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An index theorem for quotients of Bergman spaces on egg domains

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Dedicated to the memory of Ronald G. Douglas

We prove a K-homology index theorem for Toeplitz operators obtained from the multishifts of Bergman spaces on several classes of egg-like domains. This generalizes our earlier work with Douglas and Yu for the unit ball.

1. Introduction

Around a decade ago a multivariate operator theory approach to algebraic geometry was suggested by Arveson [2007] and Douglas [2006b] in the following way. Suppose that $I \subseteq A := \mathbb{C}[z_1, \dots, z_m]$ is an ideal of the ring of polynomials in m variables. To understand the geometry of the zero variety

$$V(I) := \{p \in \mathbb{C}^m : f(p) = 0, \ \forall f \in I\}$$

defined by I, algebraic geometers study the coordinate ring A/I. To find an operator theory model for A/I, one can replace A by the Bergman space $L_a^2()$ of square-integrable analytic functions on some bounded strongly pseudoconvex domain $\subseteq \mathbb{C}^m$ with smooth boundary, and mod it out by the closure $\bar{I}$ of I inside $L_a^2()$. The quotient Hilbert space $\mathcal{Q}_I := L_a^2()/\bar{I}$ has a natural Hilbert A-module structure[1] given by $p \cdot (f + \bar{I}) = pf + \bar{I}$, $p \in A$, $f \in L_a^2()$. Transporting this action to the orthogonal complement

$$L_a^2() \ominus \bar{I} = I^{\perp} \cong \mathcal{Q}_I$$

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[1] There is a one-to-one correspondence between commuting m-tuples of operators $T := (T_1, \dots, T_m)$ acting on a Hilbert space $\mathcal{H}$ and Hilbert A-module structures on $\mathcal{H}$ [Arveson 2007]. The correspondence is given by representing each polynomial $p(z_1, \dots, z_m) \in A$ by the operator $p(T_1, \dots, T_m)$. Conversely, T is identified with the m-tuple $(M_{z_1}, \dots, M_{z_m})$ of multiplication operators by coordinate functions, and is called the fundamental tuple of Toeplitz operators on the Hilbert A-module $\mathcal{H}$. Based on this correspondence, the properties of T are attributed to $\mathcal{H}$ and vice versa. For example, $\mathcal{H}$ is called *essentially normal* if all $[T_j, T_k^*]$ are compact, and *p-essentially normal* if all $[T_j, T_k^*]$ are Schatten p-summable. Also, $\sigma_e(\mathcal{H})$ denotes the essential Taylor spectrum associated to the fundamental tuple of Toeplitz operators of $\mathcal{H}$ [Taylor 1970; Müller 2007].

makes $I^\perp$ a Hilbert A-module. Alternatively, the module structure of $I^\perp$ is given by the compression of multiplication operators:

$$T_p := P_{I^\perp} M_p|_{I^\perp}, \quad p \in A,$$

where $M_p : L^2_a() \to L^2_a()$ is multiplication by p, and $P_{I^\perp}$ is orthogonal projection in $L^2_a()$ onto $I^\perp$. Let T_I be the unital C*-algebra generated by $\{T_p : p \in A\} \cup \mathsf{K}$, where K is the ideal of compact operators on $I^\perp$. Arveson, based on his work on the model theory of spherical contractions in multivariate dilation theory [Arveson 1998; Ambrozie and Müller 2015; Shalit 2015], proposed the following conjecture:

Conjecture 1.1 [Arveson 2002; 2005]. *$I^\perp$ is essentially normal. In other words, all commutators $[T_{z_j}, T^*_{z_k}]$ with $j, k = 1, \ldots m$, are compact.*

Suppose momentarily that this conjecture holds. Also, assume that I is homogeneous. Then the maximal ideal space of T_I/K is homeomorphic via the mapping $\phi \mapsto (\phi(T_{z_1}), \ldots, \phi(T_{z_m}))$ to the essential Taylor spectrum σ^I_e of $(T_{z_1}, \ldots T_{z_m})$, which coincides with $X_I := V(I) \cap \partial$ [Guo and Wang 2008, Theorem 5.1]; see also [Curto 1981, Corollary 3.10; Douglas 2006b, Theorem 4.1; Curto and Salinas 1985; Gleason et al. 2005]. The Gelfand–Naimark duality then gives the short exact sequence of C*-algebras

$$0 \to \mathsf{K} \hookrightarrow \mathsf{T}_I \to C(X_I) \to 0.$$

Let

$$\tau_I := [\mathsf{T}_I]$$

be the equivalence class represented by this exact sequence in the odd K-homology group $K_1(X_I)$ of Brown–Douglas–Fillmore [Brown et al. 1973; 1977]. Douglas [2006b] (see also [Baum and Douglas 1982, Section 25]) asked for an explicit computation of this element in other topological or geometric realizations of K-homology:

Problem 1.2 [Douglas 2006b]. Assume that I is homogeneous and $I^\perp$ is essentially normal. Identify $\tau_I \in K_1(X_I)$.

More specifically, he made the following conjecture:

Conjecture 1.3 [Douglas 2006b]. *Let I be the vanishing ideal of an algebraic variety $V \subseteq \mathbb{C}^m$ which intersects ∂ transversally. Then $I^\perp$ is essentially normal, and its induced extension class τ_I is identified with the fundamental class of X_I, namely the extension class induced by the* Spin^c *Dirac operator associated to the natural Cauchy–Riemann structure of X_I.*

By analogy with the Atiyah–Singer index theorem, one expects that this conjecture would lead to new connections between geometry and operator theory. To see what brought Arveson and Douglas to their conjecture/problem, we refer the reader to their original papers [Arveson 2005; 2000; Douglas 2006b] as well as [Shalit 2015; Jabbari 2019, Sections 1.2–3]. In particular, Conjecture 1.3 generalizes some aspects of the Boutet de Monvel index theorem for Toeplitz operators on strongly pseudoconvex domains to possibly singular algebraic varieties [Boutet de Monvel 1978/79; Baum et al. 1989].

Let us review some results about these conjectures and problem. (See also [Shalit 2015; Guo and Wang 2020].) When is the unit open ball, Conjecture 1.1 has been proved in the following cases:

(1) I is monomial [Arveson 2005; Douglas 2006a; Douglas et al. 2018].

(2) I is homogeneous and $m \le 3$ [Guo and Wang 2008].

(3) I is homogeneous and $\dim V(I) \le 1$ [Guo and Wang 2008].

(4) I is principal [Guo and Wang 2008; Douglas and Wang 2011; Fang and Xia 2013; 2018; Douglas et al. 2017; Wang and Xia 2020]. (The last two references allow for strongly pseudoconvex domains .)

(5) I has a stable generating set $\{p_1, \ldots, p_k\}$ of homogeneous polynomials in the sense that there exists $C > 0$ such that every $q \in I$ can be written as $q = \sum_{j=1}^{k} r_j p_j$ with $r_j \in A$ and $\|r_j p_j\|_{L^2()} \le C\|q\|_{L^2()}$ [Shalit 2011; Wang 2019].

(6) I is the vanishing ideal of a homogeneous variety smooth away from the origin [Engliš and Eschmeier 2015; Douglas et al. 2016; Douglas and Wang 2017; Wang and Xia 2019].

When is the unit ball, the articles [Guo and Wang 2008] and [Douglas et al. 2016] answer Problem 1.2 when $m \le 2$ and when I is the vanishing ideal of a complete intersection variety (possibly singular away from the boundary), respectively. In [Douglas et al. 2018] we gave an answer to Problem 1.2 when is the unit open ball and I is monomial:

Theorem 1.4. *Let be the unit open ball $\mathbb{B}_m$ and I a monomial ideal.*

(a) *There exist a positive integer k, essentially normal Hilbert A-modules*

$$\mathcal{A}_0 := L^2_a(), \quad \mathcal{A}_1, \ldots, \mathcal{A}_k,$$

and Hilbert A-module morphisms[2] $\varphi_q : \mathcal{A}_q \to \mathcal{A}_{q+1}$, $q = 0, \ldots, k-1$ *such that*

$$0 \to \bar{I} \hookrightarrow \mathcal{A}_0 \xrightarrow{\varphi_0} \mathcal{A}_1 \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_{k-1}} \mathcal{A}_k \to 0$$

is exact. (*This implies that $I^\perp$ is essentially normal.*)

[2] Bounded linear maps that preserve A-module structures.

(b) *For each* q, *let* $\mathcal{T}(\mathcal{A}_q)$ *be the unital C*-algebra generated by all module action operators as well as all compact operators on the Hilbert module* $\mathcal{A}_q$, *and let* $\sigma_e^q := \sigma_e^q(\mathcal{A}_q)$ *be the essential Taylor spectrum associated to* $\mathcal{A}_q$. *Then the identification*

$$\tau_I = \sum_{q=1}^{k} (-1)^{q-1}[\mathcal{T}(\mathcal{A}_q)]$$

holds in $K_1(\sigma_e^1 \cup \cdots \cup \sigma_e^k)$. σ

By its explicit construction, each $\mathcal{A}_q$ has a tractable geometry as the Hilbert space of square-integrable analytic sections of a Hermitian vector bundle on a disjoint union of subsets of $\mathbb{B}_m$.

In this paper, we generalize Theorem 1.4 to the case when is an egg domain of the form

$$ {}_1 := \Big\{ (z_1, \ldots, z_m) \in \mathbb{C}^m : \sum_{j=1}^{n} |z_j|^{2p_j} < 1 \Big\}, \quad p_j > 0, \tag{1.5}$$

or more generally of the form

$$ {}_2 := \Big\{ \Big(\sum_{j=1}^{n} |z_j|^{2p_j}\Big)^a + \Big(\sum_{k=1}^{n} |w_k|^{2q_k}\Big)^b + \cdots < 1 \Big\} \subseteq \mathbb{C}^{m+n+\cdots}, \tag{1.6}$$

where the finitely many parameters $p_j, q_k, a, b, \ldots$ are arbitrary positive reals. (When all $p_j, q_k, \ldots$ equal 1, ${}_2$ is called a generalized complex ellipsoid in [Jarnicki and Pflug 2008, page 208; Kodama et al. 1992].)

Theorem 1.7. *Let be a domain of the form* (1.5) *or* (1.6), *and* I *a monomial ideal.*

(a) *There exist a positive integer* k, *essentially normal Hilbert A-modules*

$$\mathcal{A}_0 := L_a^2(), \quad \mathcal{A}_1, \ldots, \mathcal{A}_k,$$

and Hilbert A-module morphisms $\vartheta_q : \mathcal{A}_q \to \mathcal{A}_{q+1}$, $q = 0, \ldots, k-1$ *such that*

$$0 \to \bar{I} \hookrightarrow \mathcal{A}_0 \xrightarrow{\vartheta_0} \mathcal{A}_1 \xrightarrow{\vartheta_1} \cdots \xrightarrow{\vartheta_{k-1}} \mathcal{A}_k \to 0 \tag{1.8}$$

is exact. (*This implies that* $I^\perp$ *is essentially normal.*)

(b) *For each* q, *let* $\mathcal{T}(\mathcal{A}_q)$ *be the unital C*-algebra generated by all module action operators as well as all compact operators on the Hilbert module* $\mathcal{A}_q$, *and let* $\sigma_e^q := \sigma_e^q(\mathcal{A}_q)$ *be the essential Taylor spectrum associated to* $\mathcal{A}_q$. *Then the identification*

$$\tau_I = \sum_{q=1}^{k} (-1)^{q-1}[\mathcal{T}(\mathcal{A}_q)]$$

holds in $K_1(\sigma_e^1 \cup \cdots \cup \sigma_e^k)$. σ

The explicit construction of the resolution (1.8) comes in Section 2, and the proof of Theorem 1.7 in Section 3. Our proof uses crucially the fact that monomials constitute an orthogonal basis for $L^2_a()$ if is a domain of type (1.5) or (1.6). Each $\mathcal{A}_q$ has a tractable geometry as the Hilbert space of square-integrable analytic sections of a Hermitian vector bundle on a disjoint union of subsets of .

Remark 1.9. If I is homogeneous, the C*-algebra generated by $\{1\} \cup \{T_p : p \in A\}$ is irreducible (it has no proper reducing closed subspace), and hence contains $\mathcal{K}$ if $I^\perp$ is essentially normal [Guo and Wang 2008, page 923; Douglas 1998, Theorem 5.39].

Remark 1.10. One reason why we care about monomial ideals is that a comprehensive understanding of the phenomena appearing in this generically nonradical case may lead to new results beyond the recently established ones about radical ideals [Douglas et al. 2016; Douglas and Wang 2017; Engliš and Eschmeier 2015].

Remark 1.11. A domain of type (1.5) is weakly (but not strongly) pseudoconvex and with C^2 boundary when $m > 1$, all p_j are ≥ 1 and at least one p_j is > 1 [D'Angelo 1978]. (The same is true for a domain of type (1.6) when $m + n + \cdots > 1$, all $a, b, \ldots 2p_j, 2q_k, \ldots$ are ≥ 2 and at least one of $2p_j, 2q_k, \ldots$ is > 2.) As far as we know, putting the polydiscs aside [Wang and Zhao 2018], Theorem 1.7 is the only result which discusses Conjecture 1.1 and Problem 1.2 on weakly pseudoconvex domains.

Remark 1.12. Note that a domain of type (1.6) is obtained from a domain of type (1.5) when each $|z_j|$ is replaced by an expression of the form $\sum_{k=1}^{n_j} |z_{jk}|^{p_{jk}}$, where all coordinates z_{jk} are distinct. Applying this process on a domain of type (1.6) and repeating this process finitely many times gives rise to more generalized egg domains. For example, we can get

$$\Big(\big(|z_{111}|^{p_{111}} + |z_{112}|^{p_{112}}\big)^{p_{11}} + \big(|z_{121}|^{p_{121}} + |z_{122}|^{p_{122}} + |z_{123}|^{p_{123}}\big)^{p_{12}} + |z_{13}|^{p_{13}}\Big)^{p_1} + \big(\cdots\big)^{p_2} + \cdots < 1.$$

(Compare [Egorychev 1984, Section 6.2; Boas et al. 1999].) The arguments in this paper prove Theorem 1.7 for all such domains.

Arveson's statement of his essential normality conjecture was more refined than Conjecture 1.1 in the sense that it addressed the Schatten class membership of commutators [Arveson 2005; Douglas 2006a]. In this paper, however, we merely focused on the membership of commutators in the ideal of compacts. The reason is that our proof of Theorem 1.7(b) relies crucially on the usage of the Fuglede–Putnam theorem in the proof of Proposition 2.5(b,c,d). Since the Schatten class version of the Fuglede–Putnam theorem at the quotient level is missing [Douglas 2006a; Weiss 1981; Shulman 1996], our result does not determine the Schatten class membership of the commutators for the quotients of Bergman spaces by monomial ideals. Nevertheless, it is worth pointing out that our computations (not included in this paper) show that the whole Bergman space $L_a^2(\Omega_1)$ associated with domain (1.5) is p-essentially normal exactly when

$$p > \begin{cases} \frac{1}{2} & \text{if } m = 1, \\ \max\{m, p_j(m-1) : j = 1, \dots, m\} & \text{if } m > 1. \end{cases}$$

(See also Remark 2.7.) This suggests that the Schatten class property of the commutators may be related to the convexity and geometry of the domain [Beatrous and Li 1995; Connes 1994; Douglas and Voiculescu 1981; Krantz et al. 1997; Milnor 1968]. We plan to discuss this relation in the future [Jabbari 2020].

2. The construction of the resolution in Theorem 1.7

From now on, Ω_1 and Ω_2 are domains of type (1.5) and (1.6), respectively. We develop the details for Ω_1, and Ω_2 can be treated similarly, with the only difference being Proposition 2.2 and the proof of Lemma 2.6. We always use the multi-index notation [Krantz 2001, page 3], especially $|\alpha|$ to stand for the sum of the components of the multi-index α. $\mathbb{N}$ denotes the set of nonnegative integers.

2A. ***The monomial orthonormal basis for the Bergman space.*** Monomial functions z^α, $\alpha \in \mathbb{N}^m$, are orthogonal in $L_a^2(\Omega_1)$, as the integration in polar coordinates in each variable shows. On the other hand, since Ω_1 is a complete Reinhardt domain, polynomials are dense in $L_a^2(\Omega_1)$ with respect to the topology of uniform convergence on compacts [Range 1986, page 47]. Then a standard shrinking argument [Zhu 2005, page 43; Duren and Schuster 2004, page 11] shows that the normalized monomials

$$\frac{z^\alpha}{\sqrt{\omega_1(\alpha)}}, \qquad \omega_1(\alpha) := \|z^\alpha\|^2_{L_a^2(\Omega_1)}$$

constitute an orthonormal basis for the Hilbert space $L_a^2(\Omega_1)$. Next, we are going to find an explicit formula for $\omega_1(\alpha)$ as well as $\omega_2(\alpha, \beta, \dots) := \|z^\alpha w^\beta \cdots\|^2_{L_a^2(\Omega_2)}$. In what follows, $dx := dx_1 \cdots dx_m$ denotes the Riemannian density of the Euclidean space $\mathbb{R}^m$, for a variable $x = (x_1, \dots, x_m)$ ranging over some part of $\mathbb{R}^m$. The set of positive reals is denoted by $\mathbb{R}_+$.

Lemma 2.1. *Given $\alpha := (\alpha_1, \dots, \alpha_m) \in \mathbb{R}_+^m$, we have*

$$\int_{x \in \mathbb{R}_+^m,\ \sum_{j=1}^m x_j^2 < 1} x^\alpha \, dx = \frac{B\left(\frac{1}{2}(\alpha+1)\right)}{2^m \, \frac{1}{2}(\alpha+1)},$$

$$\int_{x \in \mathbb{R}_+^m,\ \sum_{j=1}^m x_j^2 = 1} x^\alpha \, d\sigma_m(x) = \frac{B\left(\frac{1}{2}(\alpha+1)\right)}{2^{m-1}},$$

where $\frac12(\alpha+1) = \left(\frac12(\alpha_1+1), \ldots, \frac12(\alpha_m+1)\right)$,

$$B\left(\tfrac12(\alpha+1)\right) = \frac{\prod_{j=1}^m \Gamma\left(\frac12(\alpha_j+1)\right)}{\Gamma\left(\sum_{j=1}^m \frac12(\alpha_j+1)\right)}$$

is the multivariable Beta function, and $d\sigma_m$ *is the Riemannian density that* dx *induces on the unit sphere* $\mathbb{S}^{m-1} \subseteq \mathbb{R}^m$.

Proof. These are standard facts; see [Andrews et al. 1999, Section 1.8; Zhu 2005, page 13; Folland 1999, page 80].

Proposition 2.2. (a) *Given multi-index* $\alpha \in \mathbb{N}^m$, *we have*

$$\omega_1(\alpha) = \|z^\alpha\|^2_{L^2_a(\Omega_1)} = \frac{\pi^m}{\prod p_j} \frac{B\left(\frac1p(\alpha+1)\right)}{\left|\frac1p(\alpha+1)\right|},$$

where $\frac1p(\alpha+1) := \left(\frac{1}{p_1}(\alpha_1+1), \ldots, \frac{1}{p_m}(\alpha_m+1)\right)$.

(b) *Given multi-indices* $\alpha \in \mathbb{N}^m$, $\beta \in \mathbb{N}^n$, $\ldots$, *we have*

$$\begin{aligned}\omega_2(\alpha, \beta, \ldots) &= \|z^\alpha w^\beta \cdots\|^2_{L^2_a(\Omega_2)} \\ &= \frac{\pi^{m+n+\cdots}}{\prod p_j \prod q_k \cdots} \frac{1}{ab\cdots} B\left(\tfrac1p(\alpha+1)\right) B\left(\tfrac1q(\beta+1)\right) \cdots \\ &\qquad \times \frac{B\left(\left|\frac{1}{ap}(\alpha+1)\right|, \left|\frac{1}{bq}(\beta+1)\right|, \ldots\right)}{\left|\frac{1}{ap}(\alpha+1)\right| + \left|\frac{1}{bq}(\beta+1)\right| + \cdots}.\end{aligned}$$

Proof. (a) Using polar coordinates $z_j = x_j e^{\sqrt{-1}\theta_j}$ for $z = (z_1, \ldots, z_m)$, we have

$$\omega_1(\alpha) = \int_{z\in\Omega_1} x^{2\alpha} \prod x_j\, dx_j\, d\theta_j = (2\pi)^m \int_{x\in\mathbb{R}^m_+,\ \sum x_j^{2p_j}<1} x^{2\alpha+1}\, dx.$$

After the change of variables $X_j := x_j^{p_j}$, we have

$$\omega_1(\alpha) = \frac{(2\pi)^m}{\prod p_j} \int_{X\in\mathbb{R}^m_+,\ \sum X_j^2<1} X^{\frac1p(2\alpha+2)-1}\, dX.$$

We are done by Lemma 2.1.

(b) Using polar coordinates $z_j = x_j e^{\sqrt{-1}\theta_j}$, $w_k = y_k e^{\sqrt{-1}\phi_k}$, $\ldots$, we have

$$\begin{aligned}&\omega_2(\alpha, \beta, \ldots) \\ &\quad = (2\pi)^{m+n+\cdots} \int_{x\in\mathbb{R}^m_+,\, y\in\mathbb{R}^n_+,\, \ldots,\ \left(\sum x_j^{2p_j}\right)^a + \left(\sum y_k^{2q_k}\right)^b + \cdots < 1} x^{2\alpha+1} y^{2\beta+1} \cdots\, dx\, dy \cdots.\end{aligned}$$

After the change of variables $X_j := x_j^p$, $Y_k := y_k^q$, ..., we have

$$\omega_2(\alpha,\beta,\dots) = \frac{(2\pi)^{m+n+\cdots}}{\prod p_j \prod q_k \cdots} \times \int_{X\in\mathbb{R}^m_+,\, Y\in\mathbb{R}^n_+,\,\dots,\ (\sum X_j^2)^a + (\sum Y_k^2)^b + \cdots < 1} X^{\frac{1}{p}(2\alpha+2)-1}\, Y^{\frac{1}{q}(2\beta+2)-1}\cdots \, dX\, dY \cdots .$$

Now comes the trick we learned from [D'Angelo 1994]. Changing to the spherical coordinates $X = r\xi$, $Y = s\eta$, ..., where $r, s, \dots$ are positive reals and $\xi, \eta, \dots$ live on unit spheres $\mathbb{S}^{m-1}, \mathbb{S}^{n-1}, \dots$, respectively, we have

$$\begin{aligned}\omega_2(\alpha,\beta,\dots) &= \frac{(2\pi)^{m+n+\cdots}}{\prod p_j \prod q_k \cdots} \int_{r,s,\dots\in\mathbb{R}_+,\ r^{2a}+s^{2b}+\cdots<1} r^{|\frac{1}{p}(2\alpha+2)|-1} s^{|\frac{1}{q}(2\beta+2)|-1}\cdots \, dr\, ds \cdots \\ &\quad\times \int_{\xi\in\mathbb{S}^{m-1}_+,\ \eta\in\mathbb{S}^{n-1}_+,\ \dots} \xi^{\frac{1}{p}(2\alpha+2)-1}\, \eta^{\frac{1}{q}(2\beta+2)-1}\cdots \, d\sigma_m(\xi)\, d\sigma_n(\eta)\cdots ,\end{aligned}$$

where S^{m-1}_+ denotes $\mathbb{S}^{m-1}\cap\mathbb{R}^m_+$, and similarly for others. The first integral is given by the first formula in Lemma 2.1 after the change of variables $R := r^a$, $S := s^b$, ..., and the second integral is given by the second formula in Lemma 2.1.

For later use, we do the same computations in the more general context of weighted Bergman spaces. Given a domain $\Omega\subseteq\mathbb{C}^m$ with smooth boundary, $L^2_{a,s}(\Omega)$, $s > -1$ denotes the weighted Bergman space consisting of all holomorphic functions f on Ω such that $\int_\Omega |f(z)|^2\rho(z)^s\, dV(z) < \infty$, where $\rho(z)$ is a positively signed smooth defining function for Ω and dV is the Lebesgue measure. For Ω_1 and Ω_2, we use the defining functions $1-\sum|z_j|^{2p_j}$ and $1-(\sum|z_j|^{2p_j})^a - (\sum|w_k|^{2q_k})^b - \cdots$, respectively.

Proposition 2.3. (a) *Given multi-index* $\alpha\in\mathbb{N}^m$, *we have*

$$\omega_{1,s}(\alpha) := \|z^\alpha\|^2_{L^2_{a,s}(\Omega_1)} = \frac{\pi^m}{\prod p_j} B\big(\tfrac{1}{p}(\alpha+1)\big)\frac{\Gamma\big(|\tfrac{1}{p}(\alpha+1)|\big)\, s!}{\Gamma\big(|\tfrac{1}{p}(\alpha+1)|+s+1\big)}.$$

(b) *Given multi-indices* $\alpha\in\mathbb{N}^m$, $\beta\in\mathbb{N}^n$, ..., *we have*

$$\begin{aligned}\omega_{2,s}(\alpha,\beta,\dots) &:= \|z^\alpha w^\beta\cdots\|^2_{L^2_{a,s}(\Omega_2)} \\ &= \frac{\pi^{m+n+\cdots}}{\prod p_j\prod q_k\cdots}\,\frac{1}{ab\cdots}\, B\big(\tfrac{1}{p}(\alpha+1)\big)\, B\big(\tfrac{1}{q}(\beta+1)\big)\cdots \\ &\quad\times B\big(|\tfrac{1}{ap}(\alpha+1)|, |\tfrac{1}{bq}(\beta+1)|, \dots\big)\frac{s!\,\Gamma\big(|\tfrac{1}{ap}(\alpha+1)|+|\tfrac{1}{bq}(\beta+1)|+\cdots\big)}{\Gamma\big(s+1+|\tfrac{1}{ap}(\alpha+1)|+|\tfrac{1}{bq}(\beta+1)|+\cdots\big)}.\end{aligned}$$

Proof. (a) Similar to the proof of Proposition 2.2(a), we have

$$\omega_{1,s}(\alpha) = \frac{(2\pi)^m}{\prod p_j}\int_{X\in\mathbb{R}^m_+,\ \sum X_j^2<1} X^{\frac{1}{p}(2\alpha+2)-1}\Big(1-\sum X_j^2\Big)^s\,dX.$$

Changing to the spherical coordinates $X = r\xi$, $r > 0$, $\xi \in \mathbb{S}^{m-1}$, we have

$$\omega_{1,s}(\alpha) = \frac{(2\pi)^m}{\prod p_j}\int_{\xi\in\mathbb{S}^{m-1}_+} \xi^{\frac{1}{p}(2\alpha+2)-1}\,d\sigma_m(\xi) \times \int_0^1 r^{|\frac{1}{p}(2\alpha+2)|-1}(1-r^2)^s\,dr.$$

The first integral is given by the second formula in Lemma 2.1, and the second integral is given by the formula $\int_0^1 t^{a-1}(1-t)^{b-1}\,dt = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ after the change of variable $r^2 = t$.

(b) Similar to the proof of Proposition 2.2(b), we have

$$\begin{aligned}\omega_{2,s}&(\alpha,\beta,\dots)\\ &= \frac{(2\pi)^{m+n+\cdots}}{\prod p_j \prod q_k\cdots}\int_{\xi\in\mathbb{S}^{m-1}_+,\ \eta\in\mathbb{S}^{n-1}_+,\ \dots} \xi^{\frac{1}{p}(2\alpha+2)-1}\eta^{\frac{1}{q}(2\beta+2)-1}\cdots\,d\sigma_m(\xi)\,d\sigma_n(\eta)\cdots\\ &\quad\times\int_{\substack{r,s,\dots\in\mathbb{R}_+,\\ r^{2a}+s^{2b}+\cdots<1}} r^{|\frac{1}{p}(2\alpha+2)|-1}s^{|\frac{1}{q}(2\beta+2)|-1}\cdots\,dr\,ds\cdots(1-r^{2a}-s^{2b}-\cdots)^s.\end{aligned}$$

The first integral is given by the second formula in Lemma 2.1. The second integral after the change of coordinates $R := r^a$, $S := s^b$, … becomes

$$\frac{1}{ab\cdots}\int_{\substack{R,S,\dots\in\mathbb{R}_+,\\ R^2+S^2+\cdots<1}} R^{|\frac{1}{ap}(2\alpha+2)|-1}S^{|\frac{1}{bq}(2\beta+2)|-1}\cdots(1-R^2-S^2-\cdots)^s\,dR\,dS\cdots.$$

Changing to the spherical coordinates $(R,S,\dots)=r\xi$, $r > 0$, ξ in the unit sphere, this latter integral equals an integral in the second formula in Lemma 2.1 multiplied by some integral of the form $\int_0^1 t^{u-1}(1-t)^{v-1}\,dt = \Gamma(u)\Gamma(v)/\Gamma(u+v)$.

2B. ***Some notation.*** From now on, we are going to use the notation

$$z^{\mathbf{n}} := \frac{z_1^{n_1}\cdots z_m^{n_m}}{\sqrt{\omega_1(\mathbf{n})}},\qquad \mathbf{n}=(n_1,\dots,n_m)\in\mathbb{N}^m, \tag{2.4}$$

for the elements of the orthonormal basis of $L^2_a(\Omega_1)$ derived in Section 2A.

Given a positive integer q, let $S_q(m)$ denote the set of all q-shuffles of the set $\{1,\dots,m\}$, namely

$$S_q(m) := \{\mathbf{j} := (j_1,\dots,j_q)\in\mathbb{Z}^q : 1\le j_1<j_2<\cdots<j_q\le m\}.$$

Whenever necessary, we identify shuffles in $S_q(m)$ with subsets of $\{1,\dots,m\}$ of size q. This enables us to talk about the union, intersection, etc. of shuffles of $\{1,\dots,m\}$ with themselves and with other subsets of $\{1,\dots,m\}$.

2C. *Boxes and their associated Hilbert modules.* To each $\mathsf{j} = (j^1, \ldots, j^q) \in S_q(m)$ and $\mathsf{b} = (b^1, \ldots, b^q) \in \mathbb{N}^q$ we associate the *box*

$$\mathsf{B}_{\mathsf{j}}^{\mathsf{b}} := \{(n^1, \ldots, n^m) \in \mathbb{N}^m : n^{j^i} \le b^i \text{ for } i = 1, \ldots, q\},$$

and to each box $\mathsf{B}_{\mathsf{j}}^{\mathsf{b}}$ we associate the Hilbert space

$$H_{\mathsf{j}}^{\mathsf{b}} := L_a^2(\ \ ) \ominus \overline{\langle z_{j^1}^{b^1+1}, \ldots, z_{j^q}^{b^q+1} \rangle}$$

consisting of all functions $X = \sum_{\mathsf{n}\in\mathbb{N}^m} X_{\mathsf{n}} z^{\mathsf{n}} \in L_a^2(\ \ )$ such that $X_{\mathsf{n}} = 0$ for every $\mathsf{n} \in \mathbb{N}^m \setminus \mathsf{B}_{\mathsf{j}}^{\mathsf{b}}$. An element $X \in H_{\mathsf{j}}^{\mathsf{b}}$ has the Taylor expansion $X = \sum X_{n^1 \cdots n^m} z^{\mathsf{n}}$ with summation over $n^{j^1} \le b^1, \ldots, n^{j^q} \le b^q$. The general construction in Section 1 about the orthogonal complements of polynomial ideals makes $H_{\mathsf{j}}^{\mathsf{b}}$ a Hilbert A-module. (A denotes the ring of polynomials in m variables.) More explicitly, its fundamental tuple of Toeplitz operators is given by

$$T_{z_i}^{\mathsf{j},\mathsf{b}}(z^{\mathsf{n}}) := \begin{cases} z_i z^{\mathsf{n}} & \text{if } (n^1, \ldots, n^{i-1}, n^i + 1, n^{i+1}, \ldots, n^m) \in \mathsf{B}_{\mathsf{j}}^{\mathsf{b}}, \\ 0 & \text{otherwise}, \end{cases} \qquad i = 1, \ldots, m.$$

In the next proposition we gather several facts about essential normality which will be used later.

Proposition 2.5 (Arveson–Douglas). (a) *Let be an open subset of $\mathbb{C}^m$, $I \subseteq A$ be a homogeneous ideal, and P, $Q := 1 - P$ be the orthogonal projections in $L_a^2(\ )$ onto $\bar{I}$ and $I^\perp$, respectively. Suppose that $L_a^2(\ )$ is essentially normal. Then $\bar{I}$ is essentially normal (module actions are given by restrictions of multiplications in $L_a^2(\ )$) if and only if $I^\perp$ is essentially normal, if and only if all $[M_{z_\alpha}, P]$ for $\alpha = 1, \ldots, m$, are compact, if and only if all $PM_{z_\alpha}Q$ are compact, if and only if all $[M_{z_\alpha}, Q]$ are compact, if and only if all $QM_{z_\alpha}^* P$ are compact.*

(b) *Let $\mathcal{M}$ and $\mathcal{N}$ be isomorphic Hilbert A-modules. Then $\mathcal{M}$ is essentially normal if and only of $\mathcal{N}$ is; if so, then they represent the same odd K-homology class.*

(c) *Let $\mathcal{M}$ be an essentially normal Hilbert A-module, and $\mathcal{N} \subseteq \mathcal{M}$ be a submodule. Then $\mathcal{N}$ is essentially normal if and only if the quotient module $\mathcal{M}/\mathcal{N}$ is.*

(d) *Let $\Phi : \mathcal{A}_1 \to \mathcal{A}_2$ be a closed-range Hilbert A-module map between essentially normal Hilbert modules. Then the kernel and range of Φ are essentially normal.*

Proof. (a) Our reference is [Arveson 2005, Theorem 4.3]. Recall that an operator T is compact if and only if T^* is compact, if and only if TT^* is compact. Let the module action of $p \in A$ on $L_a^2(\ )$, $\bar{I}$ and $I^\perp$ be denoted by operators M_p, R_p and T_p, respectively. For brevity, set $M_\alpha := M_{z_\alpha}$, $R_\alpha := R_{z_\alpha}$ and $T_\alpha := T_{z_\alpha}$. The last four statements are easily seen to be equivalent. Here are the reasons. Since $\bar{I}$ is invariant under M_α, we have $PM_\alpha P = M_\alpha P$. Then

$$[M_\alpha, P] = M_\alpha P - PM_\alpha = PM_\alpha P - PM_\alpha = -PM_\alpha Q.$$

The equality $P + Q = 1$ gives $[M_\alpha, P] = -[M_\alpha, Q]$. Also note that

$$(PM_\alpha Q)^* = QM_\alpha^* P.$$

For the rest, we need the assumption that $L^2_a(\Omega)$ is essentially normal. With an abuse of language, one says that, as mappings from $L^2_a(\Omega)$ to $\bar{I}$, $R_\alpha P$ and $R_\beta^* P$ equal $PM_\alpha P = M_\alpha P$ and $PM_\beta^* P$, respectively. Then

$$\begin{aligned}[R_\alpha, R_\beta^*]P &= M_\alpha PM_\beta^* P - PM_\beta^* M_\alpha P \sim M_\alpha PM_\beta^* P - PM_\alpha M_\beta^* P \\ &= [M_\alpha, P]M_\beta^* P = -PM_\alpha QM_\beta^* P = -(PM_\alpha Q)(QM_\beta^* P) \\ &= -(PM_\alpha Q)(PM_\beta Q)^* = -[M_\alpha, P][M_\beta, P]^*,\end{aligned}$$

where $\sim$ denotes equality modulo compacts. This identity shows that all $[R_\alpha, R_\beta^*]$ are compact if and only if all $[M_\alpha, P]$ are. The rest of the proof is dual. As mappings from $L^2_a(\Omega)$ to $I^\perp$, $T_\alpha Q$ and $T_\beta^* Q$ equal $QM_\alpha Q$ and $QM_\beta^* Q = M_\beta^* Q$, respectively. We also have the identity

$$[T_\alpha, T_\beta^*]Q \sim [M_\beta, Q]^*[M_\alpha, Q],$$

which proves that all $[T_\alpha, T_\beta^*]$ are compact if and only if all $[M_\alpha, Q]$ are.

(b, c, d) Refer to [Douglas et al. 2016, Proposition 4.4], [Douglas 2006a, Theorem 2.1] and [Douglas 2006b, Theorem 2.2], respectively.

Lemma 2.6. *Each $\mathcal{H}_j^b$ is essentially normal.*

Proof. We first show that $L^2_a(\Omega_1)$ is essentially normal; compare [Curto and Salinas 1985]. Let $M_{z_i} \in B(L^2_a(\Omega_1))$ for $i = 1, \dots, m$, be multiplication by the coordinate function z_i. Since these operators commute with each other, according to the Fuglede–Putnam theorem, it suffices to verify that each M_{z_i} is essentially normal. A straightforward computation shows that

$$[M_{z_i}, M_{z_i}^*](z^{\mathbf{n}}) = \lambda z^{\mathbf{n}}, \quad \forall \mathbf{n} = (n_1, \dots, n_m) \in \mathbb{N}^m,$$

where

$$\lambda = \lambda' - \lambda'', \qquad \lambda' = \frac{\omega_1(n_1 \cdots n_m)}{\omega_1(n_1 \cdots n_i - 1 \cdots n_m)}, \qquad \lambda'' = \frac{\omega_1(n_1 \cdots n_i + 1 \cdots n_m)}{\omega_1(n_1 \cdots n_m)},$$

and λ' is set to be zero when $n_i = 0$. We need to check that $\lambda \to 0$ when the norm of $\mathbf{n}$ (say the l^1 norm) tends to infinity. By Proposition 2.2(a), we have

$$\lambda' = \begin{cases} \dfrac{\Gamma\left(\frac{n_i+1}{p_i}\right)}{\Gamma\left(\frac{n_i}{p_i}\right)} \dfrac{\Gamma\left(N + \frac{n_i}{p_i}\right)}{\Gamma\left(N + \frac{n_i+1}{p_i}\right)} \dfrac{N + \frac{n_i}{p_i}}{N + \frac{n_i+1}{p_i}} & \text{if } n_i > 0, \\ 0 & \text{if } n_i = 0, \end{cases}$$

where $N := \sum_{l \neq i} (n_l + 1)/p_l$.

Note that by Stirling's formula (or, more strongly, [Tricomi and Erdélyi 1951]),

$$\frac{0(x+a)}{0(x)} = x^a(1 + O(x^{-1}))$$

as the real variable x grows large. Therefore, when n_i is bounded and $N \to \infty$, λ^0 is dominated by N^{-1/p_i}, so $\lambda \to 0$. On the other hand, when $n_i \to \infty$, λ^0 asymptotically behaves like

$$1 - \frac{N}{n_i + p_i N} + O\left(\frac{1}{n_i}\right) + O\left(\frac{1}{n_i + N}\right).$$

This shows that $\lambda = \lambda^0 - \lambda^0 \to 0$ when $n_i \to \infty$. We have shown that $L^2_a(\ _1)$ is essentially normal.

Let P be the orthogonal projection in $L^2_a(\ _1)$ onto $H^{\mathrm{b}}_{\mathrm{j}}$. To prove our lemma, according to Proposition 2.5(a) it suffices to check that each $[M_{z_i}, P]$ is compact. For each $\mathrm{n} \in \mathrm{B}^{\mathrm{b}}_{\mathrm{j}}$ we have

$$PM_{z_i}(z^{\mathrm{n}}) = \begin{cases} \sqrt{\dfrac{\omega_1(n_1 \cdots n_i + 1 \cdots n_m)}{\omega_1(n_1 \cdots n_m)}}\, z^{n_1 \cdots n_i + 1 \cdots n_m} & \text{if } (n_1 \cdots n_i + 1 \cdots n_m) \in \mathrm{B}^{\mathrm{b}}_{\mathrm{j}}, \\ 0 & \text{otherwise,} \end{cases}$$

$$M_{z_i}P(z^{\mathrm{n}}) = \begin{cases} \sqrt{\dfrac{\omega_1(n_1 \cdots n_i + 1 \cdots n_m)}{\omega_1(n_1 \cdots n_m)}}\, z^{n_1 \cdots n_i + 1 \cdots n_m} & \text{if } (n_1 \cdots n_i \cdots n_m) \in \mathrm{B}^{\mathrm{b}}_{\mathrm{j}}, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the coefficients $\sqrt{\cdots}$ appear because of the normalization in definition (2.4). Therefore

$$[M_{z_i}, P](z^{\mathrm{n}}) = \begin{cases} \sqrt{\dfrac{\omega_1(n_1 \cdots b_l + 1 \cdots n_m)}{\omega_1(n_1 \cdots b_l \cdots n_m)}}\, z^{n_1 \cdots n_i + 1 \cdots n_m} & \begin{array}{l}\text{if } (n_1 \cdots n_i \cdots n_m) \in \mathrm{B}^{\mathrm{b}}_{\mathrm{j}} \\ \text{and there exists } l \text{ such} \\ \text{that } i = j_l,\ n_i = b_l,\end{array} \\ 0 & \text{otherwise.} \end{cases}$$

We need to check that the ratio

$$\rho := \frac{\omega_1(n_1 \cdots b_l + 1 \cdots n_m)}{\omega_1(n_1 \cdots b_l \cdots n_m)},$$

with l and b_l fixed, approaches zero when the norm of $(n_1, \ldots b_l, \ldots n_m)$ tends to infinity. This was verified during the proof of the essential normality of $L^2_a(\ _1)$. This finishes the proof of our lemma for domains of type (1.5). The proof for domains of type (1.6) is completely similar, having at hand the explicit formula for $\omega_2(\alpha, \beta, \ldots)$ from Proposition 2.2(b).

Remark 2.7. With arguments similar to the ones in the proof of Lemma 2.6, one can show that $L^2_a(\Omega_1)$ is p-essentially normal if and only if

$$p > \begin{cases} \frac{1}{2} & \text{if } m = 1, \\ \max\{m, p_j(m-1) : j = 1, \dots, m\} & \text{if } m > 1. \end{cases}$$

The computations will be included in our forthcoming paper [Jabbari 2020]. It is worth pointing out that it is a new phenomenon that the p-essential normality of the Bergman module depends not only on the dimension of the domain but also on its geometry. (See also [Beatrous and Li 1995; Krantz et al. 1997].) This phenomenon will also be explored in [Jabbari 2020].

2D. ***The geometry of Hilbert modules associated to boxes.*** Consider the Hilbert module $\mathcal{H}^{\mathbf{b}}_{\mathbf{j}}$ associated to the box $\mathrm{B}^{\mathbf{b}}_{\mathbf{j}}$. Set

$$\Omega_{1,\mathbf{j}} := \{(z_1, \dots, z_m) \in \Omega_1 : z_{j^1} = \cdots = z_{j^q} = 0\}.$$

Observe that $\Omega_{1,\mathbf{j}}$ is an egg domain of type (1.5) inside $\mathbb{C}^{m-q}$. Consider the Hilbert space

$$\widetilde{\mathcal{H}}^{\mathbf{b}}_{\mathbf{j}} := \bigoplus_{\substack{\mathbf{i}=(i^1,\dots,i^q)\in\mathbb{N}^q \\ i^1\le b^1,\dots,i^q\le b^q}} L^2_{a,\sum_{l=1}^q (i^l+1)/p_{j^l}}(\Omega_{1,\mathbf{j}}),$$

and the map $R^{\mathbf{b}}_{\mathbf{j}} : \mathcal{H}^{\mathbf{b}}_{\mathbf{j}} \to \widetilde{\mathcal{H}}^{\mathbf{b}}_{\mathbf{j}}$ given by sending $X \in \mathcal{H}^{\mathbf{b}}_{\mathbf{j}}$ to

$$Y = \sum Y^{\mathbf{i}},$$

$$Y^{\mathbf{i}} = \left(\frac{\pi^q}{\mathbf{i}! \prod_{l=1}^q p_{j^l}} \frac{\prod_{l=1}^q \Gamma((i^l+1)/p_{j^l})}{\Gamma\left(\sum_{l=1}^q (i^l+1)/p_{j^l}\right)} \right)^{1/2} \frac{\partial^{|\mathbf{i}|} X}{\partial z_{j^1}^{i^1} \cdots \partial z_{j^q}^{i^q}} \bigg|_{\Omega_{1,\mathbf{j}}} \in L^2_{a,\sum_{l=1}^q (i^l+1)/p_{j^l}}(\Omega_{1,\mathbf{j}}).$$

A straightforward computation with the orthonormal bases (Propositions 2.2 and 2.3) shows that $R^{\mathbf{b}}_{\mathbf{j}}$ *is an isometric isomorphism of Hilbert spaces*.

Now consider the trivial vector bundle $E^{\mathbf{b}}_{\mathbf{j}} := \mathbb{C}^{(b^1+1)\cdots(b^q+1)} \times \Omega_{1,\mathbf{j}}$ over $\Omega_{1,\mathbf{j}}$, together with the standard frame $e_{\mathbf{i}}$, $\mathbf{i} = (i^1, \dots, i^q) \in \mathbb{N}^q$, $i^1 \le b^1, \dots, i^q \le b^q$, and equip it with the Hermitian structure

$$\langle e_{\mathbf{i}}, e_{\mathbf{i}'} \rangle(z) = \left(1 - \sum_{l=1}^q |z_l|^{2p_{j^l}}\right)^{\sum_{l=1}^q (i^l+1)/p_{j^l}} \delta_{\mathbf{i},\mathbf{i}'}, \quad z \in \Omega_{1,\mathbf{j}},$$

where δ is the Kronecker tensor. This way, $\widetilde{\mathcal{H}}^{\mathbf{b}}_{\mathbf{j}}$ can be identified with the Bergman space of the L^2-holomorphic sections of $E^{\mathbf{b}}_{\mathbf{j}}$. Under the isomorphism $R^{\mathbf{b}}_{\mathbf{j}}$, one can identify the Toeplitz algebra generated by $T^{\mathbf{j},\mathbf{b}}_{z_i} \in B(\mathcal{H}^{\mathbf{b}}_{\mathbf{j}})$, $i = 1, \dots, m$, with the algebra generated by matrix-valued Toeplitz operators on the latter Bergman space of L^2-holomorphic sections of $E^{\mathbf{b}}_{\mathbf{j}}$.

2E. ***The construction of the resolution.*** This section constructs the resolution in Theorem 1.7. Let the ideal $I \subseteq A$ be generated by distinct monomials

$$z^{\alpha_i}, \quad \alpha_i := (\alpha_i^1, \dots, \alpha_i^m) \in \mathbb{N}^m, \; i = 1, \dots, l,$$

Let the *complementary space* $\mathsf{C}(I) \subseteq \mathbb{N}^m$ be the set of the exponents of those monomials which do not belong to I. Note that the set of monomials belonging to I is a basis of I as a complex vector space [Herzog and Hibi 2011, Theorem 1.1.2]. Also note that a monomial u belongs to I if and only if there is a monomial v such that $u = vz^{\alpha_i}$ for some $i = 1, \dots, l$ [Herzog and Hibi 2011, Proposition 1.1.5]. In other words, $z_1^{n^1} \cdots z_m^{n^m} \in \mathsf{C}(I)$ if and only if for every $i = 1, \dots, l$ there exists $s_i \in \{1, \dots, m\}$ such that $n^{s_i} < \alpha_i^{s_i}$. Consider the finite collection

$$S(\alpha_1, \dots, \alpha_l) = \{1, \dots, m\}^l$$

of l-tuples $\mathsf{s} = (s_1, \dots, s_l)$ of integers such that $1 \le s_i \le m$ for every i. Given s, let j_s be the shuffle associated to the set $\{s_1, \dots, s_l\}$. For each $j \in \mathsf{j}_\mathsf{s}$, let b_j be the minimum of all $\alpha_i^{s_i} - 1$, $i = 1, \dots, l$, such that $s_i = j$. Set $\mathsf{b}_\mathsf{s} := (b_j)_{j \in \mathsf{j}_\mathsf{s}}$. The following symbolic logic computation shows that $\mathsf{C}(I)$ *is the union of boxes* $\mathsf{B}_{\mathsf{j}_\mathsf{s}}^{\mathsf{b}_\mathsf{s}}$ *for* $\mathsf{s} \in S(\alpha_1, \dots, \alpha_l)$:

$$z_1^{n^1} \cdots z_m^{n^m} \in \mathsf{C}(I) \leftrightarrow (n^1 < \alpha_1^1 \vee \cdots \vee n^m < \alpha_1^m) \wedge \cdots \wedge (n^1 < \alpha_l^1 \vee \cdots \vee n^m < \alpha_l^m)$$
$$\leftrightarrow \bigvee_{(s_1, \dots, s_l) \in \{1, \dots, m\}^l} (n^{s_1} < \alpha_1^{s_1} \wedge \cdots \wedge n^{s_l} < \alpha_l^{s_l}).$$

The construction of modules $\mathcal{A}_q$. From now on, fix a finite collection of boxes

$$\mathsf{B}_{\mathsf{j}_i}^{\mathsf{b}_i}, \quad i = 1, \dots, k, \tag{2.8}$$

such that their union equals $\mathsf{C}(I)$. Given $I \subseteq \{1, \dots, k\}$, (note that we are using the symbol I for two purposes), let

$$\mathsf{B}_{\mathsf{j}_I}^{\mathsf{b}_I} := \bigcap_{i \in I} \mathsf{B}_{\mathsf{j}_i}^{\mathsf{b}_i}$$

denote the intersection of boxes $\mathsf{B}_{\mathsf{j}_i}^{\mathsf{b}_i}$, $i \in I$. (Note that the intersections of boxes are again boxes.) Each box $\mathsf{B}_{\mathsf{j}_I}^{\mathsf{b}_I}$ has a corresponding Hilbert module $H_{\mathsf{j}_I}^{\mathsf{b}_I}$ as introduced in Section 2C. For each $q = 1, \dots, k$, set

$$\mathcal{A}_q := \bigoplus_{I \in S_q(k)} H_{\mathsf{j}_I}^{\mathsf{b}_I}, \quad \mathcal{A}_0 := L_a^2(\mathbb{B}_1).$$

Note that each Hilbert space $\mathcal{A}_q$ is equipped with a Hilbert A-module structure coming from the A-module structures on its direct summands. The following proposition is immediate from Lemma 2.6.

Proposition 2.9. *Each* $\mathcal{A}_q$ *is essentially normal.*

The construction of the maps ∂_q. Thinking of the elements of $S_{q+1}(k)$ as the subsets $I_{q+1} \subseteq \{1, \ldots, k\}$, of size $q+1$, define the maps $f^i_{q+1} : S_{q+1}(k) \to S_q(k)$, $i = 1, \ldots, q+1$, by setting $f^i_{q+1}(I_{q+1})$ to be the subset of $\{1, \ldots, k\}$, obtained by dropping the i-th smallest element in I_{q+1}. The map $\partial_q : \mathcal{A}_q \to \mathcal{A}_{q+1}$ is defined by

$$X = \sum_{I_q \in S_q(k)} X^{I_q} \mapsto Y = \sum_{I_{q+1} \in S_{q+1}(k)} Y^{I_{q+1}}, \quad X^{I_q} \in \mathcal{H}_{\widehat{\jmath}_{I_q}}^{b_{I_q}}, \ Y^{I_{q+1}} \in \mathcal{H}_{\widehat{\jmath}_{I_{q+1}}}^{b_{I_{q+1}}},$$

given by

$$(Y^{I_{q+1}})_{n} = \begin{cases} \displaystyle\sum_{i=1}^{q+1} (-1)^{i-1} \left(X^{f^i_{q+1}(I_{q+1})}\right)_{n} & \text{if } n \in B_{\widehat{\jmath}_{I_{q+1}}}^{b_{I_{q+1}}}, \\ 0 & \text{otherwise}. \end{cases}$$

Remark 2.10. Similar to the explanation in Section 2D, each Hilbert module $\mathcal{A}_q$, $q = 1, \ldots, k$, can be identified with the Bergman space of the L^2-holomorphic sections of a Hermitian vector bundle on a disjoint union of subsets of Ω_1. Under this identification, the module morphisms ∂_q, $q = 0, \ldots, k-1$ can be realized as the restriction maps of jets of holomorphic sections to the subsets. Although this geometric picture is not used heavily in what follows, we believe that such an intuition will play a crucial role in the study of nonradical ideals beyond monomials.

3. The proof of Theorem 1.7

In this section we prove Theorem 1.7. Again, we develop the details for a domain Ω_1 of type (1.5), and domains of type (1.6) can be treated similarly.

3A. ***The proof of Theorem 1.7*(a).** In this section we prove that the construction of Section 2E is a resolution of Hilbert modules asserted in Theorem 1.7(a). This is an adjustment of the proof of Theorem 1.4, which first appeared in [Douglas et al. 2018, Theorem 1.1].

Proposition 3.1. *Each ∂_q is a morphism of Hilbert A-modules.*

Proof. We first verify boundedness. For each $X = \sum_{I_q \in S_q(k)} X^{I_q} \in \mathcal{A}_q$, $X^{I_q} \in \mathcal{H}_{\widehat{\jmath}_{I_q}}^{b_{I_q}}$, we defined

$$\partial_q(X) = \sum_{I^0_{q+1}} Y^{I^0_{q+1}},$$

$$Y^{I^0_{q+1}} \in B_{\widehat{\jmath}_{I^0_{q+1}}}^{b_{I^0_{q+1}}}, \quad Y_{n}^{I^0_{q+1}} = \begin{cases} \displaystyle\sum_{i=1}^{q+1} (-1)^{i-1} X_{n}^{f^i_{q+1}(I^0_{q+1})} & \text{if } n \in B_{\widehat{\jmath}_{I^0_{q+1}}}^{b_{I^0_{q+1}}}, \\ 0 & \text{otherwise}. \end{cases}$$

Therefore,

$$
\begin{aligned}
\|\vartheta_q(X)\|^2 &= \sum_{I^0_{q+1}} \sum_{\mathbf{n}\in B^{\mathbf{b}_{I^0_{q+1}}}_{\mathbf{j}_{I^0_{q+1}}}} \Big| \sum_{i=1}^{q+1} (-1)^{i-1} X_{\mathbf{n}}^{f^i_{q+1}(I^0_{q+1})} \Big|^2 \le \sum_{I^0_{q+1}} \sum_{\mathbf{n}\in B^{\mathbf{b}_{I^0_{q+1}}}_{\mathbf{j}_{I^0_{q+1}}}} (q+1) \Big| X_{\mathbf{n}}^{f^i_{q+1}(I^0_{q+1})} \Big|^2 \\
&\le \sum_{I^0_{q+1}} \sum_{\mathbf{n}\in B^{\mathbf{b}_{I_q}}_{\mathbf{j}_{I_q}}} (q+1) \big| X_{\mathbf{n}}^{I_q} \big|^2 \quad \text{since } B^{\mathbf{b}_{I^0_{q+1}}}_{\mathbf{j}_{I^0_{q+1}}} \subseteq B^{\mathbf{b}_{I_q}}_{\mathbf{j}_{I_q}} \\
&\le (k-q)(q+1) \sum_{I\in S_q(k)} \sum_{\mathbf{n}\in B^{\mathbf{b}_{I_q}}_{\mathbf{j}_{I_q}}} |X_{\mathbf{n}}^{I_q}|^2 = (k-q)(q+1)\|X\|^2.
\end{aligned}
$$

The last inequality is because every I_q is contained in at most $k-q$ copies of I^0_{q+1}.

Next, we prove that ϑ_q commutes with the module actions. For each $I \in S_q(k)$ and $X^I \in H^{\mathbf{b}_I}_{\mathbf{j}_I}$, we defined

$$
\vartheta_q(X^I) = \sum_{1\le s\le k,\ s\notin I} (-1)^{\mathrm{sign}(I,s)} Y^{I\cup\{s\}},
$$

$$
Y^{I\cup\{s\}} \in H^{\mathbf{b}_{I\cup\{s\}}}_{\mathbf{j}_{I\cup\{s\}}}, \quad Y^{I\cup\{s\}}_{\mathbf{n}} = \begin{cases} X^I_{\mathbf{n}} & \text{if } \mathbf{n}\in B^{\mathbf{b}_{I\cup\{s\}}}_{\mathbf{j}_{I\cup\{s\}}}, \\ 0 & \text{otherwise}, \end{cases}
$$

where s is the α-th smallest number in $I\cup\{s\}$, and $\mathrm{sign}(I,s) = \alpha - 1$.

Each z_p action on $H^{\mathbf{b}_I}_{\mathbf{j}_I}$ is implemented by

$$
T^{\mathbf{j}_I,\mathbf{b}_I}_{z_p}(X^I)_{n_1\cdots n_p+1\cdots n_m} = \begin{cases} \sqrt{\dfrac{\omega_1(n_1\cdots n_p+1\cdots n_m)}{\omega_1(n_1\cdots n_m)}}\, X^I_{n_1\cdots n_p\cdots n_m} & \text{if } p\notin \mathbf{j}_I, \\ \sqrt{\dfrac{\omega_1(n_1\cdots n_p+1\cdots n_m)}{\omega_1(n_1\cdots n_m)}}\, X^I_{n_1\cdots n_p\cdots n_m} & \text{if } p = j^s \in \mathbf{j}_I,\ n_p+1\le b^s, \\ 0 & \text{otherwise.} \end{cases}
$$

This shows that $T^{\mathbf{j}_I,\mathbf{b}_I}_{z_p}$ preserves the component $H^{\mathbf{b}_I}_{\mathbf{j}_I}$. Similarly, the z_p action on $H^{\mathbf{b}_{I\cup\{s\}}}_{\mathbf{j}_{I\cup\{s\}}}$ is realized by

$$
T^{\mathbf{j}_{I\cup\{s\}},\mathbf{b}_{I\cup\{s\}}}_{z_p}(Y^{I\cup\{s\}})_{n_1\cdots n_p+1\cdots n_m} = \begin{cases} \sqrt{\dfrac{\omega_1(n_1\cdots n_p+1\cdots n_m)}{\omega_1(n_1\cdots n_m)}}\, Y^{I\cup\{s\}}_{n_1\cdots n_p\cdots n_m} & \text{if } p\notin \mathbf{j}_I,\ p\neq s, \\ \sqrt{\dfrac{\omega_1(n_1\cdots n_p+1\cdots n_m)}{\omega_1(n_1\cdots n_m)}}\, Y^{I\cup\{s\}}_{n_1\cdots n_p\cdots n_m} & \text{if } p = j^t \in \mathbf{j}_{I\cup\{s\}},\ n_p+1\le b^t, \\ 0 & \text{otherwise.} \end{cases}
$$

It is straightforward to directly check that on each component $H^{\mathbf{b}_{I\cup\{s\}}}_{\mathbf{j}_{I\cup\{s\}}}$ we have

$$(\vartheta_q(T^{\mathbf{j}_I,\mathbf{b}_I}_{z_p}(X^I)))^{I\cup\{s\}} = T^{\mathbf{j}_{I\cup\{s\}},\mathbf{b}_{I\cup\{s\}}}_{z_p}(\vartheta_q(X^I)^{I\cup\{s\}}),$$

and we are done.

Proposition 3.2. $\bar{I} = \ker(\vartheta_0)$.

Proof. If $f \in I$, then f has no nonzero component in any of the boxes $\mathsf{B}^{\mathbf{b}_s}_{\mathbf{j}_s}$, $\mathbf{s} \in S(\alpha_1, \ldots, \alpha_l)$, so $f \in \ker(\vartheta_0)$. This shows that $\bar{I} \subseteq \ker(\vartheta_0)$. For the other direction, assume $f = \sum_{\mathbf{n}\in\mathbb{N}^m} f_{\mathbf{n}} z^{\mathbf{n}} \in \ker(\vartheta_0)$. Since $\vartheta_0(f) = 0$, it follows that $f_{\mathbf{n}} = 0$ for every $i = 1, \ldots, k$ and $\mathbf{n} \in \mathsf{B}^{\mathbf{b}_i}_{\mathbf{j}_i}$. Let f_M, $M = 1, 2, \ldots$, be the truncation of the Taylor expansion of f by requiring $n^1, \ldots, n^m < M$. Since f_M has no component in the boxes $\mathsf{B}^{\mathbf{b}_1}_{\mathbf{j}_1}, \ldots, \mathsf{B}^{\mathbf{b}_k}_{\mathbf{j}_k}$, we have $f_M \in I$. Thus, $f = \lim f_M \in \bar{I}$.

Proposition 3.3. $\operatorname{Im}(\vartheta_{q-1}) \subseteq \ker(\vartheta_q)$ *for every* $q = 1, \ldots, k$.

Proof. For each $I \in S_{q-1}(k)$ and $X^I \in H^{\mathbf{b}_I}_{\mathbf{j}_I}$, the image of X^I under ϑ_{q-1} is of the form

$$\sum_{1\le s\le k,\ s\notin I} (-1)^{\operatorname{sign}(I,s)} Y^{I\cup\{s\}},$$

where s is the α-th smallest number in $I \cup \{s\}$, $\operatorname{sign}(I, s) = \alpha - 1$, and the function $Y^{I\cup\{s\}} \in H^{\mathbf{b}_{I\cup\{s\}}}_{\mathbf{j}_{I\cup\{s\}}}$ is given by

$$Y^{I\cup\{s\}}_{\mathbf{n}} = \begin{cases} X^I_{\mathbf{n}} & \text{if } \mathbf{n} \in \mathsf{B}^{\mathbf{b}_{I\cup\{s\}}}_{\mathbf{j}_{I\cup\{s\}}}, \\ 0 & \text{otherwise}. \end{cases}$$

Similarly, the image of $Y^{I\cup\{s\}}$ under ϑ_q is of the form

$$\sum_{1\le t\le k,\ t\notin I\cup\{s\}} (-1)^{\operatorname{sign}(I\cup\{s\},t)} Z^{I\cup\{s,t\}},$$

where t is the β-th smallest number in $I \cup \{s, t\}$, $\operatorname{sign}(I \cup \{s\}, t) = \beta - 1$, and the function $Z^{I\cup\{s,t\}} \in H^{\mathbf{b}_{I\cup\{s,t\}}}_{\mathbf{j}_{I\cup\{s,t\}}}$ is given by

$$Z^{I\cup\{s,t\}}_{\mathbf{n}} = \begin{cases} Y^{I\cup\{s\}}_{\mathbf{n}} & \text{if } \mathbf{n} \in \mathsf{B}^{\mathbf{b}_{I\cup\{s,t\}}}_{\mathbf{j}_{I\cup\{s,t\}}}, \\ 0 & \text{otherwise}. \end{cases}$$

Therefore,

$$\begin{aligned} \vartheta_q(\vartheta_{q-1}(X^I)) &= \sum_{1\le s\ne t\le k,\ s,t\notin I} (-1)^{\operatorname{sign}(I,s)+\operatorname{sign}(I\cup\{s\},t)} Z^{I\cup\{s,t\}} \\ &= \sum_{1\le s<t\le k,\ s,t\notin I} \left((-1)^{\operatorname{sign}(I,s)+\operatorname{sign}(I\cup\{s\},t)} + (-1)^{\operatorname{sign}(I,t)+\operatorname{sign}(I\cup\{t\},s)}\right) Z^{I\cup\{s,t\}}. \end{aligned}$$

Every summand in the latter sum vanishes because

$$\operatorname{sign}(I, s) = \operatorname{sign}(I \cup \{t\}, s), \quad \operatorname{sign}(I \cup \{s\}, t) = \operatorname{sign}(I, t) + 1$$

when $s < t$.

Proposition 3.4. (a) $\mathrm{Im}(\vartheta_0) \supseteq \ker(\vartheta_1)$.

(b) $\mathrm{Im}(\vartheta_{q-1}) \supseteq \ker(\vartheta_q)$ *for every* $q = 1, \dots, k$.

Proof. (a) Assume $X := (X^1, \dots, X^p) \in \ker(\vartheta_1)$. Consider $\xi \in \mathcal{A}_0$ given by

$$\xi_{\mathbf{n}} := \begin{cases} X^s_{\mathbf{n}} & \text{if there is } s \text{ such that } \mathbf{n} \in \mathsf{B}^{\mathbf{b}_s}_{\mathbf{j}_s}, \\ 0 & \text{otherwise.} \end{cases}$$

This is well-defined because $\vartheta_1(\xi) = 0$. Note that $\xi \in \mathcal{A}_0$ because

$$\|\xi\|^2 = \|X^1\|^2 + \cdots + \|X^p\|^2.$$

Clearly, $\vartheta_0(\xi) = X$.

(b) We apply induction on k. When $k = 1$, the map $\vartheta_0 : \mathcal{A}_0 \to \mathcal{A}_1$ is surjective because computing with the orthonormal basis shows that $\mathcal{A}_1$ can be identified with a closed subspace of $\mathcal{A}_0 = L^2_a(\Gamma_1)$, with ϑ_0 being the corresponding orthogonal projection. Assuming

$$\mathrm{Im}(\vartheta_{q-1}) \supseteq \ker(\vartheta_q), \quad q = 1, \dots, k, \; 1 \le k < p,$$

we prove the statement for $k = p$. The case $q = 1$ is proved in (a), so from now on we assume $2 \le q \le k$.

Consider the following two collections of $p - 1$ boxes:

- The first $p - 1$ boxes: $\mathsf{B}^{\mathbf{b}_1}_{\mathbf{j}_1}, \dots, \mathsf{B}^{\mathbf{b}_{p-1}}_{\mathbf{j}_{p-1}}$.
  Applying the construction in Section 2E to these boxes, we get the Hilbert modules $\mathcal{A}^1_s$ together with the Hilbert module maps $\vartheta^1_s : \mathcal{A}^1_s \to \mathcal{A}^1_{s+1}$, $s = 1, \dots, p-2$. Set $\mathcal{A}^1_p := \{0\}$ and $\vartheta^1_{p-1} := 0$.
- The intersection of the first $p - 1$ boxes with the last one: $\mathsf{B}^{\mathbf{b}_{1p}}_{\mathbf{j}_{1p}}, \dots, \mathsf{B}^{\mathbf{b}_{p-1\,p}}_{\mathbf{j}_{p-1\,p}}$.
  Applying the construction in Section 2E to these boxes, we get the Hilbert modules $\mathcal{A}^2_s$ together with the Hilbert module maps $\vartheta^2_s : \mathcal{A}^2_s \to \mathcal{A}^2_{s+1}$, $s = 1, \dots, p-2$. Set $\mathcal{A}^2_p := \{0\}$ and $\vartheta^2_{p-1} := 0$.

By the induction assumption we have

$$\mathrm{Im}(\vartheta^1_{q-1}) \supseteq \ker(\vartheta^1_q), \quad \mathrm{Im}(\vartheta^2_{q-1}) \supseteq \ker(\vartheta^2_q), \quad q = 1, \dots, p-1.$$

Define a map $\delta_s : \mathcal{A}^1_s \to \mathcal{A}^2_s$ by

$$\delta_s(X^I) = Y^{I \cup \{p\}}, \quad I \in S_s(p-1),$$

where $Y^{I\cup\{p\}}$ denotes the component corresponding to the intersection of the boxes $\mathsf{B}^{\mathbf{b}_{i_1 p}}_{\mathbf{j}_{i_1 p}}, \dots, \mathsf{B}^{\mathbf{b}_{i_s p}}_{\mathbf{j}_{i_s p}}$, given by

$$Y^{I\cup\{p\}}_{\mathbf{n}} := \begin{cases} (-1)^t X^I_{\mathbf{n}} & \text{if } \mathbf{n} \in \mathsf{B}^{\mathbf{b}_{I\cup\{p\}}}_{\mathbf{j}_{I\cup\{p\}}}, \\ 0 & \text{otherwise.} \end{cases}$$

Similar to the proof of Proposition 3.1, ψ_s is an A-module map. Furthermore, we can easily check that

- $\mathcal{A}_q = \mathcal{A}^1_q \oplus \mathcal{A}^2_{q-1}$ for $q = 2, \ldots, p$,
- $\varphi_q = \begin{pmatrix} \varphi^1_q & 0 \\ \psi_q & \varphi^2_{q-1} \end{pmatrix}$ for $q = 2, \ldots, p-1$.

These identifications are used below to prove that $\operatorname{Im}(\varphi_{q-1}) \supseteq \ker(\varphi_q)$. We split the proof into three cases.

(1) $q = 2$.

Suppose $(X_1, X_2) \in \mathcal{A}^1_2 \oplus \mathcal{A}^2_1 = \mathcal{A}_2$ is in $\ker(\varphi_2)$. By the identification above for φ_q, we have

$$\varphi^1_2(X_1) = 0, \qquad \psi_2(X_1) + \varphi^2_1(X_2) = 0.$$

By the induction assumption, we have $\ker(\varphi^1_2) \subseteq \operatorname{Im}(\varphi^1_1)$, so there exists $Y_1 \in \mathcal{A}^1_1$ such that $\varphi^1_1(Y_1) = X_1$. By Proposition 3.3, for the morphism $\varphi_\bullet$, we have

$$\begin{aligned}
&(0, 0) \\
&= \varphi_2(\varphi_1(Y_1, 0)) = \varphi_2(\varphi^1_1(Y_1), \psi_1(Y_1)) \\
&= \left(\varphi^1_2(\varphi^1_1(Y_1)), \psi_2(\varphi^1_1(Y_1)) + \varphi^2_1(\psi_1(Y_1))\right) \quad \left[\varphi^1_1(Y_1) = X_1, \varphi^1_2(\varphi^1_1(Y_1)) = 0\right] \\
&= \left(0, \psi_2(X_1) + \varphi^2_1(\psi_1(Y_1))\right).
\end{aligned}$$

Therefore, $\psi_2(X_1) + \varphi^2_1(\psi_1(Y_1)) = 0$. Setting $X^0_2 := X_2 - \psi_1(Y_1)$, we have

$$\varphi^2_1(X^0_2) = \varphi^2_1(X_2) - \varphi^2_1(\psi_1(Y_1)) = \varphi^2_1(X_2) + \psi_2(X_1) = 0,$$

because $0 = \varphi_2(X_1, X_2) = (\varphi^1_2(X_1), \psi_2(X_1) + \varphi^2_1(X_2))$. Since $\varphi^2_1(X^0_2) = 0$, it follows that the following assignment is well-defined:

$$(Y_2)_n := \begin{cases} (X^{0ip}_2)_n & \text{if } n \in \mathbb{B}^{\mathfrak{b}_{ip}}_{\mathfrak{j}_{ip}} \text{ for some } i = 1, \ldots, p-1, \\ 0 & \text{otherwise.} \end{cases}$$

Arguments similar to the proof of Proposition 3.4 show that this assignment gives $Y_2 \in \mathcal{H}^{\mathfrak{b}_p}_{\mathfrak{j}_p}$ such that $\varphi^2_0(Y_2) = X^0_2$. In summary, we have found $(Y_1, Y_2) \in \mathcal{A}_1 = \mathcal{A}^1_1 \oplus \mathcal{H}^{\mathfrak{b}_p}_{\mathfrak{j}_p}$ which satisfies

$$\varphi_1(Y_1, Y_2) = \left(\varphi^1_1(Y_1), \psi_1(Y_1) + \varphi^2_0(Y_2)\right) = \left(X_1, \psi_1(Y_1) + X^0_2\right) = (X_1, X_2).$$

(2) $q = 3, \ldots, p-1$.

Suppose $(X_1, X_2) \in \mathcal{A}^1_q \oplus \mathcal{A}^2_{q-1} = \mathcal{A}_q$ is in $\ker(\varphi_q)$. By the identification above for φ_q, we have

$$\varphi^1_q(X_1) = 0, \qquad \psi_q(X_1) + \varphi^2_{q-1}(X_2) = 0.$$

Since $\operatorname{Im}(\partial^1_{q-1}) \supseteq \ker(\partial^1_q)$, there exists $Y_1 \in \mathcal{A}^1_{q-1}$ such that $X_1 = \partial^1_{q-1}(Y_1)$. Since $\partial_q(\partial_{q-1}(Y_1, 0)) = 0$, it follows that $\delta_q(X_1) + \partial^2_{q-1}(\delta_{q-1}(Y_1)) = 0$. Therefore,

$$\partial^2_{q-1}(X_2 - \delta_{q-1}(Y_1)) = 0.$$

Since $\operatorname{Im}(\partial^2_{q-2}) \supseteq \ker(\partial^2_{q-1})$, there exists $Y_2 \in \mathcal{A}^2_{q-2}$ such that

$$\partial^2_{q-2}(Y_2) = X_2 - \delta_{q-1}(Y_1).$$

In summary, we have found $(Y_1, Y_2) \in \mathcal{A}_q$ which satisfies

$$\partial_{q-1}(Y_1, Y_2) = (\partial^1_{q-1}(Y_1), \delta_{q-1}(Y_1) + \partial^2_{q-2}(Y_2)) = (X_1, X_2).$$

(3) $q = p$.

Since $\partial^2_{p-2} : \mathcal{A}^2_{p-2} \to \mathcal{A}^2_{p-1}$ is surjective, it follows that

$$\partial_{p-1} : (\mathcal{A}_{p-1} = \mathcal{A}^1_{p-1} \oplus \mathcal{A}^2_{p-2}) \to (\mathcal{A}_p = \mathcal{A}^2_{p-1})$$

is also surjective.

All cases are exhausted.

3B. ***The proof of Theorem 1.7*(b).** To deduce the index formula in Theorem 1.7(b) from the resolution in Theorem 1.7(a), we need the following proposition.

Proposition 3.5. *Let* $0 \to \mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3 \to 0$ *be a short exact sequence of essentially normal Hilbert A-modules and Hilbert A-module maps between them. Suppose that the essential spectra of* $\mathcal{M}_i$, $i = 1, 2, 3$, *is contained in* $\overline{\mathbb{B}}_1$, *and let* $\alpha_i : C(\overline{\mathbb{B}}_1) \to Q(\mathcal{M}_i)$ *be the* $*$*-representation of* $C(\overline{\mathbb{B}}_1)$ *on the Calkin algebra* $Q(\mathcal{M}_i) = B(\mathcal{M}_i)/\mathcal{K}(\mathcal{M}_i)$ *induced by the essential normality of* $\mathcal{M}_i$.

(a) *There are co-isometries* $U : \mathcal{M}_2 \to \mathcal{M}_1$ *and* $V : \mathcal{M}_2 \to \mathcal{M}_3$ *such that*

$$UV^* = 0 = VU^*, \qquad U^*U + V^*V = 1,$$

and they commute with A-module structures up to compact operators in the sense that $[U]\alpha_2[U]^* = \alpha_1$ *and* $[V]\alpha_2[V]^* = \alpha_3$, *where* $\alpha_i(p) = [T^i_p] \in Q(\mathcal{M}_i)$, $p \in A$ *is the equivalence class of the multiplication operator* $T^i_p \in B(\mathcal{M}_i)$.

(b) *We have* $[\alpha_2] = [\alpha_1] + [\alpha_3]$ *in* $K_1(\sigma^2_e)$, *where* σ^2_e *is the essential Taylor spectrum associated to the Hilbert module* $\mathcal{M}_2$, *and* $[\alpha_1]$, $[\alpha_3]$ *are identified as classes in* $K_1(\sigma^2_e)$ *by the co-isometries U and V*.

Proof. (a) This is [Douglas et al. 2018, Proposition 3.8].

(b) Set $\sigma^i_e := \sigma_e(\mathcal{M}_i)$, the essential Taylor spectrum associated to the Hilbert module $\mathcal{M}_i$, $i = 1, 2, 3$. The representation α_i factors through the $*$-monomorphism $C(\sigma^i_e) \to Q(\mathcal{M}_i)$. We have $\alpha_1 = [U]\alpha_2[U]^*$ by (a). The composition of $[U]\alpha_2[U]^*$ with α_1^{-1} is a $*$-homomorphism $C(\sigma^2_e) \to C(\sigma^1_e)$, and this induces a natural map

$\sigma_e^1 \to \sigma_e^2$. Similarly, we have a natural map $\sigma_e^3 \to \sigma_e^2$. Therefore, α_1 and α_3 induce classes $[\alpha_1]$ and $[\alpha_3]$ in $K_1(\sigma_e^2)$ by the functoriality of K_1. Putting all equations

$$UU^* = 1 = VV^*, \quad UV^* = 0 = VU^*, \quad U^*U + V^*V = 1,$$
$$[U]\alpha_2[U]^* = \alpha_1, \quad [V]\alpha_2[V]^* = \alpha_3,$$

together, we deduce that $[\alpha_2] = [\alpha_1] + [\alpha_3]$.

The proof of Theorem 1.7(b). The idea is to decompose the resolution of $\bar{I}$ in Theorem 1.7(a) into short exact sequences and then apply Proposition 3.5(b). The details follow. Consider $\mathcal{A}_q^- := \mathrm{Im}(\vartheta_{q-1}) = \ker(\vartheta_q)$ as a closed subspace of $\mathcal{A}_q$. Note that $\mathcal{A}_k^- = \mathcal{A}_k$ because ϑ_{k-1} is surjective. The morphism $\vartheta_q : \mathcal{A}_q \to \mathcal{A}_{q+1}$ of Hilbert modules induces the short exact sequence

$$0 \to \mathcal{A}_q^- \hookrightarrow \mathcal{A}_q \xrightarrow{\vartheta_q} \mathcal{A}_{q+1}^- \to 0, \quad q = 1, \dots, k-1, \tag{3.6}$$

which, according to Propositions 2.5 and 2.9, implies that $\mathcal{A}_q^-$ is essentially normal. Set $\sigma_e^q := \sigma_e(\mathcal{A}_q)$, and let α_q be the $*$-monomorphism $C(\sigma_e^q) \to \mathcal{Q}(\mathcal{A}_q)$ and α_q^- the $*$-monomorphism $C(\sigma_e^{q-}) \to \mathcal{Q}(\mathcal{A}_q^-)$ induced by essential normality. Note that the essential spectra of all terms in the exact sequence (3.6) are contained in $\overline{\Omega}_1$. By Proposition 3.5(b), we have $[\alpha_q] = [\alpha_q^-] + [\alpha_{q+1}^-]$ in $K_1(\sigma_e^q)$ for every $q = 1, \dots, k-1$. These formulas for $q = k-1$ and $q = k-2$ give

$$[\alpha_{k-1}] = [\alpha_{k-1}^-] + [\alpha_k] \in K_1(\sigma_e^{k-1}), \quad [\alpha_{k-2}] = [\alpha_{k-2}^-] + [\alpha_{k-1}^-] \in K_1(\sigma_e^{k-2}).$$

Pushing forward these equations into $K_1(\sigma_e^{k-1} \cup \sigma_e^{k-2})$ by inclusion maps

$$\sigma_e^{k-1}, \ \sigma_e^{k-2} \hookrightarrow \sigma_e^{k-1} \cup \sigma_e^{k-2}$$

gives $[\alpha_{k-1}] + [\alpha_{k-2}^-] = [\alpha_k] + [\alpha_{k-2}]$. Continuing this argument, we have

$$[\alpha_1^-] = [\alpha_1] - [\alpha_2] + \cdots + (-1)^{k-1}[\alpha_k] \quad \text{in } K_1(\sigma_e^1 \cup \cdots \cup \sigma_e^k). \tag{3.7}$$

On the other hand, the short exact sequence

$$0 \to \bar{I} \to L_a^2(\Omega_1) \to \mathcal{A}_1^- \to 0$$

establishes a natural Hilbert module isomorphism between $\mathcal{A}_1^-$ and $L_a^2(\Omega_1)/\bar{I} \cong I^\perp$, and hence $\tau_I := [I^\perp] = [\alpha_1^-]$ by Proposition 2.5(b). This, together with (3.7), gives the index formula in Theorem 1.7(b).

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