2020 American Control Conference
Denver, CO, USA, July 1-3, 2020

A Backstepping Approach to System Level Synthesis for Spatially-Invariant Systems

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***Abstract*— We consider the controller design problem for infinite-extent spatially-invariant systems composed of n^{th}-order subsystems, generalizing recent work on the special case of 1^{st}-order subsystems. We provide a parameterization of all internally stabilizing state-feedback controllers for general n^{th}-order finite-dimensional systems, and extend this result to the infinite-extent spatially-invariant setting. We apply our results to the vehicle consensus problem. We demonstrate, through this example, that the $\mathcal{H}_2$ problem for infinite-dimensional spatially-invariant systems can be formulated as a standard model-matching problem with *finitely* many transfer function parameters, when constraints on the spatial spread of the closed-loop responses are imposed. The number of transfer function parameters scales linearly with the amount of spatial spread permitted in the closed-loop mappings. Numerical results are provided.**

I. Introduction

The optimal controller design problem subject to structural constraints has long been studied in the context of distributed systems. Structural constraints can be imposed on the controller transfer matrix in a convex manner in specific problem settings, e.g. funnel causal [1] and quadratically invariant [2]. However, the structured controller design problem is non-convex and challenging to solve in more general settings. For example, the optimal controller design problem subject to a banded structure constraint on a controller transfer matrix, for a fully connected plant, is non-convex.

The recent work of System Level Synthesis (SLS) [3] suggested an alternative method for enforcing subcontroller communication constraints. A parameterization of all stabilizing controllers for a plant was provided in terms of an affine subspace constraint on the resulting closed-loop mappings; structural constraints may be imposed on these closed-loop mappings in a convex manner. Although this is not the same as imposing structure on the controller itself, it was shown that any controller resulting in sparse closed-loop responses can be implemented with limited *local* communication between subcontroller units [3].

In this paper, we follow the System Level approach, and consider the case of infinite-extent spatially-invariant systems [4]. Although most physical systems have finite spatial extent, the infinite-spatial-extent setting is often a useful idealization of the large-but-finite setting. In addition, this setting may allow for analytic solutions which can provide intuition about more general problem settings, e.g. providing insight to issues that arise in the finite vehicle platoons problem as the number of vehicles increases [6]. Recent work has extended SLS-like results to this setting [5], but these results are restrictive to distributed systems composed of 1^{st}-order subsystems. In particular, the proof techniques of [5] cannot be extended to problem settings for which the mapping from control action to state is not invertible. The vehicular platoon problem, which can be modeled as a chain of 2^{nd}-order subsystems, provides one example with this structural property.

Our main result employs a backstepping procedure [8] to extend the results of [5] to systems with higher-order dynamics. We provide a parameterization of all internally stabilizing controllers for general finite-dimensional systems, and extend these results to the infinite-extent spatially-invariant setting. We apply our results to the vehicle consensus problem; we demonstrate that, when spatial sparsity constraints on the closed loops are imposed, the infinite-extent $\mathcal{H}_2$ problem may be converted to a standard model-matching problem with *finitely* many transfer function parameters. The resulting controller has a spatially localized implementation.

The rest of this paper is structured as follows. A parameterization of all internally stabilizing controllers for general finite-dimensional systems is provided in Section II. Section III provides a framework for the infinite-dimensional setting. In Section IV, we extend the finite-dimensional results to infinite-extent spatially-invariant systems. The vehicle consensus problem is analyzed in Section V.

A. Notation & Finite-Dimensional System Preliminaries

We let I_k denote the $k \times k$ identity matrix and 0_k denote the $k \times k$ matrix of all zeros. For simplicity of notation, we often omit the subscripts when the dimensions of these matrices are clear from context.

A transfer function G is *strictly proper* if $\lim_{s\to\infty} G(s) = 0$, and G is *stable* if it has no poles in $\{\text{Re}(s) \geq 0\}$. We denote the set of all stable and strictly proper transfer functions by $\mathcal{RH}_2$, and equip this space with the $\mathcal{H}_2$ norm, defined by

$$\|G\|^2_{\mathcal{H}_2} := tr\left(\int_{-\infty}^{\infty} G(j\omega)^T G(j\omega) d\omega\right),$$

for all $G \in \mathcal{RH}_2$. An LTI state-feedback controller, $u = Kx$, is *internally stabilizing* for G if all closed-loop transfer functions resulting from G in feedback with K are elements of $\mathcal{RH}_2$.

This work is partially supported by NSF awards ECCS-1932777 and CMI-1763064.

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II. A Backstepping Approach for Finite-Dimensional System Level Synthesis

We consider finite-dimensional LTI systems of the form

$$\dot{x} = Ax + B_1 w + B_2 u, \qquad (1)$$

where $x(t) \in \mathbb{R}^N, u(t) \in \mathbb{R}^k, w(t) \in \mathbb{R}^p$ are the finite-dimensional state, control action, and exogenous disturbance, respectively. Following the SLS framework, given a (dynamic or static) controller $u = Kx$, we define the closed-loop mappings, Φ^x and Φ^u, from disturbance to state and control action as

$$\begin{aligned} \begin{bmatrix} x \\ u \end{bmatrix} &= \begin{bmatrix} \Phi^x \\ \Phi^u \end{bmatrix} B_1 w \\ &:= \begin{bmatrix} (sI - A - B_2 K(s))^{-1} \\ K(s)(sI - A - B_2 K(s))^{-1} \end{bmatrix} B_1 w. \end{aligned} \qquad (2)$$

A. Parameterization of Stabilizing Controllers

We assume (A, B_2) is controllable. Then, it can be shown that without loss of generality, $(A + I_N, B_2)$ is in *controllable-canonical form* [7], i.e of the form

$$(A + I_N) = \begin{bmatrix} -a_1 I_k & -a_2 I_k & -a_3 I_k & \cdots & -a_n I_k \\ I_k & 0_k & 0_k & \cdots & 0_k \\ 0_k & I_k & 0_k & \cdots & 0_k \\ \vdots & & \ddots & & \\ 0_k & & & I_k & 0_k \end{bmatrix},$$
$$B_2 = \begin{bmatrix} I_k & 0_k & 0_k & \cdots & 0_k \end{bmatrix}^T, \qquad (3)$$

for some real-valued coefficients $a_1, ..., a_n$, where n is defined to be the order of the system. Here, A is of dimension $N \times N$ and B_2 is of dimension $N \times k$. We note that this prescribed form (3) is somewhat nontraditional, and is chosen to simplify the main result presented in Theorem 2.2.

The main result of this section leverages SLS results to provide an *explicit* parameterization of all such closed-loop mappings, which result from stabilizing state-feedback controllers in feedback with systems of the form (3).

We begin by analyzing the specific case of 2nd-order systems ($n = 2$). In this case, the plant of interest is of the form

$$\dot{x} = Ax + B_1 w + B_2 u,$$
$$(A + I) = \begin{bmatrix} -a_1 I & -a_2 I \\ I & 0 \end{bmatrix}, \; B_2 = \begin{bmatrix} I & 0 \end{bmatrix}^T, \qquad (4)$$

Theorem 2.1: If $u = Kx$ is an internally stabilizing controller for the 2nd-order plant (4), then the resulting closed-loop mappings, defined by (2), are of the form

$$\begin{aligned} \Phi^x &= \begin{bmatrix} \frac{1}{s+1} I \\ \frac{1}{(s+1)^2} I \end{bmatrix} \begin{bmatrix} \theta_1 & \theta_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{s+1} I & 0 \\ \frac{1}{s+1} I & \frac{1}{(s+1)^2} I \end{bmatrix} \\ \Phi^u &= \frac{1}{(s+1)^2} \chi(s) \begin{bmatrix} \theta_1 & \theta_2 \end{bmatrix} ... \\ &\quad + \begin{bmatrix} \frac{1}{(s+1)^2}\chi(s) - I & \frac{1}{s+1} I \end{bmatrix}, \end{aligned} \qquad (5)$$

for some $\theta_1, \theta_2 \in \mathcal{RH}_2$, where

$$\chi(s) := (s+1) \cdot \left((s+1) + a_1 + \frac{a_2}{s+1} \right) I$$

with a_1, a_2 defined in (4). Conversely, if Φ^x and Φ^u are of the form (5) for some $\theta_1, \theta_2 \in \mathcal{RH}_2$, then these closed-loop maps are achieved by the internally stabilizing controller

$$u = Kx := \Phi^u \left(\Phi^x\right)^{-1} x.$$

A proof of Theorem 2.1 is presented in the Appendix. Theorem 2.1 is also a direct corollary of the following more general result.

Theorem 2.2: Let $u = Kx$ be an internally stabilizing controller for the n^{th}-order system (3), then the resulting closed-loop mappings are of the form

$$\Phi^x = \begin{bmatrix} \phi_1^x \\ \phi_2^x \\ \phi_3^x \\ \vdots \\ \phi_n^x \end{bmatrix} = \begin{bmatrix} I \\ \frac{1}{s+1} I \\ \frac{1}{(s+1)^2} I \\ \vdots \\ \frac{1}{(s+1)^n} I \end{bmatrix} \Theta(s) + \Lambda(s) \qquad (6)$$

$$\Phi^u = \chi\Theta(s) + \begin{bmatrix} -I & \frac{1}{s+1} I & \frac{1}{(s+1)^2} I & \cdots & \frac{1}{(s+1)^{n-1}} I \end{bmatrix},$$

for some Θ of the form $\Theta(s) := \frac{1}{s+1} \begin{bmatrix} (\theta_1 + I) & \theta_2 & \theta_3 & \cdots & \theta_n \end{bmatrix}$ with $\theta_1, ..., \theta_n \in \mathcal{RH}_2$, where we have partitioned Φ^x by block rows as

$$\Phi^x = \begin{bmatrix} \Phi_{11}^x & \Phi_{12}^x & \cdots & \Phi_{1n}^x \\ \Phi_{21}^x & \Phi_{22}^x & \cdots & \Phi_{2n}^x \\ \vdots & & & \\ \Phi_{n1}^x & \Phi_{n2}^x & \cdots & \Phi_{nn}^x \end{bmatrix} =: \begin{bmatrix} \phi_1^x \\ \phi_2^x \\ \vdots \\ \phi_n^x \end{bmatrix}$$

The transfer function parameters in (6) are defined as

$$\Lambda(s) := \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{s+1} I & 0 & \cdots & 0 \\ 0 & \frac{1}{(s+1)^2} I & \frac{1}{s+1} I & \cdots & 0 \\ \vdots & & & \ddots & \\ 0 & \frac{1}{(s+1)^{n-1}} I & \frac{1}{(s+1)^{n-2}} I & \cdots & \frac{1}{s+1} I \end{bmatrix},$$

$$\chi(s) := (s+1) + a_1 + \frac{a_2}{s+1} + \frac{a_3}{(s+1)^2} + ... + \frac{a_n}{(s+1)^{n-1}}$$

with $a_1, ..., a_n$ defined in (3). Conversely, if Φ^x and Φ^u are of the form (6), then they are achieved by the internally stabilizing controller $u = Kx := \Phi^u \left(\Phi^x\right)^{-1} x$.

Theorem 2.2 follows from a procedure similar to the backstepping approach for strict feedback systems presented in [8]. We note that similar techniques have been applied in e.g. [9]. A proof of Theorem 2.2 is presented in [10].

We next consider the infinite-extent spatially-invariant setting.

III. Infinite-Dimensional System Preliminaries and Notation

We consider infinite-extent spatially distributed systems, where the state and all external signals are functions of a spatial variable, $n \in \mathbb{Z}$, and a temporal variable, $t \in \mathbb{R}^+ := [0, \infty)$. We denote such *spatio-temporal signals* using lower-case letters:

$$x = x_n(t) = x(n, t),$$

and denote the (temporal) Laplace transform of spatio-temporal signals (transfer functions) using upper-case letters:

$$X = X_n(s) = X(n, s).$$

Note that this transform is taken only in the temporal variable, the spatial variable remains in the original domain.

Square integrable spatio-temporal signals x are elements of $L^2(\mathbb{Z} \times \mathbb{R}^+)$, i.e.

$$\|x\|_{L^2}^2 := \sum_{n\in\mathbb{Z}} \int_{t=0}^{\infty} x_n^*(t) x_n(t) dt < \infty.$$

Definition 3.1: To each $n \in \mathbb{Z}$, we associate the *spatial translation operator* $T_n : L^2(\mathbb{Z}\times\mathbb{R}^+) \to L^2(\mathbb{Z}\times\mathbb{R}^+)$ defined by

$$T_n : x_m(\cdot) \mapsto x_{m-n}(\cdot).$$

An operator A on $L^2(\mathbb{Z} \times \mathbb{R}^+)$ is *translation invariant* if A commutes with all spatial translations, i.e. for all $n \in \mathbb{Z}$, $AT_n = T_n A$.

We introduce the following translation invariant operators:

- The *differentiation operator* $\mathcal{S}$ is defined by

$$\mathcal{S} x_n(t) = \frac{d}{dt} x_n(t),$$

  and is represented in the transfer function domain as

$$\mathcal{S} X_n(s) := s \cdot X_n(s).$$

  For compact notation, we define the operator

$$\Psi := \mathcal{S} + I.$$

- A *spatially-invariant system* C is defined to be a spatio-temporal convolution operator, i.e.

$$\begin{aligned}(Cx)_n(t) &:= (c * x)_n(t) \\ &:= \sum_{m\in\mathbb{Z}} \int_{\tau=0}^{\infty} c_m(\tau) x_{n-m}(t-\tau) d\tau.\end{aligned}$$

  We refer to $c = c_n(t)$ as the *spatio-temporal impulse response* of C. For each $n \in \mathbb{Z}$, $t \in \mathbb{R}^+$, $c_n(t)$ is a real-valued finite-dimensional matrix and $x_n(t)$ a real-valued finite-dimensional vector. Special cases of spatio-temporal convolution operators include:

  • A *purely spatial convolution operator*, A, is of the form

$$(Ax)_n(t) := \sum_{m\in\mathbb{Z}} a_m x_{m-n}(t),$$

  for some matrix-valued sequence $\{a_m\}$,

  • A *pointwise multiplication operator,* A is of the form

$$(Ax)_n(t) := a x_n(t),$$

  for some constant matrix a.

A *spatially-invariant system* K can be represented by a spatial convolution in the transfer function domain, for each fixed frequency s, i.e.

$$(CX)_n(s) = \sum_{m\in\mathbb{Z}} C_m(s) X_{n-m}(s),$$

where we denote the Laplace transform (transfer function) of the impulse response of system C by $C(s)$.

A. Localized Systems

Definition 3.2: For a finite integer N, we define a spatio-temporal signal x to be *local* with spatial extent N if

$$x_n(t) \equiv 0, \quad \text{for all } |n| > N,$$

Equivalently, its transfer function representation satisfies

$$X_n(s) \equiv 0, \quad \text{for all } |n| > N.$$

A spatially-invariant system K is said to be *local* with spatial extent N if it is of the form

$$Y_n(s) = \sum_{|m|\le N} K_m(s) U_{n-m}(s),$$

where we use $K(s)$ to refer to the transfer function representation of the spatially-invariant system K.

A spatially-invariant system K is said to be an element of $\mathcal{RH}_2$ if

$$K_n(s) \in \mathcal{RH}_2 \text{ for all } n \in \mathbb{Z}.$$

The $\mathcal{H}_2$ norm can be extended to spatially-invariant systems $\in \mathcal{RH}_2$, which are local with spatial extent N, as

$$\|K\|_{\mathcal{H}_2}^2 := \sum_{|n|\le N} \|K_n(s)\|_{\mathcal{H}_2}^2$$

IV. A Backstepping Approach to System Level Synthesis for Spatially-Invariant Systems

We extend the results of Section II to the infinite-extent spatially-invariant setting. For simplicity of exposition, we restrict our attention to the special case of 2nd-order subsystems, which applies to the vehicle consensus problem. Specifically, the plant model we consider is an infinite chain of 2nd-order subsystems with dynamics at each spatial location $n \in \mathbb{Z}$ given by

$$\dot{x}_n = A x_n + B_1 w_n + B_2 u_n, \tag{7}$$

with (A, B_2) controllable, so that without loss of generality $(A + I), B_2$ are of the form of the form

$$(A+I) = \begin{bmatrix} -a_1 I & -a_2 I \\ I & 0 \end{bmatrix}, \; B_2 = \begin{bmatrix} I & 0 \end{bmatrix}^T \tag{8}$$

Remark 1: We note that the subsystem dynamics are identical and *decoupled*. The setting of finite dimensional distributed systems composed of subsystems with *heterogenous* dynamics is analyzed in [10]. The case of *coupled* subsystem dynamics is analyzed for special cases in [5], and deriving more general results for the setting of coupled subsystem dynamics is the subject of current work.

Equivalently, we write the dynamics of system (7) in operator form as

$$\begin{aligned}\mathcal{S}x &= Ax + B_1 w + B_2 w \\ y &= \begin{bmatrix} C \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ \gamma I \end{bmatrix} u,\end{aligned} \tag{9}$$

where with slight abuse of notation we use A, B_1, B_2 to denote finite-dimensional matrices as well as the pointwise multiplication operators defined by these matrices. y is the performance output of interest, where $C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$ and C_1, C_2 are spatial convolution operators that are local with finite spatial extent.

Motivated by SLS, given a spatially-invariant controller $u = Kx$, we define the resulting closed-loop mappings as[1]:

$$\begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} \Phi^x \\ \Phi^u \end{bmatrix} B_1 w := \begin{bmatrix} (\mathcal{S} - A - B_2 K)^{-1} \\ K(\mathcal{S} - A - B_2 K)^{-1} \end{bmatrix} B_1 w, \tag{10}$$

where $B_2 K$ is a well-defined spatially-invariant system, given by the composition of the pointwise multiplication operator B_2 and the spatially-invariant system K. Φ^x and Φ^u, defined in (10), are spatially-invariant systems if and only if K is, and an application of the results of [5] proves the following result.

Lemma 4.1: The spatially-invariant controller, $u = Kx$, is internally stabilizing for (9) if and only if the resulting closed loops are elements of $\mathcal{RH}_2$ and satisfy

$$\begin{bmatrix} \Psi - (A + I) & -B_2 \end{bmatrix} \begin{bmatrix} \Phi^x \\ \Phi^u \end{bmatrix} = I \tag{11}$$

If $\Phi^x, \Phi^u \in \mathcal{RH}_2$ satisfy (11), then they are achieved by the stabilizing controller $u = Kx = \Phi^u (\Phi^x)^{-1} x$.

We apply this lemma to prove the following result, which is an infinite-dimensional analogue of Theorem 2.1.

Theorem 4.2: If $u = Kx$ is a spatially-invariant, internally stabilizing controller for (9), then the corresponding closed-loop maps, Φ^x and Φ^u, are of the form

$$\begin{aligned} \Phi^x &= \begin{bmatrix} \Psi^{-1} \\ \Psi^{-2} \end{bmatrix} \begin{bmatrix} \theta_1 & \theta_2 \end{bmatrix} + \begin{bmatrix} \Psi^{-1} & 0 \\ \Psi^{-1} & \Psi^{-2} \end{bmatrix} \\ \Phi^u &= \Psi^{-2} \chi \begin{bmatrix} \theta_1 & \theta_2 \end{bmatrix} + \begin{bmatrix} \Psi^{-2}\chi - I & \Psi^{-1} \end{bmatrix}, \end{aligned} \tag{12}$$

for some $\theta_1, \theta_2 \in \mathcal{RH}_2$, where

$$\chi := \Psi^2 + a_1 \Psi + a_2 I$$

with a_1, a_2 defined in (4). Conversely, if Φ^x and Φ^u are of the form (12), then they are achieved by the internally stabilizing, spatially-invariant controller $u = Kx := \Phi^u (\Phi^x)^{-1} x$.

Proof: See Appendix.

A. Optimal Controller Design

The $\mathcal{H}_2$ design problem for system (9) may be written as:

$$\begin{aligned} &\inf_K \left\| \begin{bmatrix} C & 0 \\ 0 & \gamma I \end{bmatrix} \begin{bmatrix} \Phi^x \\ \Phi^u \end{bmatrix} \mathcal{B}_1 \right\|_{\mathcal{H}_2}^2 \\ &\text{s.t.} \quad K \text{ stabilizing} \\ &= \inf_{\theta_1, \theta_2 \in \mathcal{RH}_2} \left\| \left(\begin{bmatrix} C_1 \Psi^{-1} + C_2 \Psi^{-2} \\ \gamma \Psi^{-2} \chi \end{bmatrix} \begin{bmatrix} \theta_1 & \theta_2 \end{bmatrix} + \ldots \right.\right. \\ &\qquad \left.\left. \begin{bmatrix} C_1 \Psi^{-1} + C_2 \Psi^{-2} & C_2 \Psi^{-2} \\ \gamma \left(\Psi^{-2} \chi - I \right) & \gamma \Psi^{-1} \end{bmatrix} \right) B_1 \right\|_{\mathcal{H}_2}^2 \end{aligned} \tag{13}$$

where the equality follows from Theorem 4.2.

[1]A common assumption to ensure these inverse operators are well-defined is that the spatially-invariant operator $(\mathcal{S} - A - B_2 K)$ defines a C_0-semigroup of operators, for a full exposition of infinite-dimensional systems theory, we refer the reader to e.g. [11].

The following lemma, whose proof follows from definition (12), allows us to easily impose locality constraints on the closed-loop maps using formulation (13).

Lemma 4.3: The closed-loop mappings Φ^x, Φ^u are local with spatial extent N if and only if θ_1, θ_2 defined in (12) are local with spatial extent N.

In Section V, we will demonstrate, through an example, that when locality constraints are imposed on the closed loops, this infinite-dimensional $\mathcal{H}_2$ problem (13) may be formulated as a standard model-matching problem with *finitely* many transfer function parameters. The corresponding controller has the following *local* implementation [5]

$$\begin{aligned} u &= \mathcal{S} \Phi^u (x - \tilde{x}) \\ \tilde{x} &= (\mathcal{S} \Phi^x - I)(x - \tilde{x}). \end{aligned} \tag{14}$$

V. Application: Vehicle Consensus with Locality Constraints

We consider the problem of consensus of an infinite chain of vehicles. Following [6], we model each vehicle in the platoon as a double integrator

$$\ddot{\xi}_n = u_n + w_n, \quad n \in \mathbb{Z}, \tag{15}$$

with u_n the local control signal and w_n the local disturbance. The control objective is to maintain a specified cruising velocity, $\overline{v}$, of all vehicles, and keep the distance between neighboring vehicles at a prescribed value of δ. ξ_n represents the absolute deviations of vehicle n from the desired trajectory:

$$\overline{\xi}_n(t) := \overline{v} t + n\delta.$$

The plant dynamics can be written in the framework of Section IV as

$$\begin{aligned} \dot{x}_n &= \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} x_n + \begin{bmatrix} 1 \\ 0 \end{bmatrix} (w_n + u_n) \\ &=: A x_n + B_1 w_n + B_2 u_n, \\ y_n &= \begin{bmatrix} \xi_n - \xi_{n-1} \\ \gamma u_n \end{bmatrix} \end{aligned} \tag{16}$$

with $x_n := \begin{bmatrix} (\xi_n + v_n) & \xi_n \end{bmatrix}^T$ where $v_n = \dot{\xi}_n$ is the velocity of vehicle n, and the performance output y represents a local position error and a scaled version of the control effort. The optimal $\mathcal{H}_2$ design problem for (16) is written in terms of the closed loops:

$$\begin{aligned} &\inf_K \left\| \begin{bmatrix} C & 0 \\ 0 & \gamma I \end{bmatrix} \begin{bmatrix} \Phi^x \\ \Phi^u \end{bmatrix} B_1 \right\|_{\mathcal{H}_2}^2 \\ &\text{s.t.} \quad K \text{ stabilizing} \end{aligned} \tag{17}$$

with $C = \begin{bmatrix} C_1 & C_2 \end{bmatrix} := \begin{bmatrix} 0 & (I - T_1) \end{bmatrix}$. Without additional structural constraints, this problem can be solved using the methods of [4].

A problem of interest, that remains unsolved, is (17) subject to locality constraints on the controller K:

$$\begin{aligned} &\inf_K \left\| \begin{bmatrix} C & 0 \\ 0 & \gamma I \end{bmatrix} \begin{bmatrix} \Phi^x \\ \Phi^u \end{bmatrix} B_1 \right\|_{\mathcal{H}_2}^2 \\ &\text{s.t.} \quad K \text{ stabilizing} \\ &\qquad K \text{ local with spatial extent } N \end{aligned} \tag{18}$$

The solution to (18) would provide a solution to the best-achievable-performance of the vehicular consensus problem under constraints on the interaction and communication between the spatially distributed subcontroller units. However, (18) is nonconvex and challenging to solve. Solutions to this problem in the case of static controllers and controllers with just one internal state have been analyzed in e.g. [12], [13], [14], although the general constrained $\mathcal{H}_2$ problem (18) remains unsolved.

Motivated by SLS, we then impose locality constraints on the resulting closed-loop mappings rather than on the controller itself, solving:

$$\begin{aligned} \inf_{K} \quad & \left\| \begin{bmatrix} C & 0 \\ 0 & \gamma I \end{bmatrix} \begin{bmatrix} \Phi^x \\ \Phi^u \end{bmatrix} B_1 \right\|_{\mathcal{H}_2}^2 \\ \text{s.t.} \quad & K \text{ stabilizing} \\ & \Phi^x, \Phi^u \text{ local with spatial extent } N \end{aligned} \tag{19}$$

The main result of this section demonstrates that *with locality constraints imposed on the closed loops, this infinite-dimensional $\mathcal{H}_2$ design problem* (19) *can be formulated as a standard model-matching problem, with finitely many transfer function parameters.* In addition, the corresponding controller can be implemented in a *local* manner. We present this result for the specific vehicle consensus example, and note that similar techniques may be applied to more general settings.

To prove our result, we first apply Lemma 4.3 to write (18) in terms of the parameters θ_1, θ_2, defined in (12):

$$\begin{aligned} \inf_{\theta_1, \theta_2 \in \mathcal{RH}_2} \quad & \left\| \left(\begin{bmatrix} C_1 \Psi^{-1} + C_2 \Psi^{-2} \\ \gamma \Psi^{-2} \chi \end{bmatrix} \begin{bmatrix} \theta_1 & \theta_2 \end{bmatrix} + \ldots \right.\right. \\ & \left.\left. \begin{bmatrix} C_1 \Psi^{-1} + C_2 \Psi^{-2} & C_2 \Psi^{-2} \\ \gamma \left(\Psi^{-2} \chi - I \right) & \gamma \Psi^{-1} \end{bmatrix} \right) B_1 \right\|_{\mathcal{H}_2}^2 \\ \text{s.t.} \quad & \theta_1, \theta_2 \text{ local with spatial extent } N \end{aligned} \tag{20}$$

For the vehicle consensus problem, $C_1 = 0$, $C_2 = I - T_1$, $B_1 = \begin{bmatrix} I & 0 \end{bmatrix}^T$, and $\chi = \Psi^2 + \Psi - 2I$, and (20) can be written as

$$\begin{aligned} \inf_{\theta_1 \in \mathcal{RH}_2} \quad & \left\| (I - T_1) \Psi^{-2} (\theta_1 + I) \right\|_{\mathcal{H}_2}^2 \ldots \\ & + \left\| \gamma \left(\left(I + \Psi^{-1} - 2\Psi^{-2} \right) \theta_1 + \Psi^{-1} - 2\Psi^{-2} \right) \right\|_{\mathcal{H}_2}^2 \\ \text{s.t.} \quad & \theta_1 \text{ local with spatial extent } N \end{aligned} \tag{21}$$

For simplicity, we consider the case of $N = 1$, corresponding to nearest neighbor interactions. With this locality constraint, (21) can be written in terms of the nonzero entries of the transfer function θ_1, denoted by $\theta_{1,-1}, \theta_{1,0}, \theta_{1,1}$, as

$$\inf_{\theta_{-1}, \theta_0, \theta_1 \in \mathcal{RH}_2} \left\| H(s) + V(s) \begin{bmatrix} \theta_{-1} \\ \theta_0 \\ \theta_1 \end{bmatrix} \right\|_{\mathcal{H}_2}^2, \tag{22}$$

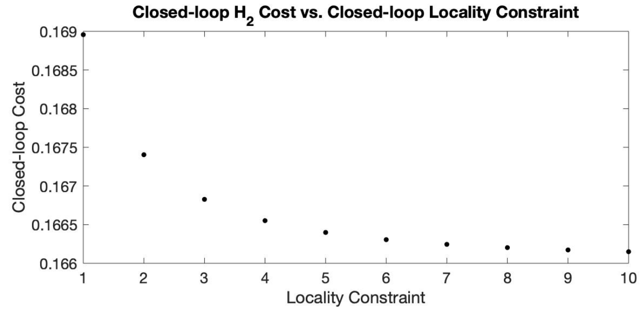

Fig. 1. Closed-loop cost of the infinite-dimensional $\mathcal{H}_2$ problem for vehicle consensus with control cost weighting $\gamma = 0.1$ is plotted against the locality constraint imposed on the closed-loop responses.

with $H(s) = \begin{bmatrix} 0 & \frac{1}{(s+1)^2} & \frac{-1}{(s+1)^2} & 0 & 0 & \beta(s) & 0 \end{bmatrix}^T$,

$$V(s) = \begin{bmatrix} \frac{1}{(s+1)^2} & 0 & 0 \\ \frac{-1}{(s+1)^2} & \frac{1}{(s+1)^2} & 0 \\ 0 & \frac{-1}{(s+1)^2} & \frac{1}{(s+1)^2} \\ 0 & 0 & \frac{-1}{(s+1)^2} \\ \gamma\alpha(s) & 0 & 0 \\ 0 & \gamma\alpha(s) & 0 \\ 0 & 0 & \gamma\alpha(s) \end{bmatrix},$$

$\alpha(s) = \left(1 + \frac{1}{s+1} - \frac{2}{(s+1)^2}\right)$, and $\beta(s) = \frac{\gamma}{s+1} - \frac{2\gamma}{(s+1)^2}$.

This is a standard model-matching problem with three transfer function parameters. The solutions for more general N follow similarly, resulting in a model-matching problem with $2N + 1$ transfer function parameters. Thus, the complexity of the resulting problem scales with the amount of spatial spread allowed in the closed-loop mappings.

We numerically solve this problem for $\gamma = 0.1$ and varying values of N; the results are illustrated in Figure 1. The closed-loop norm decreases as we allow the closed-loop mappings to have a larger spatial extent, as we are imposing a less strict constraint. We note that this convergence appears to be exponential. Formally analyzing this convergence rate, for instance using proof techniques similar to [15], [16], is the subject of future work.

VI. CONCLUSIONS

In this paper we studied the problem of optimal controller design for distributed systems, subject to subcontroller communication constraints. We provided a parameterization of the set of all stabilizing controllers for n^{th} order finite-dimensional systems, and extended these results to the infinite-extent spatially-invariant setting, with decoupled subsystem dynamics. The proof of our results followed from a nonstandard controllable-canonical formulation along with a backstepping algorithm. We applied our results to the vehicle consensus problem, demonstrating that when locality constraints are imposed on the closed loops, the $\mathcal{H}_2$ design problem for infinite-extent spatially-invariant systems composed of 2^{nd}-order subsystems may be converted to a standard model-matching problem with finitely many transfer function parameters. Current work includes extending these results to more general classes of distributed systems, composed of subsystems with *coupled* dynamics.

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Appendix

A. Proof of Corollary 2.1

To prove this result, we employ the following lemma, which follows directly from the results of [3].

Lemma 6.1: If the controller $u = Kx$ internally stabilizes (1), then the corresponding closed-loop mappings are elements of $\mathcal{RH}_2$ and satisfy:

$$\begin{bmatrix} (s+1)I - (A+I) & -B_2 \end{bmatrix} \begin{bmatrix} \Phi^x \\ \Phi^u \end{bmatrix} = I \quad (23)$$

By Lemma 6.1, it is sufficient to show that Φ^x, Φ^u are of the form (5) if and only if $\Phi^x, \Phi^u \in \mathcal{RH}_2$ and satisfy (23).

First assume that $\Phi^x, \Phi^u \in \mathcal{RH}_2$ satisfy (23), so that

$$\begin{bmatrix} ((s+1)+a_1) I & a_2 I \\ -I & (s+1)I \end{bmatrix} \Phi^x - \begin{bmatrix} I \\ 0 \end{bmatrix} \Phi^u = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \quad (24)$$

Partition Φ^x as $\Phi^x = \begin{bmatrix} \Phi^x_{11} & \Phi^x_{12} \\ \Phi^x_{21} & \Phi^x_{22} \end{bmatrix} = \begin{bmatrix} \Phi^x_1 \\ \Phi^x_2 \end{bmatrix}$. Rearranging the second block row of (24) gives

$$\Phi^x_2 = \frac{1}{s+1} \left(\Phi^x_1 + \begin{bmatrix} 0 & I \end{bmatrix} \right)$$

Substituting this for Φ^x_2 in the first block row of (24) gives

$$\Phi^u = \begin{bmatrix} -I & \frac{a_2}{s+1} I \end{bmatrix} + \left(a_1 + (s+1) + \frac{a_2}{s+1} \right) \Phi^x_1$$

Thus, Φ^x, Φ^u can be written in terms of the components Φ^x_{11}, Φ^x_{12} of Φ^x:

$$\begin{aligned} \Phi^x &= \begin{bmatrix} \Phi^x_{11} & \Phi^x_{12} \\ \frac{1}{s+1} \Phi^x_{11} & \frac{1}{s+1} \left(\Phi^x_{12} + I \right) \end{bmatrix} \\ \Phi^u &= \begin{bmatrix} \chi(s) \Phi^x_{11} - I & \frac{a_2}{s+1} I + \chi(s) \Phi^x_{12} \end{bmatrix}, \end{aligned} \quad (25)$$

with $\chi(s) := \left(a_1 + (s+1) + \frac{a_2}{s+1} \right) I$. $\Phi^x_{11}, \Phi^x_{12} \in \mathcal{RH}_2$ imply $\Phi^x \in \mathcal{RH}_2$, and it is straightforward to show that $\Phi^u \in \mathcal{RH}_2$ if and only if Φ^x_{11} and Φ^x_{12} are of the form

$$\Phi^x_{11} = \frac{1}{s+1} \left(\theta_1 + I \right), \; \Phi^x_{12} = \frac{1}{s+1} \theta_2,$$

for some $\theta_1, \theta_2 \in \mathcal{RH}_2$. The proof of the converse follows from a direct computation to confirm that (24) holds for all for Φ^x, Φ^u of the form (5). ■

B. Proof of Theorem 4.2

By Lemma 4.1 it is sufficient to prove that Φ^x, Φ^u are of the form (12) if and only if $\Phi^x, \Phi^u \in \mathcal{RH}_2$ satisfy (11). First assume that $\Phi^x, \Phi^u \in \mathcal{RH}_2$ satisfy (11), so that

$$(\mathcal{S} + I)\Phi^x - (A + I)\Phi^x - B_2 \Phi^u = I. \quad (26)$$

As A and B_2 are pointwise multiplication operators, taking a Laplace transform shows that (26) is equivalent to

$$\left((s+1)I - (A+I) \right) \Phi^x_n(s) - B_2 \Phi^u_n(s) = I, \quad \forall n \in \mathbb{Z}.$$

It then follows from Corollary 2.1 that for all n,

$$\begin{aligned} \Phi^x_n &= \begin{bmatrix} \frac{1}{s+1} I \\ \frac{1}{(s+1)^2} I \end{bmatrix} \begin{bmatrix} \theta_{1,n} & \theta_{2,n} \end{bmatrix} + \begin{bmatrix} \frac{1}{s+1} I & 0 \\ \frac{1}{s+1} I & \frac{1}{(s+1)^2} I \end{bmatrix} \\ \Phi^u_n &= \frac{1}{(s+1)^2} \chi(s) \begin{bmatrix} \theta_{1,n} & \theta_{2,n} \end{bmatrix} \ldots \\ &\quad + \begin{bmatrix} \frac{1}{(s+1)^2} \chi(s) - I & \frac{1}{s+1} I \end{bmatrix}, \end{aligned} \quad (27)$$

Then Φ^x, Φ^u can be written in terms of the spatially-invariant systems $\theta_1, \theta_2 \in \mathcal{RH}_2$ as

$$\begin{aligned} \Phi^x &= \begin{bmatrix} \Psi^{-1} \\ \Psi^{-2} \end{bmatrix} \begin{bmatrix} \theta_1 & \theta_2 \end{bmatrix} + \begin{bmatrix} \Psi^{-1} & 0 \\ \Psi^{-1} & \Psi^{-2} \end{bmatrix} \\ \Phi^u &= \Psi^{-2} \chi \begin{bmatrix} \theta_1 & \theta_2 \end{bmatrix} + \begin{bmatrix} \Psi^{-2}\chi - I & \Psi^{-1} \end{bmatrix}, \end{aligned} \quad (28)$$

where θ_i denotes both the spatially-invariant system and its transfer function. The proof of the converse is straightforward and is omitted. ■