

COHOMOLOGICAL SUPPORTS OVER DERIVED COMPLETE INTERSECTIONS AND LOCAL RINGS

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ABSTRACT. A theory of cohomological support for pairs of DG modules over a Koszul complex is investigated. These specialize to the support varieties of Avramov and Buchweitz defined over a complete intersection ring, as well as support varieties over an exterior algebra. The main objects of study are certain DG modules over a polynomial ring; these determine the aforementioned cohomological supports and are shown to encode (co)homological information about pairs of DG modules over a Koszul complex. The perspective in this article leads to new proofs of well-known results for pairs of complexes over a complete intersection. Furthermore, these cohomological supports are used to define a support theory for pairs of objects in the derived category of an arbitrary commutative noetherian local ring. Finally, we calculate several examples; one of which answers a question of D. Jorgensen in the negative.

INTRODUCTION

In this article we study the cohomological properties of a Koszul complex over a commutative noetherian ring and introduce a corresponding theory of cohomological support. This simultaneously generalizes results in two well-studied settings: (1) arbitrary deformations of a commutative noetherian ring, and (2) exterior algebras defined over a commutative noetherian ring. This theory provides a unified perspective that leads to more natural proofs. Furthermore, the calculations of certain geometric invariants constitute one of the main contributions of this paper.

The setup is the following. Let Q denote a commutative noetherian ring, $\mathbf{f} = f_1, \dots, f_n$ an *arbitrary* list of elements in Q and let E denote the Koszul complex on $\mathbf{f}$ over Q . We regard E as a DG Q -algebra in the usual way, and in doing so we recover the cases above by specializing to when (1) $\mathbf{f}$ is a Q -regular sequence, and (2) each $f_i = 0$, respectively. In the introduction we restrict to the case that Q is a regular local ring with residue field k , since this is when the strongest results hold. In this setting we refer to E as a *derived complete intersection*. See Remark 3.1.2 for a discussion of the terminology.

For DG E -modules M and N with finitely generated homology, we associate a Zariski closed subset $\mathcal{V}_E(M, N)$ of $\mathbb{P}_k^{n-1}$ (see Definition 4.2.4), called the cohomological support of the pair (M, N) whose dimension records the polynomial growth rate of the minimal number of generators of $\text{Ext}_E^*(M, N)$. The latter value is called

Date: February 17, 2021.

2010 *Mathematics Subject Classification.* 13D07, 13D09 (primary), 14M10, 18G15 (secondary).

Key words and phrases. Local ring, Complete intersection, Derived category, DG algebra, Cohomology operators, Support, Koszul complex.

The author was partly supported through National Science Foundation grants DMS 1103176, DMS 1840190, and DMS 2002173.

the complexity of (M, N) , denoted $\text{cx}_E(M, N)$ (cf. 4.2.8 for a precise definition). In Theorem 4.3.1, we show these supports satisfy the following.

Theorem A. *For DG E -modules M, M', N, N' with finitely generated homology,*

$$\mathcal{V}_E(M, N) \cap \mathcal{V}_E(M', N') = \mathcal{V}_E(M, N') \cap \mathcal{V}_E(M', N).$$

A consequence of Theorem A is the following bound and symmetry of complexity over derived complete intersections: *For DG E -modules M and N with finitely generated homology,*

$$\text{cx}_E(M, N) = \text{cx}_E(N, M) \leq n.$$

This recovers the asymptotic theorems of Avramov and Buchweitz [5, Theorem II], for local complete intersections, and Avramov and Iyengar [10, 5.3], for exterior algebras. The proof in [5] puts to use a theory of intermediate hypersurfaces which reduces the study of Ext-modules over a complete intersection to the study of Ext-modules over certain hypersurface rings; the latter are well-understood due to the foundational work of Eisenbud in [24] (see also [19]). The proof in [10] uses the Hopf-algebra structure of an exterior algebra.

Theorem A follows a different route. Namely, we relate $\mathcal{V}_E(M, N)$ to the cohomological support (c.f. Definition 2.1.3) of a certain DG module over $\mathcal{S} = Q[\chi_1, \dots, \chi_n]$. This perspective builds on ideas from [4] and works with chain level operators corresponding to the Hochschild cohomology of E over Q ; these operators are also those introduced by Gulliksen [30] and studied in various avatars by Avramov, Eisenbud, Gasharov, Mehta, Peeva, Sun and many others (see, for example, [1, 7, 13, 24, 35]). The regularity of Q is used in a fundamental way to establish isomorphisms for the DG $\mathcal{S}$ -modules which determine the supports under consideration in Theorem A; the equality of supports is a direct consequence of these isomorphisms.

The perspective adopted in this paper provides a framework for studying cohomology in settings where one cannot resort to the study of intermediate hypersurfaces or exploit a Hopf-algebra structure. For example, [26] identified a class of color commutative rings that exhibit behavior similar to that of a local complete intersection that do not enjoy enough intermediate hypersurfaces nor, *a priori*, a Hopf-algebra structure. In a recent collaboration of the present author with Ferraro and Moore [27], we prove an analog of Theorem A and arrive at symmetries in complexity for such color commutative rings by following the proof strategy of Theorem A described above.

Aside from the asymptotic information recorded in these supports, the regularity of the sequence $\mathbf{f}$, and hence, the complete intersection property is detected by these varieties. Theorem 4.2.2 establishes the following.

Theorem B. *Let $E = \text{Kos}^Q(\mathbf{f})$ be a derived complete intersection and set $R = Q/(\mathbf{f})$. The following are equivalent:*

- (1) *R is a complete intersection.*
- (2) *$\mathcal{V}_E(R, k) = \emptyset$.*
- (3) *$\mathcal{V}_E(M, k) = \emptyset$ for some nonzero finitely generated R -module M .*

The equivalence of (1) and (2) in Theorem B was first proven in [36] but an independent proof of the more general Theorem 4.2.2 is given in this article. Theorem B has been applied in [36], and more recently in [16], to establish a triangulated category characterization of locally complete intersection rings, answering a question of

Dwyer, Greenlees, and Iyengar [23]. One of the major points is that the structure of thick subcategories is also reflected in these varieties (cf. Remark 4.2.6).

Finally, we use this theory of supports over a derived complete intersection to define a support theory for pairs of complexes over an arbitrary local ring R ; we denote these by $\mathcal{V}_R(M, N)$ for complexes of R -modules M and N , see Definition 5.1.5 for details. These recover the supports from [5] for pairs of finitely generated modules over a local complete intersection, and in Theorem 5.2.4 it is shown that these also specialize to the more general support varieties in [33]. Working at the chain level with the DG $\mathcal{S}$ -modules described above allows us to answer a question from [33] in the negative (see Example 5.4.3): *If $\mathcal{V}_R(M, N) = \emptyset$ for some finitely generated R -modules M and N , is R a complete intersection?* Moreover, Theorem B can be interpreted as a corrected form of the question from [33].

It is worth noting that Theorem B says the supports $\mathcal{V}_R(M, k)$ for a finitely generated R -module M , especially $\mathcal{V}_R(R, k)$, are geometric obstructions to R itself being a complete intersection. In general, embedded deformations of R cut down $\mathcal{V}_R(R, k)$ by a hyperplane (see Proposition 5.3.7); a complete intersection being a ring with a maximal number of embedded deformations and hence, an empty support (cf. Theorem B). However, the general structure of $\mathcal{V}_R(R, k)$ remains mysterious when R is not a complete intersection. The final result we highlight makes progress on gaining insight on $\mathcal{V}_R(R, k)$ by characterizing the possible closed subsets it can be when R has small codepth (see Section 5.3).

Theorem C. *If R is not a complete intersection and $\text{codepth } R \leq 3$, then $\mathcal{V}_R(R, k) = \mathbb{P}_k^{n-1}$ except when R admits an embedded deformation. In the exceptional case, $\mathcal{V}_R(R, k)$ is a hyperplane in $\mathbb{P}_k^{n-1}$.*

Determining the possible closed subsets that can be realized by $\mathcal{V}_R(R, k)$ remains an interesting problem for arbitrary codepth. In hopes of better understanding this problem, we propose the following question: *Is $\mathcal{V}_R(R, k)$ always a union of linear spaces?* The examples in this article and [16, Section 4], as well as various calculations using Macaulay2 [29], have yet to produce an example that is not a union of linear spaces.

ACKNOWLEDGEMENTS

This paper was partially completed at the University of Nebraska-Lincoln while I was finishing my thesis under the supervision of Luchezar Avramov and Mark Walker. It is a pleasure to thank both of them for innumerable conversations relating to this work. I would also like to thank Srikanth Iyengar and the referee for helpful comments on this article.

1. DIFFERENTIAL GRADED HOMOLOGICAL ALGEBRA

1.1. Conventions and Notation. Fix a commutative noetherian ring Q . Let $A = \{A_i\}_{i \in \mathbb{Z}}$ denote a commutative DG Q -algebra. By *commutative*, we mean that A is commutative in the graded sense; namely,

$$ab = (-1)^{|a||b|}ba$$

for all a and b in A .

1.1.1. A map $\varphi : M \rightarrow N$ between DG A -modules M and N is called a *morphism of DG A -modules* provided that φ is a morphism of the underlying complexes of

Q -modules such that $\varphi(am) = a\varphi(m)$ for all $a \in A$ and $m \in M$. We use the notation $\varphi : M \xrightarrow{\sim} N$ to mean that the morphism of DG A -modules φ is a quasi-isomorphism.

1.1.2. Let M be a DG A -module. The differential of M is denoted by ∂^M . We use $|m|$ to denote the degree of an element of M , i.e., $|m| = d$ exactly when $m \in M_d$. For each $i \in \mathbb{Z}$, $\Sigma^i M$ is the DG A -module given by

$$(\Sigma^i M)_n := M_{n-i}, \quad \partial^{\Sigma^i M} := (-1)^i \partial^M, \quad \text{and } a \cdot m := (-1)^{|a||i|} am.$$

The *boundaries* and *cycles* of M are

$$B(M) := \{\text{Im } \partial_{i+1}^M\}_{i \in \mathbb{Z}} \quad \text{and} \quad Z(M) := \{\text{Ker } \partial_i^M\}_{i \in \mathbb{Z}},$$

respectively. The *homology* of M is defined to be

$$H(M) := Z(M)/B(M) = \{H_i(M)\}_{i \in \mathbb{Z}}$$

which is a graded module over the graded Q -algebra $H(A) := \{H_i(A)\}_{i \in \mathbb{Z}}$. We let $M^\natural$ denote the underlying graded Q -module obtained by forgetting the differential of M ; note that $A^\natural$ is a graded Q -algebra and $M^\natural$ is a graded $A^\natural$ -module.

1.1.3. As Ext-modules have historically been graded cohomologically, we will be consistent with this convention. In particular, when working with DG modules whose homology is a graded Ext-module of interest we will grade these objects cohomologically. An effort has been made to make it clear when working in this setting. All other graded objects will be graded homologically as indicated above. We point out that when $M = \{M^i\}_{i \in \mathbb{Z}}$ is cohomologically graded DG A -module, $\Sigma^j M$ has $(\Sigma^j M)^i = M^{i+j}$.

1.2. Semifree and Semiprojective DG modules. Besides setting terminology, the goal of this section is to establish the “moreover” statement from Proposition 1.2.8 below in the generality stated there; this will be needed for proving some of the main results of the article (for example, Theorem 4.3.1). This section may be skipped by the experts.

1.2.1. A DG A -module P is *semiprojective* if for every morphism of DG A -modules $\alpha : P \rightarrow N$ and each surjective quasi-isomorphism of DG A -modules $\gamma : M \rightarrow N$ there exists a unique up to homotopy morphism of DG A -module $\beta : P \rightarrow M$ such that $\alpha = \gamma\beta$. Equivalently, $P^\natural$ is a projective graded $A^\natural$ -module and $\text{Hom}_A(P, -)$ preserves quasi-isomorphisms. When P is semiprojective the functor $P \otimes_A -$ is also exact and preserves quasi-isomorphisms.

1.2.2. A *semiprojective resolution* of a DG A -module M is a surjective quasi-isomorphism of DG A -modules $\epsilon : P \rightarrow M$ where P is a semiprojective DG A -module. Semiprojective resolutions exist and any two semiprojective resolutions of M are unique up to homotopy equivalence [25, 6.6].

1.2.3. Assume that

$$0 \rightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \rightarrow 0$$

is an exact sequence of DG A -modules such that

$$0 \rightarrow L^\natural \xrightarrow{\alpha^\natural} M^\natural \xrightarrow{\beta^\natural} N^\natural \rightarrow 0$$

is a split exact sequence of graded $A^\natural$ -modules. By the Five Lemma, it follows that if two of the three DG A -modules are semiprojective, so is the third.

1.2.4. A DG A -module F is *semifree* if there exists a chain of DG A -submodules of F

$$0 = F(-1) \subseteq F(0) \subseteq F(1) \subseteq \dots$$

such that $\bigcup_i F(i) = F$ and for each i

$$F(i+1)/F(i) \cong \coprod_{x \in X_i} \Sigma^{|x|} A$$

where X_i is some graded set. Every semifree DG A -module is semiprojective (see [25, 6.10]).

The following construction is the standard one used to construct a semiprojective (in fact, semifree) resolution of a DG A -module; see, for example, [2, 2.2.6].

Construction 1.2.5. Let M be a DG A -module. For each $m \in M$ we let e_m be a graded variable of degree $|m|$. We define $C(m)$ to be the DG A -module with

$$C(m)^{\natural} := Ae_m^{\natural} \oplus Ae_{\partial^M(m)}^{\natural}$$

where

$$\partial(ae_m + be_{\partial^M(m)}) = \partial^A(a)m + (-1)^{|a|}ae_{\partial^M(m)} + \partial^A(b)e_{\partial^M(m)}.$$

It is straightforward to see that $C(m)$ is a contractible semifree DG A -module. Moreover, we have a morphism of DG A -modules $\pi(m) : C(m) \rightarrow M$ given by

$$ae_m + be_{\partial^M(m)} \mapsto am + b\partial^M(m).$$

For any cycle $z \in Z(M)$, we have a morphism of DG A -modules $\tau(z) : Ae_z \rightarrow M$ given by $ae_z \mapsto az$. Define

$$C^M := \left(\coprod_{m \in M} C(m) \right) \coprod \left(\coprod_{z \in Z(M)} Ae_z \right)$$

and define

$$\pi^M := \left(\sum_{m \in M} \pi(m) \right) + \left(\sum_{z \in Z(M)} \tau(z) \right) : C^M \rightarrow M.$$

Finally, set $K^M := \text{Ker } \pi^M$ and so we have a short exact sequence

$$(1.1) \quad 0 \rightarrow K^M \xrightarrow{\iota^M} C^M \xrightarrow{\pi^M} M \rightarrow 0$$

Lemma 1.2.6. *Using the notation from Construction 1.2.5, the following hold:*

- (1) C^M is a semifree DG A -module.
- (2) There exists a surjective homotopy equivalence $C^M \rightarrow \coprod_{z \in Z(M)} \Sigma^{|z|} A$.
- (3) Applying $B(-)$, $H(-)$ or $Z(-)$ to (1.1) yield exact sequences of graded $Z(A)$ -modules.

In particular, if $\partial^A = 0$, then $Z(C^M)$ and $B(C^M)$ are free DG A -modules.

Proof. It is clear that C^M is a semifree DG A -module and that π^M and $Z(\pi^M)$ are surjective. Consider the following commutative diagram of DG $Z(A)$ -modules

$$\begin{array}{ccccccc}
& & 0 & & 0 & & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & Z(K^M) & \xrightarrow{Z(\iota^M)} & Z(C^M) & \xrightarrow{Z(\pi^M)} & Z(M) \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & K^M & \xrightarrow{\iota^M} & C^M & \xrightarrow{\pi^M} & M \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \Sigma B(K^M) & \xrightarrow{\Sigma B(\iota^M)} & \Sigma B(C^M) & \xrightarrow{\Sigma B(\pi^M)} & \Sigma B(M) \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 0 & & 0 & & 0
\end{array}$$

Since $Z(-)$ is left exact, it follows that the top two rows of the diagram are exact. Also, the columns are the canonical exact sequences. Hence, by the *nine lemma* it follows that the last row is exact. A similar argument now yields that applying $H(-)$ to (1.1) yields an exact sequence.

Since $C(m)$ is contractible for each $m \in M$ and $Ae_z \cong \Sigma^{|z|} A$ for each $z \in Z(M)$, we conclude that there is a surjective homotopy equivalence $C^M \rightarrow \coprod_{z \in Z(M)} \Sigma^{|z|} A$.

Now assume that $\partial^A = 0$. For each $m \in M$,

$$B(C(m)) = Z(C(m)) = Ae_{\partial^M(m)}.$$

Thus,

$$Z(C^M) = \left(\coprod_{m \in M} Ae_{\partial^M(m)} \right) \coprod \left(\coprod_{z \in Z(M)} Ae_z \right)$$

and

$$B(C^M) = \coprod_{m \in M} Ae_{\partial^M(m)}.$$

Thus, $Z(C^M)$ and $B(C^M)$ are free DG A -modules. $\square$

Lemma 1.2.7. *Let M be a DG A -module. There exists an exact sequence of DG A -modules*

$$\cdots \rightarrow F^n \rightarrow F^{n-1} \rightarrow \cdots \rightarrow F^1 \rightarrow F^0 \rightarrow M \rightarrow 0$$

such that each F^i is a semifree free DG A -module that maps onto $\coprod_{z \in Z(K^i)} \Sigma^{|z|} A$ via a surjective homotopy equivalence where $K^i = \text{coker}(F^{i+1} \rightarrow F^i)$. Moreover, the induced sequences of graded $Z(A)$ -modules are exact:

- (1) $\cdots \rightarrow B(F^1) \rightarrow B(F^0) \rightarrow B(M) \rightarrow 0$
- (2) $\cdots \rightarrow Z(F^1) \rightarrow Z(F^0) \rightarrow Z(M) \rightarrow 0$
- (3) $\cdots \rightarrow H(F^1) \rightarrow H(F^0) \rightarrow H(M) \rightarrow 0$

Furthermore, if $\partial^A = 0$, F^i can be chosen such that $Z(F^i)$ and $B(F^i)$ are free DG A -modules.

Proof. Iteratively applying Lemma 1.2.6 to obtain exact sequences

$$0 \rightarrow K^i \rightarrow F^i \rightarrow M^i \rightarrow 0$$

where $M^0 := M$ and $M^i := K^{i-1}$ gives us the desired result. $\square$

Proposition 1.2.8. *Suppose that $\partial^A = 0$. Let M be a DG A -module where two of $B(M)$, $Z(M)$, $H(M)$ have finite projective dimension when regarded as graded A -modules. There exists an exact sequence of DG A -modules*

$$(1.2) \quad 0 \rightarrow F^n \rightarrow F^{n-1} \rightarrow \dots \rightarrow F^1 \rightarrow F^0 \rightarrow M \rightarrow 0$$

where F^n is semiprojective and F^i is semifree for $0 \leq i \leq n-1$. Moreover, if M is graded projective, then M is semiprojective.

Proof. Let

$$\dots \rightarrow F^n \rightarrow F^{n-1} \rightarrow \dots \rightarrow F^1 \rightarrow F^0 \rightarrow M \rightarrow 0$$

be the sequence obtained from Lemma 1.2.7. Set $K^n := \ker(F^n \rightarrow F^{n-1})$. The assumption and the exact sequences of DG A -modules

$$0 \rightarrow B(M) \rightarrow Z(M) \rightarrow H(M) \rightarrow 0$$

imply that $B(M)$ and $Z(M)$ have finite projective dimension when regarded as graded A -modules. Since (1) and (2) are graded free resolutions of $B(M)$ and $Z(M)$ over A , respectively, it follows that $B(K^n)$ and $Z(K^n)$ are graded projective DG A -modules for $n \gg 0$. As $\partial^A = 0$, $\partial^{Z(K^n)} = 0$, and $\partial^{B(K^n)} = 0$ it follows that

$$H(\mathrm{Hom}_A(Z(K^n), -)) \cong \mathrm{Hom}_A(Z(K^n), H(-))$$

$$H(\mathrm{Hom}_A(B(K^n), -)) \cong \mathrm{Hom}_A(B(K^n), H(-)).$$

Thus, $Z(K^n)$ and $B(K^n)$ are semiprojective DG A -modules. Finally, 1.2.3 and the graded split exact sequence

$$0 \rightarrow Z(K^n) \rightarrow K^n \rightarrow \Sigma B(K^n) \rightarrow 0$$

yield that K^n is semiprojective. Induction on the length of the exact sequence (1.2) and again applying 1.2.3 establishes that if M is graded projective then M is semiprojective. $\square$

1.3. The Derived Category of a DG Algebra.

1.3.1. Let $D(A)$ denote the derived category of A ; recall that $D(A)$, equipped with Σ , is a triangulated category. Define $D^f(A)$ to be the full subcategory of $D(A)$ consisting of all M such that $H(M)$ is a finitely generated graded module over $H(A)$. We use $\simeq$ to denote isomorphisms in $D(A)$ and reserve $\cong$ for isomorphisms of DG A -modules.

1.3.2. For a DG A -module M , define

$$\mathrm{RHom}_A(M, -) := \mathrm{Hom}_A(P, -) \text{ and } M \otimes_A^{\mathrm{L}} - = P \otimes_A -$$

where P is a semiprojective resolution of M over A . By 1.2.1 and 1.2.2, $\mathrm{RHom}_A(M, -)$ and $M \otimes_A^{\mathrm{L}} -$ are well-defined exact endo-functors on $D(A)$. For each object N of $D(A)$, define

$$\mathrm{Ext}_A^*(M, N) := H(\mathrm{RHom}_A(M, N)) \text{ and } \mathrm{Tor}_*^A(M, N) = H(M \otimes_A^{\mathrm{L}} N).$$

1.3.3. Let $\varphi : A' \rightarrow A$ be a morphism of DG Q -algebras and let M and N be DG A -modules. By [25, 6.10], If φ is a quasi-isomorphism, then φ induces an isomorphism of graded Q -modules

$$\mathrm{Ext}_\varphi^*(M, N) : \mathrm{Ext}_A^*(M, N) \rightarrow \mathrm{Ext}_{A'}^*(M, N).$$

1.3.4. Let $\mathcal{T}$ denote a triangulated category. A full subcategory $\mathcal{T}'$ of $\mathcal{T}$ is called triangulated if it is closed under suspension and has the two out of three property on exact triangles. If, in addition, $\mathcal{T}'$ is closed under direct summands, we say that $\mathcal{T}'$ is a thick subcategory of $\mathcal{T}$.

Let X be in $\mathcal{T}$. Define the thick closure of X in $\mathcal{T}$, denoted $\text{Thick}_{\mathcal{T}} X$, to be the intersection of all thick subcategories of $\mathcal{T}$ containing X . As an intersection of thick subcategories is a thick subcategory, $\text{Thick}_{\mathcal{T}} X$ is the smallest thick subcategory of $\mathcal{T}$ containing X . See [6, Section 2] for an inductive definition of $\text{Thick}_{\mathcal{T}} X$.

For the rest of the section, assume that Q is a commutative noetherian ring and A is a non-negatively graded, commutative DG Q -algebra such that $H_i(A)$ is finitely generated over Q for each i , and the canonical map $Q \rightarrow H_0(A)$ is surjective.

1.3.5. Suppose $(Q, \mathfrak{n}, k)$ is local and let M be in $D^f(A)$. In this context, we can test whether M is in $\text{Thick}_{D(A)} A$ by calculating the eventual vanishing of certain graded Ext-modules. That is, Jørgensen established in [34, 2.2] that M is in $\text{Thick}_{D(A)} A$ if and only if $\text{Ext}_A^{\geq 0}(M, k) = 0$.

1.3.6. Let M be a complex of Q -modules. The amplitude of M is defined to be

$$\text{amp } M := \sup\{i : H_i(M) \neq 0\} - \inf\{i : H_i(M) \neq 0\}.$$

By [34, 4.1], if $\text{amp } A < \infty$, then for each nontrivial object M of $\text{Thick}_{D(A)} A$

$$\text{amp } A \leq \text{amp } M.$$

2. DG MODULES OVER A GRADED COMMUTATIVE NOETHERIAN RING

In this section, we follow the convention in 1.1.3 of grading objects cohomologically. The reason being that theory in this section will be applied to studying DG modules and graded Ext-modules over a ring of cohomology operators (see Section 3 for details).

2.1. **Support.** Let $\mathcal{A} = \{\mathcal{A}^i\}_{i \geq 0}$ be a graded, commutative noetherian ring. Recall that, as a set, $\text{Proj } \mathcal{A}$ consists of the homogeneous prime ideals of $\mathcal{A}$ not containing the irrelevant ideal $\mathcal{A}^{>0} := \{\mathcal{A}^i\}_{i > 0}$. The topology on $\text{Proj } \mathcal{A}$ is the Zariski topology, which has as its closed sets those of the form

$$\{\mathfrak{p} \in \text{Proj } \mathcal{A} : \mathfrak{p} \supseteq \mathcal{I}\}$$

where $\mathcal{I}$ is a homogeneous ideal of $\mathcal{A}$.

2.1.1. We regard $\mathcal{A}$ as a DG algebra with trivial differential. For a DG $\mathcal{A}$ -module $X = \{X^i\}_{i \in \mathbb{Z}}$, $H(X) := \{H^i(X)\}_{i \in \mathbb{Z}}$ is a graded $\mathcal{A}$ -module and, as usual, X is an object of $D^f(\mathcal{A})$ if and only if $H(X)$ is a finitely generated graded $\mathcal{A}$ -module.

2.1.2. Let X be a DG $\mathcal{A}$ -module. For each $\mathfrak{p} \in \text{Proj } \mathcal{A}$ we let $X_{\mathfrak{p}}$ denote the homogeneous localization of X at $\mathfrak{p}$. We let $\kappa(\mathfrak{p}) := \mathcal{A}_{\mathfrak{p}}/\mathfrak{p}\mathcal{A}_{\mathfrak{p}}$. By [17, 1.5.7],

$$\kappa(\mathfrak{p}) \cong k[t, t^{-1}]$$

for some field k and a variable t of positive (cohomological) degree.

Definition 2.1.3. The *cohomological support* of a DG $\mathcal{A}$ -module X is defined to be

$$\text{Supp}_{\mathcal{A}}^+ X := \{\mathfrak{p} \in \text{Proj } \mathcal{A} : X \otimes_{\mathcal{A}}^L \kappa(\mathfrak{p}) \neq 0\}.$$

2.1.4. Let $S = k[t, t^{-1}]$ where k is a field and t is a variable of positive cohomological degree. As S is a graded field, for each DG S -module X there exists a surjective homotopy equivalence $X \rightarrow H(X)$ and we have an isomorphism of DG S -modules

$$H(X) \cong S^{(\beta)}$$

where β is a k -basis for $H^0(X)$. Thus, there is a natural isomorphism

$$H(\text{Hom}_S(X, -)) \cong \text{Hom}_S(H(X), H(-)).$$

Therefore, each DG S -module is semiprojective.

2.1.5. For DG $\mathcal{A}$ -modules X and X' ,

$$\text{Supp}_{\mathcal{A}}^+(X \otimes_{\mathcal{A}}^L X') = \text{Supp}_{\mathcal{A}}^+ X \cap \text{Supp}_{\mathcal{A}}^+ X'.$$

Indeed, for any homogeneous prime $\mathfrak{p}$ we have the following

$$\begin{aligned} X \otimes_{\mathcal{A}}^L X' \otimes_{\mathcal{A}}^L \kappa(\mathfrak{p}) &\simeq (X \otimes_{\mathcal{A}}^L \kappa(\mathfrak{p})) \otimes_{\kappa(\mathfrak{p})}^L (X' \otimes_{\mathcal{A}}^L \kappa(\mathfrak{p})) \\ &\simeq (X \otimes_{\mathcal{A}}^L \kappa(\mathfrak{p})) \otimes_{\kappa(\mathfrak{p})} (X' \otimes_{\mathcal{A}}^L \kappa(\mathfrak{p})) \\ &\simeq \kappa(\mathfrak{p})^{(\beta)} \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{p})^{(\beta')}. \end{aligned}$$

where β and β' are the k -bases for $H^0(X)$ and $H^0(X')$, respectively (see 2.1.4 for the last two isomorphisms).

2.1.6. Let X be an object of $D^f(\mathcal{A})$. By [21, 2.4],

$$\text{Supp}_{\mathcal{A}}^+ X = \{\mathfrak{p} \in \text{Proj } \mathcal{A} : M_{\mathfrak{p}} \neq 0\}.$$

Since localization is exact, $M_{\mathfrak{p}} \simeq 0$ if and only if $H(M)_{\mathfrak{p}} = 0$. Therefore, there is an equality

$$\begin{aligned} \text{Supp}_{\mathcal{A}}^+ X &= \text{Supp}_{\mathcal{A}}^+ H(X) \\ &= \{\mathfrak{p} \in \text{Proj } \mathcal{A} : \mathfrak{p} \supseteq \text{ann}_{\mathcal{A}}(H(X))\} \end{aligned}$$

where the second equality holds by [9, 2.2(4)]. Thus, $\text{Supp}_{\mathcal{A}}^+ X$ is a closed subset of $\text{Proj } \mathcal{A}$ whenever X is an object of $D^f(\mathcal{A})$.

The next result follows easily from the definition of cohomological support (also see [9, 2.2]).

Proposition 2.1.7. *Let $\mathcal{A} = \{\mathcal{A}^i\}_{i \geq 0}$ be a cohomologically graded, commutative noetherian ring.*

- (1) *Let X be a DG $\mathcal{A}$ -module and $n \in \mathbb{Z}$. Then $\text{Supp}_{\mathcal{A}}^+ X = \text{Supp}_{\mathcal{A}}^+(\Sigma^n X)$.*
- (2) *Let $0 \rightarrow X' \rightarrow X \rightarrow X'' \rightarrow 0$ be an exact sequence of DG $\mathcal{A}$ -modules where either:*

- (a) *it is split exact, or*
- (b) *each DG $\mathcal{A}$ -module has trivial differential and is an object of $D^f(\mathcal{A})$.*

Then

$$\text{Supp}_{\mathcal{A}}^+ X = \text{Supp}_{\mathcal{A}}^+ X' \cup \text{Supp}_{\mathcal{A}}^+ X''.$$

- (3) *If X is an object of $D^f(\mathcal{A})$, then $\text{Supp}_{\mathcal{A}}^+ X = \emptyset$ if and only if $H^{\geq 0}(X) = 0$.*
- (4) *Let X and X' be objects of $D^f(\mathcal{A})$ with trivial differential. Then*

$$\text{Supp}_{\mathcal{A}}^+(X \otimes_{\mathcal{A}} X') = \text{Supp}_{\mathcal{A}}^+ X \cap \text{Supp}_{\mathcal{A}}^+ X'.$$

2.2. Finite Generation via Koszul Objects.

2.2.1. Let $a \in \mathcal{A}$ be a homogeneous element of degree d . For a DG $\mathcal{A}$ -module X we define

$$X//a = \text{cone}(\Sigma^{-d}X \xrightarrow{a} X).$$

As $aH(X//a) = 0$, $H(X//a)$ is a graded $\mathcal{A}/a$ -module. Also, the following is an exact sequence of graded $\mathcal{A}$ -modules

$$\Sigma^{-d}H(X) \xrightarrow{a} H(X) \rightarrow H(X//a) \rightarrow \Sigma^{-d+1}H(X) \xrightarrow{a} \Sigma H(X).$$

2.2.2. Let $\mathbf{a} := a_1, \dots, a_n \in \mathcal{A}$ be a sequence of homogeneous elements. For a DG $\mathcal{A}$ -module X , we define the *Koszul object of X with respect to $\mathbf{a}$* to be

$$X//\mathbf{a} := (((X//a_1)//a_2) \dots // a_n).$$

It follows that $H(X//\mathbf{a})$ is a graded $\mathcal{A}/(\mathbf{a})$ -module. Furthermore, if $\mathbf{a}$ is a regular sequence on $X^\natural$ then we have the following isomorphism in $D(\mathcal{A})$

$$X//\mathbf{a} \xrightarrow{\sim} X/(\mathbf{a})X.$$

2.2.3. We recall the *graded version* of Nakayama's lemma: Let M be a graded $\mathcal{A}$ -module such that M is bounded below, i.e., $M^i = 0$ for $i \ll 0$. If $\mathcal{A}^{>0}M = 0$, then $M = 0$. In particular, if $M/\mathcal{A}^{>0}M$ is a finitely generated graded $\mathcal{A}$ -module, then M is a finitely generated graded $\mathcal{A}$ -module.

Theorem 2.2.4. *Let X be a DG $\mathcal{A}$ -module such that $H^{\leq 0}(X) = 0$. For each sequence of homogeneous elements $\mathbf{a} := a_1, \dots, a_n \in \mathcal{A}$ of positive degree, $H(X)$ is a finitely generated graded $\mathcal{A}$ -module if and only if $H(X//\mathbf{a})$ is a finitely generated graded $\mathcal{A}/(\mathbf{a})$ -module.*

Proof. Since $X//\mathbf{a}$ is defined inductively, it suffices to show that theorem holds when $\mathbf{a} = a$ is just a single homogeneous element a of degree $d > 0$. By 2.2.1, we have an exact sequence of DG $\mathcal{A}$ -modules

$$(2.1) \quad \Sigma^{-d}H(X) \xrightarrow{a} H(X) \rightarrow H(X//a) \rightarrow \Sigma^{-d+1}H(X) \xrightarrow{a} \Sigma H(X).$$

Assume that $H(X)$ is a finitely generated graded $\mathcal{A}$ -module. Hence, (2.1) implies that $H(X//a)$ is finitely generated over $\mathcal{A}$. Since $aH(X//a) = 0$, it follows that $H(X//a)$ is a finitely generated graded $\mathcal{A}/(a)$ -module.

Conversely, assume that $H(X//a)$ is a finitely generated graded $\mathcal{A}/(a)$ -module. Since $aH(X//a) = 0$ then $H(X//a)$ is finitely generated as a graded $\mathcal{A}$ via the canonical surjection $\mathcal{A} \rightarrow \mathcal{A}/(a)$. From (2.1), $H(X)/aH(X)$ is a submodule of $H(X//a)$ and since $H(X//a)$ is a noetherian graded $\mathcal{A}$ -module, $H(X)/aH(X)$ is a noetherian graded $\mathcal{A}$ -module. Therefore, $H(X)/aH(X)$ is a finitely generated graded $\mathcal{A}$ -module. As $H^{\leq 0}(X) = 0$, it follows that $(H(X)/aH(X))^{\leq 0} = 0$. Therefore, by 2.2.3, it follows that $H(X)$ is a finitely generated graded $\mathcal{A}$ -module. $\square$

3. COHOMOLOGICAL OPERATORS

3.1. **Koszul Complexes.** Fix a commutative noetherian ring Q . Let $\mathbf{f} = f_1, \dots, f_n$ be a list of elements in Q . Define $\text{Kos}^Q(\mathbf{f})$ to be the DG Q -algebra with $\text{Kos}^Q(\mathbf{f})^\natural$ the exterior algebra on a free Q -module with basis $\xi_1, \dots, \xi_n$ of homological degree 1, and differential $\partial\xi_i = f_i$. We write

$$\text{Kos}^Q(\mathbf{f}) = Q\langle \xi_1, \dots, \xi_n | \partial\xi_i = f_i \rangle.$$

3.1.1. Let $E := \text{Kos}^Q(\mathbf{f})$. We say that E is a *derived complete intersection* if Q is a regular ring. We say that E is a *minimal derived complete intersection* provided that $(Q, \mathfrak{n}, k)$ is a regular local ring where $\mathbf{f} \subseteq \mathfrak{n}^2$ minimally generates $(\mathbf{f})$.

Remark 3.1.2. Imposing the condition that the base ring Q is regular has strong implications which will be utilized heavily in Section 4.2. There are two perspectives which motivate the terminology in 3.1.1. First, when $\mathbf{f}$ is a Q -regular sequence then E is a cofibrant replacement of the complete intersection ring $Q/(\mathbf{f})$ in $\text{D}(Q)$. However, regardless whether $\mathbf{f}$ is assumed to be a Q -regular sequence, E is still an object of $\text{D}(Q)$ that possesses the same homological properties as a complete intersection quotient ring of Q . Second, by taking the *derived* pushout of the following diagram

$$\begin{array}{ccc} \mathbb{Z}[x_1, \dots, x_n] & \longrightarrow & Q \\ \downarrow & & \\ \mathbb{Z} & & \end{array}$$

we obtain E , where $x_i \mapsto f_i$ along the horizontal map and the vertical map is the canonical quotient. In the special case that $\mathbf{f}$ is a Q -regular sequence, we get the complete intersection $Q/(\mathbf{f})$. Either point of view suggests derived complete intersections should naturally generalize of the complete intersections rings.

3.1.3. Let $(Q, \mathfrak{n}, k)$ be a local ring. Suppose that $\mathbf{f}$ and $\mathbf{f}'$ are minimal generating set for the same ideal I . We have an isomorphism of DG Q -algebras

$$\text{Kos}^Q(\mathbf{f}) \xrightarrow{\cong} \text{Kos}^Q(\mathbf{f}').$$

3.1.4. Let

$$E = Q\langle \xi_1, \dots, \xi_n | \partial \xi_i = f_i \rangle \text{ and } E' = Q'\langle \xi'_1, \dots, \xi'_m | \partial \xi'_i = f'_i \rangle$$

be derived complete intersections. Assume further that Q and Q' are both *complete* regular local rings with $H_0(E) = H_0(E')$. There exists a derived complete intersection E'' and surjective DG algebra quasi-isomorphisms $E'' \xrightarrow{\simeq} E$ and $E'' \xrightarrow{\simeq} E'$.

Indeed, we form the following commutative diagram with surjective ring morphisms

$$\begin{array}{ccccc} P & & \xrightarrow{\pi'} & & Q' \\ & \searrow \pi & & \searrow & \downarrow \\ & Q \times_{H_0(E)} Q' & \longrightarrow & Q' & \\ & \downarrow & & \downarrow & \\ & Q & \longrightarrow & H_0(E) & \end{array}$$

where P is a regular local ring presenting the complete local ring $Q \times_{H_0(E)} Q'$. As π and π' are surjective maps between regular local rings, $\text{Ker } \pi$ and $\text{Ker } \pi'$ are generated by (linear) regular sequences; let $\mathbf{x}$ and $\mathbf{x}'$ minimally generate $\text{Ker } \pi$ and $\text{Ker } \pi'$, respectively. Finally, we let $\mathbf{g}$ and $\mathbf{g}'$ be list of elements in P such that $\mathbf{g}Q = \mathbf{f}$ and $\mathbf{g}'Q' = \mathbf{f}'$.

First, since $(\mathbf{x}, \mathbf{g}) = (\mathbf{x}', \mathbf{g}')$ and P is a local ring, we have an isomorphism of DG P -algebras

$$\text{Kos}^P(\mathbf{x}, \mathbf{g}) \xrightarrow{\cong} \text{Kos}^P(\mathbf{x}', \mathbf{g}').$$

Moreover, since $\mathbf{x}$ and $\mathbf{x}'$ are regular sequences in P , we have that

$$\mathrm{Kos}^P(\mathbf{x}, \mathbf{g}) \xrightarrow{\sim} E \text{ and } \mathrm{Kos}^P(\mathbf{x}', \mathbf{g}') \xrightarrow{\sim} E',$$

respectively. Therefore, $E'' := \mathrm{Kos}^Q(\mathbf{x}, \mathbf{g})$ is a derived complete intersection with surjective quasi-isomorphisms $E'' \xrightarrow{\sim} E$ and $E'' \xrightarrow{\sim} E'$.

3.1.5. Let $F \xrightarrow{\sim} Q/(\mathbf{f})$ be a DG Q -algebra resolution of R . We write

$$F_1 = Q\xi'_1 \oplus \dots \oplus Q\xi'_n$$

where $\partial^F(\xi'_i) = f_i$. We have the following commutative diagram of DG Q -algebras

$$\begin{array}{ccc} & & F \\ & \nearrow \varphi & \downarrow \simeq \\ E & \xrightarrow{\epsilon} & Q/(\mathbf{f}) \end{array}$$

where ϵ is the canonical augmentation map and φ is the morphism of DG Q -algebras determined by

$$\xi_i \mapsto \xi'_i.$$

In particular, F is a DG E -module via

$$\xi_i \cdot x = \xi'_i x$$

for all $x \in F$.

3.1.6. We let $E := \mathrm{Kos}^Q(\mathbf{f})$ for some list of elements $\mathbf{f} = f_1, \dots, f_n$ in Q . There is a canonical DG E -module structure on $\mathrm{Hom}_Q(E, Q)$. Moreover, we have the following isomorphism of DG E -modules

$$\mathrm{Hom}_Q(E, Q) \cong \Sigma^{-n} E.$$

3.1.7. Let $E = \mathrm{Kos}^Q(\mathbf{f})$ for some list of elements $\mathbf{f}_1, \dots, \mathbf{f}_n$ in Q . Suppose Q has finite injective dimension over itself and set

$$(-)^\dagger := \mathrm{RHom}_E(-, \mathrm{Hom}_Q(E, Q)).$$

By [28, 2.1], for each object M of $\mathbf{D}^f(E)$ the following hold:

- (1) $M^\dagger$ is an object of $\mathbf{D}^f(E)$, and
- (2) the natural map $M \rightarrow M^{\dagger\dagger}$ is an isomorphism.

By (2), it follows for each pair of objects M and N of $\mathbf{D}^f(E)$,

$$(3.1) \quad \mathrm{RHom}_E(M, N) \simeq \mathrm{RHom}_E(N^\dagger, M^\dagger).$$

Lemma 3.1.8. *We let $E := \mathrm{Kos}^Q(\mathbf{f})$ for some list of elements $\mathbf{f} = f_1, \dots, f_n$ in Q . For any object M of $\mathbf{D}(E)$,*

$$\mathrm{RHom}_E(M, E) \simeq \Sigma^n \mathrm{RHom}_Q(M, Q).$$

Proof. By 3.1.6 and adjunction, we get the first two following isomorphisms below

$$\begin{aligned} \mathrm{RHom}_E(M, E) &\simeq \mathrm{RHom}_E(M, \Sigma^n \mathrm{Hom}_Q(E, Q)) \\ &\simeq \mathrm{RHom}_Q(M \otimes_E^L E, \Sigma^n Q) \\ &\simeq \mathrm{RHom}_Q(M, \Sigma^n Q) \\ &\simeq \Sigma^n \mathrm{RHom}_Q(M, Q). \end{aligned}$$

The last two isomorphisms are standard. □

Proposition 3.1.9. *With the notation and assumptions in 3.1.7, if M is an object of $\mathcal{D}(E)$ where $H_i(M)$ is a finitely generated Q -modules for each i and $H_i(M) = 0$ for all $i \ll 0$, then*

$$M^{\dagger\dagger} \simeq M.$$

Proof. First, we state a general isomorphism in (3.2), below. Let X be a DG E -module. By 3.1.6 and Lemma 3.1.8 the first and third isomorphisms below hold, respectively,

$$\begin{aligned} X^\dagger &= \mathrm{RHom}_E(X, \mathrm{Hom}_Q(E, Q)) \\ &\simeq \mathrm{RHom}_E(X, \Sigma^{-n} E) \\ &\simeq \Sigma^{-n} \mathrm{RHom}_E(X, E) \\ &\simeq \Sigma^{-n} \Sigma^n \mathrm{RHom}_Q(X, Q) \\ &\simeq \mathrm{RHom}_Q(X, Q). \end{aligned}$$

Hence,

$$(3.2) \quad X^{\dagger\dagger} \simeq \mathrm{RHom}_Q(\mathrm{RHom}_Q(X, Q), Q).$$

Returning the setup of the proof, by assumption, there exists a semifree resolution $F \xrightarrow{\simeq} M$ such that

$$F^\natural = \prod_{j=i}^{\infty} \Sigma^j (E^{\beta_j})^\natural.$$

Thus, applying (3.2) yields

$$M^{\dagger\dagger} \simeq F^{\dagger\dagger} \simeq \mathrm{RHom}_Q(\mathrm{RHom}_Q(F, Q), Q).$$

Since Q has finite injective dimension over itself and F is bounded below complex of Q -modules that is finitely generated (over Q) in each degree, it follows that

$$\mathrm{RHom}_Q(\mathrm{RHom}_Q(F, Q), Q) \simeq F$$

(see [22, A.4.24]). Hence,

$$M^{\dagger\dagger} \simeq F \simeq M. \quad \square$$

Construction 3.1.10. Suppose $E = \mathrm{Kos}^Q(\mathcal{f})$ and set $\mathcal{S} := Q[\chi_1, \dots, \chi_n]$ to be the polynomial ring in n variables of cohomological degree two. As a graded Q -algebra, $\mathcal{S}$ has a graded basis

$$(3.3) \quad \{\chi^H := \chi_1^{h_1} \dots \chi_n^{h_n}\}_{H \in \mathbb{N}^n}.$$

We let Γ be the graded Q -linear dual of $\mathcal{S}$, i.e.,

$$\Gamma := \mathrm{Hom}_Q^*(\mathcal{S}, Q) = \bigoplus_{i \in \mathbb{N}} \mathrm{Hom}_Q(\mathcal{S}^i, Q).$$

Let $\{y^{(H)}\}_{H \in \mathbb{N}^n}$ be the graded Q -basis of Γ which is dual to (3.3). We will regard Γ as a graded $\mathcal{S}$ -module with structure determined by

$$\chi_i \cdot y^{(H)} = \begin{cases} y^{(h_1, \dots, h_{i-1}, h_i-1, h_{i+1}, \dots, h_n)} & h_i \geq 1 \\ 0 & h_i = 0. \end{cases}$$

We let $E^e := E \otimes_Q E^o$ where E^o denotes the opposite DG algebra of E and we regard E as a DG E^e -module via the multiplication map $\mu : E^e \rightarrow E$. By [4, 2.6], E admits a semiprojective resolution L over E^e defined in the following way

$$L^\natural := (E \otimes_Q \Gamma \otimes_Q E)^\natural$$

$$\partial^L = \partial^E \otimes 1 \otimes 1 + 1 \otimes \partial^E + \sum_{i=1}^n (1 \otimes \chi_i \otimes \lambda_i - \lambda_i \otimes \chi_i \otimes 1)$$

where $\lambda_i : E \rightarrow E$ is left multiplication by ξ_i , and augmentation map $\epsilon : L \rightarrow E$ is given by

$$a \otimes y^{(H)} \otimes b \mapsto \begin{cases} ab & |H| = 0 \\ 0 & |H| > 0 \end{cases}.$$

3.2. Universal Resolutions. Throughout this section, $\mathbf{f} = f_1, \dots, f_n$ is a list of elements in a commutative noetherian ring Q , $E = Q\langle \xi_1, \dots, \xi_n | \partial \xi_i = f_i \rangle$, and L is the semiprojective resolution of E over E^e defined in Construction 3.1.10.

3.2.1. Let M be a DG E -module. By 1.2.2 (or [4, 2.1] in this specific context), there exists a quasi-isomorphism of DG E -modules $F \xrightarrow{\sim} M$ where F is semiprojective as a DG Q -module when regarded as a DG Q -module via restriction of scalars along $Q \rightarrow E$. We call such a resolution a *Koszul resolution of M* . Moreover, if M is an object of $\text{Thick}_{\text{D}(Q)} Q$ when regarded as a complex of Q -modules, then there exists a Koszul resolution F where each F_i is finitely generated as a Q -module and $F_i = 0$ for all $|i| \gg 0$. Such a Koszul resolution is called *finite*.

3.2.2. Let M be a DG E -module. Fix a Koszul resolution $\delta : F \xrightarrow{\sim} M$ and define

$$U_E(F) := L \otimes_E F,$$

with augmentation map $\epsilon^M : U_E(F) \rightarrow M$ given by

$$a \otimes y^{(H)} \otimes x \mapsto \begin{cases} a\delta(x) & |H| = 0 \\ 0 & |H| > 1 \end{cases}$$

By [4, 2.4], $\epsilon^M : U_E(F) \xrightarrow{\sim} M$ is a semiprojective DG E -module resolution of M . We call $U_E(F)$ the *universal resolution of M over E (with respect to $\delta : F \rightarrow M$)* which is a DG $\mathcal{S}$ -module via the $\mathcal{S}$ -action on Γ .

3.2.3. Let M and N be DG E -modules and let U be a universal resolution of M over E associated to the Koszul resolution $F \xrightarrow{\sim} M$ (see 3.2.2). Since U is a DG $\mathcal{S}$ -module, it follows that $\text{Hom}_E(U, N)$ is a DG $\mathcal{S}$ -module. Hence,

$$\text{Ext}_E^*(M, N) = H^*(\text{Hom}_E(U, N))$$

is a graded module over $\mathcal{S}$. The $\mathcal{S}$ -module structure on $\text{Ext}_E^*(M, N)$ is independent of choice of Koszul resolution in the construction above (see [36, 3.2.2]).

Proposition 3.2.4. *Let M and N be DG E -modules. The $\mathcal{S}$ -module structure on $\text{Ext}_E^*(M, N)$ is independent of choice of Koszul resolution for M . Moreover, the $\mathcal{S}$ -module action on $\text{Ext}_E^*(M, N)$ is functorial in both M and N . Namely, we have the following isomorphisms of graded $\mathcal{S}$ -modules.*

$$\begin{aligned} \text{Ext}_E^*(M, \Sigma N) &\cong \Sigma \text{Ext}_E^*(M, N) \cong \text{Ext}_E^*(\Sigma^{-1} M, N), \\ \text{Ext}_E^*(M, N \oplus N') &\cong \text{Ext}_E^*(M, N) \oplus \text{Ext}_E^*(M, N'), \text{ and} \\ \text{Ext}_E^*(M \oplus M', N) &\cong \text{Ext}_E^*(M, N) \oplus \text{Ext}_E^*(M', N) \end{aligned}$$

and for each exact triangle $M' \rightarrow M \rightarrow M'' \rightarrow$ in $\mathcal{D}(E)$ and each object N of $\mathcal{D}(E)$ we get an exact sequences of graded $\mathcal{S}$ -modules:

$$\begin{aligned} \mathrm{Ext}_E^*(M'', N) &\rightarrow \mathrm{Ext}_E^*(M, N) \rightarrow \mathrm{Ext}_E^*(M', N) \rightarrow \Sigma \mathrm{Ext}_E^*(M'', N) \text{ and} \\ \mathrm{Ext}_E^*(N, M') &\rightarrow \mathrm{Ext}_E^*(N, M) \rightarrow \mathrm{Ext}_E^*(N, M'') \rightarrow \Sigma \mathrm{Ext}_E^*(N, M'). \end{aligned}$$

Proof. The first part of the proposition and the functoriality of the first entry of $\mathrm{Ext}_E^*(-, -)$ was shown in [36, 3.2.2]. The functoriality in the second component of $\mathrm{Ext}_E^*(-, -)$ is an immediate consequence of the $\mathcal{S}$ -module structure being determined by the first entry. $\square$

3.2.5. By [4, 2.9], we have an isomorphism of graded Q -algebras

$$(3.4) \quad \varphi : \mathrm{HH}^*(E|Q) := \mathrm{H}(\mathrm{Hom}_{E^e}(L, E)) \xrightarrow{\cong} \mathrm{H}(E)[\chi_1, \dots, \chi_n]$$

where each χ_i is a cohomologically graded variable of degree two. Using the isomorphism of complexes

$$\psi^{F,N} : \mathrm{Hom}_{E^e}(L, \mathrm{Hom}_Q(F, N)) \xrightarrow{\cong} \mathrm{Hom}_E(U, N)$$

where F is a semiprojective resolution of M over Q , it follows this $\mathcal{S}$ -module action is compatible with the action of $\mathrm{HH}^*(E|Q)$ on $\mathrm{Ext}_E^*(M, N)$. That is,

$$\mathrm{H}(\psi^{F,N})(\alpha \cdot h) = \mathrm{H}(\psi^{F,N})(\alpha) \cdot \varphi(h)$$

for every homotopy class α of a cycle from $\mathrm{Hom}_{E^e}(L, \mathrm{Hom}_Q(F, N))$ and $h \in \mathrm{HH}^*(E|Q)$.

Remark 3.2.6. A specific version of the following construction first appeared in [4, 3.2] when $\mathbf{f}$ is a Koszul-regular sequence and M and N are $Q/(\mathbf{f})$ -modules. The differential below and the coefficients of $\mathcal{S}$ have been adjusted from its form in [4, 3.2] to accommodate for N being a DG E -module instead of a $Q/(\mathbf{f})$ -module.

Construction 3.2.7. Let X and Y be DG E -modules. Define $\mathcal{C}_E(X, Y)$ to be

$$\begin{aligned} \mathcal{C}_E(X, Y)^\natural &:= \mathcal{S} \otimes_Q \mathrm{Hom}_Q(X, Y)^\natural \\ \partial^{\mathcal{C}_E(X, Y)} &:= 1 \otimes \partial^{\mathrm{Hom}_Q(X, Y)} + \sum_{i=1}^n \chi_i \otimes (\mathrm{Hom}(\lambda_i, Y) - \mathrm{Hom}(X, \lambda_i)). \end{aligned}$$

A direct calculation shows that $\mathcal{C}_E(X, Y)$ is a DG $\mathcal{S}$ -module.

Proposition 3.2.8. *Let Q be a commutative noetherian ring, $\mathbf{f} = f_1, \dots, f_n \in Q$ and E be the Koszul complex on $\mathbf{f}$. Suppose M and N are DG E -modules and let $F \xrightarrow{\cong} M$ be a Koszul resolution of M . Then we have an isomorphism of DG $\mathcal{S}$ -modules*

$$\mathrm{Hom}_E(U_E(F), N) \cong \mathcal{C}_E(F, N).$$

Proof. Consider the following isomorphisms of graded $\mathcal{S}$ -modules

$$\begin{aligned} \mathrm{Hom}_E(U_E(F), N)^\natural &\cong \mathrm{Hom}_E(E \otimes_Q \Gamma, \mathrm{Hom}_Q(F, N))^\natural \\ &\cong \mathrm{Hom}_E(E \otimes_Q \Gamma, E) \otimes_E \mathrm{Hom}_Q(F, N)^\natural \\ &\cong \mathrm{Hom}_Q(\Gamma, Q) \otimes E \otimes_E \mathrm{Hom}_Q(F, N)^\natural \\ &\cong \mathcal{S} \otimes_Q \mathrm{Hom}_Q(F, N)^\natural. \end{aligned}$$

Tracing the differential of $\mathrm{Hom}_E(U_E(F), N)$ through these isomorphisms completes the proof. $\square$

Remark 3.2.9. Let X be DG E -module and assume that N is a $Q/(\mathbf{f})$ -module such that $\partial^{\mathrm{Hom}(X,N)} = 0$. In this case, $\mathcal{C}_E(X, N)$ reduces to the following complex of graded $\mathcal{S}$ -modules,

$$\mathcal{C}_E^j(X, N) = \Sigma^{2j} \mathcal{S} \otimes_Q \mathrm{Hom}_Q(X_{-j}, N)$$

and the differential is

$$\sum_{i=1}^n \chi_i \otimes \mathrm{Hom}(\lambda_i, N).$$

When $X \xrightarrow{\sim} M$ is a Koszul resolution then by Proposition 3.2.8, the $\mathcal{S}$ -module structure of $\mathrm{Ext}_E^*(M, N)$ can be calculated as the cohomology of the $\mathcal{C}_E(X, N)$. This was first noticed in [4, 3.7].

Proposition 3.2.10. *Let E be a derived complete intersection and let M and N be objects of $\mathrm{D}^f(E)$. For any pair of Koszul resolutions $F \xrightarrow{\sim} M$ and $G \xrightarrow{\sim} N$, $\mathcal{C}_E(F, G)$ is a semiprojective DG $\mathcal{S}$ -module.*

Proof. By assumption Q is regular, and hence, $\mathcal{S}$ is regular. Also, $\mathcal{C}_E(F, G)^\natural$ is a free graded $\mathcal{S}$ -module. Thus, by Proposition 1.2.8, we conclude that $\mathcal{C}_E(F, G)$ is a semiprojective DG $\mathcal{S}$ -module, as claimed. $\square$

Remark 3.2.11. Let $E := \mathrm{Kos}^Q(f_1, \dots, f_n)$, where Q is not necessarily a regular ring, and set $\mathcal{A} := \mathcal{S} \otimes_Q k$. It follows that $\mathcal{C}_E(F, G) \otimes_Q k$ is a semiprojective DG $\mathcal{A}$ -module for DG E -modules F and G such that F and G are graded projective when regarded as DG Q -modules. Indeed, since F and G are graded projective, it follows that $\mathrm{Hom}_Q(F, G)$ is a complex of projective Q -modules. Thus, $\mathcal{C}_E(F, G) \otimes_Q k$ is a graded projective DG $\mathcal{A}$ -module. Finally, since $\mathcal{A}$ has finite global dimension, the claim follows from Proposition 1.2.8.

3.3. Finite Generation. In this section, we establish analogs to the theorems of Gulliksen and Avramov-Gasharov-Peeva (see [30, 3.1] and [7, 4.2], respectively). Theorem 3.3.2 specializes to the aforementioned results. In the following proposition, we give an elementary argument for one implication of Theorem 3.3.2 that holds provided that E is a derived complete intersection; this is essentially the same argument from [8, 4.5] transported to the derived setting.

Proposition 3.3.1. *Let E be a derived complete intersection. For each pair of objects M and N of $\mathrm{D}^f(E)$, $\mathrm{Ext}_E^*(M, N)$ is a finitely generated graded $\mathcal{S}$ -module.*

Proof. By [4, 2.1], since Q is regular there exist finite Koszul resolutions $P \xrightarrow{\sim} M$ and $P' \xrightarrow{\sim} N$. Thus, $\mathcal{C}_E(P, P')^\natural$ is a noetherian graded $\mathcal{S}$ -module. Hence, any subquotient of $\mathcal{C}_E(P, P')$ is a noetherian graded $\mathcal{S}$ -module. In particular,

$$\mathrm{H}(\mathcal{C}_E(P, P')) \cong \mathrm{Ext}_E^*(M, N)$$

is a noetherian graded $\mathcal{S}$ -module. $\square$

Theorem 3.3.2. *Let Q be a commutative noetherian ring and $\mathbf{f} = f_1, \dots, f_n$ be a sequence of elements in Q . Define E to be the Koszul complex on $\mathbf{f}$ and let $\mathcal{S} = Q[\chi_1, \dots, \chi_n]$ where each χ_i has cohomological degree two. For a pair of objects M and N of $\mathrm{D}^f(E)$, $\mathrm{Ext}_Q^*(M, N)$ is a finitely generated graded Q -module if and only if $\mathrm{Ext}_E^*(M, N)$ is a finitely generated graded $\mathcal{S}$ -module.*

Proof. Let $F \xrightarrow{\sim} M$ and $G \xrightarrow{\sim} N$ be Koszul resolutions of M and N , respectively. By Theorem 2.2.4, $\text{Ext}_E^*(M, N)$ is a finitely generated graded $\mathcal{S}$ -module if and only if $H(\mathcal{C}_E(F, G) // \chi)$ is a finitely generated graded $\mathcal{S}/(\chi) = Q$ -module where $\chi = \chi_1, \dots, \chi_n$.

Next, since F and G are free as graded Q -modules it follows that $\mathcal{C}_E(F, G)^\natural$ is a free graded $\mathcal{S}$ -module. Thus, χ is a regular sequence on $\mathcal{C}_E(F, G)$. Therefore, we have an isomorphism in $D(S)$,

$$\mathcal{C}_E(F, G) // \chi \xrightarrow{\sim} \mathcal{C}_E(F, G) / (\chi) \mathcal{C}_E(F, G)$$

(see 2.2.2). Finally, we need only observe that

$$\mathcal{C}_E(F, G) / (\chi) \mathcal{C}_E(F, G) \cong \text{Hom}_Q(F, G). \quad \square$$

3.4. Functoriality.

3.4.1. Throughout this section, Q and Q' will be commutative noetherian rings. Fix $\mathbf{f} = f_1, \dots, f_n \in Q$ and $\mathbf{f}' := f'_1, \dots, f'_m \in Q'$. Define E and E' to be the Koszul complexes on $\mathbf{f}$ and $\mathbf{f}'$, respectively. That is,

$$E := Q\langle \xi_1, \dots, \xi_n | \partial \xi_i = f_i \rangle \text{ and } E' := Q'\langle \xi'_1, \dots, \xi'_m | \partial \xi'_i = f'_i \rangle.$$

In the following discussion, we use the theory of Γ -DG algebras; see [2, Section 6] or [31, Chapter 1] for the necessary background.

Discussion 3.4.2. Let L and L' be as in Construction 3.1.10. Then L is a semifree DG Γ -extension of E^e presented by

$$L \cong E^e \langle y_1, \dots, y_n | \partial y_i = 1 \otimes \xi_i - \xi_i \otimes 1 \rangle.$$

Similarly, L' is a semifree DG Γ -extension of $(E')^e$ presented by

$$L' \cong (E')^e \langle y'_1, \dots, y'_m | \partial y'_i = 1 \otimes \xi'_i - \xi'_i \otimes 1 \rangle.$$

Suppose that $\varphi : Q \rightarrow Q'$ is a morphism of rings such that $\varphi(\mathbf{f}) \subseteq \mathbf{f}'$. For each $1 \leq i \leq n$ there exists $q'_{ij} \in Q'$ such that

$$\varphi(f_i) = \sum_{j=1}^m q'_{ij} f'_j.$$

This induces a DG algebra map $\tilde{\varphi} : E \rightarrow E'$ extending φ where

$$\tilde{\varphi}(\xi_i) = \sum_{j=1}^m q'_{ij} \xi'_j.$$

Therefore, we have a DG algebra map $\psi : E^e \rightarrow L'$ given by the composition

$$E^e \xrightarrow{\tilde{\varphi} \otimes \tilde{\varphi}} E' \otimes_Q E' \rightarrow (E')^e \hookrightarrow L'.$$

Observe that

$$\begin{aligned}
\psi(\partial(y_i)) &= \psi(1 \otimes \xi_i - \xi_i \otimes 1) \\
&= 1 \otimes \sum_{j=1}^m q'_{ij} \xi'_j - \sum_{j=1}^m q'_{ij} \xi'_j \otimes 1 \\
&= \sum_{j=1}^m q'_{ij} (1 \otimes \xi'_j - \xi'_j \otimes 1) \\
&= \sum_{j=1}^m q'_{ij} \partial(y'_j) \\
&= \partial \left(\sum_{j=1}^m q'_{ij} y'_j \right).
\end{aligned}$$

Therefore, there exists a DG Γ -algebra map $\Psi : L \rightarrow L'$ extending ψ such that

$$\Psi(y_i) = \sum_{j=1}^m q'_{ij} y'_j.$$

Moreover, Ψ is unique up to homotopy.

The composition $\chi'_i \Psi : L \rightarrow L'$ is a DG Γ -algebra map of homological degree -2 that is determined by its image on $\mathbf{y}$. Observe that

$$\chi'_i \Psi(y_h) = \chi'_i \sum_{j=1}^m q'_{hj} y'_j = q'_{hi}.$$

Also,

$$\sum_{j=1}^n q'_{ji} \Psi(\chi_j y_h) = q'_{hi}.$$

Thus,

$$\chi'_i \Psi = \sum_{j=1}^n q'_{ji} \Psi \circ \chi_j.$$

Thus, for the natural transformation

$$\text{Hom}(\Psi, -) : \text{Hom}_{(E')^e}(L', -) \rightarrow \text{Hom}_{E^e}(L, -)$$

we have that

$$(3.5) \quad \text{Hom}(\Psi, -) \circ \chi'_i = \sum_{j=1}^n \chi_j \circ \text{Hom}(\Psi, -) \circ q'_{ji}.$$

If there exists $q_{ij} \in Q$ such that $\varphi(q_{ij}) = q'_{ij}$ for each i, j then

$$(3.6) \quad \text{Hom}(\Psi, -) \circ \chi'_i = \sum_{j=1}^n q_{ij} \chi_j \circ \text{Hom}(\Psi, -).$$

Proposition 3.4.3. *With the notation of Discussion 3.4.2, let M and N be DG E -modules and M' and N' be DG E' -modules. Suppose $\alpha : M \rightarrow M'$ and $\beta : N' \rightarrow N$*

are E -linear maps. Then

$$\mathrm{Ext}_{\tilde{\varphi}}^*(\alpha, \beta) \circ \chi'_i = \sum_{j=1}^n \chi_j \circ \mathrm{Ext}_{\tilde{\varphi}}^*(\alpha, \beta) \circ q'_{ji}$$

where $\mathrm{Ext}_{\tilde{\varphi}}^*(\alpha, \beta)$ is the canonical map

$$\mathrm{Ext}_{E'}^*(M', N') \rightarrow \mathrm{Ext}_E^*(M, N).$$

Moreover, if there exists $q_{ij} \in Q$ such that $\varphi(q_{ij}) = q'_{ij}$ for each i, j then

$$\mathrm{Ext}_{\tilde{\varphi}}^*(\alpha, \beta) \circ \chi'_i = \sum_{j=1}^n q_{ij} \chi_j \circ \mathrm{Ext}_{\tilde{\varphi}}^*(\alpha, \beta).$$

Proof. Recall that

$$\mathrm{Ext}_{E^e}^*(E, \mathrm{RHom}_Q(M, N)) \cong \mathrm{Ext}_E^*(M, N).$$

This combined with (3.5) and (3.6) yield the desired results. $\square$

4. COHOMOLOGICAL SUPPORT

Notation. Throughout the entirety of Section 4, we fix the following notation. Let Q be a commutative noetherian ring, $\mathbf{f} = f_1, \dots, f_n$ a list of elements in Q and set $E := \mathrm{Kos}^Q(\mathbf{f})$. We let $\mathcal{S} := Q[\chi_1, \dots, \chi_n]$ be the ring of cohomology operators where each χ_i has cohomological degree two. Finally, let $\mathbb{P}_Q^{n-1}$ denote $\mathrm{Proj} \mathcal{S}$ equipped with the Zariski topology.

4.1. Cohomological Support for Pairs of DG E -modules. This section introduces the theory of support for pairs of DG modules over the Koszul complex E . Recall that for each pair of DG E -modules M and N , $\mathrm{Ext}_E^*(M, N)$ naturally inherits the structure of a graded $\mathcal{S}$ -module (see Section 3). That is, $\mathrm{Ext}_E^*(M, N)$ is a DG $\mathcal{S}$ -module with trivial differential. We will be importing the support theory from Section 2, below.

Definition 4.1.1. Let M and N be DG E -modules. We define the *cohomological support of M and N over E* to be

$$\mathbf{V}_E(M, N) := \mathrm{Supp}_{\mathcal{S}}^+(\mathrm{Ext}_E^*(M, N)) \subseteq \mathbb{P}_Q^{n-1}.$$

The *cohomological support of M over E* is defined to be $\mathbf{V}_E(M) := \mathbf{V}_E(M, M)$.

Proposition 4.1.2. Let M and N be DG E -modules.

- (1) $\mathbf{V}_E(\Sigma^i M, N) = \mathbf{V}_E(M, N) = \mathbf{V}_E(M, \Sigma^i N)$ for each $i \in \mathbb{Z}$.
- (2) If $M = M' \oplus M''$ in $\mathrm{D}(E)$ then
 - (a) $\mathbf{V}_E(M, N) = \mathbf{V}_E(M', N) \cup \mathbf{V}_E(M'', N)$.
 - (b) $\mathbf{V}_E(N, M) = \mathbf{V}_E(N, M') \cup \mathbf{V}_E(N, M'')$.
- (3) For each exact triangle $M^1 \rightarrow M^2 \rightarrow M^3 \rightarrow$ in $\mathrm{D}^f(E)$, we have
 - (a) $\mathbf{V}_E(M^h, N) \subseteq \mathbf{V}_E(M^i, N) \cup \mathbf{V}_E(M^j, N)$ for $\{h, i, j\} = \{1, 2, 3\}$.
 - (b) $\mathbf{V}_E(N, M^h) \subseteq \mathbf{V}_E(N, M^i) \cup \mathbf{V}_E(N, M^j)$ for $\{h, i, j\} = \{1, 2, 3\}$.
- (4) If M is an object of $\mathrm{Thick}_{\mathrm{D}(E)} M'$ then

$$\mathbf{V}_E(M, N) \subseteq \mathbf{V}_E(M', N) \text{ and } \mathbf{V}_E(N, M) \subseteq \mathbf{V}_E(N, M').$$

- (5) $\mathbf{V}_E(M, N) \subseteq \mathbf{V}_E(M, M) \cap \mathbf{V}_E(N, N)$.
- (6) If $\mathrm{Ext}_E^*(M, N)$ is bounded, then $\mathbf{V}_E(M, N) = \emptyset$.

Proof. Parts (1), (2) and (3) are easy consequences of Proposition 2.1.7 and 3.2.4. We leave these as exercises to the reader.

For (4), the full subcategory T of $\mathsf{D}(E)$ consisting of all objects X such that $\mathsf{V}_E(X, N) \subseteq \mathsf{V}_E(M', N)$ is a thick subcategory of $\mathsf{D}(E)$ since (1), (2) and (3) hold. Thus, since M' is an object of T and M is an object of $\mathsf{Thick}_{\mathsf{D}(E)} M'$ we have that $M \in \mathsf{T}$. Therefore, this proves the first containment in (4), and the second containment is similar.

The $\mathcal{S}$ -action on $\mathrm{Ext}_E^*(M, N)$ is central by 3.2.5 in the sense that it is compatible with the $\mathrm{Ext}_E^*(N, N)$ - $\mathrm{Ext}_E^*(M, M)$ bimodule action. Now (5) is an immediate consequence.

Finally, for (6), the first part is obvious as a high enough power of $\mathcal{S}^{>0}$ annihilates $\mathrm{Ext}_E^*(M, N)$. $\square$

Unsurprisingly, when $\mathrm{Ext}_E^*(M, N)$ is *finitely generated* this support theory satisfies the following important properties that will be used heavily in the sequel.

Proposition 4.1.3. *Assume that M and N are objects of $\mathsf{D}^f(E)$ and $\mathrm{Ext}_Q^{\geq 0}(M, N) = 0$.*

- (1) $\mathsf{V}_E(M, N)$ is a Zariski closed subset of $\mathbb{P}_Q^{n-1}$.
- (2) $\mathsf{V}_E(M, N) = \mathrm{Supp}_{\mathcal{S}}^+(\mathcal{C}_E(F, N))$ for any Koszul resolution $F \xrightarrow{\sim} M$.
- (3) $\mathsf{V}_E(M, N) = \emptyset$ if and only if $\mathrm{Ext}_E^{\geq 0}(M, N) = 0$.

Proof. The assumption that $\mathrm{Ext}_Q^{\geq 0}(M, N) = 0$ is equivalent to $\mathrm{Ext}_E^*(M, N)$ being finitely generated over $\mathcal{S}$ -module using Theorem 3.3.2. Hence, $\mathsf{V}_E(M, N)$ is a Zariski closed subset of $\mathbb{P}_Q^{n-1}$, finishing the proof of (1). Moreover, for any Koszul resolution $F \xrightarrow{\sim} M$, $\mathcal{C}_E(F, N)$ is an object of $\mathsf{D}^f(\mathcal{S})$ where

$$\mathrm{H}(\mathcal{C}_E(F, N)) = \mathrm{Ext}_E^*(M, N).$$

Thus, by Proposition 2.1.7(3)

$$\mathrm{Supp}_{\mathcal{S}}^+(\mathcal{C}_E(F, N)) = \mathrm{Supp}_{\mathcal{S}}^+(\mathrm{Ext}_E^*(M, N)) = \mathsf{V}_E(M, N)$$

and so (2) holds.

Finally, for (3), we note the backwards implication is Proposition 4.1.2(6). For the converse, since $\mathrm{Ext}_E^*(M, N)$ is a finitely generated graded $\mathcal{S}$ -module then applying Proposition 2.1.7(3) yields the desired result. $\square$

Example 4.1.4.

- (1) Let M be an object of $\mathsf{Thick}_{\mathsf{D}(E)} E$. For each object N of $\mathsf{D}^f(E)$,

$$\mathsf{V}_E(M, N) = \emptyset.$$

Indeed, $\mathrm{Ext}_E^*(E, N) = \mathrm{H}(N)$ and since N is an object of $\mathsf{D}^f(E)$, it follows that $\mathrm{Supp}_{\mathcal{S}}^+(\mathrm{Ext}_E^*(E, N)) = \emptyset$. That is, $\mathsf{V}_E(E, N) = \emptyset$ and so applying Proposition 4.1.2(4) yields the desired result.

- (2) Let E be a minimal derived complete intersection (see 3.1.1) with residue field k . By [36, 3.2.6], there is an isomorphism of graded $\mathcal{S}$ -modules

$$\mathrm{Ext}_E^*(k, k) \cong \mathcal{S} \otimes_Q \bigwedge (\Sigma^{-1} k^{\dim Q}).$$

In particular, if $\iota : \mathbb{P}_k^{n-1} \hookrightarrow \mathbb{P}_Q^{n-1}$ is the canonical map induced by the canonical quotient map of $\mathcal{S} \rightarrow k[\chi_1, \dots, \chi_n]$, then

$$\mathsf{V}_E(k, k) = \iota(\mathbb{P}_k^{n-1}).$$

That is, $\mathbf{V}_E(k, k) = \text{Supp}_{\mathcal{S}}^+(\mathcal{S} \otimes_Q k)$.

Proposition 4.1.5. *With the notation and assumptions from 3.1.7, we let M be an object of $\mathbf{D}^f(E)$.*

- (1) $\mathbf{V}_E(M, N) = \emptyset$ for each object N of $\text{Thick}_{\mathbf{D}(E)} E$.
- (2) $\mathbf{V}_E(M, N) = \mathbf{V}_E(N^\dagger, M^\dagger)$ for each object N of $\mathbf{D}^f(E)$.
- (3) $\mathbf{V}_E(M) = \mathbf{V}_E(M^\dagger)$.

Proof. For (1), we note that $\text{Ext}_E^*(M, E)$ is bounded by 3.1.7(1). Hence, by Proposition 4.1.2(6), $\mathbf{V}_E(M, E) = \emptyset$. Now applying Proposition 4.1.2(4) yields $\mathbf{V}_E(M, N) = \emptyset$, as claimed. Next, (3.1) implies that

$$\text{Ext}_E^*(M, N) \cong \text{Ext}_E^*(N^\dagger, M^\dagger)$$

from which (2) follows from. Finally, (3) is obtained from (2) by letting $N = M$. $\square$

4.2. Local Base Ring. Throughout this section we add the assumption that Q is local with maximal ideal $\mathfrak{n}$ and residue field k . One of the notable features of these cohomological supports is that when Q is local they detect whether a DG E -module is an object of $\text{Thick}_{\mathbf{D}(E)} E$.

Proposition 4.2.1. *Assume M is an object of $\mathbf{D}^f(E)$ such that $\text{Ext}_Q^{\geq 0}(M, k) = 0$. Then $\mathbf{V}_E(M, k) = \emptyset$ if and only if M is an object of $\text{Thick}_{\mathbf{D}(E)} E$.*

Proof. By assumption and Proposition 4.1.3(3) we conclude that $\mathbf{V}_E(M, k) = \emptyset$ if and only if $\text{Ext}_E^{\geq 0}(M, k) = 0$. Since M is an object of $\mathbf{D}^f(E)$ with $\text{Ext}_E^{\geq 0}(M, k) = 0$ then by 1.3.5, it follows that $\text{Ext}_E^{\geq 0}(M, k) = 0$ if and only if M is an object of $\text{Thick}_{\mathbf{D}(E)} E$. $\square$

Theorem 4.2.2. *Set $R := Q/(\mathbf{f})$ and suppose that $\text{pd}_Q R < \infty$. The following are equivalent:*

- (1) $\mathbf{f}$ is a Q -regular sequence.
- (2) $\mathbf{V}_E(R, k) = \emptyset$
- (3) $\mathbf{V}_E(M, k) = \emptyset$ for some nonzero finitely generated R -module M such that $\text{pd}_Q M < \infty$.

Remark 4.2.3. The equivalence of (1) and (2) was first established in [36, 3.4(3)]. As noted in [36, 3.4(3)], “(1) implies (2)” is clear from 1.3.3 since the augmentation map $E \rightarrow R$ is a quasi-isomorphism. Moreover, there is nothing to show for “(2) implies (3)” and so that leaves only the following implication to prove.

Proof of (3) $\implies$ (1). Assume $\mathbf{V}_E(M, k) = \emptyset$ where M is as in (3). By Proposition 4.1.3(3), $\text{Ext}_E^{\geq 0}(M, k) = 0$ and so by 1.3.5, M is in $\text{Thick}_{\mathbf{D}(E)} E$. Thus, 1.3.6 yields

$$\text{amp } E \leq \text{amp } M = 0.$$

Therefore, $E \xrightarrow{\sim} R$ and hence, $\mathbf{f}$ is a Q -regular sequence. $\square$

For the rest of the section we set

$$\mathcal{A} := \mathcal{S} \otimes_Q k = k[\chi_1, \dots, \chi_n].$$

Definition 4.2.4. Let M and N be objects of $\mathbf{D}(E)$. We define

$$\mathcal{V}_E(M, N) := \text{Supp}_{\mathcal{A}}^+(\text{Ext}_E^*(M, N) \otimes_Q k).$$

Set

$$\mathcal{V}_E(M) := \mathcal{V}_E(M, M).$$

4.2.5. Notice that $\mathcal{V}_E(M, N)$ is the closed fiber of $\mathbf{V}_E(M, N)$, that is,

$$\mathcal{V}_E(M, N) = \mathbf{V}_E(M, N) \times_{\mathrm{Spec} Q} \mathrm{Spec} k \subseteq \mathbb{P}_k^{n-1}.$$

Thus, the formulas in Sections 4.1, 4.2 and 4.3 all hold for $\mathcal{V}_E(-, -)$ in place of $\mathbf{V}_E(-, -)$. To save space, we do not list the corresponding formulas for $\mathcal{V}_E(-, -)$.

Remark 4.2.6. In [36], the present author defined the supports $\mathcal{V}_E(M, k)$ and provided an application for them. As briefly referenced in the introduction, this theory of supports is to be used, in conjunction with Theorem 4.2.2, to establish the following derived category characterization of a complete intersection ring: *a commutative noetherian ring R is locally a complete intersection if and only if every object of $\mathbf{D}^f(R)$ is virtually small*. The main ingredients are to show there exist objects $C(1), \dots, C(n)$ in $\mathrm{Thick}_{\mathbf{D}(R)} k$ such that

$$\mathcal{V}_E(C(1)) \cap \dots \cap \mathcal{V}_E(C(n)) = \emptyset$$

(cf. [36, 3.3.4]), and if M is virtually small then $\mathcal{V}_E(R) \subseteq \mathbf{V}_E(M)$. We also point out a similar application has recently appeared in a collaboration of the present author with Briggs and Grifo in [16].

Lemma 4.2.7. *Let E be a derived complete intersection and let M and N be a pair of objects in $\mathbf{D}^f(E)$. For Koszul resolutions $F \xrightarrow{\sim} M$ and $G \xrightarrow{\sim} N$,*

$$\mathcal{V}_E(M, N) = \mathrm{Supp}_{\mathcal{A}}^+(\mathcal{C}_E(F, G) \otimes_Q k).$$

Proof. Let $\iota : \mathbb{P}_k^{n-1} \hookrightarrow \mathbb{P}_Q^{n-1}$ be the canonical embedding induced by the projection $\mathcal{S} \rightarrow \mathcal{A}$. By definition,

$$\iota(\mathcal{V}_E(M, N)) = \mathrm{Supp}_{\mathcal{S}}^+(\mathrm{Ext}_E^*(M, N) \otimes_Q k).$$

The result follows from the equalities

$$\begin{aligned} \mathrm{Supp}_{\mathcal{S}}^+(\mathrm{Ext}_E^*(M, N) \otimes_Q k) &= \mathrm{Supp}_{\mathcal{S}}^+(\mathrm{Ext}_E^*(M, N) \otimes_{\mathcal{S}} \mathcal{A}) \\ &= \mathbf{V}_E(M, N) \cap \mathrm{Supp}_{\mathcal{S}}^+ \mathcal{A} \\ &= \mathrm{Supp}_{\mathcal{S}}^+ \mathcal{C}_E(F, G) \cap \mathrm{Supp}_{\mathcal{S}}^+ \mathcal{A} \\ &= \mathrm{Supp}_{\mathcal{S}}^+(\mathcal{C}_E(F, G) \otimes_{\mathcal{S}}^{\mathbf{L}} \mathcal{A}) \\ &= \mathrm{Supp}_{\mathcal{S}}^+(\mathcal{C}_E(F, G) \otimes_Q k) \end{aligned}$$

where the second equality is Proposition 2.1.7(4), the third equality is by Proposition 4.1.3(2), the fourth equality is 2.1.5 and the last equality follows from Proposition 3.2.10. $\square$

4.2.8. Let M and N be a pair of objects in $\mathbf{D}^f(E)$. We set $\mathcal{E} := \mathrm{Ext}_E^*(M, N) \otimes_Q k$. The *complexity of M and N over E* , denoted $\mathrm{cx}_E(M, N)$, measures the polynomial growth of the sequence $\{\dim_k \mathcal{E}^i\}_{i \in \mathbb{N}}$. More precisely,

$$\mathrm{cx}_E(M, N) := \inf\{b \in \mathbb{N} : \dim_k \mathcal{E}^i \leq ai^{b-1} \text{ for some } a > 0 \text{ and all } i \gg 0\},$$

and we define $\mathrm{cx}_E M := \mathrm{cx}_E(M, k)$.

4.2.9. Let M and N be objects of $\mathbf{D}^f(E)$ such that $\mathrm{Ext}_Q^{\gg 0}(M, N) = 0$. By assumption $\mathcal{E} := \mathrm{Ext}_E^*(M, N) \otimes_Q k$ is a finitely generated graded $\mathcal{A}$ -module. Thus, by the Hilbert-Serre theorem, $\mathrm{cx}_E(M, N)$ is exactly one more than the dimension of $\mathcal{V}_E(M, N)$ viewed as a Zariski closed subset of $\mathbb{P}_k^{n-1}$. Thus, $\mathrm{cx}_E(M, N) \leq n$.

Example 4.2.10.

- (1) Let E be a minimal derived complete intersection. By Example 4.1.4(2) and 4.2.9, it follows that $\text{cx}_E k = n$. This is analogous to, and recovers, the statement for ordinary complete intersection rings; namely, the complexity of the residue field is the codimension of the complete intersection.
- (2) Theorem 4.2.2 can be translated into the following numerical measure of a complete intersection, $Q/(\mathbf{f})$ is a complete intersection if and only if $\text{cx}_E Q/(\mathbf{f}) = 0$.

The same proof as in [11], using Koszul objects, can be adapted to this setting to establish the following result. We leave this to the reader.

Proposition 4.2.11. *Let E be a minimal derived complete intersection. For each closed subset C of $\mathbb{P}_k^{n-1}$, there exists an object M of $\mathbf{D}^f(E)$ such that $\mathcal{V}_E(M) = C$.*

4.3. Symmetry of Cohomological Support, Ext, Tor and Complexity. We now prove some of the major results of the article. The first relates the closed subsets $\mathcal{V}_E(M)$, $\mathcal{V}_E(N)$ and $\mathcal{V}_E(M, N)$ of $\mathbb{P}_Q^{n-1}$ when E is a derived complete intersection and contains Theorem A from the introduction. See the discussion in the introduction, and Remark 4.3.2, for a comparison of the proofs of Theorem 4.3.1 and [5, Theorem I] of Avramov and Buchweitz.

Theorem 4.3.1. *Let E be a derived complete intersection and assume that M , M' , N and N' are objects of $\mathbf{D}^f(E)$.*

- (1) $\mathcal{V}_E(M, N) \cap \mathcal{V}_E(M', N') = \mathcal{V}_E(M, N') \cap \mathcal{V}_E(M', N)$.
- (2) $\mathcal{V}_E(M, N) = \mathcal{V}_E(M) \cap \mathcal{V}_E(N) = \mathcal{V}_E(N, M)$.
- (3) $\text{Ext}_E^{\geq 0}(M, N) = 0$ if and only if $\mathcal{V}_E(M) \cap \mathcal{V}_E(N) = \emptyset$.

Furthermore, assume that $(Q, \mathfrak{n}, k)$ is local.

- (4) $\mathcal{V}_E(M) = \mathcal{V}_E(M, k) = \mathcal{V}_E(k, M)$.
- (5) The following inequalities hold

$$\text{cx}_E M + \text{cx}_E N - n \leq \text{cx}_E(M, N) = \text{cx}_E(N, M) \leq \max\{\text{cx}_E M, \text{cx}_E N\}.$$

Proof. First, we claim that (2)-(5) all follow from (1). Indeed,

$$(4.1) \quad \mathcal{V}_E(M) \cap \mathcal{V}_E(N) = \mathcal{V}_E(M, N) \cap \mathcal{V}_E(N, M)$$

by (1). Combining (4.1) with Proposition 4.1.2(5) provides us with

$$\mathcal{V}_E(M, N) \subseteq \mathcal{V}_E(M) \cap \mathcal{V}_E(N) = \mathcal{V}_E(M, N) \cap \mathcal{V}_E(N, M)$$

and so $\mathcal{V}_E(M, N) \subseteq \mathcal{V}_E(N, M)$. By symmetry, we obtain $\mathcal{V}_E(N, M) \subseteq \mathcal{V}_E(M, N)$. Thus, $\mathcal{V}_E(M, N) = \mathcal{V}_E(N, M)$ and now (4.1) finishes the proof that (2) is a consequence of (1). Also, (3) follows directly from (2) and Proposition 4.1.3(3).

In the local case, using (2) and 4.2.5 the third and fourth equalities below hold

$$\begin{aligned} \mathcal{V}_E(M) &= \mathcal{V}_E(M) \cap \mathbb{P}_k^{n-1} \\ &= \mathcal{V}_E(M) \cap \mathcal{V}_E(k) \\ &= \mathcal{V}_E(M, k) \\ &= \mathcal{V}_E(k, M). \end{aligned}$$

Hence, (4) holds. Finally, (2) and (4) imply that

$$\mathcal{V}_E(M, N) = \mathcal{V}_E(M, k) \cap \mathcal{V}_E(N, k),$$

and combining this with 4.2.9 shows (5) holds.¹ Hence, as claimed, it suffices to show (1) holds.

Now we begin the proof of (1). As E is a derived complete intersection, there exists finite Koszul resolutions $F \xrightarrow{\sim} M$, $F' \xrightarrow{\sim} M'$, $G \xrightarrow{\sim} N$ and $G' \xrightarrow{\sim} N'$ (c.f. 3.2.1). First, we claim that there is an isomorphism of DG $\mathcal{S}$ -modules

$$(4.2) \quad \mathcal{C}_E(F, G) \otimes_{\mathcal{S}} \mathcal{C}_E(F', G') \cong \mathcal{C}_E(F, G') \otimes_{\mathcal{S}} \mathcal{C}_E(F', G).$$

Indeed, there is an isomorphism of graded $\mathcal{S}$ -modules

$$(\mathcal{C}_E(F, G) \otimes_{\mathcal{S}} \mathcal{C}_E(F', G'))^{\natural} \cong \mathcal{S} \otimes_Q (\mathrm{Hom}_Q(F, G) \otimes_Q \mathrm{Hom}_Q(F', G'))^{\natural}.$$

We let $T(F, G; F', G')$ be the DG $\mathcal{S}$ -module with

$$T(F, G; F', G')^{\natural} \cong \mathcal{S} \otimes_Q (\mathrm{Hom}_Q(F, G) \otimes_Q \mathrm{Hom}_Q(F', G'))^{\natural}$$

and

$$\partial^T := 1 \otimes \partial^{\mathrm{Hom}(F, G) \otimes \mathrm{Hom}(F', G')} + \delta(F, G) + \delta(F', G')$$

where

$$\begin{aligned} \delta(F, G) &:= \sum_{i=1}^n \chi_i \otimes (\mathrm{Hom}(\lambda_i, 1) \otimes 1 - \mathrm{Hom}(1, \lambda_i) \otimes 1) \\ \delta(F', G') &:= \sum_{i=1}^n \chi_i \otimes 1 \otimes (\mathrm{Hom}(\lambda_i, 1) - \mathrm{Hom}(1, \lambda_i)). \end{aligned}$$

Also, we have the following isomorphism of complexes of Q -modules

$$\varphi : \mathrm{Hom}_Q(F, G) \otimes_Q \mathrm{Hom}_Q(F', G') \xrightarrow{\cong} \mathrm{Hom}_Q(F, G') \otimes_Q \mathrm{Hom}_Q(F', G).$$

Moreover, this induces a DG $\mathcal{S}$ -module isomorphism Ψ that fits into the following diagram

$$\begin{array}{ccc} \mathcal{C}_E(F, G) \otimes_{\mathcal{S}} \mathcal{C}_E(F', G') & \xrightarrow{\Psi} & \mathcal{C}_E(F, G') \otimes_{\mathcal{S}} \mathcal{C}_E(F', G) \\ \downarrow \cong & & \uparrow \cong \\ T(F, G; F', G') & \xrightarrow[\cong]{1 \otimes \varphi} & T(F, G'; F, G') \end{array}$$

Next, by Proposition 4.1.3(2),

$$\mathbf{V}_E(M, N) = \mathbf{V}_E(M, G) = \mathrm{Supp}_{\mathcal{S}}^+ \mathcal{C}_E(F, G).$$

Similarly,

$$\begin{aligned} \mathbf{V}_E(M, N') &= \mathrm{Supp}_{\mathcal{S}}^+ \mathcal{C}_E(F, G') \\ \mathbf{V}_E(M', N) &= \mathrm{Supp}_{\mathcal{S}}^+ \mathcal{C}_E(F', G) \\ \mathbf{V}_E(M', N') &= \mathrm{Supp}_{\mathcal{S}}^+ \mathcal{C}_E(F', G') \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbf{V}_E(M, N) \cap \mathbf{V}_E(M', N') &= \mathrm{Supp}_{\mathcal{S}}^+ \mathcal{C}_E(F, G) \cap \mathrm{Supp}_{\mathcal{S}}^+ \mathcal{C}_E(F', G') \\ &= \mathrm{Supp}_{\mathcal{S}}^+ (\mathcal{C}_E(F, G) \otimes_{\mathcal{S}}^{\mathbf{L}} \mathcal{C}_E(F', G')) \\ &= \mathrm{Supp}_{\mathcal{S}}^+ (\mathcal{C}_E(F, G) \otimes_{\mathcal{S}} \mathcal{C}_E(F', G')) \end{aligned}$$

¹This last deduction is essentially the same argument from [5, 5.7] for complexity over a complete intersection ring.

where the second equality uses 2.1.5 and the third equality uses Proposition 3.2.10. Similarly,

$$\mathbf{V}_E(M, N') \cap \mathbf{V}_E(M', N) = \text{Supp}_S^+(\mathcal{C}_E(F, G') \otimes_S \mathcal{C}_E(F', G))$$

Finally, by (4.2), we get the desired result. $\square$

Remark 4.3.2. The fact that the eventual vanishing of Ext-modules over a complete intersection is symmetric in the module arguments of Ext has already been established in each of the following [5, 10, 20, 32]. Theorem 4.3.1 adds a new proof to the collection that incorporates ideas from [5] and [20]. Moreover, Theorem 4.3.1 recovers the stronger result, of Avramov and Buchweitz [5], that complexity is in fact symmetric in both module arguments over a complete intersection. We end this section with one last result that is directly inspired by [5, Theorem III]. The argument for “(1) implies (3)” in Theorem 4.3.3 differs from the one in [5, Theorem III] as we do not resort to a theory of complete resolutions, Tate cohomology, and bands of vanishing over a complete intersection.

Theorem 4.3.3. *Let E be a derived complete intersection. For a pair of objects M and N of $\mathbf{D}^f(E)$, the following are equivalent:*

- (1) $\text{Ext}_E^{\geq 0}(M, N) = 0$.
- (2) $\text{Ext}_E^{\geq 0}(N, M) = 0$.
- (3) $\text{Tor}_{\geq 0}^E(M, N) = 0$.

Proof. The equivalence of (1) and (2) is a consequence of Theorem 4.3.1(3). We let $(-)^{\dagger} := \text{RHom}_E(-, \text{Hom}_Q(E, Q))$.

(3) $\implies$ (1): By assumption $M \otimes_E^{\mathbf{L}} N$ is an object of $\mathbf{D}^f(E)$. Since Q is Gorenstein, then 3.1.7 implies that $(M \otimes_E^{\mathbf{L}} N)^{\dagger}$ is an object of $\mathbf{D}^f(E)$. Using adjunction,

$$(M \otimes_E^{\mathbf{L}} N)^{\dagger} \simeq \text{RHom}_E(M, N^{\dagger}).$$

Therefore, $\text{Ext}_E^{\geq 0}(M, N^{\dagger}) = 0$ and so

$$\mathbf{V}_E(M) \cap \mathbf{V}_E(N^{\dagger}) = \emptyset.$$

Thus, Proposition 4.1.5(2) yields

$$\mathbf{V}_E(M) \cap \mathbf{V}_E(N) = \emptyset.$$

That is, $\text{Ext}_E^{\geq 0}(M, N) = 0$ (see Theorem 4.3.1(3)).

(1) $\implies$ (3): Assume that $\text{Ext}_E^{\geq 0}(M, N) = 0$. By Theorem 4.3.1(3) and Proposition 4.1.5(2),

$$\mathbf{V}_E(M) \cap \mathbf{V}_E(N^{\dagger}) = \emptyset.$$

That is, $\text{RHom}_E(M, N^{\dagger})$ is an object of $\mathbf{D}^f(E)$. Using adjunction, we obtain an isomorphism

$$(M \otimes_E^{\mathbf{L}} N)^{\dagger} \simeq \text{RHom}_E(M, N^{\dagger})$$

and so $(M \otimes_E^{\mathbf{L}} N)^{\dagger}$ is an object of $\mathbf{D}^f(E)$. Since $(M \otimes_E^{\mathbf{L}} N)^{\dagger}$ is an object of $\mathbf{D}^f(E)$ and again using that Q is Gorenstein, it follows that $(M \otimes_E^{\mathbf{L}} N)^{\dagger\dagger}$ is an object of $\mathbf{D}^f(E)$ (by 3.1.7). Finally, applying Proposition 3.1.9

$$M \otimes_E^{\mathbf{L}} N \simeq (M \otimes_E^{\mathbf{L}} N)^{\dagger\dagger},$$

and hence $M \otimes_E^{\mathbf{L}} N$ is an object of $\mathbf{D}^f(E)$. Therefore, $\text{Tor}_E^{\geq 0}(M, N) = 0$ as needed. $\square$

5. COHOMOLOGICAL SUPPORT FOR LOCAL RINGS

5.1. Definition and Properties.

5.1.1. Let $(R, \mathfrak{m}, k)$ be a commutative noetherian local ring. We let $\widehat{R}$ be the $\mathfrak{m}$ -adic completion of R . We say that $E = \text{Kos}^Q(\mathbf{f})$ is a *(minimal) derived complete intersection approximation*, or *(minimal) DCI-approximation*, provided that E is a (minimal) derived complete intersection with Q complete and $H_0(E) = \widehat{R}$. By the Cohen Structure Theorem, DCI-approximations of R exist and are unique up to quasi-isomorphism (see 3.1.4). Moreover, if $E = \text{Kos}^Q(f_1, \dots, f_n)$ is a minimal DCI-approximation of R then n is independent of choice of Q and $\mathbf{f}$; namely, $n = \dim_k H_1(K^R)$ where K^R is the Koszul complex on a minimal generating set for $\mathfrak{m}$. We call n the *derived codimension* of R .

Theorem 5.1.2. *Let $(R, \mathfrak{m}, k)$ be a local ring. If $E = \text{Kos}^Q(f_1, \dots, f_n)$ and $E' = \text{Kos}^{Q'}(f'_1, \dots, f'_m)$ are DCI approximations of R , then there exists a diagram of schemes*

$$\mathbb{P}_Q^{n-1} \xrightarrow{\alpha} \mathbb{P}_P^N \xrightarrow{\beta} \mathbb{P}_P^N \xleftarrow{\alpha'} \mathbb{P}_{Q'}^{m-1}$$

where P is a regular local ring, α and α' are embeddings and β is a linear automorphism such that

$$\beta(\alpha(\mathbb{V}_E(M, N))) = \alpha'(\mathbb{V}_{E'}(M, N)) \text{ in } \mathbb{P}_P^N$$

for any pair of objects M and N in $\mathcal{D}^f(\widehat{R})$.

Proof. We write

$$\begin{aligned} E &= Q\langle \xi_1, \dots, \xi_n | \partial \xi_i = f_i \rangle, \\ E' &= Q'\langle \xi'_1, \dots, \xi'_m | \partial \xi'_i = f'_i \rangle \end{aligned}$$

then by 3.1.4, there exists surjections of regular local rings $\pi : P \rightarrow Q$ and $\pi' : P' \rightarrow Q'$. Let

$$A := P\langle \xi_1, \dots, \xi_n, \tau_{n+1}, \dots, \tau_{N+1} | \partial \xi_i = g_i \text{ and } \partial \tau_{n+i} = x_i \rangle$$

where $\pi(g_i) = f_i$ and $\mathbf{x}$ minimally generates $\ker \pi$. Similarly, we let

$$A' := P'\langle \xi'_1, \dots, \xi'_m, \tau'_{m+1}, \dots, \tau'_{N+1} | \partial \xi'_i = g'_i \text{ and } \partial \tau'_{m+i} = x'_i \rangle$$

where $\pi'(g'_i) = f'_i$ and $\mathbf{x}'$ minimally generates $\ker \pi'$.

First, π induces a canonical surjection

$$P[\chi_1, \dots, \chi_n, \zeta_{n+1}, \dots, \zeta_{N+1}] \rightarrow Q[\chi_1, \dots, \chi_n],$$

which defines an embedding $\alpha : \mathbb{P}_Q^{n-1} \rightarrow \mathbb{P}_P^N$. Let $\epsilon : A \xrightarrow{\sim} E$ be the surjective quasi-isomorphism induced by π mapping $\xi_i \mapsto \xi_i$ and $\tau_i \mapsto 0$. Applying Proposition 3.4.3 to ϵ yields that

$$\mathbb{V}_A(M, N) = \alpha(\mathbb{V}_E(M, N)).$$

Similarly, π' induces a canonical surjection

$$P'[\chi'_1, \dots, \chi'_m, \zeta'_{m+1}, \dots, \zeta'_{N+1}] \rightarrow Q'[\chi'_1, \dots, \chi'_m],$$

which defines an embedding $\alpha' : \mathbb{P}_{Q'}^{m-1} \rightarrow \mathbb{P}_P^N$ such that

$$\mathbb{V}_{A'}(M, N) = \alpha'(\mathbb{V}_{E'}(M, N)).$$

Finally, since P is local and $\mathbf{x}, \mathbf{g}$ and $\mathbf{x}', \mathbf{g}'$ minimally generate the same ideal in P there is an isomorphism of DG P -algebras $\varphi : A \xrightarrow{\cong} A'$. By Proposition 3.4.3, this determines a linear automorphism $\beta : \mathbb{P}_P^N \rightarrow \mathbb{P}_{P'}^N$ such that

$$\beta(\mathbf{V}_A(M, N)) = \mathbf{V}_{A'}(M, N). \quad \square$$

5.1.3. Let $(R, \mathfrak{m})$ be a commutative noetherian local ring and let $\widehat{R}$ denote its $\mathfrak{m}$ -adic completion. Let $E := \text{Kos}^Q(f_1, \dots, f_n)$ be a minimal DCI-approximation of R . For each pair of objects M and N of $\mathbf{D}^f(R)$, $\mathbf{V}_E(M \otimes_R \widehat{R}, N \otimes_R \widehat{R})$ is a closed subset of $\mathbb{P}_Q^{n-1}$. For $E' := \text{Kos}^{Q'}(f'_1, \dots, f'_n)$ another minimal DCI-approximation of R , then $\mathbf{V}_{E'}(M \otimes_R \widehat{R}, N \otimes_R \widehat{R})$ is a closed subset of $\mathbb{P}_{Q'}^{n-1}$. By Theorem 5.1.2, $\mathbf{V}_E(M \otimes_R \widehat{R}, N \otimes_R \widehat{R})$ and $\mathbf{V}_{E'}(M \otimes_R \widehat{R}, N \otimes_R \widehat{R})$ determine the same closed subset of $\mathbb{P}_P^N$ where P is a regular local ring and the following is a commutative diagram of surjective ring morphisms

$$\begin{array}{ccc} & P & \\ \swarrow & & \searrow \\ Q & & Q' \\ \searrow & & \swarrow \\ & \widehat{R} & \end{array}$$

Remark 5.1.4. Since $\mathbf{V}_R(-, -)$ is defined in terms of $\mathbf{V}_E(-, -)$ where E is a minimal DCI-approximation of R , all of the results regarding the support sets from Sections 4.1, 4.2, and 4.3 hold for $\mathbf{V}_R(-, -)$, as well. Again, to save space, we do not list them all but instead list the main ones in Theorem 5.1.6, below.

Definition 5.1.5. Let $(R, \mathfrak{m})$ be a commutative noetherian local ring and let $\widehat{R}$ denote its $\mathfrak{m}$ -adic completion. For a pair of objects M and N of $\mathbf{D}^f(R)$, we define the *cohomological support of the pair* (M, N) to be

$$\mathbf{V}_R(M, N) := \mathbf{V}_E(M \otimes_R \widehat{R}, N \otimes_R \widehat{R})$$

where E is a minimal DCI-approximation of R . We define the *cohomological support of M* to be $\mathbf{V}_R(M) := \mathbf{V}_E(M, M)$.

Theorem 5.1.6. *Let $(R, \mathfrak{m}, k)$ be a commutative noetherian local ring.*

- (1) $\mathbf{V}_R(M, N) \cap \mathbf{V}_R(M', N') = \mathbf{V}_R(M, N') \cap \mathbf{V}_R(M', N)$ for all objects M, M', N, N' in $\mathbf{D}^f(R)$.
- (2) $\mathbf{V}_R(M, N) = \mathbf{V}_R(N, M)$ for all $M, N \in \mathbf{D}^f(R)$.
- (3) *The following are equivalent:*
 - (a) R is a complete intersection.
 - (b) $\mathbf{V}_R(R, k) = \emptyset$.
 - (c) $\mathbf{V}_R(R) = \emptyset$.
 - (d) $\mathbf{V}_R(M, k) = \emptyset$ for some nonzero finitely generated R -module M .
 - (e) $\mathbf{V}_R(M) = \emptyset$ for some nonzero finitely generated R -module M .

Proof. The statements (1) and (2) follow immediately from Theorem 4.3.1. For (3), let $E = \text{Kos}^Q(f_1, \dots, f_n)$ is a minimal DCI-approximation of R and let $\mathcal{S}$ be $Q[\chi_1, \dots, \chi_n]$, the ring of cohomology operators for E . Most of (3) follows from Theorem 4.2.2. However, it remains to prove the following claim.

Claim: For an object M of $D^f(E)$, $V_E(M) = \emptyset$ if and only if $V_E(M, k) = \emptyset$.

Indeed, $V_R(M, k) = \emptyset$ if and only if M is an object of $\text{Thick}_{D(E)} E$. Thus, $\text{Ext}_E^{\geq 0}(M, M) = 0$ and hence Proposition 4.1.3(3), it follows that $V_E(M) = \emptyset$.

Now assume that $V_E(M) = \emptyset$. By Proposition 4.1.3(3), $\text{Ext}_E^{\geq 0}(M, M) = 0$ and hence,

$$(\text{Ext}_E^*(M, M) \otimes_{\mathcal{S}} (\mathcal{S} \otimes_Q k))^{\geq 0} = \text{Ext}_E^{\geq 0}(M, M) \otimes_Q k = 0.$$

Thus,

$$\begin{aligned} \emptyset &= \text{Supp}_{\mathcal{S}}^+(\text{Ext}_E^*(M, M) \otimes_{\mathcal{S}} (\mathcal{S} \otimes_Q k)) \\ &= \text{Supp}_{\mathcal{S}}^+(\text{Ext}_E^*(M, M)) \cap \text{Supp}_{\mathcal{S}}^+(\mathcal{S} \otimes_Q k) \\ &= V_E(M) \cap V_E(k) \\ &= V_E(M, k) \end{aligned}$$

where the second equality holds since $\text{Ext}_E^*(M, M)$ and $\mathcal{S} \otimes_Q k$ are finitely generated graded $\mathcal{S}$ -modules, the third equality holds by Example 4.1.4(2) and Theorem 4.3.1(2) yields the last equality. $\square$

5.2. Relation to Other Supports. Let Q be a commutative noetherian ring and set $E := \text{Kos}^Q(\mathbf{f})$ where $\mathbf{f} = f_1, \dots, f_n$ is a list of elements from Q . We set $\mathcal{S} := Q[\chi_1, \dots, \chi_n]$, the ring of cohomology operators associated to E . In this section, we set $R := H_0(E) = Q/(\mathbf{f})$.

5.2.1. Assume that $\mathbf{f}$ is a Q -regular sequence. For each pair of DG R -modules, $\text{Ext}_R^*(M, N)$ is a graded $\mathcal{S}$ -module. By [4, 2.4] and 1.3.3, we have that the quasi-isomorphism $\epsilon : E \xrightarrow{\sim} R$ induces an isomorphism of graded $\mathcal{S}$ -modules

$$\text{Ext}_R^*(M, N) \xrightarrow{\text{Ext}_{\epsilon}^*(M, N)} \text{Ext}_E^*(M, N).$$

5.2.2 (Burke-Walker). Assume that $\mathbf{f}$ is a Q -regular sequence. Let M and N be a pair of objects from $D^f(R)$ and assume that M is in $\text{Thick}_{D(Q)} Q$ when regarded as a complex of Q -modules via the canonical projection $Q \rightarrow R$. In [20], Burke and Walker define the *cohomological support* of (M, N) , denoted $V_Q^f(M, N)$, which in the notation of that paper, satisfies

$$V_Q^f(M, N) = \text{Supp}_{\mathcal{S} \otimes_Q R}^+ \text{Ext}_R^*(M, N).$$

That is,

$$V_Q^f(M, N) = V_E(M, N) \times_{\text{Spec } Q} \text{Spec } R.$$

Thus, starting from $V_E(M, N)$ one can obtain the varieties defined by Avramov and Buchweitz [5], Benson, Iyengar and Krause [14], and Stevenson [37] when $\mathbf{f}$ is a Q -regular sequence (see [20, 8.1] for more details).

5.2.3 (D. Jorgensen). Let $(Q, \mathfrak{n}, k)$ be local. For simplicity, we assume that k is algebraically closed. In [33], D. Jorgensen defines the cohomological support for a pair of finitely generated R -modules to be

$$V(Q, \mathbf{f}; M, N) := \{(a_1, \dots, a_c) \in \mathbb{A}_k^n : \text{Ext}_{Q_{\mathbf{a}}}^*(M, N) \text{ is unbounded}\} \cup \{\mathbf{0}\}$$

where

$$Q_{\mathbf{a}} = Q/(\tilde{a}_1 f_1 + \dots + \tilde{a}_n f_n)$$

and $\tilde{a}_i$ is a lifting of a_i to Q . By [33, 2.1], if Q is a domain and $(\mathbf{f}) \subseteq \mathfrak{n}$, then $V(Q, \mathbf{f}; M, N)$ is a well-defined, (Zariski) closed subset of $\mathbb{A}_k^n$. By [5, 2.5] and 5.2.2, when $\mathbf{f}$ is a regular sequence and $\text{Ext}_Q^{\geq 0}(M, N) = 0$,

$$V(Q, \mathbf{f}; M, N) = \nu^{-1} \max(\mathcal{V}_E(M, N))$$

where $\nu : \mathbb{A}_k^n \setminus \{\mathbf{0}\} \rightarrow \mathbb{P}_k^{n-1}$ is given by

$$(a_1, \dots, a_n) \mapsto (a_i \chi_j - a_j \chi_i).$$

In Proposition 5.2.4, below, we establish a more general relation between $\mathcal{V}_E(M, N)$ and $V(Q, \mathbf{f}; M, N)$ for finitely generated R -modules.

Theorem 5.2.4. *Assume $(Q, \mathfrak{n}, k)$ is local and $(\mathbf{f}) \subseteq \mathfrak{n}$ contains a regular element. Let $\nu : \mathbb{A}_k^n \setminus \{\mathbf{0}\} \rightarrow \mathbb{P}_k^{n-1}$ given by*

$$(a_1, \dots, a_n) \mapsto (a_i \chi_j - a_j \chi_i).$$

- (1) *Let M and N be a pair of objects from $\mathcal{D}^f(E)$ such that $\text{Ext}_Q^{\geq 0}(M, N) = 0$. For $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{A}_k^n \setminus \{\mathbf{0}\}$, $\nu(\mathbf{a}) \in \mathcal{V}_E(M, N)$ if and only if $\text{Ext}_{E_{\mathbf{a}}}^*(M, N)$ is unbounded where*

$$E_{\mathbf{a}} := \text{Kos}^Q(\tilde{a}_1 f_1 + \dots + \tilde{a}_n f_n)$$

and $\tilde{a}_i$ is a lifting of a_i to Q .

- (2) *Assume that k is algebraically closed. Then*

$$V(Q, \mathbf{f}; M, N) = \nu^{-1}(\max \mathcal{V}_E(M, N))$$

for each pair of objects M and N of $\mathcal{D}^f(R)$ with $\text{Ext}_Q^{\geq 0}(M, N) = 0$.

Remark 5.2.5.

- (1) The proof of Theorem 5.2.4(1) is essentially the same as the proof [5, 2.5]. Instead of importing that proof with the slight, necessary modifications, we offer a new proof of Proposition 5.2.4(1) that holds when Q is regular. The difference is that the proof below works at the chain level rather than after taking homology.
- (2) Since $(\mathbf{f}) \subseteq \mathfrak{n}$ contains a regular element, for each $a \in \mathbb{A}_k^n \setminus \{\mathbf{0}\}$, $E_{\mathbf{a}} \xrightarrow{\sim} Q_{\mathbf{a}}$. Hence, for a pair of finitely generated R -modules M and N , $\text{Ext}_{E_{\mathbf{a}}}^*(M, N)$ is unbounded if and only if $\text{Ext}_{Q_{\mathbf{a}}}^i(M, N) \neq 0$ for infinitely many i . Hence, Theorem 5.2.4(2) follows from 5.2.4(1) and the graded Nullstellensatz.

Proof of Theorem 5.2.4(1) when Q is regular. By Proposition 3.4.3, we can assume that $\mathbf{a} = (1, 0, \dots, 0)$. Thus,

$$\nu(\mathbf{a}) = (\chi_2, \dots, \chi_n) \in \text{Proj } \mathcal{A}.$$

By assumption and Proposition 4.1.3(2),

$$\mathcal{V}_E(M, N) = \text{Supp}_{\mathcal{S}}^+ \mathcal{C}_E(F, G)$$

where $F \xrightarrow{\sim} M$ and $G \xrightarrow{\sim} N$ are Koszul resolutions. By Lemma 4.2.7,

$$\mathcal{V}_E(M, N) = \text{Supp}_{\mathcal{A}}^+(\mathcal{C}_E(F, G) \otimes_Q k).$$

Also, we have isomorphisms of DG $\mathcal{A}$ -modules

$$\begin{aligned} (\mathcal{C}_E(F, G) \otimes_Q k) \otimes_{\mathcal{A}}^L \kappa_{\mathcal{A}}(\nu(\mathbf{a})) &= \mathcal{C}_E(F, G) \otimes_Q k \otimes_{\mathcal{A}} \kappa_{\mathcal{A}}(\nu(\mathbf{a})) \\ &\cong (\mathcal{C}_E(F, G) \otimes_S \mathcal{S}/(\chi_2, \dots, \chi_n) \otimes_Q k)_{\chi_1} \\ &\cong (\mathcal{C}_{E_{\mathbf{a}}}(F, G) \otimes_Q k)_{\chi_1} \end{aligned}$$

where first equality holds by Proposition 3.2.10 and the third equality holds since F and G are Koszul resolutions of M and N , respectively, over $E_{\mathbf{a}}$.

In summary, $\nu(\mathbf{a}) \in \mathcal{V}_E(M, N)$ if and only if $\mathcal{C}_{E_{\mathbf{a}}}(F, G)_{\chi_1} \not\cong 0$. Since localization is exact, the latter is equivalent to

$$(5.1) \quad \text{Ext}_{E_{\mathbf{a}}}^*(M, N)_{\chi_1} \neq 0.$$

As $\text{Ext}_{E_{\mathbf{a}}}^*(M, N)$ is a finitely generated graded $Q[\chi_1]$ -module satisfying (5.1), it follows, equivalently, that $\text{Ext}_{E_{\mathbf{a}}}^*(M, N)$ is unbounded. Now using the isomorphism

$$\text{Ext}_{E_{\mathbf{a}}}^*(M, N) \cong \text{Ext}_{Q_{\mathbf{a}}}^*(M, N),$$

we obtain that $\nu(\mathbf{a}) \in \mathcal{V}_E(M, N)$ if and only if $\text{Ext}_{Q_{\mathbf{a}}}^i(M, N)$ is nonzero for infinitely many values of i . $\square$

5.3. A Study of $\mathcal{V}(R)$ for Local Rings with Small Codepth. Let $(R, \mathfrak{m}, k)$ be a local commutative noetherian ring. In this section we investigate the *cohomological support of R* ; namely, we study

$$\mathcal{V}(R) := \mathcal{V}_R(R, k).$$

By Theorem 5.1.6, $\mathcal{V}(R)$ is empty exactly when R is a complete intersection. Hence, of particular interest is describing $\mathcal{V}(R)$ when R is not a complete intersection.

For the rest of the section we fix the following notation.

Notation 5.3.1. Let $E := Q\langle \xi_1, \dots, \xi_n | \partial \xi_i = f_i \rangle$ be a minimal DCI-approximation of R . In particular, Q is a regular local ring; we let $\mathfrak{n}$ denote the maximal ideal of Q and note that $Q/\mathfrak{n} \cong k$. Fix a minimal free resolution $F \xrightarrow{\sim} \widehat{R}$ over Q . For a DG E -module X , we let λ_i denote left multiplication by ξ_i . Finally, note that $\mathcal{V}(R)$ is a subset of $\mathbb{P}_k^{n-1}$ where n is the derived codimension of $\widehat{R}$ (c.f. 5.1.1).

5.3.2. Suppose $\text{pd}_Q \widehat{R} \leq 3$. By [18], F admits a DG Q -algebra structure. As $H_0(E) = \widehat{R}$, it follows that F inherits a DG E -algebra structure. By Remark 3.2.9, $\mathcal{C}_E(F, k)$ is the following complex of graded $\mathcal{A}$ -modules

$$(5.2) \quad \dots \rightarrow \Sigma^{-4} \mathcal{A} \otimes_k \overline{F}_2 \xrightarrow{\partial_2} \Sigma^{-2} \mathcal{A} \otimes_k \overline{F}_1 \xrightarrow{\partial_1} \mathcal{A} \otimes_k \overline{F}_0 \rightarrow 0$$

where $\overline{F}_i = \text{Hom}_Q(F_i, k)$ and

$$\partial = \sum_{i=1}^n \chi_i \otimes \text{Hom}(\lambda_i, k).$$

We define the *codepth of R* to be

$$\text{codepth } R := \dim_k(\mathfrak{m}/\mathfrak{m}^2) - \text{depth } R.$$

By the Auslander-Buchsbaum formula $\text{codepth } R = \text{pd}_Q \widehat{R}$.

Example 5.3.3. Assume R is not a complete intersection, the derived codimension of R is n , and R satisfies one of the following conditions:

- (1) $\text{codepth } R = 2$, or

(2) codepth $R = 3$ and R is Gorenstein.

Then $V(R) = \mathbb{P}_k^{n-1}$.

Indeed, in either case $F_1 F_1 \subseteq \mathfrak{n} F_2$ (see [2, 2.1.2] and [2, 2.1.3], respectively) where we adopt the notation from 5.3.2. Hence, (5.2) has the following form

$$0 \rightarrow \Sigma^{-6} \mathcal{A} \otimes_k \overline{F_3} \xrightarrow{\partial_3} \Sigma^{-4} \mathcal{A} \otimes_k \overline{F_2} \xrightarrow{0} \Sigma^{-2} \mathcal{A} \otimes_k \overline{F_1} \xrightarrow{(\chi_1 \ \cdots \ \chi_n)} \mathcal{A} \otimes_k \overline{F_0} \rightarrow 0.$$

Therefore, in either case, $\Sigma^{-2} \mathcal{A}$ is a submodule of $\text{Ext}_E^*(\widehat{R}, k)$ which justifies the claim.

More generally, when the codepth of the ring is small, we have a complete description of $V(R)$. We quickly recap the DG algebra structure on F when codepth $R = 3$ (see [12] for more details).

5.3.4. Assume codepth $R = 3$; that is, $\text{pd}_Q \widehat{R} = 3$. We fix the following notation:

$$F_1 = \bigoplus_{i=1}^n Q a_i, \quad F_2 = \bigoplus_{i=1}^m Q b_i, \quad F_3 = \bigoplus_{i=1}^\ell Q c_i.$$

The DG E -module structure on F is determined by

$$\xi_i \cdot x := a_i \cdot x$$

(see 3.1.5). Therefore, (5.2) is the following complex of graded $\mathcal{A}$ -modules:

$$(5.3) \quad 0 \rightarrow \Sigma^{-6} \mathcal{A} \otimes_k \oplus k \gamma_i \xrightarrow{\partial_3} \Sigma^{-4} \mathcal{A} \otimes_k \oplus k \beta_i \xrightarrow{\partial_2} \Sigma^{-2} \mathcal{A} \otimes_k \oplus k \alpha_i \xrightarrow{\partial_1} \mathcal{A} \otimes_k k \rightarrow 0$$

where

$$\alpha_i = \text{Hom}(a_i, k), \quad \beta_i = \text{Hom}(b_i, k), \quad \gamma_i = \text{Hom}(c_i, k)$$

and

$$\partial_1 = (\chi_1 \ \cdots \ \chi_n).$$

In [12], Avramov, Kustin and Miller showed that there are five classes of algebra structures on F modulo $\mathfrak{n}$: **CI**, **TE**, **B**, **G**(r), **H**(p, q). We summarize the relevant results from [12] below:

- (1) R belongs to **CI** if and only if R is a complete intersection.
- (2) When R belongs to **G**(r), there exists $r \geq 2$ such that $a_i b_i = c_1$ modulo $\mathfrak{n}$ for all $1 \leq i \leq r$ and all other products on F are zero modulo $\mathfrak{n}$.
- (3) If R belongs to **TE**,

$$a_2 a_3 = b_1, \quad a_3 a_1 = b_2, \quad a_1 a_2 = b_3$$

and all other products in F are zero modulo $\mathfrak{n}$.

- (4) Assume R is in **B**. In this case, we have the following equations holding modulo $\mathfrak{n}$:

$$a_1 a_2 = b_3, \quad a_1 b_1 = c_1, \quad a_2 b_2 = c_1,$$

and all other products of basis elements of F are zero.

- (5) For R in **H**(p, q), $p < n$, $q \leq \ell$, and modulo $\mathfrak{n}$

$$a_{p+1} a_i = b_i \text{ for all } 1 \leq i \leq p, \quad a_{p+1} b_{p+i} = c_i \text{ for all } 1 \leq i \leq q,$$

and all other products of basis elements of F are zero.

Theorem 5.3.5. *We adopt the assumptions and notation from 5.3.4, and set $\mathcal{E} := \text{Ext}_E^*(\widehat{R}, k)$ and $\mathcal{Z} := \ker \partial_1$.*

- (1) *When R belongs to **CI**, $\mathcal{E} = k$.*

(2) When R belongs to $\mathbf{G}(r)$,

$$\mathcal{E} \cong k \oplus \mathcal{Z} \oplus \Sigma^{-4} \frac{\oplus_{i=1}^m \mathcal{A}\beta_i}{(\sum_{i=1}^r \chi_i \beta_i)} \oplus \Sigma^{-6} \mathcal{A}^{\ell-1}.$$

(3) When R belongs to $\mathbf{TE}$,

$$\mathcal{E} \cong k \oplus \frac{\mathcal{Z}}{\Sigma^{-2} \mathcal{A}(\chi_j \alpha_i - \chi_i \alpha_j)_{1 \leq i < j \leq 3}} \oplus \Sigma^{-4} \mathcal{A}^{m-2} \oplus \Sigma^{-6} \mathcal{A}^{\ell}.$$

(4) When R belongs to $\mathbf{B}$,

$$\mathcal{E} \cong k \oplus \frac{\mathcal{Z}}{\Sigma^{-2} \mathcal{A}(\chi_2 \alpha_1 - \chi_1 \alpha_2)} \oplus \Sigma^{-4} \frac{\oplus_{i=1}^m \mathcal{A}\beta_i}{\mathcal{A}(\chi_1 \beta_1 + \chi_2 \beta_2)} \oplus \Sigma^{-6} \mathcal{A}^{\ell-1}.$$

(5) When R belongs to $\mathbf{H}(p, q)$,

$$\mathcal{E} \cong k \oplus \frac{\mathcal{Z}}{\Sigma^{-2} \mathcal{A}(\chi_{p+1} \alpha_i - \chi_i \alpha_{p+1})_{1 \leq i \leq p}} \oplus \Sigma^{-4} \frac{\mathcal{A}^{m-p}}{(\chi_{p+1})} \oplus \Sigma^{-6} \mathcal{A}^{\ell-q}.$$

Proof. (1) This is immediate.

For each the following cases, recall that $\mathcal{E}$ can be calculated as the cohomology of (5.3). Hence, we need only specify ∂_2 and ∂_3 in each of the remaining cases.

(2) Using 5.3.4(2), it follows that

$$\chi_i \otimes \text{Hom}(\lambda_i, k)(1 \otimes \gamma_1) = \chi_i \otimes \beta_i$$

for all $1 \leq i \leq r$ and

$$\chi_i \otimes \text{Hom}(\lambda_i, k)(1 \otimes \beta_i) = 0$$

for all i . Thus,

$$\partial_3 = \begin{pmatrix} \chi_1 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ \chi_r & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \text{ and } \partial_2 = \mathbf{0}_{n \times m}$$

where $\mathbf{0}_{n \times m}$ is the $n \times m$ matrix consisting of all zeros.

(3) Using 5.3.4(3), we get

$$\begin{aligned} \chi_2 \otimes \text{Hom}(\lambda_2, k)(1 \otimes \beta_1) &= \chi_2 \otimes \alpha_3 \\ \chi_3 \otimes \text{Hom}(\lambda_3, k)(1 \otimes \beta_1) &= -\chi_3 \otimes \alpha_2 \\ \chi_3 \otimes \text{Hom}(\lambda_3, k)(1 \otimes \beta_2) &= \chi_3 \otimes \alpha_1 \\ \chi_1 \otimes \text{Hom}(\lambda_1, k)(1 \otimes \beta_2) &= -\chi_1 \otimes \alpha_3 \\ \chi_1 \otimes \text{Hom}(\lambda_1, k)(1 \otimes \beta_3) &= \chi_1 \otimes \alpha_2 \\ \chi_2 \otimes \text{Hom}(\lambda_2, k)(1 \otimes \beta_3) &= -\chi_2 \otimes \alpha_1 \end{aligned}$$

and the values of $\chi_i \otimes \text{Hom}(\lambda_i, k)$ on the remaining $1 \otimes \gamma_j$ and $1 \otimes \beta_j$ are all zero. Hence, $\partial_3 = \mathbf{0}_{m \times \ell}$ and

$$\partial_2 = \begin{pmatrix} 0 & \chi_3 & -\chi_2 & 0 & \dots & 0 \\ -\chi_3 & 0 & \chi_1 & 0 & \dots & 0 \\ \chi_2 & -\chi_1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

(4) Using 5.3.4(4), we get the following:

$$\begin{aligned} \chi_1 \otimes \text{Hom}(\lambda_1, k)(1 \otimes \gamma_1) &= \chi_1 \otimes \beta_1 \\ \chi_2 \otimes \text{Hom}(\lambda_2, k)(1 \otimes \beta_1) &= \chi_2 \otimes \beta_2 \\ \chi_1 \otimes \text{Hom}(\lambda_1, k)(1 \otimes \beta_3) &= \chi_1 \otimes \alpha_2 \\ \chi_2 \otimes \text{Hom}(\lambda_2, k)(1 \otimes \beta_3) &= -\chi_2 \otimes \alpha_1 \end{aligned}$$

and $\chi_i \otimes \text{Hom}(\lambda_i, k)$ vanishes on all the remaining $1 \otimes \gamma_j$ and $1 \otimes \beta_j$. In particular,

$$\partial_3 = \begin{pmatrix} \chi_1 & 0 & \dots & 0 \\ \chi_2 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & 0 \\ 0 & 0 & \dots & 0 \end{pmatrix} \text{ and } \partial_2 = \begin{pmatrix} 0 & 0 & -\chi_2 & 0 & \dots & 0 \\ 0 & 0 & \chi_1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

(5) Using 5.3.4(5), we get

$$\begin{aligned} \chi_{p+1} \otimes \text{Hom}(\lambda_{p+1}, k)(1 \otimes \gamma_i) &= \chi_{p+1} \otimes \beta_{p+i} \\ \chi_j \otimes \text{Hom}(\lambda_j, k)(1 \otimes \beta_j) &= -\chi_j \otimes \alpha_{p+1} \\ \chi_{p+1} \otimes \text{Hom}(\lambda_{p+1}, k)(1 \otimes \beta_j) &= \chi_{p+1} \otimes \alpha_j \end{aligned}$$

for $1 \leq i \leq q$ and $1 \leq j \leq p$, and the values of $\chi_i \otimes \text{Hom}(\lambda_i, k)$ on the remaining $1 \otimes \gamma_j$ and $1 \otimes \beta_j$ are all zero. Hence,

$$\begin{aligned} \partial_3 &= \begin{pmatrix} \mathbf{0}_{p \times q} & \mathbf{0}_{p \times (\ell - q)} \\ \chi_{p+1} I_q & \mathbf{0}_{q \times (\ell - q)} \\ \mathbf{0}_{(m-p-q) \times q} & \mathbf{0}_{(m-p-q) \times (\ell - q)} \end{pmatrix} \\ \partial_2 &= \begin{pmatrix} \chi_{p+1} I_p & \mathbf{0}_{p \times (m-p)} \\ -\chi & \mathbf{0}_{1 \times (m-p)} \\ \mathbf{0}_{(n-p-1) \times p} & \mathbf{0}_{(n-p-1) \times (m-p)} \end{pmatrix} \end{aligned}$$

where $-\chi = (-\chi_1 \quad \dots \quad -\chi_p)$ and I_t denotes the $t \times t$ identity matrix. $\square$

With the notation set in Notation 5.3.1, we say that $Q \rightarrow \widehat{R}$ admits an embedded deformation if there exists $g_1, \dots, g_n \in \mathfrak{n}^2$ with g_n being regular on $Q/(g_1, \dots, g_{n-1})$ and

$$\widehat{R} \cong Q/(g_1, \dots, g_n).$$

We restate Theorem C from the introduction for the ease of the reader.

Theorem 5.3.6. *Let $(R, \mathfrak{m}, k)$ be a commutative noetherian local ring of derived codimension n and codepth $R \leq 3$. The following characterizes the possible subsets of $\mathbb{P}_k^{n-1}$ that $V(R)$ can realize:*

- (1) *If R is a complete intersection, then $V(R) = \emptyset$.*
- (2) *If R is not a complete intersection and $Q \rightarrow \widehat{R}$ admits an embedded deformation, then $V(R)$ is a hyperplane in $\mathbb{P}_k^{n-1}$.*
- (3) *Otherwise, $V(R) = \mathbb{P}_k^{n-1}$.*

Proof. When R is a complete intersection, $V(R) = \text{Supp}_{\mathcal{A}}^+(k) = \emptyset$. Therefore, we assume that R is not a complete intersection for the rest of the proof (and in particular, codepth R is 2 or 3). For R a non-complete intersection with codepth $R = 2$ we appeal to Example 5.3.3(1). Thus, we assume that R is not a complete intersection and codepth $R = 3$.

As R is not a complete intersection, $Q \rightarrow \widehat{R}$ admits an embedded deformation if and only if R belongs to $\mathbf{H}(n-1, \ell)$ (c.f. [3, 3.3]). Notice that when R belongs to $\mathbf{H}(n-1, \ell)$,

$$\text{Ext}_E^*(\widehat{R}, k) \cong k \oplus \Sigma^{-2} \frac{\mathcal{A}}{(\chi_n)} (\chi_j \alpha_i - \chi_i \alpha_j)_{1 \leq i < j \leq n-1} \oplus \Sigma^{-4} \frac{\mathcal{A}^{m-n+1}}{(\chi_n)}.$$

In particular, $V(R) = \mathcal{V}(\chi_n)$ is a hyperplane in $\mathbb{P}_k^{n-1}$.

For R belonging to $\mathbf{G}(r)$, $\mathbf{TE}$, $\mathbf{B}$, or $\mathbf{H}(p, q)$ for $p \neq n-1$ or $q \neq \ell$, $\text{Ext}_E^*(\widehat{R}, k)$ contains a shift of an $\mathcal{A}$ -free summand (see Theorem 5.3.5). Hence, $V(R) = \mathbb{P}_k^{n-1}$ as claimed. $\square$

Proposition 5.3.7. *With the setup in Notation 5.3.1, if $Q \rightarrow \widehat{R}$ admits an embedded deformation then $V(R)$ is contained in a hyperplane of $\mathbb{P}_k^{n-1}$.*

Proof. By Proposition 3.4.3, up to a linear change of coordinates of $\mathbb{P}_k^{n-1}$, we can assume that f_n is regular on $R' := Q/(f_1, \dots, f_{n-1})$. Since $\text{pd}_Q R' < \infty$ and

$$f := f_n + \sum_{i < n} a_i f_i$$

is a Q -regular elements that is also R' -regular, it follows that $\text{pd}_{Q/f} R'/fR' < \infty$. Furthermore,

$$R'/fR' = Q/(\mathbf{f}) = \widehat{R}$$

and hence, $\text{pd}_{Q/f} \widehat{R} < \infty$. Thus, by Theorem 5.2.4, we conclude that

$$V(R) \subseteq \mathcal{V}(\chi_n). \quad \square$$

As indicated in Theorem 5.3.6 and Proposition 5.3.7, embedded deformations put a restriction on $V(R)$. More generally, one says that $Q \rightarrow \widehat{R}$ admits an embedded deformation of codimension c provided that there exists $g_1, \dots, g_n \in \mathfrak{n}^2$ with $g_{n-c+1}, \dots, g_n$ being a regular sequence on $Q/(g_1, \dots, g_{n-c})$ and

$$\widehat{R} \cong Q/(g_1, \dots, g_n).$$

The author is curious as to whether $V(R)$ can, in general, detect embedded deformations of arbitrary codimension; Theorem 5.3.6 and Proposition 5.3.7, as well as various specific examples, offer partial evidence for this.

Question 5.3.8. *Does the following hold for a local ring $(R, \mathfrak{m}, k)$?:*

If $Q \rightarrow \widehat{R}$ is a minimal Cohen presentation of R , then $Q \rightarrow \widehat{R}$ admits an embedded deformation of codimension c if and only if $V(R)$ is contained in a hyperplane of codimension c of $\mathbb{P}_k^{n-1}$.

5.4. Solution to a Question of D. Jorgensen. This subsection is devoted to answering the following question of D. Jorgensen.

Question 5.4.1. [33] Let Q be a regular local ring with an algebraically closed residue field k . Set $R := Q/(\mathbf{f})$ where $\mathbf{f} \subseteq \mathfrak{n}^2$ minimally generates $(\mathbf{f})$.

- (1) If $V(Q, \mathbf{f}; M, N) = \emptyset$ for some finitely generated R -modules M and N , does $\text{Ext}_R^{\geq 0}(M, N) = 0$?
- (2) If $V(Q, \mathbf{f}; M, N) = \emptyset$ for some finitely generated R -modules M and N , is R a complete intersection?

Remark 5.4.2. When N is held fixed as k , then by Theorem 5.1.6(3) and 5.2.3 the answers to Questions 5.4.1(1) and 5.4.1(2) are “yes.” However, in general, the answers to these questions are both “no” by Example 5.4.3.

Example 5.4.3. Let $Q = k[[x, y, z]]$ where k is algebraically closed. Set $\mathbf{f} = xy, yz$, $E := \text{Kos}^Q(\mathbf{f})$, $\mathcal{A} := k[\chi_1, \chi_2]$, and $R := Q/(\mathbf{f})$.

For each $p, q \in k$, define

$$M_{p,q} := R/(px + qz).$$

When $p \neq 0$, then $M_{p,q} = Q/(px + qz, yz)$. A minimal Q -free resolution of $M_{p,q}$ is given by

$$F = 0 \rightarrow Qc \xrightarrow{\begin{pmatrix} -yz \\ px + qz \end{pmatrix}} Qb_1 \oplus Qb_2 \xrightarrow{\begin{pmatrix} px + qz & yz \end{pmatrix}} Qa \rightarrow 0$$

and since $\mathbf{f}M_{p,q} = 0$ the DG E -module structure on F is given by

$$(\lambda_1)_0 = \begin{pmatrix} y/p \\ -q/p \end{pmatrix}, (\lambda_1)_1 = (q/p \quad y/p) \text{ and } (\lambda_2)_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, (\lambda_2)_1 = (-1 \quad 0)$$

where λ_i denotes left multiplication by ξ_i on F . That is, F is a Koszul resolution of $M_{p,q}$ with λ_1 and λ_2 prescribed above.

By Remark 3.2.9, $\mathcal{C}_E(F, k)$ is the following complex of graded $\mathcal{A}$ -modules

$$0 \rightarrow \Sigma^{-4} \mathcal{A}\gamma \xrightarrow{\begin{pmatrix} \frac{q}{p}\chi_1 - \chi_2 \\ 0 \end{pmatrix}} \Sigma^{-2} \mathcal{A}\beta_1 \oplus \mathcal{A}\beta_2 \xrightarrow{\begin{pmatrix} 0 & \chi_2 - \frac{q}{p}\chi_1 \end{pmatrix}} \mathcal{A}\alpha \rightarrow 0.$$

Hence,

$$\text{Ext}_E^*(M_{p,q}, k) = k[\chi_1, \chi_2]/(q\chi_1 - p\chi_2) \oplus \Sigma^{-2} k[\chi_1, \chi_2]/(q\chi_1 - p\chi_2).$$

A similar argument, holds for $q \neq 0$.

Therefore, for each point $(p, q) \in \mathbb{A}_k^2 \setminus \{(0, 0)\}$, we have that

$$\mathcal{V}_E(M_{p,q}, k) = \text{Supp}_{\mathcal{A}}^+(\mathcal{A}/(q\chi_1 - p\chi_2)).$$

In particular, for each point $(p, q) \in \mathbb{A}_k^2$

$$V(Q, \mathbf{f}, M_{p,q}, k) = \{(a, b) \in \mathbb{A}_k^2 : qa = pb\}$$

is the line through $(0, 0)$ and (p, q) in $\mathbb{A}_k^2$. Also, by 5.2.3,

$$V(Q, \mathbf{f}; M_{p,q}, k) \cap V(Q, \mathbf{f}; M_{s,t}, k) = V(Q, \mathbf{f}, M_{p,q}, M_{s,t}).$$

Therefore, for two points $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{A}_k^2 \setminus \{(0,0)\}$ such that $\mathbf{a} \neq \lambda \mathbf{b}$ for any λ , we have that

$$V(Q, \mathbf{f}; M_{\mathbf{a}}, M_{\mathbf{b}}) = \emptyset.$$

Therefore, this answers Question 5.4.1(2) in the negative. Finally, a direct calculation shows that $\text{Ext}_R^*(M_{1,0}, M_{0,1})$ is unbounded and hence, the answer to Question 5.4.1(1) is also “no,” in general.

Remark 5.4.4. When R is a complete intersection, every closed subset of $\mathbb{P}_k^{n-1}$ is realizable as $V_R(M)$ for some finitely generated R -module M (see [11, 15, 20]). If R is not a complete intersection, then $V_R(M) \neq \emptyset$ whenever M is a nonzero finitely generated R -module (see Theorem 5.1.6(3)). In Example 5.4.3, we demonstrate that nearly every non-empty closed subset of $\mathbb{P}_k^1$ is realizable as $V_R(M)$ for some finitely generated R -module M . However, in general, it is not known which closed subsets can be attained as $V_R(M)$ for some finitely generated R -module M (or object M of $D^f(R)$). In [36, 3.3.4], the author shows every hyperplane of $\mathbb{P}_k^{n-1}$ is realizable as a complex whose homology is just two copies of k in specified degrees. It is not known if one can use *finitely generated modules* to obtain every hyperplane of $\mathbb{P}_k^{n-1}$.

Problem 5.4.5. Determine what subsets of $\mathbb{P}_k^{n-1}$ are realizable as $V_R(M)$ for a finitely generated R -module M , or, more generally, for M in $D^f(R)$.

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