If  $Z$  is normal, then  $Z$  is isomorphic to  $\mathbb{k}[t]$ .*

*Proof.* Note that every domain of GK dimension one is commutative.

(1) Since  $Z$  is connected graded and  $\mathbb{k}$  is algebraically closed,  $Z$  is a subring of  $\mathbb{k}[t]$  where  $\deg t = 1$ . From this, it is easy to see that  $Z$  is finitely generated and noetherian.

(2) First of all,  $\text{Kdim } Z = \text{GKdim } Z = 1$ . By part (1),  $Z$  is noetherian. Every noetherian normal domain  $Z$  of Krull dimension one or zero is regular (namely, has finite global dimension). So  $Z$  is regular of global dimension no more than one. Since  $Z$  is connected graded, its graded maximal ideal is principal, which implies that  $Z \cong \mathbb{k}[t]$ .  $\square$

Note that a noetherian commutative maximal order is a normal domain.

**Lemma 2.5.** *Let  $A$  be a domain that is a maximal order.*

- (1) *Its center  $Z(A)$  is a maximal order in the field of fractions  $Q(Z(A))$ .*
- (2) *If  $A$  is connected graded and  $\text{GKdim } Z(A) \leq 1$ , then  $Z(A)$  is either  $\mathbb{k}$  or  $\mathbb{k}[t]$ .*

*Proof.* (1) Let  $B$  be a subring of  $Q(Z(A))$  containing  $Z(A)$  such that  $aBb \subseteq Z(A)$  for some  $a, b \in Z(A) \setminus \{0\}$ . Let  $C := AB$  be the subring of  $Q(A)$  generated by  $A$  and  $B$ . Then  $aCb \subseteq AZ(A) = A$ . Since  $A$  is a maximal order,  $C = A$ . As a consequence,  $B = Z(A)$ . The assertion follows.

(2) The assertion follows from part (1) and [Lemma 2.4](#).  $\square$

Note that if  $\text{GKdim } Z(A) = 2$ , then  $Z(A)$  may not be regular. For example let  $A = \mathbb{k}_{p_{ij}}[x_1, x_2, x_3, x_4]$ , where

$$p_{ij} = \begin{cases} 1 & \text{for } (i, j) = (1, 2), \\ -1 & \text{for } (i, j) = (1, 3), (2, 3), (1, 4), (2, 4), \\ q & \text{for } (i, j) = (3, 4), \end{cases}$$

where  $q$  is not a root of unity. Then it is easy to see that  $Z(A)$  is the second Veronese subring  $\mathbb{k}[x_1, x_2]^{(2)}$  of the commutative polynomial ring. Hence  $Z(A)$  is not regular.

**Lemma 2.6.** *Let  $A$  be a connected graded domain and  $t$  be a central element in  $A$  of positive degree  $d$ .*

- (1) *For every  $\alpha \in \mathbb{k}^\times$ ,  $A/(t - \alpha)$  contains  $(A[t^{-1}])_0$  as a subalgebra.*
- (2) *Suppose that  $A$  is generated in degree 1 and that  $d \neq 0$  in  $\mathbb{k}$ . Then*

$$\text{gldim } A/(t - \alpha) = \text{gldim}(A[t^{-1}])_0.$$

- (3) *Suppose that  $A$  is generated in degree 1 and that  $d \neq 0$  in  $\mathbb{k}$ . If  $A$  has finite global dimension, then so does  $A/(t - \alpha)$  for all  $\alpha \in \mathbb{k}^\times$ .*

*Proof.* (1) Let  $T$  denote the  $d$ -th Veronese subalgebra of  $A$ , where  $d = \deg t$ . So, in  $T$ ,  $t$  can be treated as an element of degree 1. Now

$$(2.6.1) \quad T/(t - \alpha) \cong T/(\alpha^{-1}t - 1) \cong (T[(\alpha^{-1}t)^{-1}])_0 = (T[t^{-1}])_0 \cong (A[t^{-1}])_0$$

where the second  $\cong$  is due to [Rogalski et al. 2014, Lemma 2.1].

Note that  $A/(t - \alpha)$  is a  $\mathbb{Z}/(d)$ -graded algebra with the degree 0 component being  $T/(t - \alpha)$ . By (2.6.1),  $A/(t - \alpha)$  contains  $(A[t^{-1}])_0$  as a subalgebra.

(2) Since  $A$  is generated in degree 1,  $A/(t - \alpha)$  is a strongly  $\mathbb{Z}/(d)$ -graded algebra with the degree 0 component being  $(A[t^{-1}])_0$ . Since we assume  $d \neq 0$  in  $\mathbb{k}$ , by [Yi 1994, Lemma 2.2(iii)],

$$\text{gldim } A/(t - \alpha) = \text{gldim}(A[t^{-1}])_0.$$

(3) By part (2) it suffices to show that  $(A[t^{-1}])_0$  has finite global dimension. Since  $A$  has finite global dimension,  $A$  has finite graded global dimension. Then  $A[t^{-1}]$  has finite graded global dimension. As a consequence,  $(A[t^{-1}])_0$  has finite global dimension as required.  $\square$

To conclude this section we list two well-known results.

**Lemma 2.7** [Lu et al. 2007, Lemma 7.6]. *Let  $A$  be a connected graded algebra and  $t$  be a central element of degree 1. If  $A/(t)$  has finite global dimension, then so does  $A$ .*

**Lemma 2.8** [Smith and Zhang 1998, Corollary 2]. *Let  $A$  be a finitely generated Ore domain that is not PI. Let  $Z$  be the center of  $A$ . Then  $\text{GKdim } Z \leq \text{GKdim } A - 2$ .*

### 3. Some cancellation lemmas

First we recall a classical result concerning the cancellation property.

**Lemma 3.1** [Abhyankar et al. 1972, Corollary 3.4]. *Let  $A$  be an affine domain of GK dimension at most one.*

- (1) *If  $A = \mathbb{k}$ , then it is trivially strongly retractable and strongly cancellative.*
- (2) *If  $A = \mathbb{k}[t]$ , then it is strongly cancellative.*
- (3) *If  $A \not\cong \mathbb{k}[t]$ , then it is strongly retractable, and consequently, strongly cancellative.*

The following lemma concerns cancellation properties for a tensor product  $A \otimes R$ , where  $R$  is commutative.

**Lemma 3.2.** *Let  $A$  be an algebra with trivial center and let  $R$  be a commutative algebra that is cancellative (respectively, strongly cancellative). Then the tensor product  $A \otimes R$  is both cancellative (respectively, strongly cancellative) and  $m$ -cancellative (respectively, strongly  $m$ -cancellative).*

*Proof.* The proofs for the assertions without the word “strongly” are similar by taking  $n = 1$  in the following proof. So we only prove the “strongly” version.

First we show that  $A \otimes R$  is strongly cancellative assuming that  $R$  is strongly cancellative. Let  $B$  be an algebra such that

$$\phi : (A \otimes R)[t_1, \dots, t_n] \xrightarrow{\cong} B[s_1, \dots, s_n]$$

is an isomorphism of algebras. Taking the center on both sides, we obtain an isomorphism

$$\phi_Z : R[t_1, \dots, t_n] \xrightarrow{\cong} Z(B)[s_1, \dots, s_n],$$

where  $\phi_Z$  is a restriction of  $\phi$  on the centers. Since  $R$  is strongly cancellative,  $R \cong Z(B)$ . Let  $I$  be the ideal of  $Z(B)[s_1, \dots, s_n]$  generated by  $\{s_i\}_{i=1}^n$ . Then  $J := \phi^{-1}(I)$  is an ideal of  $R[t_1, \dots, t_n]$  and

$$A \otimes (R[t_1, \dots, t_n]/J) \cong B \otimes (\mathbb{k}[s_1, \dots, s_n]/I) \cong B.$$

Taking the center on both sides of the above isomorphism and using the fact that  $Z(A) = \mathbb{k}$ , we have

$$R[t_1, \dots, t_n]/J \cong Z(B) \cong R.$$

Therefore

$$B \cong A \otimes (R[t_1, \dots, t_n]/J) \cong A \otimes R$$

as required.

Next we show that if  $R$  is strongly cancellative, then  $A \otimes R$  is strongly m-cancellative. Let  $B$  be an algebra such that

$$A' := (A \otimes R)[t_1, \dots, t_n] \text{ is Morita equivalent to } B[s_1, \dots, s_n] =: B'.$$

By [Lu et al. 2020, Lemma 3.1(iii)], there is an invertible  $(A', B')$ -bimodule  $\Omega$  and an isomorphism

$$\omega : Z(A') = R[t_1, \dots, t_n] \xrightarrow{\cong} Z(B)[s_1, \dots, s_n] = Z(B')$$

such that the left action of  $x \in Z(A')$  on  $\Omega$  agrees with the right action of  $\omega(x) \in Z(B')$  on  $\Omega$ . Since  $R$  is strongly cancellative,  $R \cong Z(B)$ . Let  $I$  be the ideal of  $Z(B)[s_1, \dots, s_n]$  generated by  $\{s_i\}_{i=1}^n$ . Then  $J := \omega^{-1}(I)$  is an ideal of  $R[t_1, \dots, t_n]$ , and by [Lu et al. 2020, Lemma 3.1(v)],

$$A \otimes (R[t_1, \dots, t_n]/J) \text{ is Morita equivalent to } B \otimes (\mathbb{k}[s_1, \dots, s_n]/I) \cong B.$$

Taking the center of the above Morita equivalence and using the fact that  $Z(A) = \mathbb{k}$ , we have

$$R[t_1, \dots, t_n]/J \cong Z(B) \cong R.$$

Hence

$$A \otimes (R[t_1, \dots, t_n]/J) \cong A \otimes R.$$

Therefore  $A \otimes R$  is Morita equivalent to  $B$  as required.  $\square$

**Corollary 3.3.** *Let  $\mathbb{k}$  be of characteristic zero and  $A$  be a commutative algebra. Let  $\delta$  be a locally nilpotent derivation of  $A$  with  $\delta(y) = 1$  for some  $y \in A$ . Suppose that  $\ker(\delta)$  is cancellative (respectively, strongly cancellative). Then  $A[x; \delta]$  is cancellative (respectively, strongly cancellative).*

*Proof.* Let  $C = \ker(\delta)$ . By [McConnell and Robson 1987, Lemma 14.6.4]  $A[x; \delta] \cong C \otimes A_1(\mathbb{k})$ . By hypothesis, then  $C$  is cancellative and  $Z(A_1(\mathbb{k})) = \mathbb{k}$ . The assertion follows from Lemma 3.2.  $\square$

With a slight modification to the previous lemma we can consider the case in which  $R$  is a (noncommutative)  $Z$ -retractable algebra.

**Lemma 3.4.** *Let  $A$  be an algebra with trivial center and let  $R$  be a  $Z$ -retractable algebra (respectively, strongly  $Z$ -retractable). Then the tensor product  $A \otimes R$  is  $Z$ -retractable (respectively, strongly  $Z$ -retractable).*

The proof of Lemma 3.4 is similar to the proof of Lemma 3.2, so it is omitted. We call an algebra  $R$  *universally right noetherian* if  $A \otimes R$  is right noetherian for every right noetherian  $\mathbb{k}$ -algebra  $A$ . This property was studied in [Artin et al. 1999]. Thus, if in the previous lemma, we suppose that  $A$  is right noetherian algebra and  $R$  is universally right noetherian, then  $A \otimes R$  is Hopfian, and by [Lezama et al. 2019, Lemma 3.6]  $A \otimes R$  is cancellative (respectively, strongly cancellative).

The following lemma is useful in some situations.

**Lemma 3.5.** *Let  $A$  be a noetherian algebra such that*

- (i) *its center  $Z(A)$  is the commutative polynomial ring  $\mathbb{k}[t]$  for some  $t \in A$ , and*
- (ii)  *$t$  is in the ideal  $[A, A]$  of  $A$  generated by the commutators and  $[A, A] \neq A$ .*

*Then  $A$  is strongly  $Z$ -retractable and strongly cancellative.*

*Proof.* Let  $\mathcal{P}$  be the property that the commutators generate the whole algebra. By (ii) the property  $\mathcal{P}$  fails for the maximal ideal  $(t)$  in  $\mathbb{k}[t]$  since  $[\bar{A}, \bar{A}] \neq \bar{A}$ , where  $\bar{A} = A/(t)$ . By (ii) again,  $(t - \alpha) + [A, A] = A$  for all  $\alpha \neq 0$ . This means that the property  $\mathcal{P}$  holds for the maximal ideal  $(t - \alpha)$  in  $\mathbb{k}[t]$  for all  $\alpha \neq 0$ . Thus the  $\mathcal{P}$ -discriminant of  $A$  is  $t$ . The assertion follows from Theorem 1.5.  $\square$

We refer to [Lezama et al. 2019; Lu et al. 2020] for the definition of  $\text{LND}_t^H$ -rigid in the proof of the following lemma.

**Lemma 3.6.** *Let  $A$  be a noetherian domain with  $Z(A) = \mathbb{k}[t]$ . Let  $\mathcal{P}$  be a property such that the  $\mathcal{P}$ -discriminant is  $t$ .*

- (1)  *$A$  is strongly  $Z$ -retractable and strongly cancellative.*
- (2) *If  $\mathcal{P}$  is a Morita invariant, then  $A$  is strongly  $Z$ -retractable, strongly  $m$ - $Z$ -retractable, strongly cancellative and strongly  $m$ -cancellative.*

*Proof.* We only prove (2).

Since  $t$  is an effective element in  $\mathbb{k}[t]$  by [Lezama et al. 2019, Example 2.8]. By [Lu et al. 2020, Theorem 3.10],  $Z$  is strongly  $\text{LND}_t^H$ -rigid. By [Lu et al. 2020, Proposition 3.7(ii)],  $A$  is both strongly  $Z$ -retractable and strongly  $m$ - $Z$ -retractable. Since  $A$  is noetherian, it is Hopfian in the sense of [Lezama et al. 2019, Definition 3.4]. By [Lu et al. 2020, Lemmas 4.4 and 4.6],  $A$  is strongly cancellative and strongly  $m$ -cancellative.  $\square$

Next we consider the connected graded case.

**Lemma 3.7.** *Let  $A$  be a noetherian connected graded domain.*

- (1) *If  $Z(A)$  has GK dimension  $\leq 1$  and  $Z(A)$  is not isomorphic to  $\mathbb{k}[t]$ , then  $A$  is strongly  $Z$ -retractable, strongly  $m$ - $Z$ -retractable, strongly cancellative and strongly  $m$ -cancellative.*

*For the following parts, we assume that  $A$  is generated in degree 1, that  $Z(A) \cong \mathbb{k}[t]$  and that  $\text{char } \mathbb{k} = 0$ .*

- (2) *If  $\text{gldim } A/(t) = \infty$  and  $\text{gldim } A/(t-1) < \infty$ , then  $A$  is strongly  $Z$ -retractable, strongly  $m$ - $Z$ -retractable, strongly cancellative and strongly  $m$ -cancellative.*
- (3) *Suppose the global dimension of  $A$  is finite and  $\text{gldim } A/(t) = \infty$ . Then  $A$  is strongly  $Z$ -retractable, strongly  $m$ - $Z$ -retractable, strongly cancellative and strongly  $m$ -cancellative.*
- (4) *Suppose  $A$  is AS-regular and  $\text{gldim } A/(t) = \infty$ . Then  $A$  is strongly  $Z$ -retractable, strongly  $m$ - $Z$ -retractable, strongly cancellative and strongly  $m$ -cancellative.*

*Proof.* (1) By Lemma 2.4(1),  $Z$  is an affine domain. By Lemma 3.1,  $Z$  is strongly retractable. By taking  $\mathcal{P}$  to be a trivial property, say being an algebra, the  $\mathcal{P}$ -discriminant is 1. By [Lezama et al. 2019, Remark 3.7(6)],  $Z$  is strongly  $\text{LND}_1^H$ -rigid. By [Lu et al. 2020, Proposition 3.7(ii)],  $A$  is both strongly  $Z$ -retractable and strongly  $m$ - $Z$ -retractable. Since  $A$  is noetherian, it is Hopfian in the sense of [Lezama et al. 2019, Definition 3.4]. By [Lu et al. 2020, Lemmas 4.4 and 4.6],  $A$  is strongly cancellative and strongly  $m$ -cancellative.

(2) Let  $\mathcal{P}$  be the property of having finite global dimension. By [Lemma 2.6](#), for every  $0 \neq \alpha \in \mathbb{k}$ ,

$$\text{gldim } A/(t - \alpha) = \text{gldim}(A[t^{-1}])_0 = \text{gldim } A/(t - 1) < \infty$$

and by the hypothesis, we have that

$$\text{gldim } A/(t) = \infty.$$

Hence the  $\mathcal{P}$ -discriminant is  $t$ . The assertion follows from [Lemma 3.6](#)(2).

(3) By part (2), it suffices to show that  $\text{gldim } A/(t-1) < \infty$ . Since  $A$  has finite global dimension, so does  $A[t^{-1}]$ . Then  $A[t^{-1}]$  has finite graded global dimension. Since  $A$  is generated in degree 1,  $A[t^{-1}]$  is strongly  $\mathbb{Z}$ -graded. Hence  $\text{gldim}(A[t^{-1}])_0$  is finite. By [Lemma 2.6](#),

$$\text{gldim } A/(t - 1) = \text{gldim}(A[t^{-1}])_0 < \infty$$

as required.

(4) The assertion follows from part (3) and the fact that an AS-regular algebra has finite global dimension.  $\square$

By [Lemma 3.7](#)(1), the case of  $\text{GKdim } Z(A) = 1$  is covered except for  $Z(A) = \mathbb{k}[t]$ .

**Lemma 3.8.** *Let  $A$  be a noetherian connected graded algebra.*

- (1) *Suppose  $Z(A) = \mathbb{k}[t]$  for some homogeneous element  $t$  of positive degree. If  $(A[t^{-1}])_0$  does not have any nonzero finite-dimensional left module, then  $A$  is strongly  $Z$ -retractable, strongly  $m$ - $Z$ -retractable, strongly cancellative and strongly  $m$ -cancellative.*
- (2) *Suppose  $B$  is a connected graded subalgebra of  $A$  satisfying*
  - (i)  *$Z(B) = \mathbb{k}[t]$  for some homogeneous element  $t \in B$  of positive degree.*
  - (ii)  *$Z(A) \cap Z(B) \neq \mathbb{k}$ .*
  - (iii)  *$(A[t^{-d}])_0$  does not have any nonzero finite-dimensional left module for some  $t^d \in Z(B) \cap Z(A)$ , where  $d$  is a positive integer.*
  - (iv)  *$A_B$  is finitely generated and  $B$  is noetherian.*
  - (v)  *$A = B \oplus C$  as a right  $B$ -module.*

*Then  $B$  is strongly  $Z$ -retractable, strongly  $m$ - $Z$ -retractable, strongly cancellative and strongly  $m$ -cancellative.*

*Proof.* (1) Let  $\mathcal{P}$  be the property of not having nonzero finite-dimensional left module over an algebra. Since  $A$  is connected graded,  $\mathcal{P}$  fails for  $A/(t)$ . We claim that  $\mathcal{P}$  holds for  $A/(t - \alpha)$  for all  $\alpha \in \mathbb{k}^\times$ . By the hypothesis,  $(A[t^{-1}])_0$  does not have any nonzero finite-dimensional left module. Since  $A/(t - \alpha)$  contains  $(A[t^{-1}])_0$  by [Lemma 2.6](#)(1),  $A/(t - \alpha)$  does not have any nonzero finite-dimensional left module.



So the claim holds. Therefore the  $\mathcal{P}$ -discriminant of  $A$  is  $t$ . Now the assertion follows from [Lemma 3.6\(2\)](#).

(2) By part (1) it suffices to show that  $(B[t^{-1}])_0$  does not have any nonzero finite-dimensional left module. Note that  $B[t^{-1}] = B[t^{-d}]$ . So it is equivalent to show that  $(B[t^{-d}])_0$  does not have any nonzero finite-dimensional left module. We prove this claim by contradiction. Suppose  $M$  is a nonzero finite-dimensional left  $(B[t^{-d}])_0$ -module. By hypotheses (iii)–(iv) and by inverting  $t^d$ ,  $A[t^{-d}] = C[t^{-d}] \oplus B[t^{-d}]$  and  $A[t^{-d}]$  is a finitely generated right  $B[t^{-d}]$ -module. Then  $(A[t^{-d}])_0 = (C[t^{-d}])_0 \oplus (B[t^{-d}])_0$  and  $(A[t^{-d}])_0$  is a finitely generated right  $(B[t^{-d}])_0$ -module. Hence  $(A[t^{-d}])_0 \otimes_{(B[t^{-d}])_0} M$  is a nonzero finite-dimensional left  $(A[t^{-d}])_0$ -module. This yields a contradiction. At this point, we have proved that  $(B[t^{-1}])_0$  does not have any nonzero finite-dimensional left module. The assertion follows from part (1).  $\square$

[Lemma 3.8](#) can be applied to many examples. Here is an easy one.

**Example 3.9.** Suppose  $\text{char } \mathbb{k} = 0$ . Let  $A$  be a generic 3-dimensional Sklyanin algebra generated by  $\{x, y, z\}$ ; see [\[Gaddis et al. 2019a, Introduction\]](#) for the relations. Then  $Z(A) = \mathbb{k}[g]$  where  $g$  is a homogeneous element of degree three. Let  $G$  be any finite group of graded algebra automorphisms of  $A$ . Let  $B$  be the fixed subring  $A^G$ . Then we claim that  $B$  is strongly  $Z$ -retractable, strongly  $m$ - $Z$ -retractable, strongly cancellative and strongly  $m$ -cancellative. It is easy to see that hypotheses (ii)–(v) in [Lemma 3.8\(2\)](#) hold. If hypothesis (i) in [Lemma 3.8\(2\)](#) holds, then it is easy to see that remaining hypotheses hold as well and the claim follows by that Lemma. If hypothesis (i) in [Lemma 3.8\(2\)](#) fails, then the result follows by [Lemma 3.7\(1\)](#).

To conclude this section we give an example of ungraded algebras that are cancellative.

**Example 3.10.** Suppose  $\text{char } \mathbb{k} = 0$ . Let  $A$  be the universal enveloping algebra  $U(\mathfrak{g})$ , where  $\mathfrak{g}$  is a 3-dimensional nonabelian Lie algebra. One can use Bianchi classification to list all 3-dimensional nonabelian Lie algebras [\[Jacobson 1962, Section 1.4\]](#) as follows.

- (1)  $\mathfrak{g} = sl_2$ .
- (2)  $\mathfrak{g}$  is the Heisenberg Lie algebra.
- (3)  $\mathfrak{g} = L \oplus \mathbb{k}z$ , where  $L$  is the 2-dimensional nonabelian Lie algebra.
- (4)  $\mathfrak{g}$  has a basis  $\{e, f, g\}$  and is subject to the relations

$$[e, f] = 0, \quad [e, g] = e, \quad [f, g] = \alpha f,$$

where  $\alpha \neq 0$ .

(5)  $\mathfrak{g}$  has a basis  $\{e, f, g\}$  and is subject to the relations

$$[e, f] = 0, \quad [e, g] = e + \beta f, \quad [f, g] = f,$$

where  $\beta \neq 0$ .

For each class, one can verify that  $A$  is strongly cancellative.

(1) See [Lu et al. 2020, Example 6.11].

(2) The universal enveloping algebra of the Heisenberg Lie algebra has the center  $Z = \mathbb{k}[t]$  with  $t$  in the ideal generated by the commutators, then the assertion follows from Lemma 3.5.

(3) In this case  $U(\mathfrak{g}) = U(L) \otimes \mathbb{k}[z]$  with  $Z(U(L)) = \mathbb{k}$ . The assertion follows from Lemma 3.2.

(4,5) In both cases, one can write  $A := U(\mathfrak{g})$  as an Ore extension  $\mathbb{k}[e, f][g; \delta]$  for some derivation  $\delta$  of the commutative polynomial ring  $\mathbb{k}[e, f]$ . If the center of  $A$  is trivial, then  $A$  is strongly cancellative by [Bell and Zhang 2017a, Proposition 1.3]. For the rest of proof we assume that  $Z(A) \neq \mathbb{k}$ . Note that the derivation  $\delta$  of  $\mathbb{k}[e, f]$  is determined by

$$(3.10.1) \quad \delta : \quad e \rightarrow -e, \quad f \rightarrow -\alpha f$$

in part (4), and by

$$(3.10.2) \quad \delta : \quad e \rightarrow -(e + \beta f), \quad f \rightarrow -f$$

in part (5). By an easy calculation, one sees that

$$Z(A) = \{x \in \mathbb{k}[e, f] \mid \delta(x) = 0\}$$

which is a graded subring of  $\mathbb{k}[e, f]$  (which is inside  $A$ ). Since  $A$  contains  $U(L)$  as a subalgebra where  $L$  is the 2-dimensional nonabelian Lie algebra,  $A$  is not PI. By Lemma 2.8,  $\text{GKdim } Z(A) \leq 1$ . Since  $Z(A) \neq \mathbb{k}$  and  $\mathbb{k}$  is algebraically closed,  $\text{GKdim } Z(A) \geq 1$ . Thus  $\text{GKdim } Z(A) = 1$ . By Lemma 2.4,  $Z(A)$  is a domain that is finitely generated over  $\mathbb{k}$ . If  $Z(A)$  is not isomorphic to  $\mathbb{k}[t]$ , then  $Z(A)$  is strongly retractable by Lemma 3.1(3). Using the proof of Lemma 3.7(1), one sees that  $A$  is strongly cancellative and strongly m-cancellative. For the rest, we assume that  $Z(A) \cong \mathbb{k}[t]$  for some homogeneous element  $t$  in  $\mathbb{k}[e, f]$ . By the form of  $\delta$  in (3.10.1)–(3.10.2), the degree of  $t$  is at least 2. This implies that  $\mathbb{k}[e, f]/(t)$  has infinite global dimension. On the other hand, if  $\alpha \neq 0$ , then  $\mathbb{k}[e, f]/(t - \alpha)$  has finite global dimension (applying Lemma 2.6 to the algebra  $\mathbb{k}[e, f]$ ). Therefore  $A/(t) = (\mathbb{k}[e, f]/(t))[g; \delta]$  has infinite dimension and  $A/(t - \alpha) = (\mathbb{k}[e, f]/(t - \alpha))[g; \delta]$  has finite global dimension. Then an argument similar to the proof of Lemma 3.7(2) shows that  $A$  is strongly cancellative.

One obvious question after [Example 3.10](#) is whether or not every universal enveloping algebra of a 4-dimensional nonabelian Lie algebra is cancellative.

#### 4. Proof of Theorems 0.2, 0.3 and Corollary 0.5

In this section we prove some of the main results listed in the introduction. We start with [Theorem 0.3](#).

*Proof of Theorem 0.3.* If  $Z(A)$  is not isomorphic to  $\mathbb{k}[t]$ , the assertion follows from [Lemma 3.7\(1\)](#). If  $Z(A)$  is isomorphic to  $\mathbb{k}[t]$ , the assertion follows from [Lemma 3.7\(3\)](#).  $\square$

*Proof of Theorem 0.2.* Note that every AS-regular algebra has finite global dimension. If  $A$  is not PI, then by [Lemma 2.8](#),

$$\text{GKdim } Z \leq \text{GKdim } A - 2 = 3 - 2 = 1.$$

If  $Z(A)$  is not isomorphic to  $\mathbb{k}[t]$ , the assertion follows from [Lemma 3.7\(1\)](#). If  $Z(A)$  is isomorphic to  $\mathbb{k}[t]$  and if  $\text{gldim } A/(t) = \infty$ , the assertion follows from [Lemma 3.7\(3\)](#). For the rest of the proof we assume that  $Z(A) = \mathbb{k}[t]$  and  $A/(t)$  has finite global dimension.

By Rees lemma,  $\text{gldim } A/(t) \leq 2$ . By a Hilbert series computation, we obtain that  $\text{GKdim } A/(t) = 2$ . This implies that  $A/(t)$  is AS-regular of global dimension two. Since we assume that  $\mathbb{k}$  is algebraically closed,  $A/(t)$  is either  $\mathbb{k}_q[x, y]$  or  $\mathbb{k}_J[x, y]$ . In particular, the Hilbert series of  $A/(t)$  is  $\frac{1}{(1-s)^2}$ . Since  $A$  is AS-regular of global dimension three, it is generated by either 3 elements or 2 elements. Next we consider these two cases.

**Case 1:**  $A$  is generated by two elements. Then the Hilbert series of  $A$  is  $\frac{1}{(1-s)^2(1-s^2)}$ . This forces that  $\deg t = 2$ . If  $A/(t) = \mathbb{k}_q[x, y]$ , then  $t = xy - qyx$  and  $A/(t-1) = \mathbb{k}\langle x, y \rangle / (xy - qyx - 1)$ . If  $q = 1$ , let  $\mathcal{P}$  be the property of *not* being Morita equivalent to  $A/(t)$ . Then the  $\mathcal{P}$ -discriminant is  $t$  by [Lemma 2.1\(1\)](#). Now the assertion follows from [Lemma 3.6\(2\)](#).

If  $q \neq 1$ , we claim that  $q$  is not a root of unity. If  $q$  is a root of unity, then  $A/(t-1)$  is PI. By [Lemma 2.6\(1\)](#),  $(A[t^{-1}])_0$  is PI. Note that  $A^{(2)}[t^{-1}] = (A[t^{-1}])_0[t^{\pm 1}]$ . So  $A^{(2)}[t^{-1}]$  is PI. Consequently,  $A^{(2)}$  is PI and whence  $A$  is PI, a contradiction. Then by the argument in the previous paragraph with [Lemma 2.1\(2\)](#) being replaced by [Lemma 2.1\(1\)](#), one sees that  $A$  is strongly cancellative and strongly m-cancellative.

If  $A/(t) = \mathbb{k}_J[x, y]$ , then

$$t = xy - yx + x^2 \quad \text{and} \quad A/(t-1) = \mathbb{k}\langle x, y \rangle / (xy - yx + x^2 - 1).$$

Let  $\mathcal{P}$  be the property of being Morita equivalent to  $A/(t-1)$ . Then the  $\mathcal{P}$ -discriminant is  $t$  by [Lemma 2.1\(3\)](#). By [Lemma 3.6\(2\)](#),  $A$  is strongly cancellative and strongly m-cancellative.

**Case 2:**  $A$  is generated by three elements. Then the Hilbert series of  $A$  is  $\frac{1}{(1-s)^3}$ . If  $A$  is isomorphic to  $A' \otimes \mathbb{k}[t]$  for some algebra  $A'$ , then  $Z(A')$  is trivial and the assertion follows from [Lemma 3.2](#). So we further assume that  $A$  is not a tensor product of two nontrivial algebras. In this case  $A$  is generated by  $x, y, t$  subject to three relations

$$(4.0.1) \quad \begin{aligned} xt - tx &= 0, \\ yt - ty &= 0, \\ xy - qyx &= ft + \epsilon t^2, \quad \text{or} \quad xy - yx + x^2 = ft + \epsilon t^2, \end{aligned}$$

where  $\epsilon$  is either 0 or 1 and  $f$  is a linear combination of  $x$  and  $y$ .

Again we have three cases. If  $q = 1$  in the first type of (4.0.1), using [Lemmas 2.1\(1\)](#) and [3.6\(2\)](#), one sees that  $A$  is strongly cancellative and strongly m-cancellative. If  $q \neq 1$  in the first type of (4.0.1), then we can assume that  $f = 0$  and  $\epsilon = 1$  after a base change. Since  $A$  is not PI,  $q$  is not a root of unity. Then we use [Lemma 2.1\(2\)](#) instead of [Lemma 2.1\(1\)](#). Otherwise we have the relation

$$xy - yx + x^2 - ft - \epsilon t^2 = 0.$$

Up to a base change, we may assume that  $\epsilon = 0$ . Then either  $xy - yx + x^2 - xt = 0$  or  $xy - yx + x^2 - yt = 0$ . In the case of  $xy - yx + x^2 - xt = 0$ , using [Lemmas 2.1\(4\)](#) and [3.6\(2\)](#), one sees that  $A$  is strongly cancellative and strongly m-cancellative. In the case of  $xy - yx + x^2 - yt = 0$ , using [Lemmas 2.1\(5\)](#) and [3.6\(2\)](#), one sees that  $A$  is strongly cancellative and strongly m-cancellative.  $\square$

For the rest of this section we study cancellation property for some graded isolated singularities. In noncommutative algebraic geometry, Ueyama gave the following definition of a graded isolated singularity.

**Definition 4.1** [[Ueyama 2013](#), Definition 2.2]. Let  $A$  be a noetherian connected graded algebra. Then  $A$  is called a *graded isolated singularity* if

- (1)  $\text{gldim } A$  is infinite.
- (2) The associated noncommutative projective scheme  $\text{Proj } A$  (in the sense of [[Artin and Zhang 1994](#)]) has finite global dimension.

Examples of graded isolated singularities are given in [[Chan et al. 2020](#); [Gaddis et al. 2019a](#); [Mori and Ueyama 2016a](#); [2016b](#); [Ueyama 2013](#)]. One nice example of graded isolated singularities is the fixed subring of the generic Sklyanin algebra under the cyclic permutation action [[Gaddis et al. 2019a](#), Theorem 5.2] which is cancellative by [Example 3.9](#).

**Lemma 4.2.** Suppose  $\text{char } \mathbb{k} = 0$ . Let  $A$  be a graded isolated singularity generated in degree one and  $t \in B$  be a central regular element of positive degree. Then  $A/(t - \alpha)$ , for every  $0 \neq \alpha \in \mathbb{k}$ , has finite global dimension.

*Proof.* By Lemma 2.6(2) it suffices to show that  $(A[t^{-1}])_0$  has finite global dimension. Since there is a localizing functor from  $\text{Proj } A$  to  $\text{GrMod } A[t^{-1}]$ ,

$$\text{gldim GrMod } A[t^{-1}] \leq \text{gldim Proj } A < \infty.$$

It is well-known that

$$\text{gldim}(A[t^{-1}])_0 = \text{gldim GrMod } A[t^{-1}]$$

as  $A[t^{-1}]$  is strongly  $\mathbb{Z}$ -graded. The assertion follows.  $\square$

**Theorem 4.3.** *Let  $A$  be a noetherian connected graded domain generated in degree 1. Suppose*

- (1)  $\text{char } \mathbb{k} = 0$ ,
- (2)  $\text{GKdim } Z(A) \leq 1$ , and
- (3)  $A$  is a graded isolated singularity.

*Then  $A$  is strongly cancellative and strongly  $m$ -cancellative.*

*Proof.* If  $Z(A) \not\cong \mathbb{k}[t]$ , then the assertion follows from Lemma 3.7(1). Now we assume that  $Z(A) \cong \mathbb{k}[t]$  where  $t$  can be chosen to be a homogeneous element of positive degree. Since  $A$  has infinite global dimension, so does  $A/(t)$  by Lemma 2.7. For every  $0 \neq \alpha \in \mathbb{k}$ , by Lemma 4.2,  $A/(t - \alpha)$  has finite global dimension. The assertion follows from Lemma 3.7(2).  $\square$

Now we are ready to prove Corollary 0.5.

*Proof of Corollary 0.5.* If  $d = 1$ , then it follows from Theorem 0.2.

Next we assume that  $d > 1$ . Note that the Hilbert series of  $A$  is either  $\frac{1}{(1-s)^3}$  or  $\frac{1}{(1-s)^2(1-s^2)}$ . By an easy computation, the Hilbert series of  $A^{(d)}$  cannot be of the form  $\frac{1}{f(s)}$  for some polynomial  $f(s)$ . By [Stephenson and Zhang 1997, Theorem 2.4 and the preceding argument],  $A^{(d)}$  does not have finite global dimension. By [Artin and Zhang 1994, Proposition 5.10(3)],  $A^{(d)}$  is a graded isolated singularity. Since  $A$  is not PI,  $\text{GKdim } Z(A) \leq 1$  by Lemma 2.8. The assertion follows from Theorem 4.3.  $\square$

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