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Localization of the LQR Feedback Kernel in Spatially-Invariant Problems over Sobolev Spaces

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***Abstract*— We consider the LQR controller design problem for spatially-invariant systems on the real line where the state space is a Sobolev space. Such problems arise when dealing with systems describing wave or beam-bending motion. We demonstrate that the optimal state feedback is a spatial convolution operator with an exponentially decaying kernel, enabling implementation with a localized architecture. We generalize analogous results for the L_2 setting and provide a rigorous explanation of numerical results previously observed in the Sobolev space setting. The main tool utilized is a transformation from a Sobolev to an L_2 space, which is constructed from a spectral factorization of the spatial frequency weighting matrix of the Sobolev norm. We show the equivalence of the two problems in terms of the solvability conditions of the LQR problem. As a case study, we analyze the wave equation; we provide analytical expressions for the dependence of the decay rate of the optimal LQR feedback convolution kernel on wave speed and the LQR cost weights.**

I. Introduction

We consider the LQR controller design problem for distributed parameter systems over the real line with fully distributed actuation, restricting to spatially-invariant dynamics. We assume the underlying state space is a *Sobolev space*, which applies to e.g. systems with wave-like dynamics and more general PDEs with higher-order temporal dynamics. We note that although most real-life systems are of finite spatial extent, infinite-spatial-extent spatially-invariant systems are often useful idealizations for large but finite systems, as shown in e.g. [1]–[4].

In the spatially-invariant setting, the optimal LQR feedback gain will be a spatial convolution operator [5], and we seek to quantify the decay rate of this convolution kernel in the Sobolev space setting. An exponentially decaying convolution kernel is desirable in practice it allows for approximation of the control policy by a spatial truncation of this kernel, providing an inherent degree of *localization* of the resulting implementation [5]. Two directions of research in this setting are i) analyzing when constraints which ensure such localization can be imposed in a tractable way, as in e.g. [6], [7] and ii) characterizing when the *unconstrained* optimal controller will have an inherent level of spatial localization. In this work, we focus on the second problem, which has been studied in e.g. [5], [8]–[11].

In the case of an underlying L_2 state space, the optimal LQR feedback convolution kernel for spatially-invariant systems decays exponentially [5]. These methods were applied to analyze the heat equation in [9]. [8] provided results beyond the spatially-invariant setting, analyzing the so called spatially decaying operators over an L_2 state space. Numerical results presented in [11] suggest that the exponential decay rate presented in [5] holds when the underlying state space is a Sobolev space as well. However, as emphasized in [12], a rigorous general framework in this setting has yet to be developed. The main contribution of this paper takes a step toward addressing this gap in the literature.

Our main result demonstrates that any LQR problem for a spatially-invariant system over the real line with a Sobolev space as the underlying state space has an equivalent formulation over an L_2 state space. The optimal feedback for the L_2 formulation is a convolution operator whose kernel *decays exponentially*, and the optimal feedback for the original Sobolev space formulation will have the same decay rate. This procedure extends the results of [5] from the L_2 setting to a more general Sobolev space setting.

The rest of this paper is structured as follows. In Section III we introduce the LQR design problem. In Section IV, analytic formulas demonstrate that the optimal LQR feedback kernel for the wave equation formulated over a Sobolev space decays exponentially. We generalize these results in Section V through a procedure that converts the LQR design problem for a spatially-invariant distributed parameter system over a Sobolev space to an equivalent problem over an L_2 space.

II. Notation & Mathematical Preliminaries

Given two Hilbert spaces $\mathcal{U}$ and $\mathcal{H}$, $\mathcal{L}(\mathcal{U},\mathcal{H})$ denotes the space of linear operators from $\mathcal{U}$ to $\mathcal{H}$; to simplify notation we write $\mathcal{L}(\mathcal{H}) = \mathcal{L}(\mathcal{H},\mathcal{H})$. A linear operator $b \in \mathcal{L}(\mathcal{U},\mathcal{H})$ is *bounded* if $\|b\|_{\mathcal{U}\to\mathcal{H}} := \sup_{\|u\|_{\mathcal{U}=1}} \|bu\|_{\mathcal{H}} < \infty$. The domain of $b \in \mathcal{L}(\mathcal{U},\mathcal{H})$ is denoted by $\mathcal{D}(b) \subset \mathcal{U}$, and its adjoint is denoted by $b^\dagger$, i.e.

$$\langle b\psi, \phi\rangle_{\mathcal{H}} = \left\langle \psi, b^\dagger \phi\right\rangle_{\mathcal{U}}.$$

b is *self-adjoint* if $b = b^\dagger$ and $\mathcal{D}(b) = \mathcal{D}(b^\dagger)$.

$L_2^n(\mathbb{R})$ denotes the set of square-integrable functions on $\mathbb{R}$ equipped with the inner product $\langle \psi, \phi\rangle_{L_2^n} := \int_{x\in\mathbb{R}} \phi^*(x)\psi(x)dx$, where $(^*)$ denotes the complex conjugate transpose. Given a matrix-valued function $W : \mathbb{R} \to$

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$\mathbb{R}^{n\times n}$ of the form

$$\begin{aligned} W(\lambda) &= \operatorname{diag}\{w_1(\lambda), ..., w_n(\lambda)\}, \\ w_\ell(\lambda) &= \sum_{j=0}^{m_\ell} c_{\ell j}\lambda^{2j}, \ c_{\ell j} > 0 \text{ for all } \ell, j, \end{aligned} \tag{1}$$

we define the *weighted* L_2 *space*, $L_W^n(\mathbb{R}) := \{\Psi : \mathbb{R} \to \mathbb{C}^n; \|\Psi\|_{L_W}^n < \infty\}$, with inner product

$$\langle \Psi, \Phi \rangle_{L_W} := \int_{\lambda\in\mathbb{R}} \Psi^*(\lambda) W_\lambda \Phi(\lambda) d\lambda.$$

$\mathcal{H}_m^n(\mathbb{R}) := \{\psi : \mathbb{R} \to \mathbb{C}^n; \|\psi\|_{\mathcal{H}_m^n} < \infty\}$ denotes the *Sobolev space* with inner product

$$\langle \psi, \phi \rangle_{\mathcal{H}_m^n} := \sum_{j=0}^{m} \left\langle \partial_x^j \psi, \partial_x^j \phi \right\rangle_{L_2^n} \tag{2}$$

When the dimension is clear from context we simply denote these spaces by L_2, L_W, and $\mathcal{H}_m$. The Fourier transform provides a structured mapping between these two spaces, as stated in the following proposition [15].

Proposition 2.1: The Fourier transform is an isometric isomorphism from the Sobolev space $\mathcal{H}$ (2) to the weighted space L_W, with $W := \operatorname{diag}\{w_1, ..., w_N\}$ $w_\ell(\lambda) = \sum_{j=0}^{m_\ell} \lambda^{2j}$, so that $\langle f, g \rangle_{\mathcal{H}} = \langle \mathcal{F}f, \mathcal{F}g \rangle_{L_W}$.

A *multiplication operator* is an operator $M_B \in \mathcal{L}(L_W^n, L_\mathcal{V}^m)$ of the form

$$(M_B F)(\lambda) := B(\lambda)F(\lambda),$$

for a measurable function $B : \mathbb{R} \to \mathbb{C}^{m\times n}$. B is the *symbol* of the operator M_B, and we often denote $B(\lambda)$ by B_λ. The adjoint of a multiplication operator $M_B \in \mathcal{L}(L_\mathcal{V}^n, L_W^m)$ is a multiplication operator $(M_B)^\dagger = M_{B^\dagger}$ with symbol $B^\dagger$ given by

$$(B^\dagger)_\lambda := V_\lambda^{-1}(B_\lambda)^* W_\lambda. \tag{3}$$

III. Problem Set-up

We consider distributed parameter systems

$$\partial_t \psi(x,t) = (a\psi)(x,t) + (bu)(x,t), \tag{4}$$

where the state ψ and control signal u are functions of a spatial variable $x \in \mathbb{R}$ and a temporal variable $t \in \mathbb{R}^+$. Lower case letters denote such (possibly vector-valued) *spatio-temporal signals*

$$\psi(x,t) \quad x \in \mathbb{R}, \ t \in \mathbb{R}^+ := [0, \infty).$$

$\psi(x,t)$ is the value of the signal ψ at time t and spatial location x. For a fixed time t, the functions $\psi(\cdot,t)$ and $u(\cdot,t)$, denoted as $\psi(t)$ and $u(t)$, represent a spatially distributed signal. Upper case letters denote the *spatial Fourier transform* of a spatio-temporal signal:

$$\begin{aligned} \Psi(\lambda,t) &:= (\mathcal{F}\psi)(\lambda,t) \\ &:= \frac{1}{2\pi}\int_{x\in\mathbb{R}} \psi(x,t)e^{-i\lambda x} dx, \quad \lambda \in \mathbb{R}, \ t \in \mathbb{R}^+, \end{aligned} \tag{5}$$

and we denote the signals $\Psi(t) = \Psi(\cdot,t)$, and $\Psi_\lambda = \Psi(\lambda,\cdot)$.

We consider the design of a control policy $u = u(x,t)$ for systems (4), noting that actuation over the continuous domain $\mathbb{R}$ is an idealized assumption and actuation will be implemented in practice with some degree of discretization. To ensure solutions are well-defined, we make the common assumption that a generates a C_0-semigroup $\{e^{at}\}$ [13] on a Sobolev space $\mathcal{H}$ and $\mathcal{D}(a)$ is dense in $\mathcal{H}$. We also assume $u(t) \in \mathcal{D}(b) \subset L_2$ and $be^{at} \in \mathcal{L}(L_2, \mathcal{H})$ is bounded for each $t \geq 0$.

Example 3.1: **(Wave Equation)** The dynamics of the undamped wave equation over the real line with fully distributed actuation u are given by

$$\partial_t^2 \xi(x,t) \ = \ c^2 \partial_x^2 \xi(x,t) + u(x,t), \tag{6}$$

where $c > 0$ is the wave speed. Defining $\psi(x,t) := \begin{bmatrix} \xi(x,t) & \partial_t \xi(x,t) \end{bmatrix}^T$, we write (6) in *state space form* (4)

$$\begin{aligned} \partial_t \psi(x,t) &= \begin{bmatrix} 0 & 1 \\ c^2\partial_x^2 & 0 \end{bmatrix} \psi(x,t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(x,t) \\ &:= (a\psi)(x,t) + (bu)(x,t). \end{aligned} \tag{7}$$

a generates a C_0-semigroup $\{e^{at}\}$ on the Sobolev space

$$\begin{aligned} H &:= \mathcal{H}_1^1(\mathbb{R}) \times L_2(\mathbb{R}), \\ \langle f, g \rangle_H &= \left\langle \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \right\rangle_H \\ &:= \langle f_1, g_1 \rangle_{L_2} + \langle f_1', g_1' \rangle_{L_2} + \langle f_2, g_2 \rangle_{L_2}, \end{aligned}$$

and $b \in \mathcal{L}(L_2, \mathcal{H})$ is bounded [12].

A. Spatially-Invariant Systems

We restrict our attention to *spatially-invariant* systems, formally defined as follows.

Definition 3.2: To each $y \in \mathbb{R}$, define the *translation operator* $(T_y\psi)(x,t) := \psi(x-y,t)$. $b \in \mathcal{L}(\mathcal{U}, \mathcal{H})$ is *spatially-invariant* if it commutes with all translation operators, i.e.

$$bT_y = T_y b, \quad \text{for all } y \in \mathbb{R}.$$

One subclass of spatially-invariant operators is that of spatial convolution operators.

Definition 3.3: A *spatial convolution operator* is a spatially-invariant operator b of the form

$$(b\psi)(x,t) = (b * \psi)(x,t) = \int_{y\in\mathbb{R}} b(y)\psi(x-y,t)dy,$$

where with some abuse of notation we use b to denote the operator and $b(y)$ to denote the value of the corresponding *convolution kernel* at spatial location y.

A distributed parameter system (4) is said *spatially-invariant system* if the operators a and b are spatially-invariant; we solve the LQR design problem for this class of systems:

$$\begin{aligned} \inf_u \ J &:= \int_{t=0}^{\infty} \langle \psi(t), q\psi(t) \rangle_{\mathcal{H}} + \langle u(t), ru(t) \rangle_{L_2} dt \\ \text{s.t.} & \ \text{ dynamics (4),} \end{aligned} \tag{8}$$

where $q \in \mathcal{L}(\mathcal{H})$ and $r \in \mathcal{L}(L_2)$ are bounded, self-adjoint, spatial convolution operators which represent the penalty on state and control, respectively. We introduce the following terminology to formalize the well-posedness of (8).

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Definition 3.4: Let a generate a C_0-semigroup $\{e^{at}\}$ on $\mathcal{H}$ and let $b \in \mathcal{L}(L_2, \mathcal{H})$. a is *exponentially stable* if there exists $M, \alpha > 0$ such that

$$\|e^{at}\|_{\mathcal{H}\to\mathcal{H}} \leq Me^{-\alpha t}, \text{ for all } t \geq 0.$$

(a, b) is *exponentially stabilizable* if there exists $k \in \mathcal{L}(\mathcal{H}, L_2)$ such that $(a - bk)$ is exponentially stable. In this case, the feedback operator k is said to be *stabilizing*. (q, a) is *exponentially detectable* if $(a^\dagger, q^\dagger)$ is exponentially stabilizable[1]. The LQR problem (8) is *well-posed* if (a, b) is exponentially stabilizable, (q, a) is exponentially detectable, and $e^{at}b \in \mathcal{L}(L_2, \mathcal{H})$ is a bounded operator for each t.

Proposition 3.5: Assume the spatially-invariant LQR problem (8) is well-posed. Then (8) has a unique stabilizing solution $u = kx$ given by the spatial convolution operator

$$k := -r^{-1}b^\dagger p \ \in \ \mathcal{L}(\mathcal{H}, L_2), \tag{9}$$

where $p = p^\dagger$ is the bounded solution of the Riccati equation:

$$\begin{aligned} &\langle ah_1, ph_2\rangle_{\mathcal{H}} + \langle ph_1, ah_2\rangle_{\mathcal{H}} + \langle h_1, qh_2\rangle_{\mathcal{H}} \\ &\quad = \left\langle b^\dagger ph_1, r^{-1}b^\dagger ph_2\right\rangle_{\mathcal{H}}, \text{ for all } h_1, h_2 \in \mathcal{D}(a) \end{aligned} \tag{10}$$

Proof: See Appendix. ■

The decay rate of (9) provides the degree of *localization* of the optimal distributed control policy. It has been shown that (9) *decays exponentially*, under appropriate assumptions on a, b, q, r, in the special case that $\mathcal{H} = L_2$ [5]. Numerical results suggest such decay rates also hold for more general choice of Sobolev space $\mathcal{H}$ [11]. The following analysis works toward rigorously proving these observed decay rates.

Definition 3.6: A spatial convolution operator b *decays exponentially with rate* $\tilde{\beta} > 0$ if its convolution kernel $b(\cdot)$ satisfies

$$b(x)e^{\tilde{\beta}|x|} \to 0 \text{ as } |x| \to \infty.$$

A spatially-invariant operator b is 'diagonalized' by a spatial Fourier transform to a multiplication operator [5], i.e. $\mathcal{F}b\mathcal{F}^{-1} =: M_B$. $\|b\| = \|M_B\|$ and b is self-adjoint if and only if M_B is.

Definition 3.7: Given a multiplication operator M_B, we define the *extension* of the symbol B to the complex plane, denoted by B_e, such that $B_e(i\lambda) = B_\lambda$. $B_e(\sigma)$ is constructed by replacing each λ in B_λ by $(-i\sigma)$.

If B_e is analytic and satisfies a polynomial growth bound on the strip

$$\Gamma_\beta := (-\beta, \beta) + i\mathbb{R} \ \subset \mathbb{C}, \tag{11}$$

then the inverse Fourier transform, b, of B is a spatial convolution operator that decays exponentially with rate $\tilde{\beta}$ for any $\tilde{\beta} < \beta$ [14, Thm 7.4.2].

IV. Application: Wave Equation

We first analyze the decay rate of the LQR feedback for the wave equation (Ex. 3.1). A spatial Fourier transform converts (7) to a parameterized family over $\lambda \in \mathbb{R}$ of finite-dimensional dynamics:

$$\begin{aligned} \partial_t \Psi(\lambda, t) &= \begin{bmatrix} 0 & 1 \\ -c^2\lambda^2 & 0 \end{bmatrix} \Psi(\lambda, t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U(\lambda, t) \\ &=: A_\lambda \Psi(\lambda, t) + B_\lambda U(\lambda, t), \end{aligned} \tag{12}$$

where $M_A = \mathcal{F}a\mathcal{F}^{-1}$ and $M_B = \mathcal{F}b\mathcal{F}^{-1}$. By Proposition 2.1, $\mathcal{F} : \mathcal{H} \to L_W$ is an isometry, where L_W is the weighted space with

$$W_\lambda := \begin{bmatrix} 1 + \lambda^2 & 0 \\ 0 & 1 \end{bmatrix}. \tag{13}$$

Thus the LQR problem (8) for the wave equation may be written in terms of the transformed dynamics (12) as:

$$\begin{aligned} &\inf_U \int_{t=0}^{\infty} \langle \Psi(t), M_Q \Psi(t)\rangle_{L_W} + \langle U(t), \gamma U(t)\rangle_{L_2} \, dt \\ &\text{s.t. dynamics (12)}, \end{aligned} \tag{14}$$

where r is taken as a multiple of the identity, i.e. $r = \gamma I$. Under the assumption that (a, q) is exponentially detectable and $M_Q \in \mathcal{L}(L_W)$, it can be shown that (14) is well-posed. The optimal solution to (14) is $U = M_K \Psi$,

$$K_\lambda := -\frac{1}{\gamma}(B^\dagger)_\lambda P_\lambda = -\frac{1}{\gamma}(B_\lambda)^* W_\lambda P_\lambda, \tag{15}$$

where $P_\lambda = (P^\dagger)_\lambda$ solves the family of Riccati equations:

$$(A_\lambda)^*\Pi_\lambda + \Pi_\lambda A_\lambda + W_\lambda Q_\lambda, = \frac{1}{\gamma}\Pi_\lambda B_\lambda (B_\lambda)^*\Pi_\lambda, \tag{16}$$

with $\Pi_\lambda := W_\lambda P_\lambda$ and M_P is a bounded operator.

Note that the penalty on state can be modified through both the LQR weight q or the choice of Sobolev norm W_λ. For instance, for any $\alpha > 0$ a generates a C_0-semigroup on

$$\begin{aligned} &H_\alpha := \{f : \mathbb{R} \to \mathbb{C}^2 \,\Big|\, f_1, f_1', f_2 \in L_2(\mathbb{R})\}, \\ &\langle f, g\rangle_{H_\alpha} := \langle f_1, g_1\rangle_{L_2} + \alpha^2 \langle f_1', g_1'\rangle_{L_2} + \langle f_2, g_2\rangle_{L_2}, \end{aligned} \tag{17}$$

Setting $\alpha = 1$ recovers the original space. $\mathcal{F} : H_\alpha \to L_{W_\alpha}$ is an isometry, where

$$(W_\alpha)_\lambda := \begin{bmatrix} 1 + \alpha^2\lambda^2 & 0 \\ 0 & 1 \end{bmatrix} \tag{18}$$

The LQR objective can be formulated as

$$J = \int_{t=0}^{\infty} \left\langle \Psi(t), M_{(Q_\alpha)}\Psi(t)\right\rangle_{L_{(W_\alpha)}} + \langle U(t), \gamma U(t)\rangle_{L_2} \, dt \tag{19}$$

Note that the cost functional of (14) is equivalent to (19) when $W_\lambda Q_\lambda = (W_\alpha)_\lambda (Q_\alpha)_\lambda$.

A. Analytic Solution of Optimal Feedback Gain

For the case that q is the identity operator on H_α for some α, we analytically compute

$$K_\lambda = \frac{-1}{\gamma}\begin{bmatrix} -\gamma c^2\lambda^2 + f(\lambda) & \sqrt{\gamma(1 - 2\gamma c^2\lambda^2 + 2f(\lambda))} \end{bmatrix}$$

[1]Equivalent definitions of exponential stability, stabilizability, and detectability in the spatial frequency domain are provided in [11]

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where $f(\lambda) := \sqrt{c^4\lambda^4\gamma^2 + \gamma(1+\alpha^2\lambda^2)}$, and $h(\lambda) := \frac{1}{\gamma} f(\lambda)\sqrt{\gamma(1 - 2c^2\gamma\lambda^2 + 2f(\lambda))}$. We compute the extension

$$K_e(\sigma) = \frac{-1}{\gamma}\left[\begin{array}{cc} \gamma c^2\sigma^2 + f_e(\sigma) & \sqrt{\gamma(1+2\gamma c^2\sigma^2 + 2f_e(\sigma))} \end{array}\right]$$

where $f_e(\sigma) := \sqrt{c^4\sigma^4\gamma^2 + \gamma(1-\alpha^2\sigma^2)}$. The branch points [16] of the multivalued function f_e are given by $\sigma = \infty$ along with the zeros of the function $g(\sigma) := c^4\sigma^4\gamma^2 + \gamma(1-\alpha^2\sigma^2)$. The zeros of $g(\sigma)$ are given by

$$\begin{cases} \sigma = \pm\sqrt{\frac{1}{2\nu}\left(\alpha^2 \pm \sqrt{\alpha^4 - 4\nu}\right)}, & 0 < \nu \le \frac{1}{4}\alpha^4 \\ \sigma = \pm\left(\frac{1}{2\sqrt{\nu}}\right)\left(\sqrt{2\sqrt{\nu}+\alpha^2} \pm i\sqrt{2\sqrt{\nu}-\alpha^2}\right), & \nu > \frac{1}{4}\alpha^4, \end{cases}$$

where

$$\nu := c^4\gamma. \tag{20}$$

For $\nu < \frac{1}{4}\alpha^4$, there are 4 distinct real-valued zeros of $g(\cdot)$; for $\nu = \frac{1}{4}\alpha^4$ there are 2 repeated real-valued zeros; for $\nu > \frac{1}{4}\alpha^4$ there are 4 complex-valued zeros (2 distinct complex conjugate pairs). The locations of the branch points in each of these 3 regimes is illustrated in Figure 1 for the case $\alpha = 1$. The shaded region is the strip Γ_β, with β the magnitude of the real part of the zeros. When there are 4 distinct real-valued zeros, β is the smaller of the 2 real component magnitudes. A precise formula for Γ_β as a function of the parameters $\nu = c^4\gamma$ and α is

$$\Gamma_\beta := \begin{cases} \left\{|\mathrm{Re}(z)| < \sqrt{\frac{1}{2\nu}(\alpha^2 - \sqrt{\alpha^4 - 4\nu})}\right\}, & \nu \in (0, \frac{1}{4}\alpha^4] \\ \left\{|\mathrm{Re}(z)| < \frac{\sqrt{2\sqrt{\nu}+\alpha^2}}{2\sqrt{\nu}}\right\}, & \nu > \frac{1}{4}\alpha^4 \end{cases} \tag{21}$$

$K_e^{(\alpha)}$ has no additional branch points in Γ_β, and can be uniquely defined as an analytic function in this region. β is dependent on the LQR cost parameters and the wave speed: $\beta \to \frac{1}{\alpha}$ for $\nu \ll \frac{1}{4}\alpha^4$, $\beta \to \frac{\sqrt{2}}{\alpha}$ as $\nu \to \frac{1}{4}\alpha^4$, and $\beta \to 0$ for $\nu \gg \frac{1}{4}\alpha^4$. For a fixed α, the largest region of analyticity (fastest rate of decay) will occur at $\nu = \frac{1}{4}\alpha^4$ and is given by $\beta = \frac{\sqrt{2}}{\alpha}$. Thus a smaller value of α allows for a quicker decay rate (see the log scale plot in Figure 2). The mapping $\nu \mapsto \beta$ is non-differentiable at the point $\nu = \frac{1}{4}\alpha^4$ (illustrated by a star); this point represents the transition from 4 distinct real-valued branch points to 4 complex-valued branch points (Figure 3 illustrates this for the case $\alpha = 1$).

We recover the convolution kernel k from its Fourier transform $K_\lambda^{(1)} = K_\lambda = \left[\begin{array}{cc} K_1(\lambda) & K_2(\lambda) \end{array}\right]$ for the case of $\nu = 1$. We write

$$K_1(\lambda) = \tilde{K}_1(\lambda) + K_1(\infty) := (K_1(\lambda) - 0.5) + 0.5,$$
$$K_2(\lambda) = \tilde{K}_2(\lambda) + K_2(\infty) := (K_2(\lambda) - \sqrt{2}) + \sqrt{2},$$

so that $k_1(x)$ and $k_2(x)$ are given by

$$k_1(x) = \tilde{k}_1(x) + 0.5 \cdot \delta(x),$$
$$k_2(x) = \tilde{k}_2(x) + \sqrt{2} \cdot \delta(x),$$

where δ is the Dirac delta distribution. The inverse Fourier transforms $\tilde{k}_1(x)$ and $\tilde{k}_2(x)$ of $\tilde{K}_1(\lambda)$ and $\tilde{K}_2(\lambda)$ are numerically computed and plotted in Figure 4 to illustrate the decay rate of the convolution kernels $\tilde{k}_1$ and $\tilde{k}_2$ for $\alpha = 1$.

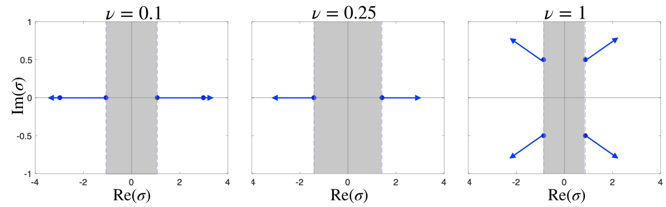

Fig. 1. For $\alpha = 1$, the blue lines denote the branch cuts for f_e for the case of $\nu = 0.1$ (left), $\nu = 0.25$ (center), and $\nu = 1$ (right). For $\nu = 0.1$ there are 4 real-valued branch points, for $\nu = 0.25$ there are 2 real-valued branch points, and for $\nu = 1$ there are 4 complex-valued branch points. The extension of the feedback, K_e, is analytic in the shaded region. The largest such region occurs for $\nu = 0.25$ and corresponds to branch cuts beginning at $\sqrt{2}$ on the Real axis.

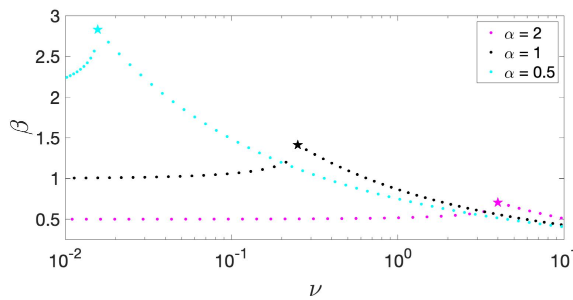

Fig. 2. The boundary of the region of analyticity ($\beta = |\mathrm{Re}(\sigma)|$) is plotted against the parameter $\nu = c^4\gamma$ for $\alpha = 2$, $\alpha = 1$, and $\alpha = 0.5$. The star denotes the non differentiable point at $\left(\nu = \frac{1}{4}\alpha^4, \beta = \frac{\sqrt{2}}{\alpha}\right)$, which corresponds to the largest region of analyticity and therefore the fastest rate of decay. Note that the axis for ν is on a log scale.

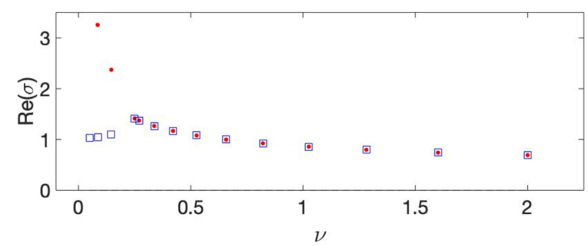

Fig. 3. For the case $\alpha = 1$, the real component of the branch points of f_e is plotted as a function of the parameter ν. We note the transition from 4 distinct (non-infinite) branch points to 2 distinct (non-infinite) branch points at the point $\nu = \frac{1}{4}$. This point corresponds to the largest region of analyticity of K_e and therefore the fastest decay rate of the feedback k.

V. Equivalence of L_2 & Sobolev Space Formulations

In this section we present an alternate method that is generalizable, as it avoids the explicit branch point computations used in the approach of Section IV. For clarity, we examine the wave equation (7) as a concrete example throughout.

Consider the LQR problem (8) where a generates a C_0-semigroup on a Sobolev space

$$\mathcal{H} := \mathcal{H}_{m_1}^{n_1}(\mathbb{R}) \times \ldots \times \mathcal{H}_{m_N}^{n_N}(\mathbb{R}), \tag{22}$$

$b \in \mathcal{L}(L_2, \mathcal{H})$, $q = q^\dagger \in \mathcal{L}(\mathcal{H})$, $r = r^\dagger \in \mathcal{L}(L_2)$. Assume:

1) The LQR problem (8) is well-posed,

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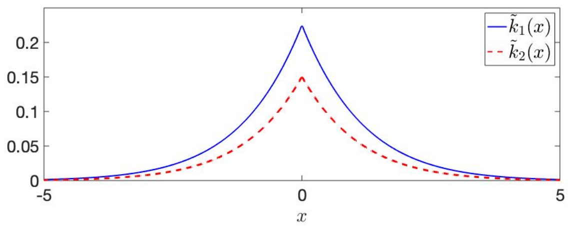

Fig. 4. The decay rates of the convolution kernels, $\tilde{k}_1$ and $\tilde{k}_2$, are represented for the case $\nu = 1, \alpha = 1$. These were computed by numerical integration the inverse Fourier transform formula of $\tilde{K}_1 = K_1(\lambda) - K_1(\infty)$ and $\tilde{K}_2 = K_2(\lambda) - K_2(\infty)$. The steady state terms of K_1 and K_2 represent a Dirac δ distribution in the convolution kernels k_1 and k_2 and were subtracted off before numerical integration.

2) a, b, q and r are spatially-invariant operators, and A_e, B_e, Q_e, and R_e, are given by analytic, rational functions on some strip Γ_β.

Under these assumptions we will demonstrate the following:

- The LQR problem (8) over the Sobolev space $\mathcal{H}$ can be formulated as an equivalent LQR problem over L_2, and this reformulation is well-posed (Thm. 5.2),
- The optimal feedback for this reformulation decays exponentially; the optimal feedback for the original Sobolev space formulation decays with the same rate (Thm 5.4).

We first demonstrate that the LQR problem (14) for the wave equation can be reformulated over an L_2 space.

Example 5.1: The LQR problem (8) for the wave equation (7) satisfies Assumptions (1), (2). We write the dynamics as

$$\begin{aligned}\partial_t \Phi(\lambda, t) &= \begin{bmatrix} 0 & 1 - i\lambda \\ \frac{-c^2\lambda^2}{1-i\lambda} & 0 \end{bmatrix} \Phi(\lambda, t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U(\lambda, t) \\ &=: \hat{A}_\lambda \Phi(\lambda, t) + \hat{B}_\lambda U(\lambda, t),\end{aligned} \tag{23}$$

where we have defined a new variable $\Phi_\lambda := S_\lambda \Psi_\lambda$, with $W_\lambda = \begin{bmatrix} 1 + i\lambda & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 - i\lambda & 0 \\ 0 & 1 \end{bmatrix} =: (S_\lambda)^* S_\lambda$ a spectral factorization of the weighting matrix W. $M_{\hat{A}}$ generates a C_0-semigroup on L_2. Note that $\hat{a} = \mathcal{F}^{-1} M_{\hat{A}} \mathcal{F}$ is *not* a differential operator. The LQR problem (14) can be written

$$\begin{aligned}\inf_U \int_{t=0}^{\infty} \left\langle \Phi(t), M_{\hat{Q}} \Phi(t) \right\rangle_{L_2} + \langle U(t), \gamma U(t) \rangle_{L_2} dt \\ \text{s.t. dynamics (23),}\end{aligned} \tag{24}$$

where $\hat{Q}_\lambda = S_\lambda Q_\lambda (S^\dagger)_\lambda$, and this formulation is well-posed. The following theorem generalizes the results of Example 5.1.

Theorem 5.2: Consider the LQR problem (8) over a Sobolev space $\mathcal{H}$ and assume that (1), (2) hold. Let L_W denote the corresponding weighted space (i.e. $\mathcal{F} : \mathcal{H} \to L_W$) and let $W_\lambda = S_\lambda^* S_\lambda$, denote a spectral factorization of the weighting matrix W_λ. Then (26) can be formulated over an L_2 space as

$$\begin{aligned}\inf_u \int_{t=0}^{\infty} \left\langle \Phi(t), M_{\hat{Q}} \Phi(t) \right\rangle_{L_2} + \langle U(t), M_R U(t) \rangle_{L_2} dt \\ \text{s.t. } \partial_t \Phi(\lambda, t) = \hat{A}_\lambda \Phi(\lambda, t) + \hat{B}_\lambda U(\lambda, t).\end{aligned} \tag{25}$$

where $\Phi(\lambda, t) := S_\lambda \Psi(\lambda, t)$, $\hat{A}_\lambda := S_\lambda A_\lambda S_\lambda^{-1}$, $\hat{B}_\lambda := S_\lambda B_\lambda$, and $\hat{Q}_\lambda = S_\lambda Q_\lambda (S^\dagger)_\lambda$. Problem (25) is well-posed.

Proof: By Proposition 2.1, the general LQR problem over a Sobolev space $\mathcal{H}$ (8) may be written in the form

$$\begin{aligned}\inf_u \int_{t=0}^{\infty} \langle \Psi(t), M_Q \Psi(t) \rangle_{L_W} + \langle U(t), M_R U(t) \rangle_{L_2} dt \\ \text{s.t. } \dot{\Psi}(\lambda, t) = A_\lambda \Psi(\lambda, t) + B_\lambda U(\lambda, t).\end{aligned} \tag{26}$$

The details demonstrating that (26) can be converted to (25) are provided in the Appendix. ■

We next relate a decay rate of the solution of the transformed problem over an L_2 space to the decay rate of the solution to the original problem over a Sobolev space. We begin by looking at the wave equation example once again.

Example 5.3: The optimal solution to (24) is

$$U = M_{\hat{K}} \Phi \ , \ \hat{K}_\lambda := -\frac{1}{\gamma} (\hat{B}^\dagger)_\lambda \hat{P}_\lambda, \tag{27}$$

where $M_{\hat{P}}$ is a bounded self-adjoint operator and $\hat{P}_\lambda = (\hat{P}^\dagger)_\lambda$ is the solution to the Riccati equation

$$(\hat{A}^\dagger)_\lambda \hat{P}_\lambda + \hat{P}_\lambda \hat{A}_\lambda + \hat{Q}_\lambda = \frac{1}{\gamma} \hat{P}_\lambda \hat{B}_\lambda (\hat{B}^\dagger)_\lambda \hat{P}_\lambda. \tag{28}$$

(24) is well-posed, so that (27) is stabilizing. The extension of $\hat{A}$ is given by the rational function $\hat{A}_e(\sigma) = \begin{bmatrix} 0 & 1 - \sigma \\ \frac{c^2\sigma^2}{1-\sigma} & 0 \end{bmatrix}$, which is analytic in Γ_1, and $\hat{Q}_e$ will be rational and analytic in some strip as well. Then, an application of [5, Thm 6] shows that $\hat{K}_e$ is analytic in Γ_η for some η and $\hat{k}$ therefore decays exponentially with rate η. The feedback policies for both formulations are equivalent, i.e. $M_K \Psi = M_{\hat{K}} \Phi$. From the relation

$$K_e(\sigma) = \hat{K}_e(\sigma) \cdot \begin{bmatrix} 1 + \sigma & 0 \\ 0 & 1 \end{bmatrix}, \tag{29}$$

we see that K_e will be analytic in the same region Γ_η as $\hat{K}_e$. Thus, k will have at least the same exponential decay rate as $\hat{k}$.

The following theorem generalizes Example 5.3.

Theorem 5.4: The optimal feedback for (25) is of the form $U = M_{\hat{K}} \Phi$, or equivalently, $u = \hat{k}\phi$. The solution to the original Sobolev space formulation is given by $u = k\psi$. These two feedbacks are equivalent, and the decay rate of the convolution kernel of k is at least as rapid as that of $\hat{k}$.

Proof: We emphasize that the relation between the decay rates of k and $\hat{k}$ is not the same as the relation between the decay rates of the Riccati equation solutions p and $\hat{p}$. The branch points of $\hat{K}_e = R_e^{-1} (\hat{B}_e)^* \hat{P}_e$ are exactly the branch points of $\hat{P}_e$ so that $\hat{k}$ has the same exponential decay rate as $\hat{p}$. In contrast, the branch points of $K_e = R_e^{-1} (B_e)^* W_e P_e$, are the branch points of $W_e P_e$, *not* the branch points of P_e. It can be shown that $W_e P_e = S_e^* \hat{P}_e S_e$, and since S_e and S_e^* are analytic, the branch points of $W_e P_e$ are the same as those of $\hat{P}_e$. ■

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VI. Conclusion & Open Problems

We demonstrated that the LQR design problem for a spatially-invariant system over a Sobolev space can be reformulated as an LQR problem over an L_2 space. The spatial decay rate of the optimal LQR feedback demonstrated in [5] was shown to apply to the more general Sobolev space setting. Future work will extend these results to the setting of *homogeneous* Sobolev spaces. Interesting and related open problems include imposing convex constraints on the decay rate of feedback to extend results of e.g. [7], [17] to the continuous spatial domain setting.

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VII. Appendix

Proof of Proposition 3.5

By the Plancherel theorem, (8) can be solved via the parameterized family of problems

$$\begin{aligned} &\inf_{U_\lambda} \int_{t=0}^{\infty} \Psi_\lambda^*(t) W_\lambda Q_\lambda \Psi_\lambda(t) + U_\lambda^*(t) R_\lambda U_\lambda(t) dt \\ &\text{s.t. } \partial_t \Psi_\lambda(t) = A_\lambda \Psi_\lambda(t) + B_\lambda U_\lambda(t) \end{aligned} \tag{30}$$

The solution of (30) is $U_\lambda^{\text{opt}} := -K_\lambda \Psi_\lambda := -R_\lambda^{-1} B_\lambda^* P_\lambda \Psi_\lambda$, where $P_\lambda = P_\lambda^*$ solves

$$A_\lambda^* P_\lambda + P_\lambda A_\lambda - P_\lambda B_\lambda R_\lambda^{-1} B_\lambda^* P_\lambda + W_\lambda Q_\lambda = 0 \tag{31}$$

(31) is written in terms of $\Pi_\lambda := W_\lambda^{-1} P_\lambda = (\Pi^\dagger)_\lambda$ as

$$0 = A_\lambda^\dagger \Pi_\lambda + \Pi_\lambda A_\lambda - \Pi_\lambda B_\lambda R_\lambda^{-1} B_\lambda^\dagger \Pi_\lambda + Q_\lambda$$

and $U_\lambda^{\text{opt}} = -R_\lambda^{-1} B_\lambda^* P_\lambda \Psi_\lambda = -R_\lambda^{-1} B_\lambda^\dagger \Pi_\lambda \Psi_\lambda$.

Proof of Theorem 5.2

The proof utilizes the following lemma.

Lemma 7.1: (a, b) is exponentially stabilizable if and only if there exists a bounded solution $p = p^\dagger \in \mathcal{L}(\mathcal{H})$ of the Riccati equation

$$\begin{aligned} &\langle a h_1, p h_2 \rangle_{\mathcal{H}} + \langle p h_1, a h_2 \rangle_{\mathcal{H}} + \langle h_1, h_2 \rangle_{\mathcal{H}} \\ &\qquad = \left\langle b^\dagger p h_1, b^\dagger p h_2 \right\rangle_{\mathcal{H}}, \ \forall h_1, h_2 \in \mathcal{D}(a). \end{aligned} \tag{32}$$

Proof of Lemma 7.1: $a \in \mathcal{L}(\mathcal{H})$ is exponentially stable if and only if there exists a bounded operator $p = p^\dagger \in \mathcal{L}(\mathcal{H})$ satisfying the Lyapunov equation

$$\langle a h, p h \rangle_{\mathcal{H}} + \langle p h, a h \rangle_{\mathcal{H}} = \langle h, g h \rangle_{\mathcal{H}}, \quad \forall h \in \mathcal{D}(a) \tag{33}$$

for some negative definite operator g [13]. Write (32) using shorthand notation as

$$a^\dagger p + p a + I = p b b^\dagger p \tag{34}$$

$$\Leftrightarrow \ (a - bf)^\dagger p + p(a - bf) = g, \tag{35}$$

where $f := b^\dagger p$, $g := -p b b^\dagger p - I \prec 0$. $(a - bf)$ is then exponentially stable as the Lyapunov equation (35) is satisfied. This completes the proof of Lemma 7.1.

The two cost functionals are equivalent if

$$\begin{aligned} 0 &= \langle \phi, \hat{q} \phi \rangle_{L_2} - \langle \psi, q \psi \rangle_H = \left\langle \Phi, M_{\hat{Q}} \Phi \right\rangle_{L_2} - \langle \Psi, M_Q \Psi \rangle_{L_W} \\ &= \int \Psi_\lambda^* \left(S_\lambda^* \hat{Q}_\lambda^* S_\lambda \ - \ Q_\lambda^* W_\lambda \right) \Psi_\lambda d\lambda. \end{aligned}$$

Equivalently, $\hat{Q}_\lambda = S_\lambda^{-*} W_\lambda Q_\lambda S_\lambda^{-1} = S_\lambda Q_\lambda (S^\dagger)_\lambda$. If (q, a) detectible, then there exists of a bounded, self-adjoint, spatially-invariant $p \in \mathcal{L}(\mathcal{H})$ such that

$$\hat{A}_\lambda \hat{P}_\lambda + \hat{P}_\lambda \hat{A}_\lambda^\dagger - \hat{P}_\lambda \hat{Q}_\lambda^\dagger \hat{Q}_\lambda \hat{P}_\lambda + I = 0, \ \forall \lambda \in \mathbb{R}, \tag{36}$$

holds with $\hat{P}_\lambda := S_\lambda P_\lambda S_\lambda^{-1}$. It can be shown that the corresponding operator $\hat{p}$ is self-adjoint and bounded, so that by Lemma 7.1, $(\hat{q}, \hat{a})$ is detectable. The proof that $(\hat{a}, \hat{b})$ is stabilizable follows similarly. Finally, let $\{e^{at}\}$ denote the C_0-semigroup on $\mathcal{H}$ generated by a. $\hat{a} = s a s^{-1}$ generates the C_0-semigroup $\{s e^{at} s^{-1}\}$ on L_2. Defining $\tau(t) := \mathcal{F} e^{at} \mathcal{F}^{-1}$,

$$\begin{aligned} \|e^{at} b\|_{L_2 \to \mathcal{H}} &= \|\tau(t) B\|_{L_2 \to L_W} \\ &= \|S \tau(t) B\|_{L_2 \to L_2} = \|s e^{at} b\|_{L_2 \to L_2}. \end{aligned}$$

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