

Luttinger sum rules and spin fractionalization in the $SU(N)$ Kondo lattice

Tamaghna Hazra

Center for Materials Theory, Rutgers University, Piscataway, New Jersey, 08854, USA

Piers Coleman

*Center for Materials Theory, Rutgers University, Piscataway, New Jersey, 08854, USA and
Department of Physics, Royal Holloway, University of London, Egham, Surrey TW20 0EX, UK*

We show how Oshikawa's theorem for the Fermi surface volume of the Kondo lattice can be extended to the $SU(N)$ symmetric case. By extending the theorem, we are able to show that the mechanism of Fermi surface expansion seen in the large N mean-field theory is directly linked to the expansion of the Fermi surface in a spin-1/2 Kondo lattice. This linkage enables us to interpret the expansion of the Fermi surface in a Kondo lattice as a fractionalization of the local moments into heavy electrons. Our method allows extension to a pure $U(1)$ spin liquid, where we find the volume of the spinon Fermi surface by applying a spin-twist, analogous to Oshikawa's flux insertion. Lastly, we discuss the possibility of interpreting the FL^* phase characterized by a small Fermi surface in the absence of symmetry breaking, as a non-topological coexistence of such a $U(1)$ spin liquid and an electronic Fermi liquid.

I. INTRODUCTION

Two decades ago, Oshikawa[1] applied the Lieb-Schultz-Mattis approach[2] to the Kondo lattice, using its response to a flux insertion to demonstrate that its Fermi surface volume counts the combined density of electrons and local moments. Although the expansion of the Fermi surface in the Kondo lattice had been informally established from arguments of continuity based on the Anderson lattice model[3], from the large N limit of the Kondo lattice[4–7], Oshikawa's result provided a rigorous foundation for the Fermi surface expansion in a strict, $S=1/2$ system Kondo lattice.

Curiously, in the twenty years that have elapsed since this hallmark development, Oshikawa's result has not been generalized to higher group symmetries. Here we show that this generalization is readily established for a family of $SU(N)$ Kondo lattices. The key result, is that for local moments in an antisymmetric representation of the group constructed from Q elementary spinons, a Fermi liquid ground-state will have an expanded Fermi surface volume V_{FS} given by

$$Nv_c \frac{V_{FS}}{(2\pi)^D} = n_e + N_S Q, \quad (1)$$

where n_e and N_S are respectively, the number of electrons and number of local moments per unit cell of volume v_c . For all N , the electronic Fermi surface expands to incorporate the number of elementary spinons forming the local moments, and by increasing N to arbitrarily large values, we can link Oshikawa's original result to the basin of attraction of large N field theoretic approaches to the Kondo lattice[4, 5, 7]. The importance of this link, is that the Kondo fractionalization of local moments into *charged* heavy fermions, inferred field theoretically, is rigorously confirmed.

One of the unexpected outcomes of our analysis, is the discovery that Oshikawa's flux attachment method can also be applied to spin liquids [8, 9]. Previously, it was assumed that since spin liquids are neutral, they are immune to flux attachment, stimulating an alternative topological interpretation of spin-liquid ground-states in co-existence with a Fermi liquid. However, because the unitary transformation that attaches a flux involves both a charge and a spin-twist of the wavefunction, a spin-liquid is sensitive to the flux attachment. This enables us to show that a $U(1)$ spin liquid in an $SU(N)$ Heisenberg model, will have a Fermi surface volume determined by purely by the number of spinons in the representation, i.e

$$Nv_c \frac{V_{FS}}{(2\pi)^D} = N_S Q, \quad (2)$$

This result suggests that fractionalization in a $U(1)$ spin liquid and the Kondo lattice does not require a topological interpretation, i.e that fractionalization and topology are not inevitably tied together.

The outline of this paper is as follows. In Section II, we derive the Luttinger sum rule for the $SU(N)$ Kondo Lattice. In Section III, we interpret the result as a signature of spin fractionalization, cementing an intuition derived from the large- N mean-field theories as a general feature of the Kondo lattice. In Section IV we show how the method can be extended to a Kondo Heisenberg model. In Section V, we discuss the role of spin-exchange interactions and identify the spinon Fermi surface volume of a $U(1)$ spin liquid. Finally in Section VI we discuss whether the co-existence of a spin and small Fermi surface conduction fluid, to form an FL^* requires a topological interpretation.

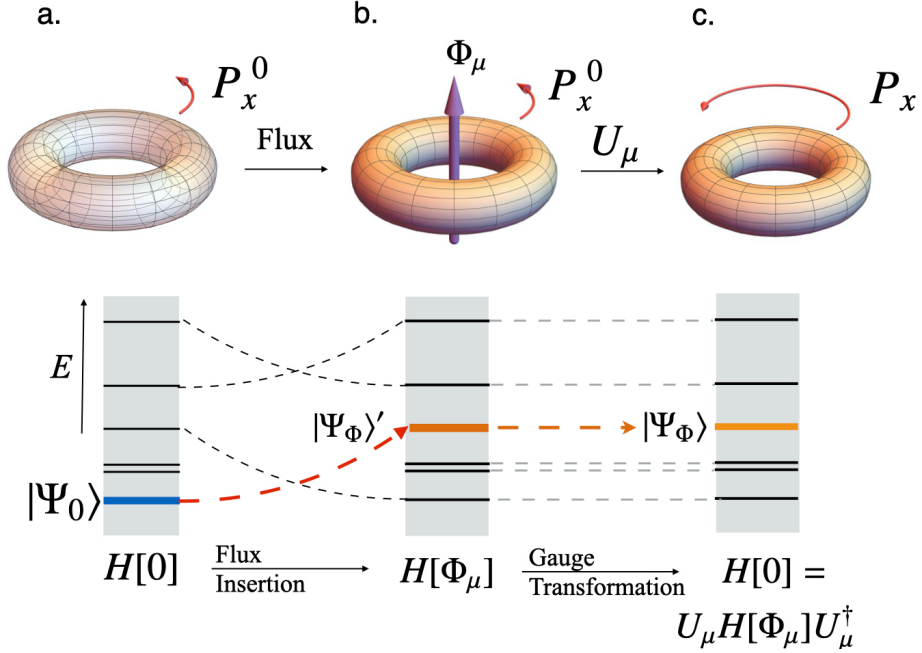


Figure 1. Flux insertion strategy: a) Initial state $|\Psi_0\rangle$ with momentum P_x^0 , b) State $|\Psi_\Phi\rangle'$ after flux insertion for electrons with spin component μ , has unchanged canonical momentum, c) after gauge transformation, $|\Psi_\Phi\rangle = U_\mu |\Psi_\Phi\rangle'$ has canonical momentum P_x . The change in momentum $\Delta P_x = P_x - P_x^0$ determines the Fermi surface volume.

II. DERIVATION

We consider the $SU(N)$ symmetric Kondo Lattice

$$H_{KL} = - \sum_{\mathbf{r}\mathbf{r}'} t_{\mathbf{r},\mathbf{r}'} c_{\mathbf{r}\sigma}^\dagger c_{\mathbf{r}'\sigma} + J_K \sum_{\mathbf{r}} \vec{\lambda}_{\mathbf{r}} \cdot \vec{\Lambda}_{\mathbf{r}}, \quad (3)$$

where $c_{\mathbf{r}\sigma}^\dagger$, ($\sigma = 1, N$) creates an electron at site $\mathbf{r}$, moving on a D -dimensional toroid with intersite hopping amplitude $t_{\mathbf{r},\mathbf{r}'}$, with dimensions $L_x, L_y \dots L_D$. $\vec{\lambda}_{\mathbf{r}} = c_{\mathbf{r}\sigma}^\dagger \vec{\lambda}_{\sigma\sigma'} c_{\mathbf{r}\sigma'}$ is the electron spin-density at $\mathbf{r}$, where the $\vec{\lambda} = (\lambda^1, \dots, \lambda^{N^2-1})$ are the $SU(N)$ Gell-Mann matrices. The $\vec{\Lambda}_{\mathbf{r}} = (\Lambda_{\mathbf{r}}^1, \dots, \Lambda_{\mathbf{r}}^{N^2-1})$ are the components of the localized moment at site $\mathbf{r}$. We shall consider local moments composed of Q elementary spinons, in an antisymmetric representation of $SU(N)$, $|\sigma_1, \dots, \sigma_Q\rangle = (-1)^P |\sigma_{P_1} \dots \sigma_{P_Q}\rangle$. The action of the spin operator Λ^a , $a = (1, N^2 - 1)$ on these states is then $\Lambda^a |\sigma_1 \dots \sigma_Q\rangle = \sum_{n=1}^Q |\sigma_1 \dots \sigma'_n \dots \sigma_Q\rangle \lambda_{\sigma'_n \sigma_n}^a$.

The $SU(N)$ Kondo lattice has a global $U(1) \times SU(N)$ symmetry, associated with the conserved electron number N_e and magnetization $M^a = \sum_{\mathbf{r}} \lambda_{\mathbf{r}}^a + \Lambda_{\mathbf{r}}^a$. Of particular interest, are the diagonal components of the magnetization, M^μ , ($\mu \in [1, N-1]$), which form the Cartan sub-algebra of the $SU(N)$ group, with Gell-Mann matrices $\lambda_{\sigma\sigma'}^\mu = (\delta^{\mu\sigma} - 1/N) \delta_{\sigma\sigma'}$.

Oshikawa's strategy (see Fig. 1) is to introduce a unit magnetic flux quantum $\Phi_\mu = \frac{h}{e}$ that couples to the μ th

spin component of the Fermi sea, giving rise to an inductive current which increases the mechanical momentum by an $\Delta P_x = 2\pi/L_x \times V/(2\pi)^D \times V_{FS}^\mu$, directly proportional to the Fermi surface volume. Since the flux insertion does not change the many-body energy eigenstates, it is equivalent to a unitary transformation U_μ of the original Hamiltonian, $H[\Phi_\mu] = U_\mu^\dagger H[0] U_\mu$. This enables a direct calculation of the change in the mechanical momentum due to flux insertion in terms of microscopic quantities. Equating the direct calculation with the Fermi liquid result determines the Fermi surface volume.

We now apply this strategy to the $SU(N)$ Kondo lattice. Flux insertion is achieved by a Peierls substitution $t_{\mathbf{r},\mathbf{r}'} \rightarrow t_{\mathbf{r},\mathbf{r}'} e^{-i\mathbf{A}^\sigma \cdot (\mathbf{r}-\mathbf{r}')}$, where $\mathbf{A}^\sigma = \delta^{\mu\sigma} \left(\frac{2\pi}{L_x} \right) \hat{\mathbf{x}}$. (Note, we are using natural units in which $e = \hbar = 1$ and the dimensions of the unit cell are rescaled to be unity, so that the unit cell volume $v_c = 1$.) This additional gauge field is generated by a large gauge transformation of the electron fields $U_\mu^\dagger c_{\mathbf{r}\sigma}^\dagger U_\mu = c_{\mathbf{r}\sigma}^\dagger e^{-i\mathbf{A}^\sigma \cdot \mathbf{r}}$. The obvious guess, $U_\mu = e^{\frac{2\pi i}{L_x} \sum_{\mathbf{r}} x_{\mathbf{r}} n_{\mathbf{r}}^\mu}$ does not leave the Kondo interaction invariant, but a modified transformation

$$U_\mu = e^{\frac{2\pi i}{L_x} \sum_{\mathbf{r}} x_{\mathbf{r}} (n_{\mathbf{r}}^\mu + \Lambda_{\mathbf{r}}^\mu + Q/N)}, \quad (4)$$

satisfies this requirement. This is a generalization of Oshikawa's original transformation, in which we have replaced the $SU(2)$ generator $S_{\mathbf{r}}^z$ by $\Lambda_{\mathbf{r}}^\mu$. We have also

added an additional gauge transformation which multiplies the wavefunction by a factor $e^{\frac{2\pi i}{L_x} x_{\mathbf{r}} Q/N}$ at each site, which ensures that the unitary transformation preserves the periodic boundary conditions. $U_{\mu}(\{x_{\mathbf{r}}\}) = U_{\mu}(\{x_{\mathbf{r}} + L_x\})$. U_{μ} is actually a product of a U(1) and an SU(N) gauge transformation: in other words, to selectively impart momentum to the μ th Fermi surface we must “twist” the wavefunction in charge *and* spin space.

To see that U_{μ} commutes with the Kondo interaction we write $n_{\mathbf{r}}^{\mu} = \lambda_{\mathbf{r}}^{\mu} + n_{\mathbf{r}}/N$ so that

$$U_{\mu} = e^{\frac{2\pi i}{L_x} \sum_{\mathbf{r}} x_{\mathbf{r}} [(n_{\mathbf{r}} + Q)/N + M_{\mathbf{r}}^{\mu}]}. \quad (5)$$

involves the electron density $n_{\mathbf{r}}$ and local magnetization

$M_{\mathbf{r}}^{\mu} = \lambda_{\mathbf{r}}^{\mu} + \Lambda_{\mathbf{r}}^{\mu}$, which both commute with the Kondo interaction. To confirm that the transformation also preserves periodic boundary conditions, we note that if we shift the x-component of the site at $\mathbf{r}_0$ by L_x , i.e $x_{\mathbf{r}_0} \rightarrow x_{\mathbf{r}_0} + L_x$, the unitary transformation picks up an additional factor $e^{2\pi i(n_{\mathbf{r}_0}^{\mu} + \Lambda_{\mathbf{r}_0}^{\mu} + q)} = e^{2\pi i(\Lambda_{\mathbf{r}_0}^{\mu} + q)}$, where $q = Q/N$ and we have used the fact that the $n_{\mathbf{r}}^{\mu}$ are integers. But under a 2π rotation, an SU(N) spin, picks up a phase factor i.e $e^{2\pi i \Lambda_{\mathbf{r}_0}^{\mu}} = e^{-2\pi i q}$, so that the factor $e^{2\pi i(\Lambda_{\mathbf{r}_0}^{\mu} + q)} = 1$ and the unitary transformation U_{μ} preserves periodic boundary conditions.

Written in full, the Hamiltonian with flux inserted is

$$H[\Phi_{\mu}] = - \sum_{\mathbf{r}\mathbf{r}'\sigma} t_{\mathbf{r},\mathbf{r}'} e^{-i\mathbf{A}^{\sigma} \cdot (\mathbf{r} - \mathbf{r}')} c_{\mathbf{r}\sigma}^{\dagger} c_{\mathbf{r}'\sigma} + J_K \sum_{\mathbf{r}} \vec{\lambda}_{\mathbf{r}} \cdot \vec{\Lambda}_{\mathbf{r}},$$

$$\mathbf{A}^{\sigma} = \delta^{\sigma\mu} \left(\frac{2\pi}{L_x} \right) \hat{\mathbf{x}}. \quad (6)$$

The process of flux insertion involves adiabatically increasing $\mathbf{A}^{\sigma}(t) = \mathbf{A}^{\sigma} e^{-|t|/\tau}$ from zero at $t = -\infty$ to its full value at $t = 0$, taking $\tau \gg (1/T_K)$ to be much longer than the inverse Kondo temperature, so that the initial eigenstate $|\psi^0\rangle$ evolves smoothly into an excited eigenstate $|\psi_{\Phi}\rangle'$ of $H_{KL}[\Phi_{\mu}]$ (see Fig. 1).

Since translational symmetry is preserved by flux insertion, and since the exponential of the canonical momentum e^{-iP_x} is the eigenstate of translation, it follows that the state retains a fixed canonical momentum $P_x(t) = P_x^0$ so that under a translation,

$$T_x |\psi_{\Phi}\rangle' = e^{-iP_x^0} |\psi_{\Phi}\rangle'. \quad (7)$$

We can obtain the mechanical momentum P_x of the final state $|\psi_{\Phi}\rangle'$ by noting that since this quantity is gauge invariant, it is unchanged when we gauge transform back into the original gauge. Now since $H_{KL}[0] = U_{\mu} H_{KL}[\Phi_{\mu}] U_{\mu}^{\dagger}$, it follows that $|\psi_{\Phi}\rangle = U_{\mu} |\psi_{\Phi}\rangle'$ is the corresponding transform of $|\psi_{\Phi}\rangle'$ back into the original gauge. But since the vector potential is now absent, the mechanical and canonical momentum coincide and can be determined from a translation,

$$T_x |\psi_{\Phi}\rangle = e^{-iP_x} |\psi_{\Phi}\rangle. \quad (8)$$

Since $T_x |\psi_{\Phi}\rangle = (T_x U_{\mu} T_x^{-1}) T_x |\psi_{\Phi}\rangle'$, it follows that

$$e^{-iP_x} |\psi_{\Phi}\rangle = (T_x U_{\mu} T_x^{-1}) e^{-iP_x^0} |\psi_{\Phi}\rangle'. \quad (9)$$

Now $T_x U_{\mu} T_x^{-1}$ describes the effect of translating the operator U_{μ} by one lattice spacing in the $\hat{x}$ direction, so that

$$(T_x U_{\mu} T_x^{-1}) = \exp \left[\frac{2\pi i}{L_x} \sum_{\mathbf{r}} x_{\mathbf{r}} (n_{\mathbf{r}+\hat{x}}^{\mu} + \Lambda_{\mathbf{r}+\hat{x}}^{\mu} + q) \right] = \exp \left[\frac{2\pi i}{L_x} \sum_{\mathbf{r}} x_{\mathbf{r}-\hat{x}} (n_{\mathbf{r}}^{\mu} + \Lambda_{\mathbf{r}}^{\mu} + q) \right], \quad (10)$$

where inside the sum, we have shifted the x-coordinate of the position vectors $\mathbf{r}$, $\mathbf{r} \rightarrow \mathbf{r} - \hat{x}$. Now naively we might expect $x_{\mathbf{r}-\hat{x}} = x_{\mathbf{r}} - 1$. However this is not the case with sites on the first layer of the crystal, for in this case $x_{1-1} = x_0$, but the periodic boundary conditions mean that $x_0 = x_{L_x} = L_x = x_1 - 1 + L_x$. Thus in general, $x_{\mathbf{r}-\hat{x}} = x_{\mathbf{r}} - 1 + L_x \delta_{r_1,1}$. Substituting this into (10) we obtain

$$(T_x U_{\mu} T_x^{-1}) = \exp \left[\frac{2\pi i}{L_x} \sum_{\mathbf{r}} (x_{\mathbf{r}} - 1 + L_x \delta_{r_1,1}) (n_{\mathbf{r}}^{\mu} + \Lambda_{\mathbf{r}}^{\mu} + q) \right]$$

$$\begin{aligned}
&= \exp \left[\frac{2\pi i}{L_x} \sum_{\mathbf{r}} (L_x \delta_{r_{1,1}} - 1) (n_{\mathbf{r}}^\mu + \Lambda_{\mathbf{r}}^\mu + q) \right] U_\mu \\
&= \exp \left[2\pi i \sum_{\mathbf{r}_\perp} (n_{1,\mathbf{r}_\perp}^\mu + \Lambda_{1,\mathbf{r}_\perp}^\mu + q) \right] \exp \left[-\frac{2\pi i}{L_x} \sum_{\mathbf{r}} (n_{\mathbf{r}}^\mu + \Lambda_{\mathbf{r}}^\mu + q) \right] U_\mu
\end{aligned} \tag{11}$$

The first term derives from the crystal boundary at $x_{\mathbf{r}} = 1$, derived from the shift of the x-coordinates by L_x . However, since we have chosen a gauge where U_μ is invariant under such co-ordinate shifts, this pre-factor is unity ($e^{2\pi i(n_{1,\mathbf{r}_\perp}^\mu + \Lambda_{1,\mathbf{r}_\perp}^\mu + q)} = 1$). Our final answer for the translated U_μ is then

$$(T_x U_\mu T_x^{-1}) = \exp \left[-\frac{2\pi i}{L_x} \sum_{\mathbf{r}} (n_{\mathbf{r}}^\mu + \Lambda_{\mathbf{r}}^\mu + q) \right] U_\mu. \tag{12}$$

We note that this answer is also obtained with Oshikawa's original choice of $U_\mu = e^{\frac{2\pi i}{L_x} \sum_{\mathbf{r}} x_{\mathbf{r}} (n_{\mathbf{r}}^\mu + \Lambda_{\mathbf{r}}^\mu)}$, but in this case, the q-dependence derives from the boundary term.

From (9) it then follows that

$$e^{-iP_x} |\psi_\Phi\rangle = e^{-iP_x^0} e^{-\frac{2\pi i}{L_x} \sum_{\mathbf{r}} (n_{\mathbf{r}}^\mu + \Lambda_{\mathbf{r}}^\mu + q)} |\psi_\Phi\rangle, \tag{13}$$

i.e, flux insertion changes the mechanical momentum by

$$\Delta P_x = \frac{2\pi}{L_x} \sum_{\mathbf{r}} (n_{\mathbf{r}}^\mu + \Lambda_{\mathbf{r}}^\mu + q). \tag{14}$$

For N_s spins per unit cell,

$$\Delta P_x = \frac{2\pi}{L_x} V [\nu^\mu + N_s (m^\mu + q)] \mod 2\pi, \tag{15}$$

where $V = L_x L_y \dots L_D$ is the system volume, while $\nu^\mu = (1/V) \sum_{\mathbf{r}} n_{\mathbf{r}}^\mu$ and $m^\mu = (1/V) \sum_{\mathbf{r}} \Lambda_{\mathbf{r}}^\mu$ are the μ -th filling fraction and magnetization respectively.

Alternatively, if we assume a Fermi liquid ground state, we can compute the change in momentum by observing that coupling to the gauge potential shifts the momentum of each μ -quasiparticle by $2\pi/L_x$, so that $\Delta P_x = \frac{2\pi}{L_x} N_F^\mu$ where N_F^μ is the number of μ -quasiparticles. The quasiparticle number operator $\tilde{n}_{\mathbf{k}\mu}$ is conserved in a Fermi liquid and jumps from 1 to 0 across the Fermi surface. This allows us to relate the shift in momentum to the volume of the μ -Fermi surface $V_{FS}^\mu = N_F^\mu (2\pi)^D / V$

$$\Delta P_x = \frac{2\pi}{L_x} V \left[\frac{V_{FS}^\mu}{(2\pi)^D} \right]. \tag{16}$$

Comparing Eq. (15) and Eq. (16) we find

$$V \frac{V_{FS}^\mu}{(2\pi)^D} = V (\nu^\mu + N_s (m^\mu + q)) + n_x L_x, \tag{17}$$

with $n_x \in \mathbb{Z}$. Now since the remainder term $n_x L_x$ can be calculated for a flux threading in any of the D directions, the remainder is also equal to $n_y L_y, \dots, n_D L_D$, where the n_i ($i = 1, D$) are distinct integers for each

direction. But since integer remainder is independent of direction, $n_x L_x = n_y L_y = \dots n_D L_D$. If we choose the $L_x, L_y, \dots, L_D$ to be coprime (no common denominators), it follows that n_x is proportional to each of the $n_y L_y, \dots, n_D L_D$, so that it follows that the remainder is a multiple of the full product, i.e the volume $V = L_x \dots L_D$. Factoring out the volume V , we obtain

$$\frac{V_{FS}^\mu}{(2\pi)^D} = \nu^\mu + N_s (m^\mu + q) + n \tag{18}$$

Since the Fermi surface volume is an intensive quantity, the remainder n is independent of the convenient choice of mutually coprime boundary lengths, and Eq. (19) is valid in the thermodynamic limit.

Finally, if we trace over all N Fermi surfaces, since the members of the Cartan sub-algebra are traceless, it follows that $\sum_{\mu} m^\mu = 0$ so that

$$N v_c \frac{V_{FS}}{(2\pi)^D} = n_e + N_s Q \tag{19}$$

where $n_e = \sum_{\mu} \nu_{\mu}$ and we have restored the engineering dimensions of the unit cell volume v_c , and have dropped the integer remainder $p = nN$, with the understanding that the Fermi surface volume is only defined mod $(2\pi)^D$.

III. THE LINK WITH FRACTIONALIZATION

Traditionally, the localized spins of a Kondo lattice are written in terms of an Abrikosov pseudo-fermion representation

$$\Lambda_{\mathbf{r}}^a = f_{\mathbf{r}\sigma}^\dagger \lambda_{\sigma\sigma'}^a f_{\mathbf{r}\sigma'}, \quad (a = 1, N^2 - 1) \tag{20}$$

with a constraint on the local f -fermion (spinon) density $n_{\mathbf{r}}^{(f)} = \sum_{\mu} f_{\mathbf{r}\mu}^\dagger f_{\mathbf{r}\mu} = Q$ which determines the number of spinons contained in the Q th antisymmetric representation of $SU(N)$. With hindsight, we now see that

since the constraint commutes with every operator involved in the proof, we could have used this representation from the outset, but by tacitly avoiding doing so, we avoided any lingering concerns about the constraint.

In the Abrikosov representation, the Kondo Lattice Hamiltonian takes the form [4]

$$H_{KL} = - \sum_{\mathbf{r}\mathbf{r}'} t_{\mathbf{r},\mathbf{r}'} c_{\mathbf{r}\sigma}^\dagger c_{\mathbf{r}'\sigma} - \frac{\tilde{J}_K}{N} \sum_{\mathbf{r}} c_{\mathbf{r}\sigma}^\dagger f_{\mathbf{r}\sigma} f_{\mathbf{r}\sigma}^\dagger c_{\mathbf{r}\sigma} \quad (21)$$

which explicitly commutes with the constraint $n_{f\mathbf{r}} = Q$ and the number of conduction electrons $n_{c\mathbf{r}}$ at site $\mathbf{r}$. With the normalization $\text{Tr}[\lambda^a \lambda^b] = (1 - \frac{1}{N})\delta^{ab}$ set by the Cartan sub-algebra, the coupling constants of the Read-Newns form and the original model (3) are related by $\tilde{J}_K = J_K(N - 1)$.

The Cartan elements are now represented by $\Lambda_{\mathbf{r}}^\mu = n_{f\mathbf{r}}^\mu - Q/N$, so that the gauge transformation (4) that imposes the flux insertion is given by

$$U^\mu = \exp \left[\frac{2\pi i}{L_x} \sum_{\mathbf{r}} x_{\mathbf{r}} (n_{c\mathbf{r}}^\mu + n_{f\mathbf{r}}^\mu) \right]. \quad (22)$$

(22) is literally, a large gauge transformation that counts the f -spinons as quasiparticles. The conduction electrons and spinons transform identically under the flux insertion,

$$U_\mu^\dagger \begin{pmatrix} c_{\mathbf{r}\sigma}^\dagger \\ f_{\mathbf{r}\sigma}^\dagger \end{pmatrix} U_\mu = e^{-i\mathbf{A}^\sigma \cdot \mathbf{r}} \begin{pmatrix} c_{\mathbf{r}\sigma}^\dagger \\ f_{\mathbf{r}\sigma}^\dagger \end{pmatrix}. \quad (23)$$

In other words, the structure of the unitary transformation, forced upon us by the Kondo coupling, means that the spinons behave exactly as charged particles under the flux attachment, consistent with a fractionalization of spins into heavy electrons in the Fermi liquid phase. Remarkably then, the seeds of fractionalization are present in the original Oshikawa gauge transformation.

The final form of the Luttinger sum rule

$$\frac{V_{FS}^\mu}{(2\pi)^D} = \nu^\mu + \nu_f^\mu, \quad \text{mod}(1) \quad (24)$$

where $\nu_f^\mu = \frac{N_s}{V} \sum_{\mathbf{r}} n_{f\mathbf{r}}^\mu = N_s(m^\mu + q)$ is the number of spinons with spin index μ per unit cell, is not a surprise, because the $U(1) \times SU(N)$ gauge transformation (22) audits every spinon entangled into the Fermi sea.

Traditionally, the Kondo Fermi surface expansion is interpreted by identifying the Kondo Hamiltonian as the strong coupling renormalization of a periodic Anderson model with the same filling[3]. However, a Kondo lattice Hamiltonian has no knowledge of its high energy origins. From a renormalization group perspective, the Kondo lattice lies on the common scaling trajectory of many high energy “microscopic” Hamiltonians. Indeed, the model is entirely agnostic as to the

origin of the local moments, and they need not have an electronic origin at all, for instance, they equally could be nuclear spins, with a Kondo interaction derived from hyperfine interactions. The main point is that since the Kondo lattice has no knowledge of its high energy origins, fractionalization in the Kondo lattice is an emergent property. This alternate interpretation allows us to contemplate the possibility that different kinds of spin fractionalization may develop in the approach to magnetism, or spin liquid behavior.

IV. KONDO HEISENBERG MODEL

We now consider an extension of our results to a Kondo Heisenberg model: a Kondo lattice with additional Heisenberg interactions, $H_{KH} = H_{KL} + H_H$, where now

$$H_H = \sum_{\langle \mathbf{r}\mathbf{r}' \rangle} J_{\mathbf{r},\mathbf{r}'} \vec{\Lambda}_{\mathbf{r}} \cdot \vec{\Lambda}_{\mathbf{r}'}. \quad (25)$$

From Doniach’s original arguments[10], we know that for large enough T_K , the Kondo interaction will stabilize a Fermi liquid, in which case, we expect Oshikawa’s result to generalize to the Kondo Heisenberg model. We are particularly interested in the case of frustrated Kondo lattices, where in the limit of small T_K , rather than forming a state of long-range magnetic order, the system develops into spin liquid, preserving the Fermi surface of the underlying spinons. We shall show that Oshikawa’s theorem can be extended to this case.

Naively, one might expect flux insertion to only affect charge particles, leaving the Heisenberg term alone. However, the unitary transformation that accomplishes flux insertion (4), $U_\mu = e^{\frac{2\pi i}{L_x} \sum_{\mathbf{r}} x_{\mathbf{r}} (n_{c\mathbf{r}}^\mu + \Lambda_{\mathbf{r}}^\mu + q)}$, adds a charge and a spin-flux to the system, thus affecting the Heisenberg interaction terms. Under the gauge transformation, the local moments transform under the adjoint representation of $SU(N)$. To keep track of these transformations, it’s simpler to switch to a Coqblin Schrieffer representation of the local moments, $\Lambda_{\mathbf{r}}^{\sigma\sigma'} = f_{\mathbf{r}\sigma}^\dagger f_{\mathbf{r}\sigma'} - \frac{Q}{N} \delta_{\sigma\sigma'}$, so that the Heisenberg interaction takes the form

$$H_H = \frac{1}{N} \sum_{\langle \mathbf{r}\mathbf{r}' \rangle} \tilde{J}_{\mathbf{r},\mathbf{r}'} \Lambda_{\mathbf{r}}^{\sigma\sigma'} \Lambda_{\mathbf{r}'}^{\sigma'\sigma}, \quad (26)$$

where $\tilde{J}_{\mathbf{r},\mathbf{r}'} = J_{\mathbf{r},\mathbf{r}'}(N - 1)$. Under the flux insertion, $f_{\mathbf{r}\sigma} \rightarrow e^{i\mathbf{A}^\sigma \cdot \mathbf{r}} f_{\mathbf{r}\sigma}$, so that under the gauge transformation (23),

$$\Lambda_{\mathbf{r}}^{\sigma\sigma'} \rightarrow U_\mu^\dagger \Lambda_{\mathbf{r}}^{\sigma\sigma'} U_\mu = e^{-i(\mathbf{A}^\sigma - \mathbf{A}^{\sigma'}) \cdot \mathbf{r}} \Lambda_{\mathbf{r}}^{\sigma\sigma'}, \quad (27)$$

which describes the transformation of the spin operator under the adjoint representation of $SU(N)$, corresponding to a slow twist of the local moments about the “ μ ”

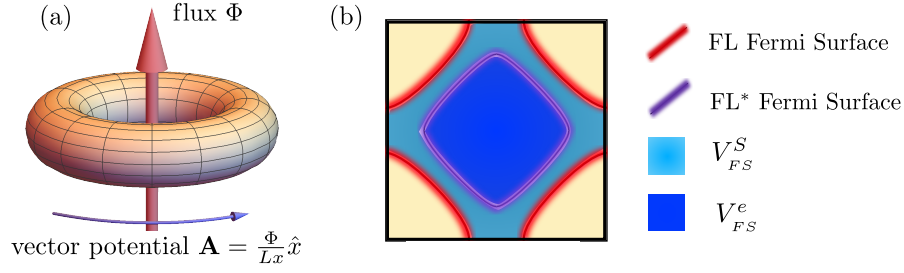


Figure 2. (a) Flux attachment in the Kondo Heisenberg model. Threading a flux results in a twist in the U(1) gauge potential and a twist in the spin orientations, imparting momentum to the electrons and the spinons. (b) The total momentum is proportional to the combined Fermi surface volume of the electrons and spinons (red). In the FL* phase the spinons decouple from the electrons to form a U(1) spin liquid, resulting in a smaller Fermi surface (purple) that only counts the electrons.

axis, created by the spin component of U_μ , through an angle $2\pi(x/L_x)$ that increases from 0 to 2π across the sample.

Using these results, we can write Heisenberg Kondo model with a flux insertion in the μ spin channel as

$$H_{KH}[\Phi_\mu] = - \sum_{\mathbf{r}, \mathbf{r}'} t_{\mathbf{r}, \mathbf{r}'} e^{-i\mathbf{A}^\sigma \cdot (\mathbf{r} - \mathbf{r}')} c_{\mathbf{r}\sigma}^\dagger c_{\mathbf{r}'\sigma} + \frac{\tilde{J}_K}{N} \sum_{\mathbf{r}} c_{\mathbf{r}\sigma}^\dagger c_{\mathbf{r}\sigma'} \Lambda_{\mathbf{r}}^{\sigma\sigma} + \frac{1}{N} \sum_{\langle \mathbf{r}\mathbf{r}' \rangle} \tilde{J}_{\mathbf{r}, \mathbf{r}'} e^{-i(\mathbf{A}^\sigma - \mathbf{A}^{\sigma'}) \cdot (\mathbf{r} - \mathbf{r}')} \Lambda_{\mathbf{r}}^{\sigma\sigma'} \Lambda_{\mathbf{r}'}^{\sigma'\sigma}. \quad (28)$$

The gauge field inside the Heisenberg term

$$e^{-i\mathbf{A} \cdot (\mathbf{r} - \mathbf{r}')(\delta_{\sigma\mu} - \delta_{\sigma'\mu})} = e^{-i\mathbf{A} \cdot (\mathbf{r} - \mathbf{r}')\delta_{\sigma\mu}} e^{i\mathbf{A} \cdot (\mathbf{r} - \mathbf{r}')\delta_{\sigma'\mu}}, \quad (\mathbf{r}' \xrightarrow[\sigma']{\sigma} \mathbf{r}) \quad (29)$$

can be interpreted as the product of two Peierls' insertions associated with a spinon exchange: an σ spinon moving from $\mathbf{r}'$ to $\mathbf{r}$, and a σ' spinon moving in the opposite direction. The derivation and final form of the Luttinger sum rule for the Fermi liquid now follows precisely the same route as in the Kondo model. In particular, the key identity (12) still holds, allowing us to generalize the Oshikawa result(1) to the Fermi liquid phases of the SU(N) Kondo Heisenberg model.

Since our flux insertion works for arbitrary N , it allows us to explicitly examine how the wavefunction $|\Psi_0\rangle$ evolves at large N , allowing us to the explicit evolution under the flux attachment and subsequent gauge transformation,

$$|\Psi_0\rangle \xrightarrow{\Phi_0} |\Psi_\Phi\rangle' \xrightarrow{U_\mu} |\Psi\rangle. \quad (30)$$

At large N , the ground-state wavefunction is accurately determined by a Gutzwiller wavefunction

$$|\Psi_0\rangle = P_G \prod_{\mathbf{k} \in \text{FS}, \sigma} (\alpha_{\mathbf{k}} c_{\mathbf{k}\sigma}^\dagger + \beta_{\mathbf{k}} f_{\mathbf{k}\sigma}^\dagger) |0\rangle \quad (31)$$

where the product runs over all wavevectors enclosed by the Fermi surface, and $P_G = \prod_{\mathbf{r}} \delta_{n_f(\mathbf{r}), Q}$ projects out the component of the wavefunction with $n_f(\mathbf{r}) = Q$ at each site, while the hybridized operators $\alpha_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma}^\dagger + \beta_{\mathbf{k}} f_{\mathbf{k}\sigma}^\dagger$ define the quasiparticles of the mean-field Hamiltonian. In fact, the Gutzwiller projection P_G can be replaced

by an average constraint in the large N limit, but here we shall keep it for greater generality.

In the large N limit, the dynamics of the wavefunction are determined by evolution under a time-dependent, translationally invariant mean-field Hamiltonian which preserves the momenta of the quasiparticle states, leaving the Fermi surface unchanged. After the flux insertion, the mean-field wavefunction then has the form

$$|\Psi_\Phi\rangle' = P_G \prod_{\mathbf{k} \in \text{FS}, \sigma} (\alpha_{\mathbf{k}\sigma}[\Phi] c_{\mathbf{k}\sigma}^\dagger + \beta_{\mathbf{k}\sigma}[\Phi] f_{\mathbf{k}\sigma}^\dagger) |0\rangle \quad (32)$$

where the coefficients $\alpha_{\mathbf{k}\sigma}[\Phi]$ and $\beta_{\mathbf{k}\sigma}[\Phi]$ differ from their zero field value by terms of order $O(1/L)$. Now if we Fourier transform (23) the transformation of the electron and spinon fields under U_μ , in momentum

space is given by

$$U_\mu \begin{pmatrix} c_{\mathbf{k}\sigma}^\dagger \\ f_{\mathbf{k}\sigma}^\dagger \end{pmatrix} U_\mu^\dagger = \begin{pmatrix} c_{\mathbf{k}+\mathbf{A}^\sigma\sigma}^\dagger \\ f_{\mathbf{k}+\mathbf{A}^\sigma\sigma}^\dagger \end{pmatrix}, \quad (33)$$

so that under the unitary transformation U_μ , $U_\mu|\psi_\Phi\rangle' = |\psi_\Phi\rangle$ is given by

$$|\psi_\Phi\rangle = P_G \prod_{\mathbf{k} \in FS, \sigma} \left(\alpha_{\mathbf{k}\sigma} [\Phi] c_{\mathbf{k}+\mathbf{A}^\sigma\sigma}^\dagger + \beta_{\mathbf{k}\sigma} [\Phi] f_{\mathbf{k}+\mathbf{A}^\sigma\sigma}^\dagger \right) |0\rangle, \quad (34)$$

If we translate this state in the x direction, then since

$$T_x \begin{pmatrix} c_{\mathbf{k}\sigma}^\dagger \\ f_{\mathbf{k}\sigma}^\dagger \end{pmatrix} T_x^{-1} = e^{-ik_x} \begin{pmatrix} c_{\mathbf{k}\sigma}^\dagger \\ f_{\mathbf{k}\sigma}^\dagger \end{pmatrix}, \quad (35)$$

it follows that the momentum of the final state P_x is given by

$$T_x |\psi_\Phi\rangle = e^{-iP_x} |\psi_\Phi\rangle, \quad (36)$$

where

$$P_x = \sum_{\mathbf{k} \in FS, \sigma} (k_x + A_x^\sigma) = P_x^{(0)} + \frac{2\pi}{L_x} \frac{V}{(2\pi)^D} V_{FS}^\mu \quad (37)$$

so we see that the shift in momentum per quasiparticle is precisely $A_x^\sigma = \frac{2\pi}{L_x}$ in the μ band.

V. U(1) SPIN LIQUID

The fascinating aspect of this result, is that it also allows us to apply the flux attachment idea to a pure Heisenberg model H_H . The Heisenberg model with a spin twist, $H_H[\Phi_\mu] = U_\mu^\dagger H_H U_\mu$ is written

$$H_H[\Phi_\mu] = \frac{1}{N} \sum_{\langle \mathbf{r}\mathbf{r}' \rangle} \tilde{J}_{\mathbf{r},\mathbf{r}'} e^{-i(\mathbf{A}^\sigma - \mathbf{A}^{\sigma'}) \cdot (\mathbf{r} - \mathbf{r}')} \Lambda_{\mathbf{r}}^{\sigma\sigma'} \Lambda_{\mathbf{r}'}^{\sigma'\sigma},$$

$$\mathbf{A}^\sigma = \delta^{\sigma\mu} \frac{2\pi}{L_x} \hat{x} \quad (38)$$

and the corresponding gauge transformation is then

$$U_\mu^s = \exp \left[\frac{2\pi i}{L_x} \sum_{\mathbf{r}} x_{\mathbf{r}} (\Lambda_{\mathbf{r}}^\mu + q) \right]. \quad (39)$$

In this case, the translated gauge transformation takes the form

$$(T_x U_\mu^s T_x^{-1}) = \exp \left[-\frac{2\pi i}{L_x} \sum_{\mathbf{r}} (\Lambda_{\mathbf{r}}^\mu + q) \right] U_\mu^s. \quad (40)$$

so the change in momentum associated with the flux insertion is then

$$\Delta P_x = \frac{2\pi}{L_x} \sum_{\mathbf{r}} (\Lambda_{\mathbf{r}}^\mu + q)$$

$$= \frac{2\pi}{L_x} V (m^\mu + q) \mod 2\pi, \quad (41)$$

where $V = L_x L_y \dots L_D$ is the volume and $m^\mu = \frac{1}{V} \sum_{\mathbf{r}} \Lambda_{\mathbf{r}}^\mu$ is the magnetization, and we have assumed $n_s = 1$ local moment per unit cell. Using Abrikosov fermions, $\Lambda_{\mathbf{r}}^\mu + q = n_{f\mathbf{r}}^\mu$ is the number of μ - spinons at site $\mathbf{r}$, so we can interpret $V(m^\mu + q)$ as the number of spinon with spin component μ . In other words, under a flux attachment, each spinon with spin component μ in the ground-state acquires a momentum $\frac{2\pi}{L}$.

A U(1) spin liquid can be thought of as an incompressible neutral Fermi liquid. In Appendix A, we demonstrate that such a state is energetically favored in the large N limit over the dimer phase and the π flux phase on a square lattice over a range of q . To see how its momentum changes under a flux attachment, consider the model ground-state provided by a Gutzwiller wavefunction

$$|\Psi_0\rangle = P_G \prod_{\mathbf{k} \in FS, \sigma} f_{\mathbf{k}\sigma}^\dagger |0\rangle, \quad (42)$$

where, as in the Kondo lattice, $P_G = \prod_{\mathbf{r}} \delta_{n_f(\mathbf{r}), Q}$ is a Gutzwiller projection onto states with Q elementary spinons at each site. Now the translation operator commutes with P_G , and since $T_x f_{\mathbf{k}\sigma}^\dagger T_x^{-1} = e^{-ik_x} f_{\mathbf{k}\sigma}^\dagger$, it follows that this state has the initial momentum $P_x^{(0)} = \sum_{\mathbf{k} \in FS, \sigma} k_x$. In the large N limit (see Appendix B), the time-evolution of the state is given by a time-dependent mean-field Hamiltonian that is explicitly translationally invariant, so that under a flux attachment, the canonical momenta of the spinons are entirely unchanged. In a one-band fluid of spinons, the corresponding Gutzwiller ground-state is then unchanged after the flux attachment $|\Psi_\Phi\rangle' = |\Psi_0\rangle$. If we now revert back to the original gauge, since $U_\mu f_{\mathbf{k}\sigma}^\dagger U_\mu^{-1} = f_{\mathbf{k}+\mathbf{A}^\sigma, \sigma}^\dagger$, it follows that

$$|\Psi_\Phi\rangle = U_\mu |\Psi_\Phi\rangle' = P_G \prod_{\mathbf{k} \in FS, \sigma} f_{\mathbf{k}+\mathbf{A}^\sigma, \sigma}^\dagger |0\rangle. \quad (43)$$

corresponding to a Fermi sea in which the spinon momenta are shifted by $\mathbf{A}^\sigma = 2\pi/L_x \delta^{\sigma\mu} \hat{x}$, i.e

$$\Delta P_x = \frac{2\pi}{L_x} V \frac{V_{FS}^\mu}{(2\pi)^D} \quad (44)$$

By comparing this result with (41), we then obtain

$$\frac{V_{FS}^\mu}{(2\pi)^D} = (m_\mu + q), \mod 1. \quad (45)$$

We emphasize that this result remains valid at arbitrary N as long as the ground state is smoothly connected to the U(1) spin liquid state in (43). For SU(2) a similar result was also obtained in Ref. [11].

VI. DISCUSSION

It is interesting to consider the implications of our results for the FL* phase of the Kondo lattice model, in which decoupled spin liquid and conduction electrons co-exist in a state of unbroken symmetry. Earlier work on $S = 1/2$ Kondo systems [12, 13] has interpreted this phase as a Z_2 spin liquid coexisting with a Fermi liquid. Flux insertion then drives a transition between two topologically degenerate ground-states characterized by the presence or absence of vizon states that carry Z_2 flux. But is the FL* phase necessarily topologically ordered?

Our result on the Kondo Heisenberg model suggests an alternate interpretation of the FL* phase as the co-existence of a U(1) spin liquid with an electronic Fermi liquid. There are in principle, two phases:

- the heavy Fermi liquid, a Higgs phase in which the U(1) gauge field of the spinons is locked to the electromagnetic U(1) fields of the conduction electrons, giving rise to a single unified Fermi surface of heavy electrons.
- the FL* in which the U(1) gauge fields of the conduction electrons and spinons are decoupled, so that one is neutral, the other charged

Oshikawa's theorem, extended to the Kondo Heisenberg model makes no judgement on which phase one is in, simply predicting that the combined volume of the Fermi surfaces

$$\frac{V_{\text{FS}}^\mu}{(2\pi)^D} = \frac{V_{\text{FS}}^{\mu,S}}{(2\pi)^D} + \frac{V_{\text{FS}}^{\mu,e}}{(2\pi)^D} = N_s(m_\mu + q) + \nu^\mu \quad (46)$$

If the spin liquid decouples from the electronic fluid, then assuming that the U(1) spin liquid is isomorphic to that of the pure Heisenberg model in Section V, the volume of the spinon Fermi surface is given by $\frac{V_{\text{FS}}^{\mu,S}}{(2\pi)^D} = N_s(m_\mu + q)$. In this case, the remaining electronic fluid has a Fermi surface volume

$$\frac{V_{\text{FS}}^{\mu,e}}{(2\pi)^D} = \nu_\mu. \quad (47)$$

From this perspective, the FL* is understood simply as two decoupled fluids, both of which respond to the flux attachment. One of the interesting aspects of this line of reasoning, is that it goes against a commonly held view-point that fractionalization in higher dimensional systems is intimately associated with a topological ground-state. It suggests instead that fractionalization does not require such inevitable linkage, and it opens the way for an interpretation of the Kondo effect as a non-topological fractionalization of local moments.

Such U(1) spin liquids are expected from large N treatments [9, 14–17] and found in variational studies

of Heisenberg-related models [18]. The breakdown of the Kondo effect, and the resulting decoupling between spinons and electrons in these spin liquids would have sharp experimental signatures owing to the coexistence of a charge-neutral spinon Fermi surface. Such spin liquids are among the prime candidates to account for the anomalous signatures in thermal conductivity [19], spin susceptibility [18] and anomalous quantum oscillations [20–22] that have been observed in various experiments. While this work makes no claim on the resolution of these experimental puzzles, an exact result on the volume of the spinon Fermi surface in a U(1) spin liquid is a valuable benchmark to compare against. Indeed, in a two-fluid picture [23], Fermi surface volumes intermediate between the large and small Fermi surface may be fit to a heuristic form $V_{\text{FS}} = N_s q + f\nu$ to estimate f , the fraction of the spins that are gauge-coupled to the electrons.

One of the unsolved questions, is whether Oshikawa's approach can be extended to other models? Central to the current derivation of the Luttinger sum rule is the identification of a U(1) gauge symmetry associated with *each* of the N spin components, and the presence of translational symmetry. There are two models that fail these requirements:

- the Kondo impurity model, where fractionalization, and the large N limit tell us that the scattering phase shift is given by $\delta = \pi Q/N$ [7].
- the family of symplectic $\text{SP}(2N)$ symmetric Kondo lattices, important for extending the notion of pairing to the large N limit [24, 25].

At first sight, the absence of a conserved momentum would seem to preclude using flux attachment on the impurity Kondo model, however however, by representing the impurities as left moving particles in a fluid of right-moving electrons, as in Bethe-Ansatz solutions of this problem [14], it may be possible to restore translational invariance required for flux attachment.

Likewise, the absence of a large number of U(1) subgroups in $\text{SP}(2N)$ appears to sabotage the application of Oshikawa's theorem to this case. However, here too, there may be a way out, for the total number of “up” electrons and spinons is still a conserved U(1) invariant, so that if we attach a flux to all the up electrons and spinons, a Fermi surface sum rule may still be possible. These topics can be considered in future work.

ACKNOWLEDGMENTS

This work was supported by NSF grant DMR-1830707 (PC, TH).

Appendix A: Stability of the U(1) spin liquid in the large N limit

The nearest-neighbor Heisenberg model is described by the path integral

$$\mathcal{Z} = \int \mathcal{D}[f^\dagger, f, \lambda] \exp \left[- \int_0^\beta d\tau \mathcal{L}(\tau) \right]$$

$$\mathcal{L} = \sum_{\mathbf{r}} (f_{\mathbf{r}\sigma}^\dagger (\partial_\tau + \lambda_{\mathbf{r}}) f_{\mathbf{r}\sigma} - \lambda_{\mathbf{r}} Q) + H_H, \quad (\text{A1})$$

with a summation convention over spin indices $\sigma = (1, N)$. The Heisenberg Hamiltonian H_H represented in terms of the Abrikosov fermi fields $f^\dagger, f$ as

$$H_H = - \frac{\tilde{J}_H}{N} \sum_{\langle \mathbf{r}\mathbf{r}' \rangle} (f_{\mathbf{r}\sigma}^\dagger f_{\mathbf{r}'\sigma}) (f_{\mathbf{r}'\sigma'}^\dagger f_{\mathbf{r}\sigma'}), \quad (\text{A2})$$

while the constraint on the local fermion number is implemented by an integral over the chemical potential $\lambda_{\mathbf{r}}$ [7]. Decoupling the four-fermion term by a Hubbard-Stratonovich transformation to the resonating valence bond fields $\chi_{\mathbf{r}\mathbf{r}'}$ and approximating the integral by the saddle point action leads to the mean-field Hamiltonian

$$H_{MF} = - \sum_{\langle \mathbf{r}\mathbf{r}' \rangle} \chi_{\mathbf{r}\mathbf{r}'} f_{\mathbf{r}\sigma}^\dagger f_{\mathbf{r}'\sigma} + \frac{N}{\tilde{J}_H} \sum_{\langle \mathbf{r}\mathbf{r}' \rangle} |\chi_{\mathbf{r}\mathbf{r}'}|^2 + \sum_{\mathbf{r}} \lambda_{\mathbf{r}} (f_{\mathbf{r}\sigma}^\dagger f_{\mathbf{r}\sigma} - Q), \quad (\text{A3})$$

which becomes exact in the limit of large N . We compare the energies of the spin-liquid (SL), dimer, and π -flux phases of this Hamiltonian on a square lattice in two dimensions with linear dimension L in units of the lattice constant. Each of these phases has $\lambda_{\mathbf{r}} = \lambda$.

For the dimer or Peierls phase [26], $\chi_{\mathbf{r}\mathbf{r}'} = 0$ on all but one of the nearest-neighbor bonds to each site, as shown in Fig. 3(a). For the π -flux phase [9], $\chi_{\mathbf{r}\mathbf{r}'} = |\chi| e^{i\pi/4}$ if $\mathbf{r} \rightarrow \mathbf{r}'$ is oriented along the arrows in Fig. 3(b). For the uniform U(1) spin liquid, $\chi_{\mathbf{r}\mathbf{r}'} = \chi \in \mathbb{R}$ for all bonds (Fig. 3(c)). Table I summarizes the results of the large N mean-field analysis for these states when $q = Q/N \leq 1/2$ and the ground state energies are compared in Fig. 3(d). Near half-filling, the Peierls phase has the lowest energy. However, for low filling upto $q \sim 0.3$, the lowest energy state is the uniform U(1) spin liquid. For intermediate filling $0.3 < q < 0.48$, the flux phase is most stable.

When $q \ll 1$, the dispersion of the filled states is approximately quadratic and we obtain the following

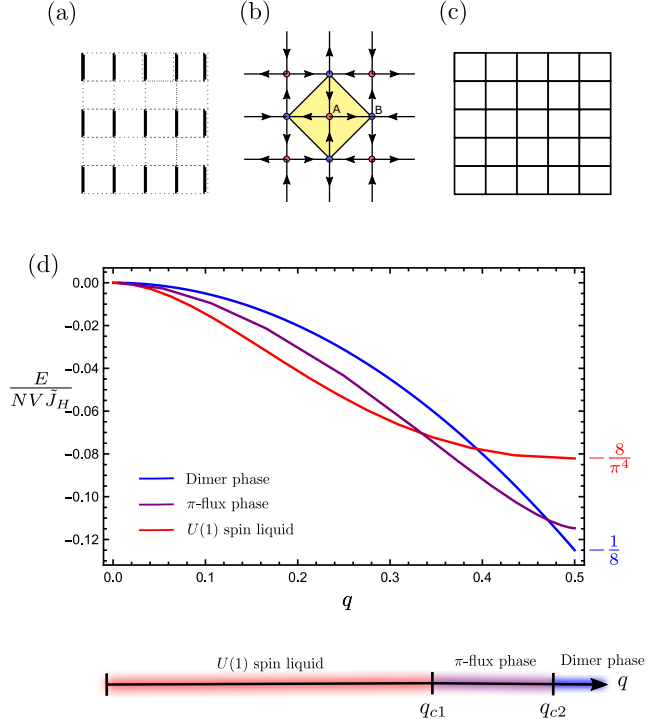


Figure 3. (a) π -flux phase: the phase of the bond order parameter χ is positive along the direction of the arrows. The unit cell (yellow) is expanded to include two inequivalent sites A and B , corresponding to a reduced Brillouin zone. (b) Peierls phase in which the spin on each site forms a dimer with its nearest-neighbor and decouples from the lattice. (c) Comparison of ground state energies for the spin liquid (red), flux phase (purple) and dimer phase (blue). The dashed curve is an analytical approximation for the spin liquid ground state energy valid at small q . For a range of filling $q < q_{c1} \sim 0.3$ the uniform U(1) spin liquid is stable with respect to the flux and dimer phases.

analytical expressions for the ground state energy

$$\frac{E}{NV\tilde{J}_H} = \begin{cases} - \left[\frac{1-(1-2\pi q)^{3/2}}{3\pi} \right]^2 \simeq -q^2, & \pi\text{-Flux} \\ -q^2/2, & \text{Dimer} \\ -2q^2 \left(1 - \frac{\pi q}{2}\right)^2 \simeq -2q^2, & \text{SL} \end{cases} \quad (\text{A4})$$

so for small q , the uniform spin liquid is the most energetically favorable state. We note that while the dimer phase is stable only near $q = 1/2$, similar phases may be present and favorable at other rational fillings. For instance, r -site ring polymer states have energy $E/(NV\tilde{J}_H) = -q^2$ at $q = 1/r$. At $q = 1/4$, the 4-site plaquette states have lower energy than the uniform U(1) spin liquid. As q becomes smaller, the likely ground state involves larger and larger decoupled clus-

Table I. Summary of large N mean-field results

State	Fermionic excitations	Gap equation	Ground state energy
Dimer	$\epsilon_{\pm} = \lambda \mp \chi$	$\chi = \tilde{J}_H q$	$\frac{2E}{\tilde{J}_H N L^2} = -q^2$
π -Flux	$\epsilon_{\mathbf{k}\pm} = \lambda \pm 2\chi \sqrt{\cos^2 k_x + \cos^2 k_y}$	$\frac{2 \chi }{\tilde{J}_H} = \int \frac{d^2 \mathbf{k}}{(2\pi)^2} n_{\mathbf{k}-} \sqrt{\cos^2 k_x + \cos^2 k_y}$	$\frac{2E}{\tilde{J}_H N L^2} = - \left(\int \frac{d^2 \mathbf{k}}{(2\pi)^2} n_{\mathbf{k}-} \sqrt{\cos^2 k_x + \cos^2 k_y} \right)^2$
U(1) SL	$\epsilon_{\mathbf{k}} = \lambda - 2\chi(\cos k_x + \cos k_y)$	$\frac{2 \chi }{\tilde{J}_H} = \int \frac{d^2 \mathbf{k}}{(2\pi)^2} (\cos k_x + \cos k_y) n_{\mathbf{k}}^f$	$\frac{2E}{\tilde{J}_H N L^2} = - \left(\int \frac{d^2 \mathbf{k}}{(2\pi)^2} (\cos k_x + \cos k_y) n_{\mathbf{k}}^f \right)^2$

ters with vanishing energy differences ΔE from the U(1) spin liquid. Above temperatures of the order of ΔE , the system behaves like a spin liquid. Additionally, on finite-sized systems, incommensuration between the cluster size and the system size may frustrate the valence bond crystal and favor the spin liquid.

Appendix B: Flux insertion in the large- N limit of Heisenberg model

In this section, we explicitly demonstrate the flux insertion and concomitant change in momentum in the Heisenberg model on a square lattice in the limit of large N , in terms of the mean-field Hamiltonian (A3). We discuss the response of the global U(1) gauge corresponding to the phase of the bond order parameters to the insertion of the flux, and explicitly show that the volume of the spinon Fermi surface is given by (2). As opposed to the main text, we consider a flux that couples to P of the N spin degrees of freedom, so that the effect of the flux threading on the relative change in the ground state energy, for instance, is non-vanishing in the large- N limit.

The Heisenberg model in presence of such a flux is given by

$$H_H[\Phi_{\{\mu\}}] = -\frac{\tilde{J}_H}{N} \sum_{\langle \mathbf{r}\mathbf{r}' \rangle} (f_{\mathbf{r}\sigma}^\dagger f_{\mathbf{r}'\sigma}) (f_{\mathbf{r}'\sigma'}^\dagger f_{\mathbf{r}\sigma'}) \times e^{-i(\mathbf{A}^\sigma - \mathbf{A}^{\sigma'}) \cdot (\mathbf{r} - \mathbf{r}')} \quad (\text{B1})$$

where $\mathbf{A}^\sigma = (2\pi/L_x)\hat{x} \sum_{\{\mu\}} \delta_{\sigma\mu}$ with the sum over P spin-channels to which the flux is coupled, where $\{\mu\} = \{\mu_1, \dots, \mu_P\}$. In the large N limit, this is exactly captured by the mean-field Hamiltonian

$$H_{MF}[\Phi_{\{\mu\}}] = - \sum_{\langle \mathbf{r}\mathbf{r}' \rangle} \chi_{\mathbf{r}\mathbf{r}'} e^{-i\mathbf{A}^\sigma \cdot (\mathbf{r} - \mathbf{r}')} f_{\mathbf{r}\sigma}^\dagger f_{\mathbf{r}'\sigma} + \frac{N}{\tilde{J}_H} \sum_{\langle \mathbf{r}\mathbf{r}' \rangle} |\chi_{\mathbf{r}\mathbf{r}'}|^2 + \sum_{\mathbf{r}} \lambda_{\mathbf{r}} (f_{\mathbf{r}\sigma}^\dagger f_{\mathbf{r}\sigma} - Q) \quad (\text{B2})$$

The saddle point condition for a uniform order param-

eter leads to the self-consistency equation

$$\chi_{\mathbf{r}\mathbf{r}'} = |\chi| e^{-i\mathbf{A} \cdot (\mathbf{r} - \mathbf{r}')} = \frac{\tilde{J}_H}{2NV} \sum_{\langle \mathbf{r}\mathbf{r}' \rangle} \langle f_{\mathbf{r}'\sigma}^\dagger f_{\mathbf{r}\sigma} \rangle e^{i\mathbf{A}^\sigma \cdot (\mathbf{r} - \mathbf{r}')} \quad (\text{B3})$$

where $\mathbf{A}$ is the global (spin-independent) U(1) gauge potential, V is the volume of the system (with the unit cell volume set to unity). With this, the mean-field Hamiltonian is diagonal in momentum space

$$H_{MF}[\Phi_{\{\mu\}}] = \sum_{\mathbf{k}} \epsilon_{\mathbf{k} + \mathbf{A} + \mathbf{A}^\sigma} f_{\mathbf{k}\sigma}^\dagger f_{\mathbf{k}\sigma} + \frac{2NV|\chi|^2}{\tilde{J}_H} - \lambda QV \quad (\text{B4})$$

where $\epsilon_{\mathbf{k}} = -2\chi(\cos k_x + \cos k_y) + \lambda$ is the dispersion of the f -fermions. Recall that the ground state is the same as before the flux insertion, since momentum is conserved throughout the process. The final state $|\psi_\Phi\rangle' = \Pi_{\mathbf{k} \in \text{FS}, \sigma} f_{\mathbf{k}\sigma}^\dagger |0\rangle$ and the canonical x-momentum $P_x = \sum_{\mathbf{k} \in \text{FS}, \sigma} \mathbf{k} = 0$ as before the flux insertion. The gauge transformation that removes the flux is now given by

$$U_{\{\mu\}}^s = \exp \left[\frac{2\pi i}{L_x} \sum_{\mathbf{r}, \{\mu\}} x_{\mathbf{r}} (\Lambda_{\mathbf{r}}^\mu + q) \right]. \quad (\text{B5})$$

To transform back to the original gauge we note that

$$\begin{aligned} U_{\{\mu\}}^s f_{\mathbf{k}\sigma}^\dagger U_{\{\mu\}}^{s\dagger} &= \frac{1}{\sqrt{V}} \sum_{\mathbf{r}} U_{\{\mu\}}^s f_{\mathbf{r}\sigma}^\dagger U_{\{\mu\}}^{s\dagger} e^{i\mathbf{k} \cdot \mathbf{r}} \\ &= \frac{1}{\sqrt{V}} \sum_{\mathbf{r}} f_{\mathbf{r}\sigma}^\dagger e^{i(\mathbf{k} + \mathbf{A}^\sigma) \cdot \mathbf{r}} = f_{\mathbf{k} + \mathbf{A}^\sigma, \sigma}^\dagger \end{aligned} \quad (\text{B6})$$

We transform the ground state to this gauge $|\psi_\Phi\rangle \equiv U_{\{\mu\}}^s |\psi_\Phi\rangle' = \Pi_{\mathbf{k} \in \text{FS}, \sigma} f_{\mathbf{k} + \mathbf{A}^\sigma, \sigma}^\dagger |0\rangle$ and evaluate the physical momentum

$$P_x = \sum_{\mathbf{k} \in \text{FS}, \sigma} (\mathbf{k} + \mathbf{A}^\sigma) = \sum_{\mathbf{k} \in \text{FS}, \sigma} \mathbf{A}^\sigma = PV \frac{V_{FS}}{(2\pi)^2} \frac{2\pi}{L_x} \quad (\text{B7})$$

when P out of N spin-components are coupled to the flux. In this case, the change in momentum on flux

insertion can be independently computed following the arguments in the main text ((38)-(41)) to yield

$$\Delta P_x = V \left(\frac{2\pi}{L_x} \right) Pq \quad (\text{B8})$$

when the ground state is unpolarized. Comparing with (B7), we find the volume of the Fermi surface to be $V_{FS} = (2\pi)^2 q$ consistent with (45).

As the flux is inserted, the global U(1) gauge potential $\mathcal{A}$ adjusts in response to preserve a zero total spinon current. Symmetry dictates that $\mathcal{A} \parallel \hat{x}$ and the new self-consistent value of $\mathcal{A}$ is determined by the saddle point condition $\partial_{\mathcal{A}} E = 0$ leading to

$$\begin{aligned} & \sum_{\mathbf{k}\sigma} (\partial_{\mathcal{A}} \epsilon_{\mathbf{k}+\mathcal{A}+\mathbf{A}^\sigma}) n_{\mathbf{k}}^f \\ &= -2|\chi| \frac{\partial}{\partial \mathcal{A}} \sum_{\mathbf{k}\sigma} \cos(\mathcal{A} + \mathbf{A}^\sigma) \cos k_x n_{\mathbf{k}}^f \\ &= 2|\chi| \left(\sum_{\mathbf{k}} \cos k_x n_{\mathbf{k}}^f \right) \left(\sum_{\sigma} \sin(\mathcal{A} + \mathbf{A}^\sigma) \right) = 0 \quad (\text{B9}) \end{aligned}$$

$$\Rightarrow P \sin \left(\mathcal{A} - \frac{2\pi}{L_x} \right) + (N - P) \sin \mathcal{A} = 0 \quad (\text{B10})$$

Since $L_x \gg 1$, we find that the saddle point value of $\mathcal{A}$ is

$$\mathcal{A} = -\frac{2\pi}{L_x} \rho, \quad (\text{B11})$$

where $\rho = \frac{P}{N}$. The global U(1) gauge potential adjusts to oppose the inserted flux and is proportional to the fraction of spin-components coupled to the flux. In fact, this keeps the net charge current fixed at 0, as expected for a response to a spin-twist. The flux imparts momentum to the μ -fermions and elicits a diamagnetic response from all the fermions. This can be seen explicitly by calculating the ground state energy in presence of the flux

$$\begin{aligned} E &= \frac{2NV|\chi|^2}{\tilde{J}_H} \\ &= -2|\chi| \sum_{\mathbf{k}\sigma} (\cos(k_x + \mathcal{A} + A_x^\sigma) + \cos k_y) n_{\mathbf{k}}^f \\ &= -2|\chi| I_0 V \sum_{\sigma} (\cos(\mathcal{A} + A_x^\sigma) + 1) \quad (\text{B12}) \end{aligned}$$

where $I_0 \equiv (1/V) \sum_{\mathbf{k}} \cos k_x n_{\mathbf{k}}^f$. When P of the N spin-components couple to the flux,

$$\begin{aligned} \frac{E}{V} &= -2|\chi| I_0 (P \cos(\mathcal{A} + A_x) \\ &\quad + (N - P) \cos \mathcal{A} - N) + \frac{2N|\chi|^2}{\tilde{J}_H} \quad (\text{B13}) \end{aligned}$$

With $\mathcal{A} \approx -\rho A_x$, the saddle point condition $\partial_{|\chi|} E = 0$ yields $|\chi| = (1/2) I_0 \tilde{J}_H (\rho \cos((1 - \rho) A_x) + (1 - \rho) \cos(\rho A_x) + 1)$. As a result, the ground state energy is

$$\begin{aligned} \frac{E}{NV} &= -2|\chi|^2 / \tilde{J}_H \\ &= -\frac{1}{2} \tilde{J}_H I_0^2 (\rho (\cos((1 - \rho) A_x) \\ &\quad + (1 - \rho) \cos(\rho A_x) + 1)^2 \\ &\approx -2 \tilde{J}_H I_0^2 \left[1 - \frac{1}{8} (\rho(1 - \rho)^2 + (1 - \rho)\rho^2) A_x^2 \right]^2 \\ &= \frac{E_0}{NV} + \frac{1}{2} \rho \tilde{D}_{\{\mu\}} A_x^2 + O(A_x^4) \quad (\text{B14}) \end{aligned}$$

where $E_0 = -2NV \tilde{J}_H I_0^2$ is the energy in absence of the flux and $\tilde{D}_{\{\mu\}} = 2 \tilde{J}_H I_0^2 (1 - \rho)$. The quantity $j_x = -(1/V) \partial E / \partial A_x = -P \tilde{D}_{\{\mu\}} A_x$ can be interpreted as the diamagnetic spin current response of the P-spin channels to the flux insertion, while $\tilde{D}_{\{\mu\}}$ is their spin stiffness.

-
- [1] M. Oshikawa, Topological Approach to Luttinger's Theorem and the Fermi Surface of a Kondo Lattice, *Physical Review Letters* **84**, 3370 (2000).
 [2] E. Lieb, T. Schultz, and D. Mattis, Two Soluble Models

- of an Antiferromagnetic Chain, *Ann. Phys. (NY)* **16**, 407 (1961).
 [3] R. M. Martin, Fermi-surface sum rule and its consequences for periodic Kondo and mixed-valence systems,

- Phys. Rev. Lett.* **48**, 362 (1982).
- [4] N. Read and D. M. Newns, On the solution of the Coqblin-Schrieffer Hamiltonian by the large-N expansion technique, *Journal of Physics C: Solid State Physics* **16**, 3273 (1983).
 - [5] A. Auerbach and K. Levin, Kondo Bosons and the Kondo Lattice: Microscopic Basis for the Heavy Fermi Liquid, *Physical Review Letters* **57**, 877 (1986).
 - [6] P. Coleman, I. Paul, and J. Rech, Sum rules and Ward identities in the Kondo lattice, *Phys. Rev. B* **72**, 094430 (2005).
 - [7] P. Coleman, *Introduction to Many-Body Physics*, 1st ed. (Cambridge University Press, New York, NY, 2016).
 - [8] P. W. Anderson, The Resonating Valence Bond State in La_2CuO_4 and Superconductivity, *Science* **235**, 1196 (1987).
 - [9] I. Affleck and J. B. Marston, Large- N limit of the Heisenberg-Hubbard model: Implications for high- T_c superconductors, *Phys. Rev. B* **37**, 3774 (1988).
 - [10] S. Doniach, Kondo lattice and weak antiferromagnetism, *Physica B+C* **91**, 231 (1977).
 - [11] D. V. Else, R. Thorngren, and T. Senthil, Non-fermi liquids as ersatz fermi liquids: General constraints on compressible metals, *Phys. Rev. X* **11**, 021005 (2021).
 - [12] T. Senthil, S. Sachdev, and M. Vojta, Fractionalized Fermi Liquids, *Physical Review Letters* **90**, 216403 (2003).
 - [13] A. Paramekanti and A. Vishwanath, Extending Luttinger's theorem to Z_2 fractionalized phases of matter, *Physical Review B* **70**, 245118 (2004).
 - [14] P. Coleman and N. Andrei, Diagonalisation of the generalised Anderson model, *J. Phys. C: Solid State Phys.* **19**, 3211 (1986).
 - [15] M. Vojta and S. Sachdev, Charge Order, Superconductivity, and a Global Phase Diagram of Doped Antiferromagnets, *Phys. Rev. Lett.* **83**, 3916 (1999).
 - [16] M. Hermele, T. Senthil, M. P. A. Fisher, P. A. Lee, N. Nagaosa, and X.-G. Wen, Stability of $U(1)$ spin liquids in two dimensions, *Phys. Rev. B* **70**, 214437 (2004).
 - [17] P. Coleman, J. B. Marston, and A. J. Schofield, Transport anomalies in a simplified model for a heavy-electron quantum critical point, *Physical Review B* **72**, 245111 (2005).
 - [18] O. I. Motrunich, Variational study of triangular lattice spin-1/2 model with ring exchanges and spin liquid state in $\kappa\text{-(ET)}_2\text{Cu}_2(\text{CN})_3$, *Phys. Rev. B* **72**, 045105 (2005).
 - [19] S.-S. Lee and P. A. Lee, $U(1)$ Gauge Theory of the Hubbard Model: Spin Liquid States and Possible Application to $\kappa\text{-(BEDT-TTF)}_2\text{Cu}_2(\text{CN})_3$, *Phys. Rev. Lett.* **95**, 036403 (2005).
 - [20] B. S. Tan, Y.-T. Hsu, B. Zeng, M. C. Hatnean, N. Harrison, Z. Zhu, M. Hartstein, M. Kiourlappou, A. Srivastava, M. D. Johannes, T. P. Murphy, J.-H. Park, L. Balicas, G. G. Lonzarich, G. Balakrishnan, and S. E. Sebastian, Unconventional Fermi surface in an insulating state, *Science* **349**, 287 (2015).
 - [21] M. Hartstein, W. H. Toews, Y.-T. Hsu, B. Zeng, X. Chen, M. C. Hatnean, Q. R. Zhang, S. Nakamura, A. S. Padgett, G. Rodway-Gant, J. Berk, M. K. Kingston, G. H. Zhang, M. K. Chan, S. Yamashita, T. Sakakibara, Y. Takano, J.-H. Park, L. Balicas, N. Harrison, N. Shitsevalova, G. Balakrishnan, G. G. Lonzarich, R. W. Hill, M. Sutherland, and S. E. Sebastian, Fermi surface in the absence of a Fermi liquid in the Kondo insulator SmB_6 , *Nature Physics* **14**, 166 (2018).
 - [22] P. Czajka, T. Gao, M. Hirschberger, P. Lampen-Kelley, A. Banerjee, J. Yan, D. G. Mandrus, S. E. Nagler, and N. P. Ong, Oscillations of the thermal conductivity in the spin-liquid state of $\alpha\text{-RuCl}_3$, *Nature Physics* (2021).
 - [23] S. Nakatsuji, D. Pines, and Z. Fisk, Two Fluid Description of the Kondo Lattice, *Physical Review Letters* **92**, 016401 (2004).
 - [24] N. Read and S. Sachdev, Large-N expansion for frustrated quantum antiferromagnets, *Phys. Rev. Lett.* **66**, 1773 (1991).
 - [25] R. Flint, M. Dzero, and P. Coleman, Heavy electrons and the symplectic symmetry of spin, *Nature Physics* **4**, 643 (2008).
 - [26] C. K. Majumdar and D. K. Ghosh, On Next-Nearest-Neighbor Interaction in Linear Chain. I, *Journal of Mathematical Physics* **10**, 1388 (1969).