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Optimal structured controllers for spatially invariant systems: a convex reformulation

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Abstract— **We consider quadratically-optimal control synthesis for systems in which controls and measurements are spatially distributed. We impose a structural constraint that the feedback can only use measurements within an a priori fixed distance from the control action. Such constraints generally lead to non-convex optimal control problems. In the context of spatially invariant systems such a constraint amounts to restricting the feedback spatial convolution kernel to have a pre-specified compact support. For systems where the state is a scalar-valued spatio-temporal field, we show that this structured controller design problem can be reformulated as a convex optimization problem, thereby identifying a new class of systems for which the optimal structured control synthesis is convex. We apply our method to design an optimal structured controller for a diffusion process on the real line.**

I. Introduction

Large-scale networks of spatially distributed dynamical systems are common in modern applications. For these, traditional optimal control techniques usually yield all-to-all (centralized) communication topologies, which are often prohibitive due to their large scale [1]. This challenge justifies the increasing interest of the research community in the design of *optimal structured controllers* with limited information exchange between subsystems. However, the optimal structured controller design is not always a straightforward task, as it generally requires solving a non-convex optimization. As a consequence, the identification of classes of systems for which optimal structured control synthesis admits a convex formulation has become of important concern. Some of these classes have already been identified and include: partially nested systems [2], [3]; cone- and funnel-causal systems [3]–[6]; quadratically-invariant systems [7]; positive systems [8]–[11]; and poset-causal systems [12].

A class of spatially distributed systems of interest is that in which the underlying dynamics are *spatially invariant* (SI), and controls and measurements are spatially distributed. This is an idealization useful for modeling applications such as platoons, smart structures, or systems with continuum mechanics in which the dynamics are described by partial differential equations (PDEs) of constant coefficients. In the seminal work [13] on optimal control of SI systems, two important structural properties of optimal *centralized* controllers were proven. Namely that optimal controllers (1) *inherit the spatially invariant structure* of the plant, and (2) have an *inherent degree of spatial localization*, as the spatial convolution kernels of the state feedback and observer gain operators decay exponentially in space. This means that the contribution of measurements for optimal feedback control decays exponentially with distance between sensor and actuator. SI systems have attracted considerable attention since, and further work followed, e.g.: the spatial structure of Kalman Filters for SI systems was studied in [14] and [15], and similar spatial decay properties of optimal controllers have been found for the more general class of spatially decaying operators in [16]. One drawback of these results is that the degree of "controller spatial localization" is determined by the plant's dynamics and performance index, but it is challenging to be constrained a priori, as this generally leads to a non-convex problem.

Several approaches to the synthesis of optimal *structured* controllers for SI systems have been introduced as well. [4] and [5] presented classes where a priori cone- and funnel-causality constraints can be imposed on the controller in a convex manner, provided that information passing in the controller travels at least as fast as disturbances propagate in the plant. Later, [17] and [18] applied the sparsity-promoting optimal control technique to SI systems. Recently, the System Level Synthesis Approach [19] was exploited in [20] to design optimal controllers for SI systems in which the spatial spread of the closed-loop response was constrained. However, none of these approaches managed to impose an *a priori* constraint on the structure of the controller for a SI system through a convex optimization, without requiring further assumptions such as funnel-causality.

In this work we design quadratically-optimal structured controllers for SI systems over L^2. In particular, our focus is on SI systems for which the state is a scalar-valued field. We focus our attention on the class of static spatially invariant controllers in which the feedback operator is a spatial convolution. For this class, spatial locality constraints on the controller can be imposed by restricting the feedback convolution kernel to be compactly supported in space (rather than exponentially decaying in space, as in the centralized setting). We show that the *optimal compactly supported stabilizing feedback convolution kernel* can be obtained through a *convex optimization* in the spatial frequency domain,

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thereby identifying a new class of systems for which optimal structured feedback control admits a convex formulation.

Our presentation is organized as follows. Section II introduces mathematical preliminaries, definitions, and notation. Section III presents the SI plant we work with. Section IV reviews relevant results on centralized LQR for SI systems, originally presented in [13]. These are required to understand our approach to the optimal structured controller design. Section V presents our main contribution, a convex formulation to synthesize the optimal feedback controller within the class of spatial convolution feedback operators with spatial locality constraints. In Section VI, we illustrate how to apply our optimal structured control synthesis method to diffusive dynamics over the real line. Finally, in Section VII we draw conclusions and discuss on-going research.

II. Mathematical Preliminaries

We study dynamical systems with translation invariances in the spatial coordinate x. We assume that x forms a locally compact abelian group $\mathbb{G}$. We focus on $\mathbb{G}=\mathbb{R}$ and $\mathbb{G}=\mathbb{Z}$. $\hat{\mathbb{G}}$ denotes the corresponding dual group, obtained after taking Fourier transforms ($\mathcal{Z}$-transforms) in the case of $\mathbb{G}=\mathbb{R}$ ($\mathbb{G}=\mathbb{Z}$), which yield $\hat{\mathbb{G}}=\mathbb{R}$ ($\hat{\mathbb{G}}=\partial\mathbb{D}$, the unit circle). We primarily consider functions f on $\mathbb{G}$ which are square integrable $f\in L^2(\mathbb{G})$ and use $\langle\cdot,\cdot\rangle$ to denote the inner product in $L^2(\mathbb{G})$. $\mathcal{D}(\mathcal{A})$ denotes the domain of an operator $\mathcal{A}$ and the symbol $\star$ denotes a convolution operator. f^* denotes the complex conjugate of f. $\mathbf{1}_{[x_1,x_2]}(x)$ is the characteristic function of the interval $[x_1,x_2]$: $\mathbf{1}_{[x_1,x_2]}(x)=1$ if $x\in[x_1,x_2]$ and $\mathbf{1}_{[x_1,x_2]}(x)=0$ otherwise.

Definition 1 *(Translation invariance).* Let $\mathcal{T}_x$ denote a translation operator for functions on $\mathbb{G}$: given any $x\in\mathbb{G}$, $(\mathcal{T}_x f)(y):=f(y-x)$. An operator $\mathcal{A}$ is *translation invariant* if $\mathcal{T}_x:\mathcal{D}(\mathcal{A})\to\mathcal{D}(\mathcal{A})$ and $\mathcal{A}\mathcal{T}_x=\mathcal{T}_x\mathcal{A}$ for every translation $\mathcal{T}_x$. Since this work is concerned with translation invariance in space, we use the terms *translation invariance* and *spatial invariance* interchangeably.

Definition 2 *(Multiplication operator).* Let Ω be some set and $A:\Omega\to\mathbb{C}$ be a measurable function. A *multiplication operator* $\mathcal{M}_A$ is defined by $(\mathcal{M}_A f)(x):=A(x)f(x)$, $\forall f\in\mathcal{D}(\mathcal{M}_A)$. The function A is called the *symbol* of the multiplication operator $\mathcal{M}_A$.

We define the *spatial* Fourier transform of a spatio-temporal signal f according to the following normalization:

$$\hat{f}(\lambda,t):=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{G}} f(x,t)e^{-jx\lambda}\mathrm{d}x, \tag{1}$$

$$f(x,t):=\frac{1}{\sqrt{2\pi}}\int_{\hat{\mathbb{G}}} \hat{f}(\lambda,t)e^{jx\lambda}\mathrm{d}\lambda, \tag{2}$$

where $\lambda\in\hat{\mathbb{G}}$ denotes the spatial frequency and j the imaginary unit. We use $\hat{f}_\lambda$ (or $\hat{f}(\lambda)$ with some notational abuse) to denote parametrization by λ. Sometimes, the dependence on λ will be omitted for simplicity. The Fourier transform (1) *diagonalizes* spatially invariant operators, transforming them into multiplication operators in the spatial frequency domain [13]. This diagonalization in frequency is an important observation that we exploit in our work.

III. Spatially invariant plants

We study systems with spatio-temporal linear dynamics in continuous time $t\in\mathbb{R}_{\geq 0}$, of the form:

$$\frac{\partial}{\partial t}\psi(x,t)=[\mathcal{A}\psi](x,t)+u(x,t), \tag{3}$$

where $x\in\mathbb{G}$ denotes the spatial coordinate. Actuators are fully distributed along this coordinate and the state ψ is fully observed. The operator $\mathcal{A}$ is time independent and translation invariant in x. We call this plant *spatially invariant*.

Assumption 3 *(Scalar-valued field)*[1]. The state ψ is a spatio-temporal *scalar-valued* field.

Assumption 4 *(L^2 space).* (i) The operator $\mathcal{A}:\mathcal{D}(\mathcal{A})\to L^2$ with $\mathcal{D}(\mathcal{A})\subseteq L^2$ is the infinitesimal generator of a strongly continuous semigroup (C_0-semigroup) on $L^2(\mathbb{G})$ and its Fourier symbol $\hat{A}_\lambda$ is continuous. (ii) The initial condition $\psi(\cdot,0)=\psi_0(\cdot)\in\mathcal{D}(\mathcal{A})$. At a fixed instant of time $t\geq 0$, the control signal is square integrable as well $u(\cdot,t)\in L^2(\mathbb{G})$. Together with Assumption 4(i), these ensure that $\psi(\cdot,t)\in L^2(\mathbb{G})$ for $t\geq 0$.

Definition 5 *(Exponential stabilizability).* The system (3) is *exponentially stabilizable* if there exists an operator $\mathcal{K}:L^2(\mathbb{G})\to L^2(\mathbb{G})$ such that $\mathcal{A}-\mathcal{K}$ generates an exponentially stable C_0-semigroup on $L^2(\mathbb{G})$.

Proposition 6 *(Exponential stabilizability of (3)).* Under Assumptions 3-4, the plant (3) is exponentially stabilizable.

Proof. Since the operator $\mathcal{A}$ is spatially invariant and generates a C_0-semigroup in L^2, then its Fourier symbol $\hat{A}_\lambda$ satisfies the half-plane condition $\sup_{\lambda\in\hat{\mathbb{G}}}\Re(\hat{A}_\lambda)\leq c$, for some finite constant c. Choosing the static feedback symbol $\hat{K}_\lambda:=\alpha$ with $\alpha>c$ gives $\sup_{\lambda\in\hat{\mathbb{G}}}\Re(\hat{A}_\lambda-\alpha)<0$. Since the Fourier symbol of the closed-loop operator is a multiplication operator, $\sup_{\lambda\in\hat{\mathbb{G}}}\Re(\hat{A}_\lambda-\alpha)<0$ implies that $\mathcal{A}-\mathcal{K}$ generates an exponentially stable C_0-semigroup in L^2. ■

IV. Centralized optimal control of spatially invariant systems

In this section, we particularize relevant results from the seminal work [13] on the *centralized* LQR for spatially invariant systems to the plant and assumptions of Section III. The LQR problem formulation in the *spatial domain* is:

$$\begin{aligned}\min_{u}\ \mathcal{J}=&\int_0^\infty \big(\langle\mathcal{Q}\psi,\psi\rangle+\langle\mathcal{R}u,u\rangle\big)\mathrm{d}t\\ \text{s.t.}\quad&\\ &\frac{\partial\psi}{\partial t}(x,t)=[\mathcal{A}\psi](x,t)+u(x,t)\\ &\psi(x,0)=\psi_0(x).\end{aligned} \tag{4}$$

Assumption 7 *(properties of operators $\mathcal{Q}$ and $\mathcal{R}$).* (i) $\mathcal{Q}:L^2\to L^2$ and $\mathcal{R}:L^2\to L^2$ are self-adjoint, spatially invariant, and time independent operators. Their respective Fourier symbols $\hat{Q}_\lambda$ and $\hat{R}_\lambda$ are continuous, bounded, and satisfy $\hat{Q}_\lambda\geq 0$ and $\hat{R}_\lambda>0$ pointwise in $\lambda\in\hat{\mathbb{G}}$. (ii) The pair $(\hat{Q}_\lambda,\hat{A}_\lambda)$ is detectable pointwise in $\lambda\in\hat{\mathbb{G}}$.

[1]Assumption 3 is motivated by the observations in [21] and [22], highlighting that PDEs over L^2 spaces generally have scalar states.

In [13] it was showed that Problem (4) is easier to solve in the *spatial frequency domain*. Take the spatial Fourier transform (1) of Problem (4). Then, exploit the facts that inner products are preserved in L^2 (by Plancherel's theorem) and that spatially invariant operators transform to multiplication operators, to obtain:

$$\begin{aligned} \min_{\hat{u}} \mathcal{J} &= \int_{\hat{\mathbb{G}}} \int_0^\infty \left(\hat{\psi}_\lambda^* \hat{Q}_\lambda \hat{\psi}_\lambda + \hat{u}_\lambda^* \hat{R}_\lambda \hat{u}_\lambda\right) \mathrm{d}t \mathrm{d}\lambda \\ \text{s.t.} & \\ & \frac{\mathrm{d}\hat{\psi}_\lambda}{\mathrm{d}t}(t) = \hat{A}_\lambda \hat{\psi}_\lambda(t) + \hat{u}_\lambda(t) \\ & \hat{\psi}_\lambda(0) = \hat{\psi}_0(\lambda), \end{aligned} \tag{5}$$

where $\lambda \in \hat{\mathbb{G}}$ denotes spatial frequency. The constraints in Problem (5) must hold pointwise in λ and the cost functional is a sum over λ. Hence, Problem (5) is *decoupled in* λ: its solution is obtained by solving a classical finite dimensional LQR at each $\lambda \in \hat{\mathbb{G}}$,

$$\begin{aligned} \min_{\hat{u}_\lambda} J_\lambda &= \int_0^\infty \left(\hat{\psi}_\lambda^* \hat{Q}_\lambda \hat{\psi}_\lambda + \hat{u}_\lambda^* \hat{R}_\lambda \hat{u}_\lambda\right) \mathrm{d}t \\ \text{s.t.} & \\ & \frac{\mathrm{d}\hat{\psi}_\lambda}{\mathrm{d}t}(t) = \hat{A}_\lambda \hat{\psi}_\lambda(t) + \hat{u}_\lambda(t) \\ & \hat{\psi}_\lambda(0) = \hat{\psi}_0(\lambda). \end{aligned} \tag{6}$$

Consequently, the optimal solution is obtained by solving a family of AREs *parametrized by the spatial frequency*,

$$\hat{A}_\lambda^* \hat{P}_\lambda + \hat{P}_\lambda \hat{A}_\lambda - \hat{P}_\lambda \hat{R}_\lambda^{-1} \hat{P}_\lambda + \hat{Q}_\lambda = 0, \ \ \lambda \in \hat{\mathbb{G}}. \tag{7}$$

At a given $\lambda \in \hat{\mathbb{G}}$ the optimal cost $J_{\lambda,*}$ of Problem (6) is $J_{\lambda,*} = |\hat{\psi}_\lambda(0)|^2 \hat{P}_\lambda$ and the corresponding optimal cost $\mathcal{J}_*$ of Problem (5) is:

$$\mathcal{J}_* = \int_{\hat{\mathbb{G}}} J_{\lambda,*} \mathrm{d}\lambda = \int_{\hat{\mathbb{G}}} |\hat{\psi}_\lambda(0)|^2 \hat{P}_\lambda \mathrm{d}\lambda. \tag{8}$$

Since by Assumption 3 we work with a scalar field ψ, $\hat{P}_\lambda$ is a scalar at each λ and can be explicitly obtained from (7). The Fourier symbol $\hat{K}_\lambda$ of the centralized state feedback operator $\mathcal{K}$ is given by $\hat{K}_\lambda = \hat{R}_\lambda^{-1} \hat{P}_\lambda$:

$$\hat{K}_\lambda = \Re(\hat{A}_\lambda) + \sqrt{\Re(\hat{A}_\lambda)^2 + \frac{\hat{Q}_\lambda}{\hat{R}_\lambda}}. \tag{9}$$

The optimal control law in the spatial frequency domain is $\hat{u}_\lambda(t) = -\hat{K}_\lambda \hat{\psi}_\lambda(t)$. Take the inverse spatial Fourier transform (2) and apply the convolution theorem to obtain the control law in the spatial domain $u(x,t) = -[\mathcal{K}\psi](x,t) = -(2\pi)^{-\frac{1}{2}}[K \star \psi](x,t)$: the optimal state feedback operator is a *spatial convolution* and hence, inherits the spatial invariance of the plant (recall that convolutions are translation invariant operators). At location x, the control law is obtained by convolving state measurements with the convolution kernel K. Therefore, the spatial spread of this kernel determines the degree of spatial localization of the optimal controller: if the spatial decay of K is fast, at each location x measurements from its neighborhood are more relevant for control than those further away; if K has slow spatial decay, measurements from further away are important for the controller too; if $K(x') = 0$, then the measurement of the state at x' does not influence the optimal controller. Under mild assumptions (Assumption 2 in [13]), [13] proved that the convolution kernel K decays at least exponentially in space, which implies that the *optimal centralized control law* has an *inherent degree of spatial localization*: the influence of each sensor depends on its position from the actuator, measurements from far away contributing less.

The rapid spatial decay of the optimal control feedback convolution kernel K suggests *spatial truncation* of the convolution kernel (see Fig. 1b) as a means to design structured controllers, in which information exchange for control is limited to a pre-specified radius T in space. However, the feedback convolution kernel $K_{trunc,T}$ obtained through spatial truncation of the centralized kernel K will not be optimal with respect to our quadratic performance criteria and might even lead to instability [16]. How to *optimally* design the structured and stabilizing control convolution kernel K_T is an open question, even for spatially invariant plants [1], [4], [23]. Next, we show that under Assumptions 3, 4, and 7, the optimal compactly supported and stabilizing control feedback convolution kernel K_T is the solution to a *convex* optimization problem.

V. STRUCTURED OPTIMAL CONTROL OF SPATIALLY INVARIANT SYSTEMS: A CONVEX FORMULATION

We aim to design an optimal *structured* feedback controller $u_T(x,t) = -[\mathcal{K}_T\psi](x,t)$ with *spatial locality constraints* for the plant (3). As shown in Section IV, the optimal centralized feedback control operator $\mathcal{K}$ is a spatial convolution. We restrict the operator $\mathcal{K}_T$ to belong to the class of stabilizing spatial convolutions as well and hence, to be time independent and *spatially invariant*. To introduce locality constraints on $\mathcal{K}_T$, we enforce its kernel K_T to be compactly supported on $[-T,T]$ ($T \geq 0$), where T is a design parameter. The corresponding structured feedback control law is of the form:

$$u_T(x,t) = -[\mathcal{K}_T\psi](x,t) = -(2\pi)^{-\frac{1}{2}}[K_T \star \psi](x,t). \tag{10}$$

At a given location x, controller (10) will only use measurements up to a prescribed distance T from x for feedback. Given the spatial invariance of $\mathcal{K}_T$, taking the spatial Fourier transform (1) of (10) yields a multiplication operator in the spatial frequency domain:

$$\hat{u}_T(\lambda,t) = -\hat{K}_T(\lambda)\hat{\psi}_\lambda(t). \tag{11}$$

In this section, we provide a method to find such a stabilizing and compactly supported kernel K_T, minimizer of the quadratic performance criterion introduced in Section IV. In the centralized control setting, exploiting spatial invariance and posing the controller optimization problem in the spatial frequency domain has considerable advantages in terms of analysis (see Section IV). Given that we aim to find a structured controller (10) restricted to a class of *spatially invariant* feedback operators, a natural question to ask is whether the optimal structured controller synthesis could have a simple and intuitive formulation in the spatial frequency domain as well. We show next that this is indeed the case. In the following subsections, we formulate the performance and constraints of the optimization problem in the spatial frequency domain and prove that the problem is *convex*.

A. Performance criterion

The performance criterion is quadratic, as in Section IV:

$$\mathcal{J}_T = \int_0^\infty \big(\langle \mathcal{Q}\psi, \psi\rangle + \langle \mathcal{R}u_T, u_T\rangle\big)\mathrm{d}t, \tag{12}$$

where operators $\mathcal{Q}$ and $\mathcal{R}$ satisfy Assumption 7. The controller (10) is spatially invariant. Hence, in the spatial frequency domain, the closed-loop dynamics are a LTI ODE at each $\lambda \in \hat{\mathbb{G}}$. This allows us to rewrite the cost (12) as a function of the Fourier symbol of the feedback convolution operator (see Appendix I for details):

$$\mathcal{J}_T(\hat{K}_R, \hat{K}_I) = \frac{1}{2}\int_{\hat{\mathbb{G}}} |\hat{\psi}_0(\lambda)|^2 \frac{\hat{Q}_\lambda + \hat{R}_\lambda(\hat{K}_R^2 + \hat{K}_I^2)}{\hat{K}_R - \Re(\hat{A}_\lambda)}\mathrm{d}\lambda, \tag{13}$$

where $\hat{K}_T(\lambda) = \hat{K}_R(\lambda) + j\hat{K}_I(\lambda)$.

B. Stability constraint

The structured closed-loop dynamics are:

$$\frac{\partial \psi}{\partial t}(x,t) = [(\mathcal{A} - \mathcal{K}_T)\psi](x,t),\ x \in \mathbb{G}. \tag{14}$$

In the spatial frequency domain, (14) transforms to:

$$\dot{\hat{\psi}}_\lambda(t) = (\hat{A}_\lambda - \hat{K}_T)\hat{\psi}_\lambda(t),\ \lambda \in \hat{\mathbb{G}}. \tag{15}$$

To ensure exponential stability of (15) at each $\lambda \in \hat{\mathbb{G}}$, we enforce the constraint:

$$\Re(\hat{A}_\lambda) - \hat{K}_R(\lambda) \le -\epsilon,\ \ \epsilon \in \mathbb{R}_{>0},\ \lambda \in \hat{\mathbb{G}}. \tag{16}$$

Proposition 8 *(Exponential stability of the structured closed-loop).* Consider that Assumptions 3, 4, and 7 hold and that $\mathcal{K}_T$ is a spatially invariant operator. Then, inequality (16) is equivalent to exponential stability of the structured closed-loop dynamics (14).

Proof. It follows from the same argument used in the proof of Proposition 6.

C. Spatial locality constraint

The structured control feedback convolution kernel K_T is supported on $[-T, T]$ in space. Hence, it must satisfy:

$$K_T(x) = K_T(x)\cdot \mathbf{1}_{[-T,T]}(x),\ \ x \in \mathbb{G}, \tag{17}$$

where $\cdot$ denotes pointwise multiplication. (17) allows for the translation of the *spatial locality* constraint of the kernel to the spatial *frequency* domain. Take the spatial Fourier transform (1) of (17), apply the convolution theorem, and separate the real and imaginary components of $\hat{K}_T(\lambda)$ to obtain:

$$\hat{K}_R(\lambda) = (2\pi)^{-\frac{1}{2}}[\hat{K}_R \star \hat{\mathbf{1}}_{[-T,T]}](\lambda), \tag{18}$$

$$\hat{K}_I(\lambda) = (2\pi)^{-\frac{1}{2}}[\hat{K}_I \star \hat{\mathbf{1}}_{[-T,T]}](\lambda). \tag{19}$$

For example, in the case $\mathbb{G} = \mathbb{R}$, the spatial locality constraint in frequency would be:

$$\hat{K}_T(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} \hat{K}_T(\omega)\,\mathrm{sinc}\big(T(\lambda - \omega)\big)\mathrm{d}\omega, \tag{20}$$

where $\mathrm{sinc}(T\lambda) := \frac{\sin(T\lambda)}{\lambda}$ is the cardinal sine function.

D. The convex optimization problem

The functional optimization problem for the Fourier symbol of the compactly supported kernel K_T of the control law (10) is:

$$\begin{aligned} \inf_{\hat{K}_R(\cdot),\hat{K}_I(\cdot)} \ & \mathcal{J}_T = \int_{\hat{\mathbb{G}}} \frac{1}{2}|\hat{\psi}_0(\lambda)|^2 \frac{\hat{Q}_\lambda + \hat{R}_\lambda(\hat{K}_R^2 + \hat{K}_I^2)}{\hat{K}_R - \Re(\hat{A}_\lambda)}\mathrm{d}\lambda \\ \text{s.t.}\ & \\ & \Re(\hat{A}_\lambda) - \hat{K}_R(\lambda) \le -\epsilon,\ \ \epsilon \in \mathbb{R}_{>0} \\ & \hat{K}_R(\lambda) = (2\pi)^{-\frac{1}{2}}[\hat{K}_R \star \hat{\mathbf{1}}_{[-T,T]}](\lambda) \\ & \hat{K}_I(\lambda) = (2\pi)^{-\frac{1}{2}}[\hat{K}_I \star \hat{\mathbf{1}}_{[-T,T]}](\lambda), \end{aligned} \tag{21}$$

with $\hat{K}_R(\cdot), \hat{K}_I(\cdot) \in L^\infty(\hat{\mathbb{G}})$ and continuous, which guarantee $u_T(\cdot, t) \in L^2(\mathbb{G})$. Once the solution to Problem (21) is obtained, the kernel in the spatial domain is recovered by taking the inverse spatial Fourier transform: $K_T(x) = \mathcal{F}^{-1}(\hat{K}_T)(x)$.

Remark *(Coupledness in λ).* Due to the convolutions (over frequency) present in the spatial locality constraints, Problem (21) is *coupled* in λ. Consequently, its solution cannot be obtained by solving a family of finite dimensional optimal control problems parametrized by λ, as happened in the centralized setting of Section IV.

Next, we present a key intermediate result upon which we build the proof of the convexity of Problem (21). Denote the integrand of the cost functional $\mathcal{J}_T$ by:

$$J_T(\lambda, \hat{K}_R, \hat{K}_I) := \frac{1}{2}|\hat{\psi}_0(\lambda)|^2 \frac{\hat{Q}_\lambda + \hat{R}_\lambda(\hat{K}_R^2 + \hat{K}_I^2)}{\hat{K}_R - \Re(\hat{A}_\lambda)}. \tag{22}$$

Denote the feasible set of Problem (21) by $\mathcal{C}$. The constraints in Problem (21) are affine and hence, $\mathcal{C}$ is *convex*. We prove next that in the feasible set $\mathcal{C}$, J_T is jointly convex in $\hat{K}_R(\lambda)$ and $\hat{K}_I(\lambda)$.

Lemma 9 *(Joint convexity pointwise in $\lambda \in \hat{\mathbb{G}}$).* Consider that Assumptions 3, 4, and 7 hold. Then, in the feasible set $\mathcal{C}$, J_T as defined in (22) is jointly convex in $\hat{K}_R(\lambda)$ and $\hat{K}_I(\lambda)$.

Proof. Pointwise joint convexity of $J_T(\lambda, \cdot, \cdot)$ is equivalent to pointwise positive semidefinitiness of its Hessian $\mathbb{H}_\lambda$,

$$\mathbb{H}_\lambda = |\hat{\psi}_0(\lambda)|^2 \begin{bmatrix} \frac{\hat{R}_\lambda \Re(\hat{A}_\lambda)^2 + \hat{R}_\lambda \hat{K}_I^2 + \hat{Q}_\lambda}{(\hat{K}_R - \Re(\hat{A}_\lambda))^3} & -\frac{\hat{R}_\lambda \hat{K}_I}{(\hat{K}_R - \Re(\hat{A}_\lambda))^2} \\ -\frac{\hat{R}_\lambda \hat{K}_I}{(\hat{K}_R - \Re(\hat{A}_\lambda))^2} & \frac{\hat{R}_\lambda}{\hat{K}_R - \Re(\hat{A}_\lambda)} \end{bmatrix}. \tag{23}$$

We evaluate the signs of the principal minors of $\mathbb{H}_\lambda$ within the feasible set $\mathcal{C}$, pointwise in $\lambda \in \hat{\mathbb{G}}$:

$$\mathbb{H}_{11} = |\hat{\psi}_0(\lambda)|^2 \frac{\hat{R}_\lambda \Re(\hat{A}_\lambda)^2 + \hat{R}_\lambda \hat{K}_I^2 + \hat{Q}_\lambda}{(\hat{K}_R - \Re(\hat{A}_\lambda))^3} \geq 0, \quad (24)$$

$$\mathbb{H}_{22} = |\hat{\psi}_0(\lambda)|^2 \frac{\hat{R}_\lambda}{\hat{K}_R - \Re(\hat{A}_\lambda)} \geq 0, \quad (25)$$

$$\det(\mathbb{H}_\lambda) = |\hat{\psi}_0(\lambda)|^4 \frac{\hat{R}_\lambda \big(\hat{R}_\lambda \Re(\hat{A}_\lambda)^2 + \hat{Q}_\lambda\big)}{(\hat{K}_R - \Re(\hat{A}_\lambda))^4} \geq 0, \quad (26)$$

as $\hat{R}_\lambda > 0$ and $\hat{Q}_\lambda \geq 0$ by Assumption 7 and in $\mathcal{C}$ the stability constraint $\hat{K}_R - \Re(\hat{A}_\lambda) \geq \epsilon > 0$ is satisfied pointwise in $\lambda \in \hat{\mathbb{G}}$. Since all the principal minors of $\mathbb{H}_\lambda$ are non-negative in $\mathcal{C}$, $\mathbb{H}_\lambda \succeq 0$ pointwise in $\lambda \in \hat{\mathbb{G}}$. ■

Theorem 10 *(Joint convexity of Problem* (21)*).* Consider that Assumptions 3, 4, and 7 hold. Then, the optimization problem (21) is jointly convex in $\hat{K}_R$ and $\hat{K}_I$.

Proof. The feasible set $\mathcal{C}$ of Problem (21) is convex. We are left to prove the convexity of the cost $\mathcal{J}_T$ in $\mathcal{C}$. The simplest sufficient condition ensuring joint convexity of the functional $\mathcal{J}_T$ is joint convexity of the integrand $J_T(\lambda, \hat{K}_R(\lambda), \hat{K}_I(\lambda))$ in $\hat{K}_R(\lambda)$ and $\hat{K}_I(\lambda)$, keeping $\lambda \in \hat{\mathbb{G}}$ fixed [24]. By Lemma 9, this pointwise condition holds for J_T in $\mathcal{C}$. The convexity of Problem (21) follows. ■

By Theorem 10, any extremal of (21) is a minimizer.

Remark *(Spatial symmetry).* The convexity of Problem (21) does *not* rely on spatial symmetry in the open-loop operator or control kernel. This is an important advantage compared to some previous structured control approaches in the literature (e.g., [23]). Furthermore, although our formulation was presented for a kernel $K_T(\cdot)$ supported on $[-T, T]$ for simplicity, asymmetric supports $[T_1, T_2]$, with $T_1 \leq T_2$, might be used without compromising convexity. When $T_1 \neq T_2$, the spatial locality constraints (18)-(19) need to be adapted to account for the imaginary part of $\hat{\mathbf{1}}_{[T_1,T_2]}$.

Finally, we highlight that there is some resemblance between Problem (21) and the *Optimal Spectral Concentration Problem* in communication theory. The latter was studied by Slepian and co-workers in the early sixties and its solution exhibits a beautiful mathematical structure. The interested reader might find [25] and references therein useful.

VI. Example: Optimal structured control of diffusion on $\mathbb{R}$

A. Problem formulation

We consider a spatially invariant diffusion process with distributed control input $u(x,t)$:

$$\frac{\partial \psi}{\partial t}(x,t) = \kappa \frac{\partial^2 \psi}{\partial x^2}(x,t) + u(x,t), \quad x \in \mathbb{R},\ t \in \mathbb{R}_{\geq 0}. \quad (27)$$

The constant $\kappa > 0$ is the diffusion coefficient and the initial condition is a Gaussian with Fourier transform $\hat{\psi}_0(\lambda) = A_0 e^{-\sigma^2\lambda^2} \in L^2(\mathbb{R})$. The control objective is to drive the state to zero while optimizing the quadratic performance criterion (12), where $\mathcal{Q} = Q$ and $\mathcal{R} = R$ with Q and R positive constants. We use a *structured* feedback control law (10), whose kernel K_T is supported on $[-T, T]$.

The Laplacian operator in (27), initial condition, and cost operators Q and R are even in x. Thus, the optimal feedback convolution kernel K_T is even in x as well and consequently, $\hat{K}_T(\lambda)$ is real and even in λ. The optimization problem for the structured feedback convolution kernel is:

$$\begin{aligned} \inf_{\hat{K}_T(\cdot), K_T(\cdot)} \quad & \mathcal{J}_T = \frac{1}{2}\int_{-\infty}^{\infty} |\hat{\psi}_0(\lambda)|^2 \frac{Q + R\hat{K}_T(\lambda)^2}{\hat{K}_T(\lambda) + \kappa\lambda^2}\, \mathrm{d}\lambda \\ \text{s.t.} \quad & \\ & -\kappa\lambda^2 - \hat{K}_T(\lambda) \leq -\epsilon, \quad \epsilon \in \mathbb{R}_{>0} \\ & \hat{K}_T(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-T}^{T} K_T(x)\cos(\lambda x)\mathrm{d}x. \end{aligned} \quad (28)$$

The introduction of the auxiliary kernel $K_T(\cdot)$ allows us to formulate the spatial locality constraint using the definition of spatial Fourier transform (1) with appropriate integration bounds, rather than (20). This alternative formulation is advantageous for quadrature as it is easier to approximate than a convolution with a cardinal sine over $\mathbb{R}$.

B. Finite dimensional approximation

We detail how to derive a finite dimensional approximation of Problem (28) suitable to be solved numerically. First, approximate the definite integral in the spatial locality constraint through the trapezoidal rule. Sample the integrand at a grid of $N_x + 1$ regularly spaced points on $[0, T]$ in the spatial coordinate x. We denote the grid points by x_i ($i = 0, \ldots, N_x$) and the grid size by $\Delta_x := T/N_x$, such that $x_i := i\Delta_x$. With the notation $k_i := K_T(x_i)$, $i = 0, \ldots, N_x$, the spatial locality constraint approximation is:

$$\begin{aligned} \hat{K}_T(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^T K_T(x)\cos(\lambda x)\mathrm{d}x \approx \\ \sqrt{\frac{2}{\pi}} \Delta_x \bigg(\frac{k_0}{2} + \frac{k_{N_x}}{2}\cos(T\lambda) + \sum_{i=1}^{N_x - 1} k_i \cos(x_i \lambda) \bigg), \end{aligned} \quad (29)$$

where we leveraged that the integrand is even in x. Substitute the RHS of (29) in the cost functional and stability constraint to obtain a finite dimensional convex optimization problem (linear transformations of the argument preserve convexity [26]). The spatial locality constraint is now embedded in the cost and the decision variables are k_i ($i = 0, \ldots, N_x$). Second, approximate the integral over frequency in the cost through the trapezoidal rule. Sample the integrand at a grid of $N_\lambda + 1$ regularly spaced points over frequency λ. We denote the grid points by λ_k ($k = 0, \ldots, N_\lambda$) and the grid size by Δ_λ, such that $\lambda_k := k\Delta_\lambda$. This procedure yields Problem (30), with a total of $N_x + 1$ decision variables.

C. Results

We set parameter values as in Table I and solve Problem (30) using CVX [27], [28]. Fig. 1c shows the optimal structured kernel K_T obtained numerically for $T = 5$.

κ	σ	A_0	Q	R	ϵ	Δ_x	Δ_λ	N_λ
4	10^{-3}	1	1	2	10^{-4}	0.025	0.05	5000

TABLE I: Parameter values used for the numerical example.

$$
\begin{aligned}
\min_{k_0,\dots,k_{N_x}} \; \Delta_\lambda \Bigg[& \frac{1}{2}|\hat{\psi}_0(0)|^2 \frac{Q + \frac{2}{\pi} R \Delta_x^2 \big(\frac{k_0}{2} + \frac{k_{N_x}}{2} + \sum_{i=1}^{N_x-1} k_i\big)^2}{\sqrt{\frac{2}{\pi}}\Delta_x \big(\frac{k_0}{2} + \frac{k_{N_x}}{2} + \sum_{i=1}^{N_x-1} k_i\big)} \\
& + \sum_{k=1}^{N_\lambda-1} |\hat{\psi}_0(\lambda_k)|^2 \frac{Q + \frac{2}{\pi} R \Delta_x^2 \big(\frac{k_0}{2} + \frac{k_{N_x}}{2}\cos(T\lambda_k) + \sum_{i=1}^{N_x-1} k_i \cos(x_i\lambda_k)\big)^2}{\sqrt{\frac{2}{\pi}}\Delta_x \big(\frac{k_0}{2} + \frac{k_{N_x}}{2}\cos(T\lambda_k) + \sum_{i=1}^{N_x-1} k_i \cos(x_i\lambda_k)\big) + \kappa\lambda_k^2} \\
& + \frac{1}{2}|\hat{\psi}_0(\lambda_{N_\lambda})|^2 \frac{Q + \frac{2}{\pi} R \Delta_x^2 \big(\frac{k_0}{2} + \frac{k_{N_x}}{2}\cos(T\lambda_{N_\lambda}) + \sum_{i=1}^{N_x-1} k_i \cos(x_i\lambda_{N_\lambda})\big)^2}{\sqrt{\frac{2}{\pi}}\Delta_x \big(\frac{k_0}{2} + \frac{k_{N_x}}{2}\cos(T\lambda_{N_\lambda}) + \sum_{i=1}^{N_x-1} k_i \cos(x_i\lambda_{N_\lambda})\big) + \kappa\lambda_{N_\lambda}^2} \Bigg]
\end{aligned} \tag{30}
$$

s.t.

$$
-\kappa\lambda_k^2 - \sqrt{\frac{2}{\pi}}\Delta_x \left(\frac{k_0}{2} + \frac{k_{N_x}}{2}\cos(T\lambda_k) + \sum_{i=1}^{N_x-1} k_i \cos(x_i\lambda_k) \right) \leq -\epsilon, \quad \text{with } \; k = 0, \dots, N_\lambda.
$$

Comparison of the different control convolution kernels provided in Fig. 1 suggests an interesting relation between the optimal centralized (K) and optimal structured (K_T) kernels for the diffusive dynamics (27). Particularly, we propose the following *ansatz* for K_T:

$$
K_T(x) = \underbrace{K(x) \cdot \mathbf{1}_{(-T,T)}(x)}_{\text{centralized kernel in } (-T,T)} + \underbrace{c_T \cdot \delta(x \pm T)}_{\text{compensation}}, \; x \in \mathbb{G}, \tag{31}
$$

where c_T is constant in x parametrized by T, and $\delta(\cdot)$ denotes the Dirac delta function. The analysis of the Euler-Lagrange equations of Problem (28) to evaluate the validity of the ansatz (31) will be presented somewhere else due to space constraints.

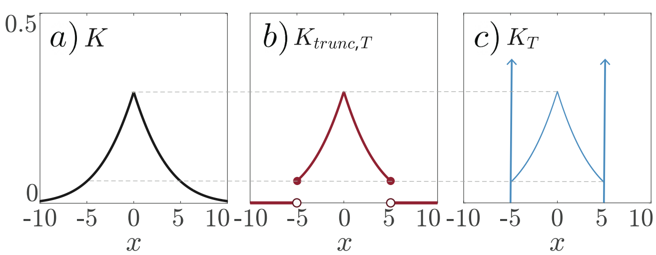

Fig. 1: Kernels of the feedback control operator for the diffusion process (27) with parameter values as given in Table I. a) Optimal centralized kernel (exponentially decaying). b) Centralized kernel truncated at $T = 5$. c) Optimal structured kernel for $T = 5$ (discontinuous components of the kernel have been re-scaled to fit in the figure). Horizontal dotted lines are guides to the eye to ease comparison. Vertical axis is the same in the three plots.

Ansatz (31) claims that for $x \in (-T, T)$ the optimal structured kernel K_T takes the values of the optimal centralized kernel K, and for $x = \pm T$ the optimal structured kernel grows a *compensation term* given by a Dirac delta weighted by c_T. This term partially compensates for the performance drop due to the spatial locality constraint (i.e., due to the missing exponential tails present in the centralized kernel that feed back measurements from spatial locations further away than a distance T from the actuator). The performance compensation of this term is apparent in Fig. 2, in which the performances of two structured controllers are compared. For small values of T, the performance of the structured controller with kernel $K_{trunc,T}$ (i.e., without compensation term) is much worse than that of the structured controller with kernel K_T (i.e., with compensation term). Finally, note that the optimal structured kernel K_T designed following our method has a very reasonable performance compared to that of the centralized kernel K (blue curve in Fig. 2).

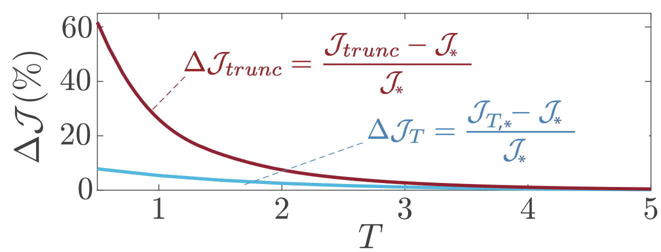

Fig. 2: Normalized suboptimality gap of the structured controllers (red for $K_{trunc,T}$ and blue for K_T) with respect to the centralized controller, as a function of T.

VII. Concluding Remarks

The primary contribution of this paper is a convex formulation for the synthesis of quadratically-optimal structured controllers for spatially invariant systems in which the state is a scalar-valued field. For spatially invariant systems, optimal centralized controllers are obtained by spatially convolving measurements with kernels that decay exponentially in space. Guided by this insight, we restricted the structured control feedback operator to belong to the class of spatial convolutions as well. In this class, we imposed spatial locality constraints (i.e., structure) on the controller by specifying a desired compact support for the convolution kernel. Since the structured closed-loop dynamics remained spatially invariant, we exploited spatial Fourier transforms to pose the optimization problem for the structured convolution kernel in the spatial frequency domain. We proved that the resulting kernel optimization problem is convex and its formulation is intuitive and interpretable in the spatial frequency domain. We applied our method to design an optimal structured controller for diffusive dynamics on $\mathbb{R}$. We used this example to highlight some considerations for the numerical optimization and to show that our optimal structured controller outperforms structured controllers designed using the simple spatial truncation strategy proposed in [13]. Furthermore, for diffusive dynamics, numerical results suggest an interesting relation between the optimal centralized and optimal structured control convolution kernels, which we are currently investigating. On-going research efforts also include the extension of our convex formulation beyond scalar spatio-temporal states and beyond L^2 spaces.

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APPENDIX I

In the spatial frequency domain, the dynamics of the closed-loop with structured controller are given by (15), a LTI ODE parametrized by $\lambda \in \hat{\mathbb{G}}$. The initial condition is $\hat{\psi}_0(\lambda)$. Hence,

$$\hat{\psi}_\lambda(t) = e^{\left(\hat{A}_\lambda - \hat{K}_T(\lambda)\right)t}\hat{\psi}_0(\lambda). \tag{32}$$

The corresponding closed-loop performance (12) is:

$$\begin{aligned}
\mathcal{J}_T(\hat{K}_R, \hat{K}_I) &= \int_0^\infty \left(\langle \mathcal{Q}\psi, \psi\rangle + \langle \mathcal{R}u_T, u_T\rangle\right)\mathrm{d}t \\
&\overset{(1)}{=} \int_{\hat{\mathbb{G}}}\int_0^\infty \left(\hat{\psi}_\lambda^* \hat{Q}_\lambda \hat{\psi}_\lambda + \hat{u}_\lambda^* \hat{R}_\lambda \hat{u}_\lambda\right)\mathrm{d}t\mathrm{d}\lambda \\
&\overset{(2)}{=} \int_{\hat{\mathbb{G}}} |\hat{\psi}_0(\lambda)|^2\left(\hat{Q}_\lambda + \hat{R}_\lambda |\hat{K}_T|^2\right)\int_0^\infty e^{2\Re(\hat{A}_\lambda - \hat{K}_T)t}\mathrm{d}t\mathrm{d}\lambda \\
&\overset{(3)}{=} \frac{1}{2}\int_{\hat{\mathbb{G}}} |\hat{\psi}_0(\lambda)|^2 \frac{\hat{Q}_\lambda + \hat{R}_\lambda(\hat{K}_R^2 + \hat{K}_I^2)}{\hat{K}_R - \Re(\hat{A}_\lambda)}\mathrm{d}\lambda.
\end{aligned} \tag{33}$$

(1): By Plancherel's and Tonelli's theorems. (2): After substitution of (11) and (32), and commuting terms (by Assumption 3, scalars). (3): After integration in time under the assumption of stable closed-loop and substitution of $\hat{K}_T = \hat{K}_R + j\hat{K}_I$.

Alternatively, (33) can be derived from a Lyapunov equation parametrized by $\lambda \in \hat{\mathbb{G}}$:

$$(\hat{A}_\lambda - \hat{K}_T)^* \hat{P}_T(\lambda) + \hat{P}_T(\lambda)(\hat{A}_\lambda - \hat{K}_T) + \hat{Q}_\lambda + \hat{K}_T^* \hat{R}_\lambda \hat{K}_T = 0.$$

Since by Assumption 3 the Lyapunov equation is scalar at each $\lambda \in \hat{\mathbb{G}}$, $\hat{P}_T(\lambda)$ can be explicitly solved for. The corresponding performance is $\mathcal{J}_T = \int_{\hat{\mathbb{G}}} |\hat{\psi}_0(\lambda)|^2 \hat{P}_T(\lambda)\mathrm{d}\lambda$, which agrees with (33).