An Input–Output Approach to Structured Stochastic Uncertainty

Bassam Bamieh, *Fellow, IEEE*, and Maurice Filo, *Member, IEEE*

***Abstract*—We consider linear time-invariant systems with exogenous stochastic disturbances, and in feedback with structured stochastic uncertainties. This setting encompasses linear systems with both additive and multiplicative noise. Our concern is to characterize second-order properties such as mean-square stability (MSS) and performance. A purely input–output treatment of these systems is given without recourse to state-space models, and, thus, the results are applicable to certain classes of distributed systems. We derive necessary and sufficient conditions for MSS in terms of the spectral radius of a linear matrix operator whose dimension is that of the number of uncertainties, rather than the dimension of any underlying state-space models. Our condition is applicable to the case of correlated uncertainties, and reproduces earlier results for uncorrelated uncertainties. For cases where state-space realizations are given, linear matrix inequality equivalents of the input-output conditions are given.**

***Index Terms*—Loop gain operator, mean-square stability, stochastic uncertainty.**

I. Introduction

LINEAR time invariant (LTI) systems driven by second-order stochastic processes are a widely used and powerful methodology for modeling and control of many physical systems in the presence of stochastic uncertainty. In the most well-known models, stochastic uncertainty enters the model equations additively. Linear systems with both additive and multiplicative stochastic signals are on the other hand relatively less studied. This problem setting is important in the study of system robustness. Although additive disturbances can represent uncertain forcing or measurement noise in a system, multiplicative disturbances are necessary to model uncertainty in system parameters and coefficients. When the multiplicative uncertainty is of the nonstochastic set-valued type, then the problem setting is the standard deterministic one of robust control [1]. The present article is concerned with the stochastic multiplicative uncertainty setting, but the approach will appear to be closer to that of robust control compared to common stochastic treatments.

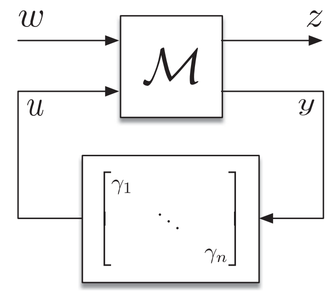

Fig. 1. General setting of linear systems with both additive and multiplicative stochastic disturbances. $\mathcal{M}$ is an LTI system, w is a stationary stochastic process that enters additively, while the multiplicative disturbances are modeled as a feedback through time-varying stochastic gains $\gamma_1, \ldots, \gamma_n$, represented here as a diagonal matrix acting on vector-valued internal signals u and y. The signal z represents an output whose variance quantifies a performance measure.

Before commenting on the background for the present article, a brief statement of the problem is given to allow for a more precise discussion. Fig. 1 illustrates the setting considered in this article. An LTI system $\mathcal{M}$ is in feedback with time-varying gains $\gamma_1, \ldots, \gamma_n$. These gains are random processes that are temporally independent, but possibly mutually correlated. Another set of stochastic disturbances are represented by the vector-valued signal w, which enters additively, while the signal z is an output whose variance quantifies a performance measure. The feedback term is, then, a diagonal matrix with the individual gains $\{\gamma_i\}$ appearing on the diagonal. Such gains are commonly referred to as structured uncertainties. We should emphasize that although $\mathcal{M}$ in Fig. 1 represents a linear model, it can also represent the linearization of a nonlinear model around a fixed point. In this case, the analysis carried out in this article leads to local results (close enough to the fixed point).

We should note the other common and related models in the literature, which are usually formulated in a state-space setting. One such model is a linear system with a random "A matrix" such as

$$x(t+1) = A(t)x(t) + Bw(t) \tag{1}$$

where A is a matrix-valued random process (with $A(t)$ independent of $\{x(\tau), \tau \le t\}$). Sometimes (1) can be rewritten in an alternative form using scalar-valued random processes $\gamma_1, \ldots, \gamma_n$ (n is not necessarily the dimension of x) as follows:

$$x(t+1) = \left(A_o + A_1\gamma_1(t) + \cdots + A_n\gamma_n(t)\right) x(t) + Bw(t). \tag{2}$$

Manuscript received March 9, 2019; revised November 21, 2019; accepted January 2, 2020. Date of publication January 30, 2020; date of current version December 3, 2020. This work was supported in part by National Science Foundation Awards CMMI-1763064 and ECCS-1932777. Recommended by Associate Editor M. Guay. *(Corresponding author: Maurice Filo.)*

B. Bamieh is with the Department of Mechanical Engineering, University of California at Santa Barbara, Santa Barbara, CA 93106 USA (e-mail: bamieh@engineering.ucsb.edu).

M. Filo is with the Department of Biosystems Science and Engineering, ETH Zürich, 8092 Zürich, Switzerland (e-mail: maurice.filo@bsse.ethz.ch).

Color versions of one or more of the figures in this article are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TAC.2020.2970393

Now this form can always be converted [1] to that of Fig. 1. The simplest case is when the matrices $A_1, \ldots, A_n$ are all of rank 1, then each $\gamma_i(t)$ in Fig. 1 is a scalar block, while otherwise, one would have so-called repeated blocks. We refer the reader to [1] and [2] for this standard construction.

The literature on systems with multiplicative noise goes back several decades. Early work considered models like (1) and (2), but primarily in continuous time and using an Ito formulation. The primary tool [3] was to derive differential equations that govern the evolution of second-order moments when the multiplicative noise is white, and conditions for the asymptotic convergence of those deterministic equations are given in terms of solvability of certain Riccati-like equations. The case of colored multiplicative noise is less tractable since equations for the second-order moments are not easy to obtain, although certain special classes have been studied [4], [5]. For the white noise case, however, more detailed analysis appeared in later work [6], [7] , which recast mean square stability (MSS) conditions in terms of linear matrix inequalities (LMIs).

Another trend [8]–[11] appeared in the 90's when systems with several sources of multiplicative noise were viewed in a similar manner to that of structured uncertainty common in robust control (as in the setting of Fig. 1). The interesting observation was made that MSS conditions can be given in terms of the spectral radius of a non-negative matrix of H^2 norms (the matrix of H^2 norms of the individual single-input–single-output (SISO) subsystems of the multi-input–multi-output (MIMO) LTI system $\mathcal{M}$). This criterion is analogous to necessary and sufficient conditions for robust stability to deterministic structured time-varying uncertainty in both the L^2 and L^∞-induced norms [12]–[19] settings.

There are two observations about the existing results [8]–[11] that motivate the current work. The first is that although the final MSS conditions are stated in terms of input–output properties (H^2 norms), the arguments and proofs rely on state-space realizations, LMIs related to those realizations, and scalings of those LMIs. Second, the existing results are for multiplicative uncertainties that are mutually uncorrelated, and it is unclear how these arguments can be generalized to the correlated uncertainties case. It should be noted that the latter case is important for several applications, such as spatially distributed systems where uncertainties enter the dynamics with spatial correlations [20].

The aim of this article is to provide a relatively elementary, and purely input–output treatment and derivation of the necessary and sufficient conditions for MSS and performance. In the process, conditions for the mutually correlated uncertainties case become transparent, as well as how special the uncorrelated case is. A new mathematical object is uncovered, which can be termed the "loop gain operator," which acts on covariances of signals in the feedback loop. We briefly describe this operator as a preview of the main result of this article (the following statement is concerned only with MSS rather than performance, so the signal w in Fig. 1 is set to zero). Let $\mathbf{\Gamma}$ be the mutual correlation matrix of the γ's, i.e., the matrix with ij'th entry $\mathbf{\Gamma}_{ij} := \mathbb{E}[\gamma_i(t)\gamma_j^*(t)]$. Let $\{M_{22}(t)\}$ be the impulse response matrix sequence of the $\mathcal{M}_{22}$ subsystem in Fig. 1, and define the *matrix-valued* linear operator

$$\mathbb{L}(X) := \mathbf{\Gamma} \circ \left(\sum_{t=0}^{\infty} M_{22}(t)\ X\ M_{22}^*(t) \right)$$

where $\circ$ is the Hadamard (element by element) product of matrices. Note that it operates on matrices X whose dimensions are the number of uncertainties, and not of any underlying state-space realization. The operator $\mathbb{L}$ is called the *loop gain operator* because it captures what happens to the covariance matrix of a signal as one goes "once around the feedback loop" in Fig. 1 in the statistical steady state. The eigenvalues and "eigen-matrices" of this operator characterize MSS as well as the fastest growing second-order statistics of the signals in the loop when MSS is lost. An examination of this operator shows that in the more general setting $\mathbf{\Gamma} \neq I$, MSS conditions require not only calculations of H^2 norms, but also inner products of the various subsystems' impulse responses (of which the H^2 norms of subsystems are a special case). The operator $\mathbb{L}$ has several nice properties including monotonicity (preserving the semidefinite ordering on matrices), and consequently a Perron-Frobenius theory for its spectral radius and the associated eigen-matrix. These properties are described and exploited in the sequel.

This article is organized as follows. Section II establishes preliminary results that are needed for the subsequent structured uncertainty analysis. One-sided random processes and their associated covariance sequences are defined. In addition, we provide a natural input–output definition of MSS in feedback systems. The main tool we use is to convert the stochastic feedback system of Fig. 1 to a deterministic feedback system that operates on matrix-valued signals, namely the covariance sequences of all the signals in the feedback loop (Fig. 4 provides a cartoon of this). LTI systems that operate on convariance-matrix-valued signals have some nice monotone properties that significantly simplify the proofs. We pay special attention to establishing these monotone properties. Our main results on MSS are established in Section III where we begin with the simple case of SISO unstructured stochastic uncertainty. This case illustrates how small-gain arguments similar to those used for deterministic perturbations [21] can be used to establish necessary and sufficient MSS conditions. We, then, consider the structured uncertainty case, introduce the loop gain operator, and show that it captures the exact necessary and sufficient structured small gain condition for MSS. Section IV examines this loop gain operator in the general case as well as several special cases. We reproduce earlier results when the uncertainties are uncorrelated, and derive conditions for repeated uncertainties. Finally, Section V treats the performance problem and Section VI translates our conditions to state-space formulae whenever such realizations are available. These can be useful for explicit computations, and, in particular, we provide a power iteration algorithm for calculating the largest eigenvalue and corresponding eigen-matrix of the loop gain operator. We close with some remarks and comments about further research questions in Section VII.

II. Preliminaries and Basic Results

All the signals considered are defined on the half-infinite, discrete-time interval $\mathbb{Z}^+ := \{0, 1, \ldots\}$. The dynamical systems considered are maps between various signal spaces over the time interval $\mathbb{Z}^+$. This is done in contrast with the standard stationary stochastic processes setting over $\mathbb{Z}$ since stability arguments involve the growth of signals starting from some initial time.

A stochastic process u is a one-sided sequence of random variables $\{u_t;\ t \in \mathbb{Z}^+\}$. The notation $u_t := u(t)$ is used whenever no confusion can occur due to the presence of other indices. Without loss of generality, we assume all processes to be zero mean. For any process, its *instantaneous variance* *sequence*$\mathbf{u}_t := \mathbb{E}[u_t^* u_t]$ is denoted by small bold font, and its *instantaneous covariance matrix sequence*$\mathbf{U}_t = \mathbb{E}[u_t u_t^*]$ is denoted by capital bold font. The entries of $\mathbf{U}_t$ are mutual correlations of the components of the vector u_t, and are sometimes referred to as *spatial correlations*. Note that $\mathbf{u}_t = \text{tr}(\mathbf{U}_t)$.

A process u is termed *second order* if it has finite covariances $\mathbf{U}_t$ for each $t \in \mathbb{Z}^+$. A process is termed *white* if it is uncorrelated at any two distinct times, i.e., $\mathbb{E}[u_t u_\tau^*] = 0$ if $t \neq \tau$. Note that in the present context, a white process u may still have spatial correlations, i.e., its instantaneous correlation matrix $\mathbf{U}_t$ need not be the identity. A process u is termed *temporally independent* if u_t and u_τ are independent when $t \neq \tau$. Although the processes considered in this article are technically not stationary (stationary processes are defined over the doubly infinite time axis), it can be shown that they are asymptotically stationary in the sense that their statistics become approximately stationary in the limit of large time, or quasi-stationary in the terminology of [22]. This fact is not used in the present treatment and the preceding comment is only included for clarification.

A. Notation Summary

1) Variance and Covariance Sequences: A stochastic process is a zero-mean, one-sided sequence of vector-valued random variables $\{u_t;\ t \in \mathbb{Z}^+\}$.

1) The *variance sequence* of u is
$$\mathbf{u}_t := \mathbb{E}\left[u_t^* u_t\right].$$
2) The *covariance sequence* of u is
$$\mathbf{U}_t := \mathbb{E}\left[u_t u_t^*\right].$$
3) When it exists, the asymptotic *limit* of a covariance sequence is denoted by an overbar
$$\bar{\mathbf{U}} := \lim_{t\to\infty} \mathbf{U}_t \tag{3}$$
with similar notation for variances $\bar{\mathbf{u}} := \lim_{t\to\infty} \mathbf{u}_t$.

We use calligraphic letters $\mathcal{M}$ to denote LTI systems as operators, and capital letters $\{M_t\}$ to denote elements of their matrix-valued impulse response sequences, i.e., $y = \mathcal{M}u$ is operator notation for $y_t = \sum_{l=0}^{t} M_{t-\tau} u_\tau$.

1) If $\mathcal{M}$ has finite H^2 norm, then the limit of the output covariance $\bar{\mathbf{Y}}$ when the input is white and has a covariance sequence with a limit $\bar{\mathbf{U}}$ is denoted by
$$\begin{array}{ccc} \bar{\mathbf{Y}} & = & \mathcal{M}\left\{\bar{\mathbf{U}}\right\} \\ \| & & \| \\ \lim_{t\to\infty} \mathbf{Y}_t & & \sum_{\tau=0}^{\infty} M_\tau \bar{\mathbf{U}} M_\tau^* = \lim_{t\to\infty} \sum_{\tau=0}^{t} M_\tau \mathbf{U}_{t-\tau} M_\tau^*. \end{array}$$
Note that $\mathcal{M}\{.\}$ is a matrix-valued linear operator.
2) The response to spatially uncorrelated white noise is denoted by
$$\mathcal{M}\left\{I\right\} = \textstyle\sum_{\tau=0}^{\infty} M_\tau M_\tau^* =: \bar{\mathbf{M}}.$$

2) Hadamard Product: For any vector v (resp. square matrix V), $\text{Diag}(v)$ (resp. $\text{Diag}(V)$) denotes the diagonal matrix with diagonal entries equal to those of v (resp. V). For any square matrix V, $\text{diag}(V)$ is the *vector* with entries equal to the diagonal entries of V.

The Hadamard, or element-by-element product of two matrices A and B is denoted by $A \circ B$. We will use the notation
$$A^{\circ 2} := A \circ A$$
for the element-by-element matrix square. Note that with this notation, for any matrix V
$$I \circ V = \text{Diag}(V).$$
A matrix-valued operator which will be needed is
$$F(V) := I \circ (AVA^*) = \text{Diag}\left(AVA^*\right). \tag{4}$$
In particular, we will need to characterize its action on diagonal matrices, which is easily shown to be
$$\text{Diag}\left(A\,\text{Diag}(v)\,A^*\right) = \text{Diag}\left(A^{\circ 2} v\right). \tag{5}$$
In other words, if $V = \text{Diag}(v)$ is diagonal, then the diagonal part of AVA^* as a vector is simply the matrix-vector product of $A^{\circ 2}$ with v.

B. Input–Output Formulation of MSS

Let $\mathcal{M}$ be a causal LTI (MIMO) system. The system $\mathcal{M}$ is completely characterized by its impulse response, which is a matrix valued sequence $\{M_t;\ t \in \mathbb{Z}^+\}$. The action of $\mathcal{M}$ on an input signal u to produce an output signal y is given by the convolution sum
$$y_t = \sum_{\tau=0}^{t} M_{t-\tau}\, u_\tau \tag{6}$$
where without loss of generality, zero initial conditions are assumed.

If the input u is a zero-mean, second-order stochastic process, then it is clear from (6) that y_t has finite covariance for any t, even in the cases where this covariance may grow unboundedly in time. If u is, in addition, white, then the following calculation is standard:
$$\begin{aligned} \mathbf{Y}_t &= \mathbb{E}\left[\left(\sum_{\tau=0}^{t} M_{t-\tau} u_\tau\right)\left(\sum_{r=0}^{t} u_r^* M_{t-r}^*\right)\right] \\ &= \sum_{\tau=0}^{t}\sum_{r=0}^{t} M_{t-\tau}\,\mathbb{E}\left[u_\tau u_r^*\right] M_{t-r}^* \\ \mathbf{Y}_t &= \sum_{\tau=0}^{t} M_{t-\tau}\,\mathbf{U}_\tau\, M_{t-\tau}^*. \end{aligned} \tag{7}$$
Note that this is a matrix convolution, which relates the instantaneous covariance sequences of the output and white input. For SISO systems, this relation simplifies to
$$\mathbf{y}_t = \sum_{\tau=0}^{t} M_{t-\tau}^2\,\mathbf{u}_\tau. \tag{8}$$
For systems with a finite number of inputs and outputs, taking the trace of (7) gives
$$\begin{aligned} \mathbf{y}_t = \text{tr}\left(\mathbf{Y}_t\right) &= \sum_{\tau=0}^{t} \text{tr}\left(M_{t-\tau}\,\mathbf{U}_\tau\, M_{t-\tau}^*\right) \\ &= \sum_{\tau=0}^{t} \text{tr}\left(M_{t-\tau}^* M_{t-\tau}\,\mathbf{U}_\tau\right) \end{aligned} \tag{9}$$

$$\leq \sum_{\tau=0}^{t} \operatorname{tr}\left(M_{t-\tau}^* M_{t-\tau}\right) \operatorname{tr}\left(\mathbf{U}_\tau\right)$$

$$\mathbf{y}_t \leq \left(\sum_{\tau=0}^{\infty} \operatorname{tr}\left(M_{t-\tau}^* M_{t-\tau}\right)\right)\left(\sup_{0\leq t<\infty} \mathbf{u}_t\right) \qquad (10)$$

where the first inequality holds because for any two positive semidefinite matrices A and B, we have $\operatorname{tr}(AB) \leq \operatorname{tr}(A)\operatorname{tr}(B)$ [23]. The above calculation motivates the following input–output definition of MSS.

Definition 1: A causal LTI system $\mathcal{M}$ is called *mean-square stable* (MSS) if for each white, second-order input process u with uniformly bounded variance, the output process $y = \mathcal{M}u$ has uniformly bounded variance

$$\mathbf{y}_t := \mathbb{E}\left[y_t^* y_t\right] \;\leq\; c\,\left(\sup_{\tau} \mathbf{u}_\tau\right) \qquad (11)$$

with c a constant independent of t and the process u.

Note that the first term in (10) is just the H^2 norm of $\mathcal{M}$

$$\|\mathcal{M}\|_2^2 := \sum_{t=0}^{\infty} \operatorname{tr}\left(M_t M_t^*\right).$$

The second term is an ℓ^∞ norm on the variance sequence, and, thus, the bound in (10) can be compactly rewritten as

$$\|\mathbf{y}\|_\infty \;\leq\; \|\mathcal{M}\|_2^2\, \|\mathbf{u}\|_\infty. \qquad (12)$$

From (9), it is easy to see that equality in (12) holds when u has constant identity covariance ($\mathbf{U}_\tau = I$). Conversely, if $\mathcal{M}$ does not have finite H^2 norm, this input causes $\mathbf{y}_t$ to grow unboundedly. Thus, a system is MSS if and only if it has finite H^2 norm.[1]

For systems that have a finite H^2 norm, the output covariance sequence has a steady-state limit when the input covariance sequence does. More precisely let $y = \mathcal{M}u$, and the input u be such that

$$\lim_{t\to\infty} \mathbf{U}_t =: \bar{\mathbf{U}}$$

exists. Then, if $\mathcal{M}$ has finite H^2 norm, it follows that the output covariance has the limit

$$\bar{\mathbf{Y}} := \lim_{t\to\infty} \mathbf{Y}_t = \sum_{\tau=0}^{\infty} M_\tau\, \bar{\mathbf{U}}\, M_\tau^* =: \mathcal{M}\left\{\bar{\mathbf{U}}\right\}. \qquad (13)$$

For covariance sequences with a well-defined limit, the overbar bold capital notation is used for the limit value as above. Also as above, the notation $\mathcal{M}\{\bar{\mathbf{U}}\}$ is used for the steady-state output covariance of an LTI system $\mathcal{M}$ with input that has steady-state covariance of $\bar{\mathbf{U}}$. When $\bar{\mathbf{U}} = I$, the following compact notation is used:

$$\bar{\mathbf{M}} := \mathcal{M}\left\{I\right\}.$$

[1]It must be emphasized that this conclusion holds only if MSS is defined with as boundedness of variance sequences when the input is white. As is well-known from the theory of stationary stochastic processes, the instantaneous variance of a signal is the integral of its power spectral density (PSD). The integrability of the output PSD cannot be concluded from the integrability of the system's magnitude squared response unless the input has a flat PSD (i.e., white). Thus for colored inputs, the boundedness of the output variance sequence cannot be concluded from only the H^2 norm.

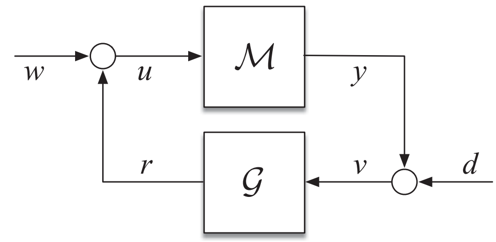

Fig. 2. Definition of MSS for a feedback interconnection. The exogenous disturbance signals are white random processes, and the requirement is that all signals in the loop have uniformly bounded variance sequences.

Thus, $\bar{\mathbf{M}}$ is the steady-state covariance of the output of an LTI system $\mathcal{M}$ when the input is white and has a steady-state covariance of identity.

C. MSS of Feedback Interconnections

The input–output setting for MSS of a feedback interconnection can be motivated using the conventional scheme [21] of injecting exogenous disturbance signals into all loops.

Definition 2: Consider the feedback system of Fig. 2 with d and w being white second-order processes, and $\mathcal{M}$ and $\mathcal{G}$ causal LTI systems. The feedback system in Fig. 2 is called MSS if all signals u, y, v, and r have uniformly bounded variance sequences, i.e., if there exists a constant c such that

$$\max\left\{\|\mathbf{u}\|_\infty, \|\mathbf{y}\|_\infty, \|\mathbf{v}\|_\infty, \|\mathbf{r}\|_\infty\right\}$$
$$\leq\; c\,\min\left\{\|\mathbf{d}\|_\infty, \|\mathbf{w}\|_\infty\right\}.$$

Remark 1: A standard argument implies that the feedback interconnection is MSS iff the four mappings $(I - \mathcal{M}\mathcal{G})^{-1}$, $\mathcal{G}(I - \mathcal{M}\mathcal{G})^{-1}$, $(I - \mathcal{M}\mathcal{G})^{-1}\mathcal{M}$, and $\mathcal{M}\mathcal{G}(I - \mathcal{M}\mathcal{G})^{-1}$ have finite H^2 norms. In general, it is not possible to bound those closed-loop norms in terms of only the H^2 norms of $\mathcal{M}$ and $\mathcal{G}$. In other words, it is not generally possible to carry out a small-gain type analysis of the feedback system of Fig. 2 using only H^2 norms. Another way to see this is that bounds like (12) are not directly applicable to Fig. 2 since the signals u and v will not in general be white.

Despite the above remark, in the present article, the concept of feedback stability is used when one of the subsystems is a temporally independent multiplicative uncertainty. As will be seen, this has the effect of "whitening" (temporally de-correlating) the signal going through it, thus enabling a type of small-gain analysis.

D. Stochastic Multiplicative Gains

The MSS problem considered in this article is for systems of the structure depicted in Fig. 3, where

$$\Gamma(t) := \operatorname{Diag}\left(\gamma_1(t), \ldots, \gamma_n(t)\right)$$

is a diagonal matrix of time-varying scalar stochastic gains acting on the vector signal v

$$r_t = \Gamma_t\, v_t$$

and $\mathcal{M}$ is a strictly causal LTI system. Without loss of generality, Γ can be assumed to be a zero mean process as the mean value can be absorbed into the known part of the dynamics.

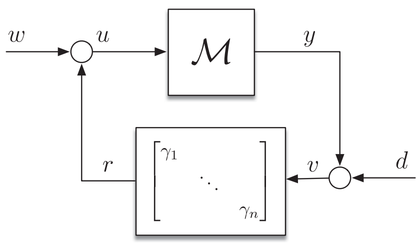

Fig. 3. Strictly causal LTI system $\mathcal{M}$ in feedback with multiplicative stochastic gains γ_i's. The exogenous stochastic signals d and w are injected to test MSS of the feedback system, which holds when all internal loop signals u, y, v, and r have bounded variance sequences.

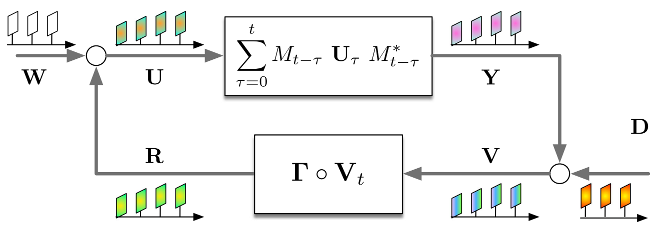

Fig. 4. Deterministic feedback system detailing how the dynamics of the stochastic feedback system in Fig. 3 operates on covariance sequences. Each signal in this diagram is positive semidefinite matrix valued. The forward block is an LTI matrix convolution, and the feedback block is the Hadamard matrix product. It is notable that all input–output mappings of this feedback system are monotone even if the original system is not. This implies in particular that all covariance sequences in the loop are nondecreasing if the exogenous inputs are nondecreasing (in the semidefinite ordering on matrices).

The assumptions we make on the additive and multiplicative uncertain signals are as follows.

Assumptions on d, w, and Γ

1) The random variables $\{d_t\}$, $\{w_t\}$, and $\{\Gamma_t\}$ are all mutually independent.
2) Both d and w have nondecreasing covariance sequences, i.e.,

$$t_2 \geq t_1 \implies \mathbf{D}_{t_2} \geq \mathbf{D}_{t_1},\ \mathbf{W}_{t_2} \geq \mathbf{W}_{t_1}.$$

The first assumption on the mutual independence of the Γ's is crucial to the techniques used in this article. Note, however, that for any one time t, individual components of Γ_t maybe correlated, and that is referred to as *spatial correlations*, which can be characterized as follows. Let $\gamma(t)$ denote the vector

$$\gamma(t) := \begin{bmatrix}\gamma_1(t) & \cdots & \gamma_n(t)\end{bmatrix}^*.$$

The instantaneous correlations of the γ_i's can be expressed with the matrix

$$\mathbf{\Gamma} := \mathbb{E}\left[\gamma(t)\,\gamma^*(t)\right] \tag{14}$$

which is assumed to be independent of t.

The mutual independence (in time) of the perturbations Γ and the strict causality of $\mathcal{M}$ have an implication for the dependencies of the various signals in the loop on the Γ s. This is expressed in the following lemma whose proof is found in the appendix.

Lemma 2.1: In the feedback diagram of Fig. 3, assume $\mathcal{M}$ is strictly causal, and that Γ_t and Γ_τ are independent for $t \neq \tau$. Then, we have the following.

1) Past and present values of v and y are independent of present and future values of Γ, i.e.,

$$\begin{aligned}&\Gamma_t,\ y_\tau, \tau \leq t, \ \text{ are independent}\\ &\Gamma_t,\ v_\tau, \tau \leq t, \ \text{ are independent.}\end{aligned} \tag{15}$$

2) Past values of r and u are independent of present and future values of Γ, i.e.,

$$\begin{aligned}&\Gamma_t,\ r_\tau, \tau < t, \quad \text{are independent}\\ &\Gamma_t,\ u_\tau, \tau < t, \quad \text{are independent.}\end{aligned} \tag{16}$$

An important consequence of these relations is that even if the input signal v may, in general, be colored, multiplication by the Γ s will cause the output r to be white. This can be seen from

$$\begin{aligned}\mathbb{E}\left[r_t r_\tau^*\right] &= \mathbb{E}\left[\Gamma_t v_t v_\tau^* \Gamma_\tau^*\right]\\ &= \mathbb{E}\left[\Gamma_t\right]\,\mathbb{E}\left[v_t v_\tau^* \Gamma_\tau^*\right] = 0, \quad \tau < t\end{aligned}$$

where the second equality follows from (15), i.e., the independence of Γ_t from v_t, v_τ, and Γ_τ, respectively. A similar argument shows that r is uncorrelated with present and past values of v and y, and uncorrelated with past values of u, but we will not need these facts in the sequel.

To calculate the instantaneous spatial correlations of r

$$\begin{aligned}\mathbb{E}\left[r_t r_t^*\right] &= \mathbb{E}\left[\Gamma_t v_t v_t^* \Gamma_t^*\right]\\ &= \mathbb{E}\left[\Gamma_t\ \left(\mathbb{E}\left[v_t v_t^*\right]\right)\ \Gamma_t^*\right]\end{aligned} \tag{17}$$

where the last equality follows from the independence of v_t and Γ_t and formula (57) in Appendix A. It is, thus, required to calculate quantities like $\mathbb{E}[\Gamma M \Gamma]$ for some constant matrix M. The case of diagonal Γ reduces to

$$\begin{aligned}\mathbb{E}\left[\Gamma M \Gamma^*\right] &= \mathbb{E}\left[\begin{bmatrix}\gamma_1 & & \\ & \ddots & \\ & & \gamma_n\end{bmatrix} M \begin{bmatrix}\gamma_1 & & \\ & \ddots & \\ & & \gamma_n\end{bmatrix}\right]\\ &= \left[m_{ij}\,\mathbb{E}\left[\gamma_i \gamma_j\right]\right] = \mathbf{\Gamma} \circ M\end{aligned}$$

which is the Hadamard (element-by-element) product of $\mathbf{\Gamma}$ and M. Applying this to (17), the above arguments lead to the following conclusion.

Lemma 2.2: Consider the feedback system of Fig. 3 with $\mathcal{M}$ a strictly causal LTI system, and Γ diagonal stochastic perturbations with spatial correlations (14). If the perturbations Γ are temporally independent, then the output r is a white process with instantaneous spatial correlations given by

$$\mathbf{R}_t = \mathbf{\Gamma} \circ \mathbf{V}_t \tag{18}$$

the Hadamard (element by element) product of $\mathbf{\Gamma}$ and $\mathbf{V}_t$.

Two special cases are worth noting. If $\Gamma = \gamma$ is a scalar perturbation, then (18) reduces to

$$\mathbf{r}_t = \mathbf{\Gamma}\,\mathbf{v}_t. \tag{19}$$

Thus, the multiplication by a scalar perturbation simply scales the variance of the input signal and "whitens" it. In the special case where the perturbations are uncorrelated and all have unit variance, i.e., $\mathbf{\Gamma} = I$, a simple expression results

$$\mathbf{R}_t = \operatorname{diag}\left(\mathbf{V}_t\right)$$

where $\mathrm{diag}(\mathbf{V}_t)$ is a diagonal matrix made up of the diagonal entries of the matrix $\mathbf{V}_t$. Thus, if the γ_i's are white and mutually uncorrelated, then the vector output signal r is temporally and spatially uncorrelated even though v may have both types of correlations. In other words, a structured perturbation with uncorrelated components will spatially and temporally whiten its input.

E. Covariance Feedback System

An important tool used in this article is to replace the analysis of the original stochastic system of Fig. 3 with an equivalent system that operates on the respective signals' instantaneous covariance matrices. This deterministic system is depicted in Fig. 4. Each signal in this feedback system is *matrix-valued*. The mathematical operations on each of the signals depicted in the individual blocks follow from (7) and (18) and the following observations.

1) u is a white process.
2) For each t

$$\mathbf{U}_t = \mathbf{R}_t + \mathbf{W}_t. \tag{20}$$

3) For each t

$$\mathbf{V}_t = \mathbf{Y}_t + \mathbf{D}_t. \tag{21}$$

Observations 1 and 2 follow from

$$\begin{aligned}\mathbb{E}\left[u_t u_\tau^*\right] &= \mathbb{E}\left[(r_t + w_t)(r_\tau^* + w_\tau^*)\right]\\ &= \mathbb{E}\left[r_t r_\tau^*\right] + \mathbb{E}\left[r_t w_\tau^*\right] + \mathbb{E}\left[w_t r_\tau^*\right] + \mathbb{E}\left[w_t w_\tau^*\right]\\ &= \begin{cases} 0+0+0+0=0, & \tau < t\\ \mathbb{E}\left[r_t r_t\right] + 0 + 0 + \mathbb{E}\left[w_t w_t^*\right] = \mathbf{R}_t + \mathbf{W}_t, & \tau = t\end{cases}\end{aligned}$$

where $\mathbb{E}[w_t r_\tau^*] = 0$ since w is uncorrelated with past system signals, and $\mathbb{E}[r_t w_\tau^*] = 0$ follows from Lemma 2.1 because $\mathbb{E}[r_t w_\tau^*] = \mathbb{E}[\Gamma_t v_t w_\tau^*] = \mathbb{E}[\Gamma_t]\mathbb{E}[v_t w_\tau^*] = 0$. For the case $\tau < t$, $\mathbb{E}[w_t w_\tau^*] = 0$ and $\mathbb{E}[r_t r_\tau^*] = 0$ since w (by assumption) and r (Lemma 2.2) are white respectively. Observation 3 follows immediately from

$$\mathbf{V}_t = \mathbb{E}\left[(y_t + d_t)(y_t^* + d_t^*)\right] = \mathbf{Y}_t + \mathbf{D}_t$$

since y_t is a function of w, Γ, and past d's, and, thus, is independent of d_t. Note that v may in general be colored.

Observation 1 implies that the forward block can indeed be written using (7) (the input needs to be white for its validity). Observations 2 and 3 imply that the summing junctions in Fig. 3 can indeed be replaced by summing junctions on the corresponding covariance sequences in Fig. 4.

F. Monotonicity

Although it is not standard to consider systems operating on matrix-valued signals, it is rather advantageous in the current setting. In this article, the order relation used on matrices is always the positive semidefinite ordering (i.e., $A \geq B$ means $(A - B)$ is positive semidefinite). The statements in this section apply to orderings with other positive cones, although this generality is not needed here.

1) Monotone Operators:

Definition 3: A matrix-valued linear operator $\mathbb{L} : \mathbb{R}^{n\times n} \to \mathbb{R}^{m\times m}$ is called *monotone* (in the terminology of [24], or *cone invariant* in the terminology of [25] and [26]) if

$$X \geq 0 \;\Rightarrow\; \mathbb{L}(X) \geq 0.$$

In other words, if it preserves the semidefinite ordering on matrices (this definition is equivalent to the statement $X \leq Y \Rightarrow \mathbb{L}(X) \leq \mathbb{L}(Y)$). There is a Perron-Frobenius theory for such operators, which gives them nice properties, some of which are now summarized.

Theorem 2.3: For a matrix-valued monotone operator $\mathbb{L}$

1) $\exists$ a real, largest eigenvalue: $\rho(\mathbb{L})$ is an eigenvalue of $\mathbb{L}$.
2) $\exists$ an eigen-matrix $X \geq 0$ for the largest eigenvalue, i.e.,

$$\mathbb{L}(X) = \rho(\mathbb{L})\; X. \tag{22}$$

3) Sums and compositions of monotone operators are monotone. For $\rho(\mathbb{L}) < \alpha$, the operator $(I - \mathbb{L}/\alpha)^{-1}$ exists and is monotone.

Proof: The first two statements are from [26, Theorem 2] or [25, Theorem 3.2]. That sums and compositions of monotone operators are monotone is immediate from the definition. Furthermore, note that the Neuman series

$$(I - \mathbb{L}/\alpha)^{-1} = \sum_{k=0}^{\infty} (\mathbb{L}/\alpha)^k$$

is made up of sums of compositions of a monotone operator $\mathbb{L}/\alpha$. This series converges in any operator norm since $\rho(\mathbb{L}/\alpha) < 1$ (this follows from Gelfand's formula, which implies that for any operator norm, there is some k such that $\|(\mathbb{L}/\alpha)^k\| < 1$). ■

Note that the "eigen-matrix" X in (22) is the counterpart of the Perron-Frobenius eigenvector for matrices with non-negative entries. Such eigenmatrices will play an important role in the sequel as a sort of worst-case covariance matrices.

2) Monotone Systems and Signals: The positive semidefinite ordering on matrices induces a natural ordering on *matrix-valued signals*, as well as a notion of monotonicity on systems [27]. For two matrix-valued signals $\mathbf{U}$ and $\mathbf{W}$, the following point-wise order relation can be defined

$$(\mathbf{U} \leq \mathbf{W}) \quad\Longleftrightarrow\quad \forall t \in \mathbb{Z}^+,\; \mathbf{U}(t) \leq \mathbf{W}(t). \tag{23}$$

For systems, the following is a restatement of the definition from [27] when the initial conditions are zero.

Definition 4: An input–output system $\mathcal{M}$ mapping on matrix-valued signals is said to be *monotone* if whenever

$$\begin{aligned}\mathbf{Y} &= \mathcal{M}(\mathbf{U})\\ \mathbf{Z} &= \mathcal{M}(\mathbf{W}),\end{aligned} \quad \text{then } \mathbf{U} \leq \mathbf{W} \;\Rightarrow\; \mathbf{Y} \leq \mathbf{Z}. \tag{24}$$

In other words, if $\mathcal{M}$ preserves the positive semidefinite ordering on matrix-valued signals. There is a further notion of monotonicity of an individual signal in the sense of mapping the time-axis ordering to that of the matrix ordering.

Definition 5: A matrix-valued signal $\{\mathbf{U}(t)\}$ is said to be *monotone* (or *nondecreasing*) if

$$t_1 \leq t_2 \;\Rightarrow\; \mathbf{U}(t_1) \leq \mathbf{U}(t_2).$$

It is simple to show (Appendix B2) that a time-invariant monotone system maps nondecreasing signals to nondecreasing signals.

3) Monotonicity of Covariance Feedback Systems: That the forward loop in Fig. 4 is monotone is immediate since

$$\forall t \in \mathbb{Z}^+,\; \mathbf{U}_t \leq \mathbf{W}_t \Longrightarrow \quad \forall t \in \mathbb{Z}^+$$

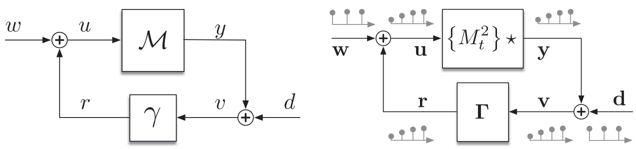

Fig. 5. (*Left*) LTI system $\mathcal{M}$ in feedback with a time-varying stochastic uncertainty $\{\gamma_t\}$, and with additive exogenous stochastic disturbances w and d. (*Right*) Equivalent LTI systems (28) operating on the variance sequences of all respective signals. $\mathcal{M}$ is replaced by convolution with the sequence $\{M_t\}$, while the stochastic gain γ is replaced by multiplication with $\mathbf{\Gamma}$, its variance.

$$\left(\sum_{\tau=0}^{t} M_{t-\tau}\,\mathbf{U}_\tau\,M_{t-\tau}^*\right) \leq \left(\sum_{\tau=0}^{t} M_{t-\tau}\,\mathbf{W}_\tau\,M_{t-\tau}^*\right).$$

Note that this is always the case *even when the original LTI system $\mathcal{M}$ is not monotone!* It is also true that the Hadamard product is monotone. This follows from the Schur Product Theorem [28, Th. 2.1], which states that for any matrices

$$A_1 \leq A_2 \text{ and } B \geq 0 \;\Rightarrow\; B \circ A_1 \,\leq\, B \circ A_2. \tag{25}$$

Thus, each of the two systems in the feedback loop of Fig. 4 is monotone.

When monotone systems are connected together with positive feedback, then all input–output mappings in the resulting feedback system are also monotone (see Theorem 7.1 in Appendix B). It, then, follows (Appendix B2) that if the covariance signals $\mathbf{D}$ and $\mathbf{W}$ are nondecreasing, then all other covariance signals in feedback system are also nondecreasing. These nondecreasing covariance sequences are depicted in Fig. 4.

III. MSS Conditions

This section contains the main result of this article characterizing MSS in terms of the spectral radius of a matrix-valued operator. MSS stability arguments in the literature are typically done with state-space models, and the present article develops an alternative input–output approach more akin to small-gain type analysis. This technique is most easily demonstrated with the SISO case, which, for clarity of exposition, is treated separately first. The MIMO structured case is, then, developed with a similar small-gain analysis. However, additional issues of spatial correlations appear in the structured case, and those are treated in detail. The section ends by demonstrating how known conditions for uncorrelated uncertainties can be derived as a special case of the general result presented here, as well as some comments about which system metrics characterize MSS in the general case of correlated uncertainties.

A. SISO Unstructured Uncertainty

Consider the simplest case of uncertainty analysis depicted in Fig. 5 (*Left*). $\mathcal{M}$ is a strictly causal LTI system, d and w are exogenous white processes with uniform variance, and γ is a white process with uniform variance $\mathbf{\Gamma}$ and independent of the signals d and w. $\mathcal{M}$ is assumed to have finite H^2 norm.

A small-gain stability analysis in the spirit of [21] can be accomplished by deriving the equivalent relations between the variance sequences of the various signals in the loop [see Fig. 5 (*Right*)]. To begin with, recall the observations made in Section II-E that u is white, and for the SISO case, the variance sequences satisfy

$$\mathbf{u}_t = \mathbf{r}_t + \mathbf{w}_t \tag{26}$$

$$\mathbf{v}_t = \mathbf{y}_t + \mathbf{d}_t. \tag{27}$$

Since u is white, the formulas for the variances sequences are particularly simple according to (8) and (19), the equivalent relations are

$$\mathbf{y}_t = \sum_{\tau=0}^{t} M_{t-\tau}^2\,\mathbf{u}_\tau, \quad \mathbf{r}_t = \mathbf{\Gamma}\,\mathbf{v}_t. \tag{28}$$

The main stability result for unstructured stochastic perturbations can now be stated.

Lemma 3.1: Consider the system in Fig. 5 with $\mathcal{M}$ a strictly causal, stable LTI system, and γ a temporally independent process with variance $\mathbf{\Gamma}$. The feedback system is MSS if and only if

$$\|\mathcal{M}\|_2^2 \,<\, 1/\mathbf{\Gamma}.$$

Proof: ("*if*") This is similar to standard sufficiency small gain arguments, but using variances rather than signal norms. Starting from $\mathbf{u}$, going backwards through the loop yields

$$\begin{aligned}\|\mathbf{u}\|_\infty &\leq \|\mathbf{r}\|_\infty + \|\mathbf{w}\|_\infty \\ &\leq \mathbf{\Gamma}\,\|\mathbf{v}\|_\infty + \|\mathbf{w}\|_\infty \\ &\leq \mathbf{\Gamma}\,\left(\|\mathbf{y}\|_\infty + \|\mathbf{d}\|_\infty\right) + \|\mathbf{w}\|_\infty \\ &\leq \mathbf{\Gamma}\,\|\mathcal{M}\|_2^2\,\|\mathbf{u}\|_\infty + \mathbf{\Gamma}\|\mathbf{d}\|_\infty + \|\mathbf{w}\|_\infty \end{aligned} \tag{29}$$

where subsequent steps follow from the triangle inequality and using (12) and (19). This bound together with the assumption $\mathbf{\Gamma}\,\|\mathcal{M}\|_2^2 < 1$ gives a bound for the internal signals u in terms of the exogenous signals d and w

$$\|\mathbf{u}\|_\infty \;\leq\; \frac{1}{1-\mathbf{\Gamma}\|\mathcal{M}\|_2^2}\left(\mathbf{\Gamma}\|\mathbf{d}\|_\infty + \|\mathbf{w}\|_\infty\right).$$

In addition, this bound gives bounds on variances of the remaining internal signals y, v, and r as follows from (12), (27), and (19), respectively.

("*only if*") See Appendix E. ■

Two remarks are in order regarding the necessity part of the previous proof. First, there was no need to construct a so-called "destabilizing" perturbation as is typical in worst-case perturbation analysis. Perturbations here are described statistically rather than members of sets, and variances will always grow when the stability condition is violated. Second, the necessity argument can be interpreted as showing that $\|\mathcal{M}\|_2 \geq 1$ implies that the transfer function $(1-\mathcal{M}(z))$ has a zero in the interval $[0,\infty)$, and, thus, $(1-\mathcal{M}(z))^{-1}$ has an unstable pole. The argument presented above, however, is more easily generalizable to the structured MIMO case considered next.

B. Structured Uncertainty

In the analysis of MIMO structured uncertainty, a certain matrix-valued operator will play a central role, and, therefore, it is first introduced and some of its properties investigated. The main result on MSS for the structured case is, then, stated and proved.

1) The Loop Gain Operator: Consider the matrix-valued operator

$$\mathbb{L}(X) := \mathbf{\Gamma} \circ \left(\sum_{t=0}^{\infty} M_t \, X \, M_t^* \right) \tag{30}$$

where $\mathbf{\Gamma}$ is the correlation matrix of the uncertainties (14), and $\{M_t\}$ is the matrix-valued impulse response sequence of a stable (finite H^2 norm), causal, LTI system $\mathcal{M}$. This is termed the *loop gain operator* since it is how the covariance matrix of stationary white noise input u is mapped to the covariance of the signal r (which will also be white) in Fig. 4, i.e., it describes what happens to the instantaneous covariance matrix of white noise as it goes once around the loop.

The loop gain operator is monotone since it is the composition of two monotone operators. There will also be a need to consider finite-horizon truncations of it

$$\mathbb{L}_T(X) := \mathbf{\Gamma} \circ \left(\sum_{t=0}^{T} M_t X \, M_t^* \right). \tag{31}$$

If $\mathcal{M}$ has finite H^2 norm, then it is immediate that

$$\lim_{T\to\infty} \mathbb{L}_T = \mathbb{L}$$

in any (finite-dimensional) operator norm.

It turns out that the spectral radius of the loop gain operator is the exact condition for MSS. This is stated next.

Theorem 3.2: Consider the system in Fig. 3 where w is a white process, both d and w have bounded, monotone covariance sequences, Γ is temporally independent multiplicative noise with instantaneous correlations $\mathbf{\Gamma}$, and $\mathcal{M}$ is a stable (finite H^2 norm), strictly-causal, LTI system. The feedback system is MSS if and only if

$$\rho\left(\mathbb{L}\right) < 1$$

where $\mathbb{L}$ is the matrix-valued "loop gain operator" defined in (30).

Proof: ("*if*") Recalling the observations made in Section II-E, an expression for $\mathbf{U}_t$ can be derived by following signals backwards through the loop in Fig. 4

$$\begin{aligned}\mathbf{U}_t &= \mathbf{R}_t + \mathbf{W}_t = \mathbf{\Gamma} \circ \mathbf{V}_t + \mathbf{W}_t \\ &= \mathbf{\Gamma} \circ (\mathbf{Y}_t + \mathbf{D}_t) + \mathbf{W}_t\end{aligned}$$

$$\mathbf{U}_t = \mathbf{\Gamma} \circ \left(\sum_{\tau=0}^{t} M_\tau \mathbf{U}_{t-\tau} M_\tau^* + \mathbf{D}_t \right) + \mathbf{W}_t \tag{32}$$

which follow from (18), (20), and (21), the fact that u is white, and (7), respectively. The monotonicity (see Section II-F) of the feedback system in Fig. 4 relating covariance sequences implies that $\mathbf{U}$ is a nondecreasing sequence. This gives the following bound:

$$\sum_{\tau=0}^{t} M_\tau \mathbf{U}_{t-\tau} M_\tau^* \le \sum_{\tau=0}^{t} M_\tau \mathbf{U}_t M_\tau^*$$

which together with Schur's theorem (25) allows for replacing (32) with the bounds

$$\begin{aligned}\mathbf{U}_t &\le \mathbf{\Gamma} \circ \left(\sum_{\tau=0}^{t} M_\tau \mathbf{U}_t M_\tau^* \right) + \mathbf{\Gamma} \circ \mathbf{D}_t + \mathbf{W}_t \\ &\le \mathbf{\Gamma} \circ \left(\sum_{\tau=0}^{\infty} M_\tau \mathbf{U}_t M_\tau^* \right) + \mathbf{\Gamma} \circ \mathbf{D}_t + \mathbf{W}_t.\end{aligned}$$

To see how this inequality gives a uniform bound on the sequence $\mathbf{U}$, rewrite it using the definition of $\mathbb{L}$ as

$$\left(I - \mathbb{L}\right)\left(\mathbf{U}_t\right) \le \mathbf{\Gamma} \circ \mathbf{D}_t + \mathbf{W}_t$$

where $(I - \mathbb{L})$ is a linear operator acting on $\mathbf{U}_t$. Now $\rho(\mathbb{L}) < 1$ implies [by Theorem 2.3 (3)] that $(I - \mathbb{L})^{-1}$ exists and is a monotone operator, and, therefore,

$$\begin{aligned}\mathbf{U}_t &\le (I - \mathbb{L})^{-1} \left(\mathbf{\Gamma} \circ \mathbf{D}_t + \mathbf{W}_t\right) \\ &\le (I - \mathbb{L})^{-1} \left(\mathbf{\Gamma} \circ \bar{\mathbf{D}} + \bar{\mathbf{W}}\right)\end{aligned}$$

where the first inequality follows from the monotonicity of $(I - \mathbb{L})^{-1}$, and the second inequality follows from its linearity, Schur's theorem, and replacing $\mathbf{D}_t$ and $\mathbf{W}_t$ by their steady-state limits. This provides a uniform upper bound on the sequence $\mathbf{U}$, and note that the stability of $\mathcal{M}$, then, implies in addition that all other signals in Fig. 4 are uniformly bounded.

(*"only if"*) In a similar manner to the necessity proof of Lemma 3.1, it is shown that $\rho(\mathbb{L}) \ge 1$ implies that $\mathbf{U}$ has an unbounded subsequence. First, observe that by setting $\mathbf{D}_t = 0$, (32) gives the following bounds:

$$\begin{aligned}\mathbf{U}_{Tk} &= \mathbf{\Gamma} \circ \left(\sum_{\tau=0}^{Tk} M_{Tk-\tau} \mathbf{U}_\tau M_{Tk-\tau}^* \right) + \mathbf{W}_{Tk} \\ &\ge \mathbf{\Gamma} \circ \left(\sum_{\tau=T(k-1)}^{Tk} M_{Tk-\tau} \mathbf{U}_\tau M_{Tk-\tau}^* \right) + \mathbf{W}_{Tk} \\ &\ge \mathbb{L}_T \left(\mathbf{U}_{T(k-1)}\right) + \mathbf{W}_{Tk}\end{aligned} \tag{33}$$

where the first inequality follows from Schur's theorem (25), and the second inequality follows from the monotonicity of the sequence $\mathbf{U}$ and the monotonicity of the operator $\mathbb{L}_T$.

A simple induction argument exploiting the monotonicity of $\mathbb{L}_T$ yields

$$\mathbf{U}_{Tk} \ge \mathbb{L}_T^k \left(\mathbf{U}_0\right) + \sum_{r=0}^{k-1} \mathbb{L}_T^r \left(\mathbf{W}_{T(k-r)}\right). \tag{34}$$

Now, set the exogenous covariance $\mathbf{W}_{Tk} = \widehat{\mathbf{U}}$, where $\widehat{\mathbf{U}}$ (the Perron–Frobenius eigen-matrix) is the nonzero semidefinite eigen-matrix such that $\mathbb{L}(\widehat{\mathbf{U}}) = \rho(\mathbb{L})\,\widehat{\mathbf{U}}$ [see Theorem 2.3 (2)]. Note that the initial covariance is, thus, $\mathbf{U}_0 = \mathbf{D}_0 = \widehat{\mathbf{U}}$. Substituting in (34) yields

$$\mathbf{U}_{Tk} \ge \sum_{r=0}^{k} \mathbb{L}_T^r \left(\widehat{\mathbf{U}}\right). \tag{35}$$

Since $\lim_{T\to\infty} \mathbb{L}_T(\widehat{\mathbf{U}}) = \mathbb{L}(\widehat{\mathbf{U}}) = \rho(\mathbb{L})\widehat{\mathbf{U}}$, then for any $\epsilon > 0$, $\exists T > 0$ such that $||\rho(\mathbb{L})\widehat{\mathbf{U}} - \mathbb{L}_T(\widehat{\mathbf{U}})|| \le \epsilon||\widehat{\mathbf{U}}||$. This inequality coupled with the fact that $0 \le \mathbb{L}_T(\widehat{\mathbf{U}}) \le \rho(\mathbb{L})\widehat{\mathbf{U}}$ allows us to apply Lemma 7.3 to obtain

$$\mathbb{L}_T(\widehat{\mathbf{U}}) \ge \left(\rho\left(\mathbb{L}\right) - \epsilon c\right) \widehat{\mathbf{U}} =: \alpha\, \widehat{\mathbf{U}} \tag{36}$$

where c is a positive constant that only depends on $\widehat{\mathbf{U}}$ (Lemma 7.3). Then, by (35), the one-step lower bound (36) becomes

$$\mathbf{U}_{Tk} \geq \left(\sum_{r=0}^{k} \alpha^r\right) \widehat{\mathbf{U}} = \frac{\alpha^{k+1}-1}{\alpha - 1}\, \widehat{\mathbf{U}}. \tag{37}$$

First consider the case when $\rho(\mathbb{L}) > 1$, then ϵ can be chosen small enough so that $\alpha > 1$ and, therefore, $\mathbf{U}$ is a geometrically growing sequence.

The case $\rho(\mathbb{L}) = 1$ can be treated in exactly the same manner as in the proof of Lemma 3.1 to conclude that $\mathbf{U}$ has a (not necessarily geometrically) unboundedly growing subsequence. ■

2) Worst Case Covariance Matrix: An interesting contrast between the SISO and MIMO cases appears in the necessity argument above. Comparing the expressions for the unbounded sequences (62) and (37), it appears that the Perron–Frobenius eigen-matrix $\widehat{\mathbf{U}}$ of $\mathbb{L}$ represents a sort of worst case growth covariance matrix. In other words, to achieve the highest rate of covariance growth in the feedback system, one needs the input w to have spatial correlations such that $\mathbf{W}_t = \widehat{\mathbf{U}}$. In an analysis where there are no exogenous inputs and one only considers growth of initial state covariances, there is a similar worst-case initial state covariance that corresponds to $\widehat{\mathbf{U}}$. Section VI elaborates on this point.

IV. Special Structures and Representations

We consider first the special case of uncorrelated uncertainties, and show how the well-known result follows as a special case. We, then, look at a Kronecker product representation of the general case, which clarifies the role played by system metrics other than H^2 norms in MSS conditions. These metrics involve what might be termed as *autocorrelations between subsystems impulse responses*. Finally, we consider the case of circulant systems in which the presence of spatial symmetries provides conditions of intermediate difficulty between the uncorrelated and the general case.

A. Uncorrelated Uncertainties

A well-known result in the literature [8]–[11] is the case of uncorrelated uncertainties $\{\gamma_i\}$, where the MSS condition is known to be given by the spectral radius of the matrix of H^2 norms of subsystems of $\mathcal{M}$. We now demonstrate how this result follows directly as a special case of Theorem 3.2.

For uncorrelated uncertainties, $\mathbf{\Gamma} = I$, and the loop gain operator (30) becomes

$$\mathbb{L}(X) := \mathrm{Diag}\left(\sum_{t=0}^{\infty} M_t\, X\, M_t^*\right) \tag{38}$$

where $\mathrm{Diag}(Y)$ is a diagonal matrix made up of the diagonal part of Y. In this case, any eigen-matrix of $\mathbb{L}$ (corresponding to a nonzero eigenvalue) must clearly be a diagonal matrix, so it suffices to consider how $\mathbb{L}$ acts on diagonal matrices. Let $V := \mathrm{Diag}(v)$ be a diagonal matrix, and recall the characterization (4) and (5) of terms like $\mathrm{Diag}(HVH^*)$ on diagonal matrices. Applying this term by term to the sum in (38) gives

$$V = \mathrm{Diag}(v), \;\Rightarrow\; \mathbb{L}(V) = \mathrm{Diag}\left(\sum_{t=0}^{\infty} M_t^{\circ 2}\, v\right) =: \mathrm{Diag}\left(\mathbf{M}^{\circ} v\right) \tag{39}$$

where $M^{\circ 2}$ is the Hadamard (element by element) square of the matrix M, and we use the notation $\mathbf{M}^{\circ}$ to denote the matrix of squared H^2 norms of subsystems of $\mathcal{M}$

$$\mathbf{M}^{\circ} := \sum_{t=0}^{\infty} M_t^{\circ 2} = \begin{bmatrix} \|\mathcal{M}_{11}\|_2^2 & \cdots & \|\mathcal{M}_{1n}\|_2^2 \\ \vdots & & \vdots \\ \|\mathcal{M}_{n1}\|_2^2 & \cdots & \|\mathcal{M}_{nn}\|_2^2 \end{bmatrix}. \tag{40}$$

We, therefore, conclude that the nonzero eigenvalues of $\mathbb{L}$ are precisely the eigenvalues of $\mathbf{M}^{\circ}$, and, in particular, their spectral radii are equal. This is summarized in the following corollary.

Corollary 4.1: For the uncertain system of Fig. 3 with uncorrelated uncertainties, the MSS condition of Theorem 3.2 becomes

$$\rho\left(\mathbf{M}^{\circ}\right) \leq 1/\boldsymbol{\gamma}$$

where $\boldsymbol{\gamma} := \mathbb{E}[\gamma_i^2]$ is the uncertainties' variance (assumed equal for all i) and $\mathbf{M}^{\circ}$ is the matrix (40) of squared H^2 norms of $\mathcal{M}$'s subsystems.

B. Repeated Perturbations

This case represents the opposite extreme to the uncorrelated perturbations case. Here, we have all the perturbations identical, i.e.,

$$\Gamma(t) := \mathrm{Diag}\left(\gamma(t), \ldots, \gamma(t)\right) = I\,\gamma(t)$$

where $\{\gamma(t)\}$ is a scalar-valued iid random process and I is the $n \times n$ identify matrix. In this case, all entries of the uncertainty correlation matrix are equal, i.e., $\mathbb{E}[\gamma_i(t)\gamma_j(t)] = \boldsymbol{\gamma}$, and, therefore

$$\mathbf{\Gamma} = \boldsymbol{\gamma}\,\mathbf{1}\mathbf{1}^*$$

where $\mathbf{1}$ is the vector of all entries equal to 1. Now the loop gain operator (30) takes on a particularly simple form

$$\mathbb{L}(X) = (\boldsymbol{\gamma}\,\mathbf{1}\mathbf{1}^*) \circ \left(\sum_{t=0}^{\infty} M_t X M_t^*\right) = \boldsymbol{\gamma} \sum_{t=0}^{\infty} M_t X M_t^* = \boldsymbol{\gamma}\,\mathcal{M}\left\{X\right\}.$$

The interpretation of $\mathbb{L}$ in this case is simple. Referring to (13), we see that $\mathbb{L}(X)$ is the steady-state covariance matrix of the output of the LTI system $\mathcal{M}$ when its input is white noise with covariance matrix $\boldsymbol{\gamma} X$. In particular, let the system $\mathcal{M}$ have a state-space realization (A, B, C), then for an eigen-matrix X of $\mathbb{L}$ with eigenvalue λ

$$\boldsymbol{\gamma}\,\mathcal{M}\left\{X\right\} = \mathbb{L}(X) = \lambda X \iff \mathcal{M}\left\{X\right\} = \frac{\lambda}{\boldsymbol{\gamma}} X \tag{41}$$

which implies that X satisfies the matrix equation

$$Y - AYA^* = BXB^*$$
$$\frac{\lambda}{\boldsymbol{\gamma}} X = CYC^*.$$

Equivalently, a single equation for Y can be written

$$Y - A\,Y\,A^* = \frac{\boldsymbol{\gamma}}{\lambda}\, BC\,Y\,C^* B^*. \tag{42}$$

This is not a standard Lyapunov equation, but it can always be thought of as a generalized eigenvalue problem as follows.

Define the linear operators $\mathbb{L}_1$ and $\mathbb{L}_2$ by

$$\mathbb{L}_1(Y) := \gamma\, BC\, Y\, C^* B^*$$
$$\mathbb{L}_2(Y) := Y \;-\; A\, Y\, A'.$$

Then, (42) can be rewritten as the generalized eigenvalue problem

$$\mathbb{L}_1(Y) = \lambda\, \mathbb{L}_2(Y). \tag{43}$$

Finally, we note an interesting interpretation of covariance matrices that arise as eignmatrices of $\mathbb{L}$ in the repeated perturbation case. As (41) shows, these eigenmatrices are exactly those covariances of input processes to an LTI system $\mathcal{M}$ with the property that the steady-state output covariance is a scalar multiple of the input covariance. These are very special processes, but there dynamical significance is not yet clear.

C. Kronecker Product Representation of the General Case

For the general case of correlated uncertainties $\mathbf{\Gamma} \neq I$, and it turns out that entries of the matrix (40) alone are insufficient to characterize $\rho(\mathbb{L})$ (and, thus, the MSS) condition. In the absence of any spatial structure in $\mathcal{M}$ and $\mathbf{\Gamma}$, one can always use a Kronecker product representation of $\mathbb{L}$. This representation gives some insight into the general case.

Let $\text{vec}(X)$ denote the "vectorization" operation of converting a matrix X into a vector by stacking up its columns. It is, then, standard to show that the loop gain operator (30) can equivalently be written as

$$\text{vec}\left(\mathbb{L}(X)\right) = \underbrace{\left(\text{Diag}\left(\text{vec}\left(\mathbf{\Gamma}\right)\right) \sum_{t=0}^{\infty} M(t) \otimes M(t)\right)}_{\text{matrix representation of } \mathbb{L}} \text{vec}(X). \tag{44}$$

Therefore, the eigenvalues (and corresponding eigenmatrices) of $\mathbb{L}$ can be found by calculating the eigenvalues/vectors of this $n^2 \times n^2$ representation using standard matrix methods. This is clearly not a desirable method even for only moderately large-scale systems. An alternative computational method for large systems based on the power iteration is presented in Section VI-A.

The formula (44), however, provides some insight into the general case when $\mathbf{\Gamma}$ is not diagonal. In that case, entries of the matrix representation will involve sums of the form

$$\sum_{t=0}^{\infty} M_{ij}(t)\, M_{kl}(t). \tag{45}$$

These are inner products of impulse responses of different SISO subsystems of $\mathcal{M}$. They can be thought of as *autocorrelations* of the MIMO system $\mathcal{M}$'s responses. In the special case of uncorrelated uncertainties $\mathbf{\Gamma} = I$, only terms for identical subsystems ($(i,j) = (k,l)$) appear, resulting in H^2 norms of subsystems. Thus, it is seen that a condition involving only H^2 norms like (40) is a highly special case. To characterize MSS in correlated uncertainty cases, one needs in addition other system metrics, like the inner product between different subsystems' impulse responses (45).

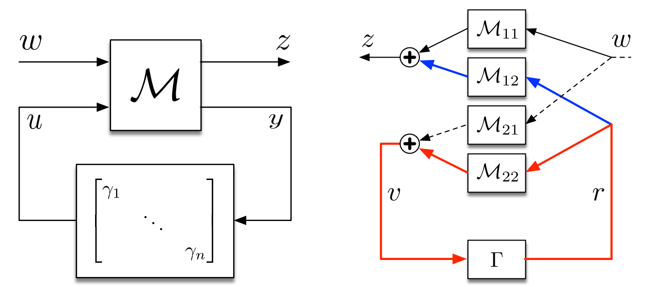

Fig. 6. (*Left*) Mean-square performance problem setting with additive noise w as an exogenous signal and multiplicative noise γ_i's as structured stochastic uncertainty. (*Right*) Details of the various signal paths. The variance of z is finite iff the feedback loop (M_{22}, Γ) is MSS.

V. Mean-Square Performance

The mean-square performance problem is a general formulation for linear systems with both additive and multiplicative noise. It is straightforward to show that any LTI system with both additive and multiplicative noise can be redrawn in the form shown in Fig. 6(*Left*), where $\mathcal{M}$ is LTI, w is the additive white noise, and the multiplicative perturbations are grouped together in the diagonal matrix gain $\Gamma := \text{diag}(\gamma_1, \ldots, \gamma_n)$. The assumption of whiteness of w is made without any loss in generality. If the additive noise is colored, then the coloring filter can be absorbed in the LTI dynamics of $\mathcal{M}$ in the standard manner.

The mean-square performance problem is to find conditions for MSS stability of the feedback system, and to calculate the steady-state covariance of the output z. It is clear from Fig. 6 (*Right*) that z has finite covariance iff the feedback subsystem $(\mathcal{M}_{22}, \Gamma)$ is MSS. The exact condition for this is that the spectral radius of the loop gain operator (30) for $\mathcal{M}_{22}$ and Γ

$$\mathbb{L}_{22}(X) := \mathbf{\Gamma} \circ \left(\sum_{t=0}^{\infty} M_{22}(t)\, X\, M_{22}^*(t)\right) = \mathbf{\Gamma} \circ \mathcal{M}_{22}\left\{X\right\} \tag{46}$$

has spectral radius less than 1.

The calculation of the covariance $\mathbf{Z}$ proceeds similarly to (32) where the first steps are to relate the covariances of the signals in the lower feedback loop. It is first noted that with assumption of MSS, all covariance sequences are bounded and have steady-state limits, so the following relations are written directly for those limits

$$\begin{aligned}\mathbf{R} &= \mathbf{\Gamma} \circ \mathbf{V}\\ &= \mathbf{\Gamma} \circ \left(\sum_{t=0}^{\infty} \begin{bmatrix} M_{21} & M_{22}\end{bmatrix}_t \begin{bmatrix}\mathbf{W} & 0\\ 0 & \mathbf{R}\end{bmatrix} \begin{bmatrix} M_{21}^*\\ M_{22}^*\end{bmatrix}_t\right)\\ &= \mathbf{\Gamma} \circ \left(\sum_{t=0}^{\infty} M_{22}(t)\mathbf{R}M_{22}^*(t) + M_{21}(t)\mathbf{W}M_{21}^*(t)\right)\end{aligned}$$

where for simplicity, we have dropped the "overbar" notation (3) for the covariance limit (e.g., in this section, $\mathbf{R}$ stands for $\lim_{t\to\infty} \mathbf{R}_t$). The expression for $\mathbf{V}$ follows from (13) and the fact that both w and r are mutually uncorrelated and white. The last equation can be rewritten in operator form using the definition (46) and the notation of (13)

$$(I - \mathbb{L}_{22})(\mathbf{R}) = \mathbf{\Gamma} \circ \mathcal{M}_{21}\{\mathbf{W}\}$$
$$\mathbf{R} = (I - \mathbb{L}_{22})^{-1}(\mathbf{\Gamma} \circ \mathcal{M}_{21}\{\mathbf{W}\}).$$

Finally, to calculate the covariance of the output, note that

$$\mathbf{Z} = \sum_{t=0}^{\infty} \begin{bmatrix} M_{11} & M_{12} \end{bmatrix}_t \begin{bmatrix} \mathbf{W} & 0 \\ 0 & \mathbf{R} \end{bmatrix} \begin{bmatrix} M_{11}^* \\ M_{12}^* \end{bmatrix}_t$$
$$= \mathcal{M}_{11}\{\mathbf{W}\} + \mathcal{M}_{12}\{\mathbf{R}\}$$
$$= \mathcal{M}_{11}\{\mathbf{W}\} + \mathcal{M}_{12}\left\{(I - \mathbb{L}_{22})^{-1}(\mathbf{\Gamma} \circ \mathcal{M}_{21}\{\mathbf{W}\})\right\}.$$

Note how this formula has a familiar feel to the linear fractional transformations (LFT) from standard transfer function block diagram manipulations. The difference being that these are operators on matrices rather than vector signals. It is instructive to compare the above formula with that of the transfer functions operating on the vector signals w and z

$$z = \left(\mathcal{M}_{11} + \mathcal{M}_{12}(I - \Gamma\mathcal{M}_{22})^{-1}\Gamma\mathcal{M}_{21}\right) w.$$

This resemblance is more immediate in the SISO case (single SISO Γ and w a scalar), where the expression simplifies to

$$\mathbf{z} = \left(\|\mathcal{M}_{11}\|_2^2 + \frac{\mathbf{\Gamma}\,\|\mathcal{M}_{12}\|_2^2\|\mathcal{M}_{12}]\|_2^2}{1 - \mathbf{\Gamma}\|\mathcal{M}_{22}\|_2^2}\right)\mathbf{w}.$$

The expression for $\mathbf{Z}$ above is written to highlight the analogy with LFT of transfer functions. For computations, it is more convenient to write it in the following form of a system of two equations:

$$\mathbf{R} - \mathbf{\Gamma} \circ \mathcal{M}_{22}\{\mathbf{R}\} = \mathbf{\Gamma} \circ \mathcal{M}_{21}\{\mathbf{W}\} \quad (47)$$
$$\mathbf{Z} = \mathcal{M}_{11}\{\mathbf{W}\} + \mathcal{M}_{12}\{\mathbf{R}\} \quad (48)$$

which indicates that in solving for $\mathbf{Z}$, one has to go through the intermediate step of solving for $\mathbf{R}$ from (47).

A. Uncorrelated Uncertainties Case

In this case, we assume the uncertainties to have correlations $\Gamma = \gamma I$. The case when different uncertainties have different variances can be accommodated through adding the appropriate multipliers in the system $\mathcal{M}$. In this case, it follows that the matrix $\mathbf{R}$ in (47) is diagonal. If we assume in addition that $\mathbf{W}$ is diagonal, and we are only interested in the diagonal part of $\mathbf{Z}$, then (47)–(48) can be rewritten in terms of matrix-vector multiplication using the notation of (39)–(40) where the vectors are the diagonal entries of the respective covariance matrices

$$\text{diag}(\mathbf{R}) - \gamma\,\mathbf{M}_{22}^{\circ}\,\text{diag}(\mathbf{R}) = \gamma\,\mathbf{M}_{21}^{\circ}\,\text{diag}(\mathbf{W})$$
$$\text{diag}(\mathbf{Z}) = \mathbf{M}_{11}^{\circ}\,\text{diag}(\mathbf{W}) + \mathbf{M}_{12}^{\circ}\,\text{diag}(\mathbf{R}).$$

Putting these two equations together by eliminating $\mathbf{R}$ gives

$$\text{diag}(\mathbf{Z}) = \left(\mathbf{M}_{11}^{\circ} + \gamma\,\mathbf{M}_{12}^{\circ}(I - \gamma\,\mathbf{M}_{22}^{\circ})^{-1}\mathbf{M}_{21}^{\circ}\right)\text{diag}(\mathbf{W}).$$

Without loss of generality, we can, in addition, assume $\mathbf{W} = I$, for which there is an even simpler expression for the total variance of z

$$\text{tr}(\mathbf{Z}) = \left|\mathbf{M}_{11}^{\circ} + \gamma\,\mathbf{M}_{12}^{\circ}(I - \gamma\,\mathbf{M}_{22}^{\circ})^{-1}\mathbf{M}_{21}^{\circ}\right| \quad (49)$$

where $|M|$ stands for the sum of all the elements of a nonnegative matrix M.

In the literature on robust stability analysis, there is often an equivalence between a robust performance condition and a robust stability condition on an augmented system with an additional uncertainty. The uncorrelated case here provides a version of such a correspondence, and we will state it without loss of generality for the case of $\gamma = 1$.

Corollary 5.1: Consider the system of Fig. 6 with uncorrelated uncertainties with variances $\mathbb{E}[\gamma_i^2(t)] = 1$, and scalar inputs and outputs w and z, respectively. Then

$$\frac{\mathbb{E}[z^2(t)]}{\mathbb{E}[w^2(t)]} < 1 \iff \rho\left(\begin{bmatrix} \mathbf{M}_{11}^{\circ} & \mathbf{M}_{12}^{\circ} \\ \mathbf{M}_{21}^{\circ} & \mathbf{M}_{22}^{\circ} \end{bmatrix}\right) < 1.$$

The proof is a simple application of the Schur complement on the 2×2 block matrix, which implies that the spectral radius condition is equivalent to the right-hand side of (49) being less than 1. Note that the variance ratio condition is a performance condition, while the spectral radius condition is an MSS stability condition for a system with an additional (fictitious) uncertainty in a feedback loop between z and w.

VI. State-Space Methods and Computations

Although the input–output setting presented in this article appears to be more expedient for analysis and statement of results, it is often (though not always) the case that actual computations are more conveniently carried out using state-space representations. From one point of view, this results in LMI conditions. For large-scale system applications, we write out a power iteration type algorithm that involves solving Lyapunov equations at each step of the iteration. Finally, we give a state-space interpretation of the "worst-case covariance" in the case where there are no exogenous inputs. This turns out to be a worst-case covariance of a random initial state.

A. MSS Conditions

Begin with the MSS problem of Theorem 3.2. Let the strictly causal LTI system $\mathcal{M}$ have the following realization:

$$x_{t+1} = Ax_t + Bu_t$$
$$y_t = Cx_t$$

from which it follows that a corresponding realization for the covariance feedback system is

$$\mathbf{X}_{t+1} = A\mathbf{X}_t A^* + B\mathbf{U}_t B^*$$
$$\mathbf{Y}_t = C\mathbf{X}_t C^*$$
$$\mathbf{U}_t = \mathbf{\Gamma} \circ \mathbf{Y}_t.$$

Taking the steady-state limit produces the following representation of the loop gain operator $\bar{\mathbf{R}} = \mathbb{L}(\bar{\mathbf{U}})$:

$$\bar{\mathbf{X}} - A\bar{\mathbf{X}}A^* = B\bar{\mathbf{U}}B^* \quad (50)$$
$$\bar{\mathbf{R}} = \mathbf{\Gamma} \circ \left(C\bar{\mathbf{X}}C^*\right). \quad (51)$$

Thus, one method of computing the action of $\mathbb{L}$ is to solve the Lyapunov equation (50) for $\bar{\mathbf{X}}$ given the input covariance $\bar{\mathbf{U}}$, and, then, calculate $\bar{\mathbf{R}}$ from (51).

1) Power Iteration Algorithm: The above procedure for calculating the action of $\mathbb{L}$ can now be used as follows in a power iteration method for calculating $\rho(\mathbb{L})$ as recommended in [26]. Starting from an arbitrary initial matrix $\mathbf{P}_0 \geq 0$

$$\mathbf{P}_{k+1} = \mathbb{L}\left(\mathbf{P}_k\right) / \|\mathbf{P}_k\|.$$

At each step, the calculation of $\mathbb{L}(P_k)$ involves (50) and (51) as follows:

$$\mathbf{X}_{k+1} - A\,\mathbf{X}_{k+1}\,A^* = B\mathbf{P}_k B^*$$

$$\mathbf{P}_{k+1} = \frac{1}{\|P_k\|}\left(\mathbf{\Gamma}\circ\left(C\mathbf{X}_{k+1}C^*\right)\right). \quad (52)$$

The major computational burden in each step is solving the Lyapunov equation (52). However, this power iteration algorithm is well-suited for use with sparse methods for solving Lyapunov equations, which are themselves iterative procedures.

2) As an LMI: To calculate the spectral radius $\rho(\mathbb{L})$, one can set $\bar{\mathbf{R}} = \lambda\bar{\mathbf{U}}$ in the above and find the largest real number λ such that

$$\lambda\left(\bar{\mathbf{X}} - A\bar{\mathbf{X}}A^*\right) - B\left(\mathbf{\Gamma}\circ\left(C\bar{\mathbf{X}}C^*\right)\right)B^* = 0$$

$$\bar{\mathbf{X}} \geq 0.$$

B. Mean Square Performance

For the mean square performance problem, begin with the following realization for the stable, strictly causal system $\mathcal{M}$

$$x_{t+1} = Ax_t + B_1 w_t + B_2 r_t$$
$$z_t = C_1 x_t$$
$$v_t = C_2 x_t.$$

Since w and r are mutually uncorrelated and white, the corresponding realization for the covariance feedback system is

$$\mathbf{X}_{t+1} = A\mathbf{X}_t A^* + B_1\mathbf{W}_t B_1^* + B_2\mathbf{R}_t B_2^*$$
$$\mathbf{Z}_t = C_1\mathbf{X}_t C_1^*$$
$$\mathbf{V}_t = C_2\mathbf{X}_t C_2^*$$
$$\mathbf{R}_t = \mathbf{\Gamma}\circ\mathbf{V}_t.$$

The corresponding steady-state equations are

$$\bar{\mathbf{X}} - A\bar{\mathbf{X}}A^* - B_2\left(\mathbf{\Gamma}\circ\left(C_2\bar{\mathbf{X}}C_2^*\right)\right)B_2^* = B_1\bar{\mathbf{W}}B_1^* \quad (53)$$

$$\bar{\mathbf{Z}} = C_1\bar{\mathbf{X}}C_1^*. \quad (54)$$

Therefore, the main step in evaluation the covariance of the output is solving the matrix equation (53) for the state covariance. This is not a standard Lyapunov equation, but iterative algorithms, akin to those designed for large-scale Lyapunov equations, can be used to tackle it.

C. Worst Case Covariances

The "Perron eigen-matrix" $\widehat{\mathbf{U}}$ of the loop gain operator $\mathbb{L}$ (30) is by definition the matrix that achieves the spectral radius of $\mathbb{L}$, i.e.,

$$\mathbb{L}\left(\widehat{\mathbf{U}}\right) = \rho(\mathbb{L})\,\widehat{\mathbf{U}}. \quad (55)$$

In the necessity proof of Theorem 3.2 (and the comment thereafter), it was shown that this matrix has an interpretation as a sort of worst-case covariance matrix. To recap, assume MSS is lost, so $\rho(\mathbb{L}) > 1$, and let the exogenous disturbances be such that $d = 0$, and w has covariance $\mathbb{E}[w_t w_t^*] = \widehat{\mathbf{U}}$. Then, a consequence of inequality (37) is that the covariance of the signal u will grow at a geometric rate of

$$\mathbb{E}\left[u_t u_t^*\right] \geq c\,\alpha^t\,\widehat{\mathbf{U}}$$

where for any $\epsilon > 0$, we can choose $\alpha = \rho(\mathbb{L}) - \epsilon$, and $c > 0$ is some constant.

An alternative interpretation, which does not require exogenous inputs, can also be given. In this scenario, the exogenous inputs w and d are set to zero, but the system $\mathcal{M}$ has some nonzero random initial state x_0 with covariance $\mathbb{E}[x_0 x_0^*] =: \mathbf{X}_0$. In this case, the evolution of the state covariance has the following dynamics:

$$\begin{aligned}\mathbf{X}_{t+1} &= A\mathbf{X}_t A^* + B\mathbf{U}_t B^*, \quad \mathbf{X}_0 = \mathbb{E}\left[x_0 x_0^*\right]\\ \mathbf{U}_t &= \mathbf{\Gamma}\circ\left(C\mathbf{X}_t C^*\right).\end{aligned} \quad (56)$$

Now let $\widehat{\mathbf{U}}$ be an eigenmatrix of $\mathbb{L}$ as (55). Then

$$\widehat{\mathbf{U}} = \frac{1}{\rho(\mathbb{L})}\mathbb{L}\left(\widehat{\mathbf{U}}\right) = \frac{1}{\rho(\mathbb{L})}\mathbf{\Gamma}\circ\left(\sum_{\tau=0}^{\infty} M_t\widehat{\mathbf{U}}M_t^*\right)$$
$$= \frac{1}{\rho(\mathbb{L})}\mathbf{\Gamma}\circ\left(C\sum_{t=0}^{\infty} A^t B\widehat{\mathbf{U}}B^* A^{*t} C^*\right)$$
$$=: \mathbf{\Gamma}\circ\left(C\hat{\mathbf{X}}C^*\right)$$

where $\hat{\mathbf{X}} := \frac{1}{\rho(\mathbb{L})}\sum_{t=0}^{\infty} A^t B\widehat{\mathbf{U}}B^* A^{*t}$ is the worst-case covariance of the state. It can be calculated from $\widehat{\mathbf{U}}$ using the following algebraic Lyapunov equation:

$$\hat{\mathbf{X}} - A\hat{\mathbf{X}}A^* = \frac{1}{\rho(\mathbb{L})}B\widehat{\mathbf{U}}B^*.$$

Note that setting $\mathbf{X_0} = \hat{\mathbf{X}}$ yields $\mathbf{U_0} = \widehat{\mathbf{U}}$. By substituting $\mathbf{W}_t = 0$ in (34) and carrying out the same argument in the necessity proof of Theorem 3.2, we obtain

$$\mathbf{U}_{Tk} \geq \mathbb{L}_T^k(\widehat{\mathbf{U}}) \geq (\rho(\mathbb{L}) - \epsilon c)^k\widehat{\mathbf{U}} =: \alpha^k\widehat{\mathbf{U}}.$$

This calculation shows that $\{\mathbf{U}_{Tk}\}$ is a geometrically growing sequence since ϵ can be chosen small enough so that $\alpha > 1$. Consequently, by (56), we have

$$\mathbf{X}_{Tk+1} = A\mathbf{X}_{Tk}A^* + B\mathbf{U}_{Tk}B^* \geq \alpha^k B\widehat{\mathbf{U}}B^*$$

and, therefore, $\{\mathbf{X}_{Tk+1}\}$ is also a geometrically growing sequence.

VII. Conclusion and Discussion

In this article, we study the MSS and performance of LTI systems in feedback with stochastic disturbances. We derive the necessary and sufficient conditions of MSS by adopting a purely input/output approach, and, thus, state-space realizations are treated as a special case. Our treatment leads to uncover a linear operator whose 1) spectral radius fully characterizes the conditions of MSS, and whose 2) "Perron–Frobenius Eigenmatrix" characterizes the fastest growing modes of the covariances when MSS is lost. The input–output approach adopted in this article has the advantages of unifying the proofs and extending the results for a broader class of linear systems that are not limited to state-space realizations only, but also distributed systems and systems with irrational transfer functions (including delays).

This article treats the discrete-time setting where the stochastic disturbances are all white in time but are allowed to have "spatial correlations." Future work in this line of research includes addressing the continuous-time setting [29] and generalizing the analysis for stochastic disturbances that are correlated in time as well.

Appendix

A. Independence

For any two independent random variables a and b, $\mathbb{E}[ab] = \mathbb{E}[a]\mathbb{E}[b]$. Let X, Y, and Z be (possibly matrix-valued) random variables. Assume Y is independent of X and Z. Then

$$\begin{aligned}\mathbb{E}\left[XYZ\right] = \mathbb{E}\left[\sum_{j,k} x_{ij}y_{jk}z_{kl}\right] &= \sum_{j,k}\mathbb{E}\left[x_{ij}y_{jk}z_{kl}\right] \\ &= \sum_{j,k}\mathbb{E}\left[x_{ij}z_{kl}\right]\,\mathbb{E}\left[y_{jk}\right] \\ &= \sum_{j,k}\mathbb{E}\left[x_{ij}\,\mathbb{E}\left[y_{jk}\right]\,z_{kl}\right] \\ &= \mathbb{E}\left[X\,\mathbb{E}\left[Y\right]\,Z\right]. \end{aligned} \tag{57}$$

B. Monotone Systems and Signals

For linear monotone systems, it is first shown that positive feedback interconnections are also monotone. It is, then, shown that a time-invariant monotone system preserves signals' temporal order.

Begin with general comments about causal discrete-time systems. Signals are identified with ℓ, the set of vector-valued sequences over $\mathbb{Z}^+$. A causal linear system $\mathcal{G} : \ell \to \ell$ is a mapping on ℓ, and it can be identified with a lower-triangular semi-infinite matrix. A (possibly unstable) positive feedback system is *well posed* if $(I - \mathcal{G})^{-1} : \ell \to \ell$ is well defined.

Let $P_T : \ell \to \ell_T$ be the "projection"

$$(P_T f)(t) := f(t), \quad t \le T$$

where $\ell_T := \{f : \{0, \ldots, T\} \to \mathbb{R}^n\}$ is the space of finite sequences of length $T+1$. With a slight abuse of notation, define the "injection" $P_T^\dagger : \ell_T \to \ell$ by

$$\left(P_T^\dagger f\right)(t) := \begin{cases} f(t) & t \le T \\ 0 & t > T. \end{cases}$$

Clearly, $P_T P_T^\dagger = I$, and for any system $\mathcal{G}$, $P_t \mathcal{G} P_T^\dagger$ is the finite matrix "upper left block" of its semi-infinite matrix representation. It can be thought of as a finite time-horizon restriction of $\mathcal{G}$. Causality of $\mathcal{G}$ implies that

$$P_T \mathcal{G}^n P_T^\dagger = \left(P_T \mathcal{G} P_T^\dagger\right)^n$$

for any power n, and if $\mathcal{G}^{-1}$ exists, then

$$P_T \mathcal{G}^{-1} P_T^\dagger = \left(P_T \mathcal{G} P_T^\dagger\right)^{-1}.$$

We finally note that while $P_T^\dagger P_T \ne I$, for any causal system $\mathcal{G}$, and any time T, we have

$$P_T\,\mathcal{G}\,P_T^\dagger P_T = P_T\,\mathcal{G}.$$

1) Feedback Interconnections:

Theorem A.1: The sum, cascade, and positive feedback interconnections of causal monotone linear systems are monotone.

Proof: Closure under sums and cascades is obvious from the definition (24). This, in particular, implies that powers $\mathcal{M}^n$ of any monotone system $\mathcal{M}$ are also monotone. Therefore, if the Neuman series

$$(I - \mathcal{M})^{-1} = \sum_{n=0}^{\infty} \mathcal{M}^n$$

can be shown to converge in an appropriate sense, then positive feedback interconnections are also monotone. A convergence argument is now given. It is similar to successive iteration schemes for Volterra operators [30] (see also [31, Appendix]).

Consider the partial series product

$$\begin{aligned} &P_T (I - \mathcal{M}) P_T^\dagger P_T \left(\sum_{n=0}^{N} \mathcal{M}^n\right) P_T^\dagger \\ &= P_T (I - \mathcal{M}) \left(\sum_{n=0}^{N} \mathcal{M}^n\right) P_T^\dagger = I - P_T \mathcal{M}^{N+1} P_T^\dagger \\ &= I - \left(P_T \mathcal{M} P_T^\dagger\right)^{N+1} \end{aligned} \tag{58}$$

where the last equality follows from the causality of $\mathcal{M}$. Strict causality of $\mathcal{M}$ means $P_T \mathcal{M} P_T^\dagger$ is just a $(T+1)\times(T+1)$ strictly lower-triangular matrix. It is, therefore, nilpotent and

$$\left(P_T \mathcal{M} P_T^\dagger\right)^{N+1} = 0, \;\; N \ge T.$$

The conclusion is, then, that

$$\begin{aligned} P_T (I - \mathcal{M})^{-1} P_T^\dagger &= \left(P_T (I - \mathcal{M}) P_T^\dagger\right)^{-1} \\ &= P_T \left(\sum_{n=0}^{T} \mathcal{M}^n\right) P_T^\dagger. \end{aligned}$$

This means that for each T, the finite horizon restriction $P_T(I - \mathcal{M})^{-1} P_T^\dagger$ is monotone, and, therefore, the system $(I - \mathcal{M})^{-1}$ itself must be monotone. ∎

2) Preserving Monotonicity of Signals:

For time invariant systems, the above definition of monotonicity has an additional implication in that nondecreasing input sequences produce nondecreasing output sequences. Consider the input–output pair $\mathbf{Y} = \mathcal{M}(\mathbf{U})$. Let $\mathcal{S}$ be the right shift operator on sequences

$$\left(\mathcal{S}\mathbf{U}\right)(t) := \begin{cases} \mathbf{U}(t-1), & t \ge 1 \\ 0, & t = 0. \end{cases}$$

The time invariance of $\mathcal{M}$ means that $\mathcal{M}\mathcal{S}^n = \mathcal{S}^n\mathcal{M}$ for all powers $n \ge 1$. Recall that a signal is said to be *monotone* (or *nondecreasing*) if

$$t_1 \le t_2 \implies \mathbf{U}(t_1) \le \mathbf{U}(t_2).$$

An equivalent condition for a signal to be monotone is

$$\forall\, n \geq 1, \quad \mathcal{S}^n \mathbf{U} \,\leq\, \mathbf{U}$$

where the relation $\leq$ is the pointwise ordering on signals (23). Now calculate that for any $n \geq 1$

$$\mathcal{S}^n \mathbf{Y} = \mathcal{S}^n \mathcal{M}(\mathbf{U}) = \mathcal{M}(\mathcal{S}^n \mathbf{U}) \,\leq\, \mathcal{M}(\mathbf{U}) = \mathbf{Y}$$

where the inequality follows from $\mathcal{M}$ being monotone together with $\mathbf{U}$ being nondecreasing. One can then conclude that *time-invariant monotone systems preserve monotonicity of signals*.

C. Proof of Lemma 2.1

First, observe that the mappings from the exogenous inputs to all signals in the loop are

$$\begin{bmatrix} u \\ y \\ v \\ r \end{bmatrix} = \begin{bmatrix} (I-\Gamma\mathcal{M})^{-1} & \Gamma(I-\mathcal{M}\Gamma)^{-1} \\ (I-\mathcal{M}\Gamma)^{-1}\mathcal{M} & \mathcal{M}\Gamma(I-\mathcal{M}\Gamma)^{-1} \\ (I-\mathcal{M}\Gamma)^{-1}\mathcal{M} & (I-\mathcal{M}\Gamma)^{-1} \\ \Gamma(I-\mathcal{M}\Gamma)^{-1}\mathcal{M} & \Gamma(I-\mathcal{M}\Gamma)^{-1} \end{bmatrix} \begin{bmatrix} w \\ d \end{bmatrix}. \quad (59)$$

Thus, one needs to investigate the causal dependencies of all four mappings in the matrix of (59). Consider first the mapping $(I-\mathcal{M}\Gamma)^{-1}$. Over the time horizon $[0,t]$, this operator can be written in partitioned matrix form as

$$\begin{bmatrix} I & & & \\ -M_1\Gamma_0 & I & & \\ & \ddots & \ddots & \\ -M_t\Gamma_0 & \cdots & -M_1\Gamma_{t-1} & I \end{bmatrix}^{-1} = \begin{bmatrix} I & & \\ & \ddots & \\ * & & I \end{bmatrix}.$$

Note the strictly block lower-triangular structure of $\mathcal{M}\Gamma$, which is a consequence of the strict causality of $\mathcal{M}$. The $*$ blocks are functions of $\Gamma_0,\ldots,\Gamma_{t-1}$, and are independent of Γ_τ, $\tau \geq t$.

Using this, we write the equation for u over the time horizon

$$\begin{bmatrix} u_0 \\ \vdots \\ u_t \end{bmatrix} = \begin{bmatrix} I & & \\ & \ddots & \\ * & & I \end{bmatrix} \begin{bmatrix} w_0 \\ \vdots \\ w_t \end{bmatrix} + \begin{bmatrix} \Gamma_0 & & \\ & \ddots & \\ & & \Gamma_t \end{bmatrix} \begin{bmatrix} I & & \\ & \ddots & \\ * & & I \end{bmatrix} \begin{bmatrix} d_0 \\ \vdots \\ d_t \end{bmatrix}.$$

Now recall that all noise terms $\{\Gamma_t\}$, $\{d_t\}$, and $\{w_t\}$ are assumed mutually independent. The equation above shows that $\{u_0,\ldots,u_t\}$ are a function of only past and present values of $\{\Gamma_t\}$, $\{d_t\}$, and $\{w_t\}$.

It is now clear that by repeating the above argument for each of the signals that whenever Γ is preceded by the operator $\mathcal{M}$, the dependence on the present value of Γ is killed by the strict causality of $\mathcal{M}$. Therefore, the following conclusion can be stated: the present values of y and v ($\mathcal{M}$ precedes Γ in their expression) are independent of the present and future values of Γ. In contrast, the present values of u and r are independent of future values of Γ only.

D. Some Properties of the Hadamard Product

Let π be a permutation matrix, this means each row and each column contains exactly one nonzero element equal to 1. A nonzero element in location ij implies that the j'th component of a vector is mapped to the i'th component of the vector. Thinking of the inverse operation, clearly $\pi^{-1} = \pi^*$. There are, in general, no simple relations between the regular matrix product and the Hadamard product. However, for permutation matrices, we have the simple relation

$$\pi_1\,(A \circ B)\,\pi_2 = (\pi_1\,A\pi_2) \circ (\pi_1 B\pi_2) \quad (60)$$

which is obviously true since for any matrix M, the matrix $\pi_1\,M\pi_2$ is simply a rearrangement of its entries.

E. Proof of Necessity in Lemma 3.1

(*"only if"*). To simplify notation, assume $\mathbf{\Gamma} = 1$. The general case follows by scaling. It will be shown next that if $\|\mathcal{M}\|_2^2 \geq 1$, w is a white, constant variance process and $d = 0$, then $\mathbf{u}$ is an unbounded sequence.

From (8), (19), (26), and (27), the sequence $\mathbf{u}$ satisfies the following recursion:

$$\mathbf{u}_t \;=\; \mathbf{y}_t + \mathbf{w}_t \;=\; \sum_{\tau=0}^{t} M_{t-\tau}^2\,\mathbf{u}_\tau + \mathbf{w}_t. \quad (61)$$

This recursion may not be of finite order (e.g., if $\{M_t\}$ is not FIR) and it is, therefore, not clear how to use it to estimate the growth of $\mathbf{u}$. However, it can be replaced with a simple recursive *inequality* for a subsequence of $\mathbf{u}$, for which a growth estimate is immediately obtained. This is the essence of the remainder of the proof.

Note that the quantity $\alpha := \sum_{\tau=0}^{T} M_\tau^2$ can be made arbitrarily close to $\|\mathcal{M}\|_2^2 \geq 1$ by choosing the time horizon T sufficiently large. It will now be shown that the subsequence $\{\mathbf{u}_{Tk};\ k \in \mathbb{Z}^+\}$ is unbounded. First, the non-negativity of all sequences in (61) gives a recursive *inequality* for the subsequence $\{\mathbf{u}_{Tk}\}$

$$\begin{aligned} \mathbf{u}_{Tk} &= \sum_{\tau=0}^{Tk} M_{Tk-\tau}^2\,\mathbf{u}_\tau + \mathbf{w}_{Tk} \\ &\geq \sum_{\tau=T(k-1)}^{Tk} M_{Tk-\tau}^2\,\mathbf{u}_\tau + \mathbf{w}_{Tk} \\ &\geq \left(\sum_{\tau=0}^{T} M_\tau^2\right) \min_{T(k-1)\leq\tau\leq Tk} \mathbf{u}_\tau + \mathbf{w}_{Tk} \\ &= \alpha\,\mathbf{u}_{T(k-1)} + \mathbf{w}_{Tk} \end{aligned}$$

where the last equality follows from the monotonicity of the sequence $\mathbf{u}$. The above is a difference inequality, which has the initial condition $\mathbf{u}_0 = \mathbf{r}_0 + \mathbf{w}_0 = \mathbf{w}_0$ ($\mathbf{r}_0 = 0$ since $d = 0$ and $\mathcal{M}$ is strictly causal). A simple induction argument gives

$$\mathbf{u}_{Tk} \;\geq\; \sum_{r=0}^{k} \alpha^r\,\mathbf{w}_{T(k-r)} \quad (62)$$

which is a convolution of $\{\alpha^k\}$ with the subsequence $\{\mathbf{w}_{Tk}\}$. Now if $\|\mathcal{M}\|_2 > 1$, then a time horizon T can be chosen such that $\sum_{\tau=0}^{T} M_\tau^2 =: \alpha > 1$. The monotonicity of the sequence $\mathbf{d}$ and (62) implies that $\{\mathbf{u}_{Tk}\}$ (and, thus, $\mathbf{u}$) is a geometrically increasing sequence.

The case $\|\mathcal{M}\|_2 = 1$ is slightly more delicate. T can be chosen such that α is as close to 1 as desired. For $\alpha < 1$, one also has

$$\lim_{k\to\infty} \left(\alpha^k + \cdots + \alpha + 1\right) = \frac{1}{1-\alpha}.$$

For any $\epsilon > 0$, k can be chosen such that[2]

$$\mathbf{u}_{Tk} \geq \frac{\mathbf{w}_0}{1-\alpha} - \epsilon.$$

Now given any lower bound B, choose T and k such that α is sufficiently close to 1 and ϵ is sufficiently small so that

$$\mathbf{u}_{Tt} \geq \frac{\mathbf{w}_0}{1-\alpha} - \epsilon > B.$$

This proves that $\mathbf{u}$ is an unbounded sequence even though it may not have geometric growth in the case $\|\mathcal{M}\|_2 = 1$.

F. Lemmas Used in the Proof of Necessity in Theorem 3.2

Throughout this appendix, let $\lambda_{\max}(A), \lambda_{\min}(A)$, and $||A||$ denote the largest eigenvalue, smallest eigenvalue, and the spectral norm of any matrix A, respectively. As the rest of the article, matrix inequalities are understood as semidefinite ordering on matrices. Note that if $A \geq 0$, then $||A|| = \lambda_{\max}(A)$. Furthermore, let $\mathcal{N}(A)$ and $\mathcal{R}(A)$ denote the null space and range space of A, respectively. We now present two lemmas that are required for the proof of necessity for Theorem 3.2.

Lemma A.2: Let $A, B \in \mathbb{R}^{n\times n}$, such that $0 \leq B \leq A$. Then, $\mathcal{N}(A) \subseteq \mathcal{N}(B)$.

Proof: Since $0 \leq B \leq A$, then $\forall v \in \mathbb{R}^n$, we have $0 \leq v^* B v \leq v^* A v$. Particularly, let $v \in \mathcal{N}(A)$, then $v^* A v = 0$, which implies that $v^* B v = 0$ as well. We are now left with proving that $Bv = 0$.

Let $B = U\Sigma U^*$ be the eigendecomposition of B, then $v^* U \Sigma U^* v = 0$. Setting $w := U^* v$ yields $w^* \Sigma w = 0$, which implies that $\Sigma w = 0$ (because Σ is diagonal with nonnegative entries). Finally, we have $Bv = U\Sigma U^* v = U \Sigma w = 0$, which completes the proof. ■

Lemma A.3: Let $A, B \in \mathbb{R}^{n\times n}$, and $\rho, \epsilon > 0$ such that $0 \leq B \leq \rho A$ and $||\rho A - B|| \leq \epsilon ||A||$. Then, $\exists c > 0$ such that $B \geq (\rho - \epsilon c) A$.

Proof: The proof is carried out for the case where $A > 0$ first. Then, the result is exploited to prove the more general case where $A \geq 0$.

("$A > 0$") $\forall v \in \mathbb{R}^n$, we have

$$v^*(\rho A - B)v \leq ||\rho A - B||\, ||v||^2 \leq \epsilon ||A||\, ||v||^2$$

where the first inequality follows by noting that $\rho A - B \geq 0$ and, thus, $\lambda_{\max}(\rho A - B) = ||\rho A - B||$. Recalling that $v^* A v \geq \lambda_{\min}(A) ||v||^2$ and $A > 0$ (i.e., $\lambda_{\min}(A) > 0$), we obtain the following upper bound:

$$\epsilon ||A||\, ||v||^2 \leq \epsilon ||A|| \frac{1}{\lambda_{\min}(A)} v^* A v =: \epsilon c\, v^* A v$$

where $c := \lambda_{\max}(A)/\lambda_{\min}(A)$ is the condition number of A. Now, $\forall v \in \mathbb{R}^n, v^*(\rho A - B) v \leq \epsilon c\, v^* A v$, which implies that $\rho A - B \leq \epsilon c A$. Finally, rearranging the last inequality completes the proof.

("$A \geq 0$"). Let $r < n$ denote the rank of A so that its eigendecomposition can be written as

$$A = U\Sigma U^* = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_1^* \\ U_2^* \end{bmatrix}$$

where U is a unitary matrix and Σ_r is a diagonal matrix with strictly positive entries. Before we continue the proof, observe that this matrix partitioning indicates that

1) $\mathcal{N}(A) = \mathcal{R}(U_2)$, and thus $AU_2 = 0$.
2) $A = U_1 \Sigma_r U_1^*$ and thus $U_1^* A U_1 > 0$ (since $U_1^* U_1 = I$).
3) Lemma A.2 guarantees that $\mathcal{N}(A) \subseteq \mathcal{N}(B)$, and thus $AU_2 = BU_2 = 0$.

Multiplying all sides of the inequality $0 \leq B \leq \rho A$ by U^* from the left and U from the right preserves its ordering, then $0 \leq U^* B U \leq \rho U^* A U$, which implies

$$0 \leq \begin{bmatrix} U_1^* B U_1 & 0 \\ 0 & 0 \end{bmatrix} \leq \begin{bmatrix} \rho U_1^* A U_1 & 0 \\ 0 & 0 \end{bmatrix}.$$

This is a consequence of $AU_2 = BU_2 = 0$ and $U_2^* A = U_2^* B = 0$. Define $A_{11} := U_1^* A U_1$ and $B_{11} := U_1^* B U_1$, then we have

$$0 \leq B_{11} \leq \rho A_{11}. \tag{63}$$

Furthermore, recalling that $||\rho A - B|| \leq \epsilon ||A||$, and knowing that the spectral norm of a matrix is preserved under multiplications by unitary matrices, we obtain $||U^*(\rho A - B)U|| \leq \epsilon ||U^* A U||$, which implies

$$||\rho A_{11} - B_{11}|| \leq \epsilon ||A_{11}||. \tag{64}$$

Since $A_{11} > 0$, the first part of the proof ("$A > 0$") can be invoked here by exploiting (63) and (64) to obtain $B_{11} \geq (\rho - \epsilon c) A_{11}$ where $c := \lambda_{\max}(A_{11})/\lambda_{\min}(A_{11})$. This implies that

$$\begin{bmatrix} U_1^* B U_1 & 0 \\ 0 & 0 \end{bmatrix} \geq (\rho - \epsilon c) \begin{bmatrix} U_1^* A U_1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Finally, multiplying both sides of the inequality by U from the left and U^* from the right completes the proof because

$$U^* A U = \begin{bmatrix} U_1^* A U_1 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow U \begin{bmatrix} U_1^* A U_1 & 0 \\ 0 & 0 \end{bmatrix} U^* = A$$

and the same reasoning holds for B. ■

[2] $\mathbf{d}_0$ is chosen as a simple lower bound on the entire sequence $\mathbf{d}$. Other choices can produce better lower bounds on $\mathbf{u}$.

References

[1] K. Zhou, J. Doyle, and K. Glover, *Robust and Optimal Control*, vol. 40. Upper Saddle River, NJ, USA: Prentice-Hall, 1996.

[2] A. Packard and J. Doyle, "The complex structured singular value," *Automatica*, vol. 29, no. 1, pp. 71–109, 1993.

[3] J. L. Willems and J. C. Willems, "Feedback stabilizability for stochastic systems with state and control dependent noise," *Automatica*, vol. 12, no. 3, pp. 277–283, 1976.
[4] J. Willems, "Stability criteria for stochastic systems with colored multiplicative noise," *Acta Mechanica*, vol. 23, no. 3, pp. 171–178, 1975.
[5] G. Blankenship, "Stability of linear differential equations with random coefficients," *IEEE Trans. Autom. Control*, vol. AC-22, no. 5, pp. 834–838, Oct. 1977.
[6] S. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*. Philadelphia, PA, USA: SIAM, 1994.
[7] L. E. Ghaoui, "State-feedback control of systems with multiplicative noise via linear matrix inequalities," *Syst. Control Lett.*, vol. 24, no. 3, pp. 223–228, 1995.
[8] N. Elia, "Remote stabilization over fading channels," *Syst. Control Lett.*, vol. 54, no. 3, pp. 237–249, 2005.
[9] D. Hinrichsen and A. Pritchard, "Stability radii for infinite dimensional systems with stochastic uncertainty," in *Proc. 3rd. Eur. Control Conf*, 1995, pp. 3291–3293.
[10] A. El Bouhtouri and A. Pritchard, "Stability radii of linear systems with respect to stochastic perturbations," *Syst. Cont. Lett.*, vol. 19, no. 1, pp. 29–33, 1992.
[11] J. Lu and R. E. Skelton, "Mean-square small gain theorem for stochastic control: discrete-time case," *IEEE Trans. Autom. Control*, vol. 47, no. 3, pp. 490–494, Mar. 2002.
[12] M. Dahleh and Y. Ohta, "A necessary and sufficient condition for robust bibo stability," *Syst. Control Lett.*, vol. 11, no. 4, pp. 271–275, 1988.
[13] A. Megretski, "Necessary and sufficient conditions of stability: A multiloop generalization of the circle criterion," *IEEE Trans. Autom. Control*, vol. 38, no. 5, pp. 753–756, May 1993.
[14] B. Bamieh and M. Dahleh, "On robust stability with structured time-invariant perturbations," *Syst. Control Lett.*, vol. 21, no. 2, pp. 103–108, 1993.
[15] J. Doyle, J. Wall, and G. Stein, "Performance and robustness analysis for structured uncertainty," in *Proc. 21st IEEE Conf. Decision Control*, 1982, vol. 21, pp. 629–636.
[16] M. Khammash and J. B. Pearson Jr, "Performance robustness of discrete-time systems with structured uncertainty," *IEEE Trans. Autom. Control*, vol. 36, no. 4, pp. 398–412, Apr. 1991.
[17] J. S. Shamma, "Robust stability with time-varying structured uncertainty," *IEEE Trans. Autom. Control*, vol. 39, no. 4, pp. 714–724, Apr. 1994.
[18] M. K. H. Fan, A. L. Tits, and J. C. Doyle, "Robustness in the presence of mixed parametric uncertainty and unmodeled dynamics," *IEEE Trans. Autom. Control*, vol. 36, no. 1, pp. 25–38, Jan. 1991.
[19] J. Doyle, "Structured uncertainty in control system design," in *Proc. 24th IEEE Conf. Decision Control*, 1985, vol. 24, pp. 260–265.
[20] M. Filo and B. Bamieh, "Investigating cochlear instabilities using structured stochastic uncertainty," in *Proc. IEEE 56th Annu. Conf. Decision Control*, 2017, pp. 1634–1640.
[21] C. A. Desoer and M. Vidyasagar, *Feedback Systems: Input-Output Properties*, vol. 55. Philadelphia, PA, USA: SIAM, 2009.
[22] L. Ljung, *System Identification*. Hoboken, NJ, USA: Wiley, 1999.
[23] I. Coope, "On matrix trace inequalities and related topics for products of hermitian matrices," *J. Math. Anal. Appl.*, vol. 188, no. 3, pp. 999–1001, 1994.
[24] H. Tigelaar, "On monotone linear operators and the spectral radius of their representing matrices," *SIAM J. Matrix Anal. Appl.*, vol. 12, p. 726, 1991.
[25] A. Berman and R. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, vol. 9. Philadelphia, PA, USA: SIAM, 1994.
[26] P. A. Parrilo and S. Khatri, "On cone-invariant linear matrix inequalities," *IEEE Trans. Autom. Control*, vol. 45, no. 8, pp. 1558–1563, Aug. 2000.
[27] D. Angeli and E. D. Sontag, "Monotone control systems," *IEEE Trans. Autom. Control*, vol. 48, no. 10, pp. 1684–1698, Oct. 2003.
[28] R. Horn and R. Mathias, "Block-matrix generalizations of schur's basic theorems on hadamard products," *Linear Algebr. Appl.*, vol. 172, pp. 337–346, 1992.
[29] M. Filo and B. Bamieh, "A block diagram approach to stochastic calculus with application to multiplicative uncertainty analysis," in *Proc. IEEE Conf. Decision Control*, 2018, pp. 3270–3275.
[30] F. Riesz and B. Sz.-Nagy, *Functional Analysis*. New York, NY, USA: Dover, 1990.
[31] B. Bamieh and P. G. Voulgaris, "A convex characterization of distributed control problems in spatially invariant systems with communication constraints," *Syst. Control Lett.*, vol. 54, no. 6, pp. 575–583, 2005.

Bassam Bamieh (Fellow, IEEE) received the B.Sc. degree in electrical engineering and physics from Valparaiso University, Valparaiso, IN, USA, in 1983, and the M.Sc. and Ph.D. degrees in electrical and computer engineering from Rice University, Houston, TX, USA, in 1986 and 1992, respectively.

From 1991 to 1998, he was an Assistant Professor with the Department of Electrical and Computer Engineering, and the Coordinated Science Laboratory, University of Illinois at Urbana-Champaign. He is currently a Professor of mechanical engineering, with the University of California at Santa Barbara. His research interests include robust and optimal control, distributed and networked control and dynamical systems, shear flow transition and turbulence, and the use of feedback in thermoacoustic energy conversion devices.

Dr. Bamieh received the IEEE Control Systems Society G. S. Axelby Outstanding Paper Award (twice), the AACC Hugo Schuck Best Paper Award, and the National Science Foundation CAREER Award. He was elected as a Distinguished Lecturer of the IEEE Control Systems Society (2005) and a Fellow of the International Federation of Automatic Control.

Maurice Filo (Member, IEEE) received the Dipl.Ing. degree in electrical engineering from the Lebanese University, Beirut, Lebanon, the M.Sc. degree from the American University of Beirut, Beirut, Lebanon, in 2010 and 2013, respectively. He also received the M.Sc. and Ph.D. degrees in mechanical engineering from the University of California, Santa Barbara, Santa Barbara, CA, USA, in 2017 and 2018, respectively.

After Ph.D., he was a Postdoctoral Researcher with the Swiss Federal Institute of Technology, ETH, Zurich. His research interests include stochastic dynamics, distributed dynamical systems, cochlear dynamics, mobile sensing and optimal control and estimation.