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Base change along the Frobenius endomorphism and the Gorenstein property

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ABSTRACT

Let R be a local ring of positive characteristic and X a complex with nonzero finitely generated homology and finite injective dimension. We prove that if the derived base change of X via the Frobenius (or more generally, via a contracting) endomorphism has finite injective dimension then R is Gorenstein. In particular, we give an affirmative answer to a question by Falahola and Marley [7, Question 3.9].

1. Introduction

Kunz [11] proved that a local ring (which will henceforth be assumed to be Noetherian) $(R, \mathfrak{m}, k)$ of positive characteristic is regular if and only if some (equivalently, every) power of the Frobenius endomorphism is flat as an R-module homomorphism. Since then analogous characterizations of other properties of local rings, such as complete intersections (by Rodicio [15]), Gorenstein (by Goto [17]) and Cohen-Macaulay (by Takahashi and Yoshino [16]), have been obtained. Many of these results have been generalized for the larger family of *contracting* endomorphisms. Following [4], an endomorphism

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$\varphi\colon R \to R$ is said to be *contracting* if $\varphi^i(\mathfrak{m}) \subseteq \mathfrak{m}^2$ for some $i > 0$. The Frobenius endomorphism is one example but there are many interesting examples even when R is of characteristic 0. Avramov, Iyengar and Miller [4] generalized Kunz's theorem to apply to any contracting endomorphism. For other results concerning contracting endomorphisms see, for example, Avramov, Hochster, Iyengar and Yao [2], Rahmati [14] and Nasseh and Sather-Wagstaff [12].

In this paper we study homological properties of modules and complexes under base change along contracting endomorphisms. Given an endomorphism $\varphi\colon R \to R$, we write R^φ for the R-bimodule with the right module structure induced by φ and the left usual R-module structure. Thus given an R-complex X the base change along φ is $R^\varphi \otimes^{\mathbf{L}}_R X$ where R acts on the left through R^φ. The main result of this work is the following, proved in Section 3:

Theorem 1.1. *Let $(R, \mathfrak{m}, k)$ be a local ring, and let $\varphi\colon R \to R$ be a contracting endomorphism. The following conditions are equivalent.*

(i) *R is Gorenstein.*
(ii) *There exists an R-complex X with nonzero finitely generated homology and finite injective dimension for which the base change $R^\varphi \otimes^{\mathbf{L}}_R X$ has finite injective dimension.*
(iii) *For every X with nonzero finitely generated homology and finite injective dimension the base change $R^\varphi \otimes^{\mathbf{L}}_R X$ has finite injective dimension.*

An R-complex is said to have *finite injective dimension* if it is quasi-isomorphic to a bounded complex of injective modules.

In the theorem above (i) $\Rightarrow$ (iii) holds because in a Gorenstein local ring complexes of finite injective dimension coincide with complexes of finite projective dimension. For (iii) $\Rightarrow$ (ii) we only need to show that every local ring has a complex of finite injective dimension with nonzero finitely generated homology. (ii) $\Rightarrow$ (i) is the crucial implication. This is proven in two steps. 1) When $H(X)$ is finitely generated, if $R^{\varphi^i} \otimes^{\mathbf{L}}_R X$ is bounded in homology for $i \gg 0$ then X has finite projective dimension; this follows from well known arguments, see 3.2 for details. 2) When X is a complex with nonzero finite length homology and finite injective dimension, if the base change $R^\varphi \otimes^{\mathbf{L}}_R X$ has finite injective dimension then the same holds for $R^{\varphi^i} \otimes^{\mathbf{L}}_R X$ for every $i > 0$, see 3.3. The key tool in the proof of the second step is a theorem of Hopkins [10] and Neeman [13] concerning perfect complexes. The idea of using the theorem of Hopkins and Neeman was inspired by the work of Dwyer, Greenlees, and Iyengar [9] who were the first to apply it in the context of commutative algebra.

Theorem 1.1 is analogous to the following characterization of Gorenstein rings by Falahola and Marley [7, Proposition 3.7].

Theorem 1.2. *Let $\varphi\colon R \to R$ be a contracting endomorphism where R is a Cohen-Macaulay local ring and ω_R is a canonical module. Then R is Gorenstein if and only if $R^\varphi \otimes_R \omega_R$ has finite injective dimension.*

In [7, Question 3.9] the authors ask: when R is a local ring with a dualizing complex D, if $R^\varphi \otimes^{\mathbf{L}}_R D$ has finite injective dimension is then R necessarily Gorenstein? Theorem 1.1 gives an affirmative answer.

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2. Homological invariants

In this Section we recall basic definitions and results we will need in Section 3. Throughout this paper R will be a commutative Noetherian ring. We write $\mathsf{D}(R)$ for the derived category of R-complexes, with the convention that complexes are graded below i.e. we write

$$X = \ldots \to X_n \to X_{n-1} \to \ldots$$

We write $X \simeq Y$ when X is isomorphic to Y in $\mathsf{D}(R)$.

Definition 2.1. Given an R-complex X we set

$$\sup H(X) = \sup\{i \mid H_i(X) \neq 0\} \text{ and } \inf H(X) = \inf\{i \mid H_i(X) \neq 0\}$$

Thus $\sup(0) = -\infty$ and $\inf(0) = \infty$. A complex X is said to be *homologically bounded above* if $\sup H(X) < \infty$. Similarly, X is *homologically bounded below* if $\inf H(X) > -\infty$ and X is *homologically bounded* if it is homologically bounded above and below.

Let S be a full subcategory of $\mathsf{D}(R)$. We make the following conventions:

(i) $\mathsf{S}_- := \{X \in \mathsf{S} \mid \sup H(X) < \infty\}$
(ii) $\mathsf{S}_+ := \{X \in \mathsf{S} \mid \inf H(X) > -\infty\}$
(iii) $\mathsf{S}_\mathrm{b} := \mathsf{S}_+ \cap \mathsf{S}_-$
(iv) The subcategory of complexes in S with degree-wise finitely generated (resp. finite length) homology is denoted S^{fg} (resp. S^{fl})

In $\mathsf{D}(R)$ we have derived functors $\mathrm{RHom}_R(\_,\_)$ and $\_\otimes^{\mathbf{L}}_R\_$. For a detailed description on how these functors are defined, we refer the reader to [18] and [1].

When X is in $\mathsf{D}_+(R)$, there is a complex P consisting of projective modules with $P_i = 0$ for $i \ll 0$ such that $P \simeq X$. Such a complex P is called a projective resolution of X. In this case we can compute $\mathrm{RHom}(X, \_)$ by setting

$$\mathrm{RHom}(X, \_) = \mathrm{Hom}_R(P, \_)$$

Flat and injective resolutions are similarly defined. Since R has enough projectives and injectives, every complex in $\mathsf{D}_+(R)$ admits projective (and therefore flat) resolutions and every complex in $\mathsf{D}_-(R)$ admits injective resolutions.

Complexes in $\mathsf{D}_\mathrm{b}^\mathrm{fg}(R)$ with finite projective dimension are the *perfect complexes*. The subcategory of $\mathsf{D}(R)$ of complexes of finite injective dimension plays a central role in this paper and we denote it $\mathsf{I}(R)$.

Definition 2.2. Let $X \in \mathsf{D}_+(R)$ and $Y \in \mathsf{D}_-(R)$. We define

$$\text{proj-dim}_R(X) := \inf \left\{ n \;\middle|\; \begin{array}{c} \text{there exists a projective resolution } P \text{ of } X \\ 0 \to P_n \to \ldots \to P_i \to 0 \text{ with } P_n \neq 0 \end{array} \right\}$$

$$\text{inj-dim}_R(Y) := \inf \left\{ n \;\middle|\; \begin{array}{c} \text{there exists an injective resolution } I \text{ of } Y \\ 0 \to I_i \to I_{i-1} \to \ldots \to I_{-n} \to 0 \text{ with } I_{-n} \neq 0 \end{array} \right\}$$

Remark 2.3. Let $(R, \mathfrak{m}, k)$ be a local ring. Following [6, A.5.7, A.7.9] we have,

(i) Let $X \in \mathsf{D}_+(R)$. When V is an R-complex such that $\mathfrak{m}V = 0$ then,

$$\sup H(V \otimes_R^{\mathbf{L}} X) = \sup H(V) + \sup H(k \otimes_R^{\mathbf{L}} X)$$
$$\inf H(V \otimes_R^{\mathbf{L}} X) = \inf H(V) + \inf H(k \otimes_R^{\mathbf{L}} X)$$

(ii) If $X \in \mathsf{D}_+^\mathrm{fg}(R)$ and $Y \in \mathsf{D}_-^\mathrm{fg}(R)$ then the following hold

$$\text{proj-dim}_R(X) = -\inf H(\mathrm{RHom}_R(X, k)) = \sup H(k \otimes_R^{\mathbf{L}} X)$$
$$\text{inj-dim}_R(Y) = -\inf H(\mathrm{RHom}_R(k, Y))$$

Definition 2.4. A local ring R is *Gorenstein* if $\text{inj-dim}_R(R)$ is finite.

The following theorem [6, 3.3.4] asserts that in a Gorenstein local ring, the categories of finite flat dimension and finite injective dimension coincide.

Theorem 2.5. *Let $(R, \mathfrak{m}, k)$ be a local Gorenstein ring, $X \in \mathsf{D}_\mathrm{b}(R)$. Then*

$$\text{flat-dim}_R(X) < \infty \iff \text{inj-dim}_R(X) < \infty.$$

It is well known that in a local ring, a complex X has finite flat dimension if and only if X has finite projective dimension, see for example [8, Corollary 3.4]. Hence, Theorem 2.5 implies that in a local Gorenstein ring the categories of finite projective dimension and finite injective dimension coincide.

Foxby [8, Corollary 4.4] also proved the converse to Theorem 2.5. Although in the original paper it was stated for modules, it is well known to be true for complexes as well. For convenience, we give a self contained proof using the terminology and properties established above.

Theorem 2.6. *Let* $(R, \mathfrak{m}, k)$ *be a local ring. If there exists a complex* $X \in \mathsf{D}^{\mathrm{fg}}_{\mathrm{b}}(R)$ *with* $H(X) \neq 0$ *such that both* $\operatorname{proj-dim}_R(X)$ *and* $\operatorname{inj-dim}_R(X)$ *are finite, then* R *is Gorenstein.*

Proof. We have quasi-isomorphisms

$$\mathrm{RHom}_R(k, X) \simeq \mathrm{RHom}_R(k, R \otimes^{\mathbf{L}}_R X) \simeq \mathrm{RHom}_R(k, R) \otimes^{\mathbf{L}}_R X$$

The first quasi-isomorphism is trivial and the second follows from [8, 1.1.4] since X is perfect. Also, since $X \in \mathsf{D}^{\mathrm{fg}}_{\mathrm{b}}(R)$ we have $\inf H(k \otimes^{\mathbf{L}}_R X)$ is finite. As $H(X) \neq 0$, we get from 2.3:

$$\begin{aligned}\operatorname{inj-dim}_R(X) &= -\inf H(\mathrm{RHom}_R(k, X)) \\ &= -\inf H(\mathrm{RHom}_R(k, R) \otimes^{\mathbf{L}}_R X) \\ &= -\inf H(\mathrm{RHom}_R(k, R)) - \inf H(k \otimes^{\mathbf{L}}_R X) \\ &= \operatorname{inj-dim}_R R - \inf H(k \otimes^{\mathbf{L}}_R X)\end{aligned}$$

Thus $\operatorname{inj-dim}_R(X) < \infty \implies \operatorname{inj-dim}_R(R) < \infty$. $\square$

2.1. Thick subcategories and generation

Thick subcategories play a critical role in the proofs in Section 3. Here we recall the definition and give some examples, following the formulation given in [3, §1].

Definition 2.7. A non-empty subcategory $\mathcal{T}$ of $\mathsf{D}(R)$ is *thick* if it is additive, closed under taking direct summands and for every exact triangle

$$X \to Y \to Z \to \Sigma X$$

if any two of X, Y, Z belong to $\mathcal{T}$, so does the third. From the definition it is clear that intersections of thick subcategories are again thick.

Example 2.8. The subcategories $\mathsf{D}^{\mathrm{fg}}(R)$, $\mathsf{D}^{\mathrm{fl}}(R)$, $\mathsf{I}(R)$ and the subcategory of perfect complexes are all thick in $\mathsf{D}(R)$, see for example [9, 3.2]. It follows immediately that $\mathsf{I}^{\mathrm{fg}}(R)$ and $\mathsf{I}^{\mathrm{fl}}(R)$ are thick as well.

Definition 2.9. The *thick subcategory generated by* $X \in \mathsf{D}(R)$, denoted $\mathrm{Thick}_R(X)$, is the smallest thick subcategory that contains X. It is the intersection of all thick subcategories of $\mathsf{D}(R)$ containing X.

Example 2.10. For any ring R, the thick subcategory $\mathrm{Thick}_R(R)$ is the subcategory of perfect complexes. When $(R, \mathfrak{m}, k)$ is a local ring we have $\mathrm{Thick}_R(k) = \mathsf{D}^{\mathrm{fl}}_{\mathrm{b}}(R)$.

For any $X \in \mathsf{D}(R)$ one can construct $\mathrm{Thick}_R(X)$ as follows: Set $\mathrm{Thick}^0_R(X) = \{0\}$. The objects of $\mathrm{Thick}^1_R(X)$ are direct summands of finite direct sums of shifts of X. For each $n \geq 2$, the objects of $\mathrm{Thick}^n_R(X)$ are direct summands of objects U such that U appears in an exact triangle

$$U' \to U \to U'' \to \Sigma U'$$

where $U' \in \mathrm{Thick}^{n-1}_R(X)$ and $U'' \in \mathrm{Thick}^1_R(X)$. The subcategory $\mathrm{Thick}^n_R(X)$ is the *nth thickening* of X. Every thickening embeds in the next one thus we have a filtration:

$$\{0\} = \mathrm{Thick}^0_R(X) \subseteq \mathrm{Thick}^1_R(X) \subseteq \mathrm{Thick}^2_R(X) \subseteq \ldots \subseteq \bigcup_{n\geq 0} \mathrm{Thick}^n_R(X)$$

It is clear that $\bigcup_{n\geq 0} \mathrm{Thick}^n_R(X)$ is a thick subcategory. By construction it is the smallest thick subcategory containing X hence

$$\mathrm{Thick}_R(X) = \bigcup_{n\geq 0} \mathrm{Thick}^n_R(X)$$

For a broader discussion see, for example, [3, §1]. This discussion motivates the following terminology: An R-complex in $\mathrm{Thick}_R(X)$ is *finitely built* from X.

Definition 2.11. The *support* of an R-complex X is

$$\mathrm{Supp}_R(X) := \{\mathfrak{p} \in \mathrm{Spec}(R) \mid H(X)_{\mathfrak{p}} \neq 0\}$$

When $X \in \mathsf{D}^{\mathrm{fg}}_{\mathrm{b}}(R)$ the support is

$$\mathrm{Supp}_R(X) = V(\mathrm{ann}_R(H(X)).$$

If $N \in \mathrm{Thick}_R(M)$, then from the construction it follows that $\mathrm{Supp}_R(N)$ is contained in $\mathrm{Supp}_R(M)$. Indeed, since localization is an exact functor, if $H(M)_{\mathfrak{p}} = 0$ for some $\mathfrak{p} \in \mathrm{Spec}(R)$ then inductively $H(N)_{\mathfrak{p}} = 0$ for every N in $\mathrm{Thick}^i_R(M)$ for all i.

Hopkins [10, 11] and Neeman [13, 1.2] proved the following result which asserts that the converse is true when both M and N are perfect complexes.

Theorem 2.12. *Let R be a commutative Noetherian ring. Given perfect R-complexes N and M, if $\operatorname{Supp}_R N \subseteq \operatorname{Supp}_R M$ then N is finitely built from M.* □

2.2. Loewy length

Another important element in this work is the Koszul complex. We recall the definition of Koszul complexes and Loewy length.

Definition 2.13. The *Koszul complex* on $x \in R$ is the R-complex

$$K(x) \coloneqq 0 \to R \xrightarrow{x} R \to 0$$

concentrated in degrees 0 and 1. Given a sequence $\mathbf{x} = (x_1, ..., x_n)$ the Koszul complex on $\mathbf{x}$ is

$$K(\mathbf{x}) \coloneqq K(x_1) \otimes_R K(x_2) \otimes_R ... \otimes_R K(x_n)$$

with the convention that $K(\varnothing) = R$.

Set K^R to be the Koszul complex on a minimal generating set of $\mathfrak{m}$. Since K^R is a perfect complex, we have that $K^R \in \operatorname{Thick}_R(R)$. It follows that $K^R \otimes^{\mathbf{L}}_R X$ is in $\operatorname{Thick}_R(X)$ for every $X \in \mathsf{D}(R)$.

Definition 2.14. Let $(R, \mathfrak{m}, k)$ be a local ring, X an R-complex. The *Loewy length* of X is defined to be

$$\ell\ell_R(X) \coloneqq \inf\{i \in \mathbb{N} \mid \mathfrak{m}^i \cdot X = 0\}$$

Following [4, 6.2], the *homotopical Loewy length* of X is defined to be

$$\ell\ell_{\mathsf{D}(R)}(X) \coloneqq \inf\{\ell\ell_R(V) \mid V \simeq X\}$$

Avramov, Iyengar and Miller [4, 6.2] prove that the homotopical Loewy length satisfies the following finiteness property.

Theorem 2.15. *Let $(R, \mathfrak{m}, k)$ be a local ring, and K^R be the Koszul complex on a minimal generating set of $\mathfrak{m}$. For any complex X we have*

$$\ell\ell_{\mathsf{D}(R)}(K^R \otimes^{\mathbf{L}}_R X) \le \ell\ell_{\mathsf{D}(R)} K^R < \infty \quad \square$$

We say that a homomorphism $\varphi\colon (R,\mathfrak{m},k) \to (S,\mathfrak{n},l)$ is a *deep local homomorphism* if $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}^c$ where $c = \ell\ell_{\mathsf{D}(S)} K^S$. The following corollary is often used in the literature to prove various results for deep local homomorphisms.

Lemma 2.16. *If $\varphi\colon (R,\mathfrak{m},k) \to (S,\mathfrak{n},l)$ is a deep local homomorphism, then there is a quasi-isomorphism $K^S \simeq H(K^S)$.*

Proof. By 2.14 there exist a complex V such that $K^S \simeq V$ and $\mathfrak{n}^c V = 0$. As $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}^c$, this yields that $\mathfrak{m}V = 0$. Hence the R action on V factors through the map $R \to R/\mathfrak{m} = k$. Since k is a field, for every $V \in \mathsf{D}(k)$ we have $V \simeq H(V)$. In particular, $K^S \simeq H(K^S)$ in $\mathsf{D}(k)$ so the same is true in $\mathsf{D}(R)$. □

3. Homological dimension and the derived base change

Let $\varphi\colon R \to S$ be a homomorphism. There is a naturally defined functor F^φ from the category of R-complexes to the category of S-complexes by setting

$$\mathrm{F}^\varphi(\_) := S \otimes_R \_$$

We write

$$\mathsf{L}\mathrm{F}^\varphi\colon \mathsf{D}(R) \to \mathsf{D}(S) \text{ by } \mathsf{L}\mathrm{F}^\varphi(\_) = S \otimes^{\mathbf{L}}_R \_$$

for the induced functor on $\mathsf{D}(R)$.

Remark 3.1. Let $\varphi\colon (R,\mathfrak{m},k) \to (S,\mathfrak{n},l)$ be a local homomorphism.

(i) For every perfect complex X the complex $\mathsf{L}\mathrm{F}^\varphi(X)$ is perfect in $\mathsf{D}(S)$. Indeed, as X is perfect, there exists a finite free resolution $F \simeq X$. Then $\mathsf{L}\mathrm{F}^\varphi(X) \simeq S \otimes_R F$ which is a finite complex of free S modules.

(ii) For every $X \in \mathsf{D}^{\mathrm{fg}}_+(R)$ such that $H(X) \neq 0$ we have $H(\mathsf{L}\mathrm{F}^\varphi(X)) \neq 0$. Indeed, after perhaps shifting, we may assume $H_0(X) \neq 0$ and $H_i(X) = 0$ for all $i < 0$. We have

$$H_0(S \otimes^{\mathbf{L}}_R X) \cong S \otimes_R H_0(X)$$

Applying $S \otimes_R \_$ to the surjection $H_0(X) \twoheadrightarrow H_0(X)/\mathfrak{m}H_0(X) \twoheadrightarrow k$ we get $H_0(S \otimes^{\mathbf{L}}_R X) \twoheadrightarrow S \otimes_R k \cong \dfrac{S}{\mathfrak{m}S} \neq 0$ as φ is a local homomorphism.

Proposition 3.2. *Let $\varphi\colon (R,\mathfrak{m},k) \to (S,\mathfrak{n},l)$ be a deep local homomorphism. For any $X \in \mathsf{D}^{\mathrm{fg}}_+(R)$ the complex $\mathsf{L}\mathrm{F}^\varphi(X)$ is homologically bounded above if and only if X has finite projective dimension in $\mathsf{D}(R)$.*

Proof. The if part is clear. For the converse, by the discussion after 2.13 the complex $K^S \otimes^{\mathbf{L}}_R X \simeq K^S \otimes^{\mathbf{L}}_S (S \otimes^{\mathbf{L}}_R X)$ is in $\mathrm{Thick}_S(S \otimes^{\mathbf{L}}_R X)$. Example 2.8 yields

$$\sup H(S \otimes^{\mathbf{L}}_R X) < \infty \implies \sup H(K^S \otimes^{\mathbf{L}}_R X) < \infty.$$

By Corollary 2.16, the complex $K^S \simeq H(K^S)$ in $\mathsf{D}(R)$ and $H(K^S)$ is a complex of k-vector spaces as an R-complex. One gets by the Künneth formula

$$\begin{aligned} H(K^S \otimes^{\mathbf{L}}_R X) &\cong H(H(K^S) \otimes^{\mathbf{L}}_R X) \\ &\cong H(H(K^S) \otimes_k (k \otimes^{\mathbf{L}}_R X)) \\ &\cong H(K^S) \otimes_k H(k \otimes^{\mathbf{L}}_R X) \end{aligned}$$

Since $H(K^S \otimes^{\mathbf{L}}_R X)$ is bounded, so is $H(k \otimes^{\mathbf{L}}_R X)$. Therefore $\text{proj-dim}_R(X) < \infty$ by Remark 2.3 (ii). □

Corollary 3.3. *Let $\varphi\colon (R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ be a deep local homomorphism. If there exists an $X \in \mathsf{I}^{\mathrm{fg}}(R)$ with $H(X) \neq 0$ and $\text{inj-dim}_S(\mathsf{LF}^{\varphi}(X)) < \infty$ then R is Gorenstein.*

Proof. By Remark 3.1 (ii), the homology $H(\mathsf{LF}^{\varphi}(X))$ is nonzero. By hypothesis X is in $\mathsf{D}^{\mathrm{fg}}_{\mathrm{b}}(R)$ so $\mathsf{LF}^{\varphi}(X) \in \mathsf{D}^{\mathrm{fg}}_{\mathrm{b}}(S)$. Hence

$$\text{inj-dim}_S(\mathsf{LF}^{\varphi}(X)) < \infty \implies \sup H(\mathsf{LF}^{\varphi}(X)) < \infty.$$

Therefore $\text{proj-dim}_R(X) < \infty$ by Proposition 3.2. Theorem 2.6 now shows that R is Gorenstein. □

Remark 3.4. In the context of Corollary 3.3, if there exists an $X \in \mathsf{D}^{\mathrm{fg}}_{\mathrm{b}}(R)$ with $H(X) \neq 0$ such that $\text{inj-dim}_R(\mathsf{LF}^{\varphi}(X)) < \infty$ then R is regular. Indeed, if $\mathsf{LF}^{\varphi}(X)$ has finite injective dimension, then following the lines of the proof of Proposition 3.2, we see that $\text{inj-dim}_R(K^S \otimes^{\mathbf{L}}_R X) < \infty$. It follows that $\text{inj-dim}_R(k \otimes^{\mathbf{L}}_R X)$ is finite and therefore $\text{inj-dim}_R(k) < \infty$ which implies that R is regular, see [5, 3.1.26]. This gives another proof of a result of Avramov, Hochster, Iyengar and Yao [2, 5.3].

Our main result concerns the finiteness of injective dimension with respect to the derived base change over contracting endomorphisms.

Definition 3.5. Let $(R, \mathfrak{m}, k)$ be a local ring. An endomorphism $\varphi\colon R \to R$ is said to be *contracting* if $\varphi^i(\mathfrak{m}) \subseteq \mathfrak{m}^2$ for some $i \geq 1$.

As mentioned in the introduction, the most prominent example of a contracting endomorphism is the Frobenius when R has positive characteristic. But there are many other interesting examples of contracting endomorphisms even in characteristic 0.

Remark 3.6. If φ is a contracting endomorphism then φ^i will be a deep local homomorphism for each $i \gg 0$.

If φ is an endomorphism on R, then we define R^φ to be R with the right module structure induced by φ. Proposition 3.2 shows that given a contracting endomorphism φ and a complex X then for large enough i the complex $\mathsf{LF}^{\varphi^i}(X)$ is bounded if and only if X has finite projective dimension. However, there are examples of complexes of infinite projective dimension for which $\mathsf{LF}^{\varphi}(X)$ is homologically bounded. For example, let

$$R = \frac{k[x,y]}{(x^3, y^3)}$$

Set $\varphi(x) = y$ and $\varphi(y) = y^2$. One can check that $\mathsf{LF}^{\varphi}(x) \simeq (y)$ but (x) has infinite projective dimension. A natural question to ask is when does $\sup H(\mathsf{LF}^{\varphi}(X)) < \infty$ imply that $\sup H(\mathsf{LF}^{\varphi^i}(X)) < \infty$ for all $i > 0$? Our goal is to show that if X has finite injective dimension then $\mathsf{LF}^{\varphi}(X)$ is homologically bounded if and only if $\mathsf{LF}^{\varphi^i}(X)$ is homologically bounded for every $i > 0$.

Definition 3.7. Let $(R, \mathfrak{m}, k)$ be a local ring and E the injective hull of the R-module k. For an R-complex M set

$$M^\vee := \operatorname{Hom}_R(M, E).$$

We will need the following lemma; we give a proof for completeness.

Lemma 3.8. *Let $(R, \mathfrak{m}, k)$ be a local ring.*

(i) *The natural map $X \to X^{\vee\vee}$ is a quasi-isomorphism for all $X \in \mathsf{D}^{\mathrm{fl}}_{\mathrm{b}}(R)$.*
(ii) *The complex $X^\vee$ is perfect with finite length homology for all $X \in \mathsf{I}^{\mathrm{fl}}(R)$.*

Proof. For (i), we observe that $\{X \in \mathsf{D}(R) \mid X \simeq X^{\vee\vee}\}$ form a thick subcategory. When $X \in \mathsf{D}^{\mathrm{fl}}_{\mathrm{b}}(R)$ one can show that by induction on the total length of $H(X)$ that $X \in \operatorname{Thick}_R(k)$. Clearly $k \simeq k^{\vee\vee}$ so it follows that $X \simeq X^{\vee\vee}$ for all $X \in \mathsf{D}^{\mathrm{fl}}_{\mathrm{b}}(R)$.

For (ii), take an injective resolution I of X. Since $\operatorname{Supp}(X) = \{\mathfrak{m}\}$, the injective resolution I is a finite complex where all the modules are direct sums of E. Hence by Matlis duality, $X^\vee$ is quasi-isomorphic to a bounded complex of free $\widehat{R}$ modules. As $\widehat{R}$ is faithfully flat over R we have,

$$\sup(k \otimes^{\mathbf{L}}_R X^\vee) = \sup(k \otimes^{\mathbf{L}}_R X^\vee \otimes^{\mathbf{L}}_R \widehat{R})$$

Since $X^\vee$ is perfect in $\mathsf{D}(\widehat{R})$ it is also perfect in $\mathsf{D}(R)$. □

Proposition 3.9. *For all $X \in \mathsf{I}^{\mathrm{fl}}(R)$ with $H(X) \neq 0$, one has $\operatorname{Thick}_R(X) = \mathsf{I}^{\mathrm{fl}}(R)$.*

Proof. By Example 2.8, $\mathrm{Thick}_R(X) \subseteq \mathsf{I}^{\mathrm{fl}}(R)$, so it suffices to show that for all $Y \in \mathsf{I}^{\mathrm{fl}}(R)$ we have Y is in $\mathrm{Thick}_R(X)$. Let $Y \in \mathsf{I}^{\mathrm{fl}}(R)$ with $H(Y) \neq 0$. By Lemma 3.8 (ii) $X^\vee$ and $Y^\vee$ are complexes with finite length homology and finite projective dimension over R. In particular, $X^\vee$ and $Y^\vee$ are both perfect, and

$$\mathrm{Supp}(X^\vee) = \{\mathfrak{m}\} = \mathrm{Supp}(Y^\vee)$$

Theorem 2.12 yields that

$$\mathrm{Thick}_R(X^\vee) = \mathrm{Thick}_R(Y^\vee)$$

Applying the Matlis dual again, we get $\mathrm{Thick}_R(X^{\vee\vee}) = \mathrm{Thick}_R(Y^{\vee\vee})$ since the Matlis dual is an exact functor. Noting that $X^{\vee\vee} \simeq X$ by Lemma 3.8 (i), we see that $\mathrm{Thick}_R(X) = \mathrm{Thick}_R(Y)$. In particular, Y is in $\mathrm{Thick}_R(X)$. □

Lemma 3.10. *Let $\varphi\colon (R,\mathfrak{m},k) \to (R,\mathfrak{m},k)$ be contracting endomorphism. If for some $X \in \mathsf{I}^{\mathrm{fl}}(R)$ with $H(X) \neq 0$ the injective dimension of $\mathsf{L}\mathrm{F}^\varphi(X)$ is finite then the injective dimension of $\mathsf{L}\mathrm{F}^{\varphi^i}(Y)$ is finite for all $i \geq 1$ and all $Y \in \mathsf{I}^{\mathrm{fl}}(R)$.*

Proof. By Remark 3.1 $H(\mathsf{L}\mathrm{F}^\varphi(X)) \neq 0$ when $H(X) \neq 0$. Proposition 3.9 shows that $\mathrm{Thick}_{\mathsf{D}(R)}(X) = \mathsf{I}^{\mathrm{fl}}(R)$. Since $\mathsf{L}\mathrm{F}^\varphi(\_)$ is an exact functor it follows that $\mathsf{L}\mathrm{F}^\varphi(Y)$ is in $\mathrm{Thick}_R(\mathsf{L}\mathrm{F}^\varphi(X))$ for every $Y \in \mathsf{I}^{\mathrm{fl}}(R)$. By hypothesis $\mathsf{L}\mathrm{F}^\varphi(X) \in \mathsf{I}^{\mathrm{fl}}(R)$, hence the functor $\mathsf{L}\mathrm{F}^\varphi(\_)$ takes $\mathsf{I}^{\mathrm{fl}}(R)$ to $\mathsf{I}^{\mathrm{fl}}(R)$, but this implies that $\mathsf{L}\mathrm{F}^{\varphi^2}(Y) \cong \mathsf{L}\mathrm{F}^\varphi(\mathsf{L}\mathrm{F}^\varphi(Y))$ has finite injective dimension for every $Y \in \mathsf{I}^{\mathrm{fl}}(R)$. By induction on i, we have that the injective dimension of $\mathsf{L}\mathrm{F}^{\varphi^i}(Y)$ is finite for all $i \geq 1$ and all $Y \in \mathsf{I}^{\mathrm{fl}}(R)$. □

The following theorem is a restatement of Theorem 1.1.

Theorem. *Let $\varphi\colon (R,\mathfrak{m},k) \to (R,\mathfrak{m},k)$ be a contracting endomorphism. The following are equivalent.*

(i) *R is Gorenstein.*
(ii) *There exists $X \in \mathsf{I}^{\mathrm{fg}}(R)$ with $H(X) \neq 0$ and $\mathsf{L}\mathrm{F}^\varphi(X) \in \mathsf{I}^{\mathrm{fg}}(R)$.*
(iii) *For every $X \in \mathsf{I}^{\mathrm{fg}}(R)$ we have $\mathsf{L}\mathrm{F}^\varphi(X) \in \mathsf{I}^{\mathrm{fg}}(R)$.*

Proof. (i) $\implies$ (iii). The discussion after Theorem 2.5 shows that $\mathsf{I}^{\mathrm{fg}}(R)$ is the subcategory of perfect complexes. So for every $X \in \mathsf{I}^{\mathrm{fg}}(R)$ the base change $\mathsf{L}\mathrm{F}^\varphi(X)$ is also perfect and hence in $\mathsf{I}^{\mathrm{fg}}(R)$.

(iii) $\implies$ (ii). We need to show that for every local ring there exists a complex $X \in \mathsf{I}^{\mathrm{fg}}(R)$ with $H(X) \neq 0$. Let E be the injective hull of the residue field, and K^R the Koszul complex of R. As $\mathfrak{m}H(K^R \otimes_R E) = 0$, all the homology modules are isomorphic to direct sums of k. As the complex $K^R \otimes_R E$ consists of Artinian modules, it follows that for every i, the module $H_i(K^R \otimes_R E)$ is Artinian, hence it has finite length.

(ii) $\implies$ (i). Let K^R be the Koszul complex of R. Let $X \in \mathsf{I}^{\mathrm{fg}}(R)$ with $\operatorname{inj-dim}_R(\mathsf{L}\mathrm{F}^{\varphi}(X)) < \infty$. Since $K^R \otimes^{\mathbf{L}}_R X$ still has finite injective dimension and nonzero finite length homology, Lemma 3.10 shows that

$$\operatorname{inj-dim}_R(\mathsf{L}\mathrm{F}^{\varphi^{i}}(K^R \otimes^{\mathbf{L}}_R X)) < \infty \quad \text{for all } i \geq 1$$

Setting $c = \ell\ell_{\mathsf{D}(R)} K^R$ we can take i large enough so that $\varphi^i(\mathfrak{m}) \subseteq \mathfrak{m}^c$. Corollary 3.3 yields that R is Gorenstein. $\square$

Theorem 1.1 is the derived analogue of the following result by Falahola and Marley [7, Theorem 3.1].

Theorem 3.11. *Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay local ring, and let φ be a contracting endomorphism. Suppose that ω_R is a canonical module for R, then* $\operatorname{inj-dim}_{R^{\varphi}} \mathrm{F}^{\varphi}(\omega_R)$ *in finite if and only if R is Gorenstein.*

Remark 3.12. Falahola and Marley [7, Example 3.8] show that Theorem 3.11 fails if we replace ω_R with a general dualizing complex C. In [7, Question 3.9] they ask if R has a dualizing complex C is it true that $\operatorname{inj-dim}_R(R^{\varphi} \otimes^{\mathbf{L}}_R C) < \infty$ if and only if R is Gorenstein? Since a dualizing complex is in $\mathsf{I}^{\mathrm{fg}}(R)$, Theorem 1.1 shows in particular that if C is a dualizing complex in $\mathsf{D}(R)$ then $\operatorname{inj-dim}_R(\mathsf{L}\mathrm{F}^{\varphi}(C))$ being finite implies that R is Gorenstein, giving an affirmative answer.

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