# STABILITY AND EXPONENTIAL DECAY FOR THE 2D ANISOTROPIC NAVIER-STOKES EQUATIONS WITH HORIZONTAL DISSIPATION

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Abstract. Solutions to the 2D Navier-Stokes equations with full dissipation in the whole space $\mathbb{R}^2$ always decay to zero in Sobolev spaces. In particular, any perturbation near the trivial solution is always asymptotically stable. In contrast, solutions to the 2D Euler equations for inviscid flows can grow rather rapidly. An intermediate situation is when the dissipation is anisotropic and only one-directional. The stability and large-time behavior problem for the 2D Navier-Stokes equations with only one-directional dissipation is not well-understood. When the spatial domain is the whole space $\mathbb{R}^2$, this problem is widely open. This paper solves this problem when the domain is $\mathbb{T}\times\mathbb{R}$ with $\mathbb{T}$ being a 1D periodic box. The idea here is to decompose the velocity $u$ into its horizontal average $\bar{u}$ and the corresponding oscillation $\widetilde{u}$. By making use of special properties of $\widetilde{u}$, we establish a uniform upper bound and the stability of $u$ in the Sobolev space $H^2$, and show that $\widetilde{u}$ in $H^1$ decays to zero exponentially in time.

## 1. Introduction

Let $\mathbb{T} = [0, 1]$ be a one-dimensional (1D) periodic box and let $\Omega = \mathbb{T}\times\mathbb{R}$. Consider the 2D incompressible Navier-Stokes equations with only horizontal dissipation,

$$\begin{cases} \partial_t u + u\cdot\nabla u = -\nabla p + \nu\partial_{11}u, \quad x\in\Omega,\ t>0, \\ \nabla\cdot u = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{1.1}$$

where $u$ denotes the velocity field of the fluid, $p$ the pressure and $\nu > 0$ the viscosity. Here $\partial_1$ is the abbreviation of the partial derivative $\partial_{x_1}$. In certain physical regimes and after suitable rescaling, the 2D Navier-Stokes equations become degenerate and reduce to the model in (1.1). One outstanding example is Prandtl's equation (see, e.g., [4, 7, 8]).

When the spatial domain is the whole space $\mathbb{R}^2$, the global existence and regularity of solutions to (1.1) relies on the Yudovich approach and the upper bound on the Sobolev norms is double exponential in time. The stability of perturbations near the trivial solution remains an open problem, let alone the precise large-time behavior of these perturbations. This paper focuses on the domain $\Omega$ specified above. The goal is two-fold. The first is to establish a uniform upper bound on the Sobolev norms of solutions to (1.1), and the second is to assess the stability of perturbations

2020 *Mathematics Subject Classification.* 35B35, 35B40, 35Q35, 76D03.
*Key words and phrases.* Navier-Stokes equations; Partial dissipation; Stability.

and pinpoint the exact large-time behavior of these perturbations. The results presented in this paper appear to be the very first ones devoted to understanding the 2D anisotropic Navier-Stokes equations. We shall explain some of the difficulties associated with the uniform upper bound and the stability problem.

Mathematically the model in (1.1) is intermediate between the 2D Euler equations and the 2D Navier-Stokes equations with full dissipation. The 2D Euler equations given by

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p, \\ \nabla \cdot u = 0 \end{cases}$$

represent the simplest but one of the most frequently used models for incompressible ideal fluids. There have been considerable recent interests on the precise large-time behavior of its solutions. One particular issue is whether the vorticity gradient can grow double exponentially in time. Here the vorticity $\omega = \nabla \times u$ is transported by the velocity field,

$$\partial_t \omega + u \cdot \nabla \omega = 0.$$

The vorticity gradient in any Lebesgue norm $L^p$ with $2 \le p \le \infty$ admits an upper bound that grows double exponentially in time. A significant problem is whether or not the double exponential growth rate is sharp [11]. Kiselev and Sverak were able to construct an explicit initial vorticity on a unit disk for which the corresponding vorticity gradient indeed grows double exponentially [5]. A general bounded domain appears to share this property [13]. Whether or not such examples can be constructed in $\mathbb{R}^2$ remains an open problem. Other important results on related issues can be found in several references (see, e.g., [2, 3, 15]). As a special consequence of these growth results, perturbations near the trivial solution of the 2D Euler equations are in general not stable in the Sobolev setting. In contrast, the Sobolev norms of solutions to the 2D incompressible Navier-Stokes equations

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p + \nu \Delta u, \quad x \in \mathbb{R}^2,\, t > 0, \\ \nabla \cdot u = 0 \end{cases} \tag{1.2}$$

always decay algebraically in time (see, e.g., [9, 10]). In particular, perturbations near the trivial solution of (1.2) are always asymptotically stable in the Sobolev space $H^2(\mathbb{R}^2)$.

When the dissipation is degenerate and is only one-directional as in (1.1), it is not clear how the solution would behave. In the case when the spatial domain is $\mathbb{R}^2$, the global existence and regularity relies on the Yudovich approach designed for the 2D Euler equations [14]. The essence of the Yudovich approach is that the vorticity $\omega = \nabla \times u$ is bounded for all time. We can show via the Yudovich approach that any $u_0 \in H^s(\mathbb{R}^2)$ with $s > 2$ leads to a unique global solution of (1.1). The solution remains in $H^s$ for all time, but the $H^s$-norm of the solution could grow rather rapidly in time. An upper bound on $\|u(t)\|_{H^s}$ is double exponential in time,

$$\|u(t)\|_{H^s} \le (\|u_0\|_{H^s})^{e^{C\,\|\omega_0\|_{L^\infty} t}}, \tag{1.3}$$

where $\omega_0 = \nabla \times u_0$ is the corresponding initial vorticity and $C$ is a pure constant. It remains an open problem whether or not the upper bound in (1.3) is sharp. Another immediate issue is whether or not we can lower the regularity of the initial data to $u_0 \in H^2(\mathbb{R}^2)$. Due to the Yudovich approach, the initial vorticity $\omega_0 = \nabla \times u_0$ is required to be in $L^\infty(\mathbb{R}^2)$, which in turn forces $u_0 \in H^s(\mathbb{R}^2)$ with $s > 2$. If we want to lower the regularity assumption to $H^2(\mathbb{R}^2)$, we need a different approach. Unfortunately the lack of dissipation in the vertical direction makes it impossible to control the growth of its solution without the boundedness of the vorticity. When we resort to the energy method to bound $\nabla\omega$, namely

$$\frac{d}{dt}\|\nabla\omega(t)\|_{L^2}^2 + 2\nu\|\partial_1\nabla\omega(t)\|_{L^2}^2 = -2\int \nabla\omega\cdot\nabla u\cdot\nabla\omega\,dx,$$

the one-directional dissipation fails to control the nonlinearity. In fact, the nonlinear part contains four component terms

$$\begin{aligned} \text{Hard} &:= -\int_{\mathbb{R}^2} \nabla\omega\cdot\nabla u\cdot\nabla\omega\,dx \\ &= -\int_{\mathbb{R}^2} \partial_1 u_1\,(\partial_1\omega)^2\,dx - \int_{\mathbb{R}^2} \partial_1 u_2\,\partial_1\omega\,\partial_2\omega\,dx \\ &\quad -\int_{\mathbb{R}^2} \partial_2 u_1\,\partial_1\omega\,\partial_2\omega\,dx - \int_{\mathbb{R}^2} \partial_2 u_2\,(\partial_2\omega)^2\,dx \end{aligned} \tag{1.4}$$

and the last two terms in (1.4) do not admit a time-integrable upper bound. This explains the difficulty of seeking a solution in $H^2$ as well as lowering the exponential upper bound. This is also the main reason why the stability problem on (1.1) in $\mathbb{R}^2$ remains a mystery.

When the spatial domain is $\Omega = \mathbb{T}\times\mathbb{R}$, this paper is able to obtain the global existence and regularity in the $H^2$-setting and provide a uniform upper bound on the $H^2$-norm of the solution. By offering an upper bound depending explicitly on the initial data, this paper also proves the stability of perturbations near the trivial solution. More importantly, we establish the precise large-time behavior of the solutions. The main idea here is to separate a physical quantity into its horizontal average and the corresponding oscillation. More precisely, for a function $f = f(x_1, x_2)$ integrable in $x_1$ on $\mathbb{T}$, we define the horizontal average

$$\bar{f} = \int_{\mathbb{T}} f(x_1, x_2)\,dx_1 \tag{1.5}$$

and write

$$f = \bar{f} + \widetilde{f}. \tag{1.6}$$

Clearly $\bar{f}$ also represents the zero-th horizontal Fourier mode of $f$. This decomposition is very useful due to some of the associated fine properties. For example, $\bar{f}$ and $\widetilde{f}$ are orthogonal in $L^2$, namely the inner product $(\bar{f}, \widetilde{f}) = 0$ and as a consequence, for any $f \in L^2(\Omega)$,

$$\|f\|_{L^2(\Omega)}^2 = \|\bar{f}\|_{L^2(\Omega)}^2 + \|\widetilde{f}\|_{L^2(\Omega)}^2.$$

In addition, a strong Poincaré type inequality holds,

$$\|\widetilde{f}\|_{L^2(\Omega)} \le C\,\|\partial_1 \widetilde{f}\|_{L^2(\Omega)}.$$

By applying this decomposition to the velocity field, namely writing $u = \bar{u} + \widetilde{u}$ and taking advantage of the special properties of $\widetilde{u}$ such as the Poincaré type inequality, we are able to establish suitable upper bounds for the nonlinear terms in (1.4), which in turn leads to a global and uniform upper bound for $\|u\|_{H^2}$. This explicit upper bound also implies the stability of perturbations near the trivial solution. In addition, by writing the evolution equations of the oscillation $\widetilde{u}$, we also able to prove that the $H^1$-norm of $\widetilde{u}$ decays to zero exponentially in time. More precisely, the following theorem holds.

**Theorem 1.1.** *Let $\mathbb{T} = [0, 1]$ be a 1D periodic box and let $\Omega = \mathbb{T} \times \mathbb{R}$. Let $\nu > 0$. Consider (1.1) in $\Omega$. Assume $u_0 \in H^2(\Omega)$ and $\nabla \cdot u_0 = 0$. Then (1.1) has a unique global solution $u$ that obeys the global and uniform $H^2$ bound,*

$$\|u(t)\|_{H^2}^2 + \nu \int_0^t \|\partial_1 u(\tau)\|_{H^2}^2\, d\tau \le \|u_0\|_{H^2}^2 e^{C(\|u_0\|_{H^1}^4 + \|u_0\|_{H^1}^2)} \tag{1.7}$$

*for some constant $C > 0$ and for all $t > 0$. In particular, (1.7) implies the stability of any perturbation near the trivial solution.*

*Assume the initial data $\|u_0\|_{H^2}$ is sufficiently small. Let $u$ be the corresponding solution. Let $\widetilde{u}$ denote the oscillation of $u$, defined as in (1.6). Then the $H^1$-norm of $\widetilde{u}$ decays to zero exponentially in time, namely*

$$\|\widetilde{u}(t)\|_{H^1} \le \|u_0\|_{H^1}\, e^{-C_0 t}$$

*for some $C_0 > 0$ and for all $t > 0$.*

The rest of this paper proves Theorem 1.1.

## 2. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. A crucial idea in the proof is to decompose the velocity $u$ into its horizontal average $\bar{u}$ and the corresponding oscillation part $\widetilde{u}$. We establish and take advantage of some special properties of this decomposition and those of $\widetilde{u}$. To facilitate the proof, we list several frequently used facts on the decomposition as lemmas.

**Lemma 2.1.** *Let $\bar{f}$ and $\widetilde{f}$ be defined as in (1.5) and (1.6). Then*

$$\overline{\widetilde{f}} = 0, \quad \overline{\partial_1 f} = \partial_1 \bar{f} = 0, \quad \overline{\partial_2 f} = \partial_2 \bar{f}, \quad \widetilde{\partial_2 f} = \partial_2 \widetilde{f}.$$

*If a vector field $F$ satisfies $\nabla \cdot F = 0$, then*

$$\nabla \cdot \bar{F} = 0 \quad \textit{and} \quad \nabla \cdot \widetilde{F} = 0.$$

*For any $f \in L^2(\Omega)$, we have*

$$(\bar{f}, \widetilde{f}) = 0, \quad \|f\|_{L^2(\Omega)}^2 = \|\bar{f}\|_{L^2(\Omega)}^2 + \|\widetilde{f}\|_{L^2(\Omega)}^2,$$

*where $(\bar{f}, \widetilde{f})$ denotes the $L^2$-inner product.*

All the items in Lemma 2.1 can be directly verified by the definition of $\bar{f}$ and $\widetilde{f}$. The next lemma assesses that the oscillation part $\widetilde{f}$ obeys a strong Poincaré type inequality with the upper bound in terms of $\partial_1 \widetilde{f}$ instead of $\nabla \widetilde{f}$.

**Lemma 2.2.** *Let $\bar{f}$ and $\widetilde{f}$ be defined as in (1.5) and (1.6). If $\|\partial_1 \widetilde{f}\|_{L^2(\Omega)} < \infty$, then*

$$\|\widetilde{f}\|_{L^2(\Omega)} \le C \|\partial_1 \widetilde{f}\|_{L^2(\Omega)},$$

*where $C$ is an absolute constant. In addition, if $\|\partial_1 \widetilde{f}\|_{H^1(\Omega)} < \infty$, then*

$$\|\widetilde{f}\|_{L^\infty(\Omega)} \le C \|\partial_1 \widetilde{f}\|_{H^1(\Omega)}.$$

The next two lemmas provide anisotropic upper bounds for the $L^\infty$-norm of a function on $\Omega$ and for the triple product integral defined on $\Omega$. They are simple but powerful tools for dealing with anisotropic models. Such anisotropic upper bounds on the whole space $\mathbb{R}^d$ with $d = 2, 3$ have been discovered and used by many authors (see, e.g., [1, 12]). For a 1D function $f = f(x_2)$ satisfying $f \in H^1(\mathbb{R})$,

$$\|f\|_{L^\infty_{x_2}(\mathbb{R})} \le \sqrt{2}\, \|f\|^{\frac{1}{2}}_{L^2_{x_2}(\mathbb{R})} \|\partial_2 f\|^{\frac{1}{2}}_{L^2_{x_2}(\mathbb{R})}. \tag{2.1}$$

However, when the domain is bounded such as $\mathbb{T}$, this type of inequalities would necessarily contain the $L^2$-part. More precisely, if $f = f(x_1)$ satisfying $f \in H^1(\mathbb{T})$,

$$\|f\|_{L^\infty_{x_1}(\mathbb{T})} \le \sqrt{2}\, \|f\|^{\frac{1}{2}}_{L^2_{x_1}(\mathbb{T})} \|\partial_1 f\|^{\frac{1}{2}}_{L^2_{x_1}(\mathbb{T})} + \|f\|_{L^2_{x_1}(\mathbb{T})}. \tag{2.2}$$

As a consequence of (2.1) and (2.2), we obtain the anisotropic upper bounds in the following lemma.

**Lemma 2.3.** *If a function $f = f(x_1, x_2)$ on $\Omega$ satisfies $f \in H^2(\Omega)$, then*

$$\begin{aligned} \|f\|_{L^\infty(\Omega)} \;&\le\; C\, \|f\|^{\frac{1}{4}}_{L^2(\Omega)} \left(\|f\|_{L^2(\Omega)} + \|\partial_1 f\|_{L^2(\Omega)}\right)^{\frac{1}{4}} \|\partial_2 f\|^{\frac{1}{4}}_{L^2(\Omega)} \\ &\qquad \times \left(\|\partial_2 f\|_{L^2(\Omega)} + \|\partial_1 \partial_2 f\|_{L^2(\Omega)}\right)^{\frac{1}{4}}. \end{aligned} \tag{2.3}$$

*In addition, the integral of the triple product over $\Omega$ is bounded by*

$$\left| \int_\Omega f\, g\, h\, dx \right| \le C\, \|f\|^{\frac{1}{2}}_{L^2} \left(\|f\|_{L^2} + \|\partial_1 f\|_{L^2}\right)^{\frac{1}{2}} \|g\|^{\frac{1}{2}}_{L^2} \|\partial_2 g\|^{\frac{1}{2}}_{L^2} \|h\|_{L^2}. \tag{2.4}$$

For the convenience of the readers, we provide a proof for this lemma.

*Proof.* Applying Hölder's inequality in each direction, Minkowski's inequality, and (2.1) and (2.2), we have

$$\begin{aligned} \left| \int_\Omega f\, g\, h\, dx \right| &\le \|f\|_{L^2_{x_2} L^\infty_{x_1}} \|g\|_{L^\infty_{x_2} L^2_{x_1}} \|h\|_{L^2} \\ &\le \|f\|_{L^2_{x_2} L^\infty_{x_1}} \|g\|_{L^2_{x_1} L^\infty_{x_2}} \|h\|_{L^2} \\ &\le C \left\| \|f\|^{\frac{1}{2}}_{L^2_{x_1}} \|\partial_1 f\|^{\frac{1}{2}}_{L^2_{x_1}} + \|f\|_{L^2_{x_1}} \right\|_{L^2_{x_2}} \\ &\quad \times \left\| \|g\|^{\frac{1}{2}}_{L^2_{x_2}} \|\partial_2 g\|^{\frac{1}{2}}_{L^2_{x_2}} \right\|_{L^2_{x_1}} \|h\|_{L^2} \end{aligned}$$

$$\le C \, \|f\|_{L^2}^{\frac12} \, (\|f\|_{L^2} + \|\partial_1 f\|_{L^2})^{\frac12} \, \|g\|_{L^2}^{\frac12} \, \|\partial_2 g\|_{L^2}^{\frac12} \, \|h\|_{L^2}.$$

Here $\|f\|_{L^2_{x_2} L^\infty_{x_1}}$ represents the $L^\infty$-norm in the $x_1$-variable, followed by the $L^2$-norm in the $x_2$-variable. To prove (2.3), we again use Hölder's inequality, Minkowski's inequality, and (2.1) and (2.2),

$$\begin{aligned}
\|f\|_{L^\infty_{x_1} L^\infty_{x_2}} &\le C \left\| \|f\|_{L^2_{x_2}}^{\frac12} \, \|\partial_2 f\|_{L^2_{x_2}}^{\frac12} \right\|_{L^\infty_{x_1}} \\
&\le C \left\| \|f\|_{L^\infty_{x_1}} \right\|_{L^2_{x_2}}^{\frac12} \, \left\| \|\partial_2 f\|_{L^\infty_{x_1}} \right\|_{L^2_{x_2}}^{\frac12} \\
&\le C \left\| \|f\|_{L^2_{x_1}}^{\frac12} \, \|\partial_1 f\|_{L^2_{x_1}}^{\frac12} + \|f\|_{L^2_{x_1}} \right\|_{L^2_{x_2}}^{\frac12} \\
&\quad \times \left\| \|\partial_2 f\|_{L^2_{x_1}}^{\frac12} \, \|\partial_1 \partial_2 f\|_{L^2_{x_1}}^{\frac12} + \|\partial_2 f\|_{L^2_{x_1}} \right\|_{L^2_{x_2}}^{\frac12} \\
&\le C \, \|f\|_{L^2}^{\frac14} \, (\|f\|_{L^2} + \|\partial_1 f\|_{L^2})^{\frac14} \, \|\partial_2 f\|_{L^2}^{\frac14} \\
&\quad \times (\|\partial_2 f\|_{L^2} + \|\partial_1 \partial_2 f\|_{L^2})^{\frac14} \, .
\end{aligned}$$

This completes the proof of Lemma 2.3. □

If we replace $f$ by the oscillation part $\widetilde{f}$, some of the lower-order parts in (2.2), (2.3) and (2.4) can be dropped, as the following lemma states.

**Lemma 2.4.** *Let $\bar f$ and $\widetilde{f}$ be defined as in (1.5) and (1.6). Then*

$$\|\widetilde{f}\|_{L^\infty_{x_1}(\mathbb{T})} \le C \, \|\widetilde{f}\|_{L^2_{x_1}(\mathbb{T})}^{\frac12} \, \|\partial_1 \widetilde{f}\|_{L^2_{x_1}(\mathbb{T})}^{\frac12}. \tag{2.5}$$

*As a special consequence,*

$$\|\widetilde{f}\|_{L^\infty(\Omega)} \le C \, \|\widetilde{f}\|_{L^2(\Omega)}^{\frac14} \|\partial_1 \widetilde{f}\|_{L^2(\Omega)}^{\frac14} \, \|\partial_2 \widetilde{f}\|_{L^2(\Omega)}^{\frac14} \, \|\partial_1 \partial_2 \widetilde{f}\|_{L^2(\Omega)}^{\frac14} \tag{2.6}$$

*and*

$$\begin{aligned}
\left| \int_\Omega \widetilde{f} \, g \, h \, dx \right| \;&\le\; C \, \|\widetilde{f}\|_{L^2}^{\frac12} \, \|\partial_1 \widetilde{f}\|_{L^2}^{\frac12} \, \|g\|_{L^2}^{\frac12} \, \|\partial_2 g\|_{L^2}^{\frac12} \, \|h\|_{L^2} \\
&\le\; C \, \|\partial_1 \widetilde{f}\|_{L^2} \, \|g\|_{L^2}^{\frac12} \, \|\partial_2 g\|_{L^2}^{\frac12} \, \|h\|_{L^2}.
\end{aligned} \tag{2.7}$$

*Proof.* (2.6) and (2.7) in this lemma can be shown similarly as those in Lemma 2.3. The only modification here is to use (2.5) instead of (2.2). Since (2.5) does not contain the lower-order part, the inequalities in this lemma do not have the lower-order terms. □

We are ready to prove Theorem 1.1.

*Proof.* To establish the global existence and stability of the solutions to (1.1), the first step is the local existence result, which can be proven by the standard contraction mapping argument together with a local-in-time *a priori* bound. The portion with the contraction mapping argument is standard and can be found in the book [6].

We shall just provide the local *a priori* bound. Taking the inner product in $H^2$ of $u$ with the first equation in (1.1), we find

$$\frac{d}{dt}\|u(t)\|_{H^2}^2 + 2\nu\|\partial_1 u\|_{H^2}^2 = -2\int_\Omega \nabla u\cdot\nabla\omega\cdot\nabla\omega\,dx. \tag{2.8}$$

where $\|u\|_{H^2}^2 = \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2$. By Hölder's inequality and (2.3) in Lemma 2.3,

$$\begin{aligned}
-2\int_\Omega \nabla u\cdot\nabla\omega\cdot\nabla\omega\,dx &\le 2\|\nabla u\|_{L^\infty}\|\nabla\omega\|_{L^2}^2\\
&\le C\,\|\nabla u\|_{L^2}^{\frac14}\left(\|\nabla u\|_{L^2}+\|\partial_1\nabla u\|_{L^2}\right)^{\frac14}\|\partial_2\nabla u\|_{L^2}^{\frac14}\\
&\quad\times\left(\|\partial_2\nabla u\|_{L^2}+\|\partial_1\partial_2\nabla u\|_{L^2}\right)^{\frac14}\|\nabla\omega\|_{L^2}^2\\
&\le C\left(\|\nabla u\|_{H^1}+\|\partial_1\nabla u\|_{H^1}\right)\|\nabla\omega\|_{L^2}^2\\
&\le \nu\|\partial_1 u\|_{H^2}^2 + C\left(\|u\|_{H^2}^3+\|u\|_{H^2}^4\right).
\end{aligned}$$

Inserting this upper bound in (2.8) leads to a differential inequality that assesses the local upper bound for $\|u\|_{H^2}$. The local well-posedness follows as a consequence.

To prove the global existence and stability result, we need to obtain the uniform in time $H^2$ estimate. It is easy to see that, due to $\nabla\cdot u = 0$,

$$\|u(t)\|_{L^2}^2 + 2\nu\int_0^t\|\partial_1 u(\tau)\|_{L^2}^2\,d\tau = \|u_0\|_{L^2}^2, \tag{2.9}$$

$$\|\nabla u(t)\|_{L^2}^2 + 2\nu\int_0^t\|\partial_1\nabla u(\tau)\|_{L^2}^2\,d\tau = \|\nabla u_0\|_{L^2}^2, \tag{2.10}$$

where we have used

$$\int_\Omega (u\cdot\nabla u)\cdot\Delta u\,dx = 0.$$

To bound the $H^2$-norm, we resort to the vorticity equation,

$$\partial_t\omega + u\cdot\nabla\omega = \nu\partial_{11}\omega. \tag{2.11}$$

Taking the inner product of $\nabla\omega$ with the gradient of (2.11), we have

$$\frac12\frac{d}{dt}\|\nabla\omega(t)\|_{L^2}^2 + \nu\|\partial_1\nabla\omega\|_{L^2}^2 = -\int \nabla u\cdot\nabla\omega\cdot\nabla\omega\,dx := N. \tag{2.12}$$

We further write $N$ into four terms,

$$\begin{aligned}
N &= -\int \partial_1 u_1\,(\partial_1\omega)^2\,dx - \int \partial_1 u_2\,\partial_1\omega\,\partial_2\omega\,dx\\
&\quad - \int \partial_2 u_1\,\partial_1\omega\,\partial_2\omega\,dx - \int \partial_2 u_2\,(\partial_2\omega)^2\,dx\\
&:= N_1 + N_2 + N_3 + N_4.
\end{aligned}$$

$N_1$ and $N_2$ can be bounded directly. By Lemma 2.1, $\partial_1\bar u = 0$ and $\partial_1 u = \partial_1\widetilde u$. By Lemma 2.2, Lemma 2.4 and Young's inequality,

$$|N_1| = \left|-\int \partial_1\widetilde u_1\,\partial_1\omega\,\partial_1\widetilde\omega\,dx\right|$$

$$
\begin{aligned}
&\le C\,\|\partial_1\widetilde{u}_1\|_{L^2}^{\frac12}\,\|\partial_1\partial_1\widetilde{u}_1\|_{L^2}^{\frac12}\,\|\partial_1\omega\|_{L^2}^{\frac12}\,\|\partial_2\partial_1\omega\|_{L^2}^{\frac12}\,\|\partial_1\widetilde{\omega}\|_{L^2}\\
&\le C\,\|\partial_1 u_1\|_{L^2}^{\frac12}\,\|\partial_1\nabla u_1\|_{L^2}^{\frac12}\,\|\partial_1\omega\|_{L^2}\,\|\partial_1\nabla\omega\|_{L^2}\\
&\le C\,\|\partial_1 u\|_{L^2}\|\partial_1\nabla u\|_{L^2}\,\|\nabla\omega\|_{L^2}^2+\frac{\nu}{12}\|\partial_1\nabla\omega\|_{L^2}^2
\end{aligned}
$$

and

$$
\begin{aligned}
|N_2| &= \left|-\int \partial_1\widetilde{u}_2\,\partial_1\widetilde{\omega}\,\partial_2\omega\,dx\right|\\
&\le C\,\|\partial_1\widetilde{u}_2\|_{L^2}^{\frac12}\,\|\partial_1\partial_1\widetilde{u}_2\|_{L^2}^{\frac12}\,\|\partial_1\widetilde{\omega}\|_{L^2}^{\frac12}\,\|\partial_2\partial_1\widetilde{\omega}\|_{L^2}^{\frac12}\,\|\partial_2\omega\|_{L^2}\\
&\le C\,\|\partial_1 u_2\|_{L^2}^{\frac12}\,\|\partial_1\nabla u_2\|_{L^2}^{\frac12}\,\|\partial_1\nabla\widetilde{\omega}\|_{L^2}\,\|\partial_2\omega\|_{L^2}\\
&\le C\,\|\partial_1 u\|_{L^2}\|\partial_1\nabla u\|_{L^2}\,\|\nabla\omega\|_{L^2}^2+\frac{\nu}{12}\|\partial_1\nabla\omega\|_{L^2}^2.
\end{aligned}
$$

The estimate of $N_3$ is slightly more delicate.

$$
\begin{aligned}
N_3 &= -\int \partial_2 u_1\,\partial_1\omega\,\partial_2\omega\,dx = -\int \partial_2(\bar{u}_1+\widetilde{u}_1)\,\partial_1\widetilde{\omega}\,\partial_2(\bar{\omega}+\widetilde{\omega})\,dx\\
&= -\int \partial_2\bar{u}_1\,\partial_1\widetilde{\omega}\,\partial_2\bar{\omega}\,dx-\int \partial_2\bar{u}_1\,\partial_1\widetilde{\omega}\,\partial_2\widetilde{\omega}\,dx\\
&\quad -\int \partial_2\widetilde{u}_1\,\partial_1\widetilde{\omega}\,\partial_2\bar{\omega}\,dx-\int \partial_2\widetilde{u}_1\,\partial_1\widetilde{\omega}\,\partial_2\widetilde{\omega}\,dx\\
&:= N_{31}+N_{32}+N_{33}+N_{34}.
\end{aligned}
$$

The first term $N_{31}$ is clearly zero,

$$
N_{31} = -\int_{\mathbb{R}} \partial_2\bar{u}_1\,\partial_2\bar{\omega}\,\int_{\mathbb{T}} \partial_1\widetilde{\omega}_1\,dx_1\,dx_2 = 0.
$$

To bound $N_{32}$, we first use (2.7) of Lemma 2.4 and then Lemma 2.2 to obtain

$$
\begin{aligned}
|N_{32}| &\le C\,\|\partial_2\bar{u}_1\|_{L^2}\,\|\partial_1\widetilde{\omega}\|_{L^2}^{\frac12}\,\|\partial_2\partial_1\widetilde{\omega}\|_{L^2}^{\frac12}\,\|\partial_2\widetilde{\omega}\|_{L^2}^{\frac12}\|\partial_1\partial_2\widetilde{\omega}\|_{L^2}^{\frac12}\\
&\le C\,\|\partial_2\bar{u}_1\|_{L^2}\,\|\partial_1\widetilde{\omega}\|_{L^2}^{\frac12}\,\|\partial_1\partial_2\widetilde{\omega}\|_{L^2}^{\frac32}\\
&\le C\,\|\partial_2 u\|_{L^2}^4\,\|\partial_1\omega\|_{L^2}^2+\,\|\partial_1\nabla\omega\|_{L^2}^2\\
&\le C\,\|u_0\|_{H^1}^2\|\nabla u\|_{L^2}^2\,\|\partial_1\nabla u\|_{L^2}^2+\frac{\nu}{12}\,\|\partial_1\nabla\omega\|_{L^2}^2
\end{aligned}
$$

and

$$
\begin{aligned}
|N_{33}| &\le C\,\|\partial_2\bar{\omega}\|_{L^2}\,\|\partial_2\widetilde{u}_1\|_{L^2}^{\frac12}\,\|\partial_1\partial_2\widetilde{u}_1\|_{L^2}^{\frac12}\,\|\partial_1\widetilde{\omega}\|_{L^2}^{\frac12}\,\|\partial_2\partial_1\widetilde{\omega}\|_{L^2}^{\frac12}\\
&\le C\,\|\partial_2\omega\|_{L^2}\,\|\partial_1\partial_1\partial_2\widetilde{u}_1\|_{L^2}^{\frac12}\,\|\partial_1\nabla\widetilde{u}\|_{L^2}\,\|\partial_2\partial_1\widetilde{\omega}\|_{L^2}^{\frac12}\\
&\le C\,\|\partial_2\omega\|_{L^2}\,\|\partial_1\nabla\widetilde{\omega}\|_{L^2}^{\frac12}\,\|\partial_1\nabla\widetilde{u}\|_{L^2}\,\|\partial_1\nabla\widetilde{\omega}\|_{L^2}^{\frac12}\\
&\le C\,\|\partial_2\omega\|_{L^2}\,\|\partial_1\nabla\widetilde{u}\|_{L^2}\|\partial_1\nabla\widetilde{\omega}\|_{L^2}\\
&\le C\,\|\partial_1\nabla u\|_{L^2}^2\,\|\nabla\omega\|_{L^2}^2+\frac{\nu}{12}\|\partial_1\nabla\omega\|_{L^2}^2.
\end{aligned}
$$

$N_{34}$ can be similarly bounded as $N_{32}$. $N_4$ can also be bounded similarly.

$$\begin{aligned}
|N_4| &= \left|\int \partial_1\widetilde{u}_1\,(\partial_2\bar{\omega}+\partial_2\widetilde{\omega})^2\,dx\right| \\
&= \left|2\int \partial_1\widetilde{u}_1\,\partial_2\bar{\omega}\,\partial_2\widetilde{\omega}\,dx + \int \partial_1\widetilde{u}_1\,(\partial_2\widetilde{\omega})^2\,dx\right| \\
&\le C\,(\|\partial_2\bar{\omega}\|_{L^2}+\|\partial_2\widetilde{\omega}\|_{L^2})\|\partial_1\widetilde{u}_1\|_{L^2}^{\frac12}\,\|\partial_2\partial_1\widetilde{u}_1\|_{L^2}^{\frac12}\,\|\partial_2\widetilde{\omega}\|_{L^2}^{\frac12}\,\|\partial_1\partial_2\widetilde{\omega}\|_{L^2}^{\frac12} \\
&\le C\,\|\partial_1 u\|_{L^2}\|\partial_1\nabla u\|_{L^2}\,\|\nabla\omega\|_{L^2}^2 + \frac{\nu}{12}\|\partial_1\nabla\omega\|_{L^2}^2.
\end{aligned}$$

Thus,

$$\begin{aligned}
|N| \le\, & C\,(\|\partial_1 u\|_{L^2}^2+\|\partial_1\nabla u\|_{L^2}^2)\,\|\nabla\omega\|_{L^2}^2 \\
&+ C\|u_0\|_{H^1}^2\,\|\partial_1\nabla u\|_{L^2}^2\,\|\nabla u\|_{L^2}^2 + \frac{\nu}{2}\|\partial_1\nabla\omega\|_{L^2}^2. \qquad (2.13)
\end{aligned}$$

Inserting (2.13) in (2.12), combining with (2.9) and (2.10) and integrating in time yields the desired inequality in (1.7),

$$\sup_{0\le\tau\le t}\|u(\tau)\|_{H^2}^2 + \nu\int_0^t \|\partial_1 u(\tau)\|_{H^2}^2\,d\tau \le\, \|u_0\|_{H^2}^2 e^{C(\|u_0\|_{H^1}^4+\|u_0\|_{H^1}^2)}.$$

Next we show the desired exponential decay. We first write the equations of $\bar{u}=(\bar{u}_1,\bar{u}_2)$. Taking the average of (1.1)

$$\begin{cases}\partial_t\bar{u}_1+\partial_2(\overline{u_1u_2})=0,\\ \partial_t\bar{u}_2+\partial_2(\overline{u_2^2})=\partial_2\bar{p},\\ \partial_2\bar{u}_2=0.\end{cases} \qquad (2.14)$$

Taking the difference of (1.1) and (2.14), we find

$$\begin{cases}\partial_t\widetilde{u}_1+\partial_1(u_1^2)+\partial_2(u_1u_2-\overline{u_1u_2})=-\partial_1\widetilde{p}+\nu\partial_{11}\widetilde{u}_1,\\ \partial_t\widetilde{u}_2+\partial_1(u_1u_2)+\partial_2(u_2^2-\overline{u_2^2})=-\partial_2\widetilde{p}+\nu\partial_{11}\widetilde{u}_2,\\ \partial_1\widetilde{u}_1+\partial_2\widetilde{u}_2=0.\end{cases} \qquad (2.15)$$

Taking the inner product of $(\widetilde{u}_1,\widetilde{u}_2)$ with (2.15) yields

$$\frac{d}{dt}\|\widetilde{u}(t)\|_{L^2}^2+2\nu\|\partial_1\widetilde{u}\|_{L^2}^2 = K_1+K_2+K_3+K_4, \qquad (2.16)$$

where

$$K_1=-\int \widetilde{u}_1\,\partial_1(u_1^2)\,dx,\quad K_2=-\int \widetilde{u}_1\,\partial_2(u_1u_2-\overline{u_1u_2})\,dx,$$
$$K_3=-\int \widetilde{u}_2\,\partial_1(u_1u_2)\,dx,\quad K_4=-\int \widetilde{u}_2\,\partial_2(u_2^2-\overline{u_2^2})\,dx.$$

By $\partial_1 u_1=\partial_1\widetilde{u}_1$ and Lemma 2.2,

$$\begin{aligned}
|K_1| &= \left|-2\int \widetilde{u}_1\,u_1\,\partial_1\widetilde{u}_1\,dx\right| \le \|u_1\|_{L^\infty}\,\|\widetilde{u}_1\|_{L^2}\,\|\partial_1\widetilde{u}_1\|_{L^2} \\
&\le C\,\|u_1\|_{H^2}\|\partial_1\widetilde{u}_1\|_{L^2}^2.
\end{aligned}$$

$K_3$ can be bounded similarly,

$$|K_3| \le 2\|\widetilde{u}_2\|_{L^2}\, \|u\|_{L^\infty}\, \|\partial_1 \widetilde{u}\|_{L^2} \le C\, \|u\|_{H^2} \|\partial_1 \widetilde{u}\|_{L^2}^2.$$

To bound $K_2$, we write $u = \bar{u} + \widetilde{u}$ and

$$\begin{aligned} u_1 u_2 - \overline{u_1 u_2} &= \bar{u}_1\, \widetilde{u}_2 + \bar{u}_2\, \widetilde{u}_1 + \widetilde{u}_1\, \widetilde{u}_2 - \overline{\widetilde{u}_1\, \widetilde{u}_2} \\ &= \bar{u}_1\, \widetilde{u}_2 + \bar{u}_2\, \widetilde{u}_1 + \widetilde{\widetilde{u}_1\, \widetilde{u}_2} \end{aligned} \tag{2.17}$$

and thus

$$\begin{aligned} &\partial_2(u_1 u_2 - \overline{u_1 u_2}) = \partial_2(\bar{u}_1\, \widetilde{u}_2) + \partial_2(\bar{u}_2\, \widetilde{u}_1) + \partial_2\widetilde{(\widetilde{u}_1\, \widetilde{u}_2)} \\ &= \widetilde{u}_2\, \partial_2 \bar{u}_1 - \bar{u}_1 \partial_1 \widetilde{u}_1 + \bar{u}_2 \partial_2 \widetilde{u}_1 + \partial_2\widetilde{(\widetilde{u}_1\, \widetilde{u}_2)}. \end{aligned}$$

Therefore, by Lemma 2.4,

$$\begin{aligned} |K_2| &= \left| -\int \widetilde{u}_1 (\widetilde{u}_2\, \partial_2 \bar{u}_1 - \bar{u}_1 \partial_1 \widetilde{u}_1 + \bar{u}_2 \partial_2 \widetilde{u}_1 + \partial_2\widetilde{(\widetilde{u}_1\, \widetilde{u}_2)})\, dx \right| \\ &\le C\, \|\partial_2 \bar{u}_1\|_{L^2}\, \|\widetilde{u}_1\|_{L^2}^{\frac12}\, \|\partial_1 \widetilde{u}_1\|_{L^2}^{\frac12}\, \|\widetilde{u}_2\|_{L^2}^{\frac12}\, \|\partial_2 \widetilde{u}_2\|_{L^2}^{\frac12} \\ &\quad + \|\widetilde{u}_1\|_{L^2}\, \|\partial_2\widetilde{(\widetilde{u}_1\, \widetilde{u}_2)}\|_{L^2}, \end{aligned}$$

where we have used

$$\int \widetilde{u}_1\, \bar{u}_1\, \partial_1 \widetilde{u}_1\, dx = 0 \quad \text{and} \quad \int \widetilde{u}_1\, \bar{u}_2\, \partial_2 \widetilde{u}_1\, dx = 0.$$

By Lemma 2.2 and $\nabla \cdot \widetilde{u} = 0$, we have

$$\begin{aligned} &\|\widetilde{u}_1\|_{L^2} \le C \|\partial_1 \widetilde{u}_1\|_{L^2} \le C\, \|\partial_1 \widetilde{u}\|_{L^2}, \\ &\|\widetilde{u}_2\|_{L^2} \le C \|\partial_1 \widetilde{u}_2\|_{L^2} \le C\, \|\partial_1 \widetilde{u}\|_{L^2}, \\ &\|\partial_2 \widetilde{u}_2\|_{L^2} = \|\partial_1 \widetilde{u}_1\|_{L^2} \le \|\partial_1 \widetilde{u}\|_{L^2}. \end{aligned}$$

Using these inequalities and $\|\widetilde{f}\|_{L^2} \le \|f\|_{L^2}$, $K_2$ is bounded by

$$\begin{aligned} |K_2| &\le C\, \|\partial_2 \bar{u}_1\|_{L^2}\, \|\partial_1 \widetilde{u}\|_{L^2}^2 + \|\widetilde{u}_1\|_{L^2}\, \|\partial_2 (\widetilde{u}_1\, \widetilde{u}_2)\|_{L^2} \\ &\le C\, \|\partial_2 \bar{u}_1\|_{L^2}\, \|\partial_1 \widetilde{u}\|_{L^2}^2 + \|\widetilde{u}_1\|_{L^2}\, \|\widetilde{u}_1\|_{L^\infty}\, \|\partial_1 \widetilde{u}_1\|_{L^2} \\ &\qquad + \|\widetilde{u}_1\|_{L^2}\, \|\partial_2 \widetilde{u}_1\|_{L^\infty_{x_1} L^2_{x_2}}\, \|\widetilde{u}_2\|_{L^2_{x_1} L^\infty_{x_2}} \\ &\le C\, (\|\partial_2 \bar{u}_1\|_{L^2} + \|\widetilde{u}_1\|_{H^2})\, \|\partial_1 \widetilde{u}\|_{L^2}^2. \end{aligned}$$

$K_4$ can be similarly estimated and

$$|K_4| \le C\, \|u\|_{H^2} \|\partial_1 \widetilde{u}\|_{L^2}^2.$$

Inserting these upper bounds in (2.16) yields

$$\frac{d}{dt} \|\widetilde{u}\|_{L^2}^2 + (2\nu - C\, \|u\|_{H^2})\, \|\partial_1 \widetilde{u}\|_{L^2}^2 \le 0. \tag{2.18}$$

According to the stability result established above, if $\varepsilon > 0$ is sufficiently small and $\|u_0\|_{H^2} \le \varepsilon$, then $\|u(t)\|_{H^2} \le C\, \varepsilon$ and

$$2\nu - C\, \|u\|_{H^2} \ge \nu.$$

(2.18) and Lemma 2.2 then yields the desired exponential decay for $\|\widetilde{u}\|_{L^2}$.

Next we show the exponential decay of $\|\nabla\widetilde{u}(t)\|_{L^2}$. We start by taking the gradient of the velocity equation in (2.15) and then dotting with $\nabla\widetilde{u}$ to obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla\widetilde{u}(t)\|_{L^2}^2+\nu\|\partial_1\nabla\widetilde{u}\|_{L^2}^2=Q_1+Q_2+Q_3+Q_4,$$

where

$$Q_1=-\int\nabla\widetilde{u}_1\cdot\nabla\partial_1(u_1^2)\,dx,\quad Q_2=-\int\nabla\widetilde{u}_1\cdot\nabla\partial_2(u_1u_2-\overline{u_1u_2})\,dx,$$
$$Q_3=-\int\nabla\widetilde{u}_2\cdot\nabla\partial_1(u_1u_2)\,dx,\quad Q_4=-\int\nabla\widetilde{u}_2\cdot\nabla\partial_2(u_2^2-\overline{u_2^2})\,dx.$$

All terms can be bounded suitably. In fact, by Lemma 2.2,

$$\begin{aligned}|Q_1| &= \left|-2\int\nabla\widetilde{u}_1\cdot(\nabla u_1\,\partial_1\widetilde{u}_1+u_1\partial_1\nabla\widetilde{u}_1)\,dx\right|\\ &\le C\,\|\nabla u_1\|_{L^2}\,\|\nabla\widetilde{u}_1\|_{L^2}^{\frac12}\|\partial_1\nabla\widetilde{u}_1\|_{L^2}^{\frac12}\,\|\partial_1\widetilde{u}_1\|_{L^2}^{\frac12}\,\|\partial_2\partial_1\widetilde{u}_1\|_{L^2}^{\frac12}\\ &\quad+\|u_1\|_{L^\infty}\,\|\nabla\widetilde{u}_1\|_{L^2}\,\|\partial_1\nabla\widetilde{u}_1\|_{L^2}\\ &\le C\,\|u_1\|_{H^2}\|\partial_1\nabla\widetilde{u}_1\|_{L^2}^2.\end{aligned}$$

$Q_3$ can be bounded similarly and

$$|Q_3|\le C\,\|u\|_{H^2}\|\partial_1\nabla\widetilde{u}\|_{L^2}^2.$$

By (2.17),

$$\begin{aligned}Q_2 &= -\int\nabla\widetilde{u}_1\cdot\nabla\partial_2(\bar{u}_1\widetilde{u}_2+\widetilde{u}_1\bar{u}_2+\widetilde{\widetilde{u}_1\widetilde{u}_2})\,dx\\ &= Q_{21}+Q_{22}+Q_{23}.\end{aligned}$$

Writing $Q_{21}$ more explicitly,

$$Q_{21}=-\int\nabla\widetilde{u}_1\cdot(\nabla\partial_2\bar{u}_1\widetilde{u}_2+\bar{u}_1\nabla\partial_2\widetilde{u}_2+\nabla\bar{u}_1\partial_2\widetilde{u}_2+\partial_2\bar{u}_1\nabla\widetilde{u}_2)\,dx$$

and applying Lemma 2.4 and then Lemma 2.2, we obtain

$$\begin{aligned}|Q_{21}| &\le \|\nabla\partial_2\bar{u}_1\|_{L^2}\|\nabla\widetilde{u}_1\|_{L^2}^{\frac12}\,\|\partial_1\nabla\widetilde{u}_1\|_{L^2}^{\frac12}\|\widetilde{u}_2\|_{L^2}^{\frac12}\|\partial_2\widetilde{u}_2\|_{L^2}^{\frac12}\\ &\quad+\|\bar{u}_1\|_{L^\infty}\|\nabla\widetilde{u}_1\|_{L^2}\,\|\nabla\partial_2\widetilde{u}_2\|_{L^2}\\ &\quad+\|\nabla\bar{u}_1\|_{L^2}\,\|\nabla\widetilde{u}_1\|_{L^2}^{\frac12}\,\|\partial_1\nabla\widetilde{u}_1\|_{L^2}^{\frac12}\|\partial_2\widetilde{u}_2\|_{L^2}^{\frac12}\|\partial_2\partial_2\widetilde{u}_2\|_{L^2}^{\frac12}\\ &\quad+\|\partial_2\bar{u}_1\|_{L^2}\,\|\nabla\widetilde{u}_1\|_{L^2}^{\frac12}\,\|\partial_1\nabla\widetilde{u}_1\|_{L^2}^{\frac12}\|\nabla\widetilde{u}_2\|_{L^2}^{\frac12}\|\partial_2\nabla\widetilde{u}_2\|_{L^2}^{\frac12}\\ &\le C\,\|u\|_{H^2}\,\|\partial_1\widetilde{u}\|_{H^1}^2.\end{aligned}$$

The estimate for $Q_{22}$ is similar and the upper bound is the same.

$$\begin{aligned}|Q_{23}| &= \left|-\int\nabla\widetilde{u}_1\cdot\nabla\partial_2(\widetilde{\widetilde{u}_1\widetilde{u}_2})\,dx\right|\\ &= \left|-\int\partial_1\widetilde{u}_1\,\partial_1\partial_2(\widetilde{\widetilde{u}_1\widetilde{u}_2})\,dx-\int\partial_2\widetilde{u}_1\,\partial_2\partial_2(\widetilde{\widetilde{u}_1\widetilde{u}_2})\,dx\right|\\ &\le \|\partial_{11}\widetilde{u}_1\|_{L^2}\,\|\partial_2(\widetilde{\widetilde{u}_1\widetilde{u}_2})\|_{L^2}+\|\partial_2\partial_2\widetilde{u}_1\|_{L^2}\|\partial_2(\widetilde{\widetilde{u}_1\widetilde{u}_2})\|_{L^2}\end{aligned}$$

$$\begin{aligned}
&\le \|\partial_{11}\widetilde{u}_1\|_{L^2}\,\|\partial_2(\widetilde{u}_1\widetilde{u}_2)\|_{L^2} + \|\partial_2\partial_2\widetilde{u}_1\|_{L^2}\|\partial_2(\widetilde{u}_1\widetilde{u}_2)\|_{L^2}\\
&\le C\,\|\partial_1\partial_1\widetilde{u}_1\|_{L^2}\,\|\widetilde{u}\|_{L^\infty}\|\partial_2\widetilde{u}\|_{L^2}\\
&\quad +\|\partial_2\partial_2\widetilde{u}_1\|_{L^2}(\|\widetilde{u}_2\|_{L^2_{x_1}L^\infty_{x_2}}\|\partial_2\widetilde{u}_1\|_{L^2_{x_2}L^\infty_{x_1}} + \|\widetilde{u}_1\|_{L^2_{x_2}L^\infty_{x_1}}\|\partial_2\widetilde{u}_2\|_{L^2_{x_1}L^\infty_{x_2}})\\
&\le C\,\|u\|_{H^2}\,\|\partial_1\widetilde{u}\|^2_{H^1}.
\end{aligned}$$

The upper bound for $Q_4$ is the same. Combining the estimates for $\|\widetilde{u}\|_{L^2}$ and $\|\nabla\widetilde{u}\|_{L^2}$, we find that

$$\frac{d}{dt}\|\widetilde{u}\|^2_{H^1} + (2\nu - C\,\|u\|_{H^2})\|\partial_1\widetilde{u}\|^2_{H^1} \le 0,$$

which leads to the desired exponential decay in $H^1$. This completes the proof of Theorem 1.1. □

**Acknowledgments**. Dong was partially supported by the National Natural Science Foundation of China (NNSFC) (No.11871346), the Natural Science Foundation (NSF) of Guangdong Province (No.2018A030313024), NSF of Shenzhen City (No. JCYJ20180305125554234) and Research Fund of Shenzhen University (No. 2017056). Wu was partially supported by the National Science Foundation under grant DMS 2104682 and the AT&T Foundation at Oklahoma State University. X. Xu was partially supported by NNSFC (Grant No. 11771045 and No.11871087). N. Zhu was partially supported by NNSFC (Grant No. 11771043 and No. 11771045).

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