

A TRANSCENDENTAL BRAUER–MANIN OBSTRUCTION TO WEAK APPROXIMATION ON A CALABI–YAU THREEFOLD

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(WITH AN APPENDIX BY NICOLAS ADDINGTON)

ABSTRACT. In this paper we investigate the \mathbb{Q} -rational points of a class of simply connected Calabi–Yau threefolds, which were originally studied by Hosono and Takagi in the context of mirror symmetry. These varieties are defined as a linear section of a double quintic symmetroid; their points correspond to rulings on quadric hypersurfaces. They come equipped with a natural 2-torsion Brauer class. Our main result shows that under certain conditions, this Brauer class gives rise to a transcendental Brauer–Manin obstruction to weak approximation.

Hosono and Takagi showed that over \mathbb{C} each of these Calabi–Yau threefolds Y is derived equivalent to a Reye congruence Calabi–Yau threefold X . We show that these derived equivalences may also be constructed over \mathbb{Q} , and we give sufficient conditions for X to not satisfy weak approximation. In the appendix, N. Addington exhibits the Brauer groups of each class of Calabi–Yau variety over \mathbb{C} .

1. INTRODUCTION

Let Y be a smooth projective variety defined over a number field K . The K -rational points of Y embed diagonally in the adelic points $Y(K) \subseteq Y(\mathbb{A}_K) = \prod_v Y(K_v)$. Suppose $Y(\mathbb{A}_K) \neq \emptyset$, that is, that $Y(K_v)$ is nonempty for all places v of K . We say that Y satisfies **weak approximation** if $\overline{Y(K)} = Y(\mathbb{A}_K)$ in the adelic topology. Let $\mathrm{Br}(Y)$ denote the Brauer group of Y . Given any element $\alpha \in \mathrm{Br}(Y)$, there is an intermediate set $Y(\mathbb{A}_K)^\alpha$ that contains the closure of the K -rational points

$$\overline{Y(K)} \subset Y(\mathbb{A}_K)^\alpha \subset Y(\mathbb{A}_K).$$

If $Y(\mathbb{A}_K)^\alpha \neq Y(\mathbb{A}_K)$, we say that α gives a **Brauer–Manin obstruction to weak approximation** on Y . Such an obstruction is **transcendental** if $\alpha_{Y_{\bar{K}}}$ is nontrivial; otherwise it is algebraic.

In this paper, we construct examples of Calabi–Yau threefolds Y/\mathbb{Q} that naturally come with a 2-torsion class $\alpha \in \mathrm{Br}(Y)[2]$ providing a transcendental Brauer–Manin obstruction to weak approximation.

Transcendental Brauer classes are often more difficult to describe explicitly than their algebraic counterparts. While the first examples of

transcendental Brauer–Manin obstructions were constructed using extra structure such as fibrations [Har96, Wit04, Ier10], Várilly-Alvarado along with collaborators Hassett–Varilly [HVAV11], Hassett [HVA13], McKinnie–Sawon–Tanimoto [MSTVA17], and Berg [BVA20] have demonstrated that a lattice-theoretic framework can be leveraged to exhibit transcendental Brauer–Manin obstructions on general K3 surfaces.

Our Calabi–Yau threefold examples are of a similar flavor to those in [HVAV11]; in particular, we also leverage the data of linear spaces in a family of quadrics and prove that evaluation of our Brauer class has a very elegant and satisfying geometric description detailed in Sections 2 and 3. Relatively few examples of transcendental obstructions on threefolds are known, and to the best of our knowledge, these are the first such examples on Calabi–Yau threefolds. The Calabi–Yau threefolds we consider are also of independent interest in string theory. They have been studied over \mathbb{C} in a series of papers of Hosono and Takagi [HT13, HT14, HT16], but the present paper initiates the study of their arithmetic.

The varieties $Y = Y_P$ we consider are double covers of the locus of singular quadrics in a 4-dimensional linear system P of quadric hypersurfaces in \mathbb{P}^4 over \mathbb{Q} , as we now describe. A singular quadric threefold in \mathbb{P}^4 is the cone over a quadric surface of the same rank in \mathbb{P}^3 . Since rank 4 quadric surfaces are doubly ruled by lines, a rank 4 quadric hypersurface in \mathbb{P}^4 is doubly ruled by 2-planes; furthermore, the 2-planes in a given ruling are parametrized by a conic. Similarly, a rank 3 quadric in \mathbb{P}^4 has a unique ruling by 2-planes, and the 2-planes in this ruling are again parametrized by a conic. Given a suitably generic (we call the precise condition *regular*, see Definition 2.1) 4-dimensional linear system P of quadric hypersurfaces in \mathbb{P}^4 , we may associate to it a variety Y_P that parameterizes the singular quadrics (of ranks 3 and 4) in P along with a ruling of 2-planes on the quadric. The incidence variety parameterizing singular quadrics in P together with a choice of 2-plane is then a conic bundle over Y_P . This étale \mathbb{P}^1 -bundle over Y_P gives rise to a nontrivial class $\alpha = \alpha_P$ in $\text{Br}(Y_P)$ [HT16, Proposition 3.2.1]. We show that α_P can provide a transcendental Brauer–Manin obstruction to weak approximation on Y_P .

It is not a priori clear whether the rational points of such Y should satisfy weak approximation: some simple obstructions to weak approximation vanish. The threefolds Y that arise in this way are simply connected [HT13, Proposition 4.3.4], so the fundamental group does not provide an obstruction to weak approximation (see [Har04, Theorem 2.4.4] for a description of this obstruction). Furthermore, there is no algebraic Brauer–Manin obstruction to weak approximation: Proposition 2.8 shows that the algebraic part of the Brauer group of Y is trivial. In fact, it is a consequence of Proposition A.1 by Addington that $\text{Br}(Y_{\bar{\mathbb{Q}}})$ is generated by $\alpha_{\bar{\mathbb{Q}}}$.

Our main result shows that with fairly mild restrictions on the choice of quadrics defining the linear system P , the resulting Brauer class α_P

on the threefold Y_P always provides a transcendental obstruction to weak approximation.

Theorem 1. *Suppose that P is a regular 4-dimensional linear system of quadrics defined over \mathbb{Q} that contains a quadric with signature $(+1, +1, +1, +1, 0)$ or $(-1, -1, -1, -1, 0)$ over \mathbb{R} . If $Y_P(\mathbb{A}_{\mathbb{Q}}) \neq \emptyset$, then we have the proper containment*

$$\emptyset \neq Y_P(\mathbb{A}_{\mathbb{Q}})^{\alpha_P} \subsetneq Y_P(\mathbb{A}_{\mathbb{Q}}).$$

In particular, the smooth Calabi–Yau threefold Y_P does not satisfy weak approximation.

The proper containment $Y_P(\mathbb{A}_{\mathbb{Q}})^{\alpha_P} \subsetneq Y_P(\mathbb{A}_{\mathbb{Q}})$ arises in these cases because the Brauer class α_P has nonconstant evaluation at the real points of Y_P : we have chosen P so that $Y_P(\mathbb{R})$ contains a point where α_P has nontrivial evaluation and we show that for *any* choice of P , the set $Y_P(\mathbb{R})$ always contains a point where α_P has trivial evaluation.

Linear systems P containing a quadric of the desired signature and giving rise to locally soluble Y_P are common; in the proof of the following corollary, we produce one such example.

Corollary 2. *Let P be a general 4-dimensional linear system of quadrics defined over \mathbb{Q} that contains the quadric $x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0$. Then $Y_P(\mathbb{A}_{\mathbb{Q}}) \neq \emptyset$ and Y_P does not satisfy weak approximation.*

The proof of Theorem 1 centers upon controlling the evaluation of α_P at real points of Y_P . For other places, we show that for a random choice of P , the class α_P has trivial evaluation at every p -adic point of Y_P for *all* finite places p with probability at least .73. To make this precise, we must explain how we generate a random P . We do this by specifying (integral) basis vectors for the 5-dimensional subspace of \mathbb{A}^{15} corresponding to this plane. We can then define the density of some condition \star as follows:

$$d(\star) = \lim_{N \rightarrow \infty} \frac{\#\{\text{bases satisfying } \star \text{ all of whose coordinates have absolute value at most } N\}}{\#\{\text{bases all of whose coordinates have absolute value at most } N\}}.$$

Theorem 3. *Among all 4-dimensional linear systems of quadrics P defined over \mathbb{Z} , those where the Brauer class α_P has trivial evaluation at all of the points in $Y_P(\mathbb{Q}_p)$ for every prime p occur with density at least .73.*

Furthermore, the following five quadrics in \mathbb{P}^4 :

$$\begin{aligned} & x_0x_1 + x_2x_3 \\ & x_0^2 + x_0x_1 + 3x_0x_2 + 2x_1^2 + x_1x_2 + x_1x_3 + 5x_2^2 + x_2x_3 + x_3^2 \\ & x_0^2 + 2x_0x_2 + x_0x_3 + x_0x_4 + x_1x_2 + x_1x_4 + x_2x_4 + 4x_3x_4 \\ & 4x_0^2 + 4x_1^2 + 3x_1x_3 + 2x_1x_4 + 3x_2^2 + x_2x_4 + x_4^2 \\ & x_0^2 + x_0x_2 + x_0x_3 + 4x_0x_4 + 3x_1x_2 + 4x_1x_4 + 2x_2x_3 + 2x_3^2 + x_4^2 \end{aligned}$$

generate a regular linear system P for which we have the containments

$$\emptyset \neq Y_P(\mathbb{Q}) \subset Y_P(\mathbb{A}_{\mathbb{Q}})^{\alpha_P} \subsetneq Y_P(\mathbb{A}_{\mathbb{Q}}),$$

and α_P has trivial evaluation at every point of $Y_P(\mathbb{Q}_p)$ for every prime p .

We also give an explicit quaternion algebra over $\mathbb{Q}(Y_P)$ in (2.3) that represents α_P in this case.

Remark 4. In Theorems 1 and 3, the fact that α_P *always* has trivial evaluation at some real point of Y_P allowed us to specify information only about the real place (i.e., the signature) for one quadric in P . Instead, we can use a finite place p to obstruct weak approximation if we specify p -adic data for *two* quadrics in P . For example, working with the prime $p = 3$, we show in Proposition 3.10 that for P a general 4-dimensional linear system of quadrics over \mathbb{Q} containing $6x_0^2 - 3x_1^2 + 2x_2^2 - x_3^2 = 0$ and $x_0x_1 + x_2x_3 = 0$, the corresponding Y_P does not satisfy weak approximation.

Our examples are also of interest in the study of derived equivalence, particularly in comparing the rational points of derived equivalent varieties. For instance, the question posed by Hassett and Tschinkel [HT17] of how the existence and density of \mathbb{Q} -rational points on derived equivalent K3 surfaces over \mathbb{Q} compares remains open. It was shown in [AAFH19] that for hyperkähler fourfolds and abelian varieties of dimension at least 2, the existence of a \mathbb{Q} -rational point is not a derived invariant. Hence Calabi–Yau threefolds are a very interesting intermediate type of variety to study with regard to this question. Hosono and Takagi showed that each of the Calabi–Yau threefolds Y_P we study here is derived equivalent, but not birational, to another Calabi–Yau threefold X_P also attached to the linear system P . We prove that these derived equivalences may be constructed over \mathbb{Q} as well, and show that in many cases the \mathbb{Q} -points of these derived equivalent varieties behave similarly. We believe that this class of examples will open many questions for future study.

In Section 2 we outline the construction of the Calabi–Yau threefolds Y_P along with the Brauer class $\alpha_P \in \mathrm{Br}(Y_P)[2]$ and give some preliminary results about these objects. In Lemma 2.5 we show that for all (Y_P, α_P) , the set $Y_P(\mathbb{R})$ contains a point at which α_P has trivial evaluation. In (2.3) we give explicit equations for the quaternion algebra in Theorem 3. In Section 3 we review the construction of the intermediate set $Y_P(\mathbb{A}_{\mathbb{Q}})^{\alpha_P}$, discuss the evaluation of α_P , and give equivalent geometric criteria for this evaluation to be trivial. We prove Theorem 1 and Corollary 2 in Section 3.3. We prove that a random P gives an α_P with constant trivial evaluation at all finite primes in Section 3.2 and discuss the explicit example of Theorem 3 in Section 3.4. Finally, we end with a discussion of the derived equivalent threefolds X_P in Section 4. We construct X_P over \mathbb{Q} and show that kernel of Hosono and Takagi’s equivalence is defined over \mathbb{Q} . We show in Lemma 4.3 that if the linear system P contains a singular quadric defined over \mathbb{Q} , then X_P does not satisfy weak approximation.

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1.3. Code Availability. The code associated to this paper can be found at the github repository [HHLV20] and on the supplementary files with the arXiv preprint of this paper.

2. THE CALABI–YAU THREEFOLD Y AND BRAUER CLASS α

We begin by reviewing the construction of the Calabi–Yau threefold Y_P associated to a regular 4-dimensional linear system P of quadrics in \mathbb{P}^4 . This construction proceeds through the auxiliary incidence variety Z_P , which gives the nontrivial 2-torsion Brauer class $\alpha_P \in \text{Br}(Y)[2]$. We then give a quaternion algebra over the function field of Y_P that represents it.

When the linear system P is unambiguous, we write $Y = Y_P$, $Z = Z_P$ and $\alpha = \alpha_P$ for brevity.

2.1. The variety Y . We now give a detailed description of the Calabi–Yau threefolds Y_P on which we will obstruct weak approximation. These varieties are constructed from the geometry of linear spaces in a family of quadrics; see [Rei72, Chapter 1] or [Har92, Lecture 22] for an introduction to this topic.

Let K be a field not of characteristic 2. Recall that Y_P is attached to a choice of linear system P . Explicitly, let Q_0, \dots, Q_4 be five independent quadrics in \mathbb{P}_K^4 . These define a linear system $P \simeq \mathbb{P}_{(t_0, \dots, t_4)}^4$ by

$$P := |t_0Q_0 + t_1Q_1 + t_2Q_2 + t_3Q_3 + t_4Q_4| \subseteq \mathbb{P}H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)) \simeq \mathbb{P}_K^{14}.$$

Equivalently, by the correspondence between symmetric bilinear forms and quadratic forms, we may view P as defining a linear system of $(1, 1)$ -divisors in $\mathbb{P}^4 \times \mathbb{P}^4$. We will always impose the following genericity assumption on the linear system.

Definition 2.1. A linear system $P \simeq \mathbb{P}_K^4$ of $(1,1)$ -divisors in $\mathbb{P}^4 \times \mathbb{P}^4$ is **regular** if their intersection is smooth and does not intersect the diagonal in $\mathbb{P}^4 \times \mathbb{P}^4$.

Remark 2.2. We adopt the above perspective of viewing P as a linear system of $(1,1)$ -divisors in $\mathbb{P}^4 \times \mathbb{P}^4$ in anticipation of the construction of the Calabi-Yau threefold X in Section 4.1. Note that there are two distinct copies of \mathbb{P}^4 , with one being dual to the other. This projective duality arises in the study of the derived equivalence between X and Y .

By [HT14, Proposition 2.1], this may be equivalently phrased purely in terms of the incidence geometry of lines in the quadrics: a linear system is regular if and only if it is base-point free, and any (geometric) line $\ell \subseteq \text{Sing}(Q)$ for some Q in $P(\overline{K})$ is not contained in a sublinear system of dimension at least 2 in P . In particular, regularity guarantees that the singular quadrics in P have rank 3 or 4.

Suppose that P is a regular linear system. Write $\mathcal{Q} \xrightarrow{f} P$ for the universal quadric over P . Over a point $t = (t_0 : t_1 : t_2 : t_3 : t_4) \in P$, we write Q_t for the corresponding quadric in \mathbb{P}^4 , with equation $\sum_{i=0}^4 t_i Q_i = 0$. Let Z_P be the relative Fano scheme of 2-planes contained in the fibers of f

$$Z_P := F(2, \mathcal{Q}/P) = \left\{ (t, [\Lambda]) \in P \times \mathbb{G}(2, \mathbb{P}^4) : \Lambda \subset Q_t, \text{ i.e., } \sum_i t_i Q_i|_{\Lambda} = 0 \right\}.$$

Then Z_P admits two natural projections

$$(2.1) \quad Z_P \xrightarrow{\pi_1} P, \quad Z_P \xrightarrow{\pi_2} \mathbb{G}(2, \mathbb{P}^4).$$

Since any 3-dimensional quadric hypersurface containing a 2-dimensional linear space is necessarily singular, the image of the first projection of Z_P to P is the locus $H \subset P$ of singular quadrics, that is, cones over quadric surfaces in \mathbb{P}^3 . As we observed in the introduction, lines on quadric surfaces come in two rulings, so the map $Z_P \rightarrow H$ does not have connected fibers. The threefold Y_P of interest can be seen in the nontrivial Stein factorization:

$$\pi_1 : Z_P \xrightarrow{\text{conic bundle}} Y_P \xrightarrow{2:1} H \subseteq P.$$

Hosono and Takagi show that if P is regular, then Y_P is smooth [HT14, Proposition 3.11]. The conic bundle Z_P gives us our 2-torsion Brauer class $\alpha_P \in \text{Br}(Y)[2]$, and $(\alpha_P)_{\overline{K}}$ is nontrivial by [HT16, Proposition 3.2.1].

2.2. Rational points on Y . For any field L/K , an L -rational point on Y_P corresponds to a singular quadric from P along with a choice of ruling both defined over L . Given a rank 4 (singular) quadric Q in $\mathbb{P}^4(L)$, we write \bar{Q} for the smooth quadric in \mathbb{P}^3 over which Q is a cone. By regularity of P , the points in Y_P are over quadrics in $\mathbb{P}^4(L)$ of ranks only 3 or 4, hence the finite morphism $Y \rightarrow H$ is ramified over the locus of quadrics of rank 3. The rulings of 2-planes on Q (equivalently, lines on \bar{Q}) are defined over $L(\sqrt{\text{disc}(\bar{Q})})$ [Wan18, Section 2.2, footnote on p. 366]. Therefore, an

L -point in the (open) rank 4 locus of H lifts to an L -point of Y_P if and only if the corresponding quadric in \mathbb{P}^3 has square discriminant. The one ruling of 2-planes on a quadric of rank 3 is defined over L , and thus any L -point in the rank 3 locus of H lifts to an L -point of Y_P .

Over the generic point of H , the corresponding quadric $\mathcal{Q}_{\mathbb{Q}(H)}$ has rank exactly 4; as such it is the cone over a smooth rank 4 quadric $\bar{\mathcal{Q}}_{\mathbb{Q}(H)}$ in \mathbb{P}^3 . Therefore the function field $\mathbb{Q}(Y)$ is the discriminant quadratic extension

$$(2.2) \quad \mathbb{Q}(Y) = \mathbb{Q}(H) \left(\sqrt{\text{disc}(\bar{\mathcal{Q}}_{\mathbb{Q}(H)})} \right)$$

of $\mathbb{Q}(H)$ over which the rulings of $\bar{\mathcal{Q}}_{\mathbb{Q}(H)}$ are defined. We make this explicit in Lemma 2.4. Moreover, Y is determined by its function field: since Y is a Stein factorization it is normal and therefore by [Gro63, I.1 §10 Proposition 10.1] it is the normalization of H inside the field extension $\mathbb{Q}(Y)$.

Definition 2.3. Given a quadratic form Q , we call $B_Q(x, y) = Q(x + y) - Q(x) - Q(y)$ the associated symmetric bilinear form. Write \mathcal{B} for the universal symmetric bilinear form corresponding to the universal quadric \mathcal{Q} .

Note that $2Q(x) = B_Q(x, x)$, and hence if $\text{char}(K) \neq 2$, the quadratic form $Q(x)$ can be recovered from the bilinear form $B_Q(x, x)$.

Given a symmetric 5×5 matrix B and subsets $I, J \subseteq \{0, \dots, 4\}$, we define $B_{I,J}$ to be the minor of B obtained by deleting the i th rows and j th columns for $i \in I, j \in J$ (i.e. $B_{I,J}$ is the determinant of the resulting submatrix).

Lemma 2.4. *For any index i , if $\mathcal{B}_{i,i}$ is not identically 0 on H , then*

$$\mathbb{Q}(Y) = \mathbb{Q}(H) \left(\sqrt{(\mathcal{B}_{\mathbb{Q}(H)})_{i,i}} \right).$$

Proof. Fix an i such that $0 \leq i \leq 4$. On the open locus in H where $\mathcal{B}_{i,i} \neq 0$, the matrix \mathcal{B} has rank exactly 4, and therefore has a 1-dimensional kernel V not contained in the hyperplane $x_i = 0$. Let W be any codimension 1 subspace that does not contain V . The restriction $\mathcal{B}|_W$ is therefore a nondegenerate bilinear form. Up to isomorphism, this bilinear form is independent of the choice of W . Therefore, for any W , the discriminant of $\mathcal{B}|_W$ is equal (modulo squares) to $\text{disc}(\bar{\mathcal{Q}}_{\mathbb{Q}(H)})$. The result follows by applying this to $W = (x_i = 0)$. \square

Geometrically, this argument shows that if $\mathcal{B}_{i,i}$ and $\mathcal{B}_{j,j}$ are nonzero, then the intersections of $\mathcal{Q}_{\mathbb{Q}(H)}$ with the hyperplanes $x_i = 0$ and $x_j = 0$, respectively, are isomorphic smooth rank 4 quadrics in \mathbb{P}^3 over which $\mathcal{Q}_{\mathbb{Q}(H)}$ is a cone.

The other projection $\pi_2 : Z_P \rightarrow \mathbb{G}(2, \mathbb{P}^4)$ is also illuminating when studying points on Y_P . Containing a fixed 2-plane $\Lambda_0 \subset \mathbb{P}^4$ is a linear condition on the quadrics in P , so the fibers of π_2 over each point $[\Lambda_0] \in \mathbb{G}(2, \mathbb{P}^4)$ are projective spaces. Let $\Sigma_P \subset \mathbb{G}(2, \mathbb{P}^4)$ denote the image $\pi_2(Z_P)$ and let \mathcal{S} be the tautological subbundle on $\mathbb{G}(2, \mathbb{P}^4)$. Then the subvariety Σ_P can be viewed as a degeneracy locus, as we now explain. The quadrics

Q_i give rise to sections s_i of the rank 6 vector bundle $\mathrm{Sym}^2 \mathcal{S}^\vee$ on $\mathbb{G}(2, \mathbb{P}^4)$ by restriction of Q_i to each $[\Lambda] \in \mathbb{G}(2, \mathbb{P}^4)$. If $[\Lambda] \in \mathbb{G}(2, \mathbb{P}^4)$ is in Σ_P , then there exists some point $(t_0 : \cdots : t_4) \in P$ such that

$$\sum_{i=0}^4 t_i Q_i|_\Lambda = 0.$$

Therefore, we may equivalently define Σ_P as the locus in $\mathbb{G}(2, \mathbb{P}^4)$ where s_0, \dots, s_4 fail to be linearly independent, and hence it is a degeneracy locus. By [EH16, Theorem 5.3(b)], the class of Σ_P in $\mathrm{CH}^*(\mathbb{G}(2, \mathbb{P}^4))$ is $c_2(\mathrm{Sym}^2 \mathcal{S}^\vee)$.

Lemma 2.5. *Let P be any 4-plane in $\mathbb{P}_{\mathbb{Q}}^{14}$ and let Y_P and Z_P be as defined in Section 2.1. Then $Z_P(\mathbb{R}) \neq \emptyset$; in particular, $Y_P(\mathbb{R}) \neq \emptyset$.*

Proof. We prove this lemma using topological arguments involving characteristic classes of real vector bundles; see [Hat17, Section 3.3] for more details. Given a choice of five quadrics Q_0, \dots, Q_4 (defined over \mathbb{Q}) that span P , we may consider the construction described above over \mathbb{R} . Analogous to the second Chern class, the class of $(\Sigma_P)_{\mathbb{R}} \subset \mathbb{G}(2, \mathbb{P}_{\mathbb{R}}^4) =: \mathbb{G}_{\mathbb{R}}$ in the cohomology $H^*(\mathbb{G}_{\mathbb{R}}, \mathbb{Z}/2\mathbb{Z})$ is given by the second Stiefel-Whitney class $w_2(\mathrm{Sym}^2 \mathcal{S}_{\mathbb{R}}^\vee)$ [Hat17, Proposition 3.21].

A standard computation using the splitting principle shows

$$w_2(\mathrm{Sym}^2 \mathcal{S}_{\mathbb{R}}^\vee) = w_1(\mathcal{S}_{\mathbb{R}}^\vee)^2 + w_2(\mathcal{S}_{\mathbb{R}}^\vee).$$

Finally, we observe that $H^2(\mathbb{G}_{\mathbb{R}}, \mathbb{Z}/2\mathbb{Z})$ is freely generated over $\mathbb{Z}/2\mathbb{Z}$ by $w_1(\mathcal{S}_{\mathbb{R}}^\vee)^2$ and $w_2(\mathcal{S}_{\mathbb{R}}^\vee)$ [MS74, Problem 6-B, Theorem 7.1]. Therefore, $[(\Sigma_P)_{\mathbb{R}}]$ is a nontrivial class in $H^2(\mathbb{G}_{\mathbb{R}}, \mathbb{Z}/2\mathbb{Z})$, and hence $(\Sigma_P)_{\mathbb{R}}(\mathbb{R})$ cannot be empty. Since the fibers of π_2 over each real point of $(\Sigma_P)_{\mathbb{R}}$ are projective spaces, this implies that $Y(\mathbb{R})$ is nonempty as well. \square

2.3. The Brauer class α . In this section, we begin by explicitly describing a quaternion algebra over the function field of $Y = Y_P$ that is the image of $\alpha = \alpha_P$ in $\mathrm{Br}(\mathbb{Q}(Y_P))$. Recall that the conic bundle $Z = Z_P$ over Y provides the nontrivial Brauer class $\alpha \in \mathrm{Br}(Y)$. We continue to fix a field K not of characteristic 2.

Lemma 2.6. *Let $Q \subseteq \mathbb{P}_K^3$ be a smooth quadric with square discriminant (i.e., whose rulings are defined over K). Let $[\Gamma]$ in $(\mathbb{P}^3)^\vee(K)$ be a hyperplane transverse to Q . Then the variety of lines on Q in a given ruling is isomorphic to the conic $\Gamma \cap Q$.*

Proof. Each line on Q intersects $\Gamma \cap Q$ in a unique point. Conversely, the tangent hyperplane section of Q at any point of $\Gamma \cap Q$ is the union of a unique line from each ruling. \square

In particular, applying Lemma 2.6 to the generic quadric, we have that the generic fiber $Z|_{\mathbb{Q}(Y)}$ of the conic bundle defining α is isomorphic to a transverse hyperplane section of $\bar{Q}_{\mathbb{Q}(Y)}$. We now make this more explicit in

the generic situation, which will include our example in Theorem 3. Recall that we write $\mathcal{B} = \mathcal{B}_P$ for the universal symmetric bilinear form over P .

Lemma 2.7. *Suppose that the principal minors $\mathcal{B}_{1234,1234}$, $\mathcal{B}_{234,234}$, $\mathcal{B}_{34,34}$, and $\mathcal{B}_{4,4}$ are not identically zero on H . Then the quaternion algebra*

$$\left(\frac{\mathcal{B}_{234,234}}{(\mathcal{B}_{1234,1234})^2}, \frac{\mathcal{B}_{34,34}}{\mathcal{B}_{234,234}\mathcal{B}_{1234,1234}} \right) \in \text{Br}(\mathbb{Q}(Y)) [2]$$

represents the restriction of α .

Proof. To show this result, we will use our hypotheses on the minors of \mathcal{B} to make a nice choice of basis with which to diagonalize $\mathcal{Q}_{\mathbb{Q}(H)}$, yielding the quaternion algebra above.

Recall that \mathcal{B} has rank 4 over $\mathbb{Q}(H)$. Because of our hypotheses on the minors of \mathcal{B} , the matrix in the linear system below is invertible, allowing us to use Cramer’s rule to solve the system. The vector v satisfying the linear system is in the kernel of the generic bilinear form $\mathcal{B}|_{\mathbb{Q}(H)}$. Applying Cramer’s rule gives us the entries of v shown on the right.

$$\begin{pmatrix} & & & & 0 \\ & & & & 0 \\ & & & & 0 \\ & & & & 0 \\ & & & & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} v \\ * \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ * \\ \mathcal{B}_{4,4} \end{pmatrix}, \quad v = \begin{pmatrix} \mathcal{B}_{4,0} \\ -\mathcal{B}_{4,1} \\ \mathcal{B}_{4,2} \\ -\mathcal{B}_{4,3} \\ \mathcal{B}_{4,4} \end{pmatrix}.$$

By hypothesis, the last entry of v is nonzero, and so $\mathcal{B}_{\mathbb{Q}(H)}$ is non-degenerate when restricted to the subspace generated by e_0, e_1, e_2, e_3 . We may therefore carry out Gram–Schmidt orthonormalization on this subspace to diagonalize $\mathcal{Q}_{\mathbb{Q}(H)}$. Since Gram–Schmidt orthonormalization only involves adding multiples of the previous columns, the diagonalized matrix must have the same leading principal minors: $\mathcal{B}_{1234,1234}, \mathcal{B}_{234,234}/\mathcal{B}_{1234,1234}, \mathcal{B}_{34,34}/\mathcal{B}_{234,234}, \mathcal{B}_{4,4}/\mathcal{B}_{34,34}, 0$.

Over $\mathbb{Q}(Y)$, the conic $Z|_{\mathbb{Q}(Y)}$ is isomorphic to

$$(\mathcal{B}_{1234,1234}) x_0^2 + \left(\frac{\mathcal{B}_{234,234}}{\mathcal{B}_{1234,1234}} \right) x_1^2 + \left(\frac{\mathcal{B}_{34,34}}{\mathcal{B}_{234,234}} \right) x_2^2 = 0,$$

which corresponds (see for instance [KKS11, Section 8.2]) to the desired quaternion algebra. \square

For the choice of five quadrics in Theorem 3, we have that α is therefore represented by the quaternion algebra $(\frac{a_1}{a_2}, \frac{b_1}{b_2})$, where

$$\begin{aligned}
 (2.3) \quad & a_1 = -t_0^2 - 2t_0t_1 + 7t_1^2 + 8t_1t_2 + 48t_1t_3 + 16t_2t_3 + 64t_3^2 + 8t_1t_4 + 16t_3t_4, \\
 & a_2 = 4t_1^2 + 8t_1t_2 + 4t_2^2 + 32t_1t_3 + 32t_2t_3 + 64t_3^2 + 8t_1t_4 + 8t_2t_4 \\
 & \quad + 32t_3t_4 + 4t_4^2, \\
 & b_1 = -10t_0^2t_1 - 14t_0t_1^2 + 38t_1^3 + 10t_0t_1t_2 + 36t_1^2t_2 + 4t_0t_2^2 - 18t_1t_2^2 - 6t_0^2t_3 \\
 & \quad - 12t_0t_1t_3 + 44t_1^2t_3 + 96t_1t_2t_3 - 40t_2^2t_3 + 928t_1t_3^2 + 96t_2t_3^2 + 384t_3^3 \\
 & \quad + 20t_0t_1t_4 + 62t_1^2t_4 + 14t_0t_2t_4 - 30t_1t_2t_4 - 14t_2^2t_4 + 112t_1t_3t_4 \\
 & \quad - 80t_2t_3t_4 + 96t_3^2t_4 + 6t_0t_4^2 - 28t_1t_4^2 - 30t_2t_4^2 - 80t_3t_4^2 - 18t_4^3, \\
 & b_2 = -2t_0^2t_1 - 4t_0t_1^2 + 14t_1^3 - 2t_0^2t_2 - 4t_0t_1t_2 + 30t_1^2t_2 + 16t_1t_2^2 - 8t_0^2t_3 \\
 & \quad - 16t_0t_1t_3 + 152t_1^2t_3 + 192t_1t_2t_3 + 32t_2^2t_3 + 512t_1t_3^2 + 256t_2t_3^2 + 512t_3^3 \\
 & \quad - 2t_0^2t_4 - 4t_0t_1t_4 + 30t_1^2t_4 + 32t_1t_2t_4 + 192t_1t_3t_4 + 64t_2t_3t_4 \\
 & \quad + 256t_3^2t_4 + 16t_1t_4^2 + 32t_3t_4^2.
 \end{aligned}$$

We conclude this section with the computation of the Brauer group of Y .

Proposition 2.8. *Assume that P is regular and write $Y = Y_P$. Then we have that*

- (i) $\text{Pic}(Y_{\bar{\mathbb{Q}}}) \simeq \mathbb{Z}$.
- (ii) $\text{Br}_1(Y)/\text{Br}(\mathbb{Q}) \simeq H^1(\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), \text{Pic}(Y_{\bar{\mathbb{Q}}})) = 0$.
- (iii) $\text{Br}(Y_{\bar{\mathbb{Q}}}) = \langle \alpha_{\bar{\mathbb{Q}}} \rangle \simeq \mathbb{Z}/2\mathbb{Z}$.

Proof.

- (i) By [HT13, Proposition 4.2.4], Y/\mathbb{C} has Picard rank 1 and $\text{Pic}^0(Y_{\mathbb{C}})$ is trivial. The natural map $H^1(\mathcal{O}_{Y_{\bar{\mathbb{Q}}}}) \otimes \mathbb{C} \rightarrow H^1(\mathcal{O}_{Y_{\mathbb{C}}})$ is an isomorphism by cohomology and base change, hence $\text{Pic}^0(Y_{\bar{\mathbb{Q}}})$ is also trivial. Thus the statement follows from the fact that the Néron-Severi group of a smooth proper variety over an algebraically closed field remains the same after base change to another algebraically closed field [MP12, Proposition 3.1].
- (ii) The identification $\text{Br}_1(Y)/\text{Br}(\mathbb{Q}) \simeq H^1(\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), \text{Pic}(Y_{\bar{\mathbb{Q}}}))$ follows from the low degree exact sequence of the Hochschild-Serre spectral sequence [Poo17, Corollary 6.7.8] since $H^3(\mathbb{Q}, \mathbb{G}_m) = 0$ [Poo17, Remark 6.7.9]. Since $\text{Pic}(Y_{\bar{\mathbb{Q}}}) = \mathbb{Z}$ and $H^1(\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), \mathbb{Z}) = 0$, the result follows.
- (iii) The natural map of Brauer groups $\text{Br}(Y_{\bar{\mathbb{Q}}}) \rightarrow \text{Br}(Y_{\mathbb{C}})$ is an isomorphism by [CTS, Propositions 5.2.2 and 5.2.3]. By Proposition A.1 in Appendix A, $\text{Br}(Y/\mathbb{C}) \simeq \mathbb{Z}/2\mathbb{Z}$ and is generated by the base change of α to \mathbb{C} , so the result follows. \square

3. AN OBSTRUCTION TO WEAK APPROXIMATION

3.1. Background. Recall that a smooth projective variety Y over \mathbb{Q} satisfies **weak approximation** if the \mathbb{Q} -rational points are dense inside of the adelic points in the adelic topology. We recall how elements of the Brauer group of Y may obstruct weak approximation.

Let $L \supset \mathbb{Q}$ be a field. A point $y \in Y(L)$ gives a map $y: \operatorname{Spec} L \rightarrow Y$. By functoriality of Brauer groups, we can pull back an Azumaya algebra \mathcal{A} under y to obtain

$$\mathcal{A}(y) := y^* \mathcal{A} \in \operatorname{Br}(\operatorname{Spec} L) = \operatorname{Br}(L).$$

We call this pullback **evaluation of \mathcal{A} at y** . When $L = \mathbb{Q}_v$ is the completion of \mathbb{Q} at a place v , we may further compose with the invariant map

$$\operatorname{inv}_v: \operatorname{Br}(\mathbb{Q}_v) \rightarrow \mathbb{Q}/\mathbb{Z}.$$

If v is a finite place, then inv_v is an isomorphism onto \mathbb{Q}/\mathbb{Z} . For ∞ the real infinite place of \mathbb{Q} , we have $\operatorname{inv}_\infty: \operatorname{Br}(\mathbb{R}) \xrightarrow{\sim} (\frac{1}{2}\mathbb{Z})/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$. Recall that the fundamental exact sequence of global class field theory [Mil03, Chapter VIII, Theorem 4.2]

$$0 \rightarrow \operatorname{Br}(\mathbb{Q}) \rightarrow \bigoplus_v \operatorname{Br}(\mathbb{Q}_v) \xrightarrow{\sum \operatorname{inv}_v} \mathbb{Q}/\mathbb{Z} \rightarrow 0$$

says that the tuples of local Brauer classes that come from $\operatorname{Br}(\mathbb{Q})$ are exactly those whose invariants sum to 0. For any subset $S \subseteq \operatorname{Br}(Y)$ we can produce an intermediate set of adelic points containing the closure of the \mathbb{Q} -rational points

$$(3.1) \quad \overline{Y(\mathbb{Q})} \subseteq Y(\mathbb{A}_{\mathbb{Q}})^S \subseteq Y(\mathbb{A}_{\mathbb{Q}}),$$

defined by

$$Y(\mathbb{A}_{\mathbb{Q}})^S := \left\{ (P_v) \in Y(\mathbb{A}_{\mathbb{Q}}) : \sum_v \operatorname{inv}_v \mathcal{A}(P_v) = 0, \text{ for all } \mathcal{A} \in S \right\}.$$

For more details on these inclusions, which arise from class field theory, see [Sko01].

We will produce obstructions to weak approximation by means of the following standard argument.

Lemma 3.1. *Let $\mathcal{A} \in \operatorname{Br}(Y)[2]$ and let v be a place of \mathbb{Q} . Suppose that $Y(\mathbb{A}_{\mathbb{Q}}) \neq \emptyset$ and that there exist two \mathbb{Q}_v points $Q_v, Q'_v \in Y(\mathbb{Q}_v)$ for which*

$$\operatorname{inv}_v(\mathcal{A}(Q_v)) = 0 \in \operatorname{Br}(\mathbb{Q}_v) \quad \text{and} \quad \operatorname{inv}_v(\mathcal{A}(Q'_v)) = \frac{1}{2} \in \operatorname{Br}(\mathbb{Q}_v).$$

Then $Y(\mathbb{A}_{\mathbb{Q}})^{\mathcal{A}}$ is not equal to $Y(\mathbb{A}_{\mathbb{Q}})$. In particular, \mathcal{A} obstructs weak approximation on Y .

Proof. Since $Y(\mathbb{A}_{\mathbb{Q}}) \neq \emptyset$, there exist local points $P_{\ell} \in Y(\mathbb{Q}_{\ell})$ for all places $\ell \neq v$. Since $\mathcal{A} \in \text{Br}(Y)[2]$,

$$\sum_{\ell \neq v} \text{inv}_{\ell}(\mathcal{A}(P_{\ell})) \in \frac{1}{2}\mathbb{Z}/\mathbb{Z}.$$

We use these local points to define an adelic point $(P_{\ell}) \in Y(\mathbb{A}_{\mathbb{Q}})$ where $P_v \in Y(\mathbb{Q}_v)$ is defined by

$$P_v := \begin{cases} Q'_v & \text{if } \sum_{\ell \neq v} \text{inv}_{\ell}(\mathcal{A}(P_{\ell})) = 0, \\ Q_v & \text{if } \sum_{\ell \neq v} \text{inv}_{\ell}(\mathcal{A}(P_{\ell})) = \frac{1}{2}. \end{cases}$$

By construction, $(P_{\ell}) \in Y(\mathbb{A}_{\mathbb{Q}}) \setminus Y(\mathbb{A}_{\mathbb{Q}})^{\mathcal{A}}$, so $\overline{Y(\mathbb{Q})} \subset Y(\mathbb{A}_{\mathbb{Q}})^{\mathcal{A}} \subsetneq Y(\mathbb{A}_{\mathbb{Q}})$, and Y fails to satisfy weak approximation. \square

3.2. Evaluation of the Brauer class α . We now specialize to the setup of our Calabi–Yau threefold $Y = Y_P$ and 2-torsion Brauer class $\alpha = \alpha_P \in \text{Br}(Y)[2]$ that are associated to a choice of linear system P of five quadrics in $\mathbb{P}_{\mathbb{Q}}^4$. We will always assume that we are working with a regular P so that Y_P is smooth.

For any extension $L \supset \mathbb{Q}$, we begin by geometrically describing the evaluation of α at a point $y \in Y(L)$. By slight abuse of notation, we will write Q_y for the quadric corresponding to the image of y in H . Since α is the Brauer class associated to an étale \mathbb{P}^1 -bundle Z , the evaluation $\alpha(y)$ is the class of the conic $Z|_y$ over $\text{Spec } L$ in $\text{Br}(L)[2]$. Therefore

$$\begin{aligned} \alpha(y) = 0 \in \text{Br}(L)[2] &\Leftrightarrow Z|_y \text{ has an } L\text{-point} \\ &\Leftrightarrow Q_y \text{ contains an } L\text{-rational plane } \Lambda \simeq \mathbb{P}_L^2 \text{ in ruling } y. \end{aligned}$$

Thus for $y \in Y(\mathbb{Q}_v)$,

$$(3.2) \quad \text{inv}_v(\alpha(y)) = \begin{cases} 0 & \text{if } Q_y \text{ contains a } \mathbb{Q}_v\text{-rational 2-plane in ruling } y, \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

We give an equivalent criterion for $\text{inv}_v(\alpha(y))$ to be trivial.

Lemma 3.2. *Let $y \in Y(L)$ be such that Q_y is of rank at least 3. Then Q_y contains an L -rational 2-plane in ruling y if and only if Q_y has a smooth L -point.*

Proof. To see the forward direction, we note that if Q_y contains an L -rational plane $\Lambda \simeq \mathbb{P}_L^2$, then as the dimension of the singular locus of Q_y is at most 1, Λ is generically contained in the smooth locus of Q_y , which hence has a smooth L -point.

In the other direction, first assume that Q_y has rank 3. Then it is a cone (with “vertex” a line ℓ defined over L) over a smooth conic in \mathbb{P}_L^2 . If Q_y has a smooth L -point, then this conic has an L -point, and hence the span of that point and the line ℓ is an L -rational plane Λ in Q_y (which has only one ruling).

Similarly, if Q_y has rank 4, then it is a cone (with vertex a point $x \in \mathbb{P}^4(L)$) over a smooth quadric \bar{Q} in \mathbb{P}_L^3 . If Q_y has a smooth L -point, then this quadric \bar{Q} has a L -point p . The tangent plane section $T_p\bar{Q} \cap \bar{Q}$ consists of two lines, one from each ruling. As y was an L -point of Y , these lines are individually defined over L , and hence the span of the one in ruling y and x gives the desired plane Λ in Q_y in ruling y . \square

Say that a pair (Y_P, α_P) has **trivial evaluation at all finite places** if

(\dagger) for all p and all $y \in Y_P(\mathbb{Q}_p)$, we have $\alpha_P(y) = 0$.

We next show that (\dagger) is the “expected” behavior as we vary the defining linear system P . We will guarantee this, using Lemma 3.2 and Hensel’s Lemma, by spreading out our family of quadrics to $\text{Spec } \mathbb{Z}$ and showing that over \mathbb{F}_p , all quadrics have smooth \mathbb{F}_p -points. Extra care is necessary when $p = 2$ since quadrics and bilinear forms do not coincide in characteristic 2.

Let \mathcal{B} be the universal 5×5 symmetric matrix with even diagonal entries over $\mathbb{A}_{\mathbb{Z}}^{15}$. We say an ideal I of a ring R is **saturated with respect to** $s \in R$ if for all $n \geq 1$, the equation $s^n x \in I$ implies $x \in I$. Given an ideal I of a ring R we can produce a new ideal I' that is saturated with respect to s in R through the process of **saturation**: $I' := \{x \in R : s^n x \in I \text{ for all } n \geq 1\}$. We consider the ideal of 3×3 minors of \mathcal{B} and saturate with respect to the element $s = 2$. Define \mathcal{I}_3 to be the resulting ideal of 3×3 minors of \mathcal{B} that is *saturated with respect to* 2. Since this is independent of scaling, it descends to the projective space \mathbb{P}^{14} of quadric hypersurfaces in \mathbb{P}^4 . Let $\mathcal{V}_3 = V(\mathcal{I}_3)$ be the associated subscheme of $\mathbb{P}_{\mathbb{Z}}^{14}$.

We can also describe the process of saturation geometrically. Let $\mathcal{V}_3^\circ \subset \mathbb{P}_{\mathbb{Z}[1/2]}^{14}$ be the subscheme of quadrics of rank at most 2; the ideal of \mathcal{V}_3° is given by the 3×3 minors of \mathcal{B} . Then the saturation of this ideal at 2 is the ideal of the closure $\mathcal{V}_3 = \overline{\mathcal{V}_3^\circ}$ in $\mathbb{P}_{\mathbb{Z}}^{14}$.

Lemma 3.3. *Suppose that $Q \subseteq \mathbb{P}_{\mathbb{F}_q}^4$ is a quadric whose associated bilinear form B_Q does not lie in $\mathcal{V}_3(\mathbb{F}_q)$. Then Q has a smooth \mathbb{F}_q -point.*

Proof. If q is not a power of 2, then we may assume that $Q = ax_0^2 + bx_1^2 + cx_2^2 + dx_3^2 + ex_4^2$ is diagonal. Since $[B_Q] \notin \mathcal{V}_3(\mathbb{F}_q)$, we may assume that a , b , and c are nonzero. The 2-plane section $x_3 = x_4 = 0$ of Q is therefore a smooth conic, which has smooth \mathbb{F}_q -points by the Chevalley–Warning Theorem. These points are smooth points of Q .

Now, we let $q = 2^r$. By [Alb38, Theorem 16], every quadric in $\mathbb{P}_{\mathbb{F}_q}^4$ is equivalent to one of the form

$$Q_0 = ax_0^2 + bx_1^2 + cx_2^2 + dx_3^2 + ex_4^2 + fx_0x_1 + gx_2x_3.$$

This defines a map from the parameter space \mathbb{A}^{15} of quadrics (or symmetric bilinear forms with even diagonal entries) to \mathbb{A}^7 with variables a, b, c, d, e, f, g . Let I denote the image of the ideal \mathcal{I}_3 under this specialization. (Since pullback and Zariski closure do not commute, this is *not* the same as the ideal of 3×3 minors of B_{Q_0} saturated at 2.)

Write $\bar{I} = I \bmod 2$. Then one can compute that explicitly that \bar{I} has generators

$$\bar{I} = (f^2c, f^2d, f^2e, f^2g, g^2a, g^2b, g^2e, g^2f).$$

It therefore suffices to show that if $a, b, \dots, g \in \mathbb{F}_q$ is a specialization such that one of the above generators of \bar{I} is nonzero in \mathbb{F}_q , then that associated quadric has a smooth \mathbb{F}_q -point. We consider the generators in turn below.

- Case 1. Generators guaranteeing both $f, g \in \mathbb{F}_q^\times$ (i.e., generators f^2g, g^2f). In this case, the hyperplane section $x_4 = 0$ is a smooth quadric surface, and hence by Chevalley-Warning, has smooth \mathbb{F}_q -points.
- Case 2. Generators guaranteeing both $e, f \in \mathbb{F}_q^\times$ or both $e, g \in \mathbb{F}_q^\times$ (i.e., generators f^2e, g^2e). In this case, the 2-plane section $x_2 = x_3 = 0$ (resp. $x_0 = x_1 = 0$) is a smooth conic, and hence by Chevalley-Warning, has smooth \mathbb{F}_q -points.
- Case 3. Generators guaranteeing both f and c or $d \in \mathbb{F}_q^\times$ or both g and a or $b \in \mathbb{F}_q^\times$ (i.e., generators f^2c, f^2d, g^2a, g^2b). In this case, the 2-plane section $x_3 = x_4 = 0$ or $x_2 = x_4 = 0$ (resp. $x_1 = x_4 = 0$ or $x_0 = x_4 = 0$) is a smooth conic, and hence by Chevalley-Warning, has smooth \mathbb{F}_q -points. \square

The key technical ingredient in understanding the density of planes satisfying (\dagger) is the Ekedahl sieve. Recall that the height of an integral point on \mathbb{A}^n is the maximum of the absolute value of the coordinates:

$$h((p_1, \dots, p_n)) = \max_i |p_i|.$$

For compactness of notation we will write $\mathbb{A}^n(\mathbb{Z})_{\leq N}$ for the set of integral points on \mathbb{A}^n of height at most N .

Lemma 3.4 (The Ekedahl sieve). *Let $Z \subset \mathbb{A}^n$ be a subscheme of codimension at least 2. Then we have*

$$\lim_{M \rightarrow \infty} \lim_{N \rightarrow \infty} \frac{\#\{P \in \mathbb{A}^n(\mathbb{Z})_{\leq N} : \exists p > M \text{ s.t. } (P \bmod p) \in Z(\mathbb{F}_p)\}}{\#\mathbb{A}^n(\mathbb{Z})_{\leq N}} = 0.$$

Proof. Since any subscheme of codimension at least 2 is contained in a complete intersection subscheme of codimension 2, the result follows from [Poo99, Lemma 21]. See also [Eke91, Theorem 1.2]. \square

This result allows one to compute the global density of points that satisfy “codimension at least 2” local conditions at all primes as a product of local densities.

As above, let $Z \subset \mathbb{A}^n$ be a subscheme of codimension at least 2. Let $S_p \subset Z(\mathbb{F}_p)$ be a subset of the \mathbb{F}_p points of Z . Write $S = (S_p)_p$ prime. Define

$$d(!S) = \lim_{N \rightarrow \infty} \frac{\#\{P \in \mathbb{A}^n(\mathbb{Z})_{\leq N} : \forall p \ (P \bmod p) \notin S_p\}}{\#\mathbb{A}^n(\mathbb{Z})_{\leq N}}.$$

Proposition 3.5. *With the notation and assumptions as above,*

$$d(!S) = \prod_p \left(1 - \frac{\#S_p}{\#\mathbb{A}^n(\mathbb{F}_p)} \right).$$

Remark 3.6. The Lang-Weil estimates guarantee that the infinite product on the right converges.

Proof. Let M be an integer. We have bounds

$$\begin{aligned} & \frac{\#\{P \in \mathbb{A}^n(\mathbb{Z})_{\leq N} : \forall p \leq M (P \bmod p) \notin S_p\}}{\#\mathbb{A}^n(\mathbb{Z})_{\leq N}} \\ & - \frac{\#\{P \in \mathbb{A}^n(\mathbb{Z})_{\leq N} : \exists p > M \text{ s.t. } (P \bmod p) \in S_p\}}{\#\mathbb{A}^n(\mathbb{Z})_{\leq N}} \\ & \leq \frac{\#\{P \in \mathbb{A}^n(\mathbb{Z})_{\leq N} : \forall p (P \bmod p) \notin S_p\}}{\#\mathbb{A}^n(\mathbb{Z})_{\leq N}} \\ & \leq \frac{\#\{P \in \mathbb{A}^n(\mathbb{Z})_{\leq N} : \forall p \leq M (P \bmod p) \notin S_p\}}{\#\mathbb{A}^n(\mathbb{Z})_{\leq N}}. \end{aligned}$$

Taking the limit as $N \rightarrow \infty$ (each term of which exists) we obtain:

$$\begin{aligned} & \lim_{N \rightarrow \infty} \frac{\#\{P \in \mathbb{A}^n(\mathbb{Z})_{\leq N} : \forall p \leq M (P \bmod p) \notin S_p\}}{\#\mathbb{A}^n(\mathbb{Z})_{\leq N}} \\ & - \lim_{N \rightarrow \infty} \frac{\#\{P \in \mathbb{A}^n(\mathbb{Z})_{\leq N} : \exists p > M \text{ s.t. } (P \bmod p) \in S_p\}}{\#\mathbb{A}^n(\mathbb{Z})_{\leq N}} \\ & \leq d(!S) \\ & \leq \lim_{N \rightarrow \infty} \frac{\#\{P \in \mathbb{A}^n(\mathbb{Z})_{\leq N} : \forall p \leq M (P \bmod p) \notin S_p\}}{\#\mathbb{A}^n(\mathbb{Z})_{\leq N}}. \end{aligned}$$

Since $S_p \subseteq Z(\mathbb{F}_p)$, by Lemma 3.4, we have

$$\lim_{M \rightarrow \infty} \lim_{N \rightarrow \infty} \frac{\#\{P \in \mathbb{A}^n(\mathbb{Z})_{\leq N} : \exists p > M \text{ s.t. } (P \bmod p) \in S_p\}}{\#\mathbb{A}^n(\mathbb{Z})_{\leq N}} = 0.$$

Therefore

$$d(!S) = \lim_{M \rightarrow \infty} \lim_{N \rightarrow \infty} \frac{\#\{P \in \mathbb{A}^n(\mathbb{Z})_{\leq N} : \forall p \leq M (P \bmod p) \notin S_p\}}{\#\mathbb{A}^n(\mathbb{Z})_{\leq N}}.$$

By the Chinese Remainder Theorem we have that

$$\lim_{N \rightarrow \infty} \frac{\#\{P \in \mathbb{A}^n(\mathbb{Z})_{\leq N} : \forall p \leq M (P \bmod p) \notin S_p\}}{\#\mathbb{A}^n(\mathbb{Z})_{\leq N}} = \prod_{p \leq M} \left(1 - \frac{\#S_p}{\#\mathbb{A}^n(\mathbb{F}_p)} \right).$$

The result follows by taking the limit as $M \rightarrow \infty$. \square

Proposition 3.5 formally implies the same result where we replace \mathbb{A}^n by an open locus in \mathbb{A}^n whose complement has codimension at least 2 (divide the density coming from the local conditions S in all of \mathbb{A}^n by the density coming from the local conditions of being in the complement of the open locus).

We now apply this to our situation. To obtain an affine space and a codimension at least 2 subscheme, we will consider the **frame space** $\text{Fr}(k, n)$ parameterizing bases of $(k+1)$ -dimensional subspaces of an $(n+1)$ -dimensional vector space. Explicitly, $\text{Fr}(k, n)$ is the complement of the vanishing of the $(k+1)$ -minors in space of $(k+1) \times (n+1)$ -matrices, which is itself $\mathbb{A}^{(k+1)(n+1)}$. If $k < n$ then the subscheme where all $(k+1)$ -minors vanish is codimension at least 2.

On the other hand, the natural map

$$\pi: \text{Fr}(k, n) \rightarrow \text{Gr}(k, n)$$

sending a basis to its span exhibits $\text{Fr}(k, n)$ as a GL_{k+1} -torsor over $\text{Gr}(k, n)$. For this reason, for any subscheme $W \subseteq \text{Gr}(k, n)$, we have

$$\frac{\#W(\mathbb{F}_p)}{\#\text{Gr}(k, n)(\mathbb{F}_p)} = \frac{\#\pi^{-1}W(\mathbb{F}_p)}{\#\text{Fr}(k, n)(\mathbb{F}_p)}.$$

The 4-planes that meet \mathcal{V}_3 form a Zariski-closed subscheme \mathcal{W} of $\mathbb{G}_{\mathbb{Z}}(4, 14)$ of codimension 2. Since $\pi^{-1}\mathcal{W}$ has codimension 2 in $\text{Fr}(4, 14)$, we can use Proposition 3.5 to compute the density of planes $[P] \in \text{Gr}(4, 14)(\mathbb{Q})$ that give rise to pairs (Y_P, α_P) that satisfy (\dagger) . Recall from the introduction that we define the density by sampling integral points on $\text{Fr}(4, 14)$ of bounded height going to infinity. This notion of density, therefore, agrees with the one used in Proposition 3.5 for subschemes of affine space.

We will write \mathcal{P} to indicate the linear system P over \mathbb{Z} . Let $S_p \subseteq \mathbb{G}(4, 14)(\mathbb{F}_p)$ denote the subset of planes $P_p \in \mathbb{G}(4, 14)(\mathbb{F}_p)$ such that there exists a point in $P_p(\mathbb{F}_p)$ for which the corresponding quadric *does not* have a smooth \mathbb{F}_p -point. By Lemma 3.3, we have that $S_p \subseteq \mathcal{W}(\mathbb{F}_p)$. (In particular, if the 4-plane \mathcal{P} and \mathcal{V}_3 do not meet in $\mathbb{P}_{\mathbb{Z}}^{14}$, then for every prime p , we have that $[\mathcal{P} \bmod p]$ is not in S_p . Since it is easy to compute with the ideal \mathcal{I}_3 , in practice we will use this criterion to rule out membership in S_p for all p simultaneously.)

Proposition 3.7. *The set of integral points of $\text{Fr}(4, 14)$ whose reductions are not contained in $\pi^{-1}(S_p)$ for any p have density*

$$d(!S) = \prod_p \left(1 - \frac{\#S_p}{\#\text{Gr}(4, 14)(\mathbb{F}_p)} \right) \geq .73.$$

Proof. The left equality follows from Proposition 3.5. It suffices now to compute a lower bound on this infinite product. Let $B_p \subset \mathbb{P}^{14}(\mathbb{F}_p)$ denote the set of quadrics over \mathbb{F}_p that *do not* have a smooth \mathbb{F}_p -point. This is exactly the union of the set of double planes and the set of quadratic conjugate planes (when $p > 2$ these are the quadrics of rank 1 and non-split quadrics of rank 2). The set of double planes has cardinality $\#\mathbb{G}(3, 4)(\mathbb{F}_p)$. Similarly, a pair of quadratic conjugate planes is a cone with vertex a codimension 2 plane over a nonsplit quadric in \mathbb{P}^1 . Fixing the vertex, we may assume that it is of the form $x_0^2 - bx_0x_1 + cx_1^2$ for some $b, c \in \mathbb{F}_p$. If $p = 2$, such a quadric is nonsplit if and only if $b = c = 1$. If $p > 2$, such a quadric is nonsplit if

and only if $\Delta = b^2 - 4c$ is not a square in \mathbb{F}_p^\times . There are therefore $(p^2 - p)/2$ such choices of b and c . In total, we have

$$\#B_p = \#\mathbb{G}(3, 4)(\mathbb{F}_p) + \#\mathbb{G}(2, 4)(\mathbb{F}_p) \cdot (p^2 - p)/2.$$

Finally, the number of 4-planes over \mathbb{F}_p that contain an \mathbb{F}_p -point in B_p is at most $\#\mathbb{G}(3, 13)\#B_p$. Therefore the fraction of 4-planes in $\mathbb{G}(4, 14)(\mathbb{F}_p)$ that contain an \mathbb{F}_p -point in B_p is at most

$$b(p) := \frac{\#\mathbb{G}(3, 13)(\mathbb{F}_p)\#B_p}{\#\mathbb{G}(4, 14)(\mathbb{F}_p)} = \frac{p^8 + p^6 + 2p^4 + p^3 + 2p^2 + p + 2}{2p^{10} + 2p^5 + 2}.$$

The function $p^2b(p)$ is monotonically decreasing for $p > 2$ and has an asymptote at $1/2$. Its value on the interval of $p \geq N$ is therefore bounded by

$$(3.3) \quad \max(N^2b(N), 1/2)$$

We have

$$\begin{aligned} d(!S) &= \prod_p (1 - b(p)) \\ &\geq \left(\prod_{p < 100} (1 - b(p)) \right) \cdot \left(1 - \sum_{p > 100} b(p) \right) \end{aligned}$$

One can compute (e.g., using the functions in the file `probability.sage`) that the first factor is at least .737. Combining this with (3.3),

$$\begin{aligned} &\geq (.737) \left(1 - \sum_{p > 100} \frac{.50005}{p^2} \right) \\ &\geq (.737) \left(1 - (.50005) \int_{100}^{\infty} \frac{1}{p^2} dp \right) \\ &\geq .73. \end{aligned}$$

□

Proposition 3.8. *Suppose that (Y_P, α_P) is defined by a regular 4-dimensional linear system $[P] \in \mathbb{G}(4, 14)(\mathbb{Q}) = \mathbb{G}(4, 14)(\mathbb{Z})$ that is not contained in the set $S_p \subset \mathbb{G}(4, 14)(\mathbb{F}_p)$ for any p . Then for any \mathbb{Q}_p -point $y \in Y_P(\mathbb{Q}_p)$, the associated quadric Q_y has a smooth \mathbb{Q}_p -point.*

Proof. We spread out $P \simeq \mathbb{P}_{\mathbb{Q}}^4$ to $\mathcal{P} \simeq \mathbb{P}_{\mathbb{Z}}^4$. By slight abuse of notation, we also write $y \in P(\mathbb{Q}_p)$ for the image of $y \in Y(\mathbb{Q}_p)$. This spreads out to a map $\text{Spec}(\mathbb{Z}_p) \rightarrow \mathcal{P}$ by properness of \mathcal{P} over $\text{Spec}(\mathbb{Z})$, and therefore gives a quadric \mathcal{Q} over $\text{Spec}(\mathbb{Z}_p)$ with generic fiber Q_y and special fiber \bar{Q} a quadric over \mathbb{F}_p . Since $[P] \notin S_p$, this quadric \bar{Q} has a smooth \mathbb{F}_p -point. Therefore, the smooth locus of $\mathcal{Q} \rightarrow \text{Spec}(\mathbb{Z}_p)$ has \mathbb{F}_p -points, and hence by Hensel's Lemma [Poo17, Theorem 3.5.63] it has \mathbb{Z}_p -points. Therefore Q_y has smooth \mathbb{Q}_p -points. □

Corollary 3.9. *A randomly generated integral basis yields a span P for which the associated pair (Y_P, α_P) satisfies trivial evaluation at all finite places (\dagger) with probability at least .73.*

Proof. Since regularity is an open condition, a random linear system P will be regular with probability 1. Hence with probability at least .73, it is a regular linear system with $[P \bmod p] \notin S_p$ for all p . By Proposition 3.8, for every $y \in Y_P(\mathbb{Q}_p)$, the corresponding quadric Q_y has a smooth \mathbb{Q}_p -point. Since regularity implies that P avoids the locus of quadrics of rank 2 or less, it now follows by Equation (3.2) and Lemma 3.2 that $\alpha_P(y) = 0$, as desired. \square

3.3. Proofs of the main theorems. In this section, we prove the main theorems guaranteeing that Y_P has a Brauer–Manin obstruction to weak approximation.

Proof of Theorem 1. Let $Y = Y_P$ be the Calabi–Yau threefold with class $\alpha = \alpha_P \in \text{Br}(Y_P)[2]$ that is associated to the choice of a regular 4-plane P in $\mathbb{P}_{\mathbb{Q}}^{14}$. By assumption, there exists a real point $h \in H(\mathbb{R})$ so that the corresponding quadric Q_h has signature $(+1, +1, +1, +1, 0)$ or $(-1, -1, -1, -1, 0)$. As such, Q_h is the cone over a smooth quadric \bar{Q}_h in $\mathbb{P}_{\mathbb{R}}^3$ and the discriminant of \bar{Q}_h is a square in \mathbb{R} . Therefore the preimage of h in Y is two points y_1 and y_2 in $Y(\mathbb{R})$. However, the quadric Q_h has no smooth \mathbb{R} -points, since $\bar{Q}_h(\mathbb{R}) = \emptyset$. Therefore, by Lemma 3.2 in combination with (3.2), we have that

$$\alpha(y_1) = \alpha(y_2) = \frac{1}{2} \in \text{Br}(\mathbb{R}).$$

However, by Lemma 2.5, there must exist a point $y_0 \in Y(\mathbb{R})$ such that $Z|_q(\mathbb{R}) \neq \emptyset$. Again by (3.2), we have

$$\alpha(y_0) = 0 \in \text{Br}(\mathbb{R}).$$

By Lemma 3.1, this implies that Y fails to satisfy weak approximation. \square

Proof of Corollary 2. The quadric $Q_0 := x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0$ defined over \mathbb{Q} is the cone over a smooth quadric surface of discriminant 1. Therefore, it gives rise to a \mathbb{Q} -point on Y_P for any linear system P containing $[Q_0]$. Furthermore, over \mathbb{R} , this quadric has signature $(+1, +1, +1, +1, 0)$. Finally, the condition of regularity (Definition 2.1) is open; therefore, it suffices to show that one such 4-plane through $[Q_0]$ is regular.

We verified the regularity of the 4-plane P given by the following additional generating quadrics with Magma, using code in the ancillary file `yexample.magma`, located in the repository [HHLV20] and in the

supplementary files.

$$\begin{aligned}
& x_0x_1 + x_0x_4 + x_1x_3 - x_1x_4 + x_2x_3 + x_3x_4, \\
& x_0^2 - x_0x_2 - x_0x_4 + x_1x_2 - x_1x_3 - x_1x_4 - x_2^2 - x_2x_4 - x_3x_4 - x_4^2, \\
& x_0x_1 + x_0x_2 + x_0x_4 + x_1^2 - x_1x_4 + x_2^2 + x_2x_3 - x_2x_4 + x_3^2, \\
& x_0^2 + x_0x_1 + x_0x_2 - x_0x_4 - x_1^2 - x_1x_2 + x_1x_3 - x_1x_4 - x_2x_3 - x_2x_4 + x_4^2. \square
\end{aligned}$$

3.4. Explicit example. In this section we prove Theorem 3, explicitly exhibiting a choice of quadrics P for which the associated Brauer class α_P on the associated Calabi–Yau threefold Y_P obstructs weak approximation. Recall that Q_0, Q_1, Q_2, Q_3 , and Q_4 are given by the equations

$$\begin{aligned}
& x_0x_1 + x_2x_3 \\
& x_0^2 + x_0x_1 + 3x_0x_2 + 2x_1^2 + x_1x_2 + x_1x_3 + 5x_2^2 + x_2x_3 + x_3^2 \\
& x_0^2 + 2x_0x_2 + x_0x_3 + x_0x_4 + x_1x_2 + x_1x_4 + x_2x_4 + 4x_3x_4 \\
& 4x_0^2 + 4x_1^2 + 3x_1x_3 + 2x_1x_4 + 3x_2^2 + x_2x_4 + x_4^2 \\
& x_0^2 + x_0x_2 + x_0x_3 + 4x_0x_4 + 3x_1x_2 + 4x_1x_4 + 2x_2x_3 + 2x_3^2 + x_4^2.
\end{aligned}$$

Furthermore, this example will exhibit the “expected” behavior of Corollary 3.9: the pair (Y_P, α_P) has constant trivial evaluation at all finite primes (\dagger).

Proof of Theorem 3. Corollary 3.9 proves the first part of the Theorem.

Let Q_0, \dots, Q_4 be the explicit choice of linearly independent quadrics given above. The quadrics Q_0, \dots, Q_4 define a regular system P , which can be verified using Magma code included in the ancillary file `yexample.magma`. Write $(Y, \alpha) = (Y_P, \alpha_P)$. We first show the containments

$$\emptyset \neq Y_P(\mathbb{Q}) \subset Y_P(\mathbb{A}_{\mathbb{Q}})^{\alpha_P} \subsetneq Y_P(\mathbb{A}_{\mathbb{Q}}).$$

The quadric Q_0 is rank 4, and is a cone with vertex $[0 : 0 : 0 : 0 : 1]$ over the smooth quadric

$$\bar{Q}_0 := x_0x_1 + x_2x_3$$

in $\mathbb{P}_{\mathbb{Q}}^3$. Since $\text{disc}(\bar{Q}_0) = 1$ is a square, the two rulings of lines on \bar{Q}_0 are defined over \mathbb{Q} . Therefore the fiber over $[Q_0] \in H(\mathbb{Q})$ in Y consists of two \mathbb{Q} -points, and so $Y(\mathbb{Q}) \neq \emptyset$ (and, in particular, Y is everywhere locally soluble). Furthermore, $Q_1 := x_0^2 + x_0x_1 + 3x_0x_2 + 2x_1^2 + x_1x_2 + x_1x_3 + 5x_2^2 + x_2x_3 + x_3^2$ has signature $(+1, +1, +1, +1, 0)$; therefore this choice of P satisfies the conditions of Theorem 1, and α obstructs weak approximation on Y .

Finally, we show that (Y_P, α_P) has constant trivial evaluation at all finite primes (\dagger). Let Q be the universal quadric $Q = t_0Q_0 + \dots + t_4Q_4$ over the projective space $P \simeq \mathbb{P}_{\mathbb{Z}}^4$. Let J_3 be the ideal of $\mathbb{Z}[t_0, \dots, t_4]$ given by specializing \mathcal{I}_3 to Q and then saturating with respect to the ideal (t_0, \dots, t_4) . A Groebner basis calculation over \mathbb{Z} shows that 1 is in J_3 , and therefore for the choice of quadrics above, we have $P \cap \mathcal{V}_3 = \emptyset$. Therefore the reduction of every quadric in P (and hence Y) modulo every prime p has a smooth \mathbb{F}_p

point by Lemma 3.3. Hence $[P \bmod p]$ is not contained in the set S_p for any prime p . Thus, by Proposition 3.8, for every \mathbb{Q}_p -point $y \in Y(\mathbb{Q}_p)$, the associated quadric Q_y has a smooth \mathbb{Q}_p -point. Hence by Lemma 3.2 and equation (3.2), $y^*\alpha = \alpha|_y = 0 \in \text{Br}(\mathbb{Q}_p)[2]$. \square

3.5. Obstructions from finite primes.

Proposition 3.10. *Let $P \subset \mathbb{P}H^0(\mathbb{P}_{\mathbb{Q}}^4, \mathcal{O}_{\mathbb{P}^4}(2))$ be a general 4-plane through the points $[Q_0 : 6x_0^2 - 3x_1^2 + 2x_2^2 - x_3^2 = 0]$ and $[Q_1 : x_0x_1 + x_2x_3 = 0]$ in $\mathbb{P}^{14}(\mathbb{Q})$. Then $\emptyset \neq Y_P(\mathbb{Q}) \subsetneq Y_P(\mathbb{A}_{\mathbb{Q}})$ and so Y_P does not satisfy weak approximation.*

Proof. The condition of regularity (Definition 2.1) is open.

The quadric Q_0 does not contain any smooth \mathbb{Q}_3 -points, as we now show. We may rewrite the equation for Q_0 as $3(2x_0^2 - x_1^2) + (2x_2^2 - x_3^2) = 0$, and directly check modulo 3 we only have the non-smooth point $(0 : 0 : 0 : 0 : 1)$. Consider solutions to $Q_0 \bmod 3^n$ for $n \in \mathbb{N}$; the valuation of x_i in the solution must increase as n does, for $i = 0, \dots, 3$.

However, Q_0 is the cone over a smooth quadric surface with discriminant 36, so Y_P has at least one \mathbb{Q}_3 -point over $[Q_0] \in H(\mathbb{Q}_3)$.

The quadric Q_1 contains \mathbb{Q} -points and, over \mathbb{Q} , is the cone over a smooth quadric surface of discriminant 1. Therefore, it also contains smooth \mathbb{Q}_3 -points and gives rise to a \mathbb{Q}_3 -point on Y_P . Thus, by Lemma 3.1, for such a choice of regular 4-plane P , the Brauer class α_P will obstruct weak approximation.

To finish, it suffices to produce a regular 4-plane P containing $[Q_0]$ and $[Q_1]$. The regularity of the 4-plane given by the following additional generating quadrics was verified with Magma, using code in the ancillary file `yexample.magma`.

$$\begin{aligned} & x_0^2 - x_0x_2 - x_0x_3 - x_1^2 + x_1x_2 - x_2^2 - x_2x_4 + x_3^2 - x_3x_4 - x_4^2, \\ & x_0^2 + x_0x_1 + x_0x_2 - x_0x_3 + x_0x_4 - x_1^2 + x_1x_2 - x_1x_3 + x_2x_3 - x_2x_4, \\ & \quad - x_3^2 - x_3x_4, \\ & x_0^2 - x_0x_3 + x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_3x_4 - x_4^2. \end{aligned} \quad \square$$

Remark 3.11.

- (a) It is possible that the example produced in the proof of Proposition 3.10 satisfies the hypotheses of Theorem 1. It has been chosen so that none of its generators, considered over \mathbb{R} , have signature $(+1, +1, +1, +1, 0)$ or $(-1, -1, -1, -1, 0)$, but some element of P may correspond to a quadric which does.
- (b) One can also ask if there are examples of linear systems P where α_P obstructs the Hasse principle on Y_P , i.e., where α_P has constant evaluation at all \mathbb{Q}_v -points for each place v , but the sum of the invariants over all places is nontrivial. It appears that P would have to be incredibly special in order for this to occur. First, by Lemma 2.5,

there always exists a real point of Y_P where α_P evaluates to 0, and so evaluation at real points would need to be trivial. Therefore, there must exist an odd number of finite places p for which the evaluation at all \mathbb{Q}_p -points is constant and nontrivial. However, by Lemmas 3.2 and 3.8, this only occurs if every such quadric over \mathbb{Q}_p (which is necessarily of rank at least 3) reduces to a quadric with rank at most 2 over \mathbb{F}_p .

4. THE DERIVED EQUIVALENT THREEFOLD X

Hosono and Takagi show in [HT14] that a choice of regular linear system P as in Definition 2.1 can also be used to construct another Calabi–Yau variety they call $X = X_P$. Hosono and Takagi proved in [HT16, Theorem 8.0.2] that, for any fixed choice of regular linear system P over \mathbb{C} , the complex varieties X_P and Y_P are derived equivalent but non-birational over \mathbb{C} . That is, there is a \mathbb{C} -linear equivalence between their bounded derived categories of coherent sheaves. See [Huy06] or [Cal00] for an introduction to these categories.

In many cases, derived equivalence is a strong relationship. For example, a smooth curve is determined uniquely by its associated derived category. Examples of varieties where some invariant is not preserved under such an equivalence is considered very interesting. Hosono and Takagi’s Calabi–Yau threefolds are one of very few known examples of derived equivalent varieties that do not have the same fundamental group [HT16] or Brauer group [Add17].

There has been recent interest in comparing the rational points of derived equivalent varieties; see [HT17]. Work of Addington, Antieau, Frei and Honigs [AAFH19] gives the first examples of derived equivalent varieties where one variety has a \mathbb{Q} -point and the other does not. The varieties in these examples are all either hyperkähler fourfolds or abelian varieties of dimension at least 2. Because Hosono and Takagi’s Calabi–Yau threefolds are already known to not share some invariants, particularly the Brauer group, they seem a likely place to look for such behavior occurring in a different class of varieties. However, to the extent that we are able to compare them, we find the same behavior in both X and Y with regard to \mathbb{Q} -rational points. This raises several questions: Is there some special case in which these examples do not share the same behavior with regard to \mathbb{Q} -points? If not, is derived equivalence a more restrictive condition on Calabi–Yau threefolds with regard to \mathbb{Q} -rational points than it is on the types of varieties in the examples of [AAFH19]? If so, what is the mechanism?

In this section we give the construction of X_P and show that if X_P and Y_P are defined over \mathbb{Q} , then Hosono and Takagi’s derived equivalence is also defined over \mathbb{Q} . Then, in order to compare X_P with Y_P , we show that for any regular P , if $X_P(\mathbb{Q}) \neq \emptyset$ then X_P does not satisfy weak approximation. Although X_P and Y_P do not have equal Brauer groups or fundamental

groups, in all examples we have analyzed, X_P and Y_P both have \mathbb{Q} -points and do not satisfy weak approximation.

4.1. The variety X . Consider a choice of regular linear system P over a field K generated by quadrics $Q_0, \dots, Q_4 \subseteq \mathbb{P}^4$. When the characteristic of K is not 2, the data of each quadric Q_i is equivalent to the data of a projective symmetric bilinear form B_i , and hence a $(1, 1)$ -divisor in $\mathbb{P}^4 \times \mathbb{P}^4$ that is symmetric under the $\mathbb{Z}/2\mathbb{Z}$ -action swapping the factors. Let \tilde{X}_P denote the complete intersection of B_0, \dots, B_4 . By the regularity of P , \tilde{X}_P does not meet the diagonal in $\mathbb{P}^4 \times \mathbb{P}^4$, and hence the $\mathbb{Z}/2\mathbb{Z}$ -action induces a fixed-point-free involution $\iota: \tilde{X}_P \rightarrow \tilde{X}_P$. In these terms, X_P is the quotient by this involution:

$$X_P = \tilde{X}_P / \iota.$$

The smoothness of X_P is equivalent to the regularity of P [HT14, Proposition 2.1].

4.2. The kernel of the equivalence. For smooth projective varieties X and Y , by a result of Orlov, equivalences between their associated derived categories are produced by Fourier-Mukai transforms [Huy06, Thm 5.14]. Such a transform is defined via an object in the derived category of $X \times Y$, which we call a Fourier-Mukai kernel. The interested reader is directed to [Huy06, Chapter 5] for a precise discussion on the construction of these functors.

Lemma 4.1. *For any choice of regular system P defined over \mathbb{Q} , there is a \mathbb{Q} -linear derived equivalence between the associated varieties X_P and Y_P .*

Proof. Write $X = X_P$ and $Y = Y_P$. As shown in Section 2.1, there is a natural projection map from $Z \rightarrow \mathbb{G}(2, \mathbb{P}^4)$. Since each point in X corresponds to a pair of distinct points in \mathbb{P}^4 , there is a natural map $X \rightarrow \mathbb{G}(1, \mathbb{P}^4)$ that sends each point in X to the line in \mathbb{P}^4 containing the corresponding two points in \mathbb{P}^4 . Consider the subscheme $\Delta \subseteq X \times Y$ formed by pulling back the flag variety $F(1, 2, V) \subseteq \mathbb{G}(1, \mathbb{P}^4) \times \mathbb{G}(2, \mathbb{P}^4)$ to $X \times Z$ and then taking its image in $X \times Y$. In $X \times Z$, points of this subscheme correspond to the data of a quadric in P , a plane it contains, and two points in \mathbb{P}^4 which lie on that plane.

Hosono and Takagi show in [HT16, Theorem 8.0.1] that, when working over \mathbb{C} , the ideal sheaf of Δ is the kernel of a Fourier-Mukai equivalence. Just as we may define X, Y over \mathbb{Q} in a way that is compatible with base change, we see from our description of Δ that it may be defined over \mathbb{Q} as well. To finish, we observe that if two smooth, projective varieties and a kernel of a Fourier-Mukai functor between their bounded derived categories of coherent sheaves are defined over a field K and the functor is shown to give a derived equivalence after base change to \bar{K} , then it is an equivalence over K [AAFH19, cf. proof of Theorem 4]. \square

4.3. Rational points and weak approximation on X . By the following lemma, any linear system P satisfying the hypotheses of Theorem 1 or Proposition 3.10 gives rise to a variety X_P where $X_P(\mathbb{Q}) \neq \emptyset$ and therefore X_P does not satisfy weak approximation.

Lemma 4.2. *If $X_P(\mathbb{A}_{\mathbb{Q}}) \neq \emptyset$, then X_P does not satisfy weak approximation.*

Proof. As in [HT13, Proposition 3.5.3], the Lefschetz hyperplane theorem guarantees that $\pi_1(\tilde{X}_P) = 0$, and so the étale double cover $\pi: \tilde{X}_P \rightarrow X_P$ represents the nontrivial element of $\pi_1(X_P) \simeq \mathbb{Z}/2\mathbb{Z}$.

By [Har04, Theorem 2.4.4], this shows X_P does not satisfy weak approximation. \square

Recall that $H \subset P$ is the discriminant locus of singular quadrics in the linear system P .

Lemma 4.3. *Given any choice of regular linear system P defined over \mathbb{Q} , $H(\mathbb{Q}) \neq \emptyset$ if and only if $X_P(\mathbb{Q}) \neq \emptyset$.*

Proof. Fix a regular linear system P generated by quadrics defined over \mathbb{Q} corresponding to bilinear forms B_0, \dots, B_4 .

Suppose $H(\mathbb{Q}) \neq \emptyset$. Suppose B_0 has rank less than 5. Then we may choose a point $v \in \mathbb{P}_{\mathbb{Q}}^4$ in its kernel. If we pick a representative v' of v in \mathbb{Q}^5 , we note $B_0 v' = 0$ and $\{B_0 v', B_1 v', B_2 v', B_3 v', B_4 v'\}$ span a subspace of \mathbb{Q}^5 of dimension at most 4. Therefore, we may pick a nonzero vector with coefficients in \mathbb{Q} in the orthogonal complement, giving an element $w \in \mathbb{P}_{\mathbb{Q}}^4$ so that $w B_i v = 0$ for all i , hence the unordered pair (w, v) determines a point on X_P . For instance, for the choice of quadrics in Theorem 3, $([0 : 0 : 0 : 0 : 1], [1 : -3 : 2 : 4 : 1])$ is a \mathbb{Q} -point of the corresponding variety X_P .

If all of the bilinear forms B_0, \dots, B_4 have full rank, then by the assumption that $H(\mathbb{Q})$ is not empty, we may choose a different basis for our linear system that includes a matrix that does not have full rank. Hence, $X_P(\mathbb{Q}) \neq \emptyset$.

Now suppose $X_P(\mathbb{Q}) \neq \emptyset$. Then there exist $v, w \in \mathbb{P}_{\mathbb{Q}}^4$ so that $w B_i v = 0$ for all i . The five vectors $B_i v$ are contained in the 4-dimensional space w^\perp and are therefore linearly dependent. Thus there exists a linear combination of the B_i where v is contained in its kernel, and so it does not have full rank. \square

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APPENDIX A. THE BRAUER GROUP OF
HOSONO AND TAKAGI'S THREEFOLD
BY NICOLAS ADDINGTON

In [HT16], Hosono and Takagi constructed a derived equivalence between a pair of Calabi–Yau threefolds X and Y with $\pi_1(X) = \mathbb{Z}/2$ and $\pi_1(Y) = 0$, and an explicit Brauer class of order 2 on Y . In [Add17] I showed that this derived equivalence and a little topological K-theory yield an exact sequence

$$0 \rightarrow \mathbb{Z}/2 \rightarrow \mathrm{Br}(Y) \rightarrow \mathrm{Br}(X) \rightarrow 0.$$

The purpose of this appendix is to prove:

Proposition A.1. $\mathrm{Br}(X) = 0$, and hence $\mathrm{Br}(Y) = \mathbb{Z}/2$.

First recall the construction of X . We choose 5 symmetric bilinear forms on \mathbb{C}^5 , which determine five divisors of type $(1, 1)$ in $\mathbb{P}^4 \times \mathbb{P}^4$, symmetric under the action of $G := \mathbb{Z}/2$ that exchanges the two factors, and let $\tilde{X} \subset \mathbb{P}^4 \times \mathbb{P}^4$ be their intersection. For generic choices, \tilde{X} is a smooth threefold that does not intersect the diagonal, so G acts freely on \tilde{X} . Then $X := \tilde{X}/G$.

Next recall that $\mathrm{Br}(X)$ is isomorphic to the torsion subgroup of $H^3(X, \mathbb{Z})$, because $H^2(\mathcal{O}_X) = 0$. We will use the Hochschild–Serre spectral sequence

$$E_2^{p,q} = H^p(G, H^q(\tilde{X}, \mathbb{Z})) \implies H^{p+q}(X, \mathbb{Z}),$$

which is the Leray spectral sequence for the fibration

$$\begin{array}{ccc} \tilde{X} & \hookrightarrow & \tilde{X} \times_G EG \simeq X \\ & & \downarrow \\ & & BG. \end{array}$$

Thus we need to compute the G -action on $H^i(\tilde{X}, \mathbb{Z})$ for $i = 0, 1, 2, 3$. In fact we only need a little information about H^3 , which is lucky because it seems hard to get complete information.

By the Lefschetz hyperplane theorem we have

$$H^i(\tilde{X}, \mathbb{Z}) = H^i(\mathbb{P}^4 \times \mathbb{P}^4, \mathbb{Z}) \quad \text{for } i < 3,$$

and similarly with H_i . Thus:

- $H^0(\tilde{X}, \mathbb{Z}) = \mathbb{Z}$ with the trivial G -action.
- $H^1(X, \mathbb{Z}) = 0$.
- $H^2(X, \mathbb{Z}) = \mathbb{Z}^2$ with the G -action that exchanges the two factors.
- $H^3(\tilde{X}, \mathbb{Z})$ is torsion free: by the universal coefficient theorem, the torsion in H^3 is isomorphic to the torsion in H_2 .

The cohomology of G with coefficients in the trivial module \mathbb{Z} is

$$H^*(G, \mathbb{Z}) = \mathbb{Z}, 0, \mathbb{Z}/2, 0, \mathbb{Z}/2, 0, \mathbb{Z}/2, \dots$$

With coefficients in the regular representation \mathbb{Z}^2 , also known as $\mathbb{Z}G$, it is

$$H^*(G, \mathbb{Z}G) = \mathbb{Z}, 0, 0, 0, 0, 0, 0, \dots$$

This can be found in many textbooks, for example [DF04, §17.2].

Now the E_2 page of the spectral sequence is

$q \uparrow$							
\vdots	\vdots						
$H^3(\tilde{X}, \mathbb{Z})^G$?	?	?	?	?	...	
\mathbb{Z}	0	0	0	0	0	...	
0	0	0	0	0	0	...	
\mathbb{Z}	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	0	...	$p \rightarrow$

where $H^3(\tilde{X}, \mathbb{Z})^G$ denotes the invariant submodule of $H^3(\tilde{X}, \mathbb{Z})$. This is the only term that can contribute to $H^3(X, \mathbb{Z})$, and it is torsion-free. On the E_4 page there may be a non-zero differential $H^3(\tilde{X}, \mathbb{Z})^G \rightarrow \mathbb{Z}/2$, but its kernel is still torsion-free. Thus $H^3(X, \mathbb{Z})$ is torsion-free, as desired.

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