Cambridge Journal of Mathematics
Volume 9, Number 1, 149–176, 2021

Uniqueness of the minimizer of the normalized volume function

Chenyang Xu* and Ziquan Zhuang

We confirm a conjecture of Chi Li which says that the minimizer of the normalized volume function for a klt singularity is unique up to rescaling. This is achieved by defining stability thresholds for valuations in the local setting, and then showing that a valuation is a minimizer if and only if it is K-semistable, and that K-semistable valuation is unique up to rescaling. As applications, we prove a finite degree formula for volumes of klt singularities and an effective bound of the local fundamental group of a klt singularity.

AMS 2000 subject classifications: Primary 14B05; secondary 13A18.
Keywords and phrases: Normalized volume, K-stability, singularity.

1. Introduction

Throughout this paper, we work over an algebraically closed field k of characteristic 0. Given a klt singularity $x \in (X, \Delta)$, Chi Li introduced in [20] the *normalized volume function* $\widehat{\mathrm{vol}}_{X,\Delta}$ on the space $\mathrm{Val}_{X,x}$ of real valuations on the function field $K(X)$ of X that are centered on x. Motivated by the study of K-stability of Fano varieties, the minimizing valuation of $\widehat{\mathrm{vol}}_{X,\Delta}$ is conjectured to have a number of deep geometric properties, which together comprise the so-called *Stable Degeneration Conjecture*, see [20, 22].

There has been a lot of progress on the solution of different parts of the Stable Degeneration Conjecture in [3, 19, 22, 23, 32]. In particular, it has been known that a minimizing valuation exists (see [3]) and it is always quasi-monomial (see [32]).

1.1. Main Theorems

In this paper, we aim to solve another part of the Stable Degeneration Conjecture, namely, the uniqueness of the minimizing valuation, as conjectured in [20, Conjecture 7.1.2].

*CX is partially supported by the NSF (DMS-1901849 and DMS-1952531).

Theorem 1.1. *Let $x \in (X, \Delta)$ be a klt singularity, then up to rescaling, there is a unique minimizer v_0 of the normalized volume function $\widehat{\mathrm{vol}}_{X,\Delta}$.*

We remark that our proof of the theorem does not rely on the fact that the minimizer is quasi-monomial.

An immediate consequence is the following, which is the local version of the K-semistable case of [33, Theorem 1.1].

Corollary 1.2. *If a klt singularity $x \in (X, \Delta)$ admits a group G-action, then any minimizer v_0 of $\widehat{\mathrm{vol}}_{X,\Delta}$ is G-invariant.*

Another direct consequence is the finite degree formula for normalized volumes.

Theorem 1.3 (Finite degree formula)**.** *Let $f\colon \big(y \in (Y, \Delta_Y)\big) \to \big(x \in (X, \Delta)\big)$ be a finite Galois morphism between klt singularities such that $f^*(K_X + \Delta) = K_Y + \Delta_Y$. Then*

$$\widehat{\mathrm{vol}}(x, X, \Delta) \cdot \deg(f) = \widehat{\mathrm{vol}}(y, Y, \Delta_Y).$$

Here $\widehat{\mathrm{vol}}(x, X, \Delta)$ denotes the volume of the klt singularity $x \in (X, \Delta)$, see Definition 2.5. We apply this to obtain the following effective bound of the local fundamental group.

Corollary 1.4. *Let $x \in (X, \Delta)$ be the germ of a klt singularity, then the order of the fundamental group of the smooth locus satisfies*

$$\#|\pi_1(x, X^{\mathrm{sm}})| \le \frac{n^n}{\widehat{\mathrm{vol}}(x, X, \Delta)},$$

where the equality holds if and only if $\Delta = 0$ and $x \in X$ is étale locally isomorphic to $\mathbb{C}^n/G$ where the action of $G \cong \pi_1(x, X^{\mathrm{sm}})$ is fixed point free in codimension one.

Combining Corollary 1.4 with the results from [24, 4] relating local and global volumes of Fano varieties, we also have the following theorem.

Theorem 1.5. *Let (X, Δ) be a log Fano variety. Then for any $x \in (X, \Delta)$, if we denote by $\pi_1^{\mathrm{loc}}(x, X^{\mathrm{sm}})$ the local fundamental group of the smooth locus of the germ $x \in (X, \Delta)$, we have the inequality*

$$\#|\pi_1^{\mathrm{loc}}(x, X^{\mathrm{sm}})| \le \frac{(n+1)^n}{\delta(X, \Delta)^n \cdot \big(-(K_X + \Delta)\big)^n}.$$

In particular, the Cartier index of X is bounded from above by the right hand side of the above inequality.

Here $\delta(X, \Delta)$ denotes the stability threshold of the log Fano pair (X, Δ), see [11, Definition 0.2] or [4].

Remark 1.6. An interesting application of Theorem 1.5 is that it gives a new proof of the boundedness of K-semistable Fano varieties of a fixed dimension and with volume bounded from below. This was originally proved in [13] as a consequence of the boundedness results proved in [2]. Applying Theorem 1.5, we only need the fact that Fano varieties with a fixed Cartier index form a bounded family, which was first proved in [12, Corollary 1.8].

1.2. Outline of the proof

Given a klt singularity $x \in (X = \mathbf{Spec}(R), \Delta)$, the uniqueness of the minimizer v (up to rescaling) of $\widehat{\mathrm{vol}}_{X,\Delta}$ is proved in [22] *under the assumption* that the graded rings associated to the minimizers are finitely generated. The finite generation assumption is used to give a degeneration of the singularity (X, Δ) to a *K-semistable* log Fano cone (X_0, Δ_0, ξ_v), where $X_0 = \mathbf{Spec}(\mathrm{gr}_v(R))$, Δ_0 the degeneration of Δ, and ξ_v is the Reeb vector induced by v. This degeneration picture allows one to degenerate any minimizer to X_0, and use the strict convexity of the volume function to conclude that ξ_v is the unique T-equivariant minimizer on (X_0, Δ_0) (see [31, Page 823]).

The main aim of this paper is to prove uniqueness of the minimizer without assuming the finite generation property, which still remains a major challenge. For this purpose, a key new input, introduced in Section 3.1, is the *K-semistability* of a general valuation $v_0 \in \mathrm{Val}_{X,x}$ centered at a klt singularity $x \in (X, \Delta)$. More generally, we will define the stability threshold $\delta(v_0)$ of a valuation v_0 with finite log discrepancy. This is done by introducing a local version of basis type divisors. Roughly speaking, a basis type divisor with respect to the chosen valuation v_0 is (up to a suitable rescaling factor) a divisor of the form $\{f_1 = 0\} + \cdots + \{f_N = 0\}$ where the images of f_i form a basis of $\mathcal{O}_{X,x}/\mathfrak{a}_m(v_0)$ (for some integer m; here $\mathfrak{a}_\bullet(v_0)$ denotes the valuation ideals) that is compatible with the filtration induced by v_0. Given another valuation $v \in \mathrm{Val}_{X,x}$, we apply the key technical observation from [1] to find basis type divisors that are compatible with both v_0 and v. This allows us to define the S-invariant and δ-invariant of a valuation v_0 with respect to another valuation v and to eventually define the local analogue of the stability notions from the global setting. To justify our definition, when

v_0 is given by a Kollár component S, we will show that ord_S is K-semistable as a valuation if and only if (S, Δ_S) is K-semistable as a log Fano pair (see Theorem 3.6).

With these new definitions, in the second step we show in Section 3.2 that a K-semistable valuation is always a minimizer, and up to scaling there is a unique K-semistable valuation. The observation here is that the log canonical thresholds (lct) of basis type divisors with respect to a K-semistable valuation v_0 is asymptotically computed by v_0. On the other hand, the asymptotic expected vanishing order of these basis type divisors along a valuation v is at least $\mathrm{vol}_{X,\Delta}(v)^{-1/n}$, with equality when $v = v_0$. Through the identity

$$\widehat{\mathrm{vol}}_{X,\Delta}(v)^{1/n} = \frac{A_{X,\Delta}(v)}{\mathrm{vol}_{X,\Delta}(v)^{-1/n}},$$

minimizing the normalized volume $\widehat{\mathrm{vol}}_{X,\Delta}(v)$ can be thought of as finding valuations that compute the lct of basis type divisors. In particular, this implies that K-semistable valuations are minimizers of $\widehat{\mathrm{vol}}_{X,\Delta}$ and the uniqueness then follows from an analysis of the equality condition.

In the last step, we show in Section 3.3 that every minimizing valuation v_0 is K-semistable. To circumvent the finite generation assumption of $\mathrm{gr}_{v_0} R$ in [22], we will generalize the derivative argument from [19]. Intuitively, given two valuations $v_0, v \in \mathrm{Val}_{X,x}$, we would like to draw a ray between them in the valuation space and use the nonnegativity of the derivative of $\widehat{\mathrm{vol}}_{X,\Delta}$ at the minimizer v_0 to prove its K-semistability. When v_0 and v are quasi-monomial with respect to a common stratum, a natural candidate is given by the line joining them in the corresponding dual complex. However, it is unclear to us how to write down such a ray in general. Our idea is to instead construct a family of graded sequences of ideals that interpolates the valuation ideals of the two given valuations. Combining the derivative formula from [19] and an analysis of the log canonical thresholds and multiplicities of these "mixed" ideal sequences, we can then show that if v_0 is a minimizer, then $\delta(v_0) \geq 1$, i.e. v_0 is K-semistable.

Acknowledgement

We want to thank Yuchen Liu for discussions, especially for showing us the preprint [25]. We also would like to thank Harold Blum for helpful comments.

2. Preliminaries

Notation and Conventions. We follow the notation as in [16, 17, 15].

We say $x \in (X = \mathbf{Spec}(R), \Delta)$ is a singularity if R is a local ring of essentially finite type over k, Δ is an effective divisor on X and $x \in X$ is the unique closed point.

A filtration $\mathcal{F}^\bullet$ on a finite dimensional vector space V is a decreasing sequence $\mathcal{F}^t V$ $(t \in \mathbb{R})$ of subspaces satisfying $\mathcal{F}^t V \subseteq \mathcal{F}^{t'} V$ whenever $t \geq t'$. It is called an $\mathbb{N}$-filtration if $\mathcal{F}^0 V = V$ and $\mathcal{F}^t V = \mathcal{F}^{\lceil t \rceil} V$ for all $t \in \mathbb{R}$. For any filtration $\mathcal{F}$ on V, we define its induced $\mathbb{N}$-filtration $\mathcal{F}^\bullet_{\mathbb{N}}$ by setting $\mathcal{F}^t_{\mathbb{N}} V := \mathcal{F}^{\lceil t \rceil} V$.

A projective klt pair (X, Δ) is called a log Fano pair if $-K_X - \Delta$ is ample.

2.1. Graded sequences of ideals

Let $(R, \mathfrak{m})$ be a local ring of essentially finite type over $k \cong R/\mathfrak{m}$. A graded sequence of ideals (see [14]) is a sequence of ideals $\mathfrak{a}_\bullet = (\mathfrak{a}_m)_{m \in \mathbb{N}}$ such that $\mathfrak{a}_m \cdot \mathfrak{a}_n \subseteq \mathfrak{a}_{m+n}$. We call it *decreasing* if $\mathfrak{a}_{m+1} \subseteq \mathfrak{a}_m$ for all $m \in \mathbb{N}$. A graded sequence $\mathfrak{b}_\bullet$ of ideals is said to be *linearly bounded by* another one $\mathfrak{a}_\bullet$, if there is a positive integer C such that such that

$$\mathfrak{b}_{Cm} \subseteq \mathfrak{a}_m$$

for any $m \in \mathbb{N}$. A finite subset $\{f_1, \ldots, f_N\}$ of $R \backslash \{0\}$ is said to be *compatible with* a decreasing graded sequence $\mathfrak{a}_\bullet$ of ideals if for all $m \in \mathbb{N}$, the nonzero images $\bar{f}_i$ of f_i in $R/\mathfrak{a}_m$ are linearly independent.

The following lemma is a local version of [1, Lemma 3.1].

Lemma 2.1. *Let $(R, \mathfrak{m})$ be a local ring of essentially finite type over $k \cong R/\mathfrak{m}$, let $\mathfrak{a}_\bullet$ and $\mathfrak{b}_\bullet$ be two decreasing graded sequences of $\mathfrak{m}$-primary ideals and let $m \in \mathbb{N}$. Then there exist some $f_i \in R \setminus \{0\}$ $(1 \leq i \leq N)$ whose images in $R/\mathfrak{a}_m$ form a basis such that $\{f_1, \ldots, f_N\}$ is compatible with both $\mathfrak{a}_\bullet$ and $\mathfrak{b}_\bullet$.*

Proof. Let $V := R/\mathfrak{a}_m$ which is a finite dimensional vector space. Then V has two filtrations given by

$$\mathcal{F}^r_{\mathfrak{a}_\bullet} V := (\mathfrak{a}_r + \mathfrak{a}_m)/\mathfrak{a}_m \quad \text{and} \quad \mathcal{F}^s_{\mathfrak{b}_\bullet} V := (\mathfrak{b}_s + \mathfrak{a}_m)/\mathfrak{a}_m.$$

By [1, Lemma 3.1], there exists a basis $\bar{f}_i$ $(1 \leq i \leq N)$ of V that is compatible with both filtrations $\mathcal{F}_{\mathfrak{a}_\bullet}$ and $\mathcal{F}_{\mathfrak{b}_\bullet}$. We can lift each $\bar{f}_i$ to some element

$f_i \in R$ such that $\{f_1, \cdots, f_N\}$ is compatible with $\mathfrak{b}_\bullet$ (it suffices to lift each $\bar{f}_i \in \mathcal{F}^s_{\mathfrak{b}_\bullet} V \backslash \mathcal{F}^{s+1}_{\mathfrak{b}_\bullet} V$ to some $f_i \in \mathfrak{b}_s$). On the other hand, since $\bar{f}_i$ is compatible with $\mathcal{F}_{\mathfrak{a}_\bullet}$, any such lift is automatically compatible with $\mathfrak{a}_\bullet$ (i.e. for all $r \le m$, $f_i \in \mathfrak{a}_r$ if and only if $\bar{f}_i \in \mathcal{F}^r_{\mathfrak{a}_\bullet} V$). □

2.2. The space of valuations

2.2.1. Valuations. Let X be a variety defined over k. A *real valuation* of its function field $K(X)$ is a non-constant map $v\colon K(X)^* \to \mathbb{R}$, satisfying:

- $v(fg) = v(f) + v(g)$;
- $v(f+g) \ge \min\{v(f), v(g)\}$;
- $v(k^*) = 0$.

We set $v(0) = +\infty$. A valuation v gives rise to a valuation ring

$$\mathcal{O}_v := \{f \in K(X) \mid v(f) \ge 0\}.$$

We say a valuation v is *centered at* a scheme-theoretic point $x = c_X(v) \in X$ if we have a local inclusion $\mathcal{O}_{X,x} \hookrightarrow \mathcal{O}_v$ of local rings. Notice that the center of a valuation, if exists, is unique since X is separated. Denote by Val_X the set of nontrivial real valuations of $K(X)$ that admit centers on X. For a closed point $x \in X$, we further denote by $\mathrm{Val}_{X,x}$ the set of real valuations of $k(X)$ centered at $x \in X$.

For each valuation $v \in \mathrm{Val}_{X,x}$ and any positive integer m, we define the valuation ideal

$$\mathfrak{a}_m(v) := \{f \in \mathcal{O}_{X,x} \mid v(f) \ge m\}.$$

It is clear that $\mathfrak{a}_\bullet = \{\mathfrak{a}_m\}_{m\in\mathbb{N}}$ form a decreasing graded sequence of $\mathfrak{m}_x$-primary ideals.

Let (X, Δ) be a pair. We denote by

$$A_{X,\Delta}\colon \mathrm{Val}_X \to \mathbb{R} \cup \{+\infty\}$$

the *log discrepancy function of valuations* as in [14] and [6, Theorem 3.1] which extends the standard definition of log discrepancies from divisors to all valuations in Val_X. It is possible that $A_{X,\Delta}(v) = +\infty$ for some $v \in \mathrm{Val}_X$, see e.g. [14, Remark 5.12]. We denote by Val^*_X the set of valuations $v \in \mathrm{Val}_X$ with $A_{X,\Delta}(v) < +\infty$ and set $\mathrm{Val}^*_{X,x} = \mathrm{Val}^*_X \cap \mathrm{Val}_{X,x}$ for a closed point $x \in X$. Note that $A_{X,\Delta}$ is strictly positive on Val_X if and only if (X, Δ) is klt.

Proposition 2.2. *Let $x \in (X, \Delta)$ be a klt singularity and let $v_0, v_1 \in \mathrm{Val}^*_{X,x}$. Then the graded sequences $\mathfrak{a}_\bullet(v_0)$ and $\mathfrak{a}_\bullet(v_1)$ of valuation ideals are linearly bounded by each other.*

Proof. This is a direct consequence of the Izumi type inequalities (see e.g. [20, Theorem 3.1]), which says that $\mathfrak{a}_\bullet(v_i)$ and $\{\mathfrak{m}_x^m\}_{m\in\mathbb{N}}$ are linearly bounded by each other. □

Definition 2.3 (Kollár Components)**.** Let $x \in (X, \Delta)$ be a klt singularity. A prime divisor S over (X, Δ) is a *Kollár component* if there is a birational morphism $\pi \colon Y \to X$ such that π is an isomorphism over $X \setminus \{x\}$, S is a prime divisor on Y, $\pi(S) = \{x\}$, $-S$ is $\mathbb{Q}$-Cartier and π-ample, and $(Y, \pi_*^{-1}\Delta + S)$ is plt. The map $\pi \colon Y \to X$ is called the plt blowup associated to the Kollár component S. By adjunction (see [15, Definition 4.2]) we may write

$$(K_Y + \pi_*^{-1}\Delta + S)|_S = K_S + \Delta_S,$$

where (S, Δ_S) is a log Fano pair.

2.2.2. Local volumes.

Definition 2.4. Let X be an n-dimensional normal variety and let $x \in X$ be a closed point. Following [10] we define the *volume* of a valuation $v \in \mathrm{Val}_{X,x}$ as

$$\mathrm{vol}(v) = \mathrm{vol}_{X,x}(v) = \limsup_{m\to\infty} \frac{\ell(\mathcal{O}_{X,x}/\mathfrak{a}_m(v))}{m^n/n!}.$$

where $\ell(\cdot)$ denotes the length of the Artinian module.

Thanks to the works of [10, 18, 9], the above limsup is actually a limit.

The following invariant, which was first defined in [20], plays a key role in our study of local stability.

Definition 2.5 ([20])**.** Let $x \in (X, \Delta)$ be an n-dimensional klt singularity. The *normalized volume function of valuations* $\widehat{\mathrm{vol}}_{(X,\Delta),x} : \mathrm{Val}_{X,x} \to (0, +\infty)$ is defined as

$$\widehat{\mathrm{vol}}_{(X,\Delta),x}(v) = \begin{cases} A_{X,\Delta}(v)^n \cdot \mathrm{vol}_{X,x}(v), & \text{if } v \in \mathrm{Val}^*_{X,x}; \\ +\infty, & \text{if } v \notin \mathrm{Val}^*_{X,x}. \end{cases}$$

We often denote it by $\widehat{\mathrm{vol}}_{X,\Delta}$ or $\widehat{\mathrm{vol}}$ when $x \in (X, \Delta)$ is clear from the context. The *volume of a klt singularity* $(x \in (X, \Delta))$ is defined as

$$\widehat{\mathrm{vol}}(x, X, \Delta) := \inf_{v\in \mathrm{Val}_{X,x}} \widehat{\mathrm{vol}}_{(X,\Delta),x}(v).$$

It has been known that the above infimum is indeed a minimum by [3] and that the minimizing valuations are always quasi-monomial by [32]. The study of $\widehat{\mathrm{vol}}_{X,\Delta}$ is closely related to K-stability of log Fano pairs, guided by the so-called *Stable Degeneration Conjecture* as formulated in [20, Conjecture 7.1] and [22, Conjecture 1.2]. See [21] for more background. Our Theorem 1.1 settles one part of this conjecture.

The following theorem from [22] motivates some of our arguments, although we do not need it in our proof.

Theorem 2.6. *Let $x \in (X = \mathbf{Spec}(R), \Delta)$ be a klt singularity, and v^{m} a minimizer of $\widehat{\mathrm{vol}}_{X,\Delta}$. Assume the associated grade ring $\mathrm{gr}_{v^{\mathrm{m}}}(R)$ is finitely generated. Denote by $X_0 = \mathbf{Spec}(\mathrm{gr}_{v^{\mathrm{m}}}(R))$ with the cone vertex o, Δ_0 the degeneration of Δ on X_0, ξ_v the Reeb orbit induced by v. Then $o \in (X_0, \Delta_0, \xi_v)$ is a K-semistable log Fano cone.*

Note that the finite generation assumption always holds when v is a divisorial valuation by [23, 3].

2.3. Log canonical thresholds

Definition 2.7. Given a klt pair (X, Δ) and a non-zero ideal $\mathfrak{a}$ on X, the *log canonical threshold* $\mathrm{lct}(X, \Delta; \mathfrak{a})$ of $\mathfrak{a}$ with respect to (X, Δ) is defined to be

$$\mathrm{lct}(X, \Delta; \mathfrak{a}) = \max\{t \geq 0 \,|\, (X, \Delta + \mathfrak{a}^t) \text{ is log canonical}\} = \inf_{v \in \mathrm{Val}_X^*} \frac{A_{X,\Delta}(v)}{v(\mathfrak{a})}.$$

For a graded sequence $\mathfrak{a}_\bullet = \{\mathfrak{a}_m\}_{m\in\mathbb{N}}$ of non-zero ideals on a klt pair (X, Δ), we can also define its log canonical threshold to be

$$\mathrm{lct}(X, \Delta; \mathfrak{a}_\bullet) := \limsup_m m \cdot \mathrm{lct}(X, \Delta; \mathfrak{a}_m) \in \mathbb{R}_{>0} \cup \{+\infty\}.$$

It is proved in [24, Theorem 27] that

$$\widehat{\mathrm{vol}}(x, X, \Delta) = \inf_{\mathfrak{a}_\bullet} \mathrm{lct}(X, \Delta; \mathfrak{a}_\bullet)^n \cdot \mathrm{mult}(\mathfrak{a}_\bullet), \tag{2.1}$$

where the infimum runs through all graded ideal sequences $\mathfrak{a}_\bullet$ of $\mathfrak{m}_x$-primary ideals, and $\mathrm{lct}(X, \Delta; \mathfrak{a}_\bullet)^n \cdot \mathrm{mult}(\mathfrak{a}_\bullet)$ is set to be $+\infty$ if $\mathrm{lct}(X, \Delta; \mathfrak{a}_\bullet) = +\infty$.

3. K-semistability of a valuation

This is the main section of this paper. We will first define the notion of K-semistability for valuations. Then we will show that for valuations, being K-semistable is the same as being a minimizer of the normalized volume function, and that there is a unique K-semistable valuation up to rescaling.

3.1. Definition of K-semistability for a valuation

In this subsection, we introduce a local version of S-invariant on the product of the valuation space $\mathrm{Val}^*_{X,x} \times \mathrm{Val}^*_{X,x}$ and use it to define the δ-invariant of valuations, which then naturally give the notion of K-semistability of a valuation.

Let $x \in (X = \mathbf{Spec}(R), \Delta)$ be a klt singularity. Fix a valuation $v_0 \in \mathrm{Val}^*_{X,x}$. By Proposition 2.2, for any valuation $v \in \mathrm{Val}^*_{X,x}$, the graded sequences of ideals $\mathfrak{a}_\bullet(v)$ and $\mathfrak{a}_\bullet(v_0)$ are linearly bounded by each other. By Lemma 2.1, for any $m \in \mathbb{N}$ there exist some $f_1, \dots, f_{N_m} \in R$ (where $N_m = \ell(R/\mathfrak{a}_m(v_0)))$ which are compatible with both $\mathfrak{a}_\bullet(v_0)$ and $\mathfrak{a}_\bullet(v)$ such that their images $\bar{f}_i$ form a basis of $R_m := R/\mathfrak{a}_m(v_0)$. We call such $\{f_1, \dots, f_{N_m}\}$ an (m, v)-*basis* (*with respect to* v_0). The valuation v induces a filtration $\mathcal{F}_v$ on R_m such that an element $\bar{f} \in R_m$ is contained in $\mathcal{F}^\lambda_v R_m$ ($\lambda \in \mathbb{R}$) if and only if there exists a lifting $f \in R$ of $\bar{f}$ such that $v(f) \geq \lambda$. (For a similar filtration in the global setting, see [5, 5.1.1]).

Lemma-Definition 3.1. The limit

$$\mathrm{vol}(v_0; v) := \lim_{m\to\infty} \frac{\ell(\mathcal{F}^m_v R_m)}{m^n/n!}$$

exists. Moreover, we have $\mathrm{vol}(v_0; v/t) = 0$ for all $t \gg 0$.

Proof. From the definition we have $\mathcal{F}^m_v R_m = (\mathfrak{a}_m(v) + \mathfrak{a}_m(v_0))/\mathfrak{a}_m(v_0) \cong \mathfrak{a}_m(v)/(\mathfrak{a}_m(v) \cap \mathfrak{a}_m(v_0))$, hence

$$\ell(\mathcal{F}^m_v R_m) = \ell(R/(\mathfrak{a}_m(v) \cap \mathfrak{a}_m(v_0))) - \ell(R/\mathfrak{a}_m(v)),$$

thus by [18, Theorem 3.8] we obtain

$$\lim_{m\to\infty} \frac{1}{m^n/n!}\ell(\mathcal{F}^m_v R_m) = \mathrm{mult}(\mathfrak{a}_\bullet(v) \cap \mathfrak{a}_\bullet(v_0)) - \mathrm{mult}(\mathfrak{a}_\bullet(v)). \tag{3.1}$$

Since $\mathfrak{a}_\bullet(v_0)$ and $\mathfrak{a}_\bullet(v)$ are linearly bounded by each other, we have $\mathfrak{a}_{Cm}(v) \subseteq \mathfrak{a}_m(v_0)$ for some constant $C > 0$. Thus $\mathcal{F}_v^{Cm} R_m = 0$ and

$$\mathrm{vol}(v_0; v/t) = \lim_{m\to\infty} \frac{\ell(\mathcal{F}_v^{tm} R_m)}{m^n/n!} = 0$$

for all $t \geq C$. □

Analogous to the global log Fano case, we set $\widetilde{S}_m(v_0; v) = \sum_{i=1}^{N_m} \lfloor v(f_i) \rfloor$, which doesn't depend on the choice of f_i; indeed it is not hard to check that

$$\widetilde{S}_m(v_0; v) = \sum_{i=0}^{+\infty} i \cdot \ell(\mathcal{F}_v^i R_m / \mathcal{F}_v^{i+1} R_m) = \sum_{i=1}^{+\infty} \ell(\mathcal{F}_v^i R_m).$$

We then define

$$S_m(v_0; v) := \frac{A_{X,\Delta}(v_0)}{\widetilde{S}_m(v_0; v_0)} \cdot \widetilde{S}_m(v_0; v),$$
$$S(v_0; v) := \frac{n+1}{n} \cdot \frac{A_{X,\Delta}(v_0)}{\mathrm{vol}(v_0)} \int_0^\infty \mathrm{vol}(v_0; v/t) \mathrm{d}t.$$

Remark 3.2. In the global non-Archimedean setting, a similar construction named the (logarithmic) relative volume of two norms is given in [7, Section 3]. However, we measure 'the relative volume' by taking a quotient instead of a difference.

Lemma 3.3. *For any $v_0, v \in \mathrm{Val}^*_{X,x}$, we have $S(v_0; v) = \lim_{m\to\infty} S_m(v_0; v)$. Moreover, the function $t \mapsto \mathrm{vol}(v_0; v/t)$ is continuous.*

Proof. We can embed (X, Δ) into a projective variety $(\bar{X}, \bar{\Delta})$. By [18, Lemma 3.9], we can find a sufficiently ample line bundle L such that the natural map

$$H^0(\bar{X}, L^m) \to H^0(\bar{X}, L^m \otimes \mathcal{O}_X/\mathfrak{a}_{2Cm}(v)) \tag{3.2}$$

is surjective for all $m \in \mathbb{N}$, where C is a positive integer such that $\mathfrak{a}_{Cm}(v) \subseteq \mathfrak{a}_m(v_0)$ and $\mathfrak{a}_{Cm}(v_0) \subseteq \mathfrak{a}_m(v)$ for all $m \in \mathbb{N}$. Note that this implies that the restriction map

$$h\colon H^0(\bar{X}, L^m) \to H^0(\bar{X}, L^m \otimes \mathcal{O}_X/\mathfrak{a}_m(v_0)) \cong R/\mathfrak{a}_m(v_0) \tag{3.3}$$

is also surjective, where the last isomorphism is given by a trivialization of L near x. For such L,

$$W_m := H^0(\bar{X}, L^m \otimes \mathfrak{a}_m(v_0)) \quad \text{and} \quad V_m := H^0(\bar{X}, L^m)$$

defines two graded linear series $W_\bullet$, $V_\bullet$ that contain ample series. The valuation v induces a filtration $\mathcal{F}_v$ on both $V_\bullet$ and $W_\bullet$ by setting $\mathcal{F}_v^\lambda V_m = \{s \in H^0(\bar{X}, L^m) \,|\, v(s) \geq \lambda\}$ and $\mathcal{F}_v^\lambda W_m = W_m \cap \mathcal{F}_v^\lambda V_m$.

Through (3.3), the image of $\mathcal{F}_v$ induces a filtration $\mathcal{F}_1$ on $R_m := R/\mathfrak{a}_m(v_0)$. We claim that it is the same as the filtration $\mathcal{F}_v^\bullet$ on R_m. Indeed, given an element $f \in \mathcal{F}_v^\lambda V_m$, it is clear that its image $\bar{f} \in R_m$ lies in $\mathcal{F}_v^\lambda R_m$. Conversely, if $0 \neq \bar{f} \in \mathcal{F}_v^\lambda R_m$, then it can be lifted to some $f \in R$ with $v(f) \geq \lambda$. Since

$$H^0(\bar{X}, L^m) \to H^0(\bar{X}, L^m \otimes \mathcal{O}_X/\mathfrak{a}_{Cm}(v))$$

is a surjective, there exists some $s \in \mathcal{F}_v^\lambda V_m$ such that s and f has the same image in $R/\mathfrak{a}_{Cm}(v)$. As $\mathfrak{a}_{Cm}(v) \subseteq \mathfrak{a}_m(v_0)$, we see that the restriction of s in R_m gives $\bar{f}$. This proves the claim.

Let $W_m^t = \mathcal{F}_v^{tm} W_m$ and $V_m^t = \mathcal{F}_v^{tm} V_m$. Then from the above claim we have $\ell(\mathcal{F}_v^{tm} R_m) = \dim V_m^t - \dim W_m^t$, hence

$$\mathrm{vol}(v_0; v/t) = \mathrm{vol}(V_\bullet^t) - \mathrm{vol}(W_\bullet^t),$$

which, by [4, Proposition 2.3], is continuous in t when $0 \leq t \leq C$ since $\mathrm{vol}(V_\bullet^C) \geq \mathrm{vol}(W_\bullet^C) > 0$ by (3.2); on the other hand, $\mathrm{vol}(v_0; v/t) = 0$ when $t \geq C$ as in Lemma 3.1, thus the function $t \mapsto \mathrm{vol}(v_0; v/t)$ is continuous everywhere.

We next prove $S(v_0; v) = \lim_{m\to\infty} S_m(v_0; v)$. We claim that

$$\lim_{m\to\infty} \frac{\widetilde{S}_m(v_0; v)}{m^{n+1}/n!} = \int_0^\infty \mathrm{vol}(v_0; v/t)\mathrm{d}t. \tag{3.4}$$

By definition, this is equivalent to

$$\lim_{m\to\infty} \frac{\sum_{i=1}^\infty \ell(\mathcal{F}_v^i R_m)}{m^{n+1}/n!} = \int_0^\infty \mathrm{vol}(v_0; v/t)\mathrm{d}t. \tag{3.5}$$

Let $\psi_m(t) = \frac{\ell(\mathcal{F}_v^{\lceil tm \rceil} R_m)}{m^n/n!}$. Then we may rewrite the expression in the above limit as $\int_0^\infty \psi_m(t)\mathrm{d}t$. Notice that $\lim_{m\to\infty} \psi_m(t) = \mathrm{vol}(v_0; v/t)$ and $\psi_m(t) =$

0 for all $t \ge C$ and all $m \in \mathbb{N}$. The equality (3.5) now follows from the dominated convergence theorem.

It is clear that $\mathrm{vol}(v_0; v_0/t) = \max\{(1-t^n)\mathrm{vol}(v_0), 0\}$ for all $t \ge 0$, thus taking $v = v_0$ in (3.4) we get

$$\lim_{m\to\infty} \frac{\widetilde{S}_m(v_0;v_0)}{m^{n+1}/n!} = \int_0^\infty \mathrm{vol}(v_0;v_0/t)\mathrm{d}t = \int_0^1 (1-t^n)\mathrm{vol}(v_0)\mathrm{d}t = \frac{n}{n+1}\mathrm{vol}(v_0),$$

hence

$$\lim_{m\to\infty} \frac{S_m(v_0;v)}{A_{X,\Delta}(v_0)} = \lim_{m\to\infty} \frac{\widetilde{S}_m(v_0;v)}{\widetilde{S}_m(v_0;v_0)} = \frac{n+1}{n} \cdot \frac{\int_0^\infty \mathrm{vol}(v_0;v/t)\mathrm{d}t}{\mathrm{vol}(v_0)}.$$

In other words, $S(v_0;v) = \lim_{m\to\infty} S_m(v_0;v)$. □

Definition 3.4. A valuation $v_0 \in \mathrm{Val}^*_{X,x}$ is said to be *K-semistable* if $A_{X,\Delta}(v) \ge S(v_0;v)$ for all $v \in \mathrm{Val}^*_{X,x}$. We also define the stability threshold $\delta(v_0)$ of a valuation $v_0 \in \mathrm{Val}^*_{X,x}$ as $\delta(v_0) = \inf_v \delta(v_0;v)$ where $\delta(v_0;v) = \frac{A_{X,\Delta}(v)}{S(v_0;v)}$ and the infimum runs over all valuations $v \in \mathrm{Val}^*_{X,x}$.

Remark 3.5. The notion of K-semistable valuation has been previously defined for valuations which are quasi-monomial, and whose associated graded rings are finitely generated (see [31, Page 819] or [21, Theorem 4.14]). Whereas it is known that minimizers of $\widehat{\mathrm{vol}}_{X,\Delta}$ are quasi-monomial by [32], the finite generation of the associated graded rings remains open. Therefore, while Definition 3.4 is conjecturally equivalent to the previous definition, we circumvent the issue of finite generation.

From the definition it is clear that $\delta(v_0) = \delta(\lambda \cdot v_0)$, thus v_0 is K-semistable if and only if λv_0 is K-semistable for some $\lambda > 0$. In the special case of divisorial valuations induced by Kollár components, we have the following equivalent characterization, which serves as the motivation of our definition.

Theorem 3.6. *Let S be a Kollár component over $x \in (X,\Delta)$ (see Definition 2.3). Then we have $\delta(\mathrm{ord}_S) \ge \min\{1, \delta(S,\Delta_S)\}$ and the valuation ord_S is K-semistable if and only if the log Fano pair (S,Δ_S) is K-semistable.*

Proof. Let $v_0 = \mathrm{ord}_S$ and let $v \in \mathrm{Val}^*_{X,x}$. We know there is an ample $\mathbb{Q}$-divisor $L \sim_{\mathbb{Q}} -S|_S$ on S such that the short exact sequences

$$0 \to \mathcal{O}_Y(-(m+1)S) \to \mathcal{O}_Y(-mS) \to \mathcal{O}_S(mL) \to 0$$

hold (see e.g. [15, Section 4.1]), where by convention $\mathcal{O}_S(mL) := \mathcal{O}_S(\lfloor mL \rfloor)$. Since $R^1\pi_*\mathcal{O}_Y(-mS) = 0$ for all $m \geq 0$ by Kawamata-Viehweg vanishing, we get isomorphisms

$$\mathfrak{a}_m/\mathfrak{a}_{m+1} \cong H^0(S, mL)$$

where $\mathfrak{a}_m := \mathfrak{a}_m(\mathrm{ord}_S)$. After identifying $\oplus_{m\in\mathbb{N}}\mathfrak{a}_m/\mathfrak{a}_{m+1}$ with $R(S, L) := \oplus_{m\in\mathbb{N}}H^0(S, mL)$, the valuation v induces a filtration $\mathcal{F}_v$ on the section ring $R(S, L)$.

We claim that

$$S(v_0; v) = A_{X,\Delta}(v_0) \cdot S(L; \mathcal{F}_v), \tag{3.6}$$

where $S(L; \mathcal{F}_v)$ denotes the S-invariant of a filtration as in [4, Section 2.5-2.6] (we will also use its approximated versions $S_m(L; \mathcal{F}_v)$ from *loc. cit.*). To see this, we note that

$$\widetilde{S}_m(v_0; v) = \sum_{i=1}^{\infty}\sum_{j=1}^{m} \ell(\mathcal{F}_v^i(\mathfrak{a}_{j-1}/\mathfrak{a}_j)) = \sum_{j=1}^{m} j \cdot h^0(S, jL) \cdot S_j(L; (\mathcal{F}_v)_{\mathbb{N}}).$$

By [4, Corollary 2.12], we have $S_j(L; (\mathcal{F}_v)_{\mathbb{N}}) \to S(L; (\mathcal{F}_v)_{\mathbb{N}}) = S(L; \mathcal{F}_v)$ $(j \to \infty)$, thus as $h^0(S, jL) = (L^{n-1})\frac{j^{n-1}}{(n-1)!} + O(j^{n-2})$, we obtain

$$\lim_{m\to\infty} \frac{\widetilde{S}_m(v_0; v)}{m^{n+1}/(n+1)!} = n(L^{n-1}) \cdot S(L; \mathcal{F}_v).$$

Thus

$$\frac{S(v_0; v)}{A_{X,\Delta}(v_0)} = \lim_{m\to\infty} \frac{\widetilde{S}_m(v_0; v)}{\widetilde{S}_m(v_0; v_0)} = \frac{S(L; \mathcal{F}_v)}{S(L; \mathcal{F}_{v_0})}.$$

On the other hand, it is clear from the definition that $S(L; \mathcal{F}_{v_0}) = 1$ (the filtration $\mathcal{F}_{v_0}$ satisfies $\mathcal{F}_{v_0}^j H^0(S, mL) = H^0(S, mL)$ if $j \leq m$ and $\mathcal{F}_{v_0}^j H^0(S, mL) = 0$ when $j \geq m + 1$), which proves (3.6).

Since $-(K_S + \Delta_S) \sim_{\mathbb{Q}} -(K_Y + \pi_*^{-1}\Delta + S)|_S \sim A_{X,\Delta}(v_0) \cdot L$, we may rewrite (3.6) as

$$S(v_0; v) = S(-(K_S + \Delta_S); \mathcal{F}_v). \tag{3.7}$$

Let $m \in \mathbb{N}$ be a sufficiently divisible integer and let $f_1, \cdots, f_N \in \mathfrak{a}_{Am}$ (where $A := A_{X,\Delta}(v_0)$) be the lift of a basis $\{\bar{f}_i\}$ of $H^0(S, -m(K_S + \Delta_S)) =$

$H^0(S, mAL)$. Let

$$N := \dim H^0(S, mAL) \quad \text{and} \quad D = \frac{1}{mN}\sum_{i=1}^{N}\{f_i = 0\}.$$

Then we have $\pi^*D = A \cdot S + \widetilde{D}$ where $\widetilde{D}|_S$ is an m-basis type $\mathbb{Q}$-divisor of the log Fano pair (S, Δ_S) (see [11, 4]). Let $\delta_m := \min\{1, \delta_m(S, \Delta_S)\}$. From the definition of stability thresholds, we know that the pair $(S, \Delta_S + \delta_m \widetilde{D}|_S)$ is lc, thus $(Y, S + \pi_*^{-1}\Delta + \delta_m \widetilde{D})$ is also lc by inversion of adjunction. We have

$$K_Y + S + \pi_*^{-1}\Delta + \delta_m D \geq \pi^*(K_X + \Delta + \delta_m D),$$

hence $(X, \Delta + \delta_m D)$ is lc, which implies that $A_{X,\Delta}(v) \geq \delta_m \cdot v(D)$ for any $v \in \mathrm{Val}^*_{X,x}$ and any D as above.

If we choose $\bar{f}_i$ to be compatible with the filtration $\mathcal{F}_v$, then $v(D) = S_m(-(K_S + \Delta_S); \mathcal{F}_v)$ and we obtain

$$A_{X,\Delta}(v) \geq \delta_m \cdot S_m(-(K_S + \Delta_S); \mathcal{F}_v).$$

Letting $m \to \infty$, we deduce $\delta(v_0) \geq \min\{1, \delta(S, \Delta_S)\}$ using (3.7). In particular, if (S, Δ_S) is K-semistable, then $v_0 = \mathrm{ord}_S$ is K-semistable.

Conversely, if v_0 is K-semistable, then we have

$$A_{X,\Delta}(v) \geq S(v_0; v) = S(-(K_S + \Delta_S); \mathcal{F}_v)$$

for any $v \in \mathrm{Val}^*_{X,x}$. Let $c := v(\mathfrak{a}_\bullet(v_0))$. We may shift the filtration $\mathcal{F}_v$ by c to get a new filtration $\mathcal{F}$ on $R(S, L)$, i.e., $\mathcal{F}^\lambda H^0(S, mL) := \mathcal{F}_v^{\lambda + cm} H^0(S, mL)$. It satisfies $\mathcal{F}^0 H^0(S, mL) = H^0(S, mL)$ as $v(\mathfrak{a}_m) \geq cm$ for all $m \in \mathbb{N}$. By [4, Corollary 2.10], there exists some ϵ_m with $\lim_{m\to\infty} \epsilon_m = 1$ such that for all $m \in \mathbb{N}$ and any $v \in \mathrm{Val}^*_{X,x}$,

$$\begin{aligned}
\epsilon_m \cdot S_m(-(K_S + \Delta_S); \mathcal{F}) &\leq S(-(K_S + \Delta_S); \mathcal{F}) \\
&= S(-(K_S + \Delta_S); \mathcal{F}_v) - A_{X,\Delta}(v_0) \cdot v(\mathfrak{a}_\bullet(v_0)) \\
&\leq A_{X,\Delta}(v) - A_{X,\Delta}(v_0) \cdot v(\mathfrak{a}_\bullet(v_0)) \ .
\end{aligned}$$

For sufficiently divisible integer m and with $\{f_i\}$, D and $\widetilde{D}$ as before, this means that $(Y, S + \pi_*^{-1}\Delta + \epsilon_m \widetilde{D})$ is lc. By adjunction we see that $(S, \Delta_S + \epsilon_m \widetilde{D}|_S)$ is lc. Since $\widetilde{D}|_S$ can be any m-basis type $\mathbb{Q}$-divisor of (S, Δ_S), we conclude that $\delta_m(S, \Delta_S) \geq \epsilon_m$. Letting $m \to \infty$ we obtain $\delta(S, \Delta_S) \geq 1$, i.e. (S, Δ_S) is K-semistable. $\square$

In general, if (S, Δ_S) is not K-semistable, then the inequality in Theorem 3.6 could be strict.

3.2. K-semistable valuation is the unique minimizer

In this subsection, we show that if $\mathrm{Val}^*_{X,x}$ contains a K-semistable valuation, then it is the unique minimizer of $\widehat{\mathrm{vol}}_{X,\Delta}$ up to rescaling.

Theorem 3.7. *Let $x \in (X = \mathbf{Spec}(R), \Delta)$ be a klt singularity and let $v_0 \in \mathrm{Val}^*_{X,x}$. Assume that v_0 is K-semistable. Then*

1. *v_0 is a minimizer of $\widehat{\mathrm{vol}}_{X,\Delta}$, i.e., $\widehat{\mathrm{vol}}(x, X, \Delta) = \widehat{\mathrm{vol}}(v_0)$;*
2. *if $v_1 \in \mathrm{Val}^*_{X,x}$ is another minimizer of $\widehat{\mathrm{vol}}_{X,\Delta}$, then $v_1 = \lambda v_0$ for some $\lambda > 0$.*

For the proof we need some auxiliary calculation. For each valuation $v \in \mathrm{Val}^*_{X,x}$ and every integer $m > 0$, we set

$$w_m(v) := \min \sum_{i=1}^{m} \lfloor v(f_i) \rfloor$$

where the minimum runs over all $f_1, \cdots, f_m \in R \setminus \{0\}$ that are compatible with $\mathfrak{a}_\bullet(v)$. Clearly the minimum is achieved by some $f_1, \cdots, f_m$ that are compatible with $\mathfrak{a}_\bullet(v)$, if and only if for the unique integer r satisfying $\ell(R/\mathfrak{a}_{r+1}(v)) > m \geq \ell(R/\mathfrak{a}_r(v))$, $f_1, \cdots, f_m$ span $R/\mathfrak{a}_r(v)$ and form a linearly independent set in $R/\mathfrak{a}_{r+1}(v)$.

Lemma 3.8. *We have*

$$\lim_{m\to\infty} \frac{w_m(v)}{m^{\frac{n+1}{n}}} = \frac{n}{n+1} \cdot \left(\frac{n!}{\mathrm{vol}(v)}\right)^{1/n}.$$

Proof. Let $\mathfrak{a}_\bullet = \mathfrak{a}_\bullet(v)$. From the above description we have

$$0 \leq w_m(v) - \sum_{i=0}^{r-1} i \cdot \ell(\mathfrak{a}_i/\mathfrak{a}_{i+1}) \leq r \cdot \ell(\mathfrak{a}_r/\mathfrak{a}_{r+1})$$

for all integers $r, m > 0$ with $\ell(R/\mathfrak{a}_r) \leq m < \ell(R/\mathfrak{a}_{r+1})$. Note that this implies

$$\lim_{r\to\infty} \frac{m}{r^n/n!} = \mathrm{vol}(v).$$

We also have $\lim_{r\to\infty}\frac{\ell(\mathfrak{a}_r/\mathfrak{a}_{r+1})}{r^n}=0$ and

$$\begin{aligned}\lim_{r\to\infty}\frac{\sum_{i=0}^r i\cdot\ell(\mathfrak{a}_i/\mathfrak{a}_{i+1})}{r^{n+1}/n!}&=\lim_{r\to\infty}\frac{r\cdot\ell(R/\mathfrak{a}_{r+1})-\sum_{i=0}^r\ell(R/\mathfrak{a}_i)}{r^{n+1}/n!}\\&=\left(1-\frac{1}{n+1}\right)\mathrm{vol}(v).\end{aligned}$$

Thus $\lim_{r\to\infty}\frac{w_m(v)}{r^{n+1}/n!}=\frac{n}{n+1}\mathrm{vol}(v)$ and

$$\lim_{m\to\infty}\frac{w_m(v)}{m^{\frac{n+1}{n}}}=\lim_{r\to\infty}\left(\frac{w_m(v)}{r^{n+1}/n!}\cdot\frac{r^n/n!}{m}\cdot\frac{r}{m^{1/n}}\right)=\frac{n}{n+1}\cdot\left(\frac{n!}{\mathrm{vol}(v)}\right)^{1/n}.$$

□

Proof of Theorem 3.7. We first prove that v_0 is a minimizer of $\widehat{\mathrm{vol}}_{X,\Delta}$, i.e. $\widehat{\mathrm{vol}}(v)\ge\widehat{\mathrm{vol}}(v_0)$ for every valuation $v\in\mathrm{Val}^*_{X,x}$. Without loss of generality we may assume that $A_{X,\Delta}(v_0)=A_{X,\Delta}(v)=1$. Let $m\in\mathbb{N}$ and let $f_1,\cdots,f_{N_m}$ be an (m,v)-basis with respect to v_0 (where $N_m=\ell(R/\mathfrak{a}_m(v_0))$).

Since v_0 is K-semistable, we have

$$1=A_{X,\Delta}(v)\ge S(v_0;v). \tag{3.8}$$

From the definition it is clear that $\widetilde{S}_m(v_0;v_0)=w_{N_m}(v_0)$ and $\widetilde{S}_m(v_0;v)\ge w_{N_m}(v)$, hence by Lemma 3.8 we get

$$S(v_0;v)\ge\lim_{m\to\infty}\frac{w_{N_m}(v)}{w_{N_m}(v_0)}=\left(\frac{\mathrm{vol}(v_0)}{\mathrm{vol}(v)}\right)^{\frac{1}{n}}.$$

Combined with (3.8) we immediately have

$$\widehat{\mathrm{vol}}(v)=\mathrm{vol}(v)\ge\mathrm{vol}(v_0)=\widehat{\mathrm{vol}}(v_0),$$

i.e. v_0 minimizes the normalized volume function $\widehat{\mathrm{vol}}_{X,\Delta}$.

Now assume $\mathrm{vol}(v_0)=\mathrm{vol}(v)$. We claim that

$$\mathrm{vol}(v_0)=\mathrm{vol}(v)=\mathrm{mult}(\mathfrak{a}_\bullet(v_0)\cap\mathfrak{a}_\bullet(v)). \tag{3.9}$$

Suppose this is not the case, then $\mathrm{vol}(v_0;v)>0$ by (3.1). Thus by the continuity part of Lemma 3.3, there exists some $\epsilon>0$ such that

$$\gamma:=\mathrm{vol}\left(v_0;\frac{v}{1+2\epsilon}\right)=\lim_{m\to\infty}\frac{\ell(\mathcal{F}_v^{(1+2\epsilon)m}(R/\mathfrak{a}_m(v_0)))}{m^n/n!}>0.$$

For each $m \in \mathbb{N}$, let k_m be the unique integer k determined by

$$\ell(R/\mathfrak{a}_{k-1}(v)) \leq N_m < \ell(R/\mathfrak{a}_k(v)).$$

Since $\mathrm{vol}(v_0) = \mathrm{vol}(v)$, we have $\lim_{m\to\infty} \frac{k_m}{m} = 1$ and thus $k_m < (1+\epsilon)m$ for sufficiently large m. Let $g_1, \cdots, g_{N_m} \in R \setminus \{0\}$ be a sequence that's compatible with $\mathfrak{a}_\bullet(v)$ such that

$$w_{N_m}(v) = \sum_{i=1}^{N_m} \lfloor v(g_i) \rfloor.$$

Then by construction we have $v(g_i) \leq k_m$ for all $1 \leq i \leq N_m$ and the inequality

$$\widetilde{S}_m(v_0; v) = \sum_{i=1}^{N_m} \lfloor v(f_i) \rfloor \geq w_{N_m}(v)$$

can be upgraded as

$$\sum_{i=1}^{N_m} \min\{\lfloor v(f_i) \rfloor, k_m\} \geq \sum_{i=1}^{N_m} \lfloor v(g_i) \rfloor = w_{N_m}(v).$$

In particular, for sufficiently large m we get

$$\begin{aligned}
\sum_{i=1}^{N_m} \lfloor v(f_i) \rfloor &= \sum_{j=0}^{\infty} j \cdot \ell(\mathcal{F}_v^j R_m / \mathcal{F}_v^{j+1} R_m) \\
&\geq \sum_{j=0}^{\infty} \min\{j, k_m\} \cdot \ell(\mathcal{F}_v^j R_m / \mathcal{F}_v^{j+1} R_m) \\
&\quad + ((1+2\epsilon)m - k_m) \cdot \ell(\mathcal{F}_v^{(1+2\epsilon)m} R_m) \\
&\geq \sum_{j=0}^{\infty} \min\{j, k_m\} \cdot \ell(\mathcal{F}_v^j R_m / \mathcal{F}_v^{j+1} R_m) + \epsilon m \cdot \frac{\gamma m^n}{n!} \\
&= \sum_{i=1}^{N_m} \min\{\lfloor v(f_i) \rfloor, k_m\} + \frac{\epsilon \gamma m^{n+1}}{n!} \\
&\geq w_{N_m}(v) + \frac{\epsilon \gamma m^{n+1}}{n!},
\end{aligned}$$

where $R_m = R/\mathfrak{a}_m(v_0)$. Dividing by $\sum\lfloor v_0(f_i)\rfloor = \widetilde{S}_m(v_0;v_0) = w_{N_m}(v_0) = O(m^{n+1})$ and letting $m\to\infty$, we obtain

$$1 \geq S(v_0;v) = \lim_{m\to\infty}\frac{\widetilde{S}_m(v_0;v)}{\widetilde{S}_m(v_0;v_0)} > \lim_{m\to\infty}\frac{w_{N_m}(v)}{w_{N_m}(v_0)} = \left(\frac{\mathrm{vol}(v_0)}{\mathrm{vol}(v)}\right)^{\frac{1}{n}}$$

where the last equality follows from Lemma 3.8, hence $\mathrm{vol}(v) > \mathrm{vol}(v_0)$, a contradiction. This proves the claim (3.9). By the following Lemma 3.9, it implies $v = v_0$ and we are done. □

The following result, which is an improvement of [23, Proposition 2.7], is used in the above proof.

Lemma 3.9. *Let $x\in X = \mathbf{Spec}(R)$ be a singularity and let $v_0, v_1 \in \mathrm{Val}^*_{X,x}$. Assume that*

$$\mathrm{vol}(v_0) = \mathrm{vol}(v_1) = \mathrm{mult}(\mathfrak{a}_\bullet(v_0)\cap\mathfrak{a}_\bullet(v_1)) > 0.$$

Then $v_0 = v_1$.

Proof. We prove by contradiction. Assume that $v_0(f)\neq v_1(f)$ for some $f\in R$. Without loss of generality we may assume that $v_0(f) = \ell_0 > \ell_1 = v_1(f)$. Replacing f by f^k for some $k\in\mathbb{N}$ we may further assume that $\ell_0\geq \ell_1+1$. For $v\in\mathrm{Val}^*_{X,x}$ and $r\geq 0$, let $\mathfrak{a}_r(v) = \{f\in R\,|\,v(f)\geq r\}$. Let $\mathfrak{b}_r = \mathfrak{a}_r(v_0)\cap\mathfrak{a}_r(v_1)$ and $\mathfrak{c}_r = \mathfrak{a}_r(v_0)\cap\mathfrak{a}_{2r}(v_1)$ where $r\geq 0$.

For every $m\in\mathbb{N}$ and every $s\in\mathfrak{b}_m$, we have $v_0(f^m s) = m\cdot v_0(f) + v_0(s) \geq m(\ell_0+1)$, thus multiplication by f^m induces a map

$$\mathfrak{b}_m \xrightarrow{f^m} \mathfrak{a}_{m(\ell_0+1)}(v_0) \to \mathfrak{a}_{m(\ell_0+1)}(v_0)/\mathfrak{b}_{m(\ell_0+1)}$$

whose kernel is contained in $\mathfrak{c}_m$ (since $v_1(f^m s)\geq m(\ell_0+1)$ implies $v_1(s)\geq m(\ell_0+1) - m\ell_1 \geq 2m$). It follows that

$$\ell(\mathfrak{a}_{m(\ell_0+1)}(v_0)/\mathfrak{b}_{m(\ell_0+1)}) \geq \ell(\mathfrak{b}_m/\mathfrak{c}_m) \tag{3.10}$$

for all $m\in\mathbb{N}$. By [23, Proposition 2.7], there exists some $0\neq g\in\mathfrak{m}_x$ such that $\ell_2 = v_1(g) > v_0(g) > 0$. For every $m\in\mathbb{N}$ and every $s\in\mathfrak{c}_m$, we then have

$$v_1(g^m s) = m\cdot v_1(g) + v_1(s) \geq m(\ell_2+2),$$

thus multiplication by g^m induces a map

$$\mathfrak{c}_m \xrightarrow{g^m} \mathfrak{a}_{m(\ell_2+2)}(v_1) \to \mathfrak{a}_{m(\ell_2+2)}(v_1)/\mathfrak{b}_{m(\ell_2+2)}$$

whose kernel is contained in $\mathfrak{b}_{2m}$ (if $v_0(g^m s) \geq m(\ell_2+2)$ then as $v_0(g) \leq \ell_2$ we get $v_0(s) \geq 2m$). It follows that

$$\ell(\mathfrak{a}_{m(\ell_2+2)}(v_1)/\mathfrak{b}_{m(\ell_2+2)}) \geq \ell(\mathfrak{c}_m/\mathfrak{b}_{2m}) \tag{3.11}$$

for all $m \in \mathbb{N}$. Combining (3.10) and (3.11) we see that

$$\begin{aligned}
&(\ell_0+1)^n(\mathrm{mult}(\mathfrak{b}_\bullet) - \mathrm{vol}(v_0)) + (\ell_2+2)^n(\mathrm{mult}(\mathfrak{b}_\bullet) - \mathrm{vol}(v_1)) \\
=& \lim_{m\to\infty} \frac{\ell(\mathfrak{a}_{m(\ell_0+1)}(v_0)/\mathfrak{b}_{m(\ell_0+1)})}{m^n/n!} + \lim_{m\to\infty} \frac{\ell(\mathfrak{a}_{m(\ell_2+2)}(v_1)/\mathfrak{b}_{m(\ell_2+2)})}{m^n/n!} \\
\geq& \lim_{m\to\infty} \frac{\ell(\mathfrak{b}_m/\mathfrak{c}_m) + \ell(\mathfrak{c}_m/\mathfrak{b}_{2m})}{m^n/n!} = \lim_{m\to\infty} \frac{\ell(\mathfrak{b}_m/\mathfrak{b}_{2m})}{m^n/n!} \\
=&(2^n-1)\mathrm{mult}(\mathfrak{b}_\bullet) > 0,
\end{aligned}$$

which contradicts our assumption. Thus $v_0(f) = v_1(f)$ for all $f \in R$ as desired. □

3.3. Every minimizer is K-semistable

In this subsection, we show that every valuation that minimizes the normalized volume function is K-semistable. Combined with Theorem 3.7, this proves the uniqueness of the minimizer.

Theorem 3.10. *Let $x \in (X = \mathbf{Spec}(R), \Delta)$ be a klt singularity and let $v_0 \in \mathrm{Val}^*_{X,x}$ be a minimizer of the normalized volume function $\widehat{\mathrm{vol}}_{X,\Delta}$. Then v_0 is K-semistable.*

In other words, we will show that $A_{X,\Delta}(v) \geq S(v_0; v)$ for every valuation $v \in \mathrm{Val}^*_{X,x}$. Inspired by the argument of [19], we consider a family $\mathfrak{b}_{\bullet,t}$ ($t \in \mathbb{R}_{\geq 0}$) of graded sequences of ideals that interpolate the valuation ideal sequences of v_0 and v, defined as follows: we set $\mathfrak{b}_{\bullet,0} = \mathfrak{a}_\bullet(v_0)$; when $t > 0$, we set

$$\mathfrak{b}_{m,t} = \sum_{i=0}^{m} \mathfrak{a}_{m-i}(v_0) \cap \mathfrak{a}_i(tv). \tag{3.12}$$

Roughly speaking, the ideal $\mathfrak{b}_{m,t}$ is generated by elements $f \in R$ with $v_0(f) + t \cdot v(f) \geq m$. By (2.1), we have

$$\mathrm{lct}(\mathfrak{b}_{\bullet,t})^n \cdot \mathrm{mult}(\mathfrak{b}_{\bullet,t}) \geq \widehat{\mathrm{vol}}(v_0) = \mathrm{lct}(\mathfrak{b}_{\bullet,0})^n \cdot \mathrm{mult}(\mathfrak{b}_{\bullet,0}).$$

To relate this to the K-semistability of v_0, the idea is to take the derivative of the above normalized multiplicities at $t = 0$, which was a technique introduced in [19]. To do so we analyze the log canonical thresholds and multiplicities of $\mathfrak{b}_{\bullet,t}$.

3.3.1. Log canonical thresholds of summations. We first establish an inequality for the log canonical thresholds of graded sequences of ideals. Given two graded sequences of ideals $\mathfrak{a}_\bullet$ and $\mathfrak{b}_\bullet$, we define $\mathfrak{c}_\bullet := \mathfrak{a}_\bullet \boxplus \mathfrak{b}_\bullet$ by setting

$$\mathfrak{c}_m = (\mathfrak{a} \boxplus \mathfrak{b})_m = \sum_{i=0}^{m} \mathfrak{a}_i \cap \mathfrak{b}_{m-i}.$$

It is easy to verify that $\mathfrak{c}_\bullet$ is also a graded sequence of ideals. Note that our definition differs from the usual sum of ideal sequences (see e.g. [27]) since we use intersections of ideals rather than taking product.

Theorem 3.11. *Under the above notation, assume $\mathfrak{a}_\bullet$ and $\mathfrak{b}_\bullet$ are graded sequences of $\mathfrak{m}_x$-primary ideals. Then we have*

$$\mathrm{lct}(\mathfrak{c}_\bullet) \le \mathrm{lct}(\mathfrak{a}_\bullet) + \mathrm{lct}(\mathfrak{b}_\bullet).$$

We denote by $\mathcal{J}(\mathfrak{a}^t)$ the multiplier ideal of a fractional ideal and similarly by $\mathcal{J}(\mathfrak{a}_\bullet^t)$ the asymptotic multiplier ideal of a graded sequence of ideals $\mathfrak{a}_\bullet$ with exponent t (see [17] for details). The above inequality will follow from a summation formula of multiplier ideals.

Lemma 3.12. *For any two graded sequences of ideals $\mathfrak{a}_\bullet$, $\mathfrak{b}_\bullet$ and any $t > 0$, we have*

$$\mathcal{J}(\mathfrak{c}_\bullet^t) \subseteq \sum_{\lambda+\mu=t} \mathcal{J}(\mathfrak{a}_\bullet^\lambda) \cap \mathcal{J}(\mathfrak{b}_\bullet^\mu) \tag{3.13}$$

where $\mathfrak{c}_\bullet = \mathfrak{a}_\bullet \boxplus \mathfrak{b}_\bullet$.

Proof. We follow the proof of [28, Proposition 4.10]. Let m be a sufficiently large and divisible integer such that $\mathcal{J}(\mathfrak{c}_\bullet^t) = \mathcal{J}(\mathfrak{c}_m^{t/m})$. By the summation formula of multiplier ideals (see [28, Theorem 0.1(2)]), which says that for any two ideals $\mathfrak{a}$ and $\mathfrak{b}$,

$$\mathcal{J}\big((\mathfrak{a}+\mathfrak{b})^t\big) = \sum_{t_1+t_2=t} \mathcal{J}(\mathfrak{a}^{t_1} \cdot \mathfrak{b}^{t_2}),$$

we have

$$\mathcal{J}(\mathfrak{c}_m^{t/m}) = \mathcal{J}\left(\left(\sum_{i=0}^{m} \mathfrak{a}_i \cap \mathfrak{b}_{m-i}\right)^{t/m}\right) = \sum_{t_0+\cdots+t_m=t/m} \mathcal{J}\left(\prod_{i=0}^{m} (\mathfrak{a}_i \cap \mathfrak{b}_{m-i})^{t_i}\right).$$

(The right hand side is a finite sum.) Since $\mathfrak{a}_i^{m!/i} \subseteq \mathfrak{a}_{m!}$, each individual term in the above right hand side is contained in

$$\mathcal{J}\left(\prod_{i=0}^{m} \mathfrak{a}_i^{t_i}\right) \subseteq \mathcal{J}\left(\prod_{i=0}^{m} \mathfrak{a}_{m!}^{\frac{it_i}{m!}}\right) = \mathcal{J}\left(\mathfrak{a}_{m!}^{\lambda/m!}\right) \subseteq \mathcal{J}(\mathfrak{a}_\bullet^{\lambda})$$

where $\lambda = \sum_{i=0}^{m} it_i$. By symmetry, it is also contained in $\mathcal{J}(\mathfrak{b}_\bullet^{\mu})$ where $\mu = \sum_{i=0}^{m}(m-i)t_i$. Note that $\lambda + \mu = \sum_{i=0}^{m} mt_i = m \cdot \frac{t}{m} = t$, thus every

$$\mathcal{J}\left(\prod_{i=0}^{m} (\mathfrak{a}_i \cap \mathfrak{b}_{m-i})^{t_i}\right) \subseteq \mathcal{J}(\mathfrak{a}_\bullet^{\lambda}) \cap \mathcal{J}(\mathfrak{b}_\bullet^{\mu})$$

is contained in the right hand side of (3.13). This completes the proof. $\square$

Proof of Theorem 3.11. Let $\alpha = \operatorname{lct}(\mathfrak{a}_\bullet)$, $\beta = \operatorname{lct}(\mathfrak{b}_\bullet)$ and let $t = \alpha + \beta$. For any $\lambda, \mu \geq 0$ with $\lambda + \mu = t$ we have either $\lambda \geq \alpha$ or $\mu \geq \beta$, therefore $\mathcal{J}(\mathfrak{a}_\bullet^{\lambda}) \cap \mathcal{J}(\mathfrak{b}_\bullet^{\mu}) \subseteq \mathfrak{m}_x$. By Lemma 3.12 we see that $\mathcal{J}(\mathfrak{c}_\bullet^{t}) \subseteq \mathfrak{m}_x$ and hence $\operatorname{lct}(\mathfrak{c}_\bullet) \leq t = \operatorname{lct}(\mathfrak{a}_\bullet) + \operatorname{lct}(\mathfrak{b}_\bullet)$. $\square$

3.3.2. Multiplicities of a family of graded sequences of ideals. We next derive a formula for the multiplicities of $\mathfrak{b}_{\bullet,t}$.

Lemma 3.13. $\operatorname{mult}(\mathfrak{b}_{\bullet,t}) = \operatorname{vol}(v_0) - (n+1)\int_0^{\infty} \operatorname{vol}(v_0; v/u)\frac{t\,du}{(1+tu)^{n+2}}$.

Proof. By definition, we have

$$\operatorname{mult}(\mathfrak{b}_{\bullet,t}) = \lim_{m\to\infty} \frac{\ell(R/\mathfrak{b}_{m,t})}{m^n/n!}.$$

However, to derive the statement of the lemma, it is better to use a different formula, which follows from the above equality:

$$\operatorname{mult}(\mathfrak{b}_{\bullet,t}) = \lim_{m\to\infty} \frac{\sum_{j=1}^{m} \ell(R/\mathfrak{b}_{j,t})}{m^{n+1}/(n+1)!}. \tag{3.14}$$

For ease of notation, let $\mathfrak{a}_\bullet = \mathfrak{a}_\bullet(v_0)$. We have

$$\big(\mathfrak{a}_{j-\ell-1} \cap \mathfrak{a}_{\ell+1}(tv)\big) \cap \sum_{i=0}^{\ell} \big(\mathfrak{a}_{j-i} \cap \mathfrak{a}_i(tv)\big) = \mathfrak{a}_{j-\ell} \cap \mathfrak{a}_{\ell+1}(tv)$$

for all $0 \le \ell < j$ and we get short exact sequences

$$0 \to \frac{\mathfrak{a}_{j-\ell-1} \cap \mathfrak{a}_{\ell+1}(tv)}{\mathfrak{a}_{j-\ell} \cap \mathfrak{a}_{\ell+1}(tv)} \to \frac{R}{\sum_{i=0}^{\ell} \mathfrak{a}_{j-i} \cap \mathfrak{a}_i(tv)} \to \frac{R}{\sum_{i=0}^{\ell+1} \mathfrak{a}_{j-i} \cap \mathfrak{a}_i(tv)} \to 0.$$

Thus from the definition of $\mathfrak{b}_{\bullet,t}$, we get

$$\ell(R/\mathfrak{b}_{j,t}) = \ell(R/\mathfrak{a}_j) - \sum_{i=1}^{j} \ell(\mathcal{F}_v^{i/t}(\mathfrak{a}_{j-i}/\mathfrak{a}_{j-i+1})).$$

Summing over $j = 0, 1, \cdots, m$ we obtain

$$\begin{aligned}\sum_{j=0}^{m} \ell(R/\mathfrak{b}_{j,t}) &= \sum_{j=1}^{m} \ell(R/\mathfrak{a}_j) - \sum_{1\le i\le j\le m} \ell(\mathcal{F}_v^{i/t}(\mathfrak{a}_{j-i}/\mathfrak{a}_{j-i+1})) \\ &= \sum_{j=1}^{m} \ell(R/\mathfrak{a}_j) - \sum_{i=1}^{m} \ell(\mathcal{F}_v^{i/t}(R/\mathfrak{a}_{m-i+1}))\end{aligned}$$

Combining with (3.14), we deduce that

$$\mathrm{mult}(\mathfrak{b}_{\bullet,t}) = \mathrm{vol}(v_0) - (n+1) \cdot \lim_{m\to\infty} \frac{W_m}{m^{n+1}/n!} \tag{3.15}$$

where $W_m := \sum_{i=1}^{m} \ell(\mathcal{F}_v^{i/t}(R/\mathfrak{a}_{m-i+1}))$. To analyze the limit in the above expression, we set (c.f. the proof of Lemma 3.3 or the argument in [19, Section 4.1.1])

$$\begin{aligned}\phi_m(y) &= \frac{\ell(\mathcal{F}_v^{\lceil my\rceil/t}(R/\mathfrak{a}_{m-\lceil my\rceil+1}))}{m^n/n!} \\ &= \frac{\ell(\mathcal{F}_v^{\lceil my\rceil/t}(R/\mathfrak{a}_{m-\lceil my\rceil+1}))}{(m-\lceil my\rceil+1)^n/n!} \cdot \frac{(m-\lceil my\rceil+1)^n}{m^n}\end{aligned}$$

where $0 < y < 1$. It is not hard to check that

$$\lim_{m\to\infty} \phi_m(y) = g\left(\frac{y}{t(1-y)}\right)(1-y)^n$$

where $g(u) = \mathrm{vol}(v_0; v/u)$, hence by the dominated convergence theorem we have

$$\lim_{m\to\infty} \frac{W_m}{m^{n+1}/n!} = \lim_{m\to\infty} \int_0^1 \phi_m(y)\mathrm{d}y$$
$$= \int_0^1 g\left(\frac{y}{t(1-y)}\right)(1-y)^n \mathrm{d}y = \int_0^\infty g(u)\frac{t\mathrm{d}u}{(1+tu)^{n+2}}.$$

Together with (3.15) this implies the statement of the lemma. □

We are now ready to give the proof of Theorem 3.10.

Proof of Theorem 3.10. Let $v \in \mathrm{Val}^*_{X,x}$. Up to rescaling, we may assume that $A_{X,\Delta}(v_0) = A_{X,\Delta}(v) = 1$. Define $\mathfrak{b}_{\bullet,t}$ $(t \geq 0)$ as in (3.12), and let

$$f(t) := (1+t)^n \cdot \mathrm{mult}(\mathfrak{b}_{\bullet,t}).$$

Clearly $f(0) = \widehat{\mathrm{vol}}(v_0)$. By Theorem 3.11 we have

$$\mathrm{lct}(\mathfrak{b}_{\bullet,t}) \leq \mathrm{lct}(\mathfrak{a}_\bullet(v_0)) + \mathrm{lct}(\mathfrak{a}_\bullet(tv)) \leq \frac{A_{X,\Delta}(v_0)}{v_0(\mathfrak{a}_\bullet(v_0))} + \frac{A_{X,\Delta}(v)}{v(\mathfrak{a}_\bullet(tv))} \leq 1+t.$$

Hence for all $t \geq 0$,

$$f(t) \geq \mathrm{lct}(\mathfrak{b}_{\bullet,t})^n \cdot \mathrm{mult}(\mathfrak{b}_{\bullet,t}) \geq \widehat{\mathrm{vol}}(v_0) = f(0),$$

where the second inequality follows from (2.1) and the assumption that v_0 is a minimizer of $\widehat{\mathrm{vol}}_{X,\Delta}$. Thus $f'(0) \geq 0$. Using Lemma 3.13, we find

$$f'(0) = n \cdot \mathrm{vol}(v_0) - (n+1)\int_0^\infty \mathrm{vol}(v_0; v/u)\mathrm{d}u,$$

thus

$$A_{X,\Delta}(v) = 1 \geq \frac{n+1}{n} \cdot \frac{\int_0^\infty \mathrm{vol}(v_0; v/u)\mathrm{d}u}{\mathrm{vol}(v_0)} = S(v_0; v).$$

Since $v \in \mathrm{Val}^*_{X,x}$ is arbitrary, it follows that v_0 is K-semistable. □

Remark 3.14. If we combine together Theorems 3.6, 3.7 and 3.10, we get a proof of the fact that a Kollár component is a minimizer if and only if it is K-semistable, which was first established in [19, 23]. While in the proof of Theorem 3.10, we still use a version of the derivative formula introduced in [19], we do not need it for the converse.

4. Applications

In this section, we prove the results mentioned in the introduction.

Proof of Theorem 1.1. By [3] (see also [32, Remark 3.8]), there exists $v_0 \in \mathrm{Val}^*_{X,x}$ such that $\widehat{\mathrm{vol}}(v_0) = \widehat{\mathrm{vol}}(x, X, \Delta)$. By Theorem 3.10, v_0 is K-semistable, thus by Theorem 3.7, it is the unique minimizer of the normalized volume function up to scaling. □

Proof of Corollary 1.2. If v_0 is a minimizer of $\widehat{\mathrm{vol}}_{X,\Delta}$, then for any $g \in G$, the valuation $g \cdot v_0$ defined by $(g \cdot v_0)(s) = v_0(g^{-1} \cdot s)$ is also a minimizer of $\widehat{\mathrm{vol}}_{X,\Delta}$. By Theorem 1.1, we have $g \cdot v_0 = \lambda v_0$ for some $\lambda > 0$; but since $A_{X,\Delta}(v_0) = A_{X,\Delta}(g \cdot v_0)$, we must have $\lambda = 1$, hence $v_0 = g \cdot v_0$ is G-invariant. □

Theorem 1.3 is then an easy consequence of Corollary 1.2.

Proof of Theorem 1.3. Let $G = \mathrm{Aut}(Y/X)$ be the Galois group. By Corollary 1.2, the minimizer v_0 of $\widehat{\mathrm{vol}}_{Y,\Delta_Y}$ is G-invariant, hence $\widehat{\mathrm{vol}}(y, Y, \Delta_Y) = \widehat{\mathrm{vol}}^G(y, Y, \Delta_Y)$, where

$$\widehat{\mathrm{vol}}^G(x, X, \Delta) := \inf_{v \in \mathrm{Val}^G_{X,x}} \widehat{\mathrm{vol}}_{(X,\Delta),x}(v)$$

as the infimum runs over all valuations $v \in \mathrm{Val}_{X,x}$ that are invariant under the G-action.

By [26, Theorem 2.7(1)], we get $\widehat{\mathrm{vol}}^G(y, Y, \Delta_Y) = |G| \cdot \widehat{\mathrm{vol}}(x, X, \Delta)$ (in *loc. cit.* it is assumed that $\Delta_Y = 0$ and f is étale in codimension one, but the proof applies in general since these assumptions are only used to guarantee that $\Delta = 0$). Thus

$$\widehat{\mathrm{vol}}(y, Y, \Delta_Y) = |G| \cdot \widehat{\mathrm{vol}}(x, X, \Delta) = \deg(f) \cdot \widehat{\mathrm{vol}}(x, X, \Delta). \qquad \square$$

In fact, the above argument implies the finite degree formula for any quasi-étale (i.e. étale in codimension one) finite morphism $Y \to X$, as we can pass to the Galois closure of Y/X, which is also quasi-étale. However, if there is a branched divisor, then the pull back of $K_X + \Delta$ to the Galois closure of Y/X might have negative coefficients.

Proof of Corollary 1.4. For the germ of a klt singularity (X, Δ), by [30, 8] (see also [29]), the fundamental group $\pi_1(x, X^{\mathrm{sm}})$ of the smooth locus X^{sm} of is finite.

Let $f\colon (Y,y)\to(X,x)$ be the universal cover of X^{sm} and let $\Delta_Y = f^*\Delta$. Then we have $K_Y+\Delta_Y = f^*(K_X+\Delta)$, hence by Theorem 1.3 we get $\widehat{\mathrm{vol}}(y,Y,\Delta_Y) = \deg(f)\cdot\widehat{\mathrm{vol}}(x,X,\Delta)$. By [26, Theorem A.4], we also have $\widehat{\mathrm{vol}}(y,Y,\Delta_Y)\le n^n$ with equality if and only if $y\in X$ is smooth and $\Delta_Y=0$. It follows that

$$\deg(f) = \#|\pi_1(x,X^{\mathrm{sm}})| \le \frac{n^n}{\widehat{\mathrm{vol}}(x,X,\Delta)}$$

and the equality holds if and only if $\big(y\in(Y,\Delta_Y)\big)\cong(0\in\mathbb{C}^n)$ (étale locally), i.e., $\Delta=0$ and $(x\in X)$ is étale locally isomorphic to $\mathbb{C}^n/G$ where $G\cong\pi_1(x,X^{\mathrm{sm}})$ and the action of G is fixed point free in codimension one. □

Proof of Theorem 1.5. By [4, Theorem D], we have

$$\widehat{\mathrm{vol}}(x,X,\Delta)\ge\left(\frac{n}{n+1}\right)^n\cdot\delta(X,\Delta)^n\cdot(-(K_X+\Delta))^n\,.$$

Thus the result follows immediately from Corollary 1.4. □

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Chenyang Xu
Department of Mathematics
Princeton University
Princeton, NJ 08544
USA
E-mail address: chenyang@princeton.edu
Department of Mathematics
MIT
Cambridge, MA 02139
USA
E-mail address: cyxu@mit.edu
Beijing International Center for Mathematical Research
Beijing 100871
China
E-mail address: cyxu@math.pku.edu.cn

Ziquan Zhuang
Department of Mathematics
MIT
Cambridge, MA 02139
USA
E-mail address: ziquan@mit.edu

Received May 21, 2020