New Packings in Grassmannian Space

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***Abstract*—We provide a new algebraic construction for packing subspaces in complex Grassmannian space with respect to the *chordal* distance metric. The proposed method extends the construction of *character-polynomial* (CP) subspace codes, recently proposed by the authors, to higher dimensions. Our results indicate the superiority of the packings derived from CP codes in the real Grassmannian space compared with existing explicit construction. Furthermore, we propose a concatenation method in Grassmannian space and characterize the rate and the minimum distance of a *concatenated* Grassmann code in terms of those of its underlying inner and outer codes. This result is then utilized to arrive at the counterpart of Zyablov bound in Grassmannian space. Finally, we construct Grassmann codes with asymptotically large blocklength simultaneously attaining non-vanishing rate and normalized minimum distance. In particular, we propose a family of concatenated Grassmann codes having CP inner codes that surpass the Zyablov bound in the low-rate regime.**

I. Introduction

One of the central problems studied extensively in the field of coding theory is the sphere packing problem [1], i.e., the arrangement of non-overlapping identical sized spheres in a given metric space. The problem can be also regarded as designing a code in the considered metric space with codewords apart from each other by at least a certain distance, measured in terms of the provided distance metric and referred to as the *minimum distance* of the code. We consider packing in the space of m-dimensional linear subspaces of a real/complex ambient vector space of dimension n with respect to the so-called *chordal* distance. For the case $m = 1$, this problem is reduced to picking a set of lines passing through the origin in an n-dimensional real/complex vector space where the distance between two lines is captured by their angular separation.

The notion of chordal distance was first introduced for real Grassmannian spaces in [2] and was extended to the complex spaces in [3]. Packing in real Grassmannian space was also studied in [2] and some of these constructions were observed to be optimal. However, the suggested construction methods in [2] are numerical, making them computationally infeasible for general parameters. Another numerical method for constructing codes based on alternative projection is proposed in [4]. In another line of work, motivated by quantum error-correcting codes, constructions based on group structures are suggested [5]–[9]. The problem of deriving bounds on the minimum chordal distance of Grassmann codes was first studied by Shannon for the special case of $m = 1$ [10]. A lower bound on the best rate of the Grassmann codes with a fixed minimum chordal distance is derived by Shannon [10], assuming $n \to \infty$ and later extended to higher dimensions, assuming a fixed m, by Barg *et al.* in [3]. They also provide an upper bound on the largest achievable rate given a fixed minimum distance and m while $n \to \infty$, which was later improved in [11]–[13]. An achievability bound for the minimum distance of the codes for finite values of n was also derived in [14].

Grassmann codes have found applications in the design of communication systems, mostly in the context of space-time code design for MIMO channels [15], [16]. More specifically, the problem of constellation design for communications over a non-coherent MIMO channel is observed to be closely related to the packing problem in the complex Grassmann space [17]–[19]. Recently, in another direction, the authors have shown that Grassmann codes enable reliable communication over wireless networks in a non-coherent fashion [20]. This is mainly inspired by the seminal work by Koetter and Kschischang [21]. In particular, it is shown in [20] that the error correcting capability of the underlying codes utilized to communicate over wireless networks is characterized by the minimum chordal distance between the subspaces spanned by the transmitted codewords. A new algebraic construction for one-dimensional complex Grassmann codes, referred to as character-polynomial (CP) codes, has also been introduced in [20].

The objective of this paper is twofold. In the first part, we extend the construction of CP codes to higher dimensions. In particular, a certain property of the structure of one-dimensional CP codes is observed and is then utilized to generalize its construction to higher dimensions. We compare the trade-off between the rate and the normalized minimum distance that the CP codes offer at different values of n with the lower bound characterized in [3], for $n \to \infty$, and lower bounds derived by Henkel [14] for finite values of n. It is observed that CP codes improve the lower bound in the finite-length regime by providing explicit constructions surpassing the one characterized in [14]. Furthermore, by utilizing a certain mapping that constructs real Grassmann codes from the complex ones, we compare the parameters of the proposed CP codes with those of the closest explicit construction of Grassmann codes proposed in [7]. Our results indicate the superiority of packings derived from CP codes compared with the aforementioned explicit construction.

In the second part, we propose a concatenation technique in the Grassmannian space that enables constructing Grassmann codes for asymptotically large blocklength by utilizing the existing explicit code constructions in the finite-length regime. Similar ideas have been utilized in various settings in coding theory to construct codes with large blocklength by concatenating off-the-shelf code constructions with shorter lengths [22]. We characterize the parameters of the overall concatenated Grassmann code in terms of those of its underlying inner and outer codes. By using this result, we derive a bound for Grassmann codes analogous to Zyablov bound for block codes [23]. Finally, it is shown that the concatenation of CP codes

This work was supported by the National Science Foundation under grants CCF–1763348, CCF–1909771, and CCF–1941633.

with algebraic-geometry (AG) codes achieve this bound in the low-rate regime.

II. PRELIMINARIES

We start by establishing notations used in this paper. Matrices are represented by bold capital letters. The row space of a matrix $\boldsymbol{X}$ is denoted by $\langle \boldsymbol{X} \rangle$. Also, for a square matrix $\boldsymbol{X}$, the trace of $\boldsymbol{X}$, denoted by $\text{tr}(\boldsymbol{X})$, is defined to be the sum of elements of $\boldsymbol{X}$ on the main diagonal. For a matrix $\boldsymbol{A}$, $\|\boldsymbol{A}\|$ denotes the Frobenius norm of $\boldsymbol{A}$. The ambient vector space is denoted by W. The parameter n is reserved for the dimension of W throughout the paper. Also, we have $W = \mathbb{L}^n$, where $\mathbb{L}$ can be either $\mathbb{R}$ or $\mathbb{C}$. Let $\mathcal{P}(W)$ denote the set of all subspaces of W. For a subspace $V \in \mathcal{P}(W)$, the dimension of V is denoted by $\dim(V)$. The set of all m-dimensional subspaces of $\mathbb{L}^n$ is denoted by $G_{m,n}(\mathbb{L})$, which is referred to as Grassmannian space in the literature.

The chordal distance $d_c : G_{m,n}(\mathbb{L}) \times G_{m,n}(\mathbb{L}) \to \mathbb{R}$ was first introduced for $\mathbb{L} = \mathbb{R}$ in [2] and was extended to $\mathbb{L} = \mathbb{C}$ in [3]. Consider two subspaces U and V in $\mathcal{P}(W)$. Let $\boldsymbol{Q}_u$ and $\boldsymbol{Q}_v$ denote matrices with orthonormal rows spanning U and V respectively, i.e.,$\langle \boldsymbol{Q}_u \rangle = U$ and $\langle \boldsymbol{Q}_u \rangle = U$. Then, the matrices $\boldsymbol{P}_U = \boldsymbol{Q}_u^{\text{H}} \boldsymbol{Q}_u$ and $\boldsymbol{P}_V = \boldsymbol{Q}_v^{\text{H}} \boldsymbol{Q}_v$ are orthogonal projection operators from $\mathbb{L}^n$ on U and V, respectively. Then the chordal distance between U and V is defined as follows:

$$d_c(U,V) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}} \|\boldsymbol{P}_U - \boldsymbol{P}_V\|. \tag{1}$$

Alternatively, for $V, U \in G_{m,n}(\mathbb{L})$ one can show that:

$$d_c(U,V) = (m - \left\|\boldsymbol{Q}_u^{\text{H}} \boldsymbol{Q}_v\right\|^2)^{\frac{1}{2}}. \tag{2}$$

Definition 1: A packing $\mathcal{C}$ in Grassmannian space $G_{m,n}(\mathbb{L})$ is a subset of $G_{m,n}(\mathbb{L})$. The size of $\mathcal{C}$ is denoted by $|\mathcal{C}|$. The minimum distance of $\mathcal{C}$ is defined as

$$d_{\min}(\mathcal{C}) \stackrel{\text{def}}{=} \min_{U,V \in \mathcal{C}, U \neq V} d_c(U,V),$$

where $d_c(\cdot,\cdot)$ is the chordal distance, defined in (1). Alternatively, the packing $\mathcal{C}$ in $G_{m,n}(\mathbb{L})$ is also referred to as an $[n, m, |\mathcal{C}|, d_{\min}(\mathcal{C})]$ Grassmann code.

As in conventional block codes, one can define rate, normalized minimum distance and normalized weight for Grassmann codes as follows.

Definition 2: Let $\mathcal{C}$ be an $[n, m, M, d_{\min}(\mathcal{C})]$ Grassmann code. The normalized weight λ, the rate R, and the normalized minimum distance δ of $\mathcal{C}$ are defined as follows:

$$\lambda \stackrel{\text{def}}{=} \frac{m}{n}, \quad R \stackrel{\text{def}}{=} \frac{\ln M}{n}, \quad \delta_c \stackrel{\text{def}}{=} \frac{d_{\min}(\mathcal{C})}{\sqrt{m}}.$$

A lower bound on the largest achievable rate R, given a fixed δ_c and m while $n \to \infty$, was derived in [3]. Next, we recall this result.

Lemma 1 (Theorem 2, [3]): Let $n \to \infty$ and m be a constant. Then there exist sequences of codes in $G_{m,n}(\mathbb{L})$ with normalized distance δ_c and asymptotic rate $R \gtrsim -\beta m \ln(\delta_c)$, where $\beta = 1, 2$ for $\mathbb{L} = \mathbb{R}, \mathbb{C}$, respectively.

This bound is referred to as Gilbert–Varshamov (GV) bound in the rest of the paper. For finite values of n, an achievability bound for $d_{\min}(\mathcal{C})$ is derived in [14] in the complex Grassmannian space which is provided in the next lemma.

Lemma 2 ([14]): There exists an $[n, m, M, d_{\min}(\mathcal{C})]$ Grassmann code in $G_{m,n}(\mathbb{C})$ for all $n \geqslant m$ such that

$$d_{\min}(\mathcal{C}) \geqslant \frac{2}{\pi} |\mathcal{C}|^{\frac{-1}{2m(n-m)}}.$$

III. CHARACTER-POLYNOMIAL GRASSMANN CODES

For a finite field $\mathbb{F}_q$ the *additive character* associated to $j \in \mathbb{F}_q$ is defined as

$$\chi_j(\alpha) = \text{e}(\frac{\text{tr}_\text{a}(j\alpha)}{p}), \tag{3}$$

where p is the characteristic of $\mathbb{F}_q$, and

$$\text{tr}_\text{a}(\gamma) \stackrel{\text{def}}{=} \gamma + \gamma^p + \cdots + \gamma^{p^{m-1}}$$

is the *absolute* trace function from $\mathbb{F}_q$ to $\mathbb{F}_p$, where $q = p^m$. Note that (3) implies that $\chi_j(\alpha) = \chi_1(j\alpha)$ and the trivial additive character is $\chi_0(\alpha) = 1$ for all $\alpha \in \mathbb{F}_q$. Then, for a polynomial $f \in \mathbb{F}_q[x]$ of degree $d \geqslant 1$ with $\gcd(d, q) = 1$, the Weil bound [24], [25] states that

$$|\sum_{\alpha \in \mathbb{F}_q} \chi(f(\alpha))| \leqslant (d-1)\sqrt{q}, \tag{4}$$

for any non-trivial character χ. This bound has been utilized in several coding theoretic contexts, e.g., to provide bounds on the minimum distance of the duals of BCH codes [26] and to estimate the covering radius of long BCH codes [27], [28]. In particular, it has inspired the design of certain families of sequences with low correlation in [29]. Also, motivated by Weil bound, we recently constructed one-dimensional Grassmann codes, referred to as *character-polynomial* (CP) codes, and showed their applications to communication over non-coherent wireless networks [20]. In other words, this construction provides a family of one-dimensional complex Grassmann codes, i.e., a packing of lines in $G_{1,n}(\mathbb{C})$. Next, we provide a slightly different variant of one-dimensional CP codes that enables us to provide a family of packings in $G_{m,n}(\mathbb{C})$ for $m > 1$. Let

$$\mathcal{F} \stackrel{\text{def}}{=} \{f \in \mathbb{F}_q[x] : f(x) = \sum_{i \in [k], i \bmod p \neq 0} f_i x^i\}, \tag{5}$$

for some $k < q$. Note that $|\mathcal{F}| = q^{\lceil k(p-1)/p \rceil}$. We fix $n = q$.

Definition 3: The code $\mathcal{C}(\mathcal{F}) \subseteq G_{1,n}(\mathbb{C})$, referred to as a character-polynomial (CP) code, is defined as follows:

$$\mathcal{C}(\mathcal{F}) \stackrel{\text{def}}{=} \{\langle (c_1, c_2, \ldots, c_n) \rangle : c_i = \chi\big(f(\alpha_i)\big), \forall f \in \mathcal{F}, \}, \tag{6}$$

where χ is a fixed nontrivial additive character of $\mathbb{F}_q$, and α_i's are distinct elements of $\mathbb{F}_q$.

The definition of CP codes provided in [20] excludes the zero element of $\mathbb{F}_q$ from the set of evaluation points. Including the zero element in (6) leads to the observation of a certain property for the structure of CP codes, defined in Definition 3, that is provided in the following lemma. For any two sets of orthonormal bases B_1 and B_2 for W, the *mutual correlation* between B_1 and B_2 is defined as

$$\Delta_{B_1,B_2} \stackrel{\text{def}}{=} \max_{\boldsymbol{v}_1 \in B_1, \boldsymbol{v}_2 \in B_2} v_1 \cdot v_2, \tag{7}$$

where $\cdot$ denotes the inner product.

Lemma 3: The set of normal vectors representing one-dimensional CP codewords defined in Definition 3 can be split into $q^{\lceil k(p-1)/p \rceil -1}$, denoted by b, collections B_i's, for $i \in [b]$, where each B_i is an orthonormal basis for W and the mutual correlation between B_i's is at most $\frac{(k-1)^2}{q}$, i.e.,

$$\max_{i,j \in [b], i \neq j} \Delta_{B_i,B_j} \leqslant \frac{(k-1)}{\sqrt{n}}. \tag{8}$$

Proof: The set of polynomials $\mathcal{F}$, defined in (5), can be split into disjoint subsets such that the polynomials belonging to the same subset differ only in the coefficient of the degree one monomial, i.e., the coefficient of x. Note that the constant coefficient of all the polynomials in $\mathcal{F}$ is equal to zero according to (5). Then, two distinct polynomials f and f' in $\mathcal{F}$ belong to the same subset if and only if $\deg(f - f') = 1$. Consequently, it can be observed that $\mathcal{F}$ is partitioned into $q^{\lceil k(p-1)/p \rceil -1}$ of such subsets each of size q. Let $\boldsymbol{c} = \frac{1}{\sqrt{n}}(c_1, \cdots, c_n)$ and $\boldsymbol{c}' = \frac{1}{\sqrt{n}}(c'_1, \cdots, c_n,)$, where $c_i = \chi\big(f(\alpha_i)\big)$ and $c'_i = \chi\big(f'(\alpha_i)\big)$ for $i \in [n]$. Then,

$$\boldsymbol{c} \cdot \boldsymbol{c}' = \sum_{\alpha \in \mathbb{F}_q} \chi^*(f(\alpha))\chi(f'(\alpha)) \overset{(a)}{=} \sum_{\alpha \in \mathbb{F}_q} \chi((f'-f)(\alpha)) \overset{(b)}{\leqslant} 0 \tag{9}$$

where (a) follows by (3) and (b) is by Weil bound, specified in (4), together with noting that $\deg(f - f') = 1$. Hence, each of these subsets corresponds to q mutually orthogonal lines in W. Consequently, the set of unit-norm vectors representing these lines is an orthonormal basis for W. The upper bound in (8) can be derived similarly by noting that $\deg(f - f') \leqslant k$ for any f and f' in $\mathcal{F}$. ■

Inspired by Lemma 3, we provide a construction for packing m-planes in $G_{m,n}(\mathbb{C})$ for $m > 1$. Let $\boldsymbol{v}_1^{(i)}, \cdots, \boldsymbol{v}_q^{(i)}$ denote the orthonormal basis vectors in B_i, for $i \in [b]$, where b is defined in Lemma 3. Also, let

$$\Phi_{ij} = \begin{bmatrix} \boldsymbol{v}_j^{(i)} \\ \vdots \\ \boldsymbol{v}_{j+m-1}^{(i)} \end{bmatrix}, \tag{10}$$

for all $i \in [b]$ and $j \in [\lfloor \frac{q}{m} \rfloor]$. Then,

$$\mathcal{C} \overset{\text{def}}{=} \{\langle \Phi_{ij} \rangle : \forall i \in [b], \forall j \in [\lfloor \frac{q}{m} \rfloor]\} \tag{11}$$

is a Grassmann code in $G_{m,n}(\mathbb{C})$. Note that we have

$$|\mathcal{C}| = q^{\lceil k(p-1)/p \rceil -1} \lfloor \frac{q}{m} \rfloor.$$

The normalized minimum distance of $\mathcal{C}$ is characterized in the next theorem.

Theorem 4: The normalized minimum distance δ_c of the code $\mathcal{C}$, defined in (11), is lower bounded as

$$\delta_c \geqslant \sqrt{1 - \frac{m(k-1)^2}{n}}. \tag{12}$$

Proof: Consider two distinct codewords $C_1 = \langle \Phi_{i,j} \rangle$ and $C_2 = \langle \Phi_{i',j'} \rangle \in \mathcal{C}$. Note that the rows in $\Phi_{i,j}$ and $\Phi_{i',j'}$ are orthonormal. Note also that for $i = i'$, $\Phi_{i,j}^{\mathrm{H}} \Phi_{i',j'} = \mathbf{0}$, since all the rows of both matrices belong to the same orthonormal basis which implies that $\delta_c = 1$ in this case. Otherwise, i.e., when $i \neq i'$, we have

$$\left\| \Phi_{i,j}^{\mathrm{H}} \Phi_{i',j'} \right\|^2 \leqslant \frac{m^2(k-1)^2}{n}, \tag{13}$$

which holds by (8). The result then follows by utilizing the alternative characterization of the chordal distance in (2) together with the definition of normalized distance provided in Definition 2. ■

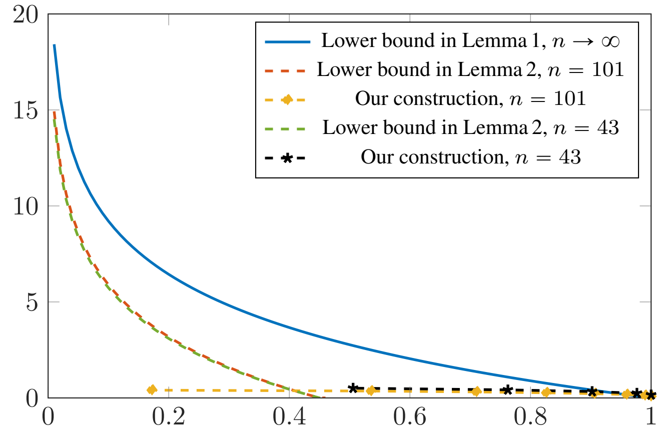

Fig. 1: Comparison of the proposed code construction with lower-bounds in terms of the trade-off between R and δ_c for $m = 2$.

In Figure 1, we compare the trade-off between the rate R and the normalized minimum distance δ_c that the codes defined in (11) offer at different values of n with the lower bounds provided in Lemma 1 and Lemma 2. Note that these lower bounds are of the same type as Gilbert-Varshamov bound and do not yield explicit constructions. Nevertheless, it can be observed that the proposed codes can outperform these lower bounds at low rates, thereby improving these bounds while providing explicit constructions.

***Remark** 1:* Given a Grassmann code $\mathcal{C} \subseteq G_{m,n}(\mathbb{C})$ one can construct a code in $G_{2m,2n}(\mathbb{R})$ by mapping a matrix $\boldsymbol{C}_i$ with orthonormal rows such that $\langle \boldsymbol{C}_i \rangle \in \mathcal{C}$ to

$$\begin{bmatrix} \Re(\boldsymbol{C}_i) & \Im(\boldsymbol{C}_i) \\ -\Im(\boldsymbol{C}_i) & \Re(\boldsymbol{C}_i) \end{bmatrix}, \tag{14}$$

where $\Re(.)$ and $\Im(.)$ represent the real part and the imaginary part of their input, respectively. It can be observed that this mapping preserves the normalized distance between the codewords. This mapping enables us to construct codes in the real Grassmannian space using the proposed codes in the complex space, while keeping the normalized minimum distance and the size of the code the same, in order to have fair comparisons with existing code constructions in the real Grassmannian space.

In Table I, we compare the parameters of our proposed codes with those of the closest explicit construction of Grassmann codes proposed in [7]. In [7], Calderbank *et al.* introduce a group-theoretic framework for packing in $G_{2^i,2^k}(\mathbb{R})$ for any pair of integers (i, k) with $i \leqslant k$. By utilizing the mapping specified in (14) for our codes in $G_{\frac{m}{2},\frac{n}{2}}(\mathbb{C})$, we compare the blocklength, logarithm of the code size, and the minimum

	Our construction			Calderbank *et al.* [7]		
m	n	$\ln(\|\mathcal{C}\|)$	$d^2_{\min}$	n	$\ln(\|\mathcal{C}\|)$	$d^2_{\min}$
4	254	28.36	2.43	256	32.62	2
4	502	43.51	2.44	512	40.25	2
4	1018	74.09	2.10	1024	48.57	2
4	2042	110.2	2.37	2048	57.59	2
8	1018	48.47	4.92	1024	49.87	4
8	2024	81.76	4.3	2048	59.57	4
8	4078	120.5	4.47	4096	69.97	4
8	8186	189.9	4.22	8192	81.07	4
16	2042	53.34	9.86	2048	59.52	8
16	4078	89.36	8.40	4096	70.61	8
32	4078	58.19	19.67	4096	69.19	16
32	8186	97.03	16.86	8192	81.67	16

TABLE I: Comparison of the parameters of packings in Grassmannian space provided in this paper to that of proposed by Calderbank *et al.* in [7, Theorem 1].

distance of the codes obtained in $G_{m,n}(\mathbb{R})$ with those of the codes in $G_{m,n}(\mathbb{R})$ from [7]. In all the instances of n and $m = 4, 8, 16, 32$ in Table I, the normalized minimum distance is equal to $\frac{1}{\sqrt{2}}$ for codes from [7] while it is at least $\frac{1}{\sqrt{2}}$ for our codes. Note that n is equal to 2^k for the construction in [7], while for our codes we pick $n = 2p$, where p is the largest prime number with $p < 2^{k-1}$, for various choices of k. It can be observed that our proposed construction offer significantly larger code size and, consequently, rate comparing to the explicit construction of [7], as n grows large.

IV. Concatenated Grassmann Codes

In this section we propose a family of Grassmann codes with non-zero normalized minimum distance and non-vanishing rate for asymptotically large blocklength by leveraging concatenation schemes that are well-explored in the context of block codes. To the best of authors' knowledge, the rate of all codes with explicit constructions in Grassmannian space, including those introduced in [7], [20], [30], approach zero as $n \to \infty$. The concatenation idea has been appeared first in the construction of Elias's product codes and further developed later by Forney [22] in Hamming space. The idea has been utilized since then in various settings in order to construct codes of *long* block length from codes of shorter *length*. In this section, we extend the concatenation idea from block coding to Grassmannian coding. To this end, we propose a method to concatenate a Grassmann code with *small* blocklength, such as those introduced in Section III with a certain family of outer codes in Hamming space and analyze the parameters of the overall concatenated Grassmann code. This result is then utilized to derive a bound for Grassmann codes analogous to Zyablov bound for block codes [23]. Finally, it is shown that this bound can be achieved in a certain regime by concatenation of CP codes, proposed in Section III, with algebraic-geometry (AG) codes.

Let $\mathcal{C}_{in}$ denote a code in $G_{m,n_1}(\mathbb{L})$ with minimum chordal distance d_c, as defined in (1). Let Φ_i denote an $m \times n_1$ matrix with orthonormal rows spanning the subspace $C_i \in \mathcal{C}$, for $i = 1, 2, \ldots, |\mathcal{C}_{in}|$. Let $\mathcal{C}_{out}$ be a $[n_2, k, d_H]_q$ linear code over a finite field $\mathbb{F}_q$ such that $q \leqslant |\mathcal{C}_{in}|$, where d_H denotes the minimum Hamming distance of $\mathcal{C}_{in}$. For all codewords $c_i \in \mathcal{C}_{out}$ where $i \in [q^k]$, let $c_i(l) \in \mathbb{F}_q$ denote the l-th coordinate of c_i. Then, the concatenation of $\mathcal{C}_{in}$ with $\mathcal{C}_{out}$ is a code in Grassmannian space defined as follows:

$$\mathcal{C}_{out} \diamond \mathcal{C}_{in} \overset{\text{def}}{=} \{\langle \Psi_i \rangle : \Psi_i = \frac{1}{\sqrt{n_2}}[\Phi_{\eta(c_i(1))}|\ldots|\Phi_{\eta(c_i(n_2))}], \tag{15}$$

$$\forall i \in [q^k]\} \tag{16}$$

It can be verified that Ψ_i's are $m \times n_1 n_2$ full-rank matrices having orthonormal rows. The rate and the minimum distance of the concatenated code $\mathcal{C}_c \overset{\text{def}}{=} \mathcal{C}_{out} \diamond \mathcal{C}_{in}$ is characterized in the following theorem in terms of parameters of $\mathcal{C}_{in}$ and $\mathcal{C}_{out}$.

Theorem 5: Let $\mathcal{C}_{in}$ be a code in Grassmannian space with the rate R and the normalized minimum chordal distance δ_c and $\mathcal{C}_{out}$ be a q-ary code with rate r and the normalized minimum Hamming distance δ_H. Then, the concatenated code $\mathcal{C}_c$, as defined in (15), is a code with rate rR and the normalized minimum chordal distance δ'_c satisfying:

$$(1 - \sqrt{1 - \delta'^2_c}) \geqslant \delta_H (1 - \sqrt{1 - \delta^2_c}).$$

Proof: Let Ψ_i,Ψ_j be matrices with orthonormal rows spanning two distinct subspaces/codewords in $\mathcal{C}_c$, as characterized in (15). Then one can write

$$\Psi_i \Psi_j^{\mathrm{H}} = \frac{1}{n_2} \sum_{l=1}^{n_2} \Phi_{\eta(c_i(l))} \Phi^{\mathrm{H}}_{\eta(c_j(l))}. \tag{17}$$

Note that

$$\left\| \Psi_i \Psi_j^{\mathrm{H}} \right\| \leqslant \frac{1}{n_2} \sum_{l=1}^{n_2} \left\| \Phi_{\eta(c_i(l))} \Phi^{\mathrm{H}}_{\eta(c_j(l))} \right\| \tag{18}$$

$$\leqslant \frac{(n_2 - d_H)\sqrt{m} + d_H \sqrt{m - d_c^2}}{n_2} \tag{19}$$

$$= \sqrt{m}(1 - \delta_H(1 - \sqrt{1 - \delta_c^2})), \tag{20}$$

where (18) holds by applying the triangle inequality to (17), and (19) is by the definition of chordal distance in (1) and recalling that the minimum distance of the underlying codes $\mathcal{C}_{in}$ and $\mathcal{C}_{out}$ are equal to d_c and d_H, respectively. By utilizing the alternative characterization of the chordal distance provided in (2), the definition of normalized chordal distance in Definition 2 and utilizing the upper bound derived in (20), one can write

$$\delta'_c = \frac{1}{\sqrt{m}}(m - \left\| \Psi_i \Psi_j^{\mathrm{H}} \right\|^2)^{\frac{1}{2}} \geqslant \sqrt{y(2-y)}, \tag{21}$$

where $y \overset{\text{def}}{=} \delta_H(1 - \sqrt{1 - \delta_c^2})$. By noting that y is less than or equal to 1, (21) implies that $y \leqslant 1 - \sqrt{1 - \delta'^2_c}$, i.e.,

$$(1 - \sqrt{1 - \delta'^2_c}) \geqslant \delta_H (1 - \sqrt{1 - \delta^2_c}),$$

which completes the proof. ■

The lower bound obtained in Theorem 5 on the normalized minimum chordal distance of the concatenated code $\mathcal{C}_c$ in Grassmannian space resembles the well-known bound on the normalized minimum Hamming distance of a concatenated block code stating that it is less than or equal to the product of the normalized minimum distances of the constituent codes.

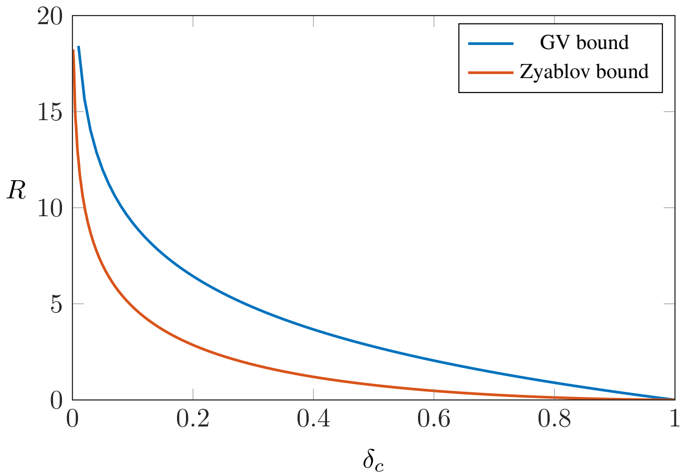

Fig. 2: Comparison of the Zyablov bound and GV bound in Grassmannian space in terms of the rate R versus the normalized minimum distance δ_c.

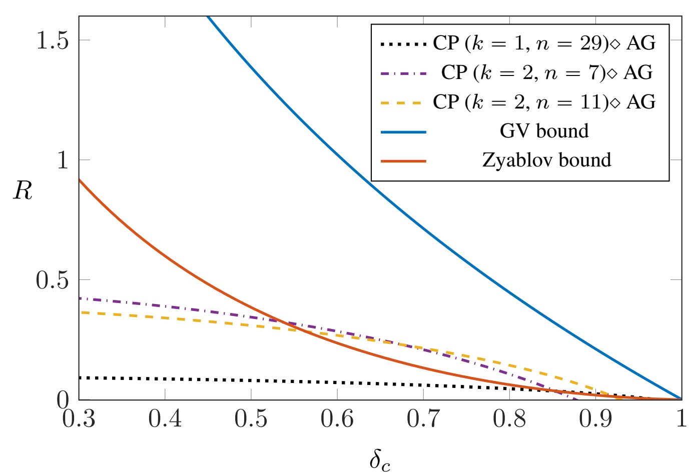

Fig. 3: Comparison of the parameters of concatenated Grassmann codes employing CP inner code and AG outer code with GV and Zyablov bound. The parameters of the CP codes used are illustrated in the plot. Other parameters are $\beta = 2$ and $m = 1$.

Next, we characterize a lower bound on achievable rates of Grassmann codes for fixed δ_c. A similar lower bound has been obtained for concatenated codes in Hamming space and is known as Zyablov bound [23]. This is done by constructing a concatenated code having an inner code with rate meeting GV-bound, as specified in Lemma 1 and an MDS outer code, e.g., Reed-Solomon code, and using the result of Theorem 5. The result is provided in the following theorem.

Theorem 6: There exist concatenated Grassmann codes in $G_{m,n}(\mathbb{L})$ having normalized chordal minimum distance δ_c and the rate

$$R \gtrsim \max_{0<\delta<1} \beta m \ln(\frac{1}{\delta})(1 - \frac{1-\sqrt{1-\delta_c^2}}{1-\sqrt{1-\delta^2}}) \qquad (22)$$

for $n \to \infty$, where $\beta = 1, 2$ for $\mathbb{L} = \mathbb{R}, \mathbb{C}$, respectively.

Proof: Let $\mathcal{C}_{in}$ be a Grassmann inner code with the normalized chordal minimum distance δ and the rate R meeting the lower bound specified in Lemma 1. Let also $\mathcal{C}_{out}$ denote an MDS code of normalized Hamming minimum distance of δ_H and the rate meeting the Singleton bound, i.e., $r = 1 - \delta_H$. Then (22) follows by noting that the result of Theorem 5 holds for all $0 < \delta < 1$. ■

We refer to the lower bound introduced in Theorem 6 as Zyablov bound in Grassmannian space or Zyablov bound in short when the underlying metric space is clear from the context. The Zyablov bound in the Grassmannian space is then regraded as a benchmark to evaluate the performance of concatenated Grassmann codes with explicit constructions in terms of the trade-off between the rate and the minimum distance. Note that the GV bound in the Grassmannian space, characterized in Lemma 1, provides a lower bound on the rate of Grassmann codes with no restriction on their construction. These two bounds are compared in Figure 2. Note that both the bounds scale linearly with β and m. Hence, we fixed $\beta = 2$ and $m = 2$ in the plot in Figure 2.

In the last part of this section, we propose a family of concatenated Grassmann codes having our proposed CP codes, introduced in Section III, as the underlying inner code together with a certain family of algebraic-geometry (AG) as its outer code. Employing this certain family of AG codes as an outer code is inspired by the fact that the concatenation of them with binary inner codes provide a family of binary codes with the largest known rate asymptotically for a fixed minimum Hamming distance.

For a survey of the results on constructions and parameters of AG codes see, e.g., [31]. In particular, we use the following result, due to Katsman *et al.* [32] that enables us to provide an explicit construction for concatenated Grassmann codes having polynomially complex encoder and decoder. Next, we recall this result.

Lemma 7 ([32]): For any $q = p^{2l}$ where p is prime and l is a positive integer, there exist AG codes with polynomial construction over $\mathbb{F}_q$, the rate r and the normalized minimum distance δ_H satisfying:

$$r + \delta_H = 1 - \frac{1}{\sqrt{q} - 1}. \qquad (23)$$

In Figure 3 we compare the performance of concatenated Grassmann codes with our CP codes as the inner code and AG codes of [32], with parameters characterized in Lemma 7, as the outer code. The plot illustrates that the concatenated Grassmann codes introduced in this section surpass Zyablov bound in low-rate/high-minimum-distance regime, implying that CP codes offer Grassmann codes with *good* parameters for this regime. Also, it shows the effectiveness of the concatenation method proposed in this section to construct Grassmann codes in the *large* blocklength regime.

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