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An index theorem for higher orbital integrals

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Abstract

Recently, two of the authors of this paper constructed cyclic cocycles on Harish–Chandra's Schwartz algebra of linear reductive Lie groups that detect all information in the K-theory of the corresponding group C^*-algebra. The main result in this paper is an index formula for the pairings of these cocycles with equivariant indices of elliptic operators for proper, cocompact actions. This index formula completely determines such equivariant indices via topological expressions.

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1 Introduction

Consider a locally compact group G, and a proper, isometric action by G on a Riemannian manifold X, such that X/G is compact. Let D be a G-equivariant, elliptic differential operator on X. Then D has an equivariant index in the K-theory of the reduced group C^*-algebra $C_r^*(G)$,

$$\mathrm{ind}_G(D) \in K_*(C_r^*(G)), \tag{1.1}$$

defined by the analytic assembly map [7]. This index generalises the equivariant index in the compact case. It has a range of applications, e.g. to geometry and topology via the Baum–Connes and Novikov conjectures; to questions about positive scalar curvature; to representation theory [24]; and to geometric quantisation [30].

It is a natural question to extract numerical invariants from the index (1.1) and compute them. This is relevant, for example, to detect the vanishing of the index, and because such numbers may have a meaning in geometry, topology or representation theory. A natural way to extract numbers from K-theory classes is by pairing them with *traces* or more general *cyclic cocycles* on the algebra in question, or an algebra with the same K-theory. For suitable convolution algebras on groups, such traces can be defined by *orbital integrals*: integrals over conjugacy classes. For the trivial conjugacy class, one then obtains the classical von Neumann trace. It was shown in various places, see e.g. [16,24,25,36], that traces defined by orbital integrals over nontrivial conjugacy classes can yield more, and relevant, information than the von Neumann trace.

In this paper, we focus on connected real linear reductive Lie groups. Connes and Moscovici [13] showed that the von Neumann trace is then only nonzero on K-theory classes defined by discrete series representations. So this trace is not enough to detect all information about K-theory classes, especially for groups with no discrete series representations. In [25], it is shown that one does detect all information in $K_*(C_r^*(G))$ for groups with discrete series representations if one uses traces defined by more general orbital integrals. In another direction, higher cyclic cocycles coming from group cohomology were used to extract and compute relevant numbers from the index (1.1) in [34].

Recently, two of the authors of this paper developed a family of cyclic cocycles on Harish–Chandra's Schwartz algebra [37], for linear reductive G. These cocycles

Table 1 Some equivariant index theorems for Lie groups G, for pairings of (1.1) with traces and higher cocycles associated to orbital integrals

	trivial conjugacy class $\{e\}$	more general conjugacy classes
Trace	Connes–Moscovici 1982 [13] (homogeneous spaces) Wang 2014 [38] (general case)	Hochs–Wang 2017 [24] (semisimple Lie groups)
Higher cocycles	Pflaum–Posthuma–Tang 2015 [34] (cocycles from group cohomology)	Theorem 2.1 (linear reductive groups)

are higher versions, in a sense, of orbital integrals. They detect all information about classes in $K_*(C_r^*(G))$, in the sense that if the pairings of a class in $K_*(C_r^*(G))$ with all these cyclic cocycles are zero, then it is the zero class. This means that an index formula for the pairing of (1.1) with these cocycles is a complete topological description of this index. Our goal in this paper is to prove such an index formula, Theorem 2.1.

At the end of [13], Connes and Moscovici write that they consider obtaining an 'intrinsic' index formula for (1.1) for Lie groups G, to be a problem that deserves further study. We believe that Theorem 2.1 is such an intrinsic index formula, detecting all relevant information, for linear reductive Lie groups. This builds on a long development, and results by many authors, see Table 1. This type of index theorem for pairings with higher cocycles, as well as the techniques involved, was heavily inspired by the work of Connes and Moscovici for discrete groups, see Theorem 5.4 in [14].

Theorem 2.1 is stated for twisted Spin^c-Dirac operators. At the end of this paper, we describe how to generalise this to arbitrary elliptic operators; obtain a higher version of Connes and Moscovici's L^2-index theorem for homogeneous spaces; obtain independent proofs of injectivity of Dirac induction and the fact that the cocycles from [37] detect all information from $K_*(C_r^*(G))$; realise the generators of $K_*(C_r^*(G))$ constructed in [37] as indices; and work out the case of complex semisimple groups.

Other applications of Theorem 2.1 are of the same type as those in [35].

- When D is the signature operator on X, the index element $\mathrm{ind}_G(D)$ is invariant under G-equivariant homotopy [15, Theorem A]. Our theorem 2.1 shows that the corresponding generalized signature number is invariant under G-equivariant homotopy.
- When D is the Spin Dirac operator on a spin manifold X, the non-vanishing of the generalized $\hat{A}$-class on the right side would be an obstruction for X to carry a G-invariant metric with positive scalar curvature.

We plan to investigate potential applications of our theorem in representation theory in the future.

Let us briefly go into the idea of the proof of Theorem 2.1. The cocycles we use are associated to parabolic subgroups $P = MAN < G$, where M is reductive, A is isomorphic to $\mathbb{R}^l$ and N is nilpotent. The proof of Theorem 2.1 is based on an equality between the pairing of (1.1) with these cocycles and an analogous pairing of an index of an M-equivariant operator on X/AN with an analogous cocycle for the parabolic

subgroup $M < M$. Because M has a compact Cartan for the parabolic subgroups we use, the latter pairing can be computed via results from [24]. It is an interesting question to what extent the transition between these G and M-equivariant indices is related to parabolic induction. Some indication for this is given in Sect. 2.4.

2 An index theorem

Let G be a connected, linear, real reductive Lie group. Let X be a Riemannian manifold on which G acts properly, cocompactly and isometrically.

Fix a G-equivariant Spin^c-structure on X, assuming it exists. Let $S \to X$ be the corresponding spinor bundle. Let $W \to X$ be a G-equivariant, Hermitian vector bundle, and write $E := S \otimes W$. Let D be a twisted Spin^c-Dirac operator acting on sections E, associated to a G-invariant Clifford connection on S and a G-invariant Hermitian connection on W. Because D is an elliptic, G-equivariant operator, and X/G is compact, we have the index

$$\text{ind}_G(D) \in K_*(C_r^*(G)), \tag{2.1}$$

defined via the analytic assembly map [7]. Here $\text{ind}_G(D)$ lies in even K-theory if X is even-dimensional, so that S admits a natural grading and D is odd, and in odd K-theory if X is odd-dimensional.

Our goal is to obtain a topological expression for the pairing of (2.1) with natural cyclic cocycles constructed in [37].

2.1 Cocycles

We use a lower case gothic letter to denote the Lie algebra of the Lie group denoted by the corresponding upper case Roman letter.

Let $K < G$ be maximal compact. Let $P < G$ be a parabolic subgroup. It has a Langlands decomposition $P = MAN$, where M is reductive, A is isomorphic to a vector space, and N is nilpotent. We assume from now on that P is a cuspidal parabolic subgroup. (See Sect. 6.4 for the non-cuspidal case.) This means that M has a compact Cartan subgroup, which implies that it has discrete series representations. In several places, we will use the facts that N normalises MA and that M and A commute.

Let dg, dk, dm, da and dn be Haar measures on the groups G, K, M, A and N, respectively, such that

$$dg = dk \, dm \, da \, dn \tag{2.2}$$

according to the decomposition $G = KMAN$. All these groups are unimodular, so these measures are left- and right-invariant.

Define the map $H \colon G \to \mathfrak{a}$ by the property that for all $g \in G$, $\exp(H(g))$ is the component of g in A according to the decomposition $G = KMAN$.

Set $l := \dim(A)$. If a basis of $\mathfrak{a}$ is chosen, then for $a_1, \ldots, a_l \in \mathfrak{a}$, the vectors $a_1, \ldots, a_l$ together define an $l \times l$ matrix. So we can take its determinant

$\det(a_1, \ldots, a_l)$. This determinant depends on the choice of basis, but if $x_1, \ldots, x_n \in \mathfrak{a}^*$ are the coordinates on $\mathfrak{a}$ defined by this basis, then the volume element $\det(a_1, \ldots, a_l) dx_1 \wedge \cdots \wedge dx_l \in \bigwedge^l \mathfrak{a}^*$, which is the relevant object in what follows, does not. For definiteness' sake, we fix a basis such that

$$dx_1 \wedge \cdots \wedge dx_l = da, \tag{2.3}$$

the Haar measure on $A \cong \mathfrak{a}$, and use this to define $\det(a_1, \ldots, a_l)$ for any $a_1, \ldots, a_l \in \mathfrak{a}$.

In [37], a cyclic cocycle on *Harish-Chandra's Schwartz algebra* $\mathcal{C}(G)$ is constructed. This algebra is a generalisation of the algebra of Schwartz functions on $\mathbb{R}^n$, and is defined as follows. Let π_0 be the unitary representation of G induced from the trivial representation of a minimal parabolic subgroup. Let ξ be a unit vector in the representation space of π_0, fixed by $\pi_0(K)$. Let Ξ be the function on G defined by

$$\Xi(g) = (\xi, \pi_0(g)\xi)$$

for all $g \in G$. The inner product on $\mathfrak{g}$ defines a G-invariant Riemannian metric on G/K. For $g \in G$, let $d(g)$ be the Riemannian distance from eK to gK in G/K. Then $\mathcal{C}(G)$ is the space of $f \in C^\infty(G)$ such that for all $m \geq 0$ and $X, Y \in U(\mathfrak{g})$,

$$\sup_{g \in G} (1 + d(g))^m \Xi(g)^{-1} |L(X)R(Y)f(g)| < \infty, \tag{2.4}$$

where L and R denote the left and right regular representations, respectively. See Section 9 in [18]. The space $\mathcal{C}(G)$ is a Fréchet algebra with respect to the seminorms (2.4).

Let $x \in M$ be a semisimple element. Let $Z := Z_M(x)$ be its centraliser in M. Because x is semisimple, the quotient M/Z has an M-invariant measure $d(hZ)$ (also denoted by $d(mZ)$) compatible with the Haar measure dm on M. In Definition 3.3 in [37], the cyclic l-cocycle Φ_x^P on $\mathcal{C}(G)$ is defined by

$$\Phi_x^P(f_0, \ldots, f_l) := \int_{M/Z} \int_{KN} \int_{G^l}$$
$$\det\big(H(g_1 g_2 \cdots g_l k), H(g_2 \cdots g_l k)), \ldots, H(g_{l-1} g_l k), H(g_l k)\big)$$
$$f_0(khxh^{-1}nk^{-1}(g_1 \cdots g_l)^{-1}) f_1(g_1) \cdots f_l(g_l) \, dg_1 \cdots dg_l \, dk \, dn \, d(hZ), \tag{2.5}$$

for $f_0, \ldots, f_l \in \mathcal{C}(G)$. Theorem 3.5 in [37] states that this is indeed a cyclic cocycle.

If G has a compact Cartan subgroup, then we can take $P = M = G$, so that If $l = 0$. Then Φ_x^P is simply the orbital integral trace associated to x:

$$\Phi_x^G(f_0) = \int_{G/Z} f_0(hxh^{-1}) d(hZ).$$

2.2 An index pairing

The Schwartz algebra $\mathcal{C}(G)$ is a dense subalgebra of $C_r^*(G)$, closed under holomorphic functional calculus, see Theorem 2.3 in [24]. So $\mathrm{ind}_G(D) \in K_*(\mathcal{C}(G))$, and we obtain the number

$$\langle \Phi_x^P, \mathrm{ind}_G(D) \rangle$$

by the pairing between cyclic cohomology and K-theory. Our main result, Theorem 2.1 below, is a topological expression for this number.

The group AN has no nontrivial compact subgroups and acts properly on X, so the action is free, and X/AN is a smooth manifold. Because M commutes with A and is normalised by N, the action by x on X preserves NA-orbits, and hence descends to an action on X/NA. Also, if $F \to X$ is a G-equivariant vector bundle, then $F/AN \to X/AN$ is an M-equivariant vector bundle.

Consider the fixed point set $(X/AN)^x \subset X/AN$. The connected components of $(X/AN)^x$ are submanifolds of X/AN of possibly different dimensions. We apply all constructions below to the connected components of $(X/AN)^x$ and add the results together.

The set $(X/AN)^x$ is preserved by the action of the centraliser Z of x in M. Let dz be the Haar measure on Z such that for all $\varphi \in C_c(M)$,

$$\int_M \varphi(m)\,dm = \int_{M/Z} \int_Z \varphi(zm)\,dz\,d(mZ).$$

Let χ_x be a smooth, compactly supported function on $(X/AN)^x$ such that for all $p \in (X/AN)^x$,

$$\int_Z \chi_x(zp)\,dz = 1.$$

The Spin^c-structure on X induces an M-equivariant Spin^c-structure on X/AN with spinor bundle S_{AN} such that

$$S/AN = S_{AN} \otimes F_{AN} \otimes S_{\mathfrak{a}} \to X/AN, \tag{2.6}$$

for an M-equivariant, graded vector bundle $F_{AN} \to X/AN$, and a vector space $S_{\mathfrak{a}}$ of dimension $2^{\lfloor \frac{\dim(\mathfrak{a})-1}{2} \rfloor}$. See Sect. 5.1 for an explicit construction, in particular (5.1). Let $L_{\mathrm{det}} \to X/AN$ be the determinant line bundle of the Spin^c-structure on X/AN with spinor bundle S_{AN}.

Let $\hat{A}((X/AN)^x)$ be the $\hat{A}$-class of $(X/AN)^x$. Let $\mathcal{N} \to (X/AN)^x$ be the normal bundle to $(X/AN)^x$ in X/AN. Let $R^{\mathcal{N}}$ be the curvature of the Levi–Civita connection on X/AN restricted to $\mathcal{N}$.

Consider the vector bundle

$$W_{AN} := F_{AN} \otimes W/AN \to X/AN.$$

If the closure T_x of the set of powers of x is a compact subgroup of M, then the restriction of W_{AN} to the compact set $\operatorname{supp}(\chi_x) \subset (X/AN)^x$ defines a class

$$[W_{AN}|_{\operatorname{supp}(\chi_x)}] \in K^0_{T_x}(\operatorname{supp}(\chi_x)) \cong K^0(\operatorname{supp}(\chi_x)) \otimes_{\mathbb{Z}} R(T_x),$$

where $R(T_x)$ is the representation ring of T_x. The last equality holds because T_x acts trivially on $(X/AN)^x$. Evaluating the factor in $R(T_x)$ at x, we obtain

$$[W_{AN}|_{\operatorname{supp}(\chi_x)}](x) \in K^0(\operatorname{supp}(\chi_x)) \otimes_{\mathbb{Z}} \mathbb{C}.$$

Consider the Chern character

$$\operatorname{ch} \colon K^0(\operatorname{supp}(\chi_x)) \otimes \mathbb{C} \to H^{\operatorname{even}}(\operatorname{supp}(\chi_x); \mathbb{C}).$$

Suppose P is a cuspidal parabolic. Let $T < K \cap M$ be a maximal torus.

Theorem 2.1 *[Index theorem for higher orbital integrals] For all $x \in T$,*

$$\langle \Phi_x^P, \operatorname{ind}_G(D) \rangle = \int_{(X/AN)^x} \chi_x \frac{\hat{A}((X/AN)^x) \operatorname{ch}([W_{AN}|_{\operatorname{supp}(\chi_x)}](x)) e^{c_1(L_{\det}|_{(X/AN)^x})}}{\det(1 - xe^{-R^N/2\pi i})^{1/2}}. \tag{2.7}$$

If P is not a maximal cuspidal parabolic subgroup or x does not lie in a compact subgroup of M, then the left hand side equals zero.

Remark 2.2 The M-equivariant Spin^c-structure on X/AN with spinor bundle S_N satisfying (2.6) is not unique: one can tensor S_{AN} by an M-equivariant line bundle and F_{AN} by the dual line bundle. But the right hand side of (2.7) does not change under this modification, so (2.6) is enough to state Theorem 2.1. For the sake of definiteness, we will use the bundles S_{AN} and F_{AN} defined in (5.1).

2.3 A reformulation on X/N

The fixed point formula in Theorem 2.1 can be reformulated in terms of the manifold X/N rather than X/AN. The manifold X/N has a Spin^c-structure with spinor bundle S_N such that

$$S/N = S_N \otimes F_N \to X/N \tag{2.8}$$

for an M-equivariant vector bundle $F_N \to X/N$. See Sect. 4.2 for an explicit construction, in particular (4.6). Analogously to Remark 2.2, any ambiguity in the choice of S_N does not affect the index formula. Let $L_{\det}^N \to X/N$ be the determinant line bundle of this Spin^c-structure. Consider the vector bundle

$$W_N := F_N \otimes W/N \to X/N.$$

For $x \in M$, let $\mathcal{N}_N \to (X/N)^x$ be the normal bundle of $(X/N)^x$ in X/N.

We view da as the volume form (2.3). Consider the projection map $q_A \colon X/N = X/AN \times A \to A$. Let $\chi_A \in C_c^\infty(A)$ be such that

$$\int_A \chi_A \, da = 1.$$

Let $q \colon X/N \to X/AN$ be the quotient map, and set $\chi_{x,A} := q^* \chi_x \, q_A^* \chi_A$.

Lemma 2.3 *If P is a maximal cuspidal parabolic subgroup and $x \in T$, then the right hand side of (2.7) equals*

$$\int_{(X/N)^x} \chi_{x,A} \frac{\hat{A}((X/N)^x) \operatorname{ch}([W_N|_{\operatorname{supp}(\chi_{x,A})}](x)) e^{c_1(L^N_{\det}|_{(X/N)^x})}}{\det(1 - x e^{-R^{\mathcal{N}_N}/2\pi i})^{1/2}} \wedge q_A^* da. \quad (2.9)$$

Remark 2.4 If $x \in T^{\mathrm{reg}}$, then $(X/N)^x \cong X^x$, and (2.9) can be rewritten as an integral over X^x. But it seems to be a nontrivial exercise to rewrite the integrand in terms of the manifold X instead of X/N.

2.4 Idea of the proof and relation with parabolic induction

An intuitive outline of the proof of Theorem 2.1 is given in Diagram (2.10). The ingredients of this diagram are described below.

$$(2.10)$$

1. The map

$$- \otimes_{C_r^*(G)} \operatorname{Ind}_P^G \colon K_*(C_r^*(G)) \to K_*(C_r^*(MA)) \quad (2.11)$$

   is defined by the tensor product over $C_r^*(G)$ from the right by a Hilbert $C_r^*(G)$-$C_r^*(MA)$ bimodule Ind_P^G defined in [10,11]. This module is closely related to parabolic induction, see Corollary 1 in [10] and Definition 4.4 in [11]. It is an interesting but seemingly nontrivial question if the top right part of Diagram (2.10) commutes.

2. There is a natural way to define a map $\operatorname{Res}_{MA}^G \colon K_*^G(X) \to K_*^{MA}(X/N)$, by first restricting group actions from G to P and then applying a construction from Section 3.2 of Valette's part of [32] or Appendix A in [21]. One could expect the top left part of (2.10) to commute, based on results in [21,31,32].

3. Let D_A be the Spin-Dirac operator on $A \cong \mathbb{R}^l$. The bottom left part of Diagram (2.10) is defined in terms of the exterior Kasparov product by the K-homology and index classes of D_A. This commutes by basic properties of the index; see also Lemma 5.2.
4. The group M has a compact Cartan subgroup. Hence its maximal cuspidal parabolic subgroup is M itself. Similarly, MA is its own maximal cuspidal parabolic subgroup. Let Φ_x^M and Φ_x^{MA} be the corresponding versions of the cocycle (2.5). The maps on the right hand side of (2.10) are defined by pairing with the respective cocycles. The bottom right part of Diagram (2.10) commutes by Lemmas 5.3 and 5.4.

The open questions in points 1 and 2 are topics of ongoing research. Positive answers would make a proof of Theorem 2.1 possible based on Diagram (2.10). Indeed, using commutativity of (2.10), one shows that the left hand side of (2.7) equals the pairing of Φ_x^M with the M-equivariant index of a certain Dirac operator on X/AN. The latter pairing can be computed via results from [24], because M has a compact Cartan subgroup.

An indication that the questions in points 1 and 2 are nontrivial is that it seems necessary for the diagram to commute (indeed, even for some of its components to be well-defined) that P is a parabolic subgroup, not just any cocompact subgroup. We give a different and more direct proof of Theorem 2.1 here, but in spirit it is analogous to Diagram 2.10. We have included this diagram to illustrate the idea of the proof, and its possible relation to the parabolic induction map from [10,11].

3 The index pairing on X

The most important step in the proof of Theorem 2.1 is the fact that the left hand side of (2.7) equals an index pairing on X/N. This is Proposition 4.8, which is analogous to commutativity of the top two diagrams in (2.10). Proving that proposition is our main goal in this section and the next.

In this section, we prepare for the proof of Proposition 4.8 by obtaining a relation between cocycles defined on G and on MA, and giving an explicit representative of $\text{ind}_G(D)$. These two ingredients lead to an expression for the left hand side of (2.7), Proposition 3.9. That proposition is a version of the commutativity of the top-right part of Diagram 2.10.

3.1 Cocycles on G and MA

Lemma 3.1 *For all $f_0, \ldots, f_l \in \mathcal{C}(G)$,*

$$\Phi_x^P(f_0, \ldots, f_l) = \int_{M/Z} \int_{K^{l+1}} \int_{N^{l+1}} \int_{(MA)^l} \det(a_1, \ldots, a_l)$$
$$f_0(k_0 n_0 h x h^{-1} m_1^{-1} a_1^{-1} k_1^{-1}) f_1(k_1 n_1 m_1 m_2^{-1} a_1 a_2^{-1} k_2^{-1}) \cdots$$
$$\cdots f_{l-1}(k_{l-1} n_{l-1} m_{l-1} m_l^{-1} a_{l-1} a_l^{-1} k_l^{-1}) f_l(k_l n_l m_l a_l k_0^{-1})$$
$$dm_1 \cdots dm_l \, da_1 \cdots da_l \, dn_0 \cdots dn_l \, dk_0 \cdots dk_l \, d(hZ). \quad (3.1)$$

Proof First substituting $k^{-1}g_jk$ for g_j in the right hand side of (2.5), and then $g_j \cdots g_l$ for g_j, we obtain

$$\Phi_x^P(f_0, \ldots, f_l) = \int_{M/Z} \int_{KN} \int_{G^l} \det\big(H(g_1), \ldots, H(g_l)\big)$$
$$f_0(khxh^{-1}ng_1^{-1}k^{-1}) f_1(kg_1g_2^{-1}k^{-1}) \cdots f_{l-1}(kg_{l-1}g_l^{-1}k^{-1}) f_l(kg_lk^{-1})$$
$$dg_1 \cdots dg_n \, dk \, dn \, d(hZ).$$

Using (2.2) and the definition of the map H, we rewrite the right hand side as

$$\int_{M/Z} \int_{KN} \int_{(KMAN)^l} \det(a_1, \ldots, a_l)$$
$$f_0(khxh^{-1}nn_1^{-1}a_1^{-1}m_1^{-1}k_1^{-1}k^{-1}) f_1(kk_1m_1a_1n_1n_2^{-1}a_2^{-1}m_2^{-1}k_2^{-1}k^{-1}) \cdots$$
$$\cdots f_{l-1}(kk_{l-1}m_{l-1}a_{l-1}n_{l-1}n_l^{-1}a_l^{-1}m_l^{-1}k_l^{-1}k^{-1}) f_l(kk_lm_la_ln_lk^{-1})$$
$$dm_1 \cdots dm_l \, da_1 \cdots da_l \, dn \, dn_1 \cdots dn_l \, dk \, dk_1 \cdots dk_l \, d(hZ).$$

The facts that N normalises MA and that M and A commute, together with substitutions in the integrals over N, imply that the above expression equals

$$\int_{M/Z} \int_{K^{l+1}} \int_{N^{l+1}} \int_{(MA)^l} \det(a_1, \ldots, a_l)$$
$$f_0(kn_0hxh^{-1}m_1^{-1}a_1^{-1}k_1^{-1}k^{-1}) f_1(kk_1n_1m_1m_2^{-1}a_1a_2^{-1}k_2^{-1}k^{-1}) \cdots$$
$$\cdots f_{l-1}(kk_{l-1}n_{l-1}m_{l-1}m_l^{-1}a_{l-1}a_l^{-1}k_l^{-1}k^{-1}) f_l(kk_ln_lm_la_lk^{-1})$$
$$dm_1 \cdots dm_l \, da_1 \cdots da_l \, dn_0 \cdots dn_l \, dk_0 \cdots dk_l \, d(hZ).$$

A substitution in the integrals over K now gives the desired equality (3.1). $\qquad\square$

If $f \in \mathcal{C}(G)$, define the function f^N on MA by

$$f^N(ma) := \int_N f(nma) \, dn. \tag{3.2}$$

It was shown in Lemma 21 in [18] that this integral converges for all $m \in M$ and $a \in A$, and that this defines a Schwartz function $f^N \in \mathcal{C}(MA)$.

Let $\mathcal{H}$ be a Hilbert space. We write $\mathcal{L}^1(\mathcal{H})$ for the algebra of trace-class operators on $\mathcal{H}$. Define seminorms on the space of smooth maps from G to $\mathcal{L}^1(\mathcal{H})$ by taking the seminorms on $\mathcal{C}(G)$ and replacing absolute values of functions by the trace norm. Let $\mathcal{C}(G, \mathcal{L}^1(\mathcal{H}))$ be the Fréchet space of all such maps for which these seminorms are finite. For a cyclic l-cocycle φ on $\mathcal{C}(G)$, let $\varphi\# \operatorname{Tr} \colon \big((\mathcal{C}(G, \mathcal{L}^1(\mathcal{H})))\big)^{l+1} \to \mathbb{C}$ be as in Theorem III.1.α.12 in [12], initially defined on the dense subspace $\mathcal{C}(G) \otimes \mathcal{L}^1(\mathcal{H})$ and extended continuously. Then the pairing of φ with an idempotent q in a matrix algebra over $\mathcal{C}(G, \mathcal{L}^1(\mathcal{H}))$, representing an element in $K_0\big(\mathcal{C}(G, \mathcal{L}^1(\mathcal{H}))\big)$, is given by

$$\langle \varphi, q \rangle = (\varphi\# \operatorname{Tr})(q, \ldots, q). \tag{3.3}$$

Here Tr denotes the tensor product of the operator trace on $\mathcal{L}^1(\mathcal{H})$ and the matrix trace.

Proposition 3.2 *Let $\mathcal{H}$ be a Hilbert space equipped with a representation of K. For $j = 0, \ldots, l$, let $f_j \in \mathcal{C}(G, \mathcal{L}^1(\mathcal{H}))$ be such that for all $g \in G$ and $k, k' \in K$,*

$$f_j(kgk') = k \circ f_j(g) \circ k' \quad \in \mathcal{L}^1(\mathcal{H}). \tag{3.4}$$

Let f_j^N be defined via the extension of (3.2) to a map $\mathcal{C}(G, \mathcal{L}^1(\mathcal{H})) \to \mathcal{C}(MA, \mathcal{L}^1(\mathcal{H}))$. Then

$$(\Phi_x^P \# \operatorname{Tr})(f_0, \ldots, f_l) = (\Phi_x^{MA} \# \operatorname{Tr})(f_0^N, \ldots, f_l^N).$$

Proof Viewing MA as a parabolic subgroup of itself, and applying Lemma 3.1, we find that that for all $f_0^{MA}, \ldots, f_l^{MA} \in \mathcal{C}(MA, \mathcal{L}^1(\mathcal{H}))$,

$$
\begin{aligned}
&(\Phi_x^{MA} \# \operatorname{Tr})(f_0^{MA}, \ldots, f_l^{MA}) = \int_{M/Z} \int_{(K \cap M)^{l+1}} \int_{(MA)^l} \det(a_1, \ldots, a_l) \\
&\operatorname{Tr}\Big(f_0^{MA}(k_0 h x h^{-1} m_1^{-1} a_1^{-1} k_1^{-1}) f_1^{MA}(k_1 m_1 m_2^{-1} a_1 a_2^{-1} k_2^{-1}) \cdots \\
&\cdots f_{l-1}^{MA}(k_{l-1} m_{l-1} m_l^{-1} a_{l-1} a_l^{-1} k_l^{-1}) f_l^{MA}(k_l m_l a_l k_0^{-1}) \Big) \\
&dm_1 \cdots dm_l \, da_1 \cdots da_l \, dk_0 \cdots dk_l \, d(hZ).
\end{aligned}
$$

Using the facts that $k_j \in M$ and that M and A commute, we can use substitutions to rewrite the right hand side as

$$
\begin{aligned}
&\int_{M/Z} \int_{(MA)^l} \det(a_1, \ldots, a_l) \operatorname{Tr}\Big(f_0^{MA}(h x h^{-1} m_1^{-1} a_1^{-1}) f_1^{MA}(m_1 m_2^{-1} a_1 a_2^{-1}) \cdots \\
&\cdots f_{l-1}^{MA}(m_{l-1} m_l^{-1} a_{l-1} a_l^{-1}) f_l^{MA}(m_l a_l) \Big) dm_1 \cdots dm_l \, da_1 \cdots da_l \, d(hZ). \tag{3.5}
\end{aligned}
$$

Explicitly, if $l \geq 4$, then one substitutes $k_0 h$ for h; $k_1 m_1 k_0^{-1}$ for m_1; $k_2 m_2 k_0^{-1}$ for m_2; $k_{l-1} m_{l-1} k_0^{-1}$ for m_{l-1}; $k_l m_l k_0^{-1}$ for m_l; and $k_j m_j$ for m_j if $3 \leq j \leq l - 2$.

Now if $f_0, \ldots, f_l \in \mathcal{C}(G, \mathcal{L}^1(\mathcal{H}))$ satisfy (3.4), then Lemma 3.1 implies that

$$
\begin{aligned}
&(\Phi_x^P \# \operatorname{Tr})(f_0, \ldots, f_l) \\
&= \int_{M/Z} \int_K \int_{(MA)^l} \det(a_1, \ldots, a_l) \operatorname{Tr}\Big(k_0 f_0^N(h x h^{-1} m_1^{-1} a_1^{-1}) f_1^N(m_1 m_2^{-1} a_1 a_2^{-1}) \cdots \\
&\cdots f_{l-1}^N(m_{l-1} m_l^{-1} a_{l-1} a_l^{-1}) f_l^N(m_l a_l) k_0^{-1} \Big) dm_1 \cdots dm_l \, da_1 \cdots da_l \, dk_0 \, d(hZ).
\end{aligned}
$$

By the trace property of the operator trace, this equals the right hand side of (3.5), with f_j^{MA} replaced by f_j^N. $\qquad \square$

3.2 A slice

Using Abels' slice theorem [1], we write

$$X = G \times_K Y, \tag{3.6}$$

for a K-invariant, compact submanifold $Y \subset X$. The index (2.1) is independent of the G-invariant Riemannian metric on X. From now on, we choose such a metric induced by a K-invariant Riemannian metric on Y and a K-invariant inner product on $\mathfrak{p}$ via

$$TX \cong G \times_K (\mathfrak{p} \oplus TY).$$

We will write dy (and dy' and dy_j) for the Riemannian density on Y. Then we can and will normalise the Haar measure dg on G so that for all $f \in C_c(X)$,

$$\int_X f(p) \, d\operatorname{vol}_p = \int_G \int_Y f(gy) \, dy \, dg,$$

where $d\operatorname{vol}$ is the Riemannian density on X. See Lemma 4.1 in [23].

Suppose that the adjoint representation $\operatorname{Ad}\colon K \to \operatorname{SO}(\mathfrak{p})$ lifts to a homomorphism

$$\widetilde{\operatorname{Ad}}\colon K \to \operatorname{Spin}(\mathfrak{p}). \tag{3.7}$$

This is not a restriction, since this is always true for a double cover $\tilde{G}$ of G, and one can then use the fact that $\tilde{G} \times_{\tilde{K}} Y = G \times_K Y = X$, for a maximal compact subgroup $\tilde{K} < \tilde{G}$. Let $S_\mathfrak{p}$ be the Spin-representation of $\operatorname{Spin}(\mathfrak{p})$, viewed as a representation of K via $\widetilde{\operatorname{Ad}}$. The slice Y in (3.6) has a Spin^c-structure with spinor bundle $S_Y \to Y$ such that

$$S = G \times_K (S_\mathfrak{p} \otimes S_Y). \tag{3.8}$$

See Proposition 3.10 in [22].

Corresponding to (3.8), we have the decomposition

$$L^2(E) = \left(L^2(G) \otimes S_\mathfrak{p} \otimes L^2(S_Y \otimes W|_Y)\right)^K. \tag{3.9}$$

Let L be the left regular representation of G on $C^\infty(G)$, and let c be the Clifford action by $\mathfrak{p}$ on $S_\mathfrak{p}$. Let $\{X_1, \ldots, X_r\}$ be an orthonormal basis of $\mathfrak{p}$ with respect to the Killing form. Consider the operator

$$D_{G,K} := \sum_j L(X_j) \otimes c(X_j)$$

on $C^\infty(G) \otimes S_\mathfrak{p}$. We have

$$D = D_{G,K} \otimes 1 + 1 \otimes D_Y \tag{3.10}$$

on

$$\Gamma^\infty(E) \cong \big(C^\infty(G) \otimes S_{\mathfrak{p}} \otimes \Gamma^\infty(S_Y \otimes W|_Y)\big)^K,$$

for a Spin^c-Dirac operator D_Y on S_Y, coupled to $W|_Y$. Here we use a graded tensor product, which means that $1 \otimes D_Y$ means the grading operator on $S_{\mathfrak{p}}$ tensored with D_Y.

Consider the vector bundle $\mathrm{Hom}(E|_Y) := E|_Y \boxtimes E|_Y^* \to Y \times Y$. Consider the action by $K \times K$ on $\mathcal{C}(G) \otimes \Gamma^\infty(\mathrm{Hom}(E|_Y))$ defined by

$$\big((k, k') \cdot (f \otimes A)\big)(g, y, y') = f(kgk'^{-1})k^{-1}A(ky, k'y')k', \qquad (3.11)$$

for $k, k' \in K$, $f \in \mathcal{C}(G)$, $A \in \Gamma^\infty(\mathrm{Hom}(E|_Y))$, $g \in G$ and $y, y' \in Y$. Let $\mathcal{C}_0(E) \subset \mathcal{C}(G) \otimes \Gamma^\infty(\mathrm{Hom}(E|_Y))$ be the space of elements invariant under this action. An element $\tilde\kappa \in \mathcal{C}_0(E)$ defines a smooth section κ of $\mathrm{Hom}(E) := E \boxtimes E^* \to X \times X$ given by

$$\kappa(gy, g'y') = g\tilde\kappa(g^{-1}g', y, y')g'^{-1}, \qquad (3.12)$$

for $g, g' \in G$ and $y, y' \in Y$.

We define $\mathcal{C}(E) := \mathcal{C}\big(G, \mathcal{L}^1(L^2(E|_Y))\big)$ as above Proposition 3.2. We identify elements of $\mathcal{C}(E)$ with the G-equivariant operators they define on $L^2(E)$ via (3.12).

3.3 Functional calculus and operators in $\mathcal{C}(E)$

We will use the fact that the K-theory class $\mathrm{ind}_G(D)$ can be represented by idempotents that lie in the unitisation of $\mathcal{C}(E)$, see Lemma 3.8. That allows us to apply Proposition 3.2, see the proof of Proposition 3.9. These arguments are based on the following fact.

Proposition 3.3 *Let f be a Schwartz function on $\mathbb{R}$, and assume that for all $n \in \mathbb{Z}_{\geq 0}$, there are $a, b > 0$ such that for all $x > 0$, the nth derivative of f at x satisfies*

$$|f^{(n)}(x)| \leq ae^{-bx}. \qquad (3.13)$$

Then the operators $f(D^2)$ and $Df(D^2)$ defined by functional calculus lie in $\mathcal{C}(E)$.

For every $V \in \widehat{K}$, let $\mathrm{spec}_V(D_Y) \subset \mathrm{spec}(D_Y)$ be the set of eigenvalues of the restriction of D_Y to the V-isotypical component $L^2(S_Y \otimes W|_Y)_V$ of $L^2(S_Y \otimes W|_Y)$. For $\lambda \in \mathrm{spec}_V(D_Y)$, let $L^2(S_Y \otimes W|_Y)_{V,\lambda} \subset L^2(S_Y \otimes W|_Y)_V$ be the corresponding eigenspace. Then by (3.9),

$$L^2(E) \cong \bigoplus_{V \in \widehat{K}} \bigoplus_{\lambda \in \mathrm{spec}_V(D_Y)} \big(L^2(G) \otimes S_{\mathfrak{p}} \otimes L^2(S_Y \otimes W|_Y)_{V,\lambda}\big)^K.$$

Via the Fourier transform, the right hand side is isomorphic to

$$\int_{\hat{G}_{\text{temp}}}^{\oplus} \bigoplus_{\substack{V \in \hat{K} \\ [\pi|_K : S_{\mathfrak{p}} \otimes V] \neq 0}} \bigoplus_{\lambda \in \text{spec}_V(D_Y)} \mathcal{H}_\pi \otimes \left(\mathcal{H}_\pi^* \otimes S_{\mathfrak{p}} \otimes L^2(S_Y \otimes W|_Y)_{V,\lambda} \right)^K d\mu(\pi),$$

$$(3.14)$$

where μ is the Plancherel measure. Every representation $\pi \in \hat{G}_{\text{temp}}$ is of the form $\pi = \text{Ind}_P^G(\sigma \otimes \nu \otimes 1)$, for a cuspidal parabolic subgroup $P = MAN < G$, a (limit of) discrete series representation σ of M, and $\nu \in \hat{A} = i\mathfrak{a}^*$. (See [27,28], or Theorem 14.91 in [26].) We denote by

$$(\Lambda(P, \sigma), \nu) \in i(\mathfrak{t} \cap \mathfrak{m})^* \otimes i\mathfrak{a}^*$$

the infinitesimal character of $\text{Ind}_P^G(\sigma \otimes \nu \otimes 1)$. Then the Casimir element Ω_G of G acts on π as the scalar

$$\Omega_G(\pi) = -\|\Lambda(P, \sigma)\|^2 + \|\nu\|^2. \tag{3.15}$$

Let f be as in Proposition 3.3. Let $\widehat{f(D^2)}$ be the operator on (3.14) corresponding to the operator $f(D^2)$ on $L^2(E)$.

Lemma 3.4 *The operator* $\widehat{f(D^2)}$ *is given by multiplication by a function on* $\hat{G}_{\text{temp}} \times \text{spec}(D_Y)$. *At* $(\pi = \text{Ind}_P^G(\sigma \otimes \nu \otimes 1), \lambda)$, *with* $\lambda \in \text{spec}_V(D_Y)$, *the absolute value of this function is at most*

$$ae^{-b\lambda^2} e^{-b(\|\nu\|^2 - \|\Lambda(P,\sigma)\|^2 + \|\mu_V + \rho_K\|^2)}, \tag{3.16}$$

where μ_V *is the highest weight of* V, *and* ρ_K *is half the sum of a choice of positive roots for* (K, T). *All derivatives of this function with respect to* ν *satisfy the same estimate, for possibly different a and b.*

Proof Because of the use of graded tensor products in (3.10),

$$D^2 = D_{G,K}^2 \otimes 1 + 1 \otimes D_Y^2$$

where the tensor products on the right hand side are ungraded. This implies that the two terms commute, so that for all $s \in \mathbb{R}$,

$$e^{isD^2} = e^{isD_{G,K}^2} \otimes e^{isD_Y^2},$$

as in (4.3) in [24]. So Proposition 10.3.5 in [19] implies that

$$f(D^2) = \int_{\mathbb{R}} e^{isD^2} \hat{f}(s) ds = \int_{\mathbb{R}} e^{isD_{G,K}^2} \otimes e^{isD_Y^2} \hat{f}(s) \, ds.$$

After Fourier transform, the operator $D^2_{G,K}$ acts on (3.14) as multiplication by a function, which takes the value

$$\|\nu\|^2 - \|\Lambda(P,\sigma)\|^2 + \|\mu_V + \rho_K\|^2$$

at the point corresponding to $V \in \widehat{K}$ and $\pi = \mathrm{Ind}^G_P(\sigma \otimes \nu \otimes 1)$. Here we use (3.15), and Proposition 3.1 in [33]; see also (3.17) in [3].

We find that $\widehat{f(D^2)}$ is indeed given by multiplication by a function on $\widehat{G}_{\mathrm{temp}} \times \mathrm{spec}(D_Y)$, and its value at $(\mathrm{Ind}^G_P(\sigma \otimes \nu \otimes 1), \lambda)$, with $\lambda \in \mathrm{spec}_V(D_Y)$, is

$$
\int_{\mathbb{R}} e^{is(\|\nu\|^2 - \|\Lambda(P,\sigma)\|^2 + \|\mu_V + \rho_K\|^2} e^{is\lambda^2} \, \hat{f}(s) \, ds
$$
$$
= f(\|\nu\|^2 - \|\Lambda(P,\sigma)\|^2 + \|\mu_V + \rho_K\|^2 + \lambda^2). \tag{3.17}
$$

By assumption on f, this function and all its derivatives with respect to ν satisfy the desired estimate.   □

Our next goal is to estimate the decay behaviour of (3.16). We use the term *rapidly decreasing* to mean decreasing faster than any rational function. Let us define a function C on $\widehat{K}$ by

$$C(V) = \sum_{\lambda \in \mathrm{spec}_V(D_Y)} e^{-b\lambda^2},$$

for $V \in \widehat{K}$.

Lemma 3.5 *The function C decreases rapidly in $\|\mu_V\|$.*

Proof Consider the special case of Proposition 3.3 where $G = K$ is compact. Then the heat kernel of D lies in

$$\mathcal{C}(E) = \left(C^\infty(K) \otimes \mathcal{L}^1(E|_Y) \right)^K,$$

simply because it is smooth. Its component in $\mathcal{L}^1(E|_Y)$ is nonnegative, so its trace norm on that component equals its trace. Applying this trace on Y, we obtain a smooth function on K, whose Fourier transform equals C. And the Fourier transform of a smooth function on K is a rapidly decreasing function on $\widehat{K}$.   □

Lemma 3.6 *The function*

$$\widetilde{C}(x) := \sum_{\substack{W \in \widehat{K} \\ \|\mu_W\| \geq x}} C(W)$$

decreases rapidly in $x \geq 0$.

Proof If $x \in \mathbb{N}$, then

$$\widetilde{C}(x) = \sum_{n=x}^{\infty} \sum_{\substack{W \in \widehat{K} \\ n \le \|\mu_W\| < n+1}} C(W)$$

$$\le \sum_{n=x}^{\infty} \#\{W \in \widehat{K}; n \le \|\mu_W\| < n+1\} \max_{\|\mu_W\| \ge n} C(W).$$

The number $\#\{W \in \widehat{K}; n \le \|\mu_W\| < n+1\}$ increases at most polynomially in n. And $\max_{\|\mu_W\| \ge n} C(W)$ decreases rapidly in n, because $C(W)$ decays rapidly in $\|\mu_W\|$ by Lemma 3.5. So the terms in the latter sum decay rapidly in n. This implies that the sum decays rapidly in x, for example via an estimate of the sum by an integral. $\square$

Lemma 3.7 *Let φ be the operator-valued function on $\widehat{G}_{\text{temp}}$ given by*

$$\varphi(\pi = \text{Ind}_P^G(\sigma \otimes \nu \otimes 1))$$
$$= \sum_{\substack{V \in \widehat{K} \\ [\pi|_K : S_{\mathfrak{p}} \otimes V] \ne 0}} \sum_{\lambda \in \text{spec}_V(D_Y)} e^{-b\lambda^2} e^{-b(\|\nu\|^2 - \|\Lambda(P,\sigma)\|^2 + \|\mu_V + \rho_K\|^2)} \cdot \text{id}_{(\pi \otimes S_{\mathfrak{p}} \otimes V)^K},$$

(3.18)

for some $b > 0$. Then the function

$$\|\varphi\|_{\text{HS}} \in C^{\infty}(\widehat{G}_{\text{temp}})$$

is a rapidly decreasing function in $\|\Lambda(P,\sigma)\|$ and $\|\nu\|$, where $\| \bullet \|_{\text{HS}}$ means the Hilbert–Schmidt norm.

Proof We study the decay behaviour of φ in ν and σ separately.

For the decay behaviour in ν, we note that positivity of $D_{G,K}^2$ implies that

$$-\|\Lambda(P,\sigma)\|^2 + \|\mu_V + \rho_K\|^2 \ge 0.$$

(3.19)

(This is known as Parthasarathy's Dirac inequality.) Hence for all σ,

$$\varphi(\text{Ind}_P^G(\sigma \otimes \nu \otimes 1)) \le \left(\sum_{V \in \widehat{K}} C(V)[\pi|_K : S_{\mathfrak{p}} \otimes V] \right) e^{-b\|\nu\|^2}.$$

(3.20)

For any $W \in \widehat{K}$, we have

$$[\pi|_K : W] \le \dim(W).$$

See for example Theorem 8.1 in [26]. (Compare also Theorem 2.9(3) and Example 2.12 in [29].) So $[\pi|_K : S_{\mathfrak{p}} \otimes V]$ grows at most polynomially in $\|\mu_V\|$, uniformly in

π. And the function C decreases rapidly by Lemma 3.5. So the sum over V in (3.20) converges, and φ decreases rapidly in $\|\nu\|$, uniformly in σ.

We now consider decay behaviour with respect to σ. For any $\pi \in \widehat{G}$, we write

$$\min_K(\pi) := \min\{\|\mu_V\|; V \in \widehat{K}, [\pi|_K : S_{\mathfrak{p}} \otimes V] \neq 0\}.$$

By Frobenius reciprocity, we have for all $W \in \widehat{K}$,

$$[\pi|_K : W] = \left[\mathrm{Ind}_P^G(\sigma \otimes \nu \otimes 1)\big|_K : W\right] = \left[\left(L^2(K) \otimes \sigma\right)^{M \cap K} : W\right]$$

$$= \left(W^*|_{M \cap K} \otimes \sigma\right)^{M \cap K}. \tag{3.21}$$

This implies that $\|\min_K(\pi)\|$ increases as a positive power of $\|\Lambda(P, \sigma)\|$ when $\|\Lambda(P, \sigma)\|$ is large enough.

Now (3.19) implies that

$$\|\varphi(\mathrm{Ind}_P^G(\sigma \otimes \nu \otimes 1))\|_{\mathrm{HS}} \leq \widetilde{C}(\min_K(\pi))[\pi|_K : S_{\mathfrak{p}} \otimes V].$$

Because $\min_K(\pi)$ increases as a positive power of $\|\Lambda(P, \sigma)\|$ and $\widetilde{C}$ decreases rapidly by Lemma 3.6, we find that $\widetilde{C}(\min_K(\pi))$ decreases rapidly in $\|\Lambda(P, \sigma)\|$ and is independent of ν. Hence $\|\varphi(\mathrm{Ind}_P^G(\sigma \otimes \nu \otimes 1))\|_{\mathrm{HS}}$ also decreases rapidly in $\|\Lambda(P, \sigma)\|$, uniformly in ν. □

Proof of Proposition 3.3 We first show that $f(D^2) \in \mathcal{C}(E)$. We consider its Fourier transform $\widehat{f(D^2)}$ on (3.14), and apply the trace norm on the component $L^2(S_Y \otimes W|_Y)$. Lemmas 3.4 and 3.7 imply that the resulting function is a Schwartz function on $\widehat{G}_{\mathrm{temp}}$. By a characterisation of $\mathcal{C}(G)$ by Arthur [2] (based on results by Harish-Chandra [17,18]), this implies that $f(D^2)$ lies in $\mathcal{C}(G) \otimes \mathrm{End}(S_{\mathfrak{p}})$ after we apply the trace norm on $L^2(S_Y \otimes W|_Y)$. Hence $f(D^2) \in \mathcal{C}(E)$.

To show that $Df(D^2) \in \mathcal{C}(E)$, we write

$$Df(D^2) = (D_{G,K} \otimes 1)f(D^2) + (1 \otimes D_Y)f(D^2),$$

using graded tensor products. The first term on the right lies in $\mathcal{C}(E)$ because the universal enveloping algebra of $\mathfrak{g}$ preserves $\mathcal{C}(G)$, so that $D_{G,K} \otimes 1$ preserves $\mathcal{C}(E)$. The second term on the right lies in $\mathcal{C}(E)$ by a similar argument as for $f(D^2)$. The difference is that for $(1 \otimes D_Y)f(D^2)$, the expression (3.17) is replaced by

$$\int_{\mathbb{R}} e^{is(\|\nu\|^2 - \|\sigma\|^2 + \|\mu_V + \rho_K^M\|^2)} \lambda e^{is\lambda^2} \widehat{f}(s) \, ds = \lambda f(\|\nu\|^2 - \|\sigma\|^2 + \|\mu_V + \rho_K^M\|^2 + \lambda^2).$$

The rest of the argument still applies, with $e^{-b\lambda^2}$ replaced by $\lambda e^{-b\lambda^2}$ everywhere. □

3.4 The index on *X*

We will use an explicit representative of the K-theory class $\mathrm{ind}_G(D)$. Suppose that G/K and X are even-dimensional. Then Y is also even-dimensional. We will later deduce the odd-dimensional case of Theorem 2.1 from the even-dimensional case; see Lemma 5.5.

Let $D^\pm$ be the restriction of D to even and odd-graded sections of E, respectively. Fix $t > 0$, and write

$$q_t := \begin{pmatrix} e^{-tD^-D^+} & e^{-\frac{t}{2}D^-D^+}\frac{1-e^{-tD^-D^+}}{D^-D^+}D^- \\ e^{-\frac{t}{2}D^+D^-}\frac{1-e^{-tD^+D^-}}{D^+D^-}D^+ & 1_{E^-} - e^{-tD^+D^-} \end{pmatrix} \in \Gamma^\infty(\mathrm{Hom}(E)), \tag{3.22}$$

and

$$p_E := \begin{pmatrix} 0 & 0 \\ 0 & 1_{E^-} \end{pmatrix} \in \mathcal{B}(L^2(E)).$$

Lemma 3.8 *The operator q_t lies in the unitisation of $\mathcal{C}(E)$.*

Proof By Proposition 3.3, the operators e^{-tD^2} and $e^{-\frac{t}{2}D^2}\frac{1-e^{-tD^2}}{D^2}D$ lie in $\mathcal{C}(E)$. □

The operator q_t is an idempotent, so we have

$$[q_t] - [p_E] \in K_*(\mathcal{C}(E)).$$

For any cyclic cocycle φ over $\mathcal{C}(G)$,

$$\langle \varphi, \mathrm{ind}_G(D) \rangle = \langle \varphi \# \mathrm{Tr}, [q_t] - [p_E] \rangle, \tag{3.23}$$

where the pairing on the right is as in (3.3). See page 356 in [13]. Combining (3.23) with Proposition 3.2, we obtain the main conclusion of this section.

Proposition 3.9 *We have*

$$\langle \Phi_x^P, \mathrm{ind}_G(D) \rangle = \langle \Phi_x^{MA} \# \mathrm{Tr}, [q_t^N] - [p_E^N] \rangle.$$

Proof By Lemma 3.8 and the $K \times K$-invariance property (3.11) of the elements of $\mathcal{C}(E)$, the element $q_t \in \mathcal{C}(G, \mathcal{L}^1(L^2(E|_Y)))$ has the $K \times K$-invariance property of the functions f_j in Proposition 3.2. By that proposition, (3.23) and (3.3), the claim follows.   □

4 An index pairing on *X*/*N*

We construct a Dirac operator on X/N and give an explicit realisation of its MA-equivariant index, Proposition 4.7. This proposition is a variation on the commutativity of the top left diagram in (2.10).

4.1 Operators on *X* and *X*/*N*

Consider the MA-invariant submanifold $X_{MA} := MA \times_{K \cap M} Y \subset X$. Then $X = NAM \times_{K \cap M} Y$, so the inclusion map $X_{MA} \hookrightarrow X$ induces $X_{MA} \cong X/N$. For $\kappa \in \mathcal{C}(E) \cap \Gamma^\infty(\mathrm{Hom}(E))$, let

$$\kappa^N \in \left(\mathcal{C}(MA) \hat\otimes \Gamma^\infty(\mathrm{Hom}(E|_Y))\right)^{K \cap M \times K \cap M} \hookrightarrow \mathrm{End}(E|_{X_{MA}}) \qquad (4.1)$$

be defined as in (3.2). (We use the completed tensor product of the Fréchet space $\mathcal{C}(MA)$ and the nuclear Fréchet space $\Gamma^\infty(\mathrm{Hom}(E|_Y))$ here, instead of the space $\mathcal{C}(MA, \mathcal{L}^1(L^2(E|_Y)))$, to make it clear that κ^N defines a smooth kernel on X_{MA}.) Explicitly, for all $m \in M$, $a \in A$ and $y, y' \in Y$,

$$\kappa^N(ma, y, y') = \int_N \kappa(nam, y, y') \, dn \quad \in \mathrm{Hom}(E_{y'}, E_y).$$

Consider the space $\Gamma^\infty(E)^{N,c}$ of smooth, N-invariant sections of E whose support has compact image in X/N. Then $\Gamma^\infty(E)^{N,c} \cong \Gamma_c^\infty(E|_{X_{MA}})$ via restriction to $X|_{MA}$. Via this identification, a section κ^N as in (4.1) defines an operator on $\Gamma^\infty(E)^{N,c}$. If s_1 and s_2 are sections of E that are not necessarily square-integrable, we will denote the integral

$$\int_X (s_1(p), s_2(p))_E \, d\,\mathrm{vol}_p$$

by $(s_1, s_2)_{L^2(E)}$ if it converges.

Lemma 4.1 *For all $\kappa \in \mathcal{C}(E)$, $\sigma \in \Gamma_c^\infty(E)$ and $s \in \Gamma^\infty(E)^{N,c}$,*

$$(\kappa^*\sigma, s)_{L^2(E)} = (\sigma, \kappa^N s)_{L^2(E)}. \qquad (4.2)$$

In particular, the integral defining the left hand side converges.

Proof For $\kappa \in \mathcal{C}(E)$, let $\tilde\kappa$ be defined as in (3.12). Writing $X = NAM \times_{K \cap M} Y$, we see that the left hand side of (4.2) equals

$$\int_{NAM} \int_Y \int_{NAM} \int_Y \left(\tilde\kappa^*(namy, n'a'm'y')\sigma(n'a'm'y'), s(namy)\right)_E$$
$$dy' \, dn' \, da' \, dm' \, dy \, dn \, da \, dm$$

By definition of the adjoint kernel $\tilde\kappa^*$, and by N-invariance of s, and G invariance of $\tilde\kappa$, this integral equals

$$\int_{NAM} \int_Y \int_{NAM} \int_Y \left(\sigma(n'a'm'y'), n\tilde\kappa(n^{-1}n'a'm'y', amy)s(amy)\right)_E$$
$$dy' \, dn' \, da' \, dm' \, dy \, dn \, da \, dm.$$

Substituting $n'^{-1}n$ for n, we rewrite this as

$$\int_{NAM}\int_Y\int_{NAM}\int_Y \left(\sigma(n'a'm'y'), n'n\tilde{\kappa}(n^{-1}a'm'y', amy)s(amy)\right)_E$$
$$dy'\,dn'\,da'\,dm'\,dy\,dn\,da\,dm. \tag{4.3}$$

If $\tilde{\kappa}^N$ is related to κ^N as in (3.12), with G replaced by MA, then for all $y, y' \in Y$, $a, a' \in A$ and $m, m' \in M$, unimodularity of N implies that

$$\int_N n\tilde{\kappa}(n^{-1}a'm'y', amy)\,dn = \tilde{\kappa}^N(a'm'y', amy).$$

So (4.3) equals

$$\int_{NAM}\int_Y \left(\sigma(n'a'm'y'), n'(\tilde{\kappa}^N s)(a'm'y')\right)_E dy'\,dn'\,da'\,dm'$$

By N-invariance of $\tilde{\kappa}^N s$, we have $n'(\tilde{\kappa}^N s)(a'm'y') = (\tilde{\kappa}^N s)(n'a'm'y')$ for all $y' \in Y$, $n' \in N$, $a' \in A$ and $m' \in M$. So the latter integral equals the right hand side of (4.2).
                                                                                                                                                                                      $\square$

Let $\kappa_t \in \mathcal{C}(E^+)$ be kernel of the heat operator $e^{-tD^-D^+}$.

Lemma 4.2 *For all $\sigma \in \Gamma_c^\infty(E^+)$ and $s \in \Gamma^\infty(E^+)^{N,c}$, and for all $t > 0$,*

$$\frac{d}{dt}(\sigma, \kappa_t^N s)_{L^2(E^+)} = (\sigma, -D^-D^+e^{-tD^-D^+}s)_{L^2(E^+)}.$$

Proof By Lemma 4.1,

$$(\sigma, \kappa_t^N s)_{L^2(E^+)} = (e^{-tD^-D^+}\sigma, s)_{L^2(E^+)}.$$

The derivative with respect to t at t equals

$$(-D^-D^+e^{-tD^-D^+}\sigma, s)_{L^2(E^+)}. \tag{4.4}$$

The operator $-D^-D^+e^{-tD^-D^+}$ is symmetric on sections in $L^2(E^+)$, so we have to take the fact into account that s is not in $L^2(E^+)$.

Let $\varepsilon > 0$. For $r > 0$, let $\psi_r \in C_c^\infty(M)$ be constant 1 within a distance r of $\mathrm{supp}(\sigma)$. Then (4.4) equals

$$\left(-D^-D^+e^{-tD^-D^+}\sigma, \psi_r s\right)_{L^2(E^+)} - \left((1-\psi_r)D^-D^+e^{-tD^-D^+}\sigma, s\right)_{L^2(E^+)}.$$

Because σ and $\psi_r s$ lie in $\Gamma_c^\infty(E^+)$, the first term equals $(\sigma, -D^-D^+e^{-tD^-D^+}\psi_r s)_{L^2(E^+)}$. And the absolute value of the second term is at most equal to

$$\|(1-\psi_r)D^-D^+e^{-tD^-D^+}\sigma\|_{L^1(E^+)}\|s\|_\infty.$$

The first factor is smaller than ε for r large enough by Gaussian decay properties of heat kernels. (Here we also use the fact that X/G is compact, so X has bounded geometry and volumes of balls in X are bounded by an exponential function of their radii.) For such a value of r, we find that

$$\left| (-D^- D^+ e^{-tD^- D^+} \sigma, s)_{L^2(E^+)} - (\sigma, -D^- D^+ e^{-tD^- D^+} \psi_r s)_{L^2(E^+)} \right| < \varepsilon \|s\|_\infty.$$

We similarly have

$$\left| (\sigma, -D^- D^+ e^{-tD^- D^+} s)_{L^2(E^+)} - (\sigma, -D^- D^+ e^{-tD^- D^+} \psi_r s)_{L^2(E^+)} \right|$$
$$= \left| \left((1 - \psi_r) D^- D^+ e^{-tD^- D^+} \sigma, s \right)_{L^2(E^+)} \right| < \varepsilon \|s\|_\infty.$$

We conclude that (4.4) equals $(\sigma, -D^- D^+ e^{-tD^- D^+} s)_{L^2(E^+)}$, and the claim follows.
$\square$

4.2 An index on X/N

We have a $K \cap M$-invariant decomposition

$$\mathfrak{p} = (\mathfrak{p} \cap \mathfrak{m}) \oplus \mathfrak{a} \oplus ((\mathfrak{p} \cap \mathfrak{m}) \oplus \mathfrak{a})^\perp,$$

where the orthogonal complement is taken inside $\mathfrak{p}$ with respect to the Killing form. The homomorphim $\widetilde{\mathrm{Ad}}$ maps $K \cap M$ into $\mathrm{Spin}(\mathfrak{p} \cap \mathfrak{m}) \times \mathrm{Spin}(\mathfrak{a}) \times \mathrm{Spin}(((\mathfrak{p} \cap \mathfrak{m}) \oplus \mathfrak{a})^\perp) \hookrightarrow \mathrm{Spin}(\mathfrak{p})$. Let $S_{\mathfrak{p} \cap \mathfrak{m}}$, $S_\mathfrak{a}$ and $S_{((\mathfrak{p} \cap \mathfrak{m}) \oplus \mathfrak{a})^\perp}$ be the corresponding Spin-representations of $K \cap M$. (The group $K \cap M$ acts trivially on $S_\mathfrak{a}$, because M centralises A.) Then we have a decomposition of representations of $K \cap M$,

$$S_\mathfrak{p} = S_{\mathfrak{p} \cap \mathfrak{m}} \otimes S_\mathfrak{a} \otimes S_{((\mathfrak{p} \cap \mathfrak{m}) \oplus \mathfrak{a})^\perp}. \tag{4.5}$$

If G/K is even-dimensional, then this includes the gradings on the respective spaces, and we use graded tensor products.

Lemma 4.3 *There is an isomorphism of graded representations of $K \cap M$*

$$S_{((\mathfrak{p} \cap \mathfrak{m}) \oplus \mathfrak{a})^\perp} \cong S_{\mathfrak{k}/(\mathfrak{k} \cap \mathfrak{m})}.$$

Proof There are $T \cap M$-equivariant linear isomorphisms

$$\mathfrak{p} \cong \mathfrak{g}/\mathfrak{k} \cong \mathfrak{m}/(\mathfrak{k} \cap \mathfrak{m}) \oplus \mathfrak{a} \oplus \mathfrak{n} \cong (\mathfrak{p} \cap \mathfrak{m}) \oplus \mathfrak{a} \oplus \mathfrak{n}.$$

If θ is the Cartan involution, then the map $y \mapsto (y + \theta y)/2 + \mathfrak{k} \cap \mathfrak{m}$ is a $T \cap M$-invariant isomorphism from $\mathfrak{n}$ onto $\mathfrak{k}/(\mathfrak{k} \cap \mathfrak{m})$. So $((\mathfrak{p} \cap \mathfrak{m}) \oplus \mathfrak{a})^\perp \cong \mathfrak{k}/(\mathfrak{k} \cap \mathfrak{m})$ as representations of $T \cap M$. This implies that their characters are equal on $K \cap M$, so they are equal as representations of $K \cap M$ as well.
$\square$

Consider the MA-equivariant vector bundles

$$S_N := MA \times_{K \cap M} (S_{\mathfrak{p} \cap \mathfrak{m}} \otimes S_{\mathfrak{a}} \otimes S_Y) \to X_{MA} = X/N;$$
$$F_N := MA \times_{K \cap M} (Y \times S_{\mathfrak{k}/(\mathfrak{k} \cap \mathfrak{m})}) \to X_{MA} = X/N. \tag{4.6}$$

Lemma 4.4 *The vector bundle S_N is the spinor bundle of an MA-equivariant Spin^c-structure on X/N, and* (2.8) *holds.*

Proof Clifford multiplication is an algebra bundle isomorphism $\mathrm{Cl}(TX) \to \mathrm{End}(S)$, where Cl stands for the Clifford bundle. The Clifford bundle

$$\mathrm{Cl}(TX_{MA}) = \mathrm{Cl}(MA \times_{K \cap M} (\mathfrak{p} \cap \mathfrak{m} \oplus \mathfrak{a} \oplus TY))$$

maps precisely onto $\mathrm{End}(S_N)$ under this isomorphism. So S_N is the spinor bundle of an MA-equivariant Spin^c-structure on X_{MA}. Lemma 4.3 and (3.8) imply (2.8). $\quad\square$

The equality (2.8) implies that

$$E|_{X_{MA}} = MA \times_{K \cap M} \big((S_{\mathfrak{p} \cap \mathfrak{m}} \otimes S_{\mathfrak{a}}) \otimes (S_Y \otimes S_{\mathfrak{k}/(\mathfrak{k} \cap \mathfrak{m})} \otimes W|_Y) \big). \tag{4.7}$$

Let $\{X_1, \ldots, X_s\}$ be a basis of $(\mathfrak{p} \cap \mathfrak{m}) \oplus \mathfrak{a}$, orthonormal with respect to the Killing form. Let L be the left regular representation of MA. Let

$$D_{MA/(K \cap M)} := \sum_j L(X_j) \otimes c(X_j) \tag{4.8}$$

be the Spin-Dirac operator on the bundle

$$MA \times_{K \cap M} (S_{\mathfrak{p} \cap \mathfrak{m}} \otimes S_{\mathfrak{a}}) \to MA/(K \cap M). \tag{4.9}$$

It extends to an operator $D_{MA, K \cap M}$ on

$$C^\infty(MA) \otimes (S_{\mathfrak{p} \cap \mathfrak{m}} \otimes S_{\mathfrak{a}}),$$

defined by the same expression (4.8).

Let $D_{Y,M}$ be the Spin^c-Dirac operator on Y coupled to $S_{\mathfrak{k}/(\mathfrak{k} \cap \mathfrak{m})} \otimes W|_Y$, acting on sections of the bundle

$$S_Y \otimes S_{\mathfrak{a}} \otimes S_{\mathfrak{k}/(\mathfrak{k} \cap \mathfrak{m})} \otimes W|_Y \to Y.$$

Consider the operator

$$D_{X_{MA}} := D_{MA, K \cap M} \otimes 1 + 1 \otimes D_{Y,M}, \tag{4.10}$$

where we use graded tensor products; i.e. $1 \otimes D_{Y,M}$ means the grading operator on (4.9) tensored with $D_{Y,M}$. Initially, it acts in the space

$$C^\infty(MA) \otimes (S_{\mathfrak{p} \cap \mathfrak{m}} \otimes S_{\mathfrak{a}}) \otimes \Gamma^\infty(S_Y \otimes S_{\mathfrak{k}/(\mathfrak{k} \cap \mathfrak{m})} \otimes W|_Y),$$

but we view it as acting on the subspace of $K \cap M$-invariant elements, which is the space of smooth sections of (4.7). Then $D_{X_{MA}}$ is a Spin^c-Dirac operator on X_{MA} for the Spin^c-structure with spinor bundle S_N,
twisted by the vector bundle $W_N := F_N \otimes W/N$.

As before, let $\kappa_t \in \mathcal{C}(E^+)$ be Schwartz kernel of the heat operator $e^{-tD^-D^+}$, and let κ_t^N be defined as in (4.1).

Lemma 4.5 *The Schwartz kernel of* $e^{-tD_{X_{MA}}^- D_{X_{MA}}^+}$ *is* κ_t^N. *The analogous statement for* $e^{-tD_{X_{MA}}^+ D_{X_{MA}}^-}$ *holds as well.*

Proof The operator D commutes with the action of κ_t^N on $\Gamma^\infty(E^+)^{N,c}$ because it is N-equivariant. So Lemmas 4.2 and 4.1 imply that for all $\sigma \in \Gamma_c^\infty(E^+)$ and $s \in \Gamma^\infty(E^+)^{N,c} \cong \Gamma_c^\infty(E^+|_{X_{MA}})$, and all $t > 0$,

$$
\begin{aligned}
\frac{d}{dt}(\sigma, \kappa_t^N s)_{L^2(E^+)} &= (e^{-tD^-D^+}\sigma, -D^-D^+ s)_{L^2(E^+)} \\
&= (\sigma, -\kappa_t^N D^-D^+ s)_{L^2(E^+)} \\
&= (\sigma, -D^-D^+ \kappa_t^N s)_{L^2(E^+)}.
\end{aligned}
$$

On N-invariant sections, D equals $D_{X_{MA}}$, so the latter inner product equals $(\sigma, -D_{X_{MA}}^- D_{X_{MA}}^+ \kappa_t^N s)_{L^2(E^+)}$.
Lemma 4.1 also implies that

$$\lim_{t \downarrow 0}(\sigma, \kappa_t^N s)_{L^2(E^+)} = (\sigma, s)_{L^2(E^+)}.$$

We conclude that the map $(p, t) \mapsto (\kappa_t^N s)(p)$ satisfies the heat equation for the operator $D_{X_{MA}}^- D_{X_{MA}}^+$, with the boundary condition $\lim_{t \downarrow 0}(\kappa_t^N s)(p) = s(p)$ for all $p \in X_{MA}$.

The statement for $e^{-tD_{X_{MA}}^+ D_{X_{MA}}^-}$ can be proved analogously. $\qquad\square$

Lemma 4.6 *Let* $\lambda_t \in \mathcal{C}(E)$ *be the smooth kernel of* $e^{-\frac{t}{2}D^-D^+}\frac{1-e^{-tD^-D^+}}{D^-D^+}D^-$. *Then* λ_t^N *is the Schwartz kernel of* $e^{-\frac{t}{2}D_{X_{MA}}^- D_{X_{MA}}^+}\frac{1-e^{-tD_{X_{MA}}^- D_{X_{MA}}^+}}{D_{X_{MA}}^- D_{X_{MA}}^+}D_{X_{MA}}^-$. *The analogous statement for* $e^{-\frac{t}{2}D^+D^-}\frac{1-e^{-tD^+D^-}}{D^+D^-}D^+$ *holds as well.*

Proof Fix $t_0 \in (0, t)$. As in Lemma 3.8, the Schwartz kernel ν_t of the operator

$$e^{-\frac{t_0}{2}D^-D^+}\frac{1 - e^{-tD^-D^+}}{D^-D^+}D^-$$

lies in $\mathcal{C}(E)$. Similarly to the proof of Lemma 4.5, we use Lemmas 4.1 and 4.2 to find that for all $\sigma \in \Gamma_c^\infty(E^+)$ and $s \in \Gamma^\infty(E^-)^{N,c}$,

$$
\begin{aligned}
\frac{d}{dt}(\sigma, \nu_t^N s)_{L^2(E^+)} &= \frac{d}{dt}\left(D^+ \frac{1 - e^{-tD^-D^+}}{D^-D^+} e^{-\frac{t_0}{2}D^-D^+}\sigma, s\right)_{L^2(E^-)} \\
&= (e^{-\frac{t_0}{2}D^-D^+} e^{-tD^-D^+}D^+\sigma, s)_{L^2(E^-)} \\
&= (\sigma, e^{-\frac{t_0}{2}D^-D^+} e^{-tD^-D^+}D^-s)_{L^2(E^+)} \\
&= (\sigma, e^{-\frac{t_0}{2}D_{X_{MA}}^- D_{X_{MA}}^+} e^{-tD_{X_{MA}}^- D_{X_{MA}}^+} D_{X_{MA}}^- s)_{L^2(E^+)}.
\end{aligned}
$$

So ν_t^N has the defining properties of the operator

$$
e^{-\frac{t_0}{2}D_{X_{MA}}^- D_{X_{MA}}^+} \frac{1 - e^{-tD_{X_{MA}}^- D_{X_{MA}}^+}}{D_{X_{MA}}^- D_{X_{MA}}^+} D_{X_{MA}}^-.
$$

Lemma 4.5 implies that if κ_t is the Schwartz kernel of $e^{-\frac{t-t_0}{2}D^-D^+}$, then κ_t^N is the Schwartz kernel of $e^{-\frac{t-t_0}{2}D_{X_{MA}}^- D_{X_{MA}}^+}$. Because N normalises M and A, the map $\kappa \mapsto \kappa^N$ is multiplicative. So $\lambda_t^N = \kappa_t^N \nu_t^N$, which by the preceding arguments is the Schwartz kernel of $e^{-\frac{t}{2}D_{X_{MA}}^- D_{X_{MA}}^+} \frac{1 - e^{-tD_{X_{MA}}^- D_{X_{MA}}^+}}{D_{X_{MA}}^- D_{X_{MA}}^+} D_{X_{MA}}^-$.

The statement for $e^{-\frac{t}{2}D^+D^-} \frac{1-e^{-tD^+D^-}}{D^+D^-} D^+$ can be proved analogously. $\qquad\square$

Proposition 4.7 *For any cyclic cocycle φ over $\mathcal{C}(MA)$,*

$$
\langle \varphi, \operatorname{ind}_{MA}(D_{X_{MA}}) \rangle = \langle \varphi \# \operatorname{Tr}, [q_t^N] - [p_{E|_{X_{MA}}}] \rangle.
$$

Proof Lemmas 4.5 and 4.6 imply that q_t^N equals (3.22), with D replaced by $D_{X_{MA}}$. So the claim follows from the analogue of (3.23) for $D_{X_{MA}}$. $\qquad\square$

Propositions 3.9 and 4.7, and the fact that $p_E^N = p_{E|_{X_{MA}}}$, lead to the following key step in the proof of Theorem 2.1.

Proposition 4.8 *We have*

$$
\langle \Phi_x^P, \operatorname{ind}_G(D) \rangle = \langle \Phi_x^{MA}, \operatorname{ind}_{MA}(D_{X_{MA}}) \rangle.
$$

5 Proof of the index theorem

Having established an analogue of commutativity of the top two diagrams in (2.10) with Proposition 4.8, we finish the proof of Theorem 2.1 by considering the bottom two diagrams in (2.10). We note that the bottom left part commutes in Lemma 5.2, and show that the bottom right part commutes in Lemmas 5.3 and 5.4. Then applying the main result of [24] to the action by M on X/AN finishes the proof.

5.1 Cococyles for *M* and *A*

Consider the cyclic l-cocycle $\tilde{\Phi}_x^M$ on $\mathcal{C}(M)$ given by

$$
\tilde{\Phi}_x^M(f_0^M, \ldots, f_l^M)
$$
$$
= \int_{M/Z} \int_{(K \cap M)^{l+1}} \int_{M^l}
$$
$$
f_0^M(k_0 h x h^{-1} m_1^{-1} a_1^{-1} k_1^{-1}) f_1^M(k_1 m_1 m_2^{-1} k_2^{-1}) \cdots
$$
$$
\cdots f_{l-1}^M(k_{l-1} m_{l-1} m_l^{-1} k_l^{-1}) f_l^M(k_l m_l k_0^{-1}) dm_1 \cdots dm_l \, dk_0 \cdots dk_l \, d(hZ).
$$

Note that $\tilde{\Phi}_x^M$ is not the same as the cocycle Φ_x^M for the cuspidal parabolic $M < M$ as defined in (2.5), because the latter is a degree zero cocycle. We do have the following fact.

Lemma 5.1 *Let $\mathcal{H}$ be a Hilbert space with a unitary representation of K. For all $q \in K_*((\mathcal{C}(M, \mathcal{L}^1(\mathcal{H})))^{K \times K})$,*

$$
\langle \tilde{\Phi}_x^M, q \rangle = \langle \Phi_x^M, q \rangle.
$$

Proof Let $q \in M_r((\mathcal{C}(M, \mathcal{L}^1(\mathcal{H})))^{K \times K})$ be an idempotent. Similarly to the last paragraph of the proof of Proposition 3.2, $K \times K$-invariance of q and the trace property of Tr imply that

$$
\langle \tilde{\Phi}_x^M, q \rangle = (\tilde{\Phi}_x^M \# \text{Tr})(q, \ldots, q) = \int_{M/Z} \int_{M^l} \text{Tr}\big(q(h x h^{-1} m_1^{-1} a_1^{-1}) q(m_1 m_2^{-1}) \cdots
$$
$$
\cdots q(m_{l-1} m_l^{-1}) q(m_l)\big) dm_1 \cdots dm_l \, d(hZ) = \int_{M/Z} \text{Tr}(q^{l+1}(h x h^{-1})) d(hZ).
$$

Because q is an idempotent, this equals

$$
\int_{M/Z} \text{Tr}(q(h x h^{-1})) d(hZ) = \langle \Phi_x^M, q \rangle.
$$

$\square$

Let $X_{MA}, S_Y, S_{\mathfrak{p} \cap \mathfrak{m}}$ and $S_{\mathfrak{k}/(\mathfrak{k} \cap \mathfrak{m})}$ be as above Lemma 4.5. Let $X_M := M \times_{K \cap M} Y \subset X_{MA}$. Then $X_{MA} = X_M \times A$, and $X_M \cong X/AN$. Consider the M-equivariant vector bundles

$$
S_{AN} = M \times_{K \cap M} (S_{\mathfrak{p} \cap \mathfrak{m}} \otimes S_Y) \to X_M \cong X/AN;
$$
$$
F_{AN} = M \times_{K \cap M} (Y \times S_{\mathfrak{k}/(\mathfrak{k} \cap \mathfrak{m})}) \to X_M \cong X/AN.
$$
(5.1)

Analogously to Lemma 4.4, S_{AN} is the spinor bundle of an M-equivariant Spin^c-structure on X/AN, and Lemma 4.3 and (3.8) imply that (2.6) holds.

Let D_A be the Spin-Dirac operator on A, acting on sections of the spinor bundle $A \times S_{\mathfrak{a}}$. Let $D_{M/(K \cap M)}$ be the Spin-Dirac operator on $M/(K \cap M)$, acting on sections of the spinor bundle $M \times_{K \cap M} S_{\mathfrak{p} \cap \mathfrak{m}}$. Let $D_{M,K \cap M}$ be its extension to an operator on $C^\infty(M) \otimes S_{\mathfrak{p} \cap \mathfrak{m}}$, defined analogously to the operator $D_{MA,K \cap M}$ in Sect. 4.2. Analogously to (4.10), consider the operator

$$D_{X_M} := D_{M,K \cap M} \otimes 1 + 1 \otimes D_{Y,M}$$

on the space of $K \cap M$-invariant elements of

$$C^\infty(M) \otimes S_{\mathfrak{p} \cap \mathfrak{m}} \otimes \Gamma^\infty(S_Y \otimes S_{\mathfrak{k}/(\mathfrak{k} \cap \mathfrak{m})} \otimes W|_Y),$$

which is the space of smooth sections of $S_{AN} \otimes F_{AN} \otimes W/AN$. Then D_{X_M} is a Spin^c-Dirac operator on X_M for the Spin^c-structure with spinor bundle S_{AN}, twisted by the vector bundle $W_{AN} = F_{AN} \otimes W/AN$.

Lemma 5.2 *We have*

$$\mathrm{ind}_{MA}(D_{X_{MA}}) = \mathrm{ind}_M(D_{X_M}) \otimes \mathrm{ind}_A(D_A).$$

On the right hand side, we used the exterior Kasparov product

$$KK(\mathbb{C}, C_r^* M) \times KK(\mathbb{C}, C_r^* A) \to KK(\mathbb{C}, C_r^*(MA)).$$

Proof We have

$$D_{X_{MA}} = D_{X_M} \otimes 1 + 1 \otimes D_A,$$

where we use graded tensor products. Hence the claim follows by Theorem 5.2 in [20].   □

Lemma 5.3

$$\langle \Phi_x^{MA}, \mathrm{ind}_{MA}(D_{X_{MA}}) \rangle = \langle \Phi_x^M, \mathrm{ind}_M(D_{X_M}) \rangle \langle \Phi_e^A, \mathrm{ind}_A(D_A) \rangle.$$

Proof Applying Lemma 3.1 and with G and P both replaced by MA, so that the integrals over K and N are not present, we find that Φ_x^{MA} decomposes as

$$\Phi_x^{MA} = \Phi_x^M \cup \Phi_e^A.$$

So by Lemma 5.2, the pairing $\langle \Phi_x^{MA}, \mathrm{ind}_{MA}(D_{X_{MA}}) \rangle$ equals

$$\langle \tilde{\Phi}_x^M, \mathrm{ind}_M(D_{X_M}) \rangle \langle \Phi_e^A, \mathrm{ind}_A(D_A) \rangle. \tag{5.2}$$

This follows from compatibility of the cup product on cyclic cohomology with the external Kasparov product, or from a direct computation involving the special case of Lemma 3.1 just mentioned. The claim then follows by Lemma 5.1.   □

Lemma 5.4 *We have*

$$\langle \Phi_e^A, \mathrm{ind}_A D_A \rangle = 1.$$

Proof This is a very special case of Theorem 4.6 in [34]. □

Lemma 5.5 *Let $D_{X \times \mathbb{R}}$ be the Spin^c-Dirac operator on $X \times \mathbb{R}$ for the Spin^c-structure with spinor bundle $S \times \mathbb{R}$, twisted by $W \times \mathbb{R}$. Then*

$$\langle \Phi_{(x,0)}^{P \times \mathbb{R}}, \mathrm{ind}_{G \times \mathbb{R}}(D_{X \times \mathbb{R}}) \rangle = \langle \Phi_x^P, \mathrm{ind}_G(D) \rangle.$$

Proof The operator $D_{X \times \mathbb{R}}$ equals

$$D_{X \times \mathbb{R}} = D \otimes 1 + 1 \otimes i \frac{d}{dt}$$

on $\Gamma^\infty(E \times \mathbb{R}) = \Gamma^\infty(E) \otimes C^\infty(\mathbb{R})$. So by Lemmas 5.3 and 5.4, and an analogue of Lemma 5.2,

$$\langle \Phi_{(x,0)}^{P \times \mathbb{R}}, \mathrm{ind}_{G \times \mathbb{R}}(D_{X \times \mathbb{R}}) \rangle = \langle \Phi_x^P, \mathrm{ind}_G(D) \rangle \left\langle \Phi_0^{\mathbb{R}}, \mathrm{ind}_{\mathbb{R}} \left(i \frac{d}{dt} \right) \right\rangle = \langle \Phi_x^P, \mathrm{ind}_G(D) \rangle.$$

□

5.2 Proofs of Theorem 2.1 and Lemma 2.3

Lemma 5.6 *The character of $S_{\mathfrak{k}/(\mathfrak{k} \cap \mathfrak{m})}$ restricted to $T \cap M$ equals*

$$\prod_\lambda (e^{\lambda/2} - e^{-\lambda/2}), \tag{5.3}$$

where λ runs over the $T \cap M$-weights of $\mathfrak{k}/(\mathfrak{k} \cap \mathfrak{m})$. If P is not a maximal cuspidal parabolic, then this equals zero.

Proof The first claim follows from Remark 2.2 in [33]. If P is not a maximal cuspidal parabolic, then $T \cap M$ is not a maximal torus in K. Hence there is a nonzero subspace of $\mathfrak{k}$ on which $T \cap M$ acts trivially. This means that the weight zero occurs in (5.3), which is therefore zero. □

Proof of Theorem 2.1 If X and G/K are even-dimensional, then applying Proposition 4.8 and Lemma 5.3, we see that

$$\langle \Phi_x^P, \mathrm{ind}_G(D) \rangle = \langle \Phi_x^{MA}, \mathrm{ind}_{MA}(D_{X_{MA}}) \rangle = \langle \Phi_e^A, \mathrm{ind}_A(D_A) \rangle \langle \Phi_x^M, \mathrm{ind}_M(D_{X_M}) \rangle. \tag{5.4}$$

Lemma 5.4 implies that the right hand side equals

$$\langle \Phi_x^M, \mathrm{ind}_M(D_{X_M}) \rangle. \tag{5.5}$$

Because M has a compact Cartan subgroup, Proposition 4.11 in [24] and the fact that X/AN is a smooth manifold equal to X_M imply that (5.5) equals the right hand side of (2.7) if x lies in a compact subgroup of M, and is zero otherwise. Here we use the fact that Φ_x^M equals the orbital integral trace τ_x used in [24]. Two points to note here are the following.

- The condition in Proposition 4.11 in [24] that x has 'finite Gaussian orbital integral' is in fact always satisfied, see Section 4.2 in [9].
- In [24], it was assumed for simplicity that the group acting is connected. The group M is not connected in general, but it belongs to Harish-Chandra's class, which is enough for the results in [24] to hold.

If P is not a maximal cuspidal parabolic, then Lemma 5.6 implies that the bundle F_{AN} in the second line of (5.1) is zero as a virtual vector bundle. So the virtual vector bundle W_{AN} is zero, and $\mathrm{ch}([W_{AN}|_{\mathrm{supp}(\chi_x)}](x))$ on the right hand side of (2.7) is zero as well.

By Lemma 5.5, the case where X and G/K are even-dimensional implies the case where X and G/K are odd-dimensional. If the dimensions of X and G/K have different parities (i.e. Y is odd-dimensional), then both sides of (2.7) are zero. □

Proof of Lemma 2.3 Recall the definition of the projection maps $q_A \colon X/N \to A$ and $q \colon X/N \to X/AN$ above Lemma 2.3. The decomposition $X/N = X/AN \times A$ implies that

$$
\begin{aligned}
\hat{A}(X/N) &= q^* \hat{A}(X/AN) q_A^* \hat{A}(A) = q^* \hat{A}(X/AN); \\
W_N &= q^* W_{AN}; \\
S_N &= q^* S_{AN} \otimes S_{\mathfrak{a}}.
\end{aligned}
$$

The third equality implies that $L_{\det}^N = q^* L_{\det}$.

Let P be a maximal cuspidal parabolic, and let $x \in T < K \cap M$. Then x acts trivially on A, so $(X/N)^x = (X/AN)^x \times A$. This implies that $\mathcal{N}_N = q^* \mathcal{N}$. We conclude that (2.9) equals

$$
\begin{aligned}
\int_{(X/N)^x} q^* &\left(\chi_x \frac{\hat{A}((X/AN)^x)\, \mathrm{ch}([W_{AN}|_{\mathrm{supp}(\chi_x)}](x)) e^{c_1(L_{\det}|_{(X/AN)^x})}}{\det(1 - xe^{-R^{\mathcal{N}}/2\pi i})^{1/2}} \right) \wedge q_A^*(\chi_A\, da) \\
&= \int_{(X/AN)^x} \chi_x \frac{\hat{A}((X/AN)^x)\, \mathrm{ch}([W_{AN}|_{\mathrm{supp}(\chi_x)}](x)) e^{c_1(L_{\det}|_{(X/AN)^x})}}{\det(1 - xe^{-R^{\mathcal{N}}/2\pi i})^{1/2}} \int_A \chi_A\, da,
\end{aligned}
$$

which equals the right hand side of (2.7). □

6 Extensions and special cases

6.1 General elliptic operators

By standard arguments, the index formula (2.7) for twisted Spin^c-Dirac operators implies an index formula for general elliptic operators. We sketch the argument here,

and refer to [8] for some details. Also compare this with the proof of Theorem 2.5 in [24].

Let $E \to X$ be any Hermitian G-vector bundle, and D a G-equivariant elliptic differential operator on E. Let ΣX be the manifold obtained from X by gluing two copies of the unit ball bundle in TX along the unit sphere bundle. This has a natural G-invariant almost complex structure. Let $p^{\Sigma X} \colon \Sigma X \to X$ be the projection map. There is a Hermitian G-vector bundle $V_D \to \Sigma X$ defined in terms of the principal symbol of D, such that the Spin^c-Dirac operator $D_{\Sigma X}^{V_D}$ on ΣX (for the Spin^c-structure defined by the almost complex structure) twisted by V_D satisfies

$$[D] = p_*^{\Sigma X}[D_{\Sigma X}^{V_D}] \in K_0^G(X),$$

see Theorem 5.0.4 in [8]. Naturality of the analytic assembly map implies that

$$\mathrm{ind}_G(D) = \mathrm{ind}_G(D_{\Sigma X}^{V_D}),$$

and Theorem 2.1 is a topological expression for the pairing of the index on the right hand side with Φ_x^P.

6.2 A higher L^2-index theorem

The case of Theorem 2.1 where $x = e$ is a higher version of Wang's L^2-index theorem, Theorem 6.10 in [38], in the case of linear reductive Lie groups. This is the equality

$$\langle \Phi_e^P, \mathrm{ind}_G(D) \rangle = \int_{X/AN} \chi_e^{AN} \hat{A}(X/AN) \, \mathrm{ch}([W_{AN}|_{\mathrm{supp}(\chi_e)}]) e^{c_1(L_{\det})}. \quad (6.1)$$

This implies a higher version of Connes and Moscovici's L^2-index theorem on homogeneous spaces, Theorem 5.2 in [13].

Let $H < K \cap M$ be a closed subgroup. On page 309 of [13], a Chern character

$$\mathrm{ch} \colon R(H) \to H^*(\mathfrak{m}, H)$$

is defined, where $H^*(\mathfrak{m}, H)$ denotes relative Lie algebra cohomology. Furthermore, an $\hat{A}$-class $\hat{A}(\mathfrak{m}, H) \in H^*(\mathfrak{m}, H)$ is constructed there from the representation $\mathfrak{m}/\mathfrak{h}$ of H. Let V be a finite-dimensional virtual representation of H. Suppose, for simplicity, that G/H has a G-invariant Spin-structure. (This assumption may be dropped as described on page 307 of [13].) Let $D_{G/H}^V$ be the Spin-Dirac operator on G/H coupled to $G \times_H V \to G/H$.

Corollary 6.1 *We have*

$$\langle \Phi_e^P, \mathrm{ind}_G(D_{G/H}^V) \rangle = \int_{\mathfrak{m}/\mathfrak{h}} \hat{A}(\mathfrak{m}, H) \, \mathrm{ch}(V \otimes S_{\mathfrak{k}/(\mathfrak{k} \cap \mathfrak{m})}). \quad (6.2)$$

Proof It is shown in Corollary 6.14 and Remark 6.15 in [38] that if $X = G/H$ and $D = D^V_{G/H}$, the right hand side of (6.1) equals the right hand side of (6.2). □

If G has a compact Cartan subgroup, then $M = G$, and Corollary 6.1 reduces to Theorem 5.2 in [13]. If G does not have a compact Cartan subgroup, then both sides of the equality in Theorem 5.2 in [13] equal zero by Theorem 6.1 in [13] and Harish-Chandra's criterion rank(G) = rank(K) for existence of discrete series represenrations; see also (1.2.5) in [6]. Corollary 6.1 is a generalisation of Theorem 5.2 in [13] that gives a nontrivial result even when G does not have a compact Cartan subgroup; see also Sect. 6.5.

6.3 Dirac induction

We assume that G/K is equivariantly Spin for simplicity. As before, let $R(K)$ be the representation ring of K. The Dirac induction map

$$\text{D-Ind}^G_K \colon R(K) \to K_*(C^*_r(G))$$

from the Connes–Kasparov conjecture maps an irreducible representation V of K to $\text{ind}_G(D^V_{G/K})$, where $D^V_{G/K}$ is the Spin-Dirac operator on G/K coupled to $G \times_K V \to G/K$. In this case, Theorem 2.1 implies that

$$\langle \Phi^P_x, \text{ind}_G(D^V_{G/K}) \rangle = (-1)^{\dim(M/(K \cap M))/2} \frac{\chi_V(x) \chi_{S_{\mathfrak{k}/(\mathfrak{k} \cap \mathfrak{m})}}(x)}{\chi_{S_{\mathfrak{p} \cap \mathfrak{m}}}(x)}, \qquad (6.3)$$

where the letter χ denotes the character of a representation. This can be deduced directly from Theorem 2.1 as in the proof of Theorem 3.1 in [25], but the easiest was to deduce this equality is to use the equalities (5.4) and then apply Theorem 3.1 in [25] directly.

The following fact was deduced from a fixed-point formula as Corollary 4.5 in [25], in the case where G has a compact Cartan subgroup. We now generalise this argument to arbitrary linear reductive groups.

Corollary 6.2 *Dirac induction for G is injective.*

Proof Let $y \in R(K)$, and suppose that D-Ind$^G_K(y) = 0$. As in the proof of Corollary 4.5 in [25], it follows from (6.3) that the character of y is zero on T. Because T is a maximal torus in G, it follows that $y = 0$. □

Another consequence of (6.3) is that pairing with Φ^P_x detects all information about classes in $K_*(C^*_r(G))$ for maximal P, and no information for non-maximal P.

Corollary 6.3 *If P is a maximal cuspidal parabolic subgroup, then pairing with Φ^P_x for regular $x \in K \cap M$ separates points in $K_*(C^*_r(G))$. If P is not a maximal cuspidal parabolic subgroup, then the pairing of Φ^P_x with any class in $K_*(C^*_r(G))$ is zero.*

Proof The first claim follows from (6.3) as in Corollary 4.1 in [25]. Here one uses surjectivity of Dirac induction. For the second claim, we use the fact that $S_{\mathfrak{k}/(\mathfrak{k} \cap \mathfrak{m})}$ is

zero as a virtual representation of $K \cap M$, if P is not maximal (see Lemma 5.6). By surjectivity of Dirac induction, (6.3) implies that the pairing of Φ_x^P with any class in $K_*(C_r^*(G)) = K^*(\mathcal{C}(G))$ is zero. $\qquad\square$

The first part of Corollary 6.3 means that Theorem 2.1 is a complete topological description of $\mathrm{ind}_G(D)$, with no loss of information.

Remark 6.4 We expect that for non-maximal P, the class of Φ_x^P in the cyclic cohomology of $\mathcal{C}(G)$ is zero. This is a stronger statement than the second part of Corollary 6.3. Proving the vanishing of these cohomology classses will likely involve detailed information on the structure of $\mathcal{C}(G)$ in terms of representation theory, as obtained in [2,11]. Vanishing of the pairings of these classes with elements of $K_*(C_r^*(G))$, as in Corollary 6.3, follows directly from Theorem 2.1, and does not require knowledge of the precise structure of $\mathcal{C}(G)$.

In Definition 5.3 in [37], a generator Q_V of $K_*(C_r^*(G))$ is defined independently of Dirac induction. Theorem 5.4 in [37] implies that

$$\langle \Phi_x^P, Q_V \rangle = (-1)^{\dim(A)} \frac{\chi_V(x)\Delta_T^K(x)}{\chi_{S_{\mathfrak{p} \cap \mathfrak{m}}}(x)\Delta_T^{K \cap M}(x)}, \tag{6.4}$$

where Δ_T^K and $\Delta_T^{K \cap M}$ are Weyl denominators for choices of positive roots for (K, T) and $(K \cap M, T)$, respectively. (Note that $T \subset M$ for a maximal parabolic.) The character of $S_{\mathfrak{k}/(\mathfrak{k} \cap \mathfrak{m})}$ restricted to $T \cap M$ equals (5.3), which also equals $\frac{\Delta_T^K}{\Delta_T^{K \cap M}}$. Because of this equality and (6.3) and (6.4), we find that for regular $x \in K \cap M$,

$$\langle \Phi_x^P, \mathrm{ind}_G(D_{G/K}^V) \rangle = (-1)^{\dim(M/(K \cap M))/2 + \dim(A)} \langle \Phi_x^P, Q_V \rangle.$$

If P is a maximal cuspidal parabolic, then the first part of Corollary 6.3 allows us to deduce that

$$\mathrm{ind}_G(D_{G/K}^V) = (-1)^{\dim(M/(K \cap M))/2 + \dim(A)} Q_V,$$

i.e. the generators defined by Dirac induction equal the generators Q_V, up to a sign.

6.4 Non-cuspidal parabolics

If $P < G$ is a parabolic subgroup but not cuspidal, then the pairing on the left hand side of (2.7) is still defined. And as in (5.4) and (5.5), this pairing equals $\langle \Phi_x^M, \mathrm{ind}_M(D_{X_M}) \rangle$. Because Φ_x^M is now the orbital integral trace on $\mathcal{C}(M)$, Theorem 3.2(b) in [25] implies that this pairing is zero. Hence for non-cuspidal P, the left hand side of (2.7) is zero. As in the proof of Corollary 6.3, this fact and surjectivity of Dirac induction imply that pairing with Φ_x^P is the zero map on K-theory.

Now suppose that G has discrete series representations. Consider the parabolic subgroup $P = G < G$. Then, because Φ_x^G is the orbital integral trace, Proposition

5.1 in [24] implies that for all regular $x \in T$, the pairing of Φ_x^G with the K-theory class of a discrete series representation π is the value of the character of π at x, and in particular nonzero for some x. By the previous paragraph, this implies that G is a cuspidal parabolic subgroup of itself. (Here we do not need the full surjectivity of Dirac induction, just the fact that discrete series classes can be realised by Dirac induction.) And that means that G has a compact Cartan subgroup. Thus we recover one half of Harish-Chandra's result that G has a discrete series if and only if it has a compact Cartan. (This could also have been deduced directly from Theorem 3.2(b) in [25].)

The authors have not checked in detail if this result by Harish-Chandra was used in the proofs of any of the results from representation theory that we have applied, or in the proofs of any of the results we refer to. So we do not claim here that our proof of necessity of existence of a compact Cartan for the existence of the discrete series is independent of Harish-Chandra's. But at least this is a possibly interesting illustration of the links of our results to representation theory.

6.5 Complex semisimple groups

Now suppose that G is a complex semisimple group. Then the maximal parabolic subgroup P is also minimal, i.e. a Borel subgroup. It is a classical fact that now $K/T \cong G/P$, as complex manifolds. And $M = T$, so we have an M-equivariant diffeomorphism

$$X/AN = X_M = M \times_{K \cap M} Y = Y.$$

So by Proposition 4.8 and Lemmas 5.3 and 5.4,

$$\langle \Phi_x^P, \operatorname{ind}_G(D) \rangle = \operatorname{ind}_T(D_Y)(x). \tag{6.5}$$

The right hand side is the classical T-equivariant index of D_Y evaluated at x. This is given by the Atiyah–Segal–Singer fixed point formula [4,5].

As an example, suppose that $D = D_{G/H}^V$ as in (6.2). Then (6.5) implies that (6.2) becomes

$$\langle \Phi_e^P, \operatorname{ind}_G(D_{G/H}^V) \rangle = \operatorname{ind}(D_{K/H}^V),$$

for a Dirac operator $D_{K/H}^V$ on K/H coupled to $K \times_H V$. The right hand side is nonzero in many cases, for example if $H = T$, and V is such that $\operatorname{ind}_K(D_{K/T}^V)$ is an irreducible representation of K by the Borel–Weil–Bott construction. Then $\operatorname{ind}(D_{K/H}^V)$ is the dimension of this representation, up to a sign. This shows that Corollary 6.1 is nontrivial, also in the non-equal rank case.

Example 6.5 Suppose that $G = \operatorname{SL}(2, \mathbb{C})$, $K = \operatorname{SU}(2)$ and $H = T = \operatorname{U}(1)$. Let $D_{G/T}$ be the Dolbeault–Dirac operator on G/T coupled to $G \times_T \mathbb{C}_n$, where $\mathbb{C}_n$ is the representation of $\operatorname{U}(1)$ in $\mathbb{C}$ with weight $n \in \mathbb{Z}_{\geq 0}$. Then for regular $x = e^{i\alpha} \in T$, the

Borel–Weil theorem and (6.5) imply that

$$\langle \Phi_x^P, \mathrm{ind}_G(D_{G/T}) \rangle = \frac{\sin((n+1)\alpha)}{\sin(\alpha)},$$

the character of the irreducible representation of SU(2) of dimension $n+1$, evaluated at x.

7 Notation

The numbers in brackets indicate on what page each symbol was introduced.

- Groups: G (5), K (5), P (5), M (5), A (5), N (5), Z (6), T (8);
- Manifolds: X (5), Y (14), X_{MA} (21), X_M (29);
- Clifford modules and Spin-representations: S (5), S_{AN} (7), S_Y (14), $S_{\mathfrak{p}}$ (14), $S_{\mathfrak{a}}$ (24), $S_{\mathfrak{p} \cap \mathfrak{m}}$ (24), $S_{\mathfrak{k}/(\mathfrak{k} \cap \mathfrak{m})}$ (24), S_N (24);
- Vector bundles: W (5), E (5), W_{AN} (8), F_{AN} (8), $\mathcal{N}$ (8), $L_{\det}$ (8), $L_{\det}^N$ (9), W_N (9), F_N (24);
- Dirac operators: D (5), $D_{G,K}$ (15), D_Y (15), $D_{X_{MA}}$ (25), $D_{MA/(K \cap M)}$ (25), $D_{MA,K \cap M}$ (25), $D_{Y,M}$ (25), D_{X_M} (29), $D_{M/(K \cap M)}$ (29), $D_{M,K \cap M}$ (29);
- Cyclic cocycles: Φ_x^P (6), $\tilde{\Phi}_x^M$ (28);
- Functions and maps: H (5), $\mathcal{C}(G)$ (6), χ_x (7), χ_A (9), q (9), q_A (9), f^N (12), κ^N (21);
- Miscellaneous: l (5), x (6), q_t (20), p_E (20).

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