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Distributed Kalman filtering for spatially-invariant diffusion processes: the effect of noise on communication requirements

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Abstract— **This work analyzes the communication requirements of Kalman filters for spatially-invariant diffusion processes with spatially-distributed sensing. In this setting Kalman filters exhibit an inherent degree of spatial localization or decentralization. We address the fundamental question of whether the statistical properties of process and measurement disturbances, namely variance and spatial-autocorrelations, can further enhance its inherent spatial localization. We show that when disturbances are spatially and temporally uncorrelated, the spatial localization of the filter depends on the ratio of model to measurement error. Building upon this result, we study exponentially-decaying spatially-autocorrelated process and measurement disturbances. We show that certain level of spatial-autocorrelation in the measurement noise reduces the communication burden of the Kalman filter: indeed, the filter is completely decentralized when a matching condition is satisfied. We also show that spatial autocorrelation of the process disturbance has no benefits in terms of communications, as the level of centralization of the filter increases with the autocorrelation length.**

I. INTRODUCTION

Optimal control and estimation of distributed parameter systems (DPSs) have received significant attention from the research community during the last decades (see [1]–[4] for some well-known monographs). For the class of linear spatially-invariant DPSs with distributed actuation, [5]–[7] proved that quadratically-optimal controllers are spatially-invariant and exhibit an *inherent degree of spatial localization*. This result was later extended in [8] for the more general class of spatially-decaying operators. These theoretical developments find immediate application in the design of decentralized controllers, since they imply that the contribution of measurements from "far-away" is negligible for optimal control as the size of the feedback decays with distance. Hence, spatial truncation of the kernel of the feedback operator is a valid and *scalable* approach to imposing local communication constraints on the structure of the controller.

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We analyze the communication requirements of the Kalman-Bucy filter (KBF) for spatially-invariant diffusion processes with distributed sensing. By duality between the LQR and MMSE estimation, the KBF is spatially localized [5] and amenable to a distributed implementation [9]. We aim to gain insight into the effect that noise statistics have on the spatial localization of the KBF, with special emphasis on the role of the spatial-autocorrelation length. Related works are e.g., [10] and [11], which consider spatially-correlated noise in optimal sensor selection and placement for DPSs.

The contributions of this work include the characterization of the spatial decay rate of the kernel of the KBF operator for three different types of disturbances: i) spatio-temporally white process and measurement noises, ii) spatially-autocorrelated measurement noise, and iii) spatially-autocorrelated process noise. The spatial-autocorrelations are assumed known a priori and characterized by an exponential decay, their decay rate being determined by the spatial-autocorrelation length. We show that good measurement fidelity, high-variance exogenous input disturbances and low diffusivity foster decentralization of the KBF. Interestingly, so does certain level of spatial-autocorrelation in the measurement noise. Indeed, the KBF is totally decentralized when a matching condition related to the statistical properties of the disturbances is satisfied. These results imply that the performance loss in a locally-constrained KBF depends upon the statistical properties of the noises. Hence, disturbances with certain statistics are more amenable than others to decentralized KBF architectures. The branch points of the analytic extension of the Fourier symbol of the KBF operator determine the degree of decentralization of the filter. We introduce the *branch-point locus* (BPL), the set of branch points parameterized by a certain system parameter, as a useful tool to identify the communication burden of the KBF. Similarly to the root locus, the graphical representation of the BPL in the complex plane allows to visually determine critical values of the parameter of interest.

Our paper is organized as follows. Section II introduces mathematical preliminaries. Section III presents the system and modeling assumptions. Section IV formulates the optimal filtering problem. The spatial-invariance property is exploited to obtain a parameterized family of decoupled Algebraic Riccati equations (ARE) in the spatial-frequency domain. Section V evaluates the role of diffusivity and noise variances in the decentralization of the KBF for spatio-temporally white disturbances. In Section VI the effect of spatial-autocorrelations is analyzed. Finally, in Section VII we draw conclusions and discuss ongoing research.

II. MATHEMATICAL PRELIMINARIES

The domain of an operator A is denoted by $\mathcal{D}(A)$. $L_2(\mathbb{R})$ represents the space of square-integrable functions in the real line $\mathbb{R}$. $\langle f, g\rangle$ with $f, g \in L_2(\mathbb{R})$ denotes the inner product in $L_2(\mathbb{R})$. The spatial Fourier transform of a function h is denoted by $\hat{h}_\lambda$, where $\lambda \in \mathbb{R}$ is the spatial frequency. $\delta(x,t)$ represents a spatio-temporal delta function. Spatial convolutions are denoted by $(f \star g)(x,t) := \int_{-\infty}^{\infty} f(x-\xi)g(\xi,t)\mathrm{d}\xi$, where $f(x)$ is the *convolution kernel*.

Definition 1. Let T_x denote a translation operator for functions on $\mathbb{R}$: given any $x \in \mathbb{R}$, $(T_x f)(y) := f(y-x)$. An operator A is *translation invariant* if $T_x : \mathcal{D}(A) \to \mathcal{D}(A)$ and $AT_x = T_x A$, for every translation T_x. Since this work is concerned with shift-invariance in space, we will use the terms *translation invariance* and *spatial invariance* interchangeably.

Definition 2. Let $A(x)$ with $x \in \mathbb{R}$ be a measurable function. A *multiplication operator* M_A is defined by $(M_A f)(x) := A(x)f(x)$, $\forall f \in \mathcal{D}(M_A)$. The function $A(x)$ is called the *symbol* of the multiplication operator M_A.

Definition 3. The *analytic extension* or *analytic continuation* of the Fourier symbol $\hat{f}_\lambda$, denoted by $\hat{f}_\sigma$, to a subset of the complex plane is defined by $\hat{f}_\sigma(\lambda j) = \hat{f}_\lambda$. If the analytic extension exists, it is unique.

III. DIFFUSION DYNAMICS

We consider a one-dimensional diffusion process in an infinite domain with spatially distributed measurement $y(x,t)$. Dynamics and measurement are both subject to spatio-temporal disturbances $w(x,t)$ and $v(x,t)$, respectively. The state $\psi(x,t)$ obeys the PDE:

$$\frac{\partial \psi}{\partial t}(x,t) = \kappa \frac{\partial^2 \psi}{\partial x^2}(x,t) + w(x,t), \tag{1}$$

$$y(x,t) = c\psi(x,t) + v(x,t), \tag{2}$$

where κ is the diffusivity constant and c is a positive scaling constant. Without loss of generality, the initial condition is taken to be $\psi(x,0) = 0$. We note that the Laplacian operator $\frac{\partial^2}{\partial x^2}$ is translation invariant. The noises $w(x,t)$ and $v(x,t)$ are assumed to be mutually uncorrelated, zero mean, and wide-sense stationary, with their autocorrelations defined as:

$$W(\Delta x, \Delta t) := \mathbb{E}[w(x,t)w(x+\Delta x, t+\Delta t)], \tag{3}$$

$$V(\Delta x, \Delta t) := \mathbb{E}[v(x,t)v(x+\Delta x, t+\Delta t)]. \tag{4}$$

We study spatially-colored disturbances, yet temporally-white. Particularizations of (3)-(4) are provided in Sections V and VI. The Fourier transforms of (3)-(4) are assumed well-defined and the power spectral density (PSD) of $v(x,t)$ strictly positive at all frequencies, $\hat{V}_\lambda > 0$.

IV. PROBLEM FORMULATION

The objective is to find the state-estimator $\tilde{\psi}^*(x,t)$ minimizer of the steady-state variance of the estimation error:

$$\tilde{\psi}^* := \underset{\tilde{\psi}}{\operatorname{argmin}} \lim_{t\to\infty} \mathbb{E}\big[\big(\psi(x,t) - \tilde{\psi}(x,t)\big)^2\big], \tag{5}$$

subject to the dynamics (1), measurements (2), and boundary and initial conditions as described in Section III. The KBF provides the optimal state-estimator for this problem [2], [12], [13]. Under the assumptions of Section III, the system and cost are translation invariant. Hence, application of the spatial Fourier transform diagonalizes the problem, as it transforms spatially-invariant operators into multiplication operators in the frequency domain [5], [14]. By the Wiener-Khinchin theorem the cost functional is:

$$\lim_{t\to\infty} \int_{-\infty}^{\infty} \hat{P}_\lambda \mathrm{d}\lambda, \tag{6}$$

where $\hat{P}_\lambda$ denotes the PSD of the estimation error. Since the problem is *decoupled in* λ its solution is obtained by solving infinitely many finite-dimensional AREs parameterized by the spatial frequency $\lambda \in \mathbb{R}$. For the diffusion dynamics (1) and autocorrelations (3)-(4) the parameterized scalar ARE and its stabilizing solution are:

$$\hat{P}_\lambda^2 + 2\frac{\kappa}{c^2}\lambda^2 \hat{V}_\lambda \hat{P}_\lambda - \frac{1}{c^2}\hat{V}_\lambda \hat{W}_\lambda = 0, \tag{7}$$

$$\hat{P}_\lambda = -\frac{\kappa}{c^2}\lambda^2 \hat{V}_\lambda + \sqrt{\frac{1}{c^2}\left(\frac{\kappa^2}{c^2}\lambda^4 \hat{V}_\lambda^2 + \hat{V}_\lambda \hat{W}_\lambda\right)}. \tag{8}$$

The corresponding Fourier symbol of the KBF operator is:

$$\hat{L}_\lambda = -\frac{\kappa}{c}\lambda^2 + \sqrt{\frac{\kappa^2}{c^2}\lambda^4 + \frac{\hat{W}_\lambda}{\hat{V}_\lambda}}. \tag{9}$$

By the convolution theorem, the optimal state-estimate $\tilde{\psi}^*$ evolves according to the following integro-differential equation:

$$\frac{\partial \tilde{\psi}^*}{\partial t}(x,t) = \underbrace{\kappa \frac{\partial^2 \tilde{\psi}^*}{\partial x^2}(x,t)}_{\text{prediction}} + \underbrace{\big(L \star (y - c\,\tilde{\psi}^*)\big)(x,t)}_{\text{correction}}. \tag{10}$$

The difference $y - c\,\tilde{\psi}^*$ is the *measurement innovation* or the *residual* [15]. From (10) we observe that the *spatial spread of the convolution kernel* $L(x)$ determines the communication requirements of the KBF, as it defines to which degree measurements from far away sensors are required to update the state-estimate. We will study the spatial decay rate of $L(x)$ through its Fourier transform. $\hat{L}_\lambda$ has an analytic extension $\hat{L}_\sigma$, the region of analyticity being a strip in the complex plane along the imaginary axis $\Gamma_\mathrm{L} + j\mathbb{R} = \{\sigma \in \mathbb{C} \mid \Re(\sigma) \in \Gamma_\mathrm{L}\}$, and $\hat{L}_\sigma$ satisfies the growth bound required by the Paley-Wiener Theorem 5 in [5]. Consequently, $L(x)$ decays exponentially in space and satisfies

$$|L(x)|e^{\xi|x|} \xrightarrow{|x|\to\infty} 0, \text{ for } 0 < \xi < \eta, \tag{11}$$

where η is defined as $\Gamma_\mathrm{L} =: (-\eta, \eta)$. We will refer to η as the spatial decay rate of $L(x)$: the larger η, the faster $L(x)$ decays and the more spatially localized the KBF is. The *branch point* of the analytic extension $\hat{L}_\sigma$ of (9) whose real part has the smallest absolute value determines η. $\hat{L}_\sigma$ will consist of two terms, a polynomial and a square root. Polynomials are entire functions. Thus, the branch points of $\hat{L}_\sigma$ are related to the square root, and must make its radicand vanish or go to infinity. Consequently, the degree of decentralization of the KBF depends upon the diffusivity constant and the *ratio* of the PSDs of process and measurement noises.

V. ANALYSIS: SPATIALLY-UNCORRELATED NOISE AND DECENTRALIZATION

In this section, we study the spatial decay properties of $L(x)$ when process and measurement disturbances are white noises with autocorrelations $W(\Delta x, \Delta t) := \sigma_w^2\, \delta(\Delta x, \Delta t)$ and $V(\Delta x, \Delta t) := \sigma_v^2 \delta(\Delta x, \Delta t)$, where σ_w^2 and σ_v^2 are finite constants representing their respective PSDs. We define the *characteristic lengthscale* l_* of the KBF dynamics:

$$l_* := \left(2\frac{\kappa\,\sigma_v}{c\,\sigma_w}\right)^{\frac{1}{2}}. \qquad (12)$$

The ratio σ_v/c is the inverse of measurement fidelity and σ_w/κ represents model error. Consequently, l_* can be interpreted as a metric for relative measurement to model distrust. We will show next that the decay rate of $L(x)$ is determined by l_*. Define the analytic extension of (9) $\hat{L}_\sigma$:

$$\hat{L}_\sigma := \frac{\kappa}{c}\sigma^2 + \sqrt{\frac{\kappa^2}{c^2}\,\sigma^4 + \frac{\sigma_w^2}{\sigma_v^2}}, \;\text{ with }\; \sigma \in \mathbb{C}. \qquad (13)$$

Its branch points are given by $|\sigma_\infty| = \infty$ and the roots:

$$\frac{\kappa^2}{c^2}\,\sigma^4 + \frac{\sigma_w^2}{\sigma_v^2} = 0 \;\rightarrow\; \sigma_n = \frac{\sqrt{2}}{l_*}\, e^{(2n-1)\frac{\pi}{4}j}, \qquad (14)$$

with $n = 1, 2, 3, 4$. Consequently, the region Γ_u of the complex plane in which (13) is analytic is defined by:

$$\Gamma_u := \{\sigma \in \mathbb{C} : |\Re(\sigma)| < \frac{1}{l_*} =: \eta_u\}, \qquad (15)$$

where η_u is the spatial decay rate of the convolution kernel $L(x)$. Fig. 1 represents the branch points (14), branch cuts, analyticity strip (15) and convolution kernel of the KBF.

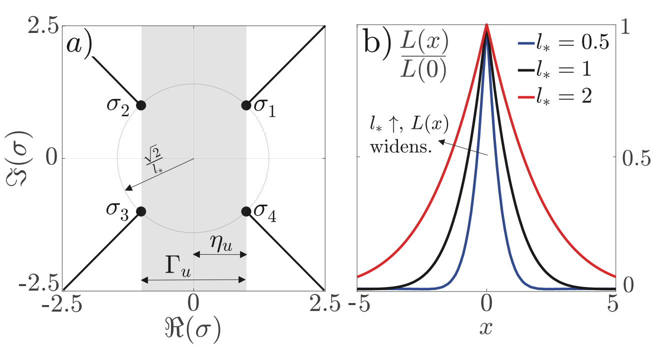

Fig. 1: a) Branch points, branch cuts and analyticity strip of $\hat{L}_\sigma$ for $l_* = 1$. The branch points $\sigma_{1,2,3,4}$ are represented in black circles. A possible set of branch cuts is given by the black lines. The analyticity strip $\Gamma_u + j\mathbb{R}$ is shaded in grey. b) Normalized convolution kernels $L(x)/L(0)$ of the KBF for different values of l_* as indicated in the legend.

The spatial decay rate η_u of the convolution kernel $L(x)$ is strictly decreasing in the characteristic lengthscale l_*. This implies that low values of l_* are preferred in order to foster decentralization and reduce the communication burden of the KBF. These might be achieved by decreasing the diffusivity constant κ (exploiting the physics of the system such that its response remains spatially-localized) or by improving the fidelity of the measurements. Hence, improving the quality of the sensing does not only yield a better performance (6) of the KBF, but also a filter with enhanced spatial localization. When σ_w is high, the KBF also relies on local measurements. However, large model error degrades the quality of the state-estimate.

Remark 1. By duality between the LQR and MMSE estimation, similar results hold for the decentralization of optimal controllers [16]. Consider the LQR problem with spatially-distributed actuation $u(x, t)$. The diffusion dynamics are:

$$\frac{\partial\psi}{\partial t}(x, t) = \kappa\frac{\partial^2\psi}{\partial x^2}(x, t) + b\, u(x, t), \qquad (16)$$

with $u(\cdot, t)$ and $\psi(\cdot, t) \in L_2(\mathbb{R})$. The objective is to find the controller $u^*(x, t)$ that minimizes the functional

$$J = \int_0^\infty \big(\langle q\psi, \psi\rangle + \langle ru, u\rangle\big)\mathrm{d}t, \qquad (17)$$

subject to (16) and where q and r are positive constants. The optimal controller is $u^*(x, t) = -(K \star \psi)(x, t)$, where $K(x)$ decays exponentially in space with a rate:

$$\eta_c = \sqrt{\frac{1}{2}\frac{b}{\kappa}\left(\frac{q}{r}\right)^{\frac{1}{2}}}. \qquad (18)$$

The decay rate (18) suggests a trade-off between actuator authority and controller centralization: as actuation becomes cheaper the controller decentralization increases [5]. The ratio $b\big(\frac{q}{r}\big)^{\frac{1}{2}}$ in the LQR is equivalent to $c\frac{\sigma_w}{\sigma_v}$ in the KBF.

VI. ANALYSIS: SPATIALLY-AUTOCORRELATED NOISE AND DECENTRALIZATION

Biological and physical diffusion processes exhibit complicated spatial variabilities. For stationary physical fields, these can be modeled by spatial correlations [17], [18]. To study the role of spatial autocorrelations of the disturbances on the localization of the KBF, we consider exponentially decaying spatial autocorrelations of Ornstein-Uhlenbeck type:

$$f(\Delta x) := \frac{\sigma_f^2}{2\, l_c} e^{-\frac{|\Delta x|}{l_c}}. \qquad (19)$$

Exponentials posses a characteristic lengthscale l_c that we will refer to as the *spatial autocorrelation length* or the *characteristic lengthscale of the noise*. When l_c is large, noise autocorrelates over a long distance; when $l_c \rightarrow 0^+$, spatially-white noise is recovered. [19] showed that autocorrelations tune the topology of optimal noise-canceling networks. Similarly, we expect l_c to heavily influence the spatial localization properties of the KBF.

We scale the spatial frequency and kernel decay rate, and denote them by $\tilde{\sigma} := \sigma\, l_*$ and $\tilde{\eta} := \eta\, l_*$, respectively. This scaling is convenient to ease interpretation and visualization of the results in the next subsections, as it allows to write them as a function of a single parameter $\gamma := \frac{l_c}{l_*}$. The decay rate obtained in the previous section is $\tilde{\eta}_u = 1$. We analyze next the effect of spatially-autocorrelated measurement and process noise independently.

A. Spatially-autocorrelated measurement noise

The process disturbance $w(x,t)$ is spatio-temporal white noise, while the measurement disturbance $v(x,t)$ is spatially-autocorrelated. Their autocorrelations are:

$$W(\Delta x, \Delta t) := \sigma_w^2\, \delta(\Delta x, \Delta t), \tag{20}$$

$$V(\Delta x, \Delta t) \;:= \frac{\sigma_v^2}{2\, l_c} e^{-\frac{|\Delta x|}{l_c}} \delta(\Delta t), \tag{21}$$

respectively. Taking the spatial Fourier transform of (20)-(21) and substituting in (9) yields the symbol of the KBF operator:

$$\hat{L}_\lambda = -\frac{\kappa}{c}\lambda^2 + \sqrt{\frac{\kappa^2}{c^2}\lambda^4 + \frac{\sigma_w^2}{\sigma_v^2}\, l_c^2\, \lambda^2 + \frac{\sigma_w^2}{\sigma_v^2}}. \tag{22}$$

Define its analytic extension as:

$$\hat{L}_\sigma := \frac{\kappa}{c}\sigma^2 + \sqrt{\frac{\kappa^2}{c^2}\sigma^4 - \frac{\sigma_w^2}{\sigma_v^2}\, l_c^2\, \sigma^2 + \frac{\sigma_w^2}{\sigma_v^2}}, \text{ with } \sigma \in \mathbb{C}. \tag{23}$$

$|\sigma_\infty| = \infty$ are branch points of (23). The remaining branch points must satisfy:

$$\frac{\kappa^2}{c^2}\sigma^4 - \frac{\sigma_w^2}{\sigma_v^2}\, l_c^2\, \sigma^2 + \frac{\sigma_w^2}{\sigma_v^2} = 0, \tag{24}$$

or equivalently in $\tilde{\sigma}$:

$$\tilde{\sigma}^4 - 4\gamma^2\, \tilde{\sigma}^2 + 4 = 0, \text{ where } \gamma := \frac{l_c}{l_*}. \tag{25}$$

The nature of the roots depends on the value of γ, the critical value being $\gamma^* = 1$ at which they transition from complex conjugates to real (Appendix A). The branch points are:

$$\tilde{\sigma}_{1,2,3,4} = \begin{cases} \pm\sqrt{1+\gamma^2} \pm j\sqrt{1-\gamma^2}, & \text{if } 0 \le \gamma < 1, \\ \pm\sqrt{2(\gamma^2 \pm \sqrt{\gamma^4 - 1})}, & \text{if } \gamma > 1. \end{cases} \tag{26}$$

A graphical representation of the branch points (26), branch cuts, and analyticity region of (23) is provided in Fig. 2a). Fig. 2a$_4$) shows the *branch-point locus* (BPL), the trajectories of the branch points in the complex plane as γ is varied. These trajectories help to visually identify the values of γ that yield a highly spatially-localized KBF. The spatial decay rate $\tilde{\eta}_m$ of the convolution kernel (22) is:

$$\tilde{\eta}_m := \begin{cases} \sqrt{1+\gamma^2}, & \text{if } 0 \le \gamma < 1, \\ \sqrt{2(\gamma^2 - \sqrt{\gamma^4 - 1})}, & \text{if } \gamma > 1. \end{cases} \tag{27}$$

(27) shows that $\tilde{\eta}_m > 1 = \tilde{\eta}_u$ for certain values of γ: there exists a range of autocorrelation length values for which spatially autocorrelated measurement noise fosters decentralization of the KBF. Fig. 2c) shows the spatial decay rate $\tilde{\eta}_m$ of the convolution kernel as a function of γ, plotted together with $\tilde{\eta}_u$ for comparison.

The critical value $\gamma^* = 1$ represents a *matching condition*, i.e., $l_* = l_c$. When this matching condition is satisfied, the Fourier symbol of the KBF operator $\hat{L}_\lambda$ is *analytic in the whole complex plane*, see Fig. 2a$_2$). Furthermore, $\hat{L}_\lambda$ takes a constant value, which implies that the convolution kernel $L(x)$ is a delta function: the KBF is *totally decentralized*.

The non-monotonic behavior of the spatial decay rate of the kernel $\tilde{\eta}_m$ with the parameter γ (see Fig. 2c) admits physical interpretation. When $\gamma \lll 1$ the KBF trusts the prediction (model for diffusion) and l_* sets the lengthscale of useful measurements for the filter: note from (27) that for $\gamma \lll 1$, $\eta_m \sim 1/l_*$. At the critical value $\gamma^* = 1$ both, the prediction and correction terms in (10) are relevant for the update of the state-estimate and the KBF is totally decentralized. When $\gamma \ggg 1$ the correction term (measurement innovation) is trusted more than the prediction and l_c defines the lengthscale of useful measurements in this regime: note again from (27) that for $\gamma \ggg 1$, $\eta_m \sim 1/l_c$. Thus, the centralization of the filter increases with l_c in this regime, as correlations carry information that the filter can exploit to estimate the state more accurately.

B. Spatially-autocorrelated process noise

Process noise $w(x,t)$ is assumed spatially-autocorrelated, while the measurement disturbance $v(x,t)$ is spatio-temporal white noise. Their autocorrelations are given by:

$$W(\Delta x, \Delta t) := \frac{\sigma_w^2}{2\, l_c} e^{-\frac{|\Delta x|}{l_c}}\, \delta(\Delta t), \tag{28}$$

$$V(\Delta x, \Delta t) \;:= \sigma_v^2\, \delta(\Delta x, \Delta t), \tag{29}$$

respectively. Take Fourier transforms of (28)-(29) and substitute in (9) to obtain:

$$\hat{L}_\lambda = -\frac{\kappa}{c}\lambda^2 + \sqrt{\frac{\kappa^2}{c^2}\lambda^4 + \frac{\sigma_w^2}{\sigma_v^2}\frac{1}{1 + l_c^2\lambda^2}}, \tag{30}$$

and define its analytic extension $\hat{L}_\sigma$ as:

$$\hat{L}_\sigma := \frac{\kappa}{c}\sigma^2 + \sqrt{\frac{\kappa^2}{c^2}\sigma^4 + \frac{\sigma_w^2}{\sigma_v^2}\frac{1}{1 - l_c^2\sigma^2}} \text{ with } \sigma \in \mathbb{C}. \tag{31}$$

$|\tilde{\sigma}_\infty| = \infty$ make the radicand in (31) go to infinity. So do the poles $\tilde{\sigma}_{1,2} = \pm\gamma^{-1}$. The remaining branch points satisfy:

$$-\gamma^2\tilde{\sigma}^6 + \tilde{\sigma}^4 + 4 = 0, \text{ where } \gamma := \frac{l_c}{l_*}. \tag{32}$$

For notational convenience, we define:

$$\alpha := \left(1 + 54\gamma^4 + 2\gamma^2\sqrt{27(1 + 27\gamma^4)}\right)^{\frac{1}{3}}. \tag{33}$$

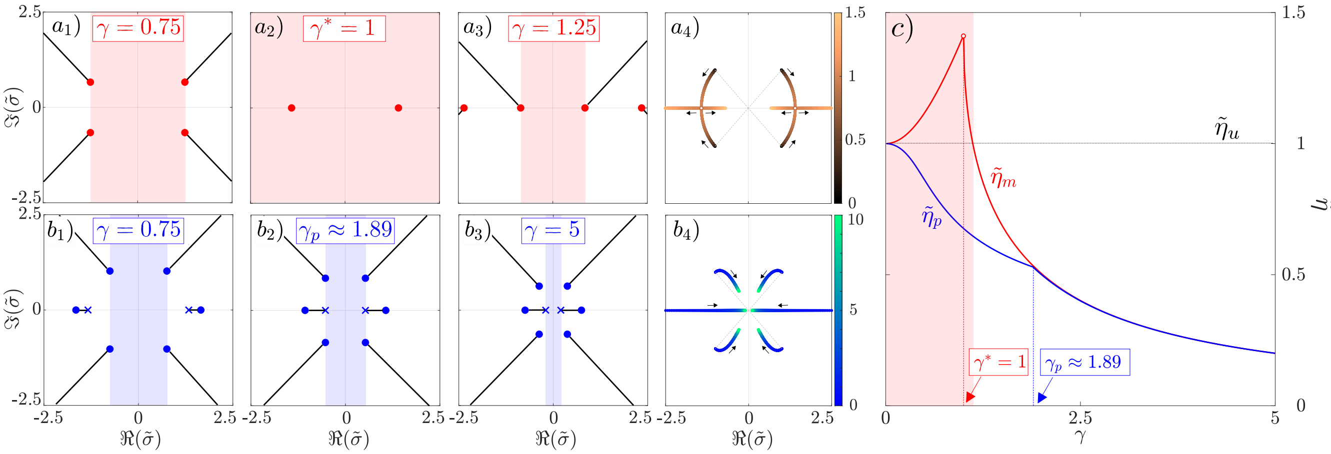

Fig. 2: a_1-b_4) Branch points (BPs), branch cuts and *branch-point locus* in the complex plane $\tilde{\sigma} := \sigma\, l_*$ for increasing $\gamma := l_c/l_*$. Shaded strips represent the analyticity region of $\hat{L}_{\tilde{\sigma}}$. Possible sets of branch cuts are provided in black. a) (red) Spatially-autocorrelated measurement noise. a_1) $\gamma = 0.75 < \gamma^*$, BPs are complex conjugates. a_2) $\gamma = \gamma^* = 1$, roots are real and collapse pairwise. The convolution kernel is analytic in the whole complex plane and the KBF totally decentralized. a_3) $\gamma = 1.25 > \gamma^*$, BPs are real. b) (blue) Idem for spatially-autocorrelated process noise. The zeros of the radicand in $\hat{L}_\sigma$ are represented with circles, and its poles with crosses. b_1) $\gamma = 0.75 < \gamma_p$, the complex BPs define the analyticity strip. b_2) $\gamma = \gamma_p \approx 1.89$, the real part of the complex BPs matches the poles. b_3) $\gamma = 5 > \gamma_p$, the poles define the region of analyticity. a_4) and b_4) *Branch-point locus* for spatially-autocorrelated measurement and process noise, respectively. BPs are plotted as γ is varied according to the respective colorbar. Black arrows indicate the direction of increasing γ. c) Spatial decay rates: $\tilde{\eta}_u = 1$ spatially-uncorrelated process and measurement noise, $\tilde{\eta}_m$ spatially-autocorrelated measurement noise, and $\tilde{\eta}_p$ spatially-autocorrelated process noise. The area shaded in red corresponds to measurement noise autocorrelation lengths that enhance the decentralization of the KBF.

The solutions to (32) are (Appendix B):

$$\tilde{\sigma}_{3,4} = \pm \frac{1}{\sqrt{3}}\gamma^{-1}\sqrt{1+\alpha+\alpha^{-1}}, \tag{34}$$

$$\tilde{\sigma}_{5,6,7,8} = \pm\frac{1}{2}\gamma^{-1}\frac{\alpha-\alpha^{-1}}{\sqrt{\alpha^{-1}+\alpha-2+2\sqrt{\alpha^2+\alpha^{-2}-\alpha-\alpha^{-1}}}}$$
$$\pm\frac{j}{2\sqrt{3}}\gamma^{-1}\sqrt{\alpha^{-1}+\alpha-2+2\sqrt{\alpha^2+\alpha^{-2}-\alpha-\alpha^{-1}}}. \tag{35}$$

Fig. 2b) provides a representation of the branch points, branch cuts and analyticity region of $\hat{L}_\sigma$ as a function of γ. The decay rate $\tilde{\eta}_p$ of the convolution kernel is determined by the complex branch points when $0 \le \gamma < \gamma_p \approx 1.89$ or by the poles otherwise (Appendix B):

$$\tilde{\eta}_p := \begin{cases} \frac{1}{2}\frac{\gamma^{-1}(\alpha-\alpha^{-1})}{\sqrt{\alpha^{-1}+\alpha-2+2\sqrt{\alpha^2+\alpha^{-2}-\alpha-\alpha^{-1}}}}, & \text{if } 0 \le \gamma < \gamma_p, \\ \gamma^{-1}, & \text{if } \gamma \ge \gamma_p. \end{cases} \tag{36}$$

$\tilde{\eta}_p$ is monotonically decreasing with γ, see Fig. 2c). We conclude that spatially-autocorrelated process noise is not beneficial to reduce the communication burden of the KBF: indeed, the longer the autocorrelation length, the higher the communication requirements. This result is particularly relevant when the design of a decentralized KBF is to be performed through spatial truncation of the convolution kernel. If the analysis assumes white noise while disturbances are spatially-autocorrelated, an excessive amount of spatial "clipping" might be carried out, considerably degrading the performance of the filter.

VII. CONCLUSIONS AND FUTURE DIRECTIONS

In this work we have investigated the spatial localization properties of the KBF for spatially-invariant diffusion processes, under different process and measurement disturbances. First, we modeled disturbances as spatio-temporal white noise and analyzed the spatial decay rate of the convolution kernel of the Kalman operator. We found that the communication requirements depend upon l_*, which represents the ratio between measurement and model inaccuracy. Invoking duality between the LQR and MMSE estimation, we extended these results to quadratically-optimal controllers and identified the role that cost hyperparameters play in their decentralization. Second, we focused on spatially-autocorrelated noises with exponentially-decaying autocorrelations. We showed that the spatial autocorrelation length impacts the degree of decentralization of the KBF: while exponentially-decaying spatial autocorrelations in measurement disturbances can foster decentralized filter architectures, spatially-autocorrelated process noise has the opposite effect. We also showed that when a matching condition is satisfied, the KBF becomes totally decentralized. Hence, the main conclusion of the study is that in addition to determining whether the prediction or correction term is trusted more for the state-estimate update, the relative trust on model and measurements together with spatial autocorrelations of the disturbances define the degree of decentralization of the filter in spatially-invariant diffusion processes.

Ongoing work includes studying the role that noise statistics play in the performace-decentralization trade-off of Kalman-Bucy filters for general spatially-invariant operators. The ultimate goal is to characterize the properties of disturbances that make the plant amenable to decentralized KBF architectures with little to no performance loss.

APPENDIX A

We compute the roots of (25). Since (25) is a biquadratic polynomial, define $\tilde{\epsilon} := \tilde{\sigma}^2$ and solve $\tilde{\epsilon}^2 - 4\gamma^2\tilde{\epsilon} + 4 = 0$ to obtain the roots: $\tilde{\epsilon}_{1,2} := 2\big(\gamma^2 \pm \sqrt{\gamma^4 - 1}\big)$. $\gamma^* = 1$ is a critical value at which the nature of the roots changes. For $0 \leq \gamma < 1$, $\tilde{\epsilon}_{1,2}$ are complex conjugates. For $\gamma \geq 1$ the roots are real. The branch points of (23) need to be defined consequently. For $\gamma > 1$, the four branch points are real and given by: $\tilde{\sigma}_{1,2,3,4} := \pm\sqrt{2\big(\gamma^2 \pm \sqrt{\gamma^4 - 1}\big)}$. For $\gamma < 1$, the branch points are complex conjugates. For convenience, define $\tilde{\sigma} := r + jy$, with $r, y \in \mathbb{R}$. Then, r and y at the branch points satisfy: $r^2 - y^2 = 2\gamma^2$ and $ry = \pm\sqrt{1 - \gamma^4}$. Solving these two equations yields:

$$r = \pm\sqrt{1 + \gamma^2} \;\text{ and }\; y = \pm\sqrt{1 - \gamma^2}. \tag{37}$$

APPENDIX B

We compute the roots of (32). Define $\tilde{\epsilon} := \tilde{\sigma}^2$ and solve for the roots of: $-\gamma^2\tilde{\epsilon}^3 + \tilde{\epsilon}^2 + 4 = 0$. The discriminant is $\Delta := -16(1 + 27\gamma^4) < 0 \; \forall\gamma$, which implies that regardless of the value of γ, there is one real root (positive, by Descartes' rule of signs) and two complex conjugate roots. This differs from the autocorrelated measurement noise case, in which as the value of γ was increased the branch points (26) transitioned from complex to real. The roots are:

$$\tilde{\epsilon}_1 \;= \frac{1}{3\gamma^2}(1 + \alpha + \alpha^{-1}), \tag{38}$$

$$\tilde{\epsilon}_{2,3} = \frac{1}{3\gamma^2}\left(1 - \frac{1}{2}(\alpha + \alpha^{-1}) \pm j\frac{\sqrt{3}}{2}(\alpha - \alpha^{-1})\right), \tag{39}$$

where α is as defined in (33). (38) provides a pair of real branch points $\tilde{\sigma}_{3,4} = \pm\frac{1}{\sqrt{3}}\gamma^{-1}\sqrt{1 + \alpha + \alpha^{-1}}$. From (39) four complex branch points are obtained. Define $\tilde{\sigma} := r + jy$ with $r, y \in \mathbb{R}$. Denote by R and I the real and imaginary parts of (39), respectively. Then, r and y are solutions to $r^2 - y^2 = R$ and $2ry = I$, given by:

$$r = \pm\sqrt{\frac{1}{2}\left(\sqrt{R^2 + I^2} + R\right)}, \;\; y = \pm\sqrt{\frac{1}{2}\left(\sqrt{R^2 + I^2} - R\right)}. \tag{40}$$

Substitution of R and I yields (35). To define the spatial decay rate $\tilde{\eta}_p$ of the convolution kernel $L(x)$, we need to determine the branch points with smallest real part (in absolute value) for each γ. We start by showing that:

Proposition: the absolute value of the poles $\tilde{\sigma}_{1,2}$ given in Section VI-B is a lower bound of the absolute value of the real branch points $\tilde{\sigma}_{3,4}$ (34).

Proof: It suffices to prove that $\alpha + \alpha^{-1} \geq 2$. Define the function $g(\alpha) := \alpha + \alpha^{-1}$, with $\alpha \in [1, \infty)$. $g''(\alpha) > 0$ in its domain, hence $g(\alpha)$ is convex. At $\alpha^* = 1$, $g(\alpha^*) = 2$ and $g'(\alpha^*) = 0$. This implies that $\alpha^* = 1$ is a global minimum and $g(\alpha) \geq 2$. □

Consequently, the real branch points $\tilde{\sigma}_{3,4}$ do not define $\tilde{\eta}_p$, regardless of the value of γ. For $\gamma < \gamma_p$, $\tilde{\eta}_p$ is defined by the complex branch points (35) and for $\gamma \geq \gamma_p$ the poles define it. In order to find γ_p we numerically solve:

$$2 = \frac{\alpha - \alpha^{-1}}{\sqrt{\alpha + \alpha^{-1} - 2 + 2\sqrt{\alpha^2 + \alpha^{-2} - \alpha^{-1} - \alpha}}}. \tag{41}$$

(41) provides the value of α at which the absolute value of the poles matches the absolute value of the real part of the complex branch points. Substitution in (33) yields $\gamma_p \approx 1.89$.

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