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Frequency Response Analysis of Parametric Resonance and Vibrational Stabilization

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Abstract— **Periodically time-varying models are found across nature and engineered systems, from fluid dynamics, structures and MEMS devices to quantum mechanics and astrophysics. Such systems are known to exhibit *parametric resonance*, a kind of instability caused by fluctuating model parameters. Under conditions of instability, they can also be *vibrationally stabilized* with the right forcing. The question of interest here is variation in behavior within these two stable regimes, and whether certain parameter configurations are preferred from a design perspective. This motivation leads us to consider Mathieu's equation with harmonic forcing as a canonical model. To address these questions, we use a *lifting* based approach to obtain a representation of the frequency response operator that is amenable to methods from LTI systems. We study the poles of the system as a function of its parameters, and obtain a description of the free response of Mathieu's equation as the product of two simple functions. We also investigate the dependence of the $\mathcal{H}_2$ norm of Mathieu's equation on its parameters. A considerable difference in $\mathcal{H}_2$ norm between the two regimes is found, as well as interesting behavior within each domain.**

I. INTRODUCTION

Systems with periodically time-varying parameters are widely observed in engineered systems and in nature. In fluid dynamics, the stability of Faraday waves on the surface of a fluid in an oscillated container is modeled by a periodic system [1], as is the Kelvin-Helmholtz instability with time-periodic shear [2]. In structural mechanics, models of columns loaded axially by time-periodic loads have this feature [3], as do MEMS devices subjected to alternating voltage under appropriate conditions [4], [5]. Such models also describe the behavior of parametric amplifiers and the dynamics of electrons in Penning traps [6].

Periodic systems display the phenomenon of parametric instability or resonance for certain parametric forcing frequencies. Parametric resonance is often undesirable, but is used in a favorable way in low-noise parametric amplifiers such as Varactors, as well as in MEMS mass-sensing devices [7], [8].

On the other hand, certain unstable time-periodic systems can be stabilized by the introduction of parametric forcing, a technique referred to as Vibrational Control (or Vibrational Stabilization). While Vibrational Control has traditionally been thought of as a high frequency phenomenon [9], [10], recent work [11] has elucidated that this impression is primarily due to the use of averaging methods to analyze such systems, and that vibrational stabilization can be achieved with parametric oscillation at lower frequencies with carefully selected amplitudes. This work [11] also clarifies how vibrational control can be thought of as the "flip side" of parametric resonance, and is the starting point of our investigations.

An interesting question is the sensitivity to noise of time periodic systems as a function of their time-periodic forcing parameters. For instance, a quantitative description of the same is useful in the design of devices that use parametric amplification for sensing. It is also salient to ask how robust vibrational stabilization schemes are to noise. In order to address this question, this paper investigates the frequency response of a linear time-periodic system as applied to Mathieu's equation.

In Section II we define the system under consideration and review the most common solution method and the well-known results of stability analysis. In Section III we describe the process of *lifting*, a method to represent a time-periodic system as LTI but with higher dimensional input/output spaces. In Section IV we define the frequency response operator for this time-periodic system and estimate its singular values numerically. The $\mathcal{H}_2$ norm, another measure of the action of the system on white noise, is investigated in Section V. Finally, we make a brief mention of the statistical properties of the system.

II. PERIODICITY, PARAMETRIC RESONANCE AND VIBRATIONAL STABILIZATION

The canonical example of a periodic linear system is Mathieu's equation,

$$\ddot{x}(t) + (\pm\omega^2 + \epsilon \cos t)x(t) = 0 \tag{1}$$

The positive case $(+\omega^2)$ can be thought of an oscillator with a spring constant that varies periodically. The negative case $(-\omega^2)$ is exponentially unstable in the absence of parametric forcing and corresponds to a system like the Kapitza pendulum, an inverted pendulum with a sinusoidally vibrating base linearized about its unstable equilibrium [12]. The form is ubiquitous and arises in many systems driven by periodic forcing, including most phenomena listed in the previous section.

Mathieu's equation is a specific case of the more general Hill ODE,

$$\ddot{x}(t) + f(t)x(t) = w(t),\ f(t+T) = f(t) \tag{2}$$

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in that the periodic forcing f contains only a single harmonic. In the following sections we focus on Mathieu's equation, although qualitatively similar results hold for the Hill equation.

Mathieu's equation is formally solved using Floquet theory [12], a general method for the analysis of linear time periodic systems developed by Floquet in 1883 [13]. Analyzing the stability of the system using this method involves looking at the eigenvalues of the monodromy map $\Phi(T, 0)$, where $\Phi(t, \tau)$ is the state transition matrix. In Section III-B we obtain an equivalent picture using lifting.

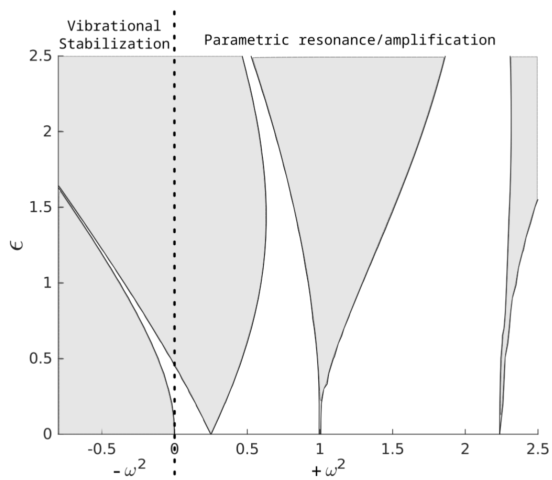

Fig. 1: Arnold Tongues for Mathieu's Equation. In the space of the parameters ω and ϵ, we see regions of instability (gray), as well as a stable region for the nominally unstable system corresponding to $-\omega^2$. The two related phenomena are commonly referred to as parametric resonance and vibrational stabilization, respectively.

The phenomenon of parametric resonance and vibrational stabilization in Mathieu's Equation are both illustrated in Fig. 1. We make two observations:

1) For the $+\omega^2$ equation, we see that a nominally stable (*i.e* with $\epsilon = 0$) harmonic oscillator can be destabilized with parametric forcing of very low amplitude if the forcing frequency is chosen carefully. This is the principle behind parametric amplification and sensing.
2) The $-\omega^2$ equation is unstable and exhibits exponential growth with no forcing. However we see that the right combination of ϵ and forcing frequency can stabilize the system. This is the principle of vibrational stabilization.

Both of these applications lead us to ask questions apropos of design. For instance, given a viable range of parameters ω and ϵ, how do they differ in potential performance? How robust is the vibrationally stabilized region of parameter space to disturbances? How sensitive are different possible operating points of a parametric amplifier to noise? To illustrate the possible variation in behavior within a single parametric regime, Fig. 2 shows the trajectories of the impulse response in state space of Mathieu's equation at three different operating points in the vibrationally stabilized regime from Fig. 1, and there is considerable variation both in the magnitudes and their qualitative features.

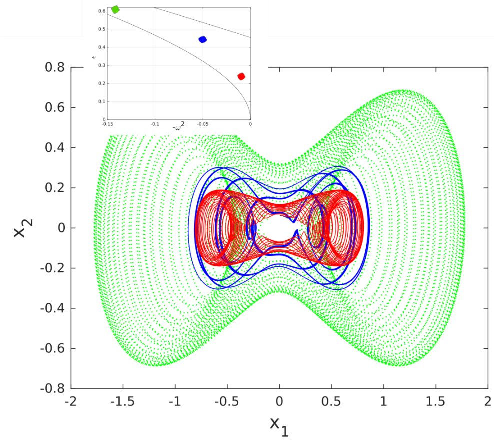

Fig. 2: Trajectories of the impulse response of vibrationally stabilized Mathieu's equation at three different operating points. They are different in both magnitude and qualitative features. The points are: $(\omega, \epsilon) = (0.1, 0.25)$ $(0.21, 0.44)$ and $(0.37, 0.62)$.

Addressing these questions requires studying the input/output characteristics of the system, which in turn requires us to consider Mathieu's equation with harmonic forcing:

$$\ddot{x}(t) + (\pm\omega^2 + \epsilon \cos t)x(t) = w(t) \tag{3}$$

Since we cannot readily apply the tools of LTI systems to this problem, we first apply the technique of lifting to recast the problem into a more tractable form.

III. DESCRIPTION OF THE LIFTING PROCESS

The lifting technique allows us to represent a time-periodic system as a time-invariant one in a space with higher dimensional input and output spaces [14]. A brief description of the procedure is as follows.

Let $L^p_{N,e}[0, \infty)$, $1 \le p < \infty$ be the extended space of continuous time $N-$vector signals, henceforth shortened to L^p. For any Banach space X, let l_X be the space of sequences which take values in X, so that $l_X := \{\{f_i\} : \mathbb{N} \to X\}$.

Then for each T, we define the lifting operator W_T : $L^p[0, \infty) \to l_{L^p[0,T]}$ by

$$\hat{f} = W_T f, \;\; \hat{f}_i(t) = f(Ti + t), \quad 0 \le t \le T \tag{4}$$

$\hat{f}$ is a sequence, each element of which is a function on $[0, T]$. The action of W_T can be visualized as breaking up the signal $f \in L^p[0, \infty)$ into an infinite number of pieces, each of which is a copy of f restricted to an interval of length T. W_T is linear and can be shown to be bijective and an isometry. W_T^{-1} thus glues together the elements of $\{f_i\}$, $f_i \in L^p[0, T]$ into a function $f \in L^p[0, \infty)$.

Let G : $L^p[0, \infty) \to L^p[0, \infty)$ be any linear operator. With W_T in hand we define the lifting for systems as

$$\hat{G} = W_T G W_T^{-1}, \quad \hat{G} : l^p_{L^p[0,T]} \to l^p_{L^p[0,T]} \tag{5}$$

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$\hat{G}$ is linear and $\left\|\hat{G}\right\| = \|G\|$ with appropriate norms.

It can be shown that if G is T-periodic, then $\hat{G}$ is shift-invariant. $\hat{G}$ thus has a convolution representation [14]: If $y = Gw$, then $\hat{y}_i = \sum_{j=0}^{i} \hat{G}_{i-j}\hat{w}_j$.

A. Example of lifting: the product of a periodic and almost-periodic function

As an illustration of the lifting process, consider a signal $y(t)$ formed from the product of the copies of any function $p(t) \in \mathcal{L}_2[0,T]$ (not necessarily continuous) and a complex exponential that is constant over each period:

$$y(t) = p(t)\exp{(j\bar{\theta}\lfloor\frac{t}{T}\rfloor)},\ p(t+T) = p(t) \tag{6}$$

The two pieces are illustrated in Fig. 3.

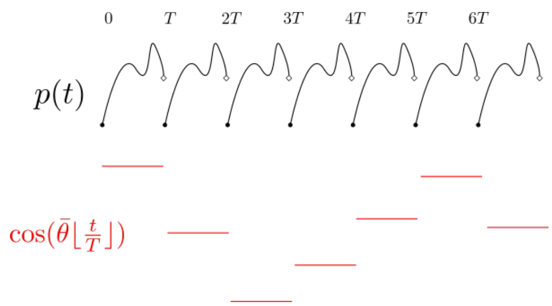

Fig. 3: Pieces of $y(t)$ in the example in Section III-A. The product of copies of $p(t)$ and the almost-periodic exponential generates a function that has a particularly simple form when lifted, and appears as the free response of the Mathieu's equation in sec. IV

If $\bar{\theta}$ is rational, the exponential is a periodic function and so is $y(t)$, otherwise we refer to the exponential here as "almost-periodic". Such a function and its $\mathcal{Z}$-transform will appear in the frequency response analysis in Section IV. When lifted, this function has a simple form:

$$\hat{y}_k = pe^{j\bar{\theta}k} \tag{7}$$

where the implicit dependence of y_k and p on t is suppressed.

Note that $\hat{y} \in l_{L_2[0,T]}$ so that for each k, $\hat{y}_k \in L^2[0,T]$. It is also instructive to look at the $\mathcal{Z}$-transform $\hat{Y}(z)$ of $\hat{y}_k$:

$$\begin{aligned}\hat{y}_k = pe^{j\bar{\theta}k} &\Leftrightarrow \hat{Y}(z) = \frac{pz}{z - e^{j\bar{\theta}}}, \text{ or}\\ \hat{y}_l = pe^{j\bar{\theta}(l-1)} &\Leftrightarrow \hat{Y}(z) = \frac{p}{z - e^{j\bar{\theta}}}\end{aligned} \tag{8}$$

B. Applying lifting to Mathieu's equation

To apply this to Mathieu's equation with input, we first write equation (1) in standard form:

$$\begin{aligned}\dot{x}(t) &= \begin{pmatrix} 0 & 1\\ \omega^2 - \epsilon\cos(t) & -2k\omega\end{pmatrix} x(t) + \begin{pmatrix}0\\1\end{pmatrix} u\\ &= A(t)x(t) + Bu\\ y(t) &= \begin{pmatrix}1\\0\end{pmatrix} x(t) = Cx(t)\end{aligned} \tag{9}$$

This system is $T = 2\pi$ periodic. Let $\Phi(t,\tau)$ be the state transition (or fundamental) matrix for the system. We define

$$\hat{x}_k = x(kT), \quad \hat{w}_k(t) = u(kT+t), \quad \hat{y}_k(t) = y(kT+t)$$

so that $\{\hat{x}_k\} \in l^p_{\mathbb{R}^2}$, and $\{\hat{w}_k\}, \{\hat{y}_k\} \in l^p_{L^2[0,T]}$.

Then it follows from the periodicity of Φ that

$$\hat{x}_{k+1} = \hat{A}\hat{x}_k + \hat{B}\hat{u}_k \tag{10}$$

$$\hat{y}_k = \hat{C}\hat{x}_k + \hat{D}\hat{u}_k \tag{11}$$

$$\begin{aligned}\hat{A} &: \mathbb{R}^2 \to \mathbb{R}^2 & \hat{A}\hat{x}_k &= \Phi(T,0)\hat{x}_k\\ \hat{B} &: L^2[0,T] \to \mathbb{R}^2 & \hat{B}\hat{w}_k &= \int_0^T \Phi(T,s)B\hat{w}_k(s)\mathrm{d}s\\ \hat{C} &: \mathbb{R}^2 \to L^2[0,T] & \left[\hat{C}\hat{x}_k\right](t) &= C\Phi(t,0)\hat{x}_k\\ \hat{D} &: L^2[0,T] \to L^2[0,T] & \left[\hat{D}\hat{u}_k\right](t) &= \int_0^t C\Phi(t,s)B\hat{u}_k(s)\mathrm{d}s\end{aligned} \tag{12}$$

This system is shift-invariant (in k) and thus has a simple solution in terms of its Markov parameters:

$$\hat{y}_k = \sum_{j=0}^{k-1} \hat{C}\hat{A}^{k-j-1}\hat{B}\hat{w}_j + \hat{D}\hat{w}_k \tag{13}$$

C. Numerical solution through lifting

To apply this method numerically, we begin with the Hill ODE rewritten as

$$\dot{x}(t) = \begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{bmatrix}0 & 1\\ -f(t) & 0\end{bmatrix} x(t) + \begin{bmatrix}0\\1\end{bmatrix} w(t) \tag{14}$$

$$y(t) = \begin{bmatrix}1 & 0\end{bmatrix} x(t) \tag{15}$$

and discretize it. The Hill equation is a Hamiltonian system with $H(x_1, x_2; t) = \frac{1}{2}(x_2^2 + f(t)x_1^2)$. Its flow is thus symplectic and must be discretized symplectically [15] to preserve the character of the eigenvalues and thus the stability. See the appendix for details.

The discretized equation is used to estimate $\Phi(t,0)$, $0 \le t \le T$, and then lifted as above, with the modification that $\hat{w}, \hat{y} \in \mathbb{R}^N$ instead of L^2. The resulting operators $\hat{B}$ through $\hat{D}$ are all matrices instead of operators with L^2 as their domain/codomain.

The result is a MIMO system that is equivalent to Mathieu's equation in the sense of the lifting described above and whose solution $\hat{y}$ is related to the solution y of the discretized Mathieu's equation by $\hat{y} = W_T y$.

IV. FREQUENCY RESPONSE OF MATHIEU'S EQUATION

LTI systems map complex exponential input signals at a given frequency to complex exponential outputs at the same frequency but a different amplitude and phase, which allows us to define the idea of frequency response. The output of a linear time periodic (LTP) system to a similar input will contain a multitude of harmonics, and thus defining a transfer function in a similar way is difficult. However, the lifting approach from the previous section allows us to represent it as a discrete-time infinite-dimensional LTI system.

The poles of the lifted system (10) can be found as the eigenvalues of the Monodromy map $\hat{A} = \Phi(T,0)$. These are a function of both the system's natural frequency ω and the

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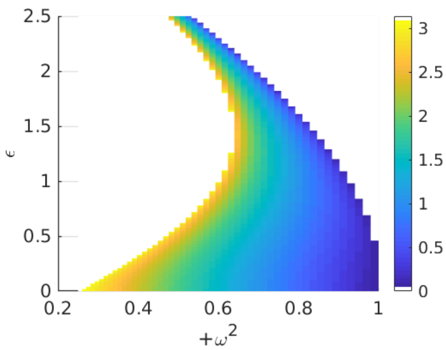
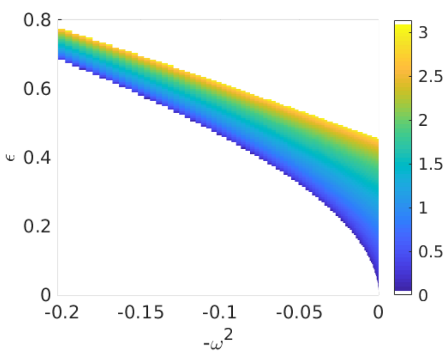
forcing amplitude ϵ. An important property of the Hill ODE (and thus Mathieu's equation) is that the dynamics are measure preserving (appendix), implying that $\frac{d}{dt} \det \Phi(t, 0) = 0$. It follows that the product of the eigenvalues of $\Phi(T, 0)$ is always 1, since $\det \Phi(0, 0) = 1$. So the eigenvalue pair of the monodromy map for Mathieu's equation is restricted to the unit circle (when stable) and the real axis (when unstable).The argument of the eigenvalues is thus always 0 or π at the stability boundaries. In the stable regions the poles are $e^{\pm j\bar{\omega}}$, $0 \leq \bar{\omega} \leq \pi$.



(a) Parametric resonance regime $+\omega^2$. ω^2 varies between 0.25 and 1.0, *i.e.* between the first two tongues. The poles are always at ± 1 at the boundaries of the Arnold tongues, but the behavior in the stable region is not necessarily monotonic with increasing ϵ.



(b) Vibrationally stabilized $-\omega^2$ regime. At given ω, increasing ϵ monotonically moves the poles from $+1$ to -1 along the unit circle. Beyond either of these points one of the poles is outside the unit circle and the system is unstable.

Fig. 4: The variation in pole location of the discretized, lifted Mathieu's equation as a function of ω and ϵ.

Fig. 4 shows the variation of the argument of the poles of the lifted system on the unit circle as a function of ω and ϵ for both ($\pm\omega^2$) regimes.

The first ten singular values of the discretized and lifted system for both regimes ($\pm\omega^2$) are shown in Fig. 5. It is observed that the lifted system has a single mode that grows at resonance. The peak location corresponds to $\bar{\omega}$ on the unit circle, and thus changes with (ω, ϵ) as indicated in Fig. 4.

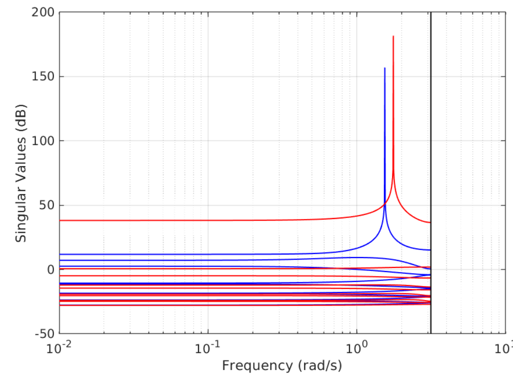


Fig. 5: First ten singular values of the transfer function of the discretized, lifted Mathieu's equation evaluated for $\pm\omega^2$. Red: $(\omega, \epsilon) = (0.2, 0.44)$, Vibrational stabilization $(-\omega^2)$ regime, Blue: $(\omega, \epsilon) = (0.8, 0.5)$, parametric resonance $(+\omega^2)$ regime. Both systems were discretized with $N = 100$. In both regimes there is a single dominant pair of singular values, with resonance corresponding to system poles at $e^{\pm\bar{\omega}}$. This lets us formulate the form of the free response of Mathieu's equation with a low rank approximation.

A. Harmonic resonance in Mathieu's equation

The single resonant mode in the lifted Mathieu's equation suggests that its frequency response close to resonance is similar to that of a second order system. This line of thinking is supported by considering the $Z-$transform of the operator:

$$\hat{G}(z) = \hat{C}(zI - \hat{A})^{-1}\hat{B} + \hat{D} \tag{16}$$

From (12), we see that the first term has at most rank 2 (since $\hat{A}$ is always 2×2) while $\hat{D}$ is potentially infinite-rank but has fixed, bounded norm. Sufficiently close to resonance, however, $(zI - \hat{A})^{-1}$ is near-singular and so outstrips $\hat{D}$ in magnitude. The behavior of the system near resonance is thus almost entirely determined by the first term, which depends on z through factors $\left(z - e^{\pm j\bar{\omega}}\right)^{-1}$.

This observation lets us characterize the response of Mathieu's equation in terms of the singular functions of $\hat{G}$ for any harmonic forcing that has a non-zero Fourier component near $\bar{\omega}$. Here we consider the free response of the system, for which we can neglect the contribution of $\hat{D}$ entirely.

Let $\hat{Y}(z) = \mathcal{Z}[\hat{y}_k](z)$, the $\mathcal{Z}$-transform of the free response, with $\hat{W}(z)$ the $\mathcal{Z}$-transform of the input. Let $\hat{u}_1$ be the first left singular function of the frequency operator $\hat{G}$, which has poles at $e^{\pm j\bar{\omega}}$. Since effectively only $\hat{u}_1(j\bar{\omega})$ (which is function or vector-valued depending on the setting) features in the output near resonance, we have the free response

$$\hat{Y}(z) \approx \frac{c\hat{u}_1(e^{j\bar{\omega}})}{z - e^{j\bar{\omega}}} + \frac{c^\star\hat{u}_1(e^{-j\bar{\omega}})}{z - e^{-j\bar{\omega}}} \tag{17}$$

for some constant c. Following the discussion in Section III-A, this corresponds to a lifted function $\hat{y}_k$ of the form $\Re\left[c\hat{u}_1(e^{j\bar{\omega}})\exp\left(j\bar{\omega}(k-1)\right)\right]$. As a result we get

$$y(t) \approx \Re\left[c\ \hat{u}_{1,t}(e^{j\bar{\omega}})\ \exp\left\{j\bar{\omega}\left(\lfloor\frac{t}{T}\rfloor - 1\right)\right\}\right] \tag{18}$$

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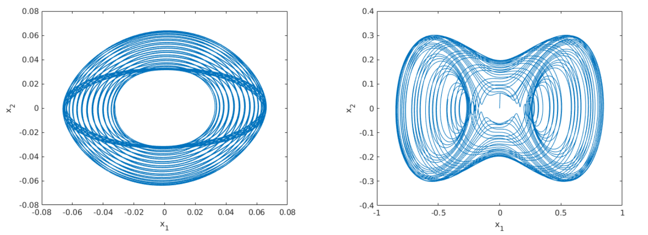
for some constant c. Here t is a discrete or continuous index depending on the nature of the original system that is lifted.

Thus the free response of Mathieu's equation can be characterized as the product of copies of a "constant" function $\hat{u}_1$ (in the sense of the lifting) and a complex exponential that is generally almost-periodic, as in Fig. 3. Some visually rich

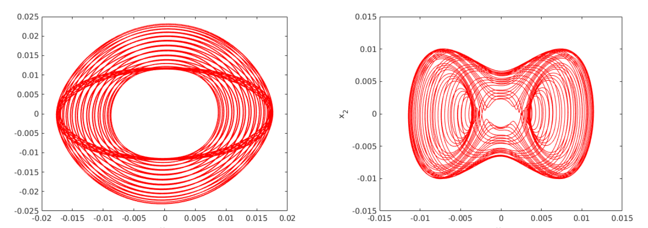


(a) Impulse response in the parametric resonance ($+\omega^2$) regime, $(\omega, \epsilon) = (0.20, 0.44)$.

(b) Impulse response in the vibrationally stabilized ($-\omega^2$) regime, $(\omega, \epsilon) = (0.8, 0.5)$

Fig. 6: Trajectories (x vs $\dot{x}$) of impulse response of Mathieu's equation calculated using the discretized, lifted model. The behavior appears to be non-repeating and also very different in the two regimes.

behaviors can result from this combination. This essentially rank-2 approximation to $\hat{G}$ is pictured in Fig. 7 and should be compared to the actual impulse response calculated from the discretization in Fig. 6.



(a) LSV based estimate in the parametric resonance ($+\omega^2$) regime, $(\omega, \epsilon) = (0.20, 0.44)$

(b) LSV based estimate in the vibrationally stabilized ($-\omega^2$) regime, $(\omega, \epsilon) = (0.8, 0.5)$

Fig. 7: Approximation to the impulse response trajectories calculated from the first left singular function of $\hat{G}$. Note that this does not include the calculation of the scaling factor c, following the argument in sec. IV-A. Compare with the actual impulse response in Fig. 6

V. $\mathcal{H}_2$ NORM OF MATHIEU'S EQUATION AS A FUNCTION OF THE SYSTEM PARAMETERS

The $\mathcal{H}_2$ norm of a system is a measure of its stability, as well as the steady-state variance of its output when fed white noise as input. For a MIMO LTI system the $\mathcal{H}_2$ norm can be interpreted as the the square average of the norms of the responses to a set of unit inputs that excite all 'parts' of the system. [16]

This interpretation can be used to generalize the definition of the $\mathcal{H}_2$ norm to time-periodic systems. Suppose the kernel of the T-periodic system G is given by $G(t, s)$. Since the response to an impulse applied at s is different for each s $(0 \leq s < T)$, we can think of these inputs as exciting different parts of the kernel $G(t, s)$. We thus define the $\mathcal{H}_2$ norm for a periodic system as

$$\|G\|_{\mathcal{H}_2}^2 := \frac{1}{T} \int_0^T \mathrm{Tr} \left(\int_0^\infty G'(t, s) G(t, s) \mathrm{d}s \right) \mathrm{d}t \qquad (19)$$

Following the lifting procedure $G \rightarrow \hat{G}$, this can be rewritten as $\|G\|_{\mathcal{H}_2}^2 = \frac{1}{T} \sum_{k=0}^\infty \mathrm{Tr} \left(\hat{G}_k^\star \hat{G}_k \right)$, where $\hat{G}_k(\tau, s) = G(\tau + Tk, s)$, $0 \leq \tau < T$ is the k^{th} lifted component of G.

When applied to the lifting of Mathieu's equation as described in Section III-B, this leads to the expression

$$\|G\|_{H_2}^2 = \mathrm{Tr} \left[\hat{C} W \hat{C}^\star + \hat{D} \hat{D}^\star \right] \qquad (20)$$

Here W is the infinite time reachability Grammian $\sum_{k=1}^\infty \hat{A}^{k-1} \hat{B} \hat{B}^\star \hat{A}^{\star(k-1)}$ for the system and can be found as the solution to the Lyapunov equation $\hat{A} W \hat{A}^\star - W = -\hat{B} \hat{B}^\star$ [17].

This calculation is performed for the discretized, lifted system from Section III-C and the results are displayed in Fig. 8. We see a minimum in the $\mathcal{H}_2$ norm in the center of the vibrationally stabilized parametric region, as well as a trend of decreasing norm with increasing ω in the stable region. This lends credence to the hypothesis that viable operating points differ in their susceptibility to noise.

VI. CONCLUSION

In an effort to understand the properties of linear periodically time-varying systems beyond stability, we consider Mathieu's equation with harmonic forcing as a representative example. The frequency response operator of Mathieu's equation obtained via lifting possesses interesting properties. The poles of the system vary non-monotonically across the stable parametric regimes of the equation.

From studying its singular values, it is also natural to approximate the frequency response operator as a rank-2 object, which leads to a description of the free response of Mathieu's equation. The free response is shown to have a simple form as a product of repeated copies of the first singular function of the operator with an almost-periodic complex exponential, which can explain visually rich dynamical oscillator behaviors that depend qualitatively on the operating point.

The $\mathcal{H}_2$ norm of Mathieu's equation is computed across the stable regimes using the discretized, lifted system. It is seen that the susceptibility to noise across the parametric space is non-uniform, with a clear local minimum in the vibrationally stabilized region and a slower variation under the Arnold tongues in the parametric resonance region.

When fed stochastic stationary signals as input, periodically time-varying systems produce cyclostationary output [18]. The operator-valued object in frequency domain used to analyze such output is the cyclic spectrum. The spectrum of the cyclostationary output of Mathieu's equation can shed further light on how it amplifies noise, as well as on the spectral content of the response. Work on this approach is ongoing.

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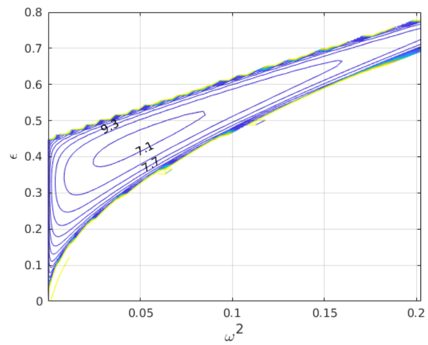
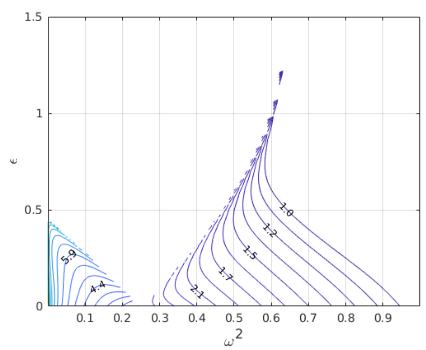


(a) Vibrationally stabilized regime, $(-\omega^2)$ case. The $\mathcal{H}_2$ norm has a minimum in the middle of the stabilizable region in parameter space.



(b) Parametric resonance regime, $(+\omega^2)$ case. This image covers the stable region under the first and second Arnold tongues. The $\mathcal{H}_2$ norms in the two sub-regions differ by factor of 2 or more.

Fig. 8: Contour plot of the $\mathcal{H}_2$ norm of Mathieu's equation in the vibrationally stabilized and stable regions. The vibrationally stabilized regime is significantly more susceptible to noise. Calculations were done with a damping coefficient of 0.1.

REFERENCES

[1] Benjamin and Ursell, "The stability of the plane free surface of a liquid in vertical periodic motion," *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, vol. 225, no. 1163, pp. 505–515, sep 1954. [Online]. Available: https://doi.org/10.1098%2Frspa.1954.0218

[2] R. E. Kelly, "The stability of an unsteady kelvin–helmholtz flow," *Journal of Fluid Mechanics*, vol. 22, no. 3, p. 547–560, 1965. [Online]. Available: https://doi.org/10.1017%2Fs0022112065000964

[3] T. Iwatsubo, Y. Sugiyama, and S. Ogino, "Simple and combination resonances of columns under periodic axial loads," *Journal of Sound Vibration*, vol. 33, pp. 211–221, 1974.

[4] K. L. Turner, S. A. Miller, P. G. Hartwell, N. C. MacDonald, S. H. Strogatz, and S. G. Adams, "Five parametric resonances in a microelectromechanical system," *Nature*, vol. 396, no. 6707, p. 149, 1998.

[5] D. I. Caruntu and I. Martinez, "Reduced order model of parametric resonance of electrostatically actuated MEMS cantilever resonators," *International Journal of Non-Linear Mechanics*, vol. 66, pp. 28–32, nov 2014. [Online]. Available: https://doi.org/10.1016%2Fj.ijnonlinmec.2014.02.007

[6] J. Tan and G. Gabrielse, "Synchronization of parametrically pumped electron oscillators with phase bistability," *Physical Review Letters*, vol. 67, no. 22, pp. 3090–3093, nov 1991. [Online]. Available: https://doi.org/10.1103%2Fphysrevlett.67.3090

[7] W. Zhang and K. L. Turner, "Application of parametric resonance amplification in a single-crystal silicon micro-oscillator based mass sensor," *Sensors and Actuators A: Physical*, vol. 122, no. 1, pp. 23–30, jul 2005. [Online]. Available: https://doi.org/10.1016%2Fj.sna.2004.12.033

[8] W. Zhang and G. Meng, "Nonlinear dynamical system of micro-cantilever under combined parametric and forcing excitations in MEMS," *Sensors and Actuators A: Physical*, vol. 119, no. 2, pp. 291–299, apr 2005. [Online]. Available: https://doi.org/10.1016%2Fj.sna.2004.09.025

[9] F. Bullo, "Averaging and vibrational control of mechanical systems," *SIAM Journal on Control and Optimization*, vol. 41, no. 2, pp. 542–562, 2002.

[10] S. Meerkov, "Principle of vibrational control: theory and applications," *IEEE Transactions on Automatic Control*, vol. 25, no. 4, pp. 755–762, 1980.

[11] J. M. Berg and I. M. Wickramasinghe, "Vibrational control without averaging," *Automatica*, vol. 58, pp. 72–81, aug 2015. [Online]. Available: https://doi.org/10.1016%2Fj.automatica.2015.04.028

[12] A. H. Nayfeh and D. T. Mook, *Nonlinear Oscillations*. Wiley, may 1995. [Online]. Available: https://doi.org/10.1002%2F9783527617586

[13] G. Floquet, "Sur les équations différentielles linéaires à coefficients périodiques," in *Annales scientifiques de l'École normale supérieure*, vol. 12, 1883, pp. 47–88.

[14] B. A. Bamieh and J. B. Pearson, "A general framework for linear periodic systems with applications to h/sup infinity/sampled-data control," *IEEE transactions on automatic control*, vol. 37, no. 4, pp. 418–435, 1992.

[15] S. J. M. Sanz-Serna, "Symplectic integrators for hamiltonian problems: an overview," *Acta Numerica*, vol. 1, pp. 243–286, jan 1992. [Online]. Available: https://doi.org/10.1017%2Fs0962492900002282

[16] B. Bamieh and J. B. Pearson, "The h2 problem for sampled-data systems," *Systems & Control Letters*, vol. 19, no. 1, pp. 1–12, jul 1992. [Online]. Available: https://doi.org/10.1016%2F0167-6911%2892%2990033-o

[17] J. P. Hespanha, *Linear Systems Theory*. Princeton University Press, feb 2018. [Online]. Available: https://doi.org/10.2307%2Fj.ctvc772kp

[18] G. Giannakis, "Cyclostationary signal analysis," in *Digital Signal Processing Fundamentals*. CRC Press, nov 2009, pp. 1–32. [Online]. Available: https://doi.org/10.1201%2F9781420046076-c17

APPENDIX

Hamiltonian systems like the Hill equation are characterized by symplectic flows that, in two dimensions, preserve area [15]. The determinant of the Jacobian of the flow ϕ_t ($\phi_t(x(0)) = x(t)$) should therefore equal 1 identically, for all t. Since the discretization of such a system approximates the flow over one time step, we require the scheme to be symplectic as well.

To discretize the Hill equation (14), let the timestep Δ and $N \in \mathbb{N}$ be chosen such that $\Delta N = T$, the period of $f(t)$. Then x and f can be discretized as $x_n = x(n\Delta)$, $f_n = f(n\Delta)$, as can w and y.

We use a first order Euler scheme that is backward in time in the equation for x_1 and forward in time for x_2. This gives

$$\begin{aligned}\frac{x_{1,k+1} - x_{1,k}}{\Delta} &= x_{2,k+1} \\ \frac{x_{2,k+1} - x_{2,k}}{\Delta} &= -f_k x_{1,k} + w_k\end{aligned}$$

or

$$\begin{aligned}\begin{bmatrix} x_{1,k+1} \\ x_{2,k+1} \end{bmatrix} &= \begin{bmatrix} 1 - f_k\Delta^2 & \Delta \\ -f_k\Delta & 1 \end{bmatrix} \begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} + \begin{bmatrix} \Delta^2 \\ \Delta \end{bmatrix} w_k \\ &= \phi_k x_k + \beta w_k \end{aligned} \tag{21}$$

Where $\det \phi_k = 1$. Lifting does not affect this property since $\det\left(\phi_{k_1}\phi_{k_2}\phi_{k_3}\ldots\phi_{k_m}\right) = 1$ for any set of indices $k_1 \ldots k_m$.

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