STABILITY AND OPTIMAL DECAY FOR A SYSTEM OF 3D ANISOTROPIC BOUSSINESQ EQUATIONS

JIAHONG WU[1] AND QIAN ZHANG[2]

Abstract. This paper focuses on a system of three-dimensional (3D) Boussinesq equations modeling anisotropic buoyancy-driven fluids. The goal here is to solve the stability and large-time behavior problem on perturbations near the hydrostatic balance, a prominent equilibrium in fluid dynamics, atmospherics and astrophysics. Due to the lack of the vertical kinematic dissipation and the horizontal thermal diffusion, this stability problem is difficult. When the spatial domain is $\Omega = \mathbb{R}^2 \times \mathbb{T}$ with $\mathbb{T} = [-1/2, 1/2]$ being a 1D periodic box, this paper establishes the desired stability for fluids with certain symmetries. The approach here is to distinguish the vertical averages of the velocity and temperature from their corresponding oscillation parts. In addition, the oscillation parts are shown to decay exponentially to zero in time.

1. Introduction

The hydrostatic balance or hydrostatic equilibrium refers to the equilibrium when the fluid is static with all external forces balanced out. Our atmosphere is mainly in hydrostatic equilibrium, between the upward-directed pressure gradient force and the downward-directed force of gravity. Understanding the stability of perturbations near the hydrostatic equilibrium may help gain insight into some weather phenomena. This paper intends to rigorously establish the stability and large-time behavior of perturbations near the hydrostatic equilibrium for a special system of the 3D Boussinesq equations. The Boussinesq systems are the most frequently used models for atmospheric and oceanographic flows (see [6, 8, 12, 15, 24]).

More precisely, the 3D Boussinesq system considered here is given by

$$\begin{cases} \partial_t U + U \cdot \nabla U = -\nabla P + \nu(\partial_{11} + \partial_{22})U + \Theta\, e_3, \quad x \in \Omega,\, t > 0, \\ \nabla \cdot U = 0, \\ \partial_t \Theta + U \cdot \nabla \Theta = \eta \partial_{33} \Theta, \end{cases} \tag{1.1}$$

where U denotes the fluid velocity, P the pressure, Θ the temperature and $e_3 = (0, 0, 1)$. Here $\nu > 0$ and $\eta > 0$ are parameters representing the kinematic viscosity and the thermal diffusivity, respectively. For notational convenience, we have written ∂_{ii} for $\partial_{x_i x_i}$ with $i = 1, 2, 3$, and shall use $\Delta_h = \partial_{11} + \partial_{22}$ and $\nabla_h = (\partial_1, \partial_2)$. Here the spatial domain Ω is taken to be

$$\Omega = \mathbb{R}^2 \times \mathbb{T} \tag{1.2}$$

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with $\mathbb{T} = [-\frac{1}{2}, \frac{1}{2}]$ being a 1D periodic box. Mathematically the hydrostatic equilibrium $(U_{he}, P_{he}, \Theta_{he})$ is given by

$$U_{he} = 0, \quad P_{he} = \frac{1}{2}x_3^2, \quad \Theta_{he} = x_3.$$

$(U_{he}, P_{he}, \Theta_{he})$ is clearly a steady-state solution of (1.1). Any perturbation (u, p, θ) near the hydrostatic equilibrium with

$$u = U - U_{he}, \quad p = P - P_{he}, \quad \theta = \Theta - \Theta_{he}$$

obeys

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p + \nu \Delta_h u + \theta\, e_3, \quad x \in \Omega,\, t > 0, \\ \nabla \cdot u = 0, \\ \partial_t \theta + u \cdot \nabla \theta + u_3 = \eta\, \partial_{33}\theta, \\ u(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x). \end{cases} \tag{1.3}$$

The aim of this paper is to rigorously establish the stability of solutions to (1.3) in a suitable functional setting and give a precise account of their large-time behavior.

We explain the physical relevance of the spatial domain and the associated periodic boundary condition, and provide physical circumstances that the Boussinesq system considered here may model. The Boussinesq systems have been studied in various spatial domains with different type of boundary conditions. The whole space and bounded domains with either the Dirichlet or the Navier-type boundary condition are the most popular setups in the study of the Boussinesq systems. The periodic boundary condition and various combinations of the periodic boundary condition with other types of boundary conditions are also relevant in the stability analysis of perturbations near the hydrostatic equilibrium.

The hydrostatic equilibrium solves the Boussinesq system (1.1), although it does not satisfy the periodic boundary condition imposed on the perturbation. There appears to be an inconsistency in the non-periodic temperature profile (linear in x_3) with periodic perturbations, but this setup actually connects with the real atmosphere [14]. Over a finite range of latitudes from, say, 30 degrees north to 60 degrees north, the temperature is approximately linear, and the perturbations may look approximately sinusoidal. So it is a local approximation in a certain region, away from the boundary (or north/south pole and equator) and a full/true solution could possibly be built by matching with other solutions near the equator and poles. There are many other examples of this type of setting, the equilibrium state being linear while perturbations are sinusoidal. One significant example is the passive tracer in a mean gradient (see [3]).

The relevance of the periodic boundary condition on perturbations near the hydrostatic equilibrium is also reflected in several research projects on stratified flows. Embid and Majda [10] used the periodic boundary condition when they studied the low Froude number limiting dynamics for stably stratified flow. In [19] Simon and Nadiga of Los Alamos Laboratory investigated the instability of a periodic flow in geostrophic and hydrostatic balance.

We also briefly comment on the relation between the model considered here and the model in the whole space. In the whole space case, the background temperature profile mathematically extends from minus infinity to plus infinity. Of course, in a real atmosphere, the range of values is actually finite. But, in order to allow analytic solutions, it is convenient to assume the range is infinite. The Boussinesq system governing the perturbations is equipped with zero boundary conditions at infinity.

The stability problem considered here is difficult. Due to the lack of thermal diffusion in two directions, the temperature gradient can potentially grow in time if the Lipschitz norm of the velocity field is not uniformly integrable in time. The velocity equation doesn't have vertical dissipation and the buoyancy forcing can propel the velocity gradients to grow in time. In fact, when the spatial domain is the whole space $\mathbb{R}^3$, the stability problem (1.3) remains an important open problem.

This paper focuses on the domain Ω in (1.2). The vertical periodic boundary condition imposed here has some advantages over the zero Dirichlet boundary condition or the no-penetration boundary condition. There are two main difficulties associated with the latter two boundary conditions. The first difficulty is that boundary terms would emerge when we estimate vertical derivatives of the solution. The second is that the pressure term on the boundary relies on the vertical derivatives of the velocity field on the boundary, which are unknown. These two difficulties prevent us from establishing necessary upper bounds on the derivatives of the solution.

Another significant advanatge of the domain Ω is that it allows us to separate a physical quantity into its vertical average and the corresponding oscillation part. More precisely, for a sufficient smooth function $f = f(x_1, x_2, x_3)$ on Ω, we define the vertical average by

$$\bar{f}(x_1, x_2) = \int_{\mathbb{T}} f(x_1, x_2, x_3)\, dx_3,$$

and set the oscillation part as

$$\widetilde{f} = f - \bar{f}.$$

It is clear that the horizontal average $\bar{f}$ represents the zeroth vertical Fourier mode while $\widetilde{f}$ consists of all non-zero vertical frequencies.

The decomposition $f = \bar{f} + \widetilde{f}$ is very special. First of all, this decomposition is orthogonal in the Sobolev space $H^k(\Omega)$ for any $k \geq 0$. As a special consequence, the H^k-norms of $\bar{f}$ and $\widetilde{f}$ are bounded by the H^k-norm of f. Furthermore, this decomposition commutes with derivatives, and $\bar{f}$ and $\widetilde{f}$ of a divergence-free vector field f are also divergence-free. A crucial property to be frequently used in our estimates is that $\widetilde{f}$ satisfies a strong Poincare type inequality

$$\|\widetilde{f}\|_{L^2(\Omega)} \leq C\, \|\partial_3 \widetilde{f}\|_{L^2(\Omega)}.$$

Besides these special properties, this decomposition also allows us to distinguish the different behaviors of the different parts of the solutions to (1.3). For example, the decomposition $\theta = \bar{\theta} + \widetilde{\theta}$ helps distinguish the behavior of $\bar{\theta}$ and $\widetilde{\theta}$. It is not difficult to see from (1.3) that the vertical dissipation actually vanishes for $\bar{\theta}$ due to the zeroth Fourier mode. In contrast, the vertical dissipation damps $\widetilde{\theta}$ and may cause $\widetilde{\theta}$ to decay, even exponentially in time. This decomposition is employed in the estimates of the nonlinear terms.

We assume the initial velocity $u_0 = (u_{01}, u_{02}, u_{03})$ and the initial temperature θ_0 in (1.3) have the following symmetries:

$$u_{01}, u_{02} \text{ are even in } x_3, \text{ and } u_{03} \text{ and } \theta_0 \text{ are odd in } x_3.$$

As demonstrated in Corollary 3.2, these symmetries are preserved in time and the corresponding solution (u, θ) with $u = (u_1, u_2, u_3)$ obeys the same symmetries

$$u_1, u_2, p \text{ are even in } x_3, \text{ and } u_3 \text{ and } \theta \text{ are odd in } x_3.$$

As a special consequence of these symmetries,

$$\bar{u}_3 = \int_{\mathbb{T}} u_3(x_1, x_2, x_3, t)\, dx_3 = 0, \quad \bar{\theta} = \int_{\mathbb{T}} \theta(x_1, x_2, x_3, t)\, dx_3 = 0$$

Therefore,

$$u_3 = \widetilde{u}_3, \quad \theta = \widetilde{\theta}. \tag{1.4}$$

The equations in (1.4) facilitate the estimates of several terms when we bound the derivatives of θ.

With the basic ingredients at our disposal, we now state our main results.

Theorem 1.1. *Consider (1.3) with $\nu > 0$ and $\eta > 0$. Assume that $(u_0, \theta_0) \in H^2(\Omega)$ satisfies $\nabla \cdot u_0 = 0$, and*

$$u_{01}, u_{02} \text{ are even in } x_3, \text{ and } u_{03} \text{ and } \theta_0 \text{ are odd in } x_3, \tag{1.5}$$

where u_{01}, u_{02} and u_{03} are the three components of u_0. Then there exists $\varepsilon = \varepsilon(\nu, \eta) > 0$ such that, if

$$\|u_0\|_{H^2} + \|\theta_0\|_{H^2} \le \varepsilon(\nu, \eta),$$

then (1.3) has a unique global solution $(u, \theta) \in L^\infty(0, \infty; H^2)$ satisfying

$$\|u(t)\|_{H^2}^2 + \|\theta(t)\|_{H^2}^2 + \nu \int_0^t \|\nabla_h u\|_{H^2}^2\, d\tau + \eta \int_0^t \|\partial_3 \theta\|_{H^2}^2 d\tau \le C\, \varepsilon^2, \tag{1.6}$$

$$u_1, u_2, p \text{ are even in } x_3, \text{ and } u_3 \text{ and } \theta \text{ are odd in } x_3. \tag{1.7}$$

Furthermore, if the initial datum is in a more regular Sobolev space, then the corresponding solution is also more regular. More precisely, if $(u_0, \theta_0) \in H^3$ is sufficiently small and has the symmetries in (1.5), then the solution (u, θ) remains small in H^3, and satisfies (1.6) with H^2 replaced by H^3 and (1.7).

We remark that, as explained in the proof of Theorem 1.1, $\varepsilon(\nu, \eta)$ is of the form $\varepsilon(\nu, \eta) = c \min\{\nu, \eta\}$ for some pure small constant c independent of ν and η. Theorem 1.1 rigorously assesses that any small initial perturbation satisfying the symmetries specified in (1.5) leads to a unique global solution of (1.3) that preserves the symmetries and remains small in H^2 for all time. This result appears to be the very first stability result for a three-dimensional Boussinesq equations with anisotropic velocity dissipation and with only one directional thermal diffusion. The stability and large-time behavior problems on perturbations near several physically important steady states have recently attracted considerable interest due to their practical applications and mathematical significance. Progress has been made on the stability of two special steady states, the hydrostatic equilibrium and shear flows (see, e.g., ([5, 7, 9, 21, 25, 27, 28]). The work of Doering, Wu, Zhao and Zheng [9] investigated the stability of the hydrostatic equilibrium to the 2D Boussinesq system with only kinematic dissipation (without thermal diffusion) and rigorously proved the global asymptotic stability of any perturbation near the hydrostatic equilibrium [9]. In addition, extensive numerical simulations are performed in [9] to corroborate the analytical results and predict some phenomena that are not proven. The work of Tao, Wu, Zhao and Zheng [21] resolves several important issues left open in [9]. In particular, [21] provides a precise description of the final buoyancy distribution in the case of general initial conditions and the explicit decay rate of the velocity field or the total mechanical energy. The paper of Castro, Córdoba and Lear successfully established the stability and large time behavior on the 2D Boussinesq equations with velocity damping instead of dissipation [5]. More recent work on the hydrostatic equilibrium can be found in [1, 11, 23, 25]. There are very significant recent developments on the stability of shear flow to the Boussinesq equations with various partial dissipation [2, 7, 20, 27–29].

Efforts are also made here to understand the large-time behavior of the perturbations. Mathematically this is a challenging problem when the velocity equation in (1.3) lacks the vertical dissipation and the temperature equation lacks the dissipation in two horizontal directions. Powerful classical tools such as the Fourier splitting methods designed for the systems with full dissipation no longer apply here [16–18]. Our approach here is to treat the vertical average of the solution $(\bar{u}, \bar{\theta})$ differently from the oscillation part $(\widetilde{u}, \widetilde{\theta})$. Unfortunately this process would break down if the velocity equation does not involve the vertical dissipation. To successfully implement our strategy, we consider the following Boussinesq system with full velocity dissipation,

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p + \nu \Delta u + \theta\, e_3, \quad x \in \Omega,\, t > 0, \\ \nabla \cdot u = 0, \\ \partial_t \theta + u \cdot \nabla \theta + u_3 = \eta\, \partial_{33} \theta, \\ u(x, 0) = u_0(x), \quad \theta(x, 0) = \theta_0(x). \end{cases} \tag{1.8}$$

The only difference between (1.3) and (1.8) is that (1.8) also involves $\partial_{33} u$, which plays a crucial role in the decay rates. Clearly, Theorem 1.1 carries over to the system in (1.8). We are now ready to state our result on the large-time behavior of solutions to (1.8).

Theorem 1.2. *Assume that the initial datum* $(u_0, \theta_0) \in H^3(\Omega)$ *satisfies the smallness and the symmetry conditions stated in Theorem 1.1, namely*

$$\|(u_0, \theta_0)\|_{H^3} \le \varepsilon(\nu, \eta) \quad \text{for sufficiently small } \varepsilon(\nu, \eta) > 0$$

and

$$u_{01}, u_{02} \text{ are even in } x_3, \text{ and } u_{03} \text{ and } \theta_0 \text{ are odd in } x_3.$$

Let (u,b) be the corresponding solution of (1.8). Let $(\widetilde{u},\widetilde{\theta})$ be the oscillation part of (u,θ). Then $(\widetilde{u},\widetilde{\theta})$ decays exponentially in time in H^1, namely

$$\|(\widetilde{u},\widetilde{\theta})(t)\|_{H^1} \le \|(u_0,\theta_0)\|_{H^1}\, e^{-ct}, \tag{1.9}$$

where $c=\min\{\nu,\eta\}$. As a consequence, the limiting system of (1.8) is the following system of $(\bar u_1,\bar u_2)$,

$$\begin{cases}\partial_t \bar u_1 + \partial_1 \overline{u_1^2} + \partial_2 \overline{u_1 u_2} = -\partial_1 \bar p + \nu \Delta_h \bar u_1,\\ \partial_t \bar u_2 + \partial_1 \overline{u_1 u_2} + \partial_2 \overline{u_2^2} = -\partial_2 \bar p + \nu \Delta_h \bar u_2,\\ \partial_1 \bar u_1 + \partial_2 \bar u_2 = 0.\end{cases}$$

Theorem 1.2 states that the oscillation part of any perturbation governed by the Boussinesq system in (1.8) decays exponentially in time to zero and the eventual system is a 2D flow obeying the 2D Navier–Stokes equation. This is consistent with the mathematics and physics of the system in (1.8) governing the buoyancy-driven fluids. Mathematically, according to the governing equations on perturbations in (1.8), the dissipation associated with the vertical average or the zeroth vertical frequency vanishes while the dissipation for the non-zero vertical frequencies plays the role of damping. The vertical dissipation plays a crucial role in damping those non-zero vertical frequencies. We also remark that, as shown in the proof of Theorem 1.2, $\varepsilon(\nu,\eta)$ is of the form $\varepsilon(\nu,\eta) = c\,\min\{\nu,\eta\}$ for some pure small constant c independent of ν and η.

We briefly outline the proofs of Theorems 1.1 and 1.2. Since the local (in time) well-posedness on (1.3) in the Sobolev setting $H^2(\Omega)$ or $H^3(\Omega)$ can be shown via standard approaches (see, e.g., [13]), the proof of Theorem 1.1 is reduced to establishing the global (in time) bounds for the solutions. The tool is the bootstrapping argument. An abstract bootstrapping argument can be found in T. Tao's book [22, p.20]. To set it up, we define the following energy functional for the H^2-solution,

$$E(t) = \sup_{0\le\tau\le t} \|(u(\tau),\theta(\tau))\|_{H^2}^2 + \nu\int_0^t \|\nabla_h u(\tau)\|_{H^2}^2\, d\tau + \eta\int_0^t \|\partial_3\theta(\tau)\|_{H^2}^2 d\tau.$$

Our main efforts are devoted to proving the inequality

$$E(t) \le E(0) + C\, E(t)^{\frac32}, \tag{1.10}$$

where $C=C(\nu,\eta)>0$ is a constant depending on ν and η. More explicit dependence will be provided in the proof of Theorem 1.1. The bootstrapping argument then implies that if

$$E(0) = \|(u_0,\theta_0)\|_{H^2}^2 \le \varepsilon^2$$

for suitable $\varepsilon=\varepsilon(\nu,\eta)>0$, then $E(t)$ remains uniformly bounded for all time, for $0<t<\infty$,

$$E(t) \le C\varepsilon^2 \tag{1.11}$$

for some pure constant $C>0$. In particular, (1.11) yields the desired global H^2-bound on the solution (u,θ). The proof of (1.10) makes use of the decomposition

$$u = \bar u + \widetilde{u}, \qquad \theta = \bar\theta + \widetilde{\theta}$$

in order to distinguish different behaviors of $\bar{u}$ and $\widetilde{u}$, and of $\bar{\theta}$ and $\widetilde{\theta}$. We develop various anisotropic inequalities to deal with the triple products resulting from the nonlinear terms. In particular, the strong Poincaré inequality

$$\|\widetilde{f}\|_{L^2(\Omega)} \le C\,\|\partial_3 \widetilde{f}\|_{L^2(\Omega)}$$

and the anisotropic upper bound on the triple product

$$\int_\Omega f\, g\, \widetilde{h}\, dx \le C\,\|f\|_{L^2}^{\frac12} \|\partial_1 f\|_{L^2}^{\frac12}\, \|g\|_{L^2}^{\frac12}\, \|\widetilde{h}\|_{L^2}^{\frac12}\, \|\partial_3 \widetilde{h}\|_{L^2}^{\frac12}$$

are frequently used. More anisotropic inequalities and their proofs can be found in Section 2. In addition, we also use special properties on the averages of functions with symmetries such as $\bar{f} = 0$ if f is odd in x_3. Details of the proof of Theorem 1.1 are provided in Section 2. The global upper bound on the solution in H^3 is obtained similarly.

To prove Theorem 1.2, we take the difference of (1.3) and its average to obtain the system governing the oscillation $(\widetilde{u}, \widetilde{\theta})$,

$$\begin{cases} \partial_t \widetilde{u}_1 + \partial_1(u_1^2 - \overline{u_1^2}) + \partial_2(u_1 u_2 - \overline{u_1 u_2}) + \partial_3(\widetilde{u}_3 u_1) = -\partial_1 \widetilde{p} + \nu \Delta \widetilde{u}_1, \\ \partial_t \widetilde{u}_2 + \partial_1(u_1 u_2 - \overline{u_1 u_2}) + \partial_2(u_2^2 - \overline{u_2^2}) + \partial_3(\widetilde{u}_3 u_2) = -\partial_2 \widetilde{p} + \nu \Delta \widetilde{u}_2, \\ \partial_t \widetilde{u}_3 + u \cdot \nabla \widetilde{u}_3 = -\partial_3 \widetilde{p} + \nu \Delta \widetilde{u}_3 + \widetilde{\theta}, \\ \partial_t \widetilde{\theta} + u \cdot \nabla \widetilde{\theta} = \eta \partial_{33} \widetilde{\theta} - \widetilde{u}_3, \\ \nabla \cdot \widetilde{u} = 0. \end{cases}$$

The estimate of the H^1-norm of $(\widetilde{u}, \widetilde{\theta})$ is separated into controlling the L^2-norm of $(\widetilde{u}, \widetilde{\theta})$ and that of $(\nabla \widetilde{u}, \nabla \widetilde{\theta})$. By invoking various anisotropic inequalities stated in Section 2, we are able to show that

$$\frac{d}{dt} \|(\widetilde{u}, \widetilde{\theta})\|_{H^1}^2 + (2 \min\{\nu, \eta\} - C\|(u,\theta)\|_{H^3})\, \|(\widetilde{u}, \widetilde{\theta})\|_{H^1}^2 \le 0,$$

which leads to the desired exponential decay in Theorem 1.2.

The rest of this paper is divided into three sections. The second section develops several properties associated with the decomposition $f = \bar{f} + \widetilde{f}$, the Poincaré and various anisotropic inequalities. This section serves as preparation. Section 3 proves Theorem 1.1, while Section 4 proves Theorem 1.2.

2. Decomposition and anisotropic inequalities

This section serves as preparation for the proofs of Theorems 1.1 and 1.2. First, we provide several properties associated with the aforementioned decomposition. In particular, a strong version of the Poincaré inequality is supplied. Second, anisotropic inequalities for the whole space $\mathbb{R}^3$ and for the domain $\Omega = \mathbb{R}^2 \times \mathbb{T}$ are presented and compared.

We start by recalling $\bar{f}$ and $\widetilde{f}$. Let $\Omega = \mathbb{R}^2 \times \mathbb{T}$. Assume that, for every $(x_1, x_2) \in \mathbb{R}^2$, $f(x_1, x_2, x_3)$ is integrable in x_3 on $\mathbb{T}$. Then, $\bar{f} = \bar{f}(x_1, x_2)$ is defined by

$$\bar{f}(x_1, x_2) = \int_{\mathbb{T}} f(x_1, x_2, x_3)\, dx_3 \tag{2.1}$$

and we decompose f as

$$f = \bar{f} + \widetilde{f}. \tag{2.2}$$

$\widetilde{f}$ will be called the oscillation part of f. $\bar{f}$ also represents the zeroth vertical Fourier mode while $\widetilde{f}$ contains all other vertical frequencies. The decomposition in (2.2) possesses many fine properties. First of all, (2.2) is an orthogonal decomposition in $H^k(\Omega)$ for any integer $k \geq 0$. Clearly, the L^2-inner product $(\bar{f}, \widetilde{f})$ satisfies

$$\begin{aligned}
(\bar{f}, \widetilde{f}) &= \int_{\Omega} \bar{f}(x_1, x_2)\, \widetilde{f}(x_1, x_2, x_3)\, dx \\
&= \int_{\mathbb{R}^2} \bar{f}(x_1, x_2) \int_{\mathbb{T}} \widetilde{f}(x_1, x_2, x_3)\, dx_3\, dx_1 dx_2 \\
&= 0
\end{aligned}$$

due to the fact that the average of $\widetilde{f}$ is zero. Similarly, for any differential operator $D^\alpha := \partial_1^{\alpha_1} \partial_2^{\alpha_2} \partial_3^{\alpha_3}$, the L^2-inner product

$$(D^\alpha \bar{f}, D^\alpha \widetilde{f}) = 0.$$

That is, $\bar{f}$ and $\widetilde{f}$ are orthogonal in any Sobolev space $H^k(\Omega)$ with $k \geq 0$ being an integer. In summary, the following lemma holds.

Lemma 2.1. *Let $k \geq 0$ be an integer. The decomposition $f = \bar{f} + \widetilde{f}$ is orthogonal in $H^k(\Omega)$,*

$$(\bar{f}, \widetilde{f})_{H^k} = 0, \qquad \|f\|_{H^k}^2 = \|\bar{f}\|_{H^k}^2 + \|\widetilde{f}\|_{H^k}^2.$$

As a special consequence,

$$\|\bar{f}\|_{H^k} \leq \|f\|_{H^k}, \quad \|\widetilde{f}\|_{H^k} \leq \|f\|_{H^k}.$$

It is a direct consequence of the definition in (2.2) that the average operator and the oscillation operator commute with the derivatives.

Lemma 2.2. *The average operator and the oscillation operator commute with the derivatives, namely, for $k = 1, 2, 3$,*

$$\overline{\partial_k f} = \partial_k \bar{f}, \quad \widetilde{\partial_k f} = \partial_k \widetilde{f}.$$

As a special consequence, if $\nabla \cdot u = 0$, then

$$\nabla \cdot \bar{u} = 0, \quad \nabla \cdot \widetilde{u} = 0.$$

One very important property about the oscillation part is that $\widetilde{f}$ obeys a strong version of the Poincaré type inequality.

Lemma 2.3. *Let $\bar{f}$ and $\widetilde{f}$ be defined as in (2.1) and (2.2). Let $k \geq 0$ be an integer. If $\partial_3 \widetilde{f} \in H^k(\Omega)$. Then $\widetilde{f} \in H^k(\Omega)$ and*

$$\|\widetilde{f}\|_{H^k(\Omega)} \leq C \, \|\partial_3 \widetilde{f}\|_{H^k(\Omega)},$$

where $C > 0$ is a constant depending on Ω and k only.

Proof of Lemma 2.3. By Lemma 2.2, it suffices to consider the case when $k = 0$. Since any function in $L^2(\Omega)$ can be approximated by smooth functions, we can assume f is smooth without loss of generality. Since, for each $(x_1, x_2) \in \mathbb{R}^2$, the average of $\widetilde{f}$ is zero, there exists $\rho = \rho(x_1, x_2)$ such that

$$\widetilde{f}(x_1, x_2, \rho) = 0.$$

Then

$$\begin{aligned} \widetilde{f}(x_1, x_2, x_3) &= \widetilde{f}(x_1, x_2, \rho) + \int_\rho^{x_3} \partial_z \widetilde{f}(x_1, x_2, z) \, dz \\ &= \int_\rho^{x_3} \partial_z \widetilde{f}(x_1, x_2, z) \, dz. \end{aligned}$$

By Hölder's inequality,

$$\left| \widetilde{f}(x_1, x_2, x_3) \right| \leq \left[\int_{\mathbb{T}} (\partial_z \widetilde{f}(x_1, x_2, z))^2 \, dz \right]^{\frac{1}{2}}.$$

Squaring each side and integrating over Ω yields

$$\|\widetilde{f}\|_{L^2(\Omega)} \leq C \, \|\partial_3 \widetilde{f}\|_{L^2(\Omega)}.$$

This completes the proof of Lemma 2.3. □

For a one-dimensional function $f \in H^1(\mathbb{R})$, we have the elementary inequality

$$\|f\|_{L^\infty(\mathbb{R})} \leq \sqrt{2} \, \|f\|_{L^2(\mathbb{R})}^{\frac{1}{2}} \, \|Df\|_{L^2(\mathbb{R})}^{\frac{1}{2}}, \tag{2.3}$$

where Df denotes the derivative of f. When the spatial domain is $\mathbb{T}$ instead of $\mathbb{R}$, (2.3) needs to be modified. More precisely, for any $f \in H^1(\mathbb{T})$, we have the following lemma.

Lemma 2.4. *Let $f \in H^1(\mathbb{T})$ and let $\widetilde{f}$ be its oscillation part. Then*

$$\|f\|_{L^\infty(\mathbb{T})} \leq \sqrt{2} \, \|f\|_{L^2(\mathbb{T})}^{\frac{1}{2}} \left(\|f\|_{L^2(\mathbb{T})} + \|Df\|_{L^2(\mathbb{T})} \right)^{\frac{1}{2}}, \tag{2.4}$$

$$\|\widetilde{f}\|_{L^\infty(\mathbb{T})} \leq \sqrt{2} \, \|\widetilde{f}\|_{L^2(\mathbb{T})}^{\frac{1}{2}} \, \|D\widetilde{f}\|_{L^2(\mathbb{T})}^{\frac{1}{2}}. \tag{2.5}$$

Proof of Lemma 2.4. For any $x_3 \in \mathbb{T}$,

$$f^2(x_3) = f^2(\rho) + \int_\rho^{x_3} D(f^2(z)) \, dz. \tag{2.6}$$

Integrating in ρ over $\mathbb{T}$ yields

$$f^2(x_3) \;\; \leq \;\; \int_{\mathbb{T}} f^2(\rho) \, d\rho + 2 \left[\int_{\mathbb{T}} |f(z)|^2 \, dz \right]^{\frac{1}{2}} \left[\int_{\mathbb{T}} |Df(z)|^2 \, dz \right]^{\frac{1}{2}}.$$

Integrating over $\mathbb{T}$ and then applying Hölder's inequality for the last term on the right yields (2.4). To prove (2.5), we replace f by $\widetilde{f}$ in (2.6) and choose ρ such that $f(\rho) = 0$. Then

$$(\widetilde{f})^2(x_3) = \int_\rho^{x_3} D((\widetilde{f})^2(z))\,dz.$$

Integrating over $\mathbb{T}$ and applying Hölder's inequality yields (2.5). □

Several anisotropic upper bounds on the integrals of triple products have been extremely useful in dealing with partial differential equations with anisotropic dissipation. The following two inequalities for the spatial domains $\mathbb{R}^2$ and $\mathbb{R}^3$ are two outstanding examples of such upper bounds,

$$\left|\int_{\mathbb{R}^2} f\,g\,h\,dx\right| \le C\,\|f\|_{L^2(\mathbb{R}^2)}\,\|g\|^{\frac12}_{L^2(\mathbb{R}^2)}\,\|\partial_1 g\|^{\frac12}_{L^2(\mathbb{R}^2)}\,\|h\|^{\frac12}_{L^2(\mathbb{R}^2)}\,\|\partial_2 h\|^{\frac12}_{L^2(\mathbb{R}^2)},$$

$$\begin{aligned}\left|\int_{\mathbb{R}^3} f\,g\,h\,dx\right| \le\ & C\,\|f\|^{\frac12}_{L^2(\mathbb{R}^3)}\,\|\partial_1 f\|^{\frac12}_{L^2(\mathbb{R}^3)}\|g\|^{\frac12}_{L^2(\mathbb{R}^3)}\,\|\partial_2 g\|^{\frac12}_{L^2(\mathbb{R}^3)}\\ & \times\|h\|^{\frac12}_{L^2(\mathbb{R}^3)}\,\|\partial_3 h\|^{\frac12}_{L^2(\mathbb{R}^3)}.\end{aligned}$$

These inequalities can be found in [4] and [26]. When the spatial domain is $\Omega = \mathbb{R}^2 \times \mathbb{T}$, these inequalities need to be modified suitably.

Lemma 2.5. *Let $\Omega = \mathbb{R}^2 \times \mathbb{T}$. Assume that $f, \partial_1 f, g\,\partial_2 g, h, \partial_3 h \in L^2(\Omega)$. Then,*

$$\left|\int_\Omega f\,g\,h\,dx\right| \le C\,\|f\|^{\frac12}_{L^2}\,\|\partial_1 f\|^{\frac12}_{L^2}\,\|g\|^{\frac12}_{L^2}\,\|\partial_2 g\|^{\frac12}_{L^2}\,\|h\|^{\frac12}_{L^2}\,\left(\|h\|_{L^2} + \|\partial_3 h\|_{L^2}\right)^{\frac12}.$$

When h is replaced by its oscillation part $\widetilde{h}$, then the lower-order term $\|h\|_{L^2}$ in the last part of the inequality above can be dropped, namely

$$\left|\int_\Omega f\,g\,\widetilde{h}\,dx\right| \le C\,\|f\|^{\frac12}_{L^2}\,\|\partial_1 f\|^{\frac12}_{L^2}\,\|g\|^{\frac12}_{L^2}\,\|\partial_2 g\|^{\frac12}_{L^2}\,\|\widetilde{h}\|^{\frac12}_{L^2}\,\|\partial_3 \widetilde{h}\|^{\frac12}_{L^2}.$$

This lemma is a direct consequence of Lemma 2.4 and Minkowski's inequality.

3. Proof of Theorem 1.1

This section proves Theorem 1.1. Since the local (in time) well-posedness of (1.3) can be established via a standard approach (see [13]), our attention is focused on the global bound of (u, θ). We need to prepare two key ingredients. The first is the uniqueness of two H^2-solutions to (1.3). As a special consequence, the symmetries of the initial data in (1.5) are preserved for all time, and the corresponding solution possesses the same symmetries. The second main ingredient is the global *a priori* estimates stated in Propositions 3.3 and 3.4 below. Once these two ingredients are at our disposal, the proof of Theorem 1.1 is then completed via a bootstrapping argument.

We first establish the uniqueness of H^2-solutions to (1.3).

Proposition 3.1. *Assume that* $(u^{(1)}, \theta^{(1)})$ *and* $(u^{(2)}, \theta^{(2)})$ *are two solutions of (1.3) in the regularity class*

$$(u^{(1)}, \theta^{(1)}), (u^{(2)}, \theta^{(2)}) \in L^\infty(0,T; H^2).$$

Then, for any $0 < t \le T$,

$$(u^{(1)}, \theta^{(1)}) = (u^{(2)}, \theta^{(2)}).$$

The proof of Proposition 3.1 is not difficult, but the uniqueness is important and would guarantee the preservation of the symmetries of the initial data.

Corollary 3.2. *Assume* $(u_0, \theta_0) \in H^2(\Omega)$ *satisfies* $\nabla \cdot u_0 = 0$ *and the symmetries in (1.5). Let* $T > 0$. *Let* $(u, \theta) \in L^\infty(0,T; H^2)$ *be the corresponding solution of (1.3). Then, for any* $t \le T$, $(u(t), \theta(t))$ *obeys the same symmetries as in (1.5).*

It is easy to check that Corollary 3.2 follows from Proposition 3.1. In fact, if $(u, p, \theta) = (u_1, u_2, u_3, p, \theta)$ is a solution (1.3), then (U, P, Θ) with

$$\begin{aligned}
&U_1 = u_1(x_1, x_2, -x_3, t), \quad U_2 = u_2(x_1, x_2, -x_3, t), \quad U_3 = -u_3(x_1, x_2, -x_3, t),\\
&P = p(x_1, x_2, -x_3, t), \quad \Theta = -\theta(x_1, x_2, -x_3, t)
\end{aligned}$$

also satisfies the same Boussinesq equations with the initial datum (U_0, Θ_0) given by

$$\begin{aligned}
&U_{01} = u_{01}(x_1, x_2, -x_3), \quad U_{02} = u_{02}(x_1, x_2, -x_3), \quad U_{03} = -u_{03}(x_1, x_2, -x_3),\\
&\Theta_0 = -\theta_0(x_1, x_2, -x_3).
\end{aligned}$$

Due to the symmetries of the initial datum, we have

$$(U_0, \Theta_0) = (u_0, \theta_0).$$

By the uniqueness stated in Proposition 3.1,

$$(U, P, \Theta) = (u, p, \theta)$$

or

$$\begin{aligned}
&u_1(x_1, x_2, x_3, t) = u_1(x_1, x_2, -x_3, t),\\
&u_2(x_1, x_2, x_3, t) = u_2(x_1, x_2, -x_3, t),\\
&u_3(x_1, x_2, x_3, t) = -u_3(x_1, x_2, -x_3, t),\\
&p(x_1, x_2, x_3, t) = p(x_1, x_2, -x_3, t),\\
&\theta(x_1, x_2, x_3, t) = -\theta(x_1, x_2, -x_3, t).
\end{aligned}$$

Therefore, (u, p, θ) has the desired symmetries.

We now turn to the proof of Proposition 3.1.

Proof of Proposition 3.1. The difference $(\delta u, \delta\theta)$ with

$$\delta u := u^{(1)} - u^{(2)} \quad \text{and} \quad \delta\theta = \theta^{(1)} - \theta^{(2)}$$

satisfies

$$\begin{cases} \partial_t \delta u + u^{(1)} \cdot \nabla \delta u + \delta u \cdot \nabla u^{(2)} = \nu \Delta_h \delta u - \nabla \delta p + \delta\theta\, e_3, \\ \nabla \cdot \delta u = 0, \\ \partial_t \delta\theta + u^{(1)} \cdot \nabla \delta\theta + \delta u \cdot \nabla \theta^{(2)} + (\delta u)_3 = \eta \partial_{33} \delta\theta, \\ \delta u(x,0) = 0, \quad \delta\theta(x,0) = 0, \end{cases} \tag{3.1}$$

where $\delta p = p^{(1)} - p^{(2)}$ represents the pressure difference. Testing (3.1) with $(\delta u, \delta\theta)$ yields

$$\frac{1}{2}\frac{d}{dt}\left(\|\delta u\|_{L^2}^2 + \|\delta\theta\|_{L^2}^2\right) + \nu \|\nabla_h \delta u\|_{L^2}^2 + \eta \|\partial_3 \delta\theta\|_{L^2}^2 = I_1 + I_2, \tag{3.2}$$

where we have used $\int (\delta\theta\, e_3) \cdot \delta u \, + (\delta u)_3 \delta\theta \, dx = 0$, and

$$I_1 = -\int \delta u \cdot \nabla u^{(2)} \cdot \delta u \, dx, \quad I_2 = -\int \delta u \cdot \nabla \theta^{(2)}\, \delta\theta \, dx.$$

By Lemma 2.5,

$$\begin{aligned} |I_1| &\le C \,\|\nabla u^{(2)}\|_{L^2}^{\frac12} \left(\|\nabla u^{(2)}\|_{L^2} + \|\partial_3 \nabla u^{(2)}\|_{L^2}\right)^{\frac12} \\ &\quad \times \|\delta u\|_{L^2}^{\frac12} \,\|\partial_1 \delta u\|_{L^2}^{\frac12} \,\|\delta u\|_{L^2}^{\frac12} \,\|\partial_2 \delta u\|_{L^2}^{\frac12} \\ &\le C \,\|u^{(2)}\|_{H^2} \,\|\delta u\|_{L^2} \,\|\nabla_h \delta u\|_{L^2} \\ &\le \frac14 \nu \|\nabla_h \delta u\|_{L^2}^2 + C \,\|u^{(2)}\|_{H^2}^2 \,\|\delta u\|_{L^2}^2. \end{aligned}$$

Similarly,

$$\begin{aligned} |I_2| &\le \|\nabla \theta^{(2)}\|_{L^2}^{\frac12} \,\|\partial_1 \nabla \theta^{(2)}\|_{L^2}^{\frac12} \,\|\delta u\|_{L^2}^{\frac12} \,\|\partial_2 \delta u\|_{L^2}^{\frac12} \\ &\quad \times \|\delta\theta\|_{L^2}^{\frac12} \left(\|\delta\theta\|_{L^2} + \|\partial_3 \delta\theta\|_{L^2}\right)^{\frac12} \\ &\le C \,\|\nabla \theta^{(2)}\|_{L^2}^{\frac12} \,\|\partial_1 \nabla \theta^{(2)}\|_{L^2}^{\frac12} \Big(\|\delta u\|_{L^2}^{\frac12} \,\|\delta\theta\|_{L^2} \,\|\partial_2 \delta u\|_{L^2}^{\frac12} \\ &\qquad\qquad + \|\delta u\|_{L^2}^{\frac12} \,\|\delta\theta\|_{L^2}^{\frac12} \,\|\partial_2 \delta u\|_{L^2}^{\frac12} \|\partial_3 \delta\theta\|_{L^2}^{\frac12}\Big) \\ &\le \frac14 \nu \|\partial_2 \delta u\|_{L^2}^2 + \frac12 \eta \|\partial_3 \delta u\|_{L^2}^2 \\ &\quad + C\,(1 + \|\theta^{(2)}\|_{H^2}^2)\,(\|\delta u\|_{L^2}^2 + \|\delta\theta\|_{L^2}^2) \end{aligned}$$

Inserting the upper bounds for I_1 and I_2 in (3.2) yields

$$\begin{aligned} &\frac{d}{dt}\left(\|\delta u\|_{L^2}^2 + \|\delta\theta\|_{L^2}^2\right) + \nu \|\nabla_h \delta u\|_{L^2}^2 + \eta \|\partial_3 \delta\theta\|_{L^2}^2 \\ &\le C\,(1 + \|u^{(2)}\|_{H^2}^2 + \|\theta^{(2)}\|_{H^2}^2)\,(\|\delta u\|_{L^2}^2 + \|\delta\theta\|_{L^2}^2). \end{aligned}$$

Gronwall's inequality implies

$$\|\delta u(t)\|_{L^2}^2 + \|\delta\theta(t)\|_{L^2}^2 \le (\|\delta u(0)\|_{L^2}^2 + \|\delta\theta(0)\|_{L^2}^2)\, e^{\int_0^t M(\tau)\, d\tau},$$

where

$$M(t) = C\,(1 + \|u^{(2)}\|_{H^2}^2 + \|\theta^{(2)}\|_{L^2}^2).$$

For $(u^{(2)}, \theta^{(2)}) \in L^\infty(0,T;H^2)$, the time integral of $M(t)$ with $0 \le t \le T$ is bounded,

$$\int_0^t M(\tau)\,d\tau < \infty$$

and therefore, $\delta u(t) = \delta\theta(t) = 0$. This completes the proof of Proposition 3.1. □

Next we state and prove our main propositions.

Proposition 3.3. *Assume the initial datum (u_0, θ_0) satisfies the regularity and symmetry conditions in Theorem 1.1. Let $T > 0$. Let (u, θ) be the corresponding solution of (1.3) on $[0, T]$. Define the energy functional $E(t)$ by*

$$E(t) = \sup_{0\le\tau\le t} \|(u,\theta)(\tau)\|_{H^2}^2 + \nu\int_0^t \|\nabla_h u\|_{H^2}^2\,d\tau + \eta\int_0^t \|\partial_3\theta\|_{H^2}^2 d\tau.$$

Then, for a constant $C > 0$ and for $0 \le t \le T$,

$$E(t) \le E(0) + C\,E(t)^{\frac{3}{2}}. \tag{3.3}$$

Proof of Proposition 3.3. According to Corollary 3.2, (u, θ) obeys the following symmetries

$$\begin{cases} u_1(x_1,x_2,x_3,t) = u_1(x_1,x_2,-x_3,t),\\ u_2(x_1,x_2,x_3,t) = u_2(x_1,x_2,-x_3,t),\\ u_3(x_1,x_2,x_3,t) = -u_3(x_1,x_2,-x_3,t),\\ \theta(x_1,x_2,x_3,t) = -\theta(x_1,x_2,-x_3,t). \end{cases} \tag{3.4}$$

As in (2.1), we define $\bar{u}$ and $\bar{\theta}$ to be the horizontal averages of u and θ, respectively, and $\widetilde{u} = u - \bar{u}$ and $\widetilde{\theta} = \theta - \bar{\theta}$. As a special consequence of the symmetries in (3.4),

$$\bar{u}_3 = \int_{\mathbb{T}} u_3(x_1,x_2,x_3,t)\,dx_3 = 0, \quad \bar{\theta} = \int_{\mathbb{T}} \theta(x_1,x_2,x_3,t)\,dx_3 = 0$$

and thus

$$u_3 = \widetilde{u}_3, \quad \theta = \widetilde{\theta}.$$

We now prove (3.3). Due to the equivalence of the norms

$$\|(u,\theta)\|_{H^2}^2 \quad \sim \quad \|(u,\theta)\|_{L^2}^2 + \sum_{i=1}^3 \|(\partial_i^2 u, \partial_i^2\theta)\|_{L^2}^2,$$

it suffices to estimate $\|(u,\theta)\|_{L^2}$ and $\sum_{i=1}^2 \|(\partial_i^2 u, \partial_i^2\theta)\|_{L^2}$. First of all, we have the global L^2-bound. Dotting the equations in (1.3) by (u,θ) and integrating by parts, we find

$$\|(u,\theta)\|_{L^2}^2 + 2\nu\int_0^t \|\nabla_h u\|_{L^2}^2\,d\tau + 2\eta\int_0^t \|\partial_3\theta\|_{L^2}^2 d\tau \le \|(u_0,\theta_0)\|_{L^2}^2. \tag{3.5}$$

Applying the differential operator ∂_i^2 to the equations in (1.3), testing the resulting equations by $(\partial_i^2 u, \partial_i^2\theta)$, and integrating by parts, we have

$$\frac{d}{dt}\sum_{i=1}^3 \left(\|\partial_i^2 u\|_{L^2}^2 + \|\partial_i^2\theta\|_{L^2}^2\right) + 2\nu\sum_{i=1}^3 \|\nabla_h\partial_i^2 u\|_{L^2}^2 + 2\eta\sum_{i=1}^3 \|\partial_3\partial_i^2\theta\|_{L^2}^2$$

$$= J_1 + J_2, \tag{3.6}$$

where we have used the fact that

$$\int (\partial_i^2 \theta e_3 \cdot \partial_i^2 u - \partial_i^2 u_3 \partial_i^2 \theta)\, dx = 0$$

and

$$J_1 = -\sum_{i=1}^{3} \int \partial_i^2 (u \cdot \nabla u) \cdot \partial_i^2 u \, dx,$$

$$J_2 = -\sum_{i=1}^{3} \int \partial_i^2 (u \cdot \nabla \theta) \cdot \partial_i^2 \theta \, dx.$$

Due to the anisotropic dissipation in (1.3), we need to decompose the terms into component terms to distinguish the derivatives in the horizontal direction from those in the vertical direction. In addition, due to $\nabla \cdot u = 0$,

$$\int (u \cdot \nabla \partial_i^2 u) \cdot \partial_i^2 u \, dx = 0, \quad i = 1, 2, 3.$$

Therefore, J_1 can be written as

$$\begin{aligned}
J_1 =& -\sum_{i=1}^{2} \int \partial_i^2 (u \cdot \nabla u) \cdot \partial_i^2 u \, dx - \int \partial_3^2 (u \cdot \nabla u) \cdot \partial_3^2 u \, dx \\
=& -\sum_{i=1}^{2} \int \partial_i^2 (u \cdot \nabla u) \cdot \partial_i^2 u \, dx - \sum_{k=1}^{2} \int \partial_3^2 (u_k \cdot \partial_k u) \cdot \partial_3^2 u \, dx \\
& - \int \partial_3^2 (u_3 \cdot \partial_3 u) \cdot \partial_3^2 u \, dx \\
=& -\sum_{i=1}^{2} \sum_{m=1}^{2} \mathcal{C}_2^m \int \partial_i^m u \cdot \partial_i^{2-m} \nabla u \cdot \partial_i^2 u \, dx \\
& - \sum_{k=1}^{2} \sum_{m=1}^{2} \mathcal{C}_2^m \int \partial_3^m u_k \cdot \partial_3^{2-m} \partial_k u \cdot \partial_3^2 u \, dx \\
& - \sum_{m=1}^{2} \mathcal{C}_2^m \int \partial_3^m u_3 \cdot \partial_3^{2-m} \partial_3 u \cdot \partial_3^2 u \, dx \\
:=& J_{11} + J_{12} + J_{13},
\end{aligned}$$

where $\mathcal{C}_2^m$ denotes the combinatorial number,

$$\mathcal{C}_2^m = \frac{2!}{m!(2-m)!}.$$

Since the derivatives in J_{11} are all in the horizontal direction, we can directly apply Lemma 2.5 to obtain

$$|J_{11}| \le C \sum_{i=1}^{2} \sum_{m=1}^{2} \|\partial_i^m u\|_{L^2}^{\frac{1}{2}} \left(\|\partial_i^m u\|_{L^2} + \|\partial_3 \partial_i^m u\|_{L^2} \right)^{\frac{1}{2}}$$

$$
\begin{aligned}
&\times \|\partial_1^{2-m}\nabla u\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_1^{2-m}\nabla u\|_{L^2}^{\frac{1}{2}}\,\|\partial_i^2 u\|_{L^2}^{\frac{1}{2}}\,\|\partial_2\partial_i^2 u\|_{L^2}^{\frac{1}{2}}\\
&\le C\,\|u\|_{H^2}\,\|\nabla_h u\|_{H^2}^2.
\end{aligned}
$$

To deal with J_{12}, we realize that the middle term in the integral $\partial_3^{2-m}\partial_k u$ with $k = 1$ or 2 has at least one horizontal derivative. Thus, we can still use Lemma 2.5 to generate enough time integrability parts,

$$
\begin{aligned}
|J_{12}| \le C &\sum_{k=1}^{2}\sum_{m=1}^{2} \|\partial_3^m u_k\|_{L^2}^{\frac{1}{2}}\,\|\partial_1\partial_3^m u_k\|_{L^2}^{\frac{1}{2}}\,\|\partial_3^{2-m}\partial_k u\|_{L^2}^{\frac{1}{2}}\\
&\times \left(\|\partial_3^{2-m}\partial_k u\|_{L^2} + \|\partial_3\partial_3^{2-m}\partial_k u\|_{L^2}\right)^{\frac{1}{2}}\,\|\partial_3^2 u\|_{L^2}^{\frac{1}{2}}\,\|\partial_2\partial_3^2 u\|_{L^2}^{\frac{1}{2}}\\
\le C\,&\|u\|_{H^2}\,\|\nabla_h u\|_{H^2}^2.
\end{aligned}
$$

To deal with J_{13}, we use the divergence-free condition $\partial_3 u_3 = -\nabla_h\cdot u_h$ and Lemma 2.5 to obtain

$$
\begin{aligned}
|J_{13}| \le C &\sum_{m=1}^{2}\sum_{k=1}^{2} \mathcal{C}_2^m\|\partial_3^{m-1}\nabla_h\cdot u_h\|_{L^2}^{\frac{1}{2}}\left(\|\partial_3^{m-1}\nabla_h\cdot u_h\|_{L^2} + \|\partial_3\partial_3^{m-1}\nabla_h\cdot u_h\|_{L^2}\right)^{\frac{1}{2}}\\
&\times \|\partial_3^{3-m} u\|_{L^2}^{\frac{1}{2}}\,\|\partial_1\partial_3^{3-m} u\|_{L^2}^{\frac{1}{2}}\,\|\partial_3^2 u\|_{L^2}^{\frac{1}{2}}\,\|\partial_2\partial_3^2 u\|_{L^2}^{\frac{1}{2}}\\
\le C\,&\|u\|_{H^2}\,\|\nabla_h u\|_{H^2}^2.
\end{aligned}
$$

In summary, we have shown that

$$
|J_1| \le C\,\|u\|_{H^2}\,\|\nabla_h u\|_{H^2}^2. \tag{3.7}
$$

We now turn to J_2. First, we distinguish the horizontal derivatives from the vertical derivatives to decompose J_2 as

$$
\begin{aligned}
J_2 &= -\sum_{i=1}^{2}\int \partial_i^2(u\cdot\nabla\theta)\cdot\partial_i^2\theta\,dx - \int \partial_3^2(u\cdot\nabla\theta)\cdot\partial_3^2\theta\,dx\\
&= -\sum_{i=1}^{2}\sum_{k=1}^{2}\int \partial_i^2(u_k\cdot\partial_k\theta)\cdot\partial_i^2\theta\,dx - \sum_{i=1}^{2}\int \partial_i^2(u_3\cdot\partial_3\theta)\cdot\partial_i^2\theta\,dx\\
&\quad - \sum_{k=1}^{2}\int \partial_3^2(u_k\cdot\partial_k\theta)\cdot\partial_3^2\theta\,dx - \int \partial_3^2(u_3\cdot\partial_3\theta)\cdot\partial_3^2\theta\,dx\\
&= -\sum_{i=1}^{2}\sum_{k=1}^{2}\sum_{m=1}^{2}\mathcal{C}_2^m\int \partial_i^m u_k\cdot\partial_i^{2-m}\partial_k\theta\cdot\partial_i^2\theta\,dx\\
&\quad - \sum_{i=1}^{2}\sum_{m=1}^{2}\mathcal{C}_2^m\int \partial_i^m u_3\cdot\partial_i^{2-m}\partial_3\theta\cdot\partial_i^2\theta\,dx\\
&\quad - \sum_{k=1}^{2}\sum_{m=1}^{2}\mathcal{C}_2^m\int \partial_3^m u_k\cdot\partial_3^{2-m}\partial_k\theta\cdot\partial_3^2\theta\,dx
\end{aligned}
$$

$$- \sum_{m=1}^{2} \mathcal{C}_2^m \int \partial_3^m u_3 \cdot \partial_3^{2-m} \partial_3 \theta \cdot \partial_3^2 \theta \, dx$$
$$:= J_{21} + J_{22} + J_{23} + J_{24},$$

where we have used the fact that, due to $\nabla \cdot u = 0$,

$$\int (u \cdot \nabla \partial_i^2 \theta) \, \partial_i^2 \theta \, dx = 0, \quad i = 1, 2, 3.$$

Since the temperature equation involves only vertical dissipation, we need to make use of the decomposition

$$\theta = \bar{\theta} + \widetilde{\theta} = \widetilde{\theta}$$

where we have used $\bar{\theta} = 0$ due to the symmetry in θ. Therefore,

$$J_{21} = - \sum_{i=1}^{2} \sum_{k=1}^{2} \sum_{m=1}^{2} \mathcal{C}_2^m \int \partial_i^m u_k \cdot \partial_i^{2-m} \partial_k \widetilde{\theta} \cdot \partial_i^2 \widetilde{\theta} \, dx.$$

It then follows from the second inequality in Lemma 2.5 that

$$|J_{21}| \le \sum_{i=1}^{2} \sum_{k=1}^{2} \sum_{m=1}^{2} \mathcal{C}_2^m \|\partial_i^m u_k\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_i^m u_k\|_{L^2}^{\frac{1}{2}} \|\partial_i^{2-m} \partial_k \widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_i^{2-m} \partial_k \widetilde{\theta}\|_{L^2}^{\frac{1}{2}}$$
$$\times \|\partial_i^2 \widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \|\partial_3 \partial_i^2 \widetilde{\theta}\|_{L^2}^{\frac{1}{2}}$$

By the strong Poincaré type inequality in Lemma 2.3,

$$\|\partial_i^2 \widetilde{\theta}\|_{L^2} \le C \, \|\partial_3 \partial_i^2 \widetilde{\theta}\|_{L^2}.$$

Therefore, by the basic facts in Lemma 2.1,

$$|J_{21}| \le C \, \|\nabla_h u\|_{H^2} \, \|\theta\|_{H^2} \, \|\partial_3 \theta\|_{H^2}.$$

The estimate of J_{22} is similar,

$$|J_{22}| \le C \sum_{i=1}^{2} \sum_{m=1}^{2} \mathcal{C}_2^m \|\partial_i^m u_3\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_i^m u_3\|_{L^2}^{\frac{1}{2}} \|\partial_i^{2-m} \partial_3 \widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_i^{2-m} \partial_3 \widetilde{\theta}\|_{L^2}^{\frac{1}{2}}$$
$$\times \|\partial_i^2 \widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \|\partial_3 \partial_i^2 \widetilde{\theta}\|_{L^2}^{\frac{1}{2}}$$
$$\le C \, \|\nabla_h u\|_{H^2} \, \|\theta\|_{H^2} \, \|\partial_3 \theta\|_{H^2}.$$

To bound J_{23}, we first change θ to $\widetilde{\theta}$,

$$J_{23} = - \sum_{k=1}^{2} \sum_{m=1}^{2} \mathcal{C}_2^m \int \partial_3^m u_k \cdot \partial_3^{2-m} \partial_k \widetilde{\theta} \cdot \partial_3^2 \widetilde{\theta} \, dx.$$

By Lemma 2.5,

$$|J_{23}| \le C \sum_{k=1}^{2} \sum_{m=1}^{2} \mathcal{C}_2^m \|\partial_3^m u_k\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_3^m u_k\|_{L^2}^{\frac{1}{2}} \|\partial_3^{2-m} \partial_k \widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \|\partial_3 \partial_3^{2-m} \partial_k \widetilde{\theta}\|_{L^2}^{\frac{1}{2}}$$
$$\times \|\partial_3^2 \widetilde{\theta}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_3^2 \widetilde{\theta}\|_{L^2}^{\frac{1}{2}}$$
$$\le C \, \|u\|_{H^2}^{\frac{1}{2}} \, \|\nabla_h u\|_{H^2}^{\frac{1}{2}} \, \|\theta\|_{H^2}^{\frac{1}{2}} \, \|\partial_3 \theta\|_{H^2}^{\frac{3}{2}}$$

$$\leq C\,(\|u\|_{H^2}+\|\theta\|_{H^2})\,\left(\|\nabla_h u\|_{H^2}^2+\|\partial_3\theta\|_{H^2}^2\right).$$

The estimate of J_{24} is similar to that for J_{23} and

$$|J_{24}|\leq C\,(\|u\|_{H^2}+\|\theta\|_{H^2})\,\left(\|\nabla_h u\|_{H^2}^2+\|\partial_3\theta\|_{H^2}^2\right).$$

Collecting the bounds for J_2, we obtain

$$|J_2|\leq C\,(\|u\|_{H^2}+\|\theta\|_{H^2})\,\left(\|\nabla_h u\|_{H^2}^2+\|\partial_3\theta\|_{H^2}^2\right). \tag{3.8}$$

Inserting (3.10) and (3.11) in (3.6), integrating in time over $[0,t]$ and adding to (3.5), we deduce

$$\begin{aligned} E(t) \leq E(0) + C\int_0^t \Big(&\|u\|_{H^2}\,\|\nabla_h u\|_{H^2}^2 \\ &+(\|u\|_{H^2}+\|\theta\|_{H^2})\,\left(\|\nabla_h u\|_{H^2}^2+\|\partial_3\theta\|_{H^2}^2\right)\Big)\,d\tau \\ \leq E(0) + C\,E(t)^{\frac{3}{2}},& \end{aligned}$$

which is the desired inequality (3.3). Here $C=C(\nu,\eta)>0$ is a constant depending on ν and η. According to the definition of $E(t)$, $C=c/\min\{\nu,\eta\}$ for a pure constant $c>0$ independent of ν and η. This completes the proof of Proposition 3.3. □

Our last proposition concerns itself with an *a priori* bound for the H^3 solutions of (1.3).

Proposition 3.4. *Assume the initial datum $(u_0,\theta_0)\in H^3$ satisfies the symmetry conditions (1.5) in Theorem 1.1. Let $T>0$. Let (u,θ) be the corresponding solution of (1.3) on $[0,T]$. Define the energy functional $E(t)$ by*

$$E(t)=\sup_{0\leq\tau\leq t}\|(u,\theta)(\tau)\|_{H^3}^2+\nu\int_0^t\|\nabla_h u\|_{H^3}^2\,d\tau+\eta\int_0^t\|\partial_3\theta\|_{H^3}^2 d\tau.$$

Then, for a constant $C>0$ and for $0\leq t\leq T$,

$$E(t)\leq E(0)+C\,E(t)^{\frac{3}{2}}. \tag{3.9}$$

Proof of Proposition 3.4. Due to the norm equivalence

$$\|f\|_{H^3}\quad\sim\quad\|f\|_{L^2}+\sum_{i=1}^3\|\partial_i^3 f\|_{L^2}$$

and the global L^2-bound in (3.5), it suffices to estimate

$$\sum_{i=1}^3\left(\|\partial_i^3 u\|_{L^2}^2+\|\partial_i^3\theta\|_{L^2}^2\right).$$

By the equations of (u,θ) in (1.3),

$$\begin{aligned}&\frac{d}{dt}\sum_{i=1}^3\left(\|\partial_i^3 u\|_{L^2}^2+\|\partial_i^3\theta\|_{L^2}^2\right)+2\nu\sum_{i=1}^3\|\nabla_h\partial_i^3 u\|_{L^2}^2+2\eta\sum_{i=1}^3\|\partial_3\partial_i^3\theta\|_{L^2}^2\\ &=K_1+K_2,\end{aligned}$$

where

$$K_1 = -\sum_{i=1}^{3} \int \partial_i^3 (u\cdot\nabla u)\cdot \partial_i^3 u\,dx,$$
$$K_2 = -\sum_{i=1}^{3} \int \partial_i^3 (u\cdot\nabla \theta)\cdot \partial_i^3 \theta\,dx.$$

To cope with the anisotropic dissipation, we decompose K_1 into three terms, as we did in the previous proof. The situation here is more complex due to the higher-order derivatives.

$$\begin{aligned} K_1 = &-\sum_{i=1}^{2}\sum_{m=1}^{3} \mathcal{C}_3^m \int \partial_i^m u\cdot \partial_i^{3-m}\nabla u\cdot \partial_i^3 u\,dx \\ &-\sum_{k=1}^{2}\sum_{m=1}^{3} \mathcal{C}_3^m \int \partial_3^m u_k\cdot \partial_3^{3-m}\partial_k u\cdot \partial_3^3 u\,dx \\ &-\sum_{m=1}^{3} \mathcal{C}_3^m \int \partial_3^m u_3\cdot \partial_3^{3-m}\partial_3 u\cdot \partial_3^3 u\,dx \\ :=\ & K_{11}+K_{12}+K_{13}. \end{aligned}$$

By Lemma 2.5,

$$\begin{aligned} |K_1| \le\ & C\sum_{i=1}^{2}\sum_{m=1}^{3} \|\partial_i^m u\|_{L^2}^{\frac12}\left(\|\partial_i^m u\|_{L^2}+\|\partial_3\partial_i^m u\|_{L^2}\right)^{\frac12} \\ &\times \|\partial_1^{2-m}\nabla u\|_{L^2}^{\frac12}\|\partial_1\partial_1^{2-m}\nabla u\|_{L^2}^{\frac12}\,\|\partial_i^3 u\|_{L^2}^{\frac12}\,\|\partial_2\partial_i^3 u\|_{L^2}^{\frac12} \\ \le\ & C\,\|u\|_{H^3}\,\|\nabla_h u\|_{H^3}^2. \end{aligned}$$

$$\begin{aligned} |K_{12}| \le\ & C\sum_{k=1}^{2}\sum_{m=1}^{3} \|\partial_3^m u_k\|_{L^2}^{\frac12}\,\|\partial_1\partial_3^m u_k\|_{L^2}^{\frac12}\,\|\partial_3^{3-m}\partial_k u\|_{L^2}^{\frac12} \\ &\times \left(\|\partial_3^{3-m}\partial_k u\|_{L^2}+\|\partial_3\partial_3^{3-m}\partial_k u\|_{L^2}\right)^{\frac12}\|\partial_3^3 u\|_{L^2}^{\frac12}\,\|\partial_2\partial_3^3 u\|_{L^2}^{\frac12} \\ \le\ & C\,\|u\|_{H^3}\,\|\nabla_h u\|_{H^3}^2. \end{aligned}$$

By the divergence-free condition $\partial_3 u_3 = -\nabla_h\cdot u_h$ and Lemma 2.5,

$$\begin{aligned} |K_{13}| \le\ & C\sum_{m=1}^{3}\sum_{k=1}^{2} \|\partial_3^{m-1}\nabla_h\cdot u_h\|_{L^2}^{\frac12}\left(\|\partial_3^{m-1}\nabla_h\cdot u_h\|_{L^2}+\|\partial_3\partial_3^{m-1}\nabla_h\cdot u_h\|_{L^2}\right)^{\frac12} \\ &\times \|\partial_3^{3-m}u\|_{L^2}^{\frac12}\,\|\partial_1\partial_3^{3-m}u\|_{L^2}^{\frac12}\,\|\partial_3^3 u\|_{L^2}^{\frac12}\,\|\partial_2\partial_3^3 u\|_{L^2}^{\frac12} \\ \le\ & C\,\|u\|_{H^3}\,\|\nabla_h u\|_{H^3}^2. \end{aligned}$$

In summary, we have shown that

$$|K_1| \le C\,\|u\|_{H^3}\,\|\nabla_h u\|_{H^3}^2. \tag{3.10}$$

To deal with K_2, we divide K_2 into four terms,

$$\begin{aligned}
K_2 = &-\sum_{i=1}^{2}\sum_{k=1}^{2}\sum_{m=1}^{3}\mathcal{C}_3^m\int \partial_i^m u_k\cdot\partial_i^{3-m}\partial_k\theta\cdot\partial_i^3\theta\,dx\\
&-\sum_{i=1}^{2}\sum_{m=1}^{3}\mathcal{C}_3^m\int \partial_i^m u_3\cdot\partial_i^{3-m}\partial_3\theta\cdot\partial_i^3\theta\,dx\\
&-\sum_{k=1}^{2}\sum_{m=1}^{3}\mathcal{C}_3^m\int \partial_3^m u_k\cdot\partial_3^{3-m}\partial_k\theta\cdot\partial_3^3\theta\,dx\\
&-\sum_{m=1}^{3}\mathcal{C}_3^m\int \partial_3^m u_3\cdot\partial_3^{3-m}\partial_3\theta\cdot\partial_3^3\theta\,dx\\
:= &\ K_{21}+K_{22}+K_{23}+K_{24},
\end{aligned}$$

To bound these terms, we invoke the fact that

$$\bar{\theta}=0,\quad \theta=\widetilde{\theta}$$

and apply Lemma 2.5 to obtain

$$\begin{aligned}
|K_{21}| \le &\ C\sum_{i=1}^{2}\sum_{k=1}^{2}\sum_{m=1}^{3}\mathcal{C}_3^m\|\partial_i^m u_k\|_{L^2}^{\frac12}\|\partial_1\partial_i^m u_k\|_{L^2}^{\frac12}\|\partial_i^{3-m}\partial_k\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_2\partial_i^{3-m}\partial_k\widetilde{\theta}\|_{L^2}^{\frac12}\\
&\times\|\partial_i^3\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_3\partial_i^3\widetilde{\theta}\|_{L^2}^{\frac12}\\
\le &\ C\sum_{i=1}^{2}\sum_{k=1}^{2}\sum_{m=1}^{3}\mathcal{C}_3^m\|\partial_i^m u_k\|_{L^2}^{\frac12}\|\partial_1\partial_i^m u_k\|_{L^2}^{\frac12}\|\partial_i^{3-m}\partial_k\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_2\partial_i^{3-m}\partial_k\widetilde{\theta}\|_{L^2}^{\frac12}\\
&\times\|\partial_3\partial_i^3\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_3\partial_i^3\widetilde{\theta}\|_{L^2}^{\frac12}\\
\le &\ C\,\|\nabla_h u\|_{H^3}\,\|\theta\|_{H^3}\,\|\partial_3\theta\|_{H^3},
\end{aligned}$$

$$\begin{aligned}
|K_{22}| \le &\ C\sum_{i=1}^{2}\sum_{m=1}^{3}\mathcal{C}_3^m\|\partial_i^m u_3\|_{L^2}^{\frac12}\|\partial_1\partial_i^m u_3\|_{L^2}^{\frac12}\|\partial_i^{3-m}\partial_3\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_2\partial_i^{3-m}\partial_3\widetilde{\theta}\|_{L^2}^{\frac12}\\
&\times\|\partial_i^2\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_3\partial_i^2\widetilde{\theta}\|_{L^2}^{\frac12}\\
\le &\ C\,\|\nabla_h u\|_{H^3}\,\|\theta\|_{H^3}\,\|\partial_3\theta\|_{H^3},
\end{aligned}$$

$$\begin{aligned}
|K_{23}| \le &\ C\sum_{k=1}^{2}\sum_{m=1}^{3}\mathcal{C}_3^m\|\partial_3^m u_k\|_{L^2}^{\frac12}\|\partial_1\partial_3^m u_k\|_{L^2}^{\frac12}\|\partial_3^{3-m}\partial_k\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_3\partial_3^{3-m}\partial_k\widetilde{\theta}\|_{L^2}^{\frac12}\\
&\times\|\partial_3^3\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_2\partial_3^2\widetilde{\theta}\|_{L^2}^{\frac12}\\
\le &\ C\,\|u\|_{H^2}^{\frac12}\,\|\nabla_h u\|_{H^2}^{\frac12}\,\|\theta\|_{H^2}^{\frac12}\,\|\partial_3\theta\|_{H^2}^{\frac32}\\
\le &\ C\,\left(\|u\|_{H^3}+\|\theta\|_{H^3}\right)\left(\|\nabla_h u\|_{H^3}^2+\|\partial_3\theta\|_{H^3}^2\right)
\end{aligned}$$

and

$$|K_{24}| \le C\,\left(\|u\|_{H^3}+\|\theta\|_{H^3}\right)\left(\|\nabla_h u\|_{H^3}^2+\|\partial_3\theta\|_{H^3}^2\right).$$

Collecting the bounds for J_2, we obtain

$$|K_2| \le C\,(\|u\|_{H^3} + \|\theta\|_{H^3})\,\left(\|\nabla_h u\|_{H^3}^2 + \|\partial_3\theta\|_{H^3}^2\right). \tag{3.11}$$

Collecting the upper bounds for K_1 and K_2 and integrating in time lead to the desired inequality (3.9). This completes the proof of Proposition 3.4. ☐

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. As we mentioned at the beginning of this section, it suffices to establish a global *a priori* bound on the norm of (u,θ) to prove Theorem 1.1. This is achieved by applying the bootstrapping argument on (3.3) in Proposition 3.3. An abstract bootstrapping argument can be found in T. Tao's book [22, p.20].

We rely on the inequality (3.3), namely

$$E(t) \le E(0) + C_0\, E(t)^{\frac{3}{2}}, \tag{3.12}$$

where $C_0 = c/\min\{\nu,\eta\} > 0$ for a pure constant $c > 0$ independent of $\nu > 0$ and $\eta > 0$. We take $\|(u_0,\theta_0)\|_{H^2}$ to be sufficiently small, say

$$E(0) = \|(u_0,\theta_0)\|_{H^2}^2 \le \frac{1}{16C_0^2} := \varepsilon^2.$$

The bootstrapping argument starts with the ansatz that

$$E(t) \le \frac{1}{4C_0^2}. \tag{3.13}$$

It then follows from (3.12) that

$$E(t) \le E(0) + C_0\, E(t)^{\frac{1}{2}}\, E(t) \le E(0) + C_0 \frac{1}{2C_0}\, E(t) = E(0) + \frac{1}{2} E(t)$$

or

$$E(t) \le 2E(0) \le \frac{1}{8C_0^2},$$

which is half of the upper bound in (3.13). The bootstrapping argument then implies that, for any $t \ge 0$,

$$E(t) \le \frac{1}{8C_0^2}.$$

In particular,

$$\|(u(t),\theta(t))\|_{H^2} \le \frac{1}{2\sqrt{2}C_0} = \sqrt{2}\,\varepsilon.$$

The global existence and stability of H^3 solutions are obtained similarly by using the inequality (3.9) in Proposition 3.4. This completes the proof of Theorem 1.1. ☐

4. Proof of Theorem 1.2

This section is devoted to the proof of the decay estimates in Theorem 1.2.

Proof of Theorem 1.2. Assume that the initial datum $(u_0, \theta_0) \in H^3$ satisfies the regularity, symmetry and smallness assumptions stated in Theorem 1.1. Let (u, θ) be the corresponding solution of (1.8), namely

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p + \nu \Delta u + \theta\, e_3, \quad x \in \Omega,\, t > 0, \\ \nabla \cdot u = 0, \\ \partial_t \theta + u \cdot \nabla \theta + u_3 = \eta\, \partial_{33} \theta. \end{cases} \tag{4.1}$$

As established by Theorem 1.1, (u, θ) remains small in H^3 for all time and obeys the same symmetries. As a special consequence of the symmetries, the vertical averages of u_3 and θ are zero, namely

$$\bar{u}_3 = \bar{\theta} = 0. \tag{4.2}$$

By taking the vertical average of (4.1), and using (4.2) and the basic properties in Lemma 2.1, we obtain the equations of $\bar{u}$,

$$\begin{cases} \partial_t \bar{u}_1 + \partial_1 \overline{u_1^2} + \partial_2 \overline{u_1 u_2} = -\partial_1 \bar{p} + \nu \Delta_h \bar{u}_1, \\ \partial_t \bar{u}_2 + \partial_1 \overline{u_1 u_2} + \partial_2 \overline{u_2^2} = -\partial_2 \bar{p} + \nu \Delta_h \bar{u}_2, \\ \partial_1 \bar{u}_1 + \partial_2 \bar{u}_2 = 0. \end{cases} \tag{4.3}$$

Taking the difference of (4.1) and (4.3), we find that $(\widetilde{u}, \widetilde{\theta})$ satisfies

$$\begin{cases} \partial_t \widetilde{u}_1 + \partial_1 (u_1^2 - \overline{u_1^2}) + \partial_2 (u_1 u_2 - \overline{u_1 u_2}) + \partial_3 (\widetilde{u}_3 u_1) = -\partial_1 \widetilde{p} + \nu \Delta \widetilde{u}_1, \\ \partial_t \widetilde{u}_2 + \partial_1 (u_1 u_2 - \overline{u_1 u_2}) + \partial_2 (u_2^2 - \overline{u_2^2}) + \partial_3 (\widetilde{u}_3 u_2) = -\partial_2 \widetilde{p} + \nu \Delta \widetilde{u}_2, \\ \partial_t \widetilde{u}_3 + u \cdot \nabla \widetilde{u}_3 = -\partial_3 \widetilde{p} + \nu \Delta \widetilde{u}_3 + \widetilde{\theta}, \\ \partial_t \widetilde{\theta} + u \cdot \nabla \widetilde{\theta} = \eta \partial_{33} \widetilde{\theta} - \widetilde{u}_3, \\ \nabla \cdot \widetilde{u} = 0. \end{cases} \tag{4.4}$$

As we shall see below, we do not really need the full dissipation in the velocity equation, but the dissipation in the vertical direction is crucial. The nonlinear terms will be controlled without using the dissipation in the x_1-direction. We estimate the L^2-norms of $(\widetilde{u}, \widetilde{\theta})$ and $(\nabla \widetilde{u}, \nabla \widetilde{\theta})$ separately. Our goal is to achieve the following inequalities

$$\frac{d}{dt} \|(\widetilde{u}, \widetilde{\theta})\|_{L^2}^2 + (2\nu - C\|u\|_{H^2}) \|(\partial_2, \partial_3) \widetilde{u}\|_{L^2}^2 + 2\eta \|\partial_3 \widetilde{\theta}\|_{L^2}^2 \leq 0$$

and

$$\begin{aligned} &\frac{d}{dt} \|(\nabla \widetilde{u}, \nabla \widetilde{\theta})\|_{L^2}^2 + 2\nu \|\partial_1 \nabla \widetilde{u}\|_{L^2}^2 \\ &\quad + (2\nu - C_1(\|u\|_{H^2} + \|\theta\|_{H^2}))\, \|(\partial_2, \partial_3) \nabla \widetilde{u}\|_{L^2}^2 \\ &\quad + (2\eta - C_2(\|u\|_{H^3} + \|\theta\|_{H^2}))\, \|\partial_3 \nabla \widetilde{\theta}\|_{L^2}^2 \leq 0. \end{aligned}$$

Dotting (4.4) by $(\widetilde{u}, \widetilde{\theta})$ and integrating by parts, we obtain

$$\begin{aligned}&\frac{d}{dt}\|(\widetilde{u}, \widetilde{\theta})\|_{L^2}^2 + 2\nu\|\nabla\widetilde{u}\|_{L^2}^2 + 2\eta\|\partial_3\widetilde{\theta}\|_{L^2}^2 \\ &:= L_1 + L_2 + L_3 + L_4 + L_5 + L_6,\end{aligned}$$

where

$$\begin{aligned}L_1 &= -\int \widetilde{u}_1\partial_1(u_1^2 - \overline{u_1^2})\,dx, \quad L_2 = -\int \widetilde{u}_1\partial_2(u_1u_2 - \overline{u_1u_2})\,dx,\\ L_3 &= -\int \widetilde{u}_1\partial_3(\widetilde{u}_3u_1)\,dx, \quad L_4 = -\int \widetilde{u}_2\partial_2(u_1u_2 - \overline{u_1u_2})\,dx,\\ L_5 &= -\int \widetilde{u}_2\partial_2(u_2^2 - \overline{u_2^2})\,dx, \quad L_6 = -\int \widetilde{u}_2\partial_3(\widetilde{u}_3u_2)\,dx.\end{aligned}$$

It is easy to check that

$$u_1^2 - \overline{u_1^2} = 2\bar{u}_1\,\widetilde{u}_1 + (\widetilde{u}_1)^2 - \overline{(\widetilde{u}_1)^2} = 2\bar{u}_1\,\widetilde{u}_1 + \widetilde{(\widetilde{u}_1)^2}, \tag{4.5}$$

$$u_1u_2 - \overline{u_1u_2} = \bar{u}_1\,\widetilde{u}_2 + \bar{u}_2\,\widetilde{u}_1 + \widetilde{\widetilde{u}_1\,\widetilde{u}_2}. \tag{4.6}$$

Therefore, we can further decompose L_1 into three parts,

$$\begin{aligned}L_1 &= 2\int \widetilde{u}_1\partial_1\bar{u}\,\widetilde{u}_1\,dx + 2\int \widetilde{u}_1\partial_1\widetilde{u}\,\bar{u}_1\,dx + \int \widetilde{u}_1\,\partial_1\widetilde{(\widetilde{u}_1)^2}\,dx\\ &:= L_{11} + L_{12} + L_{13}.\end{aligned}$$

By Lemma 2.5 and Lemma 2.3,

$$\begin{aligned}L_{11} &\le 2\|\widetilde{u}_1\|_{L^2}^{\frac12}\,\|\partial_2\widetilde{u}_1\|_{L^2}^{\frac12}\,\|\partial_1\bar{u}_1\|_{L^2}^{\frac12}\,\|\partial_1\partial_1\bar{u}_1\|_{L^2}^{\frac12}\,\|\widetilde{u}_1\|_{L^2}^{\frac12}\,\|\partial_3\widetilde{u}_1\|_{L^2}^{\frac12}\\ &\le C\,\|\bar{u}_1\|_{H^2}\,\|\partial_2\widetilde{u}_1\|_{L^2}^{\frac12}\,\|\partial_3\widetilde{u}_1\|_{L^2}^{\frac32}\\ &\le C\,\|\bar{u}_1\|_{H^2}\,(\|\partial_2\widetilde{u}_1\|_{L^2}^2 + \|\partial_3\widetilde{u}_1\|_{L^2}^2).\end{aligned}$$

By Hölder's inequality and Lemma 2.3,

$$\begin{aligned}L_{12} &\le 2\|\widetilde{u}_1\|_{L^2}\,\|\bar{u}_1\|_{L^\infty}\|\partial_1\widetilde{u}_1\|_{L^2}\\ &\le C\,\|\partial_3\widetilde{u}_1\|_{L^2}\,\|\bar{u}_1\|_{H^2}\,\|\partial_2\widetilde{u}_2 + \partial_3\widetilde{u}_3\|_{L^2}\\ &\le C\,\|\bar{u}_1\|_{H^2}\,(\|\partial_2\widetilde{u}\|_{L^2}^2 + \|\partial_3\widetilde{u}\|_{L^2}^2).\end{aligned}$$

By Lemma 2.1 and Lemma 2.3,

$$\begin{aligned}L_{13} &\le \|\widetilde{u}_1\|_{L^2}\,\|\partial_1\widetilde{(\widetilde{u}_1)^2}\|_{L^2} \le \|\widetilde{u}_1\|_{L^2}\,\|\partial_1(\widetilde{u}_1)^2\|_{L^2}\\ &\le C\,\|\widetilde{u}_1\|_{L^2}\,\|\widetilde{u}_1\|_{L^\infty}\,\|\partial_1\widetilde{u}_1\|_{L^2}\\ &\le C\,\|\partial_3\widetilde{u}_1\|_{L^2}\,\|\widetilde{u}_1\|_{H^2}\,\|\partial_2\widetilde{u}_2 + \partial_3\widetilde{u}_3\|_{L^2}\\ &\le C\,\|\widetilde{u}_1\|_{H^2}\,(\|\partial_2\widetilde{u}\|_{L^2}^2 + \|\partial_3\widetilde{u}\|_{L^2}^2).\end{aligned}$$

Therefore,

$$|L_1| \le C\,\|u_1\|_{H^2}\,(\|\partial_2\widetilde{u}\|_{L^2}^2 + \|\partial_3\widetilde{u}\|_{L^2}^2).$$

Invoking (4.6), we can rewrite L_2 as

$$L_2 = -\int \widetilde{u}_1\partial_2(\bar{u}_1\widetilde{u}_2 + \widetilde{u}_1\bar{u}_2 + \widetilde{\widetilde{u}_1\,\widetilde{u}_2})\,dx.$$

The three terms in L_2 can be estimated similarly to those terms in L_1 and the upper bound is

$$|L_2| \le C\,\|u\|_{H^2}\,(\|\partial_2\widetilde{u}\|_{L^2}^2 + \|\partial_3\widetilde{u}\|_{L^2}^2).$$

By integration by parts,

$$\begin{aligned} L_3 &= -\int \widetilde{u}_3\, u_1\, \partial_3\widetilde{u}_1\, dx \\ &\le \|u_1\|_{L^\infty}\, \|\widetilde{u}_3\|_{L^2}\, \|\partial_3\widetilde{u}_1\|_{L^2} \\ &\le C\,\|u_1\|_{H^2}\, \|\partial_3\widetilde{u}_3\|_{L^2}\, \|\partial_3\widetilde{u}_1\|_{L^2} \\ &\le C\,\|u_1\|_{H^2}\, \|\partial_3\widetilde{u}\|_{L^2}^2. \end{aligned}$$

Similarly we have

$$|L_4|, |L_5|, |L_6| \le C\,\|u\|_{H^2}\,(\|\partial_2\widetilde{u}\|_{L^2}^2 + \|\partial_3\widetilde{u}\|_{L^2}^2).$$

Collecting the upper bounds for L_1 through L_6, we find

$$\frac{d}{dt}\|(\widetilde{u},\widetilde{\theta})\|_{L^2}^2 + 2\nu\|\nabla\widetilde{u}\|_{L^2}^2 + 2\eta\|\partial_3\widetilde{\theta}\|_{L^2}^2 \le C\,\|u\|_{H^2}\,(\|\partial_2\widetilde{u}\|_{L^2}^2 + \|\partial_3\widetilde{u}\|_{L^2}^2).$$

or

$$\frac{d}{dt}\|(\widetilde{u},\widetilde{\theta})\|_{L^2}^2 + (2\nu - C\|u\|_{H^2})\|(\partial_2,\partial_3)\widetilde{u}\|_{L^2}^2 + 2\eta\|\partial_3\widetilde{\theta}\|_{L^2}^2 \le 0.$$

When the initial data (u_0, θ_0) is sufficiently small such that

$$C\|u\|_{H^2} \le \nu,$$

we have

$$\frac{d}{dt}\|(\widetilde{u},\widetilde{\theta})\|_{L^2}^2 + \nu\,\|\partial_3\widetilde{u}\|_{L^2}^2 + 2\eta\|\partial_3\widetilde{\theta}\|_{L^2}^2 \le 0.$$

Invoking the Poincaré inequality in Lemma 2.3,

$$\|\widetilde{u}\|_{L^2} \le C\,\|\partial_3\widetilde{u}\|_{L^2}, \quad \|\widetilde{\theta}\|_{L^2} \le C\,\|\partial_3\widetilde{\theta}\|_{L^2}$$

leads to

$$\frac{d}{dt}\|(\widetilde{u},\widetilde{\theta})\|_{L^2}^2 + C\,\min\{\nu,\eta\}\|(\widetilde{u},\widetilde{\theta})\|_{L^2}^2 \le 0. \tag{4.7}$$

We now estimate the H^1-norm. Taking the gradient of (4.4) and then dotting the resulting equations with $(\nabla\widetilde{u}, \nabla\widetilde{\theta})$, we have

$$\begin{aligned} &\frac{d}{dt}\|(\nabla\widetilde{u},\nabla\widetilde{\theta})\|_{L^2}^2 + 2\nu\|\nabla^2\widetilde{u}\|_{L^2}^2 + 2\eta\|\partial_3\nabla\widetilde{\theta}\|_{L^2}^2 \\ &:= M_1 + \cdots + M_8, \end{aligned}$$

where

$$M_1 = -\int \nabla\widetilde{u}_1\cdot\nabla\partial_1(u_1^2 - \overline{u_1^2})\,dx, \quad M_2 = -\int \nabla\widetilde{u}_1\cdot\nabla\partial_2(u_1u_2 - \overline{u_1u_2})\,dx,$$

$$M_3 = -\int \nabla\widetilde{u}_1\cdot\nabla\partial_3(\widetilde{u}_3u_1)\,dx, \quad M_4 = -\int \nabla\widetilde{u}_2\cdot\nabla\partial_2(u_1u_2 - \overline{u_1u_2})\,dx,$$

$$M_5 = -\int \nabla\widetilde{u}_2\cdot\nabla\partial_2(u_2^2 - \overline{u_2^2})\,dx, \quad M_6 = -\int \nabla\widetilde{u}_2\cdot\nabla\partial_3(\widetilde{u}_3u_2)\,dx,$$

$$M_7 = -\int \nabla\widetilde{u}_3 \cdot \nabla(u\cdot\nabla\widetilde{u}_3)\,dx, \quad M_8 = -\int \nabla\widetilde{\theta}\cdot\nabla(u\cdot\nabla\widetilde{\theta})\,dx.$$

To estimate M_1, we first invoke (4.5) to write M_1 as

$$M_1 = -2\int \nabla\widetilde{u}_1 \cdot \partial_1\nabla(\bar{u}_1\,\widetilde{u}_1)\,dx - \int \nabla\widetilde{u}_1\cdot\partial_1\nabla\widetilde{(\widetilde{u}_1)^2}\,dx := M_{11}+M_{12}.$$

By integration by parts and $\nabla\cdot\widetilde{u}=0$,

$$\begin{aligned} M_{11} &= 2\int \nabla\partial_1\widetilde{u}_1\cdot(\nabla\widetilde{u}_1\,\bar{u}_1 + \widetilde{u}_1\nabla\bar{u}_1)\,dx \\ &= -2\int \nabla(\partial_2\widetilde{u}_2+\partial_3\widetilde{u}_3)\cdot(\nabla\widetilde{u}_1\,\bar{u}_1+\widetilde{u}_1\nabla\bar{u}_1)\,dx. \end{aligned}$$

Noticing that $\bar{u}_1$ is a 2D function independent of x_3, and applying Hölder's inequality and Lemma 2.3, we have

$$\begin{aligned} M_{12} &\le C\,\|(\partial_2,\partial_3)\nabla\widetilde{u}\|_{L^2}\Big(\|\nabla\widetilde{u}_1\|_{L^2}\,\|\bar{u}_1\|_{L^\infty} + \|\widetilde{u}_1\|_{L^4_hL^2_{x_3}}\|\nabla\bar{u}_1\|_{L^4_h}\Big) \\ &\le C\,\|(\partial_2,\partial_3)\nabla\widetilde{u}\|_{L^2}\,\|\nabla\widetilde{u}_1\|_{L^2}\,\|\bar{u}_1\|_{H^2} \\ &\quad +C\,\|(\partial_2,\partial_3)\nabla\widetilde{u}\|_{L^2}\,\|\widetilde{u}_1\|_{L^2}^{\frac12}\|\nabla_h\widetilde{u}_1\|_{L^2}^{\frac12}\|\bar{u}_1\|_{H^2} \\ &\le C\,\|(\partial_2,\partial_3)\nabla\widetilde{u}\|_{L^2}\,\|\partial_3\nabla\widetilde{u}_1\|_{L^2}\,\|\bar{u}_1\|_{H^2} \\ &\quad +C\,\|(\partial_2,\partial_3)\nabla\widetilde{u}\|_{L^2}\,\|\partial_3\partial_3\widetilde{u}_1\|_{L^2}^{\frac12}\|\partial_3\nabla_h\widetilde{u}_1\|_{L^2}^{\frac12}\|\bar{u}_1\|_{H^2}, \end{aligned}$$

where we have used the following Sobolev inequalities

$$\|f\|_{L^4_h} \le C\,\|f\|_{L^2_h}^{\frac12}\,\|\nabla_h f\|_{L^2_h}^{\frac12} \le C\,\|f\|_{H^1_h}.$$

Here $\|f\|_{L^q_h}$ denotes the L^q-norm of f over the horizontal 2D space and $\|f\|_{L^4_hL^2_{x_3}} = \|\|f\|_{L^2_{x_3}}\|_{L^4_h}$. By Lemma 2.1 and then Lemma 2.3,

$$\begin{aligned} |M_{12}| &\le C\,\|\partial_1\nabla\widetilde{u}_1\|_{L^2}\|\nabla\widetilde{(\widetilde{u}_1)^2}\|_{L^2} \\ &\le C\,\|(\partial_2,\partial_3)\nabla\widetilde{u}\|_{L^2}\|\nabla(\widetilde{u}_1)^2\|_{L^2} \\ &\le C\,\|(\partial_2,\partial_3)\nabla\widetilde{u}\|_{L^2}\,\|\nabla\widetilde{u}_1\|_{L^2}\|\widetilde{u}_1\|_{L^\infty} \\ &\le C\,\|(\partial_2,\partial_3)\nabla\widetilde{u}\|_{L^2}\,\|\partial_3\nabla\widetilde{u}_1\|_{L^2}\|u_1\|_{H^2}. \end{aligned}$$

Therefore,

$$|M_1| \le C\,\|u\|_{H^2}\,\||(\partial_2,\partial_3)\nabla\widetilde{u}\|_{L^2}^2.$$

To estimate M_2, we use (4.6) to split M_2 into three terms,

$$M_2 = -\int \nabla\widetilde{u}_1\cdot\nabla\left(\bar{u}_1\,\widetilde{u}_2 + \bar{u}_2\,\widetilde{u}_1 + \widetilde{\widetilde{u}_1\,\widetilde{u}_2}\right)\,dx := M_{21}+M_{22}+M_{23}.$$

These terms can be bounded similarly as M_1. The upper bound for M_2 is

$$|M_2| \le C\,\|u\|_{H^2}\,\||(\partial_2,\partial_3)\nabla\widetilde{u}\|_{L^2}^2.$$

By integrating by parts, applying Sobolev's inequality and Lemma 2.3,

$$\begin{aligned} M_3 &= \int \partial_3\nabla\widetilde{u}_1\cdot\nabla(\widetilde{u}_3u_1)\,dx \\ &\le C\,\|\partial_3\nabla\widetilde{u}_1\|_{L^2}\,\left(\|\nabla\widetilde{u}_3\|_{L^2}\|u_1\|_{L^\infty} + \|\widetilde{u}_3\|_{L^4}\|\nabla u_1\|_{L^4}\right) \end{aligned}$$

$$\begin{aligned}
&\le C\,\|\partial_3\nabla\widetilde{u}_1\|_{L^2}\,\left(\|\nabla\widetilde{u}_3\|_{L^2}\|u_1\|_{H^2}+\|\widetilde{u}_3\|_{H^1}\|\nabla u_1\|_{H^1}\right)\\
&\le C\,\|\partial_3\nabla\widetilde{u}_1\|_{L^2}\,\|\partial_3\nabla\widetilde{u}_3\|_{L^2}\|u_1\|_{H^2}\\
&\le C\,\|u\|_{H^2}\,\|\partial_3\nabla\widetilde{u}\|_{L^2}^2,
\end{aligned}$$

where we have used the inequalities

$$\begin{aligned}
&\|f\|_{L^4}\le C\,\|f\|_{H^1},\\
&\|\widetilde{u}_3\|_{H^1}=\|\widetilde{u}_3\|_{L^2}+\|\nabla\widetilde{u}_3\|_{L^2}\le\|\partial_3\partial_3\widetilde{u}_3\|_{L^2}+\|\partial_3\nabla\widetilde{u}_3\|_{L^2}.
\end{aligned}$$

M_4 can be estimated similarly as M_2, M_5 as M_1 and M_6 as M_3. It remains to bound M_7 and M_8. Because $\nabla\cdot u=0$,

$$M_7=-\sum_{k,m=1}^{3}\int\partial_k\widetilde{u}_3\partial_k u_m\partial_m\widetilde{u}_3\,dx.$$

Again we intend to bound this nonlinear term without using the dissipation in the x_1-direction. We decompose the terms in the summation into several parts,

$$\begin{aligned}
M_7=&-\int\partial_1\widetilde{u}_3\partial_1 u_1\partial_1\widetilde{u}_3\,dx-\sum_{k=2}^{3}\int\partial_k\widetilde{u}_3\partial_k u_1\partial_1\widetilde{u}_3\,dx\\
&-\sum_{k=1}^{3}\sum_{m=2}^{3}\int\partial_k\widetilde{u}_3\partial_k u_m\partial_m\widetilde{u}_3\,dx\\
:=&\,M_{71}+M_{72}+M_{73}.
\end{aligned}$$

By $\nabla\cdot u=0$ or $\partial_1 u_1=-\partial_2 u_2-\partial_3 u_3$ and integrating by parts,

$$\begin{aligned}
M_{71}&=\int(\partial_1\widetilde{u}_3)^2\,(\partial_2 u_2+\partial_3 u_3)\,dx\\
&=-\int u_2\,\partial_1\widetilde{u}_3\,\partial_2\partial_1\widetilde{u}_3\,dx-\int u_3\partial_1\widetilde{u}_3\,\partial_3\partial_1\widetilde{u}_3\,dx.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|M_{71}|&\le\|u_2\|_{L^\infty}\|\partial_1\widetilde{u}_3\|_{L^2}\,\|\partial_2\partial_1\widetilde{u}_3\|_{L^2}+\|u_3\|_{L^\infty}\|\partial_1\widetilde{u}_3\|_{L^2}\,\|\partial_3\partial_1\widetilde{u}_3\|_{L^2}\\
&\le C\,\|u\|_{H^2}\|\|(\partial_2,\partial_3)\nabla\widetilde{u}\|_{L^2}^2.
\end{aligned}$$

By Lemma 2.5 and then Lemma 2.3,

$$\begin{aligned}
|M_{72}|\le&\, C\,\sum_{k=2}^{3}\|\partial_k\widetilde{u}_3\|_{L^2}^{\frac12}\,\|\partial_1\partial_k\widetilde{u}_3\|_{L^2}^{\frac12}\,\|\partial_1\widetilde{u}_3\|_{L^2}^{\frac12}\,\|\partial_2\partial_1\widetilde{u}_3\|_{L^2}^{\frac12}\\
&\times\|\partial_k u_1\|_{L^2}^{\frac12}\,\left(\|\partial_k u_1\|_{L^2}+\|\partial_3\partial_k u_1\|_{L^2}\right)^{\frac12}\\
\le&\, C\,\|u_1\|_{H^2}\,\sum_{k=2}^{3}\|\partial_3\partial_k\widetilde{u}_3\|_{L^2}^{\frac12}\,\|\partial_1\partial_k\widetilde{u}_3\|_{L^2}^{\frac12}\,\|\partial_3\partial_1\widetilde{u}_3\|_{L^2}^{\frac12}\,\|\partial_2\partial_1\widetilde{u}_3\|_{L^2}^{\frac12}\\
\le&\, C\,\|u\|_{H^2}\|\|(\partial_2,\partial_3)\nabla\widetilde{u}\|_{L^2}^2.
\end{aligned}$$

M_{73} can be bounded similarly as M_{72}, and

$$|M_{73}|\le C\,\|u\|_{H^2}\|(\partial_2,\partial_3)\nabla\widetilde{u}\|_{L^2}^2.$$

We now estimate the last term M_8. There is only dissipation in the x_3-direction in the θ-equation, so M_8 is estimated differently. Because $\nabla\cdot u=0$,

$$M_8=-\int\nabla\widetilde{\theta}\cdot\nabla u\cdot\nabla\widetilde{\theta}\,dx.$$

To pinpoint the main difficulty, we decompose M_8 into three parts,

$$\begin{aligned}M_8=&-\sum_{k=1}^{2}\sum_{m=1}^{2}\int\partial_k\widetilde{\theta}\,\partial_k u_m\,\partial_m\widetilde{\theta}\,dx-\sum_{k=1}^{2}\int\partial_k\widetilde{\theta}\,\partial_k u_3\,\partial_3\widetilde{\theta}\,dx\\&-\sum_{m=1}^{2}\int\partial_3\widetilde{\theta}\,\partial_3 u_m\,\partial_m\widetilde{\theta}\,dx-\int\partial_3\widetilde{\theta}\,\partial_3 u_3\,\partial_3\widetilde{\theta}\,dx\\:=&\,M_{81}+M_{82}+M_{83}+M_{84}.\end{aligned}$$

The terms M_{82}, M_{83} and M_{84} all contain at least one $\partial_3\widetilde{\theta}$ and they are relatively easy to estimate. By Lemma 2.5 and then Lemma 2.3,

$$\begin{aligned}|M_{82}|\le&\,C\sum_{k=1}^{2}\|\partial_k\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_3\partial_k\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_k u_3\|_{L^2}^{\frac12}\|\partial_2\partial_k u_3\|_{L^2}^{\frac12}\|\partial_3\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_1\partial_3\widetilde{\theta}\|_{L^2}^{\frac12}\\\le&\,C\,\|u\|_{H^2}\,\|\partial_3\nabla\widetilde{\theta}\|_{L^2}^2.\end{aligned}$$

Similarly,

$$|M_{83}|\le C\,\|u\|_{H^2}\,\|\partial_3\nabla\widetilde{\theta}\|_{L^2}^2,\quad |M_{84}|\le C\,\|u\|_{H^2}\,\|\partial_3\nabla\widetilde{\theta}\|_{L^2}^2.$$

The terms in M_{81} do not contain the favorable derivative $\partial_3\widetilde{\theta}$. We write

$$\partial_k u_m=\partial_k\bar{u}_m+\partial_k\widetilde{u}_m$$

and M_{81} becomes

$$\begin{aligned}M_{81}=&-\sum_{k=1}^{2}\sum_{m=1}^{2}\int\partial_k\widetilde{\theta}\,\partial_k\widetilde{u}_m\,\partial_m\widetilde{\theta}\,dx-\sum_{k=1}^{2}\sum_{m=1}^{2}\int\partial_k\widetilde{\theta}\,\partial_k\bar{u}_m\,\partial_m\widetilde{\theta}\,dx\\:=&\,M_{811}+M_{812}.\end{aligned}$$

By Lemma 2.5 and then Lemma 2.3,

$$\begin{aligned}|M_{811}|\le&\,C\sum_{k=1}^{2}\sum_{m=1}^{2}\|\partial_k\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_3\partial_k\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_k\widetilde{u}_m\|_{L^2}^{\frac12}\|\partial_2\partial_k\widetilde{u}_m\|_{L^2}^{\frac12}\\&\times\|\partial_m\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_1\partial_m\widetilde{\theta}\|_{L^2}^{\frac12}\\\le&\,C\sum_{k=1}^{2}\sum_{m=1}^{2}\|\partial_3\partial_k\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_3\partial_k\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_k\widetilde{u}_m\|_{L^2}^{\frac12}\|\partial_2\partial_k\widetilde{u}_m\|_{L^2}^{\frac12}\\&\times\|\partial_3\partial_m\widetilde{\theta}\|_{L^2}^{\frac12}\|\partial_1\partial_m\widetilde{\theta}\|_{L^2}^{\frac12}\\\le&\,C\,\|\nabla u\|_{L^2}^{\frac12}\|\widetilde{\theta}\|_{H^2}^{\frac12}\|\partial_3\nabla\widetilde{\theta}\|_{L^2}^{\frac32}\|\partial_2\nabla\widetilde{u}\|_{L^2}^{\frac12}\\\le&\,C\,(\|u\|_{H^2}+\|\theta\|_{H^2})\,(\|\partial_2\nabla\widetilde{u}\|_{L^2}^2+\|\partial_3\nabla\widetilde{\theta}\|_{L^2}^2).\end{aligned}$$

M_{812} has to be estimated differently. Since $\bar{u}_m$ is only a function of x_1 and x_2,

$$\begin{aligned}|M_{812}| &\le \sum_{k=1}^{2}\sum_{m=1}^{2} \|\partial_k \bar{u}_m\|_{L_h^\infty} \|\partial_k \widetilde{\theta}\|_{L^2} \|\partial_m \widetilde{\theta}\|_{L^2} \\ &\le \sum_{k=1}^{2}\sum_{m=1}^{2} \|\partial_k \bar{u}_m\|_{H^2} \|\partial_3\partial_k \widetilde{\theta}\|_{L^2} \|\partial_3\partial_m \widetilde{\theta}\|_{L^2} \\ &\le C\,\|u\|_{H^3}\,\|\partial_3 \nabla \widetilde{\theta}\|_{L^2}^2.\end{aligned}$$

It is this last term that needs the H^3-norm of u. The other upper bounds only involve $\|u\|_{H^2}$-norm. Putting together the bounds for M_1 through M_8, we obtain

$$\begin{aligned}&\frac{d}{dt}\|(\nabla\widetilde{u}, \nabla\widetilde{\theta})\|_{L^2}^2 + 2\nu\|\partial_1\nabla\widetilde{u}\|_{L^2}^2 \\ &\quad + (2\nu - C_1(\|u\|_{H^2} + \|\theta\|_{H^2}))\,\|(\partial_2,\partial_3)\nabla\widetilde{u}\|_{L^2}^2 \\ &\quad + (2\eta - C_2(\|u\|_{H^3} + \|\theta\|_{H^2}))\,\|\partial_3\nabla\widetilde{\theta}\|_{L^2}^2 \le 0.\end{aligned}$$

When the initial data $(u_0, \theta_0) \in H^3$ is sufficiently small such that

$$C_1(\|u\|_{H^2} + \|\theta\|_{H^2}) \le \nu, \quad C_2(\|u\|_{H^3} + \|\theta\|_{H^2}) \le \eta,$$

we have

$$\frac{d}{dt}\|(\nabla\widetilde{u}, \nabla\widetilde{\theta})\|_{L^2}^2 + 2\nu\|\partial_1\nabla\widetilde{u}\|_{L^2}^2 + \nu\|(\partial_2,\partial_3)\nabla\widetilde{u}\|_{L^2}^2 + \eta\|\partial_3\nabla\widetilde{\theta}\|_{L^2}^2 \le 0.$$

Invoking the Poincaré inequalities in Lemma 2.3,

$$\|\nabla\widetilde{u}\|_{L^2} \le C\,\|\partial_3\nabla\widetilde{u}\|_{L^2}, \quad \|\nabla\widetilde{\theta}\|_{L^2} \le C\,\|\partial_3\nabla\widetilde{\theta}\|_{L^2}$$

leads to

$$\frac{d}{dt}\|(\nabla\widetilde{u}, \nabla\widetilde{\theta})\|_{L^2}^2 + C\,\min\{\nu,\eta\}\|(\nabla\widetilde{u}, \nabla\widetilde{\theta})\|_{L^2}^2 \le 0. \tag{4.8}$$

(4.7) and (4.8) then imply (1.9). This completes the proof of Theorem 1.2. □

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[1] Department of Mathematics, Oklahoma State University, Stillwater, OK 74078, United States

Email address: jiahong.wu@okstate.edu

[2] Hebei Key Laboratory of Machine Learning and Computational Intelligence, School of Mathematics and Information Science, Hebei University, Baoding, 071002, P.R. China

Email address: zhangqian@hbu.edu.cn