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Front propagation and blocking of reaction–diffusion systems in cylinders

Hongjun Guo[1,3], Jennifer Forbey[2] and Rongsong Liu[1,*]

[1] Department of Mathematics and Statistics, University of Wyoming, Laramie, United States of America

[2] Department of Biological Sciences, Boise State University, Boise, United States of America

E-mail: Rongsong.liu@uwyo.edu

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Abstract

In this paper, we consider a bistable monotone reaction–diffusion system in cylindrical domains. We first prove the existence of the entire solution emanating from a planar front. Then, it is proved that the entire solution converges to a planar front if the propagation is complete and the domain is bilaterally straight. Finally, we give some geometrical conditions on the domain such that the propagation of the entire solution is complete or incomplete, respectively.

Keywords: reaction–diffusion systems, entire solutions, complete propagation, incomplete propagation

Mathematics Subject Classification numbers: 35B08, 35C07, 35K40.

1. Introduction

In this paper, we consider the following reaction–diffusion system

$$\begin{cases} \mathbf{u}_t = \mathbf{D}\Delta\mathbf{u} + \mathbf{F}(\mathbf{u}), & t \in \mathbb{R},\ x \in \Omega \\ \partial_\nu \mathbf{u} = \mathbf{0}, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\mathbf{u}(t,x) = (u_1(t,x), u_2(t,x))$, $\mathbf{0} = (0,0)$ and Ω is an unbounded open connected set of $\mathbb{R}^N$ defined by

$$\Omega = \{(x_1, x') \in \mathbb{R}^N; x_1 \in \mathbb{R}, x' \in \omega(x_1) \subset \mathbb{R}^{N-1}\}, \tag{1.2}$$

[3]Current address: School of Mathematical Sciences, Tongji University, Shanghai, People's Republic of China
[*]Author to whom any correspondence should be addressed.

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where ω is independent of x_1 for $x_1 < 0$. Notice that the left side of Ω is a half straight cylinder. Here, $\nu(x)$ is the unit outward normal on $\partial\Omega$ and the homogeneous Neumann boundary condition $\partial_\nu \mathbf{u} = (\partial_\nu u_1, \partial_\nu u_2) = \mathbf{0}$ implies that there is no flux cross the boundary $\partial\Omega$.

Let us first clarify some notions. For any two vectors $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$, the symbol $\mathbf{a} \ll \mathbf{b}$ means $a_i < b_i$ for $i = 1, 2$, and $\mathbf{a} \leqslant \mathbf{b}$ means $a_i \leqslant b_i$ for $i = 1, 2$. The interval $[\mathbf{a}, \mathbf{b}]$ denotes the set of $\mathbf{q} \in \mathbb{R}^2$ such that $\mathbf{a} \leqslant \mathbf{q} \leqslant \mathbf{b}$. Throughout this paper, we assume that

(A1) $\mathbf{D}$ is a 2×2 diagonal matrix with elements $D_1 > 0$, $D_2 > 0$.

(A2) $\mathbf{F}$ has two stable equilibria $\mathbf{0} = (0, 0)$ and $\mathbf{1} = (1, 1)$, that is, $\mathbf{F}(\mathbf{0}) = \mathbf{F}(\mathbf{1}) = \mathbf{0}$ and all the eigenvalues of $\mathbf{F}'(\mathbf{0})$ and $\mathbf{F}'(\mathbf{1})$ lie in the open left-half complex plane. We also assume that the matrixes $\mathbf{F}'(\mathbf{0})$ and $\mathbf{F}'(\mathbf{1})$ are irreducible.

(A3) The reaction term $\mathbf{F}(\mathbf{u}) = (F^1(\mathbf{u}), F^2(\mathbf{u}))$ is defined on an open domain E of $\mathbb{R}^2$ and of class C^1 in $\mathbf{u}$. Moreover, $\mathbf{F}$ satisfies the following conditions

$$\frac{\partial F^i}{\partial u_j}(\mathbf{u}) \geqslant 0 \quad \text{for all } u \in [\mathbf{0}, \mathbf{1}] \subset E \text{ and for all } i \neq j.$$

Actually, the technique used in this paper can be trivially extended to an n-dimensional system for $n \geqslant 2$. Here, we only deal with $n = 2$ for convenience. From above assumptions, one knows that (1.1) is a monotone system. It therefore implies that a comparison principle holds for system (1.1), see [15, 18].

The system (1.1) arises in various fields of sciences such as mathematical ecology, population genetics, chemical reactor theory, etc. Particularly, this system can describe the population distribution of two species which are interacting with each other in a certain manner and simultaneously diffusing over the domain. In the study of the cross-diffusion of two species, the travelling front plays a key role. For instance, the travelling front can describe the invasion of the species into a fresh region in the cooperative system or the invasion of one species to another in the competition system. Some evidence of existence of travelling fronts can be found in [8] for a competition model and in [6] for a model which describes chemical phenomenon on isothermal catalyst surface. For the general monotone reaction–diffusion system, we refer to [17, 18] for some conditions ensuring the existence of travelling fronts and refer to [5] for some abstract results. It is worth to mention that the authors of [17, 18] used the topological methods, while the authors of [5] used the dynamical theory. Since the system (1.1) may contain stable equilibrium other than $\mathbf{0}$ and $\mathbf{1}$, it may not exist travelling fronts connecting $\mathbf{0}$ and $\mathbf{1}$ in general. Therefore, in this paper, we always assume that (1.1) in one dimension, that is, $\Omega = \mathbb{R}$, admits a unique (up to shifts) travelling front $\Phi(x - ct) = (\phi_1(x - ct), \phi_2(x - ct))$ satisfying

$$\begin{cases} -D_i\phi_i'' - c\phi_i' - F^i(\Phi) = 0, \\ \Phi(+\infty) = \mathbf{0}, \qquad \Phi(-\infty) = \mathbf{1}, \\ \phi_i' < 0 \quad \text{on } \mathbb{R} \text{ for } i = 1,\ 2. \end{cases} \tag{1.3}$$

It is known from [18, chapter 3] that there exist $C > 0$ and $\beta > 0$ such that

$$\phi_i(\xi) \leqslant C\,\mathrm{e}^{-\beta\xi} \quad \text{for } \xi \geqslant 0, 1 - \phi_i(\xi) \leqslant C\,\mathrm{e}^{\beta\xi} \quad \text{for } \xi < 0 \quad \text{and} \quad |\phi_i'(\xi)|, |\phi_i''(\xi)| \leqslant C\,\mathrm{e}^{\beta|\xi|} \quad \text{for } \xi \in \mathbb{R}.$$

Indeed, some stability results for the travelling front $\Phi(x - ct)$ can be referred to [1, 13, 14, 18, 19]. In the study of travelling fronts, the propagation speed c is also an important aspect. For instance, the sign of c represents who is the winner in the competition system. Some results about the relationship between the parameters and the sign of the speed can be referred to [11]. Throughout this paper, we assume that

$$c > 0.$$

Otherwise, we can replace the roles of **0** and **1**.

Since we are interested in how the geometry of the domain effects the diffusion, we first recall some results for the scalar bistable reaction–diffusion equation

$$u_t = \Delta u + f(u), \quad t > 0,\ x \in \Omega, \tag{1.4}$$

where Ω is a smooth unbounded open connected set of $\mathbb{R}^N$ and the function f is of bistable, that is, it satisfies

$$f(0) = f(1) = 0, \quad f < 0 \text{ on } (0, \theta) \text{ and } f > 0 \text{ on } (\theta, 1) \text{ for some } \theta \in (0, 1),$$

and $f'(0) < 0$, $f'(1) < 0$. From [7], it is known that (1.4) in one dimension admits travelling fronts $\phi(x \cdot e - ct)$ connecting 0 and 1 and hence, (1.4) in high dimensional space $\mathbb{R}^N$ ($N \geqslant 2$) admits planar fronts $\phi(x \cdot e - ct)$ for any $e \in \mathbb{S}^{N-1}$. In [3], Berestycki, Hamel and Matano have studied the propagation phenomenon when the domain is an exterior domain, that is, $\Omega = \mathbb{R}^N \setminus K$ where K is a bounded connected open set of $\mathbb{R}^N$ and K is called an obstacle. They proved that the planar front $\phi(x - ct)$ can propagate infinitely far from the obstacle, that is, there is an entire solution $u(t, x)$ such that

$$u(t, x) \to \phi(x \cdot e - ct) \quad \text{as } t \to -\infty \text{ uniformly in } \overline{\Omega}.$$

From their work, one knows that if K is star-shaped or directionally convex with respect to some hyperplane[4], then the propagation of u is complete, that is, satisfying

$$u \to 1 \quad \text{as } t \to +\infty \text{ locally uniformly in } \overline{\Omega}.$$

They also gave an example such that u cannot propagate completely. Such complete and incomplete propagation phenomena appear in cylindrical domains too. In [2], Berestycki, Bouhours and Chapuisat investigated the propagation of a planar front in the domain which is a cylinder with different kind of cross sections, that is, Ω is defined by (1.2). They first proved that a planar front $\phi(x_1 - ct)$ can propagate from the left side of the cylinder, that is, there is an entire solution $u(t, x)$ satisfying

$$u(t, x) \to \phi(x_1 - ct) \quad \text{as } t \to -\infty \text{ uniformly in } \overline{\Omega}.$$

They also gave some conditions such that the propagation of u is complete or incomplete.

In this paper, we aim to extend the results of [2] to our system (1.1). We emphasize here that the authors of [2] dealt with a scalar reaction–diffusion equation and we are dealing with a system of reaction–diffusion equations. Due to this, many modifications and new techniques are needed. For example, theorem 1.5 and lemma 4.1 in the scalar case are known to be proved

[4] The obstacle K is called star-shaped if either $K = \emptyset$ or there is x in the interior $\mathrm{Int}(K)$ of K such that $x + t(y - x) \in \mathrm{Int}(K)$ for all $y \in \partial K$ and $t \in [0, 1)$. In the latter case, we say that K is star-shaped with respect to the point x. The obstacle K is called directionally convex with respect to a hyperplane $H = \{x \in \mathbb{R}^N : x \cdot e = a\}$, with $e \in \mathbb{S}^{N-1}$ and $a \in \mathbb{R}$, if for every line Σ parallel to e, the set $K \cap \Sigma$ is either a single line segment or empty and if $K \cap H$ is equal to the orthogonal projection of K onto H.

by the energy functional which is invalid in our case. Indeed, since we assumed the existence of travelling front $\Phi(x-ct)$ satisfying (1.3). It means that there exist planar fronts $\Phi(x\cdot e-ct)$ of (1.1) with $\Omega=\mathbb{R}^N$ for any unit vector $e\in\mathbb{S}^{N-1}$. We fist prove the existence and uniqueness of the entire solution $\mathbf{u}(t,x)$ of (1.1) emanating from the planar front $\Phi(x_1-ct)$.

Theorem 1.1. *Let $\Omega\subset\mathbb{R}^N$ satisfying (1.2). Then, system (1.1) admits a unique entire solution $\mathbf{u}(t,x)$ satisfying*

$$\mathbf{u}(t,x)\to\Phi(x_1-ct)\quad \text{as } t\to-\infty \text{ uniformly in } \overline{\Omega}. \tag{1.5}$$

Moreover, $\mathbf{u}(t,x)$ is increasing in t, $\mathbf{0}\ll\mathbf{u}(t,x)\ll\mathbf{1}$ for all $(t,x)\in\mathbb{R}\times\overline{\Omega}$.

The existence of the entire solution $\mathbf{u}(t,x)$ emanating from the planar front $\Phi(x_1-ct)$ implies that the planar front enters in the left side of the cylinder. We now investigate the large time behaviour of the entire solution $\mathbf{u}(t,x)$. The following theorem shows that if the right side of Ω is straight, then the entire solution $\mathbf{u}(t,x)$ converges to a planar front as $t\to+\infty$ provided by the complete propagation.

Theorem 1.2. *If Ω is bilaterally straight, that is, $\omega(x_1)$ is independent of x_1 when $x_1\leqslant 0$ and $x_1\geqslant L$ for some $L>0$, and the propagation of $\mathbf{u}(t,x)$ is complete, that is, it satisfies*

$$\mathbf{u}(t,x)\to\mathbf{1}\quad \text{locally uniformly in } \overline{\Omega} \text{ as } t\to+\infty, \tag{1.6}$$

then there is a constant σ such that

$$\mathbf{u}(t,x)\to\Phi(x_1-ct+\sigma)\quad \text{as } t\to+\infty \text{ uniformly in } \overline{\Omega}.$$

Remark 1.3. Theorem 1.2 implies that in the bilaterally straight cylinder, the entire solution $\mathbf{u}(t,x)$ is a transition front connecting $\mathbf{0}$ and $\mathbf{1}$ and has a global mean speed equal to c, see [10] for definitions of the transition front and the global mean speed.

The following theorem shows some geometrical conditions of Ω such that $\mathbf{u}(t,x)$ can propagate completely.

Theorem 1.4. *The propagation of $\mathbf{u}(t,x)$ is complete, if Ω satisfies one of the following conditions*

(*a*) *For all $x\in\partial\Omega$, $\nu_1(x)\geqslant 0$ where $\nu_1(x)$ is the first component of the outward unit normal at x;*
(*b*) *For a sufficiently large R_0, $\mathbb{R}\times B'_{R_0}\subset\Omega$ where B'_{R_0} denotes the ball of $\mathbb{R}^{N-1}$ with centre 0 and radius R_0 and Ω is axially star-shaped, that is, for any $x=(x_1,x')\in\partial\Omega$ and $\nu=(\nu_1,\nu')$, $\nu'\cdot x'\geqslant 0$ for all $x\in\partial\Omega$;*
(*c*) *Ω is a dilated domain with any shift $x_0\in\mathbb{R}^N$ by a cylinder Ω' and a large constant R_0, that is, $\Omega=R_0\Omega'+x_0$.*

Finally, we give some conditions of Ω such that the propagation of $\mathbf{u}(t,x)$ is incomplete or blocked.

Theorem 1.5. *Let a and b be two constants such that $-\infty<a<b<+\infty$. There exists $\varepsilon>0$ small enough depending on the distance $b-a$ and*

$$\Omega\cap\{x\in\mathbb{R}^N,\ b<x_1<b+1\}$$

such that if

$$|\Omega\cap\{x\in\mathbb{R}^N,\ x_1\in(a,b)\}|<\varepsilon,$$

then the propagation of $\mathbf{u}$ *is blocked, that is,*

$$\mathbf{u}(t,x) \to \mathbf{u}_\infty \quad \textit{in}\,\Omega\,\textit{as}\,t \to +\infty\,\textit{with}\,\mathbf{u}_\infty(x) \to \mathbf{0}\,\textit{as}\,x_1 \to +\infty.$$

We organize this paper as follows. In section 2, we give some comparison principles which are key tools in the sequel and prove the existence of the entire solution $\mathbf{u}(t,x)$ emanating from a planar front. Section 3 is devoted to the proof of that $\mathbf{u}(t,x)$ converges to a planar front when the right side of Ω is straight and the propagation of $\mathbf{u}(t,x)$ is complete. In section 4, we give some geometrical conditions on Ω such that the propagation of the entire solution $\mathbf{u}(t,x)$ is complete or incomplete, respectively.

2. Existence of front-like solutions

In this section, we prove the existence of the entire solution emanating from a planar front. The idea of the proof is inspired by [3]. However, since we are dealing with a system, some adaption should be made upon a system. We will need some comparison principles for our system (1.1) and prove theorem 1.1 by constructing sub- and supersolutions.

2.1. Comparison principles

In this subsection, we list some comparison principles. We first state the definitions of sub- and supersolutions.

Definition 2.1. If a function $\mathbf{u} = (u_1, u_2)$ satisfies $u_1, u_2 \in C^{1,1}(\Gamma \times \overline{\Omega}) \cap C^{1,2}(\Gamma \times \Omega)$ where $\Gamma \subset \mathbb{R}$ and that

$$\begin{cases} \mathbf{u}_t - \Delta\mathbf{u} - F(\mathbf{u}) \geqslant 0, & \text{in}\,\Gamma \times \Omega \\ \partial_\nu \mathbf{u} \geqslant \mathbf{0}, & \text{on}\,\Gamma \times \partial\Omega, \end{cases} \tag{2.1}$$

then $\mathbf{u}$ is called a supersolution of (1.1) in $\Gamma \times \Omega$. If $\mathbf{u}$ satisfies the reversed differential inequalities, then it is called a subsolution of (1.1) in $\Gamma \times \Omega$. If $\mathbf{u}$ and $\mathbf{v}$ are supersolutions (subsolutions) of (1.1) in $\Gamma \times \Omega$, then

$$\min(\mathbf{u}, \mathbf{v})(\max(\mathbf{u}, \mathbf{v})),$$

(min and max are to be understood componentwise) is still a supersolution (subsolution) of (1.1) in $\Gamma \times \Omega$.

Definition 2.2. Let S be a smooth hypersurface dividing Ω into disjoint regions Ω_1, Ω_2; namely, $\Omega = \Omega_1 \cup S \cup \Omega_2$. Suppose $\mathbf{u} = (u_1, u_2) \in C^{1,0}(\Gamma \times \overline{\Omega}) \times C^{1,0}(\Gamma \times \overline{\Omega})$ is C^1 on each $\overline{\Omega}_1$, $\overline{\Omega}_2$ and C^2 on each Ω_1, Ω_2. Suppose further that (2.1) holds except on S and that

$$\frac{\partial u_i}{\partial \xi} + \frac{\partial u_i}{\partial \zeta} \leqslant 0 \quad \text{on } S,$$

where ξ, ζ denote the inner normal on $\partial\Omega_1 \cap S$, $\partial\Omega_2 \cap S$ respectively. Then, $\mathbf{u}$ is called a supersolution of (1.1) in $\Gamma \times \Omega$. If $\mathbf{u}$ satisfies the reversed differential inequalities, then it is called a subsolution of (1.1) in $\Gamma \times \Omega$.

Then, the following proposition follows from [12, 15, 18].

Proposition 2.3. *If* $\underline{\mathbf{u}}$ *and* $\overline{\mathbf{u}}$ *are sub- and supersolutions of* (1.1) *in* $\mathbb{R}^+ \times \Omega$ *and it holds* $\underline{\mathbf{u}}(0,x) \leqslant \overline{\mathbf{u}}(0,x)$ *for* $x \in \Omega$, *then we have* $\underline{\mathbf{u}}(t,x) \leqslant \overline{\mathbf{u}}(t,x)$ *for all* $(t,x) \in \mathbb{R}^+ \times \Omega$.

We can also have the following proposition.

Proposition 2.4. *Let $\underline{\mathbf{u}}$ and $\overline{\mathbf{u}}$ be sub and supersolutions of (1.1) in $\overline{Q}$ where $Q = (0, \tau) \times \Omega$, $\tau \in (0, +\infty)$ and $\underline{\mathbf{u}}(t, x) \leqslant \overline{\mathbf{u}}(t, x)$ for all $(t, x) \in Q$. If $\underline{u}_i(t_0, x_0) = \overline{u}_i(t_0, x_0)$ for some $(t_0, x_0) \in Q$, then $\underline{u}_i(t, x) = \overline{u}_i(t, x)$ for all $(t, x) \in \overline{Q}$, $t \leqslant t_0$. If $\underline{\mathbf{u}}(t, x) \not\equiv \overline{\mathbf{v}}(t, x)$ and $\underline{u}_i(t_0, x_0) = \overline{u}_i(t_0, x_0)$ for some $(t_0, x_0) \in \partial Q = (0, \tau) \times \partial\Omega$, then $\partial_\nu(\underline{u}_i(t_0, x_0) - \overline{u}_i(t_0, x_0)) > 0$ where ν is the unit outward normal on $\partial\Omega$.*

Proof. The first part of this proposition can directly follow from [6]. We only prove the last assert.

Let $\mathbf{v}(t, x) = \overline{\mathbf{u}}(t, x) - \underline{\mathbf{u}}(t, x)$. We have

$$(v_i)_t - \Delta v_i - F^i(\overline{\mathbf{u}}) + F^i(\underline{\mathbf{u}}) \geqslant 0, \quad \text{in } Q.$$

By assumption (A3), one has

$$F^i(\overline{\mathbf{u}}) - F^i(\underline{\mathbf{u}}) = \sum_{j=1,2} \frac{\partial F^i}{\partial u_j}(\theta(t, x))(\overline{u}_j - \underline{u}_j) \geqslant \frac{\partial F^i}{\partial u_i}(\theta(t, x))(\overline{u}_i - \underline{u}_i),$$

where $\underline{\mathbf{u}}(t, x) \leqslant \theta(t, x) \leqslant \overline{\mathbf{u}}(t, x)$. Then, it follows that

$$(v_i)_t - \Delta v_i - \frac{\partial F^i}{\partial u_i}(\theta(t, x))v_i \geqslant 0, \quad \text{in } Q.$$

Since $v_i(t, x) \geqslant 0$ in Q and $v_i(t_0, x_0) = 0$ for $(t_0, x_0) \in \partial\Omega$ and by Hopf lemma, one has that

$$\partial_\nu v_i(t_0, x_0) < 0.$$

This completes the proof. □

2.2. Construction of sub- and supersolutions

We construct sub- and supersolutions for (1.1) by the idea inspired by [16]. By our assumptions and Perron–Frobenius theorem, one knows that the principal eigenvalues of $\mathbf{F}'(\mathbf{0})$ and $\mathbf{F}'(\mathbf{1})$ are negative and the corresponding eigenvectors are positive. It then can be easily found an irreducible constant matrixes $\mathbf{A}^\pm = (\mu_{ij}^\pm)$ such that $\partial F^i/\partial u_j(\mathbf{0}) < \mu_{ij}^+$, $\partial F^i/\partial u_j(\mathbf{1}) < \mu_{ij}^-$ for all $i, j = 1, 2$, and its principal eigenvalues are negative. Let $\mathbf{p}^\pm = (p_1^\pm, p_2^\pm)$ be the positive eigenvectors corresponding to the principal eigenvalues of $\mathbf{A}^\pm$. Then, there exist $\delta_0 > 0$ and $k > 0$ such that

$$\frac{\partial F^i}{\partial u_j}(\mathbf{u}) \leqslant \mu_{ij}^+ \quad \text{for } \mathbf{u} \in B_{\delta_0}(\mathbf{0}) \quad \text{and} \quad \frac{\partial F^i}{\partial u_j}(\mathbf{u}) \leqslant \mu_{ij}^- \quad \text{for } \mathbf{u} \in B_{\delta_0}(\mathbf{1}) \text{ and } i, j = 1,\ 2, \tag{2.2}$$

and

$$\sum_{j=1}^{2} \mu_{ij}^\pm r_j \leqslant -kr_i \quad \text{for all } \mathbf{r} = (r_1, r_2) \in \mathbb{R}_+^2 \cap B_{\delta_0}(\mathbf{p}^\pm). \tag{2.3}$$

Take a positive constant M and a C^2 decreasing function $\zeta(s)$ such that

$$\zeta(s) = 1 \quad \text{for } s \leqslant -M \quad \text{and} \quad \zeta(s) = 0 \quad \text{for } s \geqslant M.$$

Define

$$p_i(\xi) := \zeta(\xi)p_i^- + (1-\zeta(\xi))p_i^+, \quad \text{for } i = 1,\ 2,$$

and $\mathbf{p}(\xi) := (p_1(\xi), p_2(\xi))$. Notice that

$$p_i(\xi) \equiv p_i^- \quad \text{for } \xi \leqslant -M \quad \text{and} \quad p_i(\xi) \equiv p_i^+ \quad \text{for } \xi \geqslant M. \tag{2.4}$$

Since $\Phi(+\infty) = \mathbf{0}$, $\Phi(-\infty) = \mathbf{1}$, one can take a small positive constant δ and a large positive constant $C \geqslant M$ such that

$$\max_{i=1,2}\{D_i\mu^2\} \leqslant k, \qquad \max_{i=1,2}\{\delta p_i^{\pm}\} \leqslant \frac{\delta_0}{2} \quad \text{for all } i = 1,\ 2, \tag{2.5}$$

where $\mu := \delta/c$ and

$$\begin{aligned}\|\Phi(\xi)\| \leqslant \delta \min_{i=1,2}\{p_i^+\} \quad &\text{for all } \xi \geqslant C \text{ and } \|\Phi(\xi) - \mathbf{1}\| \\ \leqslant \delta \min_{i=1,2}\{p_i^-\} \quad &\text{for all } \ \xi \leqslant -C,\end{aligned} \tag{2.6}$$

where k and δ_0 are defined by (2.3) and (2.2) respectively. Let

$$C_1 := \max\left\{\sup_{\xi\in\mathbb{R}}\|\mathbf{p}(\xi)\|, \sup_{\xi\in\mathbb{R}}\|\mathbf{p}'(\xi)\|, \sup_{\xi\in\mathbb{R}}\|\mathbf{p}''(\xi)\|\right\}, \tag{2.7}$$

where $\|\cdot\|$ is the Euclidean norm of $\mathbb{R}^2$ and

$$C_2 := \sup\left\{\sum_{i,j=1}^{2}\left|\frac{\partial F^i}{\partial u_j}(\mathbf{u})\right|, \sum_{i,j,l=1}^{2}\left|\frac{\partial^2 F^i}{\partial u_j \partial u_l}(\mathbf{u})\right| : \mathbf{u} \in [\mathbf{0}, \mathbf{1}]\right\}. \tag{2.8}$$

Since $\Phi'(\xi) < 0$ for all $\xi \in \mathbb{R}$, there is $C_3 > 0$ such that

$$-\phi_i'(\xi) \geqslant C_3 \quad \text{for all } |\xi| \leqslant C \text{ and } i = 1,\ 2. \tag{2.9}$$

Let $\alpha > 0$ be a large enough constant such that

$$C_3\alpha \geqslant \max_{i=1,2}\left(c + 1 + D_i + 2D_i\mu + D_i\mu^2 + C_2\right) C_1\, \mathrm{e}^{\mu(C+1)}. \tag{2.10}$$

For $(t, x) \in \mathbb{R} \times \overline{\Omega}$, define

$$\mathbf{u}^-(t,x) := \begin{cases} \max\left(\Phi(\underline{\xi}(t,x_1)) - \delta\mathbf{p}(\underline{\xi}(t,x_1))\mathrm{e}^{\mu x_1}, \mathbf{0}\right), & \text{if } x_1 \leqslant 0 \\ \mathbf{0}, & \text{if } x_1 > 0, \end{cases} \tag{2.11}$$

where $\underline{\xi}(t, x_1) = x_1 - ct + \alpha\,\mathrm{e}^{\delta t}$.

Lemma 2.5. *There exists $T < 0$ such that $\mathbf{u}^-(t, x)$ is a subsolution of (1.1) for all $t \leqslant T$ and $x \in \overline{\Omega}$.*

Proof. Take $T < 0$ such that

$$\alpha\delta\,\mathrm{e}^{\delta t} \leqslant 1 \quad \text{for all } t \leqslant T \text{ and } -cT \geqslant C.$$

Then, $\underline{\xi}(t, x_1) \geqslant -ct + \alpha\, e^{\delta t} \geqslant -cT \geqslant C \geqslant M$ for all $t \leqslant T$ and $x_1 \geqslant 0$. By (2.6) and (2.4), it follows that $\|\Phi(\underline{\xi}(t, x_1))\| \leqslant \delta \min_{i=1,2}\{p_i^+\}$ and $p_i(\underline{\xi}(t, x_1)) = p_i^+$ for all $t \leqslant T$ and $x_1 \geqslant 0$. Thus,

$$\max\left(\Phi(\underline{\xi}(t, x_1)) - \delta \mathbf{p}(\underline{\xi}(t, x_1))e^{\mu x_1}, \mathbf{0}\right) = \mathbf{0} \quad \text{for all } t \leqslant T \text{ and } x_1 \geqslant 0.$$

It means that $\mathbf{u}^-(t, x)$ is well-defined and continuous for $t \leqslant T$ and $x \in \overline{\Omega}$. Since $\mathbf{0}$ is a solution of (1.1) and obviously $\partial_\nu \mathbf{u}^-(t, x) = \mathbf{0}$ for $x \in \partial\Omega$, one only has to check that

$$N_i[\mathbf{u}^-](t, x) := (u_i^-)_t - D_i \Delta u_i^- - F^i(\mathbf{u}^-) \leqslant 0,$$

for $i = 1, 2$ and $(t, x) \in (-\infty, T] \times \overline{\Omega}$ such that $\mathbf{u}^-(t, x) = \Phi(\underline{\xi}(t, x_1)) - \delta \mathbf{p}(\underline{\xi}(t, x_1))e^{\mu x_1} \geqslant \mathbf{0}$.

After some calculation and by (1.3), one can get that

$$\begin{aligned} N_i[\mathbf{u}^-](t, x) &:= (u_i^-)_t - D_i \Delta u_i^- - F^i(\mathbf{u}^-) \\ &= \alpha\delta\, e^{\delta t} \phi_i'(\underline{\xi}(t, x_1)) - \big((-c + \alpha\delta\, e^{\delta t})\delta p_i'(\underline{\xi}(t, x_1)) \\ &\quad - D_i \delta p_i''(\underline{\xi}(t, x_1)) - 2D_i \delta\mu p_i'(\underline{\xi}(t, x_1)) - D_i \delta \mu^2 p_i(\underline{\xi}(t, x_1))\big)\, e^{\mu x_1} \\ &\quad + F^i(\Phi(\underline{\xi}(t, x_1))) - F^i(\mathbf{u}^-(t, x)), \end{aligned} \tag{2.12}$$

for $i = 1, 2$. For $(t, x) \in (-\infty, T] \times \overline{\Omega}$ such that $|\underline{\xi}(t, x_1)| \leqslant C$, it follows from the mean-value theorem that there exist $\theta_i(t, x) \in (0, 1)$, $i = 1, 2$ such that

$$\begin{aligned} F^i(\Phi(\underline{\xi}(t, x_1))) - F^i(\mathbf{u}^-(t, x)) &= \sum_{j=1}^{2} \frac{\partial}{\partial u_j} F^i\left(\Phi(\underline{\xi}(t, x_1)) - \theta_i(t, x)\right. \\ &\quad \left.\times\, \delta \mathbf{p}(\underline{\xi}(t, x_1))e^{\mu x_1}\right) \delta p_j(\underline{\xi}(t, x_1))e^{\mu x_1} \\ &\leqslant C_1 C_2 \delta\, e^{\mu x_1}, \end{aligned}$$

where C_1 and C_2 are defined by (2.7) and (2.8). Notice that $\underline{\xi}(t, x) \leqslant C$ implies that $x_1 \leqslant C + ct$ and hence $e^{\mu x_1} \leqslant e^{\mu C}\, e^{c\mu t} = e^{\mu C}\, e^{\delta t}$ by $c\mu = \delta$. Then, it follows from (2.9), (2.10) and (2.12) that

$$\begin{aligned} N_i[\mathbf{u}^-](t, x) &\leqslant -\alpha\delta\, e^{\delta t} C_3 + \left(c + 1 + D_i + 2D_i\mu + D_i\mu^2 + C_2\right) C_1 \delta\, e^{\mu x_1} \\ &\leqslant -C_3 \alpha\delta\, e^{\delta t} + \left(c + 1 + D_i + 2D_i\mu + D_i\mu^2 + C_2\right) C_1\, e^{\mu C} \delta\, e^{\delta t} \leqslant 0. \end{aligned}$$

On the other hand, for $(t, x) \in (-\infty, T] \times \overline{\Omega}$ such that $\underline{\xi}(t, x_1) \geqslant C$, it follows that $p_i(\underline{\xi}(t, x_1)) \equiv p_i^+$ by $C \geqslant M$ and (2.4). Then, by (2.2) and (2.3), one has that

$$\begin{aligned} F^i(\Phi(\underline{\xi}(t, x_1))) - F^i(\mathbf{u}^-(t, x)) &= \sum_{j=1}^{2} \frac{\partial}{\partial u_j} F^i\left(\Phi(\underline{\xi}(t, x_1)) - \theta_i(t, x)\right. \\ &\quad \left.\times\, \delta \mathbf{p}(\underline{\xi}(t, x_1))e^{\mu x_1}\right) \delta p_j^+\, e^{\mu x_1} \\ &\leqslant -k p_i^+ \delta\, e^{\mu x_1}, \end{aligned}$$

for $\theta_i(t,x) \in (0,1)$ and $i = 1, 2$. Since $\Phi_i'(\xi) < 0$ for all $\xi \in \mathbb{R}$ and $p_i'(\xi) = 0$, $p_i''(\xi) = 0$ for $|\xi| \geqslant C$ and $i = 1, 2$, it follows from (2.5) and (2.12) that

$$N_i[\mathbf{u}^-](t,x) \leqslant D_i\delta\mu^2 p_i^+ \,\mathrm{e}^{\mu x_1} - kp_i^+\delta\,\mathrm{e}^{\mu x_1} \leqslant (D_i\mu^2 - k)p_i^+\delta\,\mathrm{e}^{\mu x_1}$$
$$\leqslant 0, \quad \text{for } i = 1, 2.$$

Similarly, one can prove that $N_i[\mathbf{u}^-](t,x) \leqslant 0$ for $(t,x) \in (-\infty, T] \times \overline{\Omega}$ such that $\underline{\xi}(t,x) \leqslant -C$.

Consequently, $\mathbf{u}^-(t,x)$ is a subsolution of (1.1) for $(t,x) \in (-\infty, T] \times \overline{\Omega}$. □

Let $h(x_1)$ be a C^2 nondecreasing function such that

$$h(x_1) = \begin{cases} 1, & \text{if } x_1 \geqslant -1, \\ \mathrm{e}^{\mu x_1}, & \text{if } x_1 \leqslant -3, \end{cases}$$

where $\mu = \delta/c$. Even if it means decreasing $\delta > 0$, assume

$$D_i h''(x_1) \leqslant kh(x_1) \quad \text{for } -3 \leqslant x_1 \leqslant -1 \text{ and } i = 1,\ 2. \tag{2.13}$$

Let $\pi(x_1)$ be a C^2 nonincreasing function such that

$$\pi(x_1) = \begin{cases} 0, & \text{if } x_1 \geqslant 0 \\ 1, & \text{if } x_1 \leqslant -1. \end{cases}$$

Remember that $\Phi(+\infty) = \mathbf{0}$. Then, for any $\varepsilon > 0$, there exists $R_\varepsilon \geqslant C$ large enough such that

$$C_2 \sum_{j,l=1,2} \phi_l(\xi)\phi_j(\xi) \leqslant \frac{kp_i^+}{2}\varepsilon, \quad \text{for all } \xi \geqslant R_\varepsilon \text{ and } i = 1, 2, \tag{2.14}$$

and

$$D_i\left(2|\phi_i'(\xi)|\|\pi'\|_{L^\infty} + |\phi_i(\xi)|\|\pi''\|_{L^\infty}\right) \leqslant \frac{kp_i^+}{2}\varepsilon \quad \text{for all } \xi \geqslant R_\varepsilon \text{ and } i = 1, 2. \tag{2.15}$$

For $(t,x) \in \mathbb{R} \times \overline{\Omega}$, define

$$\mathbf{u}^+(t,x) := \pi(x_1)\Phi(\overline{\xi}(t,x_1)) + \varepsilon\mathbf{p}(\overline{\xi}(t,x_1))h(x_1), \tag{2.16}$$

where $\overline{\xi}(t,x_1) = x_1 - ct - \alpha\,\mathrm{e}^{\delta t}$.

Lemma 2.6. *For any $0 < \varepsilon \leqslant \delta$, there exists $T_\varepsilon < 0$ such that $\mathbf{u}^+(t,x)$ is a supersolution of (1.1) for all $t \leqslant T_\varepsilon$ and $x \in \overline{\Omega}$.*

Proof. Take $T_\varepsilon < 0$ such that

$$\alpha\,\mathrm{e}^{\delta t} \leqslant 1 \quad \text{for } t \leqslant T_\varepsilon \quad \text{and} \quad -cT_\varepsilon \geqslant R_\varepsilon + 4. \tag{2.17}$$

For $x \in \overline{\Omega}$ such that $x_1 \geqslant 0$, it follows from the definitions of $h(x_1)$ and $\pi(x_1)$ that $h(x_1) = 1$ and $\pi(x_1) = 0$. Notice that $\overline{\xi}(t,x_1) \geqslant -ct - 1 \geqslant R_\varepsilon \geqslant M$ for all $t \leqslant T_\varepsilon$ and $x_1 \geqslant 0$. Thus, by (2.4), $\mathbf{p}(\overline{\xi}(t,x_1)) = \mathbf{p}^+$ and $\mathbf{u}^+(t,x) = \varepsilon\mathbf{p}^+$ for all $t \leqslant T_\varepsilon$ and $x_1 \geqslant 0$. In this case, it follows from (2.3) that

$$\begin{aligned}N_i[\mathbf{u}^+](t,x) &:= (u_i^+)_t - D_i\Delta u_i^+ - F^i(\mathbf{u}^+)\\ &= -F^i(\varepsilon\mathbf{p}^+) = -\sum_{j=1}^{2}\frac{\partial}{\partial u_j}F^i(\theta_i(t,x)\varepsilon\mathbf{p}^+)\varepsilon p_j^+ \geqslant kp_i^+\varepsilon \geqslant 0,\end{aligned}$$

where $\theta_i(t,x)\in(0,1)$ and $i=1,2$. Also notice that $\partial_\nu\mathbf{u}^+(t,x)=\mathbf{0}$ for all $x\in\partial\Omega$.

For $(t,x)\in(-\infty,T_\varepsilon]\times\overline{\Omega}$ such that $x_1\leqslant 0$, it follows from some calculation that

$$\begin{aligned}N_i[\mathbf{u}^+](t,x) = &-\pi(x_1)\phi_i'(\overline{\xi}(t,x))\alpha\delta\,\mathrm{e}^{\delta t} - (c+\alpha\delta\,\mathrm{e}^{\delta t})\varepsilon p_i'(\overline{\xi}(t,x_1))h(x_1)\\ &- D_i\left(2\phi_i'(\overline{\xi}(t,x_1))\pi'(x_1) + \pi''(x_1)\phi_i(\overline{\xi}(t,x_1))\right.\\ &+ \varepsilon p_i''(\overline{\xi}(t,x_1))h(x_1) + 2\varepsilon p_i'(\overline{\xi}(t,x_1))h'(x_1)\\ &\left.+ \varepsilon p_i(\overline{\xi}(t,x_1))h''(x_1)\right) + \pi(x_1)F^i(\Phi(\overline{\xi}(t,x_1))) - F^i(\mathbf{u}^+(t,x)).\end{aligned}$$

For $-1\leqslant x_1\leqslant 0$, one has that $h(x_1)=1$, $h'(x_1)=0$ and $h''(x_1)=0$. Meantime, by (2.17), $\overline{\xi}(t,x_1)\geqslant -1-ct-1\geqslant R_\varepsilon\geqslant C\geqslant M$ for $t\leqslant T_\varepsilon$ and $x_1\geqslant -1$. Then, $\mathbf{p}(\overline{\xi}(t,x_1))=\mathbf{p}^+$ and one has that

$$\begin{aligned}N_i[\mathbf{u}^+](t,x) = &-\pi(x_1)\phi_i'(\overline{\xi}(t,x))\alpha\delta\,\mathrm{e}^{\delta t} - D_i\left(2\phi_i'(\overline{\xi}(t,x_1))\pi'(x_1)\right.\\ &\left.+ \pi''(x_1)\phi_i(\overline{\xi}(t,x_1))\right) + \pi(x_1)F^i(\Phi(\overline{\xi}(t,x_1))) - F^i(\mathbf{u}^+(t,x)).\end{aligned}$$

By the mean-value theorem, it follows from (2.3) and (2.14) that

$$\begin{aligned}&\pi(x_1)F^i(\Phi(\overline{\xi}(t,x_1))) - F^i(\mathbf{u}^+(t,x))\\ &\quad= \pi(x_1)F^i(\Phi(\overline{\xi}(t,x_1))) - F^i(\pi(x_1)\Phi(\overline{\xi}(t,x_1)))\\ &\qquad+ F^i(\pi(x_1)\Phi(\overline{\xi}(t,x_1))) - F^i(\mathbf{u}^+(t,x))\\ &\quad= \pi(x_1)(1-\pi(x_1))\sum_{j=1}^{2}\sum_{l=1}^{2}\frac{\partial^2F^i}{\partial u_j\partial u_l}(\theta_2(t,x)\pi(x_1)\Phi(\overline{\xi}(t,x_1)))\\ &\qquad\times\theta_1(t,x)\phi_l(\overline{\xi}(t,x_1))\phi_j(\overline{\xi}(t,x_1))\\ &\qquad-\sum_{j=1}^{2}\frac{\partial F^i}{\partial u_j}(\pi(x_1)\Phi(\overline{\xi}(t,x_1)) + \theta_3(t,x)\varepsilon\mathbf{p}^+)\varepsilon p_j^+\\ &\quad\geqslant -C_2\sum_{j=1}^{2}\sum_{l=1}^{2}\phi_l(\overline{\xi}(t,x_1))\phi_j(\overline{\xi}(t,x_1)) + k\varepsilon p_i^+ \geqslant \frac{kp_i^+}{2}\varepsilon,\end{aligned}$$

where $\theta_{i1},\theta_{i2},\theta_{i3}\in(0,1)$ and $i=1,2$. Then, by $\phi_i'<0$ and (2.15), one has that

$$N_i[\mathbf{u}^+](t,x)\geqslant -\frac{kp_i^+}{2}\varepsilon + \frac{kp_i^+}{2}\varepsilon = 0.$$

For $-3\leqslant x_1\leqslant -1$, one has that $\pi(x_1)=1$, $\pi'(x_1)=0$ and $\pi''(x_1)=0$. Meantime, by (2.17), $\overline{\xi}(t,x_1)\geqslant -3-ct-1\geqslant R_\varepsilon\geqslant C\geqslant M$ for $t\leqslant T_\varepsilon$ and $x_1\geqslant -3$. Then, $\mathbf{p}(\overline{\xi}(t,x_1))=\mathbf{p}^+$ and one has that

$$\begin{aligned}N_i[\mathbf{u}^+](t,x) = &-\phi_i'(\overline{\xi}(t,x))\alpha\delta\,\mathrm{e}^{\delta t} - D_i\varepsilon p_i^+h''(x_1)\\ &+ F^i(\Phi(\overline{\xi}(t,x_1))) - F^i(\mathbf{u}^+(t,x)).\end{aligned}$$

By the mean-value theorem and (2.3), it follows that

$$F^i(\Phi(\overline{\xi}(t,x_1))) - F^i(\mathbf{u}^+(t,x)) = F^i(\Phi(\overline{\xi}(t,x_1))) - F^i(\Phi(\overline{\xi}(t,x_1)) + \varepsilon \mathbf{p}^+ h(x_1))$$
$$= -\sum_{j=1}^{2} \frac{\partial F^i}{\partial u_j} \left(\Phi(\overline{\xi}(t,x_1)) + \theta_i(t,x)\varepsilon \mathbf{p}^+ h(x_1)\right) \varepsilon p_j^+ h(x_1)$$
$$\geqslant k\varepsilon p_i^+ h(x_1).$$

Then, by $\phi_i' < 0$ and (2.13), one has that

$$N[\mathbf{u}^+](t,x) \geqslant -\varepsilon k p_i^+ h(x_1) + \varepsilon k p_i^+ h(x_1) = 0.$$

In the following, we consider for $(t,x) \in (-\infty, T_\varepsilon] \times \overline{\Omega}$ such that $x_1 \leqslant -3$. It means that $\pi(x_1) = 1$ and $h(x_1) = \mathrm{e}^{\mu x_1}$. Then, it follows that

$$\begin{aligned} N_i[\mathbf{u}^+](t,x) &= -\phi_i'(\overline{\xi}(t,x))\alpha\delta\,\mathrm{e}^{\delta t} - (c + \alpha\delta\,\mathrm{e}^{\delta t})\varepsilon p_i'(\overline{\xi}(t,x_1))\mathrm{e}^{\mu x_1} \\ &\quad - D_i\left(\varepsilon p_i''(\overline{\xi}(t,x_1)) + 2\varepsilon p_i'(\overline{\xi}(t,x_1))\mu + \varepsilon p_i(\overline{\xi}(t,x_1))\mu^2\right)\mathrm{e}^{\mu x_1} \\ &\quad + F^i(\Phi(\overline{\xi}(t,x_1))) - F^i(\mathbf{u}^+(t,x)). \end{aligned} \tag{2.18}$$

For $(t,x) \in (-\infty, T_\varepsilon] \times \overline{\Omega}$ such that $|\overline{\xi}(t,x_1)| \leqslant C$, it follows from the mean-value theorem that there exists $\theta_i \in (0,1)$, $i = 1, 2$ such that

$$F^i(\Phi(\overline{\xi}(t,x_1))) - F^i(\mathbf{u}^+(t,x)) = -\sum_{j=1}^{2} \frac{\partial F^i}{\partial u_j} \left(\Phi(\overline{\xi}(t,x_1)) + \theta_i(t,x) \times \varepsilon \mathbf{p}(\overline{\xi}(t,x_1))\mathrm{e}^{\mu x_1}\right) \varepsilon p_j(\overline{\xi}(t,x_1))\mathrm{e}^{\mu x_1}$$
$$\geqslant -C_1 C_2 \varepsilon\,\mathrm{e}^{\mu x_1},$$

where C_1 and C_2 are defined by (2.7) and (2.8). Notice that $\overline{\xi}(t,x_1) \leqslant C$ implies that $x_1 \leqslant C + ct + 1$ and hence $\mathrm{e}^{\mu x_1} \leqslant \mathrm{e}^{\mu(C+1)}\mathrm{e}^{c\mu t} = \mathrm{e}^{\mu(C+1)}\mathrm{e}^{\delta t}$ by $c\mu = \delta$. Then, it follows from (2.9), (2.10), (2.17) and $\varepsilon \leqslant \delta$ that

$$\begin{aligned} N_i[\mathbf{u}^+](t,x) &\geqslant C_3\alpha\delta\,\mathrm{e}^{\delta t} - (c+1)\varepsilon C_1\,\mathrm{e}^{\mu x_1} - D_i(\varepsilon C_1 + 2\varepsilon C_1\mu + \varepsilon C_1\mu^2)\mathrm{e}^{\mu x_1} \\ &\quad - C_1 C_2 \varepsilon\,\mathrm{e}^{\mu x_1} \geqslant 0. \end{aligned}$$

For $(t,x) \in (-\infty, T] \times \overline{\Omega}$ such that $\overline{\xi}(t,x_1) \geqslant C$, one has that $p_i^+(\overline{\xi}(t,x_1)) = p_i^+$. It follows from the mean-value theorem that

$$F^i(\Phi(\overline{\xi}(t,x_1))) - F^i(\mathbf{u}^+(t,x)) \geqslant k p_i^+ \varepsilon\,\mathrm{e}^{\mu x_1}.$$

Then, by (2.5), one has that

$$N_i[\mathbf{u}^+](t,x) \geqslant -D_i\varepsilon p_i^+ \mu^2\,\mathrm{e}^{\mu x_1} + k p_i^+ \varepsilon\,\mathrm{e}^{\mu x_1} \geqslant 0.$$

Similarly, one can prove that $N_i[\mathbf{u}^+](t,x) \geqslant 0$ for (t,x) such that $\overline{\xi}(t,x_1) \leqslant -C$.

This completes the proof. □

2.3. Existence, monotonicity and uniqueness of the entire solution

By referring to section 3.1 of [2] and considering a sequence of solutions $\mathbf{u}_n$ of (1.1) for $t > -n$ with initial value

$$\mathbf{u}_n(-n, x) = \mathbf{u}^-(-n, x),$$

one can easily get the existence of an entire solution $\mathbf{u}(t, x)$ of (1.1) satisfying

$$\mathbf{u}^-(t, x) \leqslant \mathbf{u}(t, x) \leqslant \mathbf{u}^+(t, x) \quad \text{for all } t \in (-\infty, T) \text{ and } x \in \Omega.$$

By definition of $\mathbf{u}^-$, $\mathbf{u}^+$ and remembering that ε can be arbitrary small, one then has that

$$|\mathbf{u}(t, x) - \Phi(x_1 - ct)| \to 0, \quad \text{as } t \to -\infty \text{ uniformly in } \overline{\Omega}.$$

Thereby, we have proved the existence of the entire solution of (1.1) satisfying (1.5).

By proposition 2.3, one can immediately get that $\mathbf{0} \ll \mathbf{u}(t, x) \ll \mathbf{1}$. Since $\mathbf{u}^-(t, x)$ is increasing in t for t negative enough, it follows from proposition 2.3 that $\mathbf{u}_n(t, x)$ is increasing in t. Then, letting $n \to +\infty$ and by proposition 2.3 together with (1.5), one has that $\mathbf{u}(t, x)$ is increasing in t.

For any $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$ such that $\mathbf{0} \ll \mathbf{a} \leqslant \mathbf{b} \ll \mathbf{1}$, define

$$\Omega_{[\mathbf{a},\mathbf{b}]}(t) := \{x \in \Omega, \mathbf{a} \leqslant \mathbf{u}(t, x) \leqslant \mathbf{b}\}.$$

By applying the same argument in the proof of [2, lemma 3.1] to every component of $\mathbf{u}(t, x)$, one can get the following lemma. Here, we should notice that the Hopf lemma is still true for $(u_i)_t(t, x)$ by assumption (A3).

Lemma 2.7. *For any $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$ such that $\mathbf{0} \ll \mathbf{a} \leqslant \mathbf{b} \ll \mathbf{1}$, there exist $T < 0$ and $K > 0$ such that*

$$(u_i)_t(t, x) \geqslant K \quad \text{for all } i = 1, 2,\ t \in (-\infty, T] \text{ and } x \in \Omega_{[\mathbf{a},\mathbf{b}]}(t).$$

The proof of the uniqueness of the entire solution satisfying (1.5) is basically similar to section 3.3 of [2] and section 3 of [3]. Only some slight modifications should be made for constructing sub- and supersolutions by applying the idea of Tsai [16]. We present the uniqueness in the following lemma and omit the proof.

Lemma 2.8. *If $\mathbf{u}(t, x)$ and $\mathbf{v}(t, x)$ are two entire solutions of (1.1) satisfying (1.5), then $\mathbf{u}(t, x) \equiv \mathbf{v}(t, x)$.*

3. Convergence to planar fronts on the right side

In this section, we prove theorem 1.2. We assume that Ω is a bilaterally straight cylinder, that is, there is $L > 0$ such that

$$\Omega = \{x_1 \in \mathbb{R}; x_1 \geqslant L\} \times \omega,$$

where $\omega \subset \mathbb{R}^{N-1}$. We first investigate the large time behaviour of the entire solution $\mathbf{u}(t, x)$ satisfying (1.5) on the right side of Ω.

Lemma 3.1. *There exist $t_1 \in \mathbb{R}$, $t_2 \in \mathbb{R}$, $\tau_1 \in \mathbb{R}$, $\delta_0 > 0$, $\delta > 0$ and $\mu > 0$ such that*

$$\mathbf{u}(t, x) \leqslant \Phi(x_1 - c(t - t_1) + \tau_1) + \delta_0 \mathbf{1} e^{-\delta(t-t_1)} + \delta_0 \mathbf{1} e^{-\mu(x_1 - L)} \tag{3.1}$$

for $t \geqslant t_1$ *and* $x \in \overline{\Omega}$ *such that* $x_1 \geqslant L$ *and*

$$\mathbf{u}(t,x) \geqslant \Phi(x_1 - c(t-t_2) - L) - \delta_0 \mathbf{1}\, e^{-\delta(t-t_2)} - \delta_0 \mathbf{1}\, \mathrm{e}^{-\mu(x_1-L)} \tag{3.2}$$

for $t \geqslant t_2$ *and* $x \in \overline{\Omega}$ *such that* $x_1 \geqslant L$.

The proof of lemma 3.1 is similar as the proof of lemma 3.1 of [9]. Only some slight modifications should be made for constructing sub- and supersolutions as we can see in (3.1) and (3.2). So, we omit the details of the proof.

By similar proofs as of lemma 3.3 and lemma 3.4 of [9] and virtue of lemma 3.1, we have the following lemmas.

Lemma 3.2. *For any* $\varepsilon > 0$, *there exists* $t_\varepsilon \in \mathbb{R}$ *such that*

$$\mathbf{u}(t,x) \geqslant \Phi(x_1 - c(t-t_\varepsilon) - L) - \varepsilon \mathbf{1}\, e^{-\delta(t-t_\varepsilon)} - \varepsilon \mathbf{1}\, \mathrm{e}^{-\mu(x_1-L)}$$

for all $t \geqslant t_\varepsilon$ *and* $x \in \overline{\Omega}$ *such that* $x_1 \geqslant L$, *with the same constants* $\delta > 0$ *and* $\mu > 0$ *as in lemma* 3.1.

Lemma 3.3. *There is* $M \geqslant 0$ *such that, if there are* $\varepsilon > 0$, $t_0 \in \mathbb{R}$ *and* $\tau \in \mathbb{R}$ *such that*

$$\sup_{x\in\overline{\Omega},\, x_1\geqslant L} \|\mathbf{u}(t_0,x) - \Phi(x_1 - ct_0 + \tau)\| \leqslant \varepsilon$$

together with $\|\mathbf{1} - \Phi(L - ct_0 + \tau)\| \leqslant \varepsilon$ *and* $\|\mathbf{1} - \mathbf{u}(t,x)\| \leqslant \varepsilon$ *for all* $t \geqslant t_0$ *and* $x \in \overline{\Omega}$ *with* $x_1 = L$, *then it holds*

$$\sup_{x\in\overline{\Omega},\, x_1\geqslant L} \|\mathbf{u}(t,x) - \Phi(x_1 - ct + \tau)\| \leqslant M\varepsilon \quad \textit{for all } t \geqslant t_0.$$

Proof of theorem 1.2. Let $t_1 \in \mathbb{R}$, $t_2 \in \mathbb{R}$, $\tau_1 \in \mathbb{R}$, $\delta_0 > 0$, $\delta > 0$ and $\mu > 0$ be as in lemma 3.1. For $t \geqslant \max(t_1, t_2)$ and $x \in \overline{\Omega}$ with $x \geqslant L$, one has that

$$\begin{aligned}&\Phi(x_1 - c(t-t_2) - L) - \delta_0 \mathbf{1}\, \mathrm{e}^{-\delta(t-t_2)} - \delta_0 \mathbf{1}\, \mathrm{e}^{-\mu(x_1-L)} \\ &\quad \leqslant \mathbf{u}(t,x) \leqslant \Phi(x_1 - c(t-t_1) + \tau_1) + \delta_0 \mathbf{1}\, \mathrm{e}^{-\delta(t-t_1)} + \delta_0 \mathbf{1}\, \mathrm{e}^{-\mu(x_1-L)}.\end{aligned} \tag{3.3}$$

Take a sequence $\{t_n\}_{n\in\mathbb{N}}$ such that $t_n \to +\infty$ as $n \to +\infty$. Let $\Omega_n = \Omega - (ct_n, 0)$ and it converges to a straight open cylinder Ω_∞ as $n \to +\infty$. Let $\mathbf{u}_n(t,y) = \mathbf{u}(t + t_n, y_1 + ct_n, y')$ which is defined in $\mathbb{R} \times \overline{\Omega_n}$. By standard parabolic estimates applied to every component of $\mathbf{u}(t,x)$, up to extraction of a subsequence, $\mathbf{u}_n(t,y)$ converge locally uniformly in $(t,y) \in \mathbb{R} \times \overline{\Omega_\infty}$ to a solution $\mathbf{u}_\infty(t,y)$ of

$$\begin{cases} (\mathbf{u}_\infty)_t - \Delta \mathbf{u}_\infty = \mathbf{F}(\mathbf{u}_\infty), & t \in \mathbb{R},\ y \in \Omega_\infty, \\ (\mathbf{u}_\infty)_\nu = 0, & t \in \mathbb{R},\ y \in \partial\Omega_\infty. \end{cases}$$

It follows from (3.3) that

$$\Phi(y_1 - c(t-t_2) - L) \leqslant \mathbf{u}_\infty(t,y) \leqslant \Phi(y_1 - c(t-t_1) + \tau_1)$$

for all $(t,y) \in \mathbb{R} \times \overline{\Omega_\infty}$. We need the following claim.

Claim 3.4. If Ω is a straight cylinder, that is, $\Omega = \mathbb{R} \times \omega$ where is $\omega \subset \mathbb{R}^{N-1}$, and the solution $\mathbf{u}(t, x)$ of (1.1) satisfies

$$\Phi(x_1 - ct + \sigma_1) \leqslant \mathbf{u}(t, x) \leqslant \Phi(x_1 - ct + \sigma_2),$$

for some $\sigma_1, \sigma_2 \in \mathbb{R}$, then there is $\sigma \in \mathbb{R}$ such that $\mathbf{u}(t, x) \equiv \Phi(x_1 - ct + \sigma)$.

We postpone the proof of this claim at the end of this section.

Therefore, there is $\tau \in \mathbb{R}$ such that

$$\mathbf{u}_n(t, y) = \mathbf{u}(t + t_n, y_1 + ct_n, y') \to \Phi(y_1 - ct + \tau) \quad \text{locally uniformly in} \\ \times \mathbb{R} \times \overline{\Omega_\infty} \text{ as } n \to +\infty.$$

The rest of the proof can similarly follow that of theorem 1.7 of [9] to get that

$$\mathbf{u}(t, x) \to \Phi(x_1 - c_f t + \tau) \quad \text{uniformly for } x \in \overline{\Omega} \text{ as } t \to +\infty.$$

This completes the proof. □

Proof of claim 3.4. Consider $\mathbf{v}(t, z, x') = \mathbf{u}(t, z + ct, x')$ where $z = x_1 - ct$. Then, $\mathbf{v}(t, z, x')$ satisfies

$$\begin{cases} \mathbf{v}_t = \mathbf{D}\Delta\mathbf{v} - c\partial_z\mathbf{v} + \mathbf{F}(\mathbf{v}), & t \in \mathbb{R},\ (z, x') \in \Omega \\ \partial_\nu \mathbf{v} = 0, & (z, x') \in \partial\Omega, \end{cases} \tag{3.4}$$

and

$$\Phi(z + \sigma_1) \leqslant \mathbf{v}(t, z, x') \leqslant \Phi(z + \sigma_2). \tag{3.5}$$

Notice that Ω is a straight cylinder and it is invariant with shifts in x_1-axis.

Define

$$\sigma^* = \sup\{\sigma \in [\sigma_2, \sigma_1]; \mathbf{v}(t, z, x') \leqslant \Phi(z + \sigma') \quad \text{in } \overline{\Omega} \text{ for any } \sigma' \in [\sigma_2, \sigma]\}.$$

We prove that $\mathbf{v}(t, z, x') \equiv \Phi(z + \sigma^*)$.

Assume that $\mathbf{v}(t, z, x') < \Phi(z + \sigma^*)$ for $(z, x') \in \overline{\Omega}$ such that $-C - \sigma_1 \leqslant z \leqslant C - \sigma_2$ where C is defined by (2.6). Then, $\sigma^* < \sigma_1$ and there is $\eta > 0$ such that $\sigma^* + \eta \leqslant \sigma_1$ and

$$\mathbf{v}(t, z, x') \leqslant \Phi(z + \sigma^* + \eta) \quad \text{for } (z, x') \in \overline{\Omega} \text{ such that } -C - \sigma_1 \leqslant z \leqslant C - \sigma_2. \tag{3.6}$$

Define

$$\Omega_t^- = \{(t, z, x') \in \mathbb{R} \times \Omega, z \leqslant -C - \sigma_1\} \quad \text{and} \quad \Omega_t^+ = \{(t, z, x') \in \mathbb{R} \times \Omega, z \geqslant C - \sigma_2\}.$$

By (3.5), one has that $\|\mathbf{1} - \mathbf{v}\| \leqslant \delta_0$ for $(t, z, x') \in \overline{\Omega_t^-}$ and $\|\mathbf{v}\| \leqslant \delta_0$ for $(t, z, x') \in \overline{\Omega_t^+}$. Define

$$\varepsilon^* = \inf\{\varepsilon > 0, \mathbf{v}(t, z, x') \leqslant \Phi(z + \sigma^* + \eta) + \varepsilon\mathbf{p}^- \quad \text{in } \overline{\Omega_t^-}\}.$$

Since $\|\mathbf{1} - \Phi(z + \sigma^* + \eta)\| \leqslant \delta \min_{i=1,2}\{p_i^\pm\}$ for $(t, z, x') \in \overline{\Omega_t^-}$ and $\mathbf{v}(t, x) \leqslant \mathbf{1}$, it means that $\varepsilon^* \leqslant \delta$. Assume that $\varepsilon^* > 0$. Then, there exist sequences ε_n and (t_n, z_n, x_n') such that $\varepsilon_n \xrightarrow{<} \varepsilon^*$ as $n \to +\infty$ and

$$\mathbf{v}(t_n, z_n, x_n') > \Phi(z_n + \sigma^* + \eta) + \varepsilon_n\mathbf{p}^- \quad \text{in } \overline{\Omega_t^-}. \tag{3.7}$$

If $z_n \to -\infty$ as $n \to +\infty$, one has that $\mathbf{v}(t_n, z_n, x'_n) \to \mathbf{1}$ and $\Phi(z_n + \sigma^* + \eta) \to \mathbf{1}$. By (3.7) and passing $n \to +\infty$, one gets that

$$\mathbf{1} \geqslant \mathbf{1} + \varepsilon_* \mathbf{p}^-$$

which is a contradiction. Thus, z_n is bounded. Since x'_n is also bounded, there are z_* and x'_* such that $z_n \to z_*$ and $x'_n \to x'_*$ as $n \to +\infty$. Assume without loss of generality that $t_n \to t_*$ as $n \to +\infty$. Otherwise, one can discuss for the limit of $v(t + t_n, z, x')$. Thus, it follows from (3.7) and the definition of ε^*

$$v(t_*, z_*, x'_*) = \Phi(z_* + \sigma^* + \eta) + \varepsilon^* \mathbf{p}^-.$$

One can easily check that $\Phi(z + \sigma^* + \eta) + \varepsilon^* \mathbf{p}^-$ is a supersolution of (3.4) in Ω_t^-. By proposition 2.4, we have that

$$\mathbf{v}(t, z, x') = \Phi(z + \sigma^* + \eta) + \varepsilon^* \mathbf{p}^-$$

which is a contradiction since $\mathbf{v}(t, z, x') \to \mathbf{1}$ and $\Phi(z + \sigma^* + \eta) \to \mathbf{1}$ as $z \to -\infty$. Therefore, $\varepsilon^* = 0$ and $\mathbf{v}(t, z, x') \leqslant \Phi(z + \sigma^* + \eta)$ in $\overline{\Omega_t^-}$.

Similarly, one can prove that $\mathbf{v}(t, z, x') \leqslant \Phi(z + \sigma^* + \eta)$ in $\overline{\Omega_t^+}$. Then, together with (3.6), it follows that

$$\mathbf{v}(t, z, x') \leqslant \Phi(z + \sigma^* + \eta) \quad \text{for } (t, z, x') \in \mathbb{R} \times \overline{\Omega}.$$

It contradicts the definition of σ^*.

Therefore,

$$\inf \|\mathbf{v}(t, z, x') - \Phi(z + \sigma^*)\| = 0 \quad \text{for } (z, x') \in \overline{\Omega} \text{ such that } -C - \sigma_1 \leqslant z \leqslant C - \sigma_2.$$

Then, there is $(t_0, z_0, x'_0) \in \mathbb{R} \times \{(z, x') \in \overline{\Omega}; -C - \sigma_1 \leqslant z \leqslant C - \sigma_2\}$ such that $\mathbf{v}(t_0, z_0, x'_0) = \Phi(z_0 + \sigma^*)$. By proposition 2.4, one has that

$$\mathbf{v}(t, z, x') \equiv \Phi(z + \sigma^*).$$

This completes the proof of claim 3.4. □

4. Some geometrical conditions

In this section, we give some geometrical conditions on Ω such that the propagation of the entire solution $\mathbf{u}(t, x)$ satisfying (1.5) is complete or incomplete, respectively. Since $\mathbf{u}(t, x)$ is increasing in t and by applying parabolic estimates to every component of $\mathbf{u}(t, x)$, there exists a solution $\mathbf{u}_\infty(x)$ of the following system

$$\begin{cases} \mathbf{D}\Delta \mathbf{u}_\infty + \mathbf{F}(\mathbf{u}_\infty) = \mathbf{0}, & x \in \Omega \\ \partial_\nu \mathbf{u}_\infty = \mathbf{0}, & x \in \partial\Omega, \end{cases}$$

such that $\lim_{t \to +\infty} \mathbf{u}(t, x) = \mathbf{u}_\infty(x)$ in $C^1_{\mathrm{loc}}(\Omega)$. To prove the completeness or incompleteness of the propagation, one only has to prove whether $\mathbf{u} \equiv 1$ or not.

4.1. Some geometrical conditions for complete propagation

This subsection is devoted to the proof of theorem 1.4, that is, under every condition (a)–(c), we prove that $\mathbf{u}_\infty(x) \equiv 1$. We will need the positive solution of the following problem

$$\begin{cases} \mathbf{D}\Delta\mathbf{u} + \mathbf{F}(\mathbf{u}) = 0, & \text{in } B(0,R), \\ \mathbf{u} = 0, & \text{on } \partial B(0,R), \end{cases} \tag{4.1}$$

which is well known for the scalar case by [4].

Lemma 4.1. *There exists a positive constant R_0 such that for any $R \geqslant R_0$, the system (4.1) has a positive symmetric solution $\mathbf{u}(x) = \mathbf{z}(|x|)$ and $\mathbf{z}(r)$ is decreasing in r for $0 < r < R$.*

Proof. This result is equivalent to find a solution for the following problem

$$\begin{cases} \mathbf{D}\left(\dfrac{\partial^2\mathbf{z}}{\partial r^2} + \dfrac{\partial\mathbf{z}}{\partial r}\dfrac{N-1}{r}\right) + \mathbf{F}(\mathbf{z}) = 0, & \text{in } 0 < r < R, \\ \mathbf{z}' = 0, & \text{on } r = 0, \\ \mathbf{z} = 0, & \text{on } r = R. \end{cases} \tag{4.2}$$

Let δ_0, k, $\mathbf{p}(\xi)$, δ, C, C_1, C_2 and C_3 all be defined as in section 2.2. Even if it means picking δ smaller, assume additionally that

$$\delta(1 + C_1C_2) \leqslant cC_3. \tag{4.3}$$

Take a constant $\delta_1 \leqslant \delta$ small enough such that

$$\delta_1(2 + 2C_2 + C_2 \max_{i=1,2}\{p_i^-\}) \leqslant k\delta \min_{i=1,2}\{p_i^\pm\}. \tag{4.4}$$

Let $R_1 \geqslant C > 0$ large enough such that

$$\|\mathbf{1} - \Phi(\xi)\| \leqslant \delta_1 \quad \text{for } \xi \leqslant -R_1. \tag{4.5}$$

Take an increasing C^2 function $h_1(r)$ and a constant $R_2 > 0$ such that

$$h_1(r) = 0 \quad \text{for } r \leqslant 0 \quad \text{and} \quad h_1(r) = 1 \quad \text{for } r \geqslant R_2.$$

One can make R_2 large enough such that $h'(r)$ and $h''(r)$ are small enough satisfying

$$D_i\left(\|h_1''\|_{L^\infty} + 2\|h_1'\|_{L^\infty}\|\phi_i'\|_{L^\infty} + \delta_1 p_i^-\|h_1''\|_{L^\infty}\right) \leqslant \delta_1 \quad \text{for all } i = 1,\ 2. \tag{4.6}$$

Let $R_3 > 0$ large enough such that

$$D_i(\|\phi_i'\|_{L^\infty} + \|h_1'\|_{L^\infty}\delta_1 p_i^-)\frac{N-1}{r} \leqslant \delta_1, \quad \text{for } r \geqslant R_3 \text{ and } i = 1, 2. \tag{4.7}$$

Let $R_0 = C + R_1 + R_2$. Take any $R \geqslant R_0 + R_3$. Define

$$h(r) = h_1(r - (R - R_0)).$$

For $r \in B(0,R)$, define a function $\mathbf{z}^-(r)$ by

$$\mathbf{z}^-(r) := \max\left(h(r)\Phi(r - R + C) + (1 - h(r))(\mathbf{1} - \delta_1\mathbf{p}^-) - \delta\mathbf{p}(r - R + C), \mathbf{0}\right).$$

Let us check that $\mathbf{z}^-(x)$ is a subsolution of (4.2). After some computation, one has that

$$N_i := D_i\left(\frac{\partial^2 z_i^-}{\partial r^2} + \frac{\partial z_i^-}{\partial r}\frac{N-1}{r}\right) + F^i(\mathbf{z}^-)$$
$$= D_i\left(h''(r)\phi_i + 2h'(r)\phi_i' + h(r)\phi_i'' - h''(r)\delta_1 p_i^- + \delta p_i''(r-R+C)\right.$$
$$\left. - (h'(r)\phi_i + h(r)\phi_i' - h'(r)\delta_1 p_i^- + \delta p_i'(r-R+C))\frac{N-1}{r}\right)$$
$$+ F^i(h(r)\Phi + (1-h(r))(\mathbf{1}-\delta_1\mathbf{p}^-) - \delta\mathbf{p}(r-R+C)).$$

For $0 \leqslant r \leqslant R-R_0$, one has that $h(r) = h_1(r-(R-R_0)) \equiv 0$ and $r-R+C \leqslant -R_1 - R_2 \leqslant -M$. Then, by (2.4), $\mathbf{p}(r-R+C) = \mathbf{p}^-$ and $\mathbf{z}^-(r) = \mathbf{1} - (\delta_1+\delta)\mathbf{p}^-$ for $0 \leqslant r \leqslant R-R_0$. Thus, $\mathbf{z}^-(r)$ satisfies the boundary condition $\frac{\mathrm{d}\mathbf{z}^-}{\mathrm{d}r}(0) = 0$. Notice that

$$\|\mathbf{1} - \mathbf{z}^-(r)\| \leqslant (\delta_1+\delta)\max_{i=1,\ 2}\{p_i^-\} \leqslant 2\delta\max_{i=1,\ 2}\{p_i^-\} \leqslant \delta_0$$

for $0 \leqslant r \leqslant R-R_0$ and hence

$$F^i(\mathbf{z}^-(r)) = -\sum_{j=1}^{2}\frac{\partial F^i}{\partial u_j}(\mathbf{1} - \theta_i(\delta_1+\delta)\mathbf{p}^-)(\delta_1+\delta)p_j^- \geqslant (\delta_1+\delta)kp_i^- \geqslant 0,$$

where $\theta_i \in (0,1)$ by (2.3). Then, it follows that

$$N_i = F_i(\mathbf{z}^-) \geqslant 0 \quad \text{for } 0 \leqslant r \leqslant R-R_0.$$

For $R-R_0 \leqslant r \leqslant R-R_0+R_2$, one has that $r-R+C \leqslant -R_1 \leqslant -M$ and hence $\mathbf{p}(r-R+C) = \mathbf{p}^-$. Moreover, by (4.5), one has $\|\mathbf{1} - \Phi(r-R+C)\| \leqslant \delta_1$. Then, if follows from (1.3) that

$$N_i = D_i\left(h''(r)\phi_i + 2h'(r)\phi_i' + h(r)\phi_i'' - h''(r)\delta_1 p_i^-\right.$$
$$\left. + (h'(r)\phi_i + h(r)\phi_i' - h'(r)\delta_1 p_i^-)\frac{N-1}{r}\right)$$
$$+ F^i(h(r)\Phi + (1-h(r))(1-\delta_1\mathbf{p}^-) + \delta\mathbf{p}^-)$$
$$\geqslant D_i\left(-\|h''\|_{L^\infty} - 2\|h'\|_{L^\infty}\|\phi_i'\|_{L^\infty} - \delta_1 p_i^-\|h''\|_{L^\infty}\right.$$
$$\left. - (\|\phi_i'\|_{L^\infty} + \delta_1 p_i^-\|h'\|_{L^\infty})\frac{N-1}{r}\right)$$
$$- h(r)c\phi_i' - h(r)F^i(\Phi) + F^i\left(h(r)\Phi(r-R+C)\right.$$
$$\left. + (1-h(r))(\mathbf{1}-\delta_1\mathbf{p}^-) - \delta\mathbf{p}^-\right).$$

The mean-value theorem implies that

$$F^i(\Phi) \leqslant \sum_{j=1}^{2}\frac{\partial F^i}{\partial u_j}(\mathbf{1} - \theta_i(\mathbf{1}-\Phi))(1-\phi_j) \leqslant C_2\delta_1,$$

and

$$
\begin{aligned}
&F^i(h(r)\Phi + (1-h(r))(\mathbf{1}-\delta_1\mathbf{p}^-) - \delta\mathbf{p}^-)\\
&\quad = -\sum_{j=1}^{2}\frac{\partial F^i}{\partial u_j}(1-\theta_i(h(r)(\mathbf{1}-\Phi) - (1-h(r))\delta_1\mathbf{p}^- - \delta\mathbf{p}^-))\\
&\qquad \times [h(r)(1-\phi_j) + (1-h(r))\delta_1 p_j^- + \delta p_j^-]\\
&\quad \geqslant -C_2\delta_1 - C_2\delta_1 \max_{j=1,2}\{p_j^-\} + \delta k p_i^-.
\end{aligned}
$$

Since $r \geqslant R - R_0 \geqslant R_3$, it follows from (4.4), (4.6), (4.7) and $\phi_i' < 0$ that

$$
N_i \geqslant -\delta_1 - \delta_1 - C_2\delta_1 - C_2\delta_1 - C_2\delta_1 \max_{j=1,2}\{p_j^-\} + \delta k p_i^- \geqslant 0.
$$

Now, for $R - R_0 + R_2 \leqslant r \leqslant R - 2C$, one has $h(r) = 1$, $r - R + C \leqslant -C \leqslant -M$ and hence $\mathbf{p}(r - R + C) = \mathbf{p}^-$. Thus, $\mathbf{z}^-(r) = \Phi(r - R + C) + \delta\mathbf{p}^-$. Then, it follows from (1.3) that

$$
\begin{aligned}
N_i &= D_i\left(\phi_i'' + \phi'\frac{N-1}{r}\right) + F^i(\Phi(r-R+C) + \delta\mathbf{p}^-)\\
&\geqslant -D_i\|\phi_i'\|_{L^\infty}\frac{N-1}{r} - c\phi_i' + F^i(\Phi(r-R+C) - \delta\mathbf{p}^-) - F^i\,(\Phi(r-R+C).
\end{aligned}
\tag{4.8}
$$

By the mean-value theory, one has that

$$
\begin{aligned}
&F^i(\Phi(r-R+C) - \delta\mathbf{p}^-) - F^i(\Phi(r-R+C))\\
&\quad = -\sum_{j=1}^{2}\frac{\partial}{\partial u_j}F(\Phi - \theta_i\delta\mathbf{p}^-)\delta p_j^- \geqslant k\delta p_i^-,
\end{aligned}
$$

where $\theta_i \in (0, 1)$. By (4.4), (4.7) and $\phi_i' < 0$, one gets that

$$
N_i \geqslant -\delta_1 + \delta k p_i^- \geqslant 0.
$$

For $R - 2C \leqslant r \leqslant R$, one has that $-C \leqslant r - R + C \leqslant C$ and $-\phi_i'(r - R + C) \geqslant C_3$. Then, by (4.3) and (4.8), one has that

$$
\begin{aligned}
N_i &\geqslant -D_i\|\phi_i'\|_{L^\infty}\frac{N-1}{r} - c\phi_i' - \delta C_1 C_2\\
&\geqslant -\delta_1 + cC_3 - \delta C_1 C_2 \geqslant 0.
\end{aligned}
$$

Notice that $r - R + C = C \geqslant M$ for $r = R$. Hence, $\Phi(r - R + C) \leqslant \delta \min_{i=1,2}\{p_i^+\}$ and $\mathbf{p}(r - R + C) = \mathbf{p}^+$ for $r = R$. Therefore, one has that $\mathbf{z}^-(R) = 0$. In conclusion, $\mathbf{z}^-(r)$ is a subsolution of (4.2).

Now we consider the Cauchy problem

$$
\begin{cases}
\mathbf{z}_t - \mathbf{D}\left(\dfrac{\partial^2\mathbf{z}}{\partial r^2} + \dfrac{\partial\mathbf{z}}{\partial r}\dfrac{N-1}{r}\right) = \mathbf{F}(\mathbf{z}), & \text{in } t > 0,\ 0 < r < R,\\
\mathbf{z}' = 0, & \text{on } r = 0,\\
\mathbf{z} = 0, & \text{on } r = R,\\
\mathbf{z}(0, r) = \mathbf{z}^-(r).
\end{cases}
\tag{4.9}
$$

Since $\mathbf{z}^{-}(r)$ is a subsolution and by proposition 2.3, one has that $\mathbf{z}(r)$ is increasing in r. By parabolic estimates, one has that $\mathbf{z}_\infty(r) := \mathbf{z}(+\infty, r)$ is a solution of (4.2) and it satisfies $\mathbf{0} \ll \mathbf{z}_\infty(r) \ll \mathbf{1}$ for $0 < r < R$.

Then, we prove that $\mathbf{z}_\infty(r)$ is decreasing in r. We simply denote $\mathbf{z}_\infty(r)$ by $\mathbf{z}(r)$. Since $\mathbf{z}(R) = \mathbf{0}$, it follows from proposition 2.4 that $\mathbf{z}_i'(R) < 0$ for $i = 1, 2$. Therefore, for $a \in (0, R)$ close to R enough, one has that

$$\mathbf{z}(r) \leqslant \mathbf{z}(2a - r) \quad \text{for } a \leqslant r \leqslant a + \min\{R - a, a\}. \tag{4.10}$$

For $a \geqslant R/2$, one has that $\min\{R - a, a\} = R - a$ and $\mathbf{z}(r) = \mathbf{0} \ll \mathbf{z}(2a - r)$ for $r = a + \min\{R - a, a\}$. By proposition 2.4, it implies that

$$\mathbf{z}(r) \ll \mathbf{z}(2a - r) \quad \text{for } a < r \leqslant a + \min\{R - a, a\} \quad \text{and} \quad z_i'(a) < 0 \quad \text{for } i = 1, 2, \tag{4.11}$$

for a close to R enough. Then, one can decrease a a little such that (4.10) still holds. By above argument and again by proposition 2.4, one can get that (4.11) still holds for the decreased a. By iteration, it follows that (4.11) holds for all $a \geqslant R/2$. Then, there is $\eta > 0$ small enough such that (4.10) holds for $a = R/2 - \eta$. Notice that in this case, $\min\{R - a, a\} = a$ and $\mathbf{z}'(2a - r) = \mathbf{z}'(0) = \mathbf{0}$ for $r = a + \min\{R - a, a\}$. While, $\mathbf{z}'(r) < 0$ for $r = a + \min\{R - a, a\} = 2a$ since $a = R/2 - \eta$ and η is small. Then, by proposition 2.4, one has that (4.11) holds for $a = R/2 - \eta$. By similar arguments as above, one can decrease δ again and finally get that (4.11) holds for all $a > 0$. Thus, $\mathbf{z}(r)$ is decreasing in r for $0 < r < R$.

This completes our proof. □

Proof of theorem 1.4.

(a) Let δ_0, k, $\mathbf{p}(\xi)$ and δ be defined as in section 2.2. Since the entire solution $\mathbf{u}(t, x)$ satisfies (1.5), there is T_{δ_0} such that

$$|\mathbf{u}(T_{\delta_0}, x) - \Phi(x_1 - cT_{\delta_0})| \leqslant \delta_0. \tag{4.12}$$

Define

$$\mathbf{v}(t, x) := \Phi(\xi(t, x)) - \delta \mathbf{p}(\xi(t, x)) \mathrm{e}^{-\delta t},$$

where $\xi(t, x) = x_1 - c(t + T_{\delta_0}) + \alpha(1 - \mathrm{e}^{-\delta t})$ and α can be taken sufficiently large. Notice that $\mathbf{u}(T_{\delta_0}, x) \geqslant \mathbf{v}(0, x)$ by (4.12). It can be easily checked that $\mathbf{v}(t, x)$ satisfies

$$\mathbf{v}_t - \Delta \mathbf{v} - F(\mathbf{v}) \leqslant \mathbf{0}, \quad \text{for } t \geqslant 0 \text{ and } x \in \Omega.$$

Since $\nu_1(x) \geqslant 0$ for any $x \in \partial\Omega$, one has that $(v_i)_\nu(t, x) = \phi_i'(\xi(t, x))\nu_1(x) \leqslant 0$ for $x \in \partial\Omega$ and $i = 1, 2$. Therefore, $\mathbf{v}(t, x)$ is a subsolution of (1.1). It follows from proposition 2.3 that $\mathbf{u}(t + T_{\delta_0}, x) \geqslant \mathbf{v}(t, x)$ for $t \geqslant 0$ and $x \in \Omega$. Thus, by $\Phi(-\infty) = \mathbf{1}$, one has that

$$\mathbf{u}(t, x) \to \mathbf{1} \quad \text{locally uniformly in } \overline{\Omega} \text{ as } t \to +\infty.$$

(b) Take R_0 large enough such that lemma 4.1 holds. Let $\mathbb{R} \times B'_{R_0} \subset \Omega$. Since $\mathbf{u}(t, x)$ satisfies (1.5) and $\Phi(-\infty) = \mathbf{1}$ and $\mathbf{u}(t, x)$ is increasing in t, there is $a_0 > 0$ large enough such that for any $a \geqslant a_0$,

$$\mathbf{u}_\infty(x_1 - a, x') \geqslant \mathbf{z}_{R_0}(|x|) \quad \text{for all } x \in B_{R_0}.$$

Let $\mathbf{u}^a := \mathbf{u}_\infty(x_1 - a, x')$. We claim that

$$\mathbf{u}^a(x) \geqslant \mathbf{z}_{R_0}(|x|), \quad \text{for any } a \in \mathbb{R}. \tag{4.13}$$

Define that

$$a^* = \inf\{a \in \mathbb{R}; \mathbf{u}_\infty(x_1 - a, x') \geqslant \mathbf{z}_{R_0}(|x|) \text{ in } B_{R_0}\}.$$

We only have to prove that $a^* = -\infty$. Assume, by contradiction, that $a^* > -\infty$. Then, there exist sequences $\{a_n\}_{n\in\mathbb{N}}$ of $\mathbb{R}$ and $\{x_n\}_{n\in\mathbb{N}}$ of B_{R_0} such that $a_n \overset{<}{\longrightarrow} a^*$ and $\mathbf{u}^{a_n}(x_n) < \mathbf{z}_{R_0}(|x_n|)$. Since $x_n \in B_{R_0}$, there is $y \in \overline{B_{R_0}}$ such that $x_n \to y$. By the definition of a^*, one has that $\mathbf{u}^{a^*}(y) = \mathbf{z}_{R_0}(|y|)$. Since $\mathbf{u}^{a^*}(x) \gg \mathbf{0}$ and $z_{R_0}(x) = 0$ for $x \in \partial B_{R_0}$, one has that $y \in B_{R_0}$. It then contradicts to the maximum principle. Therefore, claim (4.13) holds.

By claim (4.13) and proposition 2.4, we have

$$\mathbf{u}_\infty(x_1, x') \gg \mathbf{z}_{R_0}(|x'|) \quad \text{for all } x \in \Omega. \tag{4.14}$$

For $h > 0$ and $e \in \mathbb{S}^{N-2}$, define

$$\psi_{h,e}(x') := \mathbf{z}_{R_0}(|x' - he|) \quad \text{and} \quad \varphi_h := \max_{e\in\mathbb{S}^{N-2}} \psi_{h,e}(x').$$

By (4.14), we know that $\mathbf{u}_\infty(x) \gg \varphi_0(x')$. Since Ω is axially star-shaped and by proposition 2.4 and following the same arguments as in [2, section 7], one can prove that

$$\mathbf{u}_\infty(x) \geqslant \varphi_h(x') \quad \text{for all } x \in \Omega \text{ and all } h \geqslant 0.$$

Therefore, one has that

$$\mathbf{u}_\infty(x) \geqslant \max_{e\in\mathbb{S}^{N-2}} z_{R_0}(|x' - he|) \quad \text{for all } \ x \in \overline{\Omega}.$$

By the proof of lemma 4.1, one knows that $\max \mathbf{z}_{R_0}(r) = \mathbf{z}_{R_0}(0) \geqslant \mathbf{1} - 2\delta\mathbf{p}^-$. Moreover, one can easily check that $\mathbf{1} - 2\delta\mathbf{p}^-\,\mathrm{e}^{-\delta t}$ is a subsolution of (1.1) for $t \geqslant 0$. Then, $\mathbf{u}_\infty(x) \geqslant \mathbf{1} - 2\delta\mathbf{p}^-\,\mathrm{e}^{-\delta t}$ and hence $\mathbf{u}_\infty(x) \equiv \mathbf{1}$.

(c) Once we have lemma 4.1 and proposition 2.4, we can follow the proof of [9, corollary 1.12] to prove the conclusion. Here, we omit the details. □

4.2. Some geometrical conditions for incomplete propagation

This subsection is devoted to the proof of theorem 1.5. We first announce some notions. By (2.2) and (2.3), there are $\mathbf{p} \gg 0$, positive constants h, δ and a region

$$R := \{\mathbf{u} \in \mathbb{R}^2, |\mathbf{u} - \mathbf{p}| \leqslant \delta\},$$

such that

$$F^i(\mathbf{u}) \leqslant -h, \quad \text{in } R \text{ for } i = 1,\ 2.$$

This implies that $\mathbf{p}$ is close to $\mathbf{0}$. Let $M > 0$ such that

$$|F^i(\mathbf{u})| \leqslant M \quad \text{for } \mathbf{u} \in [\mathbf{0}, \mathbf{1}] \text{ and } i = 1, 2. \tag{4.15}$$

Let

$$B_i = 1 + \frac{M}{D_i} \quad \text{for } i = 1, 2.$$

Take a bistable function $g(u)$ satisfying

$$g(0) = g(1) = g(\theta) = 0, \quad g(s) < 0 \text{ for } 0 < s < \theta, \quad g(s) > 0 \text{ for } \theta < s < 1 \text{ and } \int_0^1 g(s)\mathrm{d}s > 0,$$

where $\theta \in (0, 1)$. Take constants $-\infty < a < b < +\infty$ and $\varepsilon > 0$. Let Ω satisfy theorem 1.4. Define

$$\Omega' = \{x \in \Omega, x_1 > a\}.$$

Then, from [2], one knows that there exists $\varepsilon > 0$ small enough such that the following equation

$$\begin{cases} \Delta z + g(z) = 0, & \text{in } \Omega', \\ \partial z = 0, & \text{on } \partial\Omega' \setminus \{x_1 = a\}, \\ z = 1, & \text{on } \{x_1 = a\} \end{cases} \tag{4.16}$$

has a positive solution $z(x)$ satisfying

$$\lim_{x_1 \to +\infty} z(x) = 0.$$

Moreover, $z(x)$ is close to $\omega_0(x)$ defined by

$$\omega_0(x) = \begin{cases} \dfrac{|x_1 - b|}{b - a} & \text{for } x_1 \in [a, b), \ x \in \Omega' \\ 0 & \text{for } x_1 \in [b, +\infty], \ x \in \Omega', \end{cases}$$

in $H^1(\Omega')$ norm. Let $A_i := \min\{2\delta, h/D_i \|g\|_{L^\infty([0,1])}\}$ for $i = 1, 2$. One can make $|b - a|$ small enough such that

$$-\frac{\partial z}{\partial x_1} \geqslant \frac{1}{A_i}\left(1 + \frac{M}{D_i}\right) \quad \text{on } x_1 = a.$$

Define functions $\overline{w}_i(x)$ as following

$$\overline{w}_i(x) = \begin{cases} p_i + \delta - A_i(1 - z(x)) & \text{if } x_1 \geqslant a, \\ p_i + \delta - \dfrac{M}{D_i}(x_1 - a)^2 - \beta_i(x_1 - a) & \text{if } x_1 < a, \end{cases}$$

for $i = 1, 2$ and $\overline{\mathbf{w}} := (\overline{w}_1, \overline{w}_2)$. Obviously, $\overline{\mathbf{w}}(x)$ is well defined and continuous in x_1. Then, let

$$\mathbf{w}^+(x) = \begin{cases} \mathbf{1}, & \text{for } x_1 \leqslant a - 1, \\ \min\{\mathbf{1}, \overline{\mathbf{w}}(x)\}, & \text{for } x_1 \geqslant a - 1. \end{cases}$$

Notice that for $x_1 = a - 1$,

$$\overline{w}_i(x) = p_i + \delta - \frac{M}{D_i} + B_i = 1 + p_i + \delta > 1.$$

Thus, $\mathbf{w}^+(x)$ is well defined and continuous in x_1.

We then check that $\mathbf{w}^+(x)$ satisfies

$$D_i \Delta w_i^+ + F^i(\mathbf{w}^+) \leqslant 0 \quad \text{for } i = 1, 2.$$

For $a - 1 \leqslant x_1 \leqslant a$, one has that $\overline{w}_i(x) = p_i + \delta - \frac{M}{D_i}(x_1 - a)^2 - B_i(x_1 - a)$. It follows from (4.15) that

$$D_i \Delta \overline{w}_i + F^i(\overline{\mathbf{w}}) = -2M + F^i(\overline{\mathbf{w}}) \leqslant 0 \quad \text{for } i = 1, 2.$$

For $x_1 \geqslant a$, one has that $\overline{w}_i(x) = p_i + \delta - A_i(1 - z(x))$. Since $A_i \leqslant 2\delta$ and $0 \leqslant z(x) \leqslant 1$, it means that $p_i - \delta \leqslant \overline{w}_i(x) \leqslant p_i + \delta$. Hence, $F^i(\overline{\mathbf{w}}) \leqslant -h$. It follows from (4.16) and $A_i \leqslant h/D_i \|g\|_{L^\infty([0,1])}$ that

$$\begin{aligned} D_i \Delta \overline{w}_i + F^i(\overline{\mathbf{w}}) &= D_i A_i \Delta z + F^i(\overline{\mathbf{w}}) \\ &= -D_i A_i g(z) + F^i(\overline{\mathbf{w}}) \leqslant h - h = 0. \end{aligned}$$

Finally, we can notice that $\overline{w}_i(x)$ is decreasing in x_1 for $a - 1 \leqslant x_1 < a$ and

$$\lim_{x_1 \overset{>}{\longrightarrow} a} \frac{\overline{w}_i}{\partial x_1}(a) = A_i \frac{\partial z}{\partial x_1}(a) \leqslant -\left(1 + \frac{M}{D_i}\right) \quad \text{and}$$

$$-\lim_{x_1 \overset{<}{\longrightarrow} a} \frac{\overline{w}_i}{\partial x_1}(a) = B_i = \left(1 + \frac{M}{D_i}\right).$$

By definition 2.2, $\mathbf{w}^+(x)$ is a supersolution of (1.1). Then, $\mathbf{u}(t, x) \leqslant \mathbf{w}^+(x)$ for all $t \in \mathbb{R}$ and $x \in \overline{\Omega}$.

Therefore, $\mathbf{u}_\infty(x) \leqslant \mathbf{w}^+(x)$ for all $x \in \overline{\Omega}$. By the definition of $\mathbf{w}^+(x)$, we then have

$$\limsup_{x_1 \to +\infty} \mathbf{u}_\infty(x) \leqslant \mathbf{p} + \delta \mathbf{1}. \tag{4.17}$$

Take any sequence $\{x_{1n}\}_{n \in \mathbb{N}}$ such that $x_{1n} \to +\infty$ as $n \to +\infty$. Let $\mathbf{v}_n(x) = \mathbf{u}_\infty(x_1 + x_{1n}, x')$ and $\Omega_n = \Omega - x_{1n}$. Then, there is Ω_∞ such that $\Omega_n \to \Omega_\infty$ as $n \to +\infty$ and $\mathbf{v}_n(x)$ converge to a solution of

$$\begin{cases} \mathbf{D}\Delta \mathbf{v}_\infty + \mathbf{F}(\mathbf{v}_\infty) = \mathbf{0}, & x \in \Omega_\infty \\ \partial_\nu \mathbf{v}_\infty = \mathbf{0}, & x \in \partial\Omega_\infty. \end{cases}$$

By (4.17), we have $\mathbf{v}_\infty(x) \leqslant \mathbf{p} - \delta \mathbf{1}$ which implies that $\mathbf{v}_\infty(x) \equiv \mathbf{0}$. Therefore, $\mathbf{u}_\infty(x) \to \mathbf{0}$ as $x_1 \to +\infty$. □

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ORCID iDs

Rongsong Liu https://orcid.org/0000-0002-0751-5547

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