



Tilting modules arising from knot invariants

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ABSTRACT

We construct tilting modules over Jacobian algebras arising from knots. To a two-bridge knot $L[a_1, \dots, a_n]$, we associate a quiver Q with potential and its Jacobian algebra A . We construct a family of canonical indecomposable A -modules $M(i)$, each supported on a different specific subquiver $Q(i)$ of Q . Each of the $M(i)$ is expected to parametrize the Jones polynomial of the knot. We study the direct sum $M = \oplus_i M(i)$ of these indecomposables inside the module category of A as well as in the cluster category.

In this paper we consider the special case where the two-bridge knot is given by two parameters a_1, a_2 . We show that the module M is rigid and τ -rigid, and we construct a completion of M to a tilting (and τ -tilting) A -module T . We show that the endomorphism algebra $\text{End}_A T$ of T is isomorphic to A , and that the mapping $T \mapsto A[1]$ induces a cluster automorphism of the cluster algebra $\mathcal{A}(Q)$. This automorphism is of order two. Moreover, we give a mutation sequence that realizes the cluster automorphism. In particular, we show that the quiver Q is mutation equivalent to an acyclic quiver of type $T_{p,q,r}$ (a tree with three branches). This quiver is of finite type if $(a_1, a_2) = (a_1, 2), (1, a_2)$, or $(2, 3)$, it is tame for $(a_1, a_2) = (2, 4)$ or $(3, 3)$, and wild otherwise.

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1. Introduction

We construct tilting modules over Jacobian algebras that are motivated by knot invariants of two-bridge knots and their relation to cluster algebras.

A relation between knot invariants and cluster algebras has been established recently in [14], where the authors give a realization of the Jones polynomial of a two-bridge link in terms of the Laurent expansion of a certain cluster variable. In this approach, one first constructs a quiver R of Dynkin type $\mathbb{A}$ from the link and then considers the cluster algebra $\mathcal{A}(R)$ determined by it. The cluster variable x_M that realizes the Jones polynomial is the one associated with the unique indecomposable module M of the path algebra of R that is one-dimensional at each vertex. In this special situation, the cluster variable x_M is a Laurent

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polynomial whose terms are parametrized by the submodules of the module M [8]. Equivalently, it can be computed via a perfect matching formula of an associated snake graph [16]. In particular, the number of terms (counted with multiplicities) in the Jones polynomial is equal to the number of submodules of M and equal to the number of perfect matchings of the snake graph.

We point out that a similar result for the Alexander polynomial was given more recently in [17] using ancestral triangles, and for the HOMFLY polynomial in [19] using path posets. See also [15] for an interesting reformulation of the results in [14].

On the other hand, the authors of [11] established a dimer model for links, using the (overlaid) Tait graph G of the link, to interpret the Alexander polynomial as a partition function. In this method, one has to remove two adjacent regions from the graph G and thereafter one can compute the Alexander polynomial as the determinant of the (weighted) adjacency matrix of the reduced graph. This formula is reminiscent of Alexander's original definition of the polynomial as the determinant of the incidence matrix from which one removes two columns that correspond to two adjacent regions in the graph [2].

It is crucial to note that the result does not depend on the choice of the two adjacent regions in G , and this observation motivated the research project in this paper.

If the link is a two-bridge link, we can associate a quiver Q with potential W to the graph G , and we denote by A the corresponding Jacobian algebra. The removal of two adjacent regions in G corresponds to the removal of a connected subquiver of Q , and, for a certain choice of adjacent regions, the resulting quiver is precisely the quiver R used in [14] to compute the Jones polynomial of the link. If we chose a different pair of adjacent regions then the subquiver may look very different, but it is natural to expect that a variation of the methods of [14] will also apply for these subquivers and give another way to compute the Jones polynomial. This problem will be considered in a different paper.

In this paper, we want to consider all possible choices of adjacent pairs at the same time. Each such choice i will determine a subquiver Q^i of the quiver Q which we need to remove. On the remaining quiver $Q(i) = Q \setminus Q^i$, we define a canonical indecomposable A -module $M(i)$ of dimension 1 at every vertex of $Q(i)$, and we let $M = \oplus_i M(i)$ be the direct sum of these indecomposables.

Motivated by the results mentioned above, we want to study the structure of M inside the module category of A . Thereby we combine all possible choices of adjacent regions into one object. It is natural to expect that this module M has a particularly nice structure.

By construction, M is a sincere A -module, meaning it is supported at every vertex of Q , each indecomposable summand $M(i)$ of M is supported on a different subquiver $Q(i)$ of Q and each $M(i)$ should parametrize the Jones and the Alexander polynomial of the two-bridge link via its submodules.

Recall that a two-bridge link $L[a_1, a_2, \dots, a_n]$ is given by a sequence of positive integers $a_1, a_2, \dots, a_n$ and the number of terms in the Jones (and Alexander) polynomial is equal to the numerator of the continued fraction

$$[a_1, a_2, \dots, a_n] = a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_n}}}.$$

In this paper, we consider the special case where $n = 2$. Thus we have two positive integers a_1, a_2 and the number of terms in the Jones polynomial is $a_1 a_2 + 1$. The case $n = 1$ is contained in this case because the link $L[a_1]$ is equivalent to the link $L[1, a_1 - 1]$. For $n = 2$, the quiver Q is of the form illustrated in Fig. 1. It consists of an oriented cycle of length $a_2 + 1$ whose vertices are labeled $r_0, r_1, \dots, r_{a_2}$, and two branches, both of which are linearly oriented subquivers of type $\mathbb{A}_{a_1}$, one entering the oriented cycle at vertex r_{a_2} and the other leaving the cycle at vertex r_0 . This quiver is actually mutation equivalent to an acyclic quiver Q'

and the potential is obtained from the trivial potential via this mutation sequence. In particular, the cluster category $\mathcal{C}$ is the cluster category of Q' of [7] and the algebra A is cluster-tilted of type Q' .

Furthermore, the removal of two adjacent regions in G corresponds to the removal of a vertex r_i on the oriented cycle of Q together with the branch at the vertex r_i if $i = 0$ or $i = a_2$. We let $M(r_i)$ be the unique indecomposable A -module that is one-dimensional at every vertex of the remaining subquiver. We prove that each $M(r_i)$ has precisely $a_1 a_2 + 1$ submodules as expected. Note that the zero module is always a submodule of $M(r_i)$.

We then construct an indecomposable A -module $M(x)$ for each of the remaining vertices $x = s_1, s_2, \dots, s_{a_1-1}, t_1, t_2, \dots, t_{a_1-1}$ of Q , and we define $T = \bigoplus_{x \in Q_0} M(x)$ to be the direct sum of all these indecomposables.

We show that T is a tilting A -module. This means that $\text{Ext}_A^1(T, T)$ vanishes, that the projective dimension of T is at most one and that there exists a short exact sequence of the form $0 \rightarrow A \rightarrow T^0 \rightarrow T^1 \rightarrow 0$ with T^0, T^1 in the additive closure of T . Tilting modules and their endomorphism algebras play a central role in representation theory, see for example [6,3]. In particular, it follows that T is a τ -tilting A -module as defined in [1]. This means that the number of indecomposable summands of T is equal to the number of vertices of Q and $\text{Hom}(T, \tau T) = 0$ where τ is the Auslander-Reiten translation.

Our τ -tilting module T is particularly nice, since we can show that its endomorphism algebra $\text{End}_A T$ is isomorphic to the dual of A , and A is self-dual.

We then consider this situation at the level of the corresponding cluster algebra $\mathcal{A}(Q)$ defined in [13]. We show that the τ -tilting module T induces a cluster-tilting object in the cluster category $\mathcal{C}$ of acyclic type as introduced in [7] (and [9], for Dynkin type $\mathbb{A}$). Therefore, the results in [10] imply that each of $A[1]$ and T corresponds to a cluster $\mathbf{x}_{A[1]}$ and $\mathbf{x}_T$ in the cluster algebra (here $[1]$ denotes the shift in the cluster category). Using a result from [5], we see that the correspondence $T \mapsto A[1]$ induces a cluster automorphism σ of $\mathcal{A}(Q)$, that maps the cluster $\mathbf{x}_T$ to the cluster $\mathbf{x}_{A[1]}$ and induces an isomorphism between the quiver Q of the cluster $\mathbf{x}_{A[1]}$ and the opposite of the quiver of the cluster $\mathbf{x}_T$. We further show that σ has order 2. We then construct a sequence of mutations μ that realizes the automorphism σ .

Our main results are summarized in the following theorem.

Theorem 1.1. *Let A be the Jacobian algebra of $Q = Q[a_1, a_2]$ and let $T = \bigoplus_{x \in Q_0} M(x)$.*

- (1) *T is a tilting A -module and hence a τ -tilting A -module.*
- (2) *The endomorphism algebra $\text{End}_A T$ is isomorphic to A .*
- (3) *The correspondence $T \mapsto A[1]$ induces a cluster automorphism σ of order two of the cluster algebra $\mathcal{A}(Q)$.*
- (4) *The automorphism σ is given by the sequence of mutations*

$$\mu = \mu_R^{-1} \circ \mu_S \circ \mu_T \circ \mu_R,$$

where

$$\mu_R = \mu_{r_{a_2-1}} \circ \dots \circ \mu_{r_2} \circ \mu_{r_1},$$

$$\mu_S = \mu_{s_1} \mu_{s_2} \mu_{s_3} \dots \mu_{r_{a_2}} \circ \dots \circ \mu_{s_1} \mu_{s_2} \mu_{s_3} \circ \mu_{s_1} \mu_{s_2} \circ \mu_{s_1},$$

$$\mu_T = \mu_{t_{a_1-1}} \mu_{t_{a_1-2}} \mu_{t_{a_1-3}} \dots \mu_{r_0} \circ \dots \circ \mu_{t_{a_1-1}} \mu_{t_{a_1-2}} \mu_{t_{a_1-3}} \circ \mu_{t_{a_1-1}} \mu_{t_{a_1-2}} \circ \mu_{t_{a_1-1}}.$$

- (5) *The cluster algebra $\mathcal{A}(Q)$ is of acyclic type $\mu_R Q = T_{p,q,r}$, a tree with three branches. Moreover, the cases where it is of finite or tame type are the following.*

$$\begin{array}{ll}
\mathbb{A}_{2a_1+1} & \text{if } a_2 = 2; \\
\mathbb{D}_{a_2+1} & \text{if } a_1 = 1; \\
\mathbb{E}_6 & \text{if } (a_1, a_2) = (2, 3); \\
\widetilde{\mathbb{E}}_7 & \text{if } (a_1, a_2) = (3, 3); \\
\widetilde{\mathbb{E}}_6 & \text{if } (a_1, a_2) = (2, 4);
\end{array}$$

and it is of wild acyclic type in all other cases.

The case $n \geq 3$ is more complicated and will require different methods. We expect that we can still construct the tilting module T and a mutation sequence. However, already in small examples, we do not obtain a cluster automorphism of order two.

The paper is organized as follows. We define the A -module T in Section 2 and compute the number of submodules in Section 3. After computing the Auslander-Reiten translate of T in Section 4, we show that T is a tilting and a τ -tilting A -module in Section 5. Section 6 is devoted to the study of the endomorphism algebra of T and to the proof of $\text{End}_A T \cong A$. In Section 7, we construct the mutation sequence that transforms A into T and show that the corresponding cluster automorphism has order two. We illustrate the results in an example in Section 8.

We thank the anonymous referee for suggesting a simpler proof of Lemma 6.1 as well as several other improvements. We also thank Lidia Angeleri Hügel for pointing out to us that every tilting module is τ -tilting.

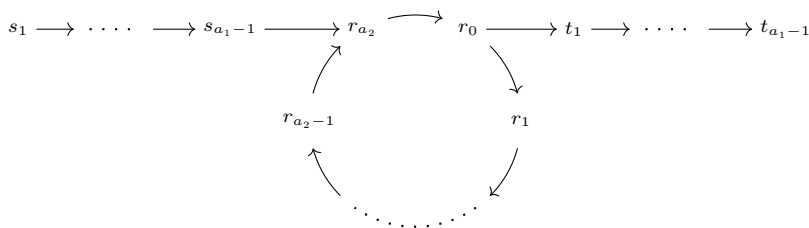


Fig. 1. The quiver $Q = Q[a_1, a_2] = (Q_0, Q_1)$.

2. Definition of T

Throughout the paper, let k denote an algebraically closed field. Let a_1, a_2 be positive integers and let $Q = Q[a_1, a_2]$ be the quiver shown in Fig. 1. Thus Q consists of an oriented cycle of length $a_2 + 1$ and a path of length $2a_1 - 1$ whose halfway arrow is part of the cycle. Let Q_0 be the set of vertices of Q and Q_1 the set of arrows. We label the vertices that lie on the cycle by $r_0, r_1, \dots, r_{a_2}$, those on the incoming branch by $s_1, s_2, \dots, s_{a_1-1}$ and those on the outgoing branch by $t_1, t_2, \dots, t_{a_1-1}$.

We equip Q with the potential $W = r_0 \rightarrow r_1 \rightarrow \dots \rightarrow r_{a_2} \rightarrow r_0$ and denote the corresponding Jacobian algebra by A . Thus $A = kQ/I$ is the quotient of the path algebra of Q by the two-sided ideal I generated by all subpaths of W of length a_2 .

The opposite quiver Q^{op} is obtained from Q by reversing the direction of each arrow. Let $D = \text{Hom}_k(-, k)$ denote the standard duality.

Remark 2.1. The quiver Q is isomorphic to its opposite quiver Q^{op} and, hence, the algebra A is isomorphic to its opposite algebra A^{op} .

For every $x \in Q_0$, let $P(x), I(x)$ and $S(x)$ denote the indecomposable projective, injective and simple A -module at x , respectively. Every A -module L is described as a representation of the quiver Q satisfying the relations in I , and we shall frequently use the notation $L = (L_x, \varphi_\alpha)_{x \in Q_0, \alpha \in Q_1}$, where L_x is the k -vector

space at vertex x and φ_α is the linear map on the arrow α of the representation corresponding to L . For further details on representation theory we refer to [6,18].

In this section, we define an indecomposable A -module $M(x)$ for every vertex $x \in Q_0$, and denote by T the direct sum $T = \bigoplus_{x \in Q_0} M(x)$. The definition of $M(x)$ is given in three separate cases depending on x being a vertex in the cycle or in one of the two branches.

Definition 2.2. For all vertices labeled r_i , we define the module $M(r_i) = (M(r_i)_x, \varphi_\alpha)$ by

$$M(r_i) = \begin{array}{cc} r_{i+1} & s_1 \\ r_{i+2} & s_2 \\ \vdots & \vdots \\ r_{a_2-1} & s_{a_1-1} \\ & r_{a_2} \\ r_1 & t_1 \\ r_2 & t_2 \\ \vdots & \vdots \\ r_{i-1} & t_{a_1-1} \end{array}, \quad M(r_i)_x := \begin{cases} 0 & \text{if } x = r_i; \\ 0 & \text{if } r_i = r_0 \text{ and } x = t_j \text{ for any } j; \\ 0 & \text{if } r_i = r_{a_2} \text{ and } x = s_j \text{ for any } j; \\ k & \text{otherwise.} \end{cases}$$

Given any arrow $x \xrightarrow{\alpha} y$ in Q_1 , the map $M(r_i)_x \xrightarrow{\varphi_\alpha} M(r_i)_y$ is defined by

$$\varphi_\alpha := \begin{cases} 1 & \text{if } M(r_i)_x = k = M(r_i)_y; \\ 0 & \text{otherwise.} \end{cases}$$

Thus the support of $M(r_i)$ is given by removing the vertex r_i from our quiver Q , together with vertices labeled s_j when $i = a_2$, and respectively vertices labeled t_j when $i = 0$. Equivalently, the support of $M(r_i)$ is given by removing the vertex r_i from our quiver Q and taking the component containing r_{i+1} .

Remark 2.3. For any minimal path w between a vertex x and a vertex y of Q , let φ_w be the composition of maps in the module $M(r_i)$ along the path w . If $M(r_i)_x = k$, then

$$\varphi_w = \begin{cases} 1 & \text{if } r_i \text{ is not in } w; \\ 0 & \text{if } r_i \text{ is in } w. \end{cases}$$

Definition 2.4. For all vertices labeled t_i , we define the module $M(t_i) = (M(t_i)_x, \varphi_\alpha)$ by the short exact sequence

$$0 \rightarrow M(t_i) \rightarrow I(r_{a_2}) \xrightarrow{f} I(s_i) \rightarrow 0,$$

with f given by the path $s_i \rightarrow s_{i+1} \rightarrow \cdots \rightarrow r_{a_2}$. For any vertex $x \in Q_0$, we have $M(t_i)_x = k$ if and only if $I(s_i)_x = 0$ and $I(r_{a_2})_x = k$. Also, $M(t_i)_x = 0$ otherwise. Therefore

$$M(t_i) = \begin{array}{cc} r_1 & s_{i+1} \\ r_2 & s_{i+2} \\ \vdots & \vdots \\ r_{a_2-1} & s_{a_1-1} \\ & r_{a_2} \end{array}, \quad M(t_i)_x = \begin{cases} 0 & \text{if } x = r_0; \\ 0 & \text{if } x = t_j \text{ for any } j; \\ 0 & \text{if } x = s_j \text{ for any } j \leq i; \\ k & \text{otherwise.} \end{cases}$$

Given any arrow $x \xrightarrow{\alpha} y$ in Q_1 , the map $M(t_i)_x \xrightarrow{\varphi_\alpha} M(t_i)_y$ is given by

$$\varphi_\alpha = \begin{cases} 1 & \text{if } M(t_i)_x = k = M(t_i)_y; \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.5. For all vertices labeled s_i , we define the module $M(s_i) = (M(s_i)_x, \varphi_\alpha)$ by the short exact sequence

$$0 \rightarrow P(t_i) \xrightarrow{g} P(r_0) \rightarrow M(s_i) \rightarrow 0,$$

with g given by the path $r_0 \rightarrow t_1 \rightarrow \cdots \rightarrow t_i$. For any vertex $x \in Q_0$, we have $M(s_i)_x = k$ if and only if $P(t_i)_x = 0$ and $P(r_0)_x = k$. Also, $M(s_i)_x = 0$ otherwise. Therefore

$$M(s_i) = \begin{matrix} & r_0 & \\ r_1 & & t_1 \\ r_2 & & t_2 \\ \vdots & & \vdots \\ r_{a_2-1} & & t_{i-1} \end{matrix}, \quad M(s_i)_x = \begin{cases} 0 & \text{if } x = r_{a_2}; \\ 0 & \text{if } x = t_j \text{ for any } j \geq i; \\ 0 & \text{if } x = s_j \text{ for any } j; \\ k & \text{otherwise.} \end{cases}$$

Given any arrow $x \xrightarrow{\alpha} y$ in Q_1 , the map $M(s_i)_x \xrightarrow{\varphi_\alpha} M(s_i)_y$ is given by

$$\varphi_\alpha = \begin{cases} 1 & \text{if } M(s_i)_x = k = M(s_i)_y; \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.6. We define the module T to be the sum of all modules defined in Definitions 2.2, 2.4, and 2.5. That is to say

$$T := \left(\bigoplus_{i=0}^{a_2} M(r_i) \right) \oplus \left(\bigoplus_{i=1}^{a_1-1} M(s_i) \right) \oplus \left(\bigoplus_{i=1}^{a_1-1} M(t_i) \right) = \bigoplus_{x \in Q_0} M(x).$$

3. Submodules of $M(r_i)$

As mentioned in the introduction, we want that *each* indecomposable module $M(r_i)$ gives a parametrization of the Alexander polynomial of the link $L[a_1, a_2]$ via its submodules. In particular, since the Alexander polynomial has $a_1 a_2 + 1$ terms, we should expect that the number of submodules of $M(r_i)$ also equals $a_1 a_2 + 1$ and does not depend on i . We prove this fact in this section.

Lemma 3.1. *For every $i = 0, 1, \dots, a_2$, the module $M(r_i)$ has exactly $a_1(a_2 - i)$ submodules with support at r_{a_2} .*

Proof. If $i = a_2$, then there is nothing to show because $M(r_{a_2})$ is not supported at r_{a_2} . Otherwise, for each pair (j, ℓ) with $i + 1 \leq j \leq a_2$ and $1 \leq \ell \leq a_1$, we have a submodule

$$\begin{matrix} r_j & & s_\ell \\ \vdots & & \vdots \\ r_{a_2-1} & & s_{a_1-1} \\ & r_{a_2} & \\ & r_0 & \\ r_1 & & t_1 \\ r_2 & & t_2 \\ \vdots & & \vdots \\ r_{i-1} & & t_{a_1-1} \end{matrix},$$

where we set $s_{a_1} = r_{a_2}$ if $\ell = a_1$. There are exactly $a_1(a_2 - i)$ such submodules. $\square$

Lemma 3.2. For every $i = 1, 2, \dots, a_2$, the module $M(r_i)$ has exactly 1 submodule that is supported at r_0 and not supported at r_{a_2} .

Proof. The submodule is given by

$$\begin{array}{ccc} & r_0 & \\ r_1 & & t_1 \\ r_2 & & t_2 \\ \vdots & & \vdots \\ r_{i-1} & & t_{a_1-1} \end{array} \cdot \quad \square$$

Lemma 3.3. For every $i = 1, 2, \dots, a_2$, the module $M(r_i)$ has exactly $(i a_1)$ submodules that are zero at r_0 and r_{a_2} .

Proof. For each pair (j, ℓ) such that $1 \leq j \leq i$ and $1 \leq \ell \leq a_1$, we have a submodule

$$\begin{array}{ccc} r_j & & t_\ell \\ \vdots & \oplus & \vdots \\ r_{i-1} & & t_{a_1-1} \end{array} =: L_1 \oplus L_2.$$

Note that if $j = i$, then we take $L_1 = 0$. Similarly, if $\ell = a_1$, then we take $L_2 = 0$. There are exactly $i a_1$ such submodules. $\square$

Proposition 3.4. Let r_i be any vertex on the cycle in Q . Then $M(r_i)$ has exactly $a_1 a_2 + 1$ submodules.

Proof. If $i = 0$, then $M(r_0)$ has exactly $a_1 a_2$ submodules supported at r_{a_2} , by Lemma 3.1. Note that $M(r_0) = I(r_{a_2})$ is the injective at r_{a_2} . Thus the only other submodule of $M(r_0)$ is the zero module.

Now suppose $i \neq 0$. Then Lemmata 3.1–3.3 yield an exhaustive list of the $a_1 a_2 + 1$ submodules of $M(r_i)$. $\square$

4. Computation of τT

In this section, we compute the Auslander-Reiten (AR) translate of each indecomposable summand of the A -module T defined in Section 2.

First, we compare the indecomposable summands of T the indecomposable projectives and injectives of A , and we see the following.

Remark 4.1. We have

$$\begin{array}{lll} M(r_{a_2-1}) = P(s_1), & M(r_{a_2}) = P(r_0), & M(t_{a_1-1}) = P(r_1), \\ M(r_0) = I(r_{a_2}), & M(r_1) = I(t_{a_1-1}), & M(s_1) = I(r_{a_2-1}). \end{array}$$

For the rest of this section, we use the notation $L = (L_x, \varphi_\alpha)_{x \in Q_0, \alpha \in Q_1}$ for an A -module L .

Lemma 4.2. The modules $\tau M(r_{a_2})$ and $\tau M(r_{a_2-1})$ are zero, and for $i = 0, 1, \dots, a_2 - 2$ we have

$$\tau M(r_i) = \begin{array}{ccc} s_2 & & r_{i+2} \\ s_3 & & r_{i+3} \\ \vdots & & \vdots \\ s_{a_1-1} & & r_{a_2-1} \end{array}, \quad \tau M(r_i)_x = \begin{cases} 0 & \text{if } x = r_j \text{ for any } 0 \leq j \leq i+1; \\ 0 & \text{if } x = t_j \text{ for any } j; \\ 0 & \text{if } x = s_1; \\ k & \text{otherwise.} \end{cases}$$

Furthermore, the projective dimension of $M(r_i)$ is 1.

Proof. Remark 4.1 implies that $\tau M(r_{a_2})$ and $\tau M(r_{a_2-1})$ are zero. Suppose now that $0 \leq i \leq a_2 - 2$. We have the projective resolution

$$0 \rightarrow P(r_{a_2}) \xrightarrow{f} P(r_{i+1}) \oplus P(s_1) \rightarrow M(r_i) \rightarrow 0.$$

This shows that $\text{pd } M(r_i) = 1$. By applying the Nakayama functor ν , we get the exact sequence

$$0 \rightarrow \tau M(r_i) \rightarrow I(r_{a_2}) \xrightarrow{\nu f} I(r_{i+1}) \oplus I(s_1).$$

The map $I(r_{a_2}) \rightarrow I(s_1)$ is nonzero only at the vertex s_1 , so $\tau M(r_i)_{s_i} = 0$. The map $I(r_{a_2}) \rightarrow I(r_{i+1})$ is nonzero at the vertices $r_1, \dots, r_{i+1}$, so $\tau M(r_i)$ is zero at these vertices. Also, both modules $I(r_{a_2})$ and $I(r_{i+1}) \oplus I(s_1)$ have no support at vertices labeled t_j , so $\tau M(r_i)$ is zero at these vertices. Finally, $\tau M(r_i)$ is zero at vertex r_0 , because $I(r_{a_2})$ is zero at that vertex. This completes the proof. $\square$

Lemma 4.3. *The module $\tau M(t_{a_1-1})$ is zero, and for $i = 0, 1, \dots, a_1 - 2$ we have*

$$\tau M(t_i) = \begin{matrix} & s_{i+2} & & r_2 \\ & s_{i+3} & & r_3 \\ & \vdots & & \vdots \\ s_{a_1-1} & & r_{a_2-1} & \\ & & r_{a_2} & \end{matrix}, \quad \tau M(t_i)_x = \begin{cases} 0 & \text{if } x = r_0 \text{ or } x = r_1; \\ 0 & \text{if } x = t_j \text{ for any } j; \\ 0 & \text{if } x = s_j \text{ for any } 1 \leq j \leq i+1; \\ k & \text{otherwise.} \end{cases}$$

Furthermore, the projective dimension of $M(t_i)$ is 1.

Proof. Remark 4.1 implies that $\tau M(t_{a_1-1})$ is zero. Suppose now that $0 \leq i \leq a_2 - 2$. We have the projective resolution

$$0 \rightarrow P(r_{a_2}) \xrightarrow{f} P(r_1) \oplus P(s_{i+1}) \rightarrow M(t_i) \rightarrow 0.$$

This shows that $\text{pd } M(t_i) = 1$. The rest of the proof is analogous to Lemma 4.2. $\square$

Lemma 4.4. *For $i = 1, 2, \dots, a_1 - 1$, we have*

$$\tau M(s_i) = \begin{matrix} & t_1 \\ & t_2 \\ & \vdots \\ & t_i \end{matrix}, \quad \tau M(s_i)_x = \begin{cases} 0 & \text{if } x = r_j \text{ for any } j; \\ 0 & \text{if } x = t_j \text{ for any } j > i; \\ 0 & \text{if } x = s_j \text{ for any } j; \\ k & \text{otherwise.} \end{cases}$$

Furthermore, the projective dimension of $M(s_i)$ is 1.

Proof. By definition of $M(s_i)$, we have the projective resolution

$$0 \rightarrow P(t_i) \xrightarrow{f} P(r_0) \rightarrow M(s_i) \rightarrow 0.$$

This shows that $\text{pd } M(s_i) = 1$. The rest of the proof is analogous to Lemma 4.2. $\square$

Proposition 4.5. *The module T from Definition 2.6 has projective dimension 1.*

Proof. This follows immediately from Lemmata 4.2–4.4. $\square$

5. T is a tilting and a τ -tilting module

In this section, we show that T is a tilting A -module and a τ -tilting A -module. We then show that T induces a cluster-tilting object in the cluster category $\mathcal{C}$.

Recall that an A -module T is called a *tilting module* if $\text{Ext}^1(T, T) = 0$, $\text{pd } T \leq 1$, and the number $|T|$ of isoclasses of indecomposable summands of T is equal to the number of vertices $|Q_0|$ of Q . Furthermore, T is said to be τ -tilting in the sense of [1] if $\text{Hom}(T, \tau T) = 0$ and $|T| = |Q_0|$. It is shown in [1] that every tilting module is τ -tilting. Tilting modules and their endomorphism algebras play a central role in representation theory, see [6] for an introduction.

Recall that an A -module $L = (L_x, \varphi_\alpha)$ is called *thin* if all vector spaces L_x are of dimension at most one. Let $Q(L)$ denote the subquiver of Q containing all vertices $x \in Q_0$ such that $L_x \neq 0$ and all arrows $\alpha \in Q_1$ such that $\varphi_\alpha \neq 0$. A thin module L is indecomposable if and only if $Q(L)$ is connected.

Remark 5.1. For every vertex x , the A -module $M(x)$ is thin and indecomposable.

Lemma 5.2. Let $L = (L_x, \varphi_\alpha)$ be a thin A -module and let $N = (N_x, \varphi'_\alpha)$ be any A -module. Let $f \in \text{Hom}(L, N)$ and let w be a nonzero path in $Q(L)$ from a vertex x to a vertex y . If $f_x = 0$, then $f_y = 0$.

Proof. We have a commutative diagram:

$$\begin{array}{ccc} L_x & \xrightarrow{\varphi_w} & L_y \\ f_x \downarrow & & \downarrow f_y \\ N_x & \xrightarrow{\varphi'_w} & N_y \end{array}$$

Since L is thin, the map φ_w is invertible, so $f_y = \varphi'_w f_x \varphi_w^{-1}$. $\square$

Lemma 5.3. Let $L = M(r_i) = (M(r_i)_x, \varphi_\alpha)$ with $i \neq a_2$ and let $N = (N_x, \psi_\alpha)$ be indecomposable and thin. Suppose $\text{Hom}(L, N) \neq 0$. Then $N_{s_1} \neq 0$ or there exists a vertex r_j with $j \neq a_2 - 1, a_2$ such that $S(r_j)$ is a summand of the socle of N .

Proof. Suppose first that $i = a_2 - 1$. Then $L = M(r_{a_2-1}) = P(s_1)$. Therefore the condition $\text{Hom}(L, N) \neq 0$ implies that $N_{s_1} \neq 0$, and we are done.

Suppose now that $i < a_2 - 1$. Let $f \in \text{Hom}(L, N)$ be a nonzero morphism. Suppose $N_{s_1} = 0$. Then Lemma 5.2 implies $f_y = 0$ for all vertices y that can be reached from s_1 by a path in $Q(L)$. In particular, $f_{r_{a_2}} = 0$.

We have $\text{top } L = S(r_{i+1}) \oplus S(s_1)$. Now consider the path $w : r_{i+1} \rightarrow \cdots \rightarrow r_{a_2}$. Note that w runs through all vertices of $Q(L)$ which cannot be reached by a path from s_1 in $Q(L)$, plus the vertex r_{a_2} . Since f is nonzero, it must be nonzero at some vertex on w . By Lemma 5.2, we see that f must be nonzero at the vertex r_{i+1} . Now let r_j be the last vertex in w such that $f_{r_j} \neq 0$. Denote by w' the subpath of w from r_{i+1} to r_j . Then we have a commutative diagram:

$$\begin{array}{ccccc} L_{r_{i+1}} & \xrightarrow{\varphi_{w'}} & L_{r_j} & \xrightarrow{\varphi_\alpha} & L_{r_{j+1}} \\ 0 \neq f_{r_{i+1}} \downarrow & & \downarrow f_{r_j} \neq 0 & & \downarrow f_{r_{j+1}} = 0 \\ N_{r_{i+1}} & \xrightarrow{\psi_w} & N_{r_j} & \xrightarrow{\psi_\alpha} & N_{r_{j+1}} \end{array}$$

where α is the arrow $r_j \rightarrow r_{j+1}$. The left square implies $\psi_w \neq 0$ and the right square implies $\psi_\alpha = 0$. The arrow α is the only arrow whose source is r_j , since $1 \leq i+1 \leq j < a_2 - 1$. This implies that $S(r_j)$ is a summand of the socle of N . $\square$

Lemma 5.4. *Let $L = M(r_{a_2})$ or $M(s_i)$ with $i = 1, 2, \dots, a_1 - 1$ and let N be indecomposable and thin. Suppose $\text{Hom}(L, N) \neq 0$. Then $N_{r_0} \neq 0$.*

Proof. We have $\text{top } L = S(r_0)$ and every vertex in $Q(L)$ can be reached by a path from r_0 in $Q(L)$. By Lemma 5.2, if $f \in \text{Hom}(L, N)$ is a nonzero morphism, then $f_{r_0} \neq 0$. $\square$

Lemma 5.5. *Let $L = M(t_i)$ with $i = 1, 2, \dots, a_1 - 1$ and let N be indecomposable and thin. Suppose $\text{Hom}(L, N) \neq 0$. Then $N_{r_1} \neq 0$ or there exists a vertex s_j with $i + 1 \leq j \leq a_1 - 1$ such that $S(s_j)$ is a summand of the socle of N .*

Proof. The proof is similar to the proof in Lemma 5.3 with r_1 in the role of s_1 and the path $s_{i+1} \rightarrow \dots \rightarrow s_{a_1-1} \rightarrow r_{a_2}$ in the role of $r_{i+1} \rightarrow \dots \rightarrow r_{a_2-1} \rightarrow r_{a_2}$. $\square$

Lemma 5.6. *Let L be an indecomposable summand of the module T defined in Definition 2.6 and let N be an indecomposable summand of τT . Then*

$$\text{Hom}(L, N) = 0.$$

In particular,

$$\text{Hom}(T, \tau T) = 0 \quad \text{and} \quad \text{Ext}^1(T, T) = 0.$$

Proof. From our computations in Section 4, we see that each indecomposable summand N of τT is thin, and zero at the vertices s_1 , r_1 , and r_0 . Using Lemma 5.4 and $N_{r_0} = 0$, we see that $\text{Hom}(L, N) = 0$ for $L = M(r_{a_2})$ or $M(s_i)$. Using Lemma 5.2 and $N_{s_1} = 0$, we see that $\text{Hom}(L, N) = 0$ for $L = M(r_i)$, $i \neq a_2$ because none of the summands of τT has $S(r_j)$ with $1 \leq j \leq a_2 - 1$ in its socle. Indeed, our computations in Section 4 show that $\text{soc } N$ can only contain $S(r_{a_2})$ and $S(t_i)$ as summands. Similarly, using Lemma 5.5 and $N_{r_i} = 0$, we see that $\text{Hom}(N, L) = 0$ for $L = M(t_i)$ because none of the summands of τT admit $S(s_j)$ as a summand of its socle. This shows $\text{Hom}(L, N) = 0$, and thus $\text{Hom}(T, \tau T) = 0$.

Because of the Auslander-Reiten formula

$$\text{Ext}^1(T, T) \cong D \overline{\text{Hom}}(T, \tau T)$$

this also implies the vanishing of $\text{Ext}^1(T, T)$. $\square$

We are now ready for the main results of this section.

Theorem 5.7. *The module T is a tilting A -module and a τ -tilting A -module.*

Proof. The projective dimension of T is 1 by Proposition 4.5 and T is rigid by Lemma 5.6. In particular, we have shown that $\text{Hom}(T, \tau T) = 0$. The result now follows since the number of non-isomorphic indecomposable summands of T is equal to the number of vertices in Q . $\square$

Theorem 5.8. *The module T induces a cluster-tilting object in the cluster category $\mathcal{C}$.*

Proof. Since T does not share any indecomposable summands with $A[1] = \bigoplus_{x \in Q_0} P(x)[1]$, we see that Theorem 5.7 and [1, Theorem 4.1] complete the proof. $\square$

6. The endomorphism algebra $\text{End } T$

In this section, we study the endomorphism algebra of our A -module T by computing the Hom spaces between its indecomposable summands.

Lemma 6.1. *Given two vertices r_i, r_j on the oriented cycle in Q , we have*

$$\text{Hom}(M(r_i), M(r_j)) = \begin{cases} 0 & \text{if } j = i + 1; \\ k & \text{if } j \neq i + 1, \end{cases}$$

where we use the convention that $r_{a_2+1} = r_0$.

Proof. If $i = a_2$ we have $M(r_i) = P(r_0)$ and hence

$$\text{Hom}(M(r_i), M(r_j)) \cong M(r_j)_{r_0} = \begin{cases} 0 & \text{if } j = 0; \\ k & \text{if } j \neq 0. \end{cases}$$

Suppose now $0 \leq i \leq a_2 - 1$ and consider the projective resolution

$$0 \rightarrow P(r_{a_2}) \rightarrow P(r_{i+1}) \oplus P(s_1) \rightarrow M(r_i) \rightarrow 0.$$

Applying the functor $\text{Hom}(-, M(r_j))$ yields an exact sequence of the form

$$0 \rightarrow \text{Hom}(M(r_i), M(r_j)) \rightarrow \text{Hom}(P(r_{i+1}) \oplus P(s_1), M(r_j)) \rightarrow \text{Hom}(P(r_{a_2}), M(r_j)) \rightarrow 0,$$

where the last term is zero, since $\text{Ext}^1(M(r_i), M(r_j)) = 0$, by Lemma 5.6. Now using the fact that $\text{Hom}(P(i), X) \cong X_i$ for any representation X , we obtain an exact sequence

$$0 \rightarrow \text{Hom}(M(r_i), M(r_j)) \rightarrow M(r_j)_{r_{i+1}} \oplus M(r_j)_{s_1} \rightarrow M(r_j)_{r_{a_2}} \rightarrow 0.$$

Since $j \neq a_2$, we have $M(r_j)_{s_1} = k$ and $M(r_j)_{a_2} = k$. Therefore

$$\text{Hom}(M(r_i), M(r_j)) \cong M(r_j)_{r_{i+1}} = \begin{cases} 0 & \text{if } j = i + 1; \\ k & \text{if } j \neq i + 1. \end{cases} \quad \square$$

Lemma 6.2. *Given two vertices labeled $t_i, t_j \in Q_0$, we have*

$$\text{Hom}(M(t_i), M(t_j)) = \begin{cases} 0 & \text{if } i < j; \\ k & \text{if } i \geq j. \end{cases}$$

Proof. This follows from the observation that if $i \geq j$, then $M(t_i)$ is a submodule of $M(t_j)$, so $\text{Hom}(M(t_i), M(t_j))$ is generated by the inclusion map. $\square$

Lemma 6.3. *Given two vertices labeled $s_i, s_j \in Q_0$, we have*

$$\text{Hom}(M(s_i), M(s_j)) = \begin{cases} 0 & \text{if } i < j; \\ k & \text{if } i \geq j. \end{cases}$$

Proof. Since Q is isomorphic to Q^{op} , we see that this is the dual argument to Lemma 6.2. $\square$

Lemma 6.4. *Let $x \in Q_0$ be any vertex labeled r_j or t_j . Given any vertex labeled $s_i \in Q_0$, we have*

$$\text{Hom}(M(x), M(s_i)) = \text{Hom}(M(x), M(r_{a_2})).$$

Proof. From Definition 2.5, we have a short exact sequence

$$0 \rightarrow P(t_i) \rightarrow M(r_{a_2}) \rightarrow M(s_i) \rightarrow 0.$$

We apply the functor $\text{Hom}(M(x), -)$ to the sequence to get the exact sequence

$$\begin{aligned} 0 \rightarrow \text{Hom}(M(x), P(t_i)) &\rightarrow \text{Hom}(M(x), M(r_{a_2})) \rightarrow \text{Hom}(M(x), M(s_i)) \\ &\rightarrow \text{Ext}^1(M(x), P(t_i)). \end{aligned}$$

We complete the proof by showing that $\text{Hom}(M(x), P(t_i)) = 0$ and $\text{Ext}^1(M(x), P(t_i)) = 0$.

First, let $f \in \text{Hom}(M(x), P(t_i))$. By Definition 2.2 and Definition 2.4, we have

$$\text{top } M(x) = \begin{cases} S(s_1) \oplus S(r_{j+1}) & \text{if } x = r_j; \\ S(r_1) \oplus S(s_{j+1}) & \text{if } x = t_j. \end{cases}$$

In both cases, there is a nonzero path from a vertex y in the top of $M(x)$ to t_i , and $P(t_i)$ is not supported on y . Lemma 5.2 implies that $f_{t_i} = 0$, hence $f = 0$.

Second, we see that $\text{Hom}(P(t_i), \tau M(x)) = 0$ by Lemma 5.6. Hence by the AR formula, $\text{Ext}^1(M(x), P(t_i)) = D\overline{\text{Hom}}(P(t_i), \tau M(x)) = 0$. $\square$

Lemma 6.5. *Let $x \in Q_0$ be any vertex labeled r_j or s_j . Given any vertex labeled $t_i \in Q_0$, we have*

$$\text{Hom}(M(t_i), M(x)) \cong \text{Hom}(M(t_i), M(r_0)).$$

Proof. Since Q is isomorphic to Q^{op} , the argument is dual to Lemma 6.4. $\square$

Lemma 6.6. *Given two vertices labeled $r_i, t_j \in Q_0$, we have*

$$\text{Hom}(M(r_i), M(t_j)) = 0.$$

Proof. This follows from Lemma 5.2 and Lemma 5.4. $\square$

Lemma 6.7. *Given two vertices labeled $r_i, s_j \in Q_0$, we have*

$$\text{Hom}(M(s_j), M(r_i)) = 0.$$

Proof. The argument is dual to Lemma 6.6. $\square$

Lemma 6.8. *Given two vertices labeled $s_i, t_j \in Q_0$, we have*

$$\text{Hom}(M(s_i), M(t_j)) = 0.$$

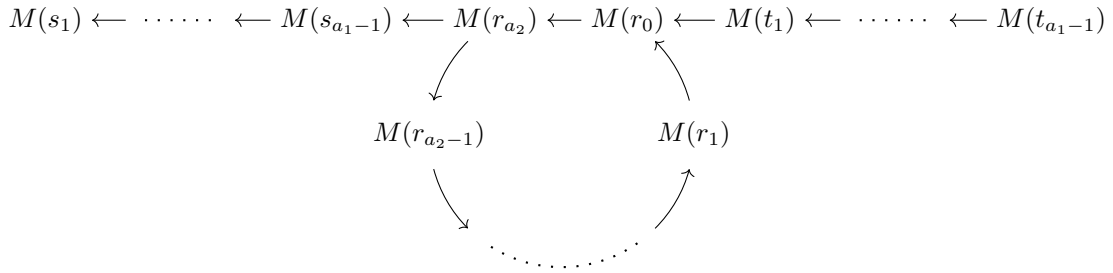
Proof. This follows from the fact that the top of $M(s_i)$ is $S(r_0)$ and the vertex r_0 does not lie in the support of $M(t_j)$. $\square$

The results of this subsection can be combined in the following theorem. Recall that A is isomorphic to its dual.

Theorem 6.9. *The endomorphism algebra of T is isomorphic to (the dual of) A ,*

$$\text{End}_A T \cong A.$$

Furthermore, the mapping $x \mapsto M(x)$ induces an isomorphism of quivers $Q_A^{\text{op}} \rightarrow Q_{\text{End}_A T}$. In particular, the quiver $Q_{\text{End}_A T}$ of $\text{End}_A T$ is given by



Proof. By Lemma 6.1, the full subquiver with vertices $M(r_i)$ is an oriented cycle of length $a_2 + 1$ in which every subpath of length a_2 is zero. Lemmata 6.2 and 6.3 imply that the full subquivers with vertices $M(s_i)$ and $M(t_i)$, respectively, are equioriented of type $\mathbb{A}$ and there are no relations on these branches. Lemmata 6.4 and 6.5 imply that there is an arrow $M(r_{a_2}) \rightarrow M(s_{a_1-1})$ and an arrow $M(t_1) \rightarrow M(r_0)$ and that there are no relations between the cycle and the two branches. The Lemmata 6.6–6.8 show that there are no other arrows. This shows that the quiver of $\text{End}_A T$ is the one in the theorem and its relations are those coming from the potential. $\square$

7. Mutation and the cluster algebra $\mathcal{A}(Q)$

In Theorem 5.8, we have shown that T induces a cluster-tilting object in the cluster category $\mathcal{C}$ introduced in [7]. From Theorem 6.9, the quiver given by T is the opposite quiver Q^{op} , so using Lemma 2.3 in [5] we see that the correspondence $T \mapsto A[1]$ induces a cluster automorphism σ of the cluster algebra $\mathcal{A}(Q)$ associated to the quiver Q .

In this section, we define a mutation sequence μ on the quiver Q and show that Q is of acyclic type. The cluster automorphism σ is given by μ , which we prove by showing that μ sends T to $A[1] = \bigoplus_{x \in Q_0} P(x)[1]$ in the cluster category. Furthermore, we show that σ has order 2.

Definition 7.1. We define the mutation sequence μ on the quiver Q by

$$\mu := \mu_R^{-1} \circ \mu_S \circ \mu_T \circ \mu_R,$$

where

$$\mu_R = \mu_{r_{a_2-1}} \circ \cdots \circ \mu_{r_2} \circ \mu_{r_1},$$

$$\mu_S = \mu_{s_1} \mu_{s_2} \mu_{s_3} \cdots \mu_{r_{a_2}} \circ \cdots \circ \mu_{s_1} \mu_{s_2} \mu_{s_3} \circ \mu_{s_1} \mu_{s_2} \circ \mu_{s_1},$$

$$\mu_T = \mu_{t_{a_1-1}} \mu_{t_{a_1-2}} \mu_{t_{a_1-3}} \cdots \mu_{r_0} \circ \cdots \circ \mu_{t_{a_1-1}} \mu_{t_{a_1-2}} \mu_{t_{a_1-3}} \circ \mu_{t_{a_1-1}} \mu_{t_{a_1-2}} \circ \mu_{t_{a_1-1}}.$$

We see the effect of these mutation sequences on the quiver illustrated in Figs. 2–5. The quiver $\mu_R Q$ is seen in the left picture of Fig. 3.

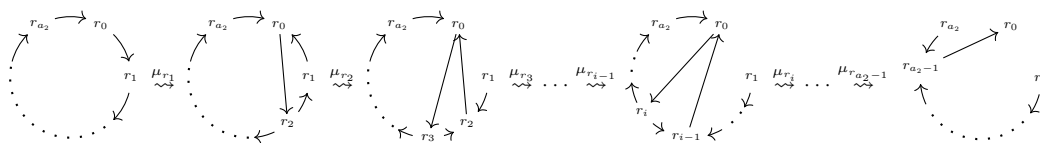


Fig. 2. The mutation sequence μ_R on the cycle $r_0 \rightarrow r_1 \rightarrow \dots \rightarrow r_{a_2} \rightarrow r_0$.

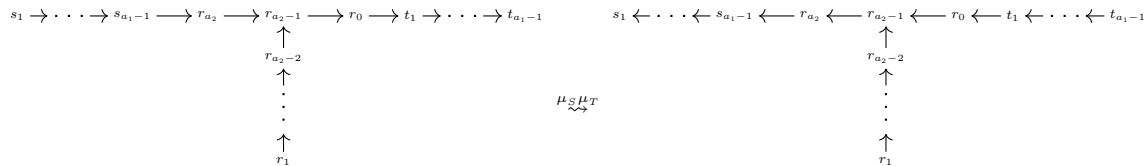


Fig. 3. The mutation sequence $\mu_S \mu_T$ on the quiver $\mu_R Q$.

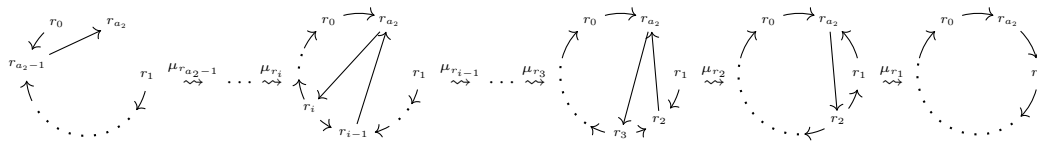


Fig. 4. The mutation sequence μ_R on the cycle $r_1 \rightarrow r_2 \rightarrow \dots \rightarrow r_{a_2-1} \rightarrow r_0 \rightarrow r_{a_2} \rightarrow r_1$, with μ_R^{-1} shown.

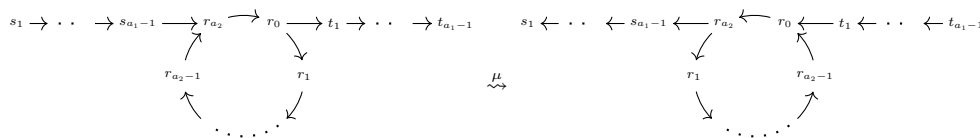


Fig. 5. The mutation sequence $\mu = \mu_R^{-1} \mu_T \mu_S \mu_R$ on the quiver Q .

Theorem 7.2. *The cluster algebra $\mathcal{A}(Q)$ is of acyclic type $\mu_R Q$. Moreover, the cases where it is of finite or tame type are the following.*

$\mathbb{A}_{2a_1+1}$	if $a_2 = 2$;
$\mathbb{D}_{a_2+1}$	if $a_1 = 1$;
$\mathbb{E}_6$	if $(a_1, a_2) = (2, 3)$;
$\tilde{\mathbb{E}}_7$	if $(a_1, a_2) = (3, 3)$;
$\tilde{\mathbb{E}}_6$	if $(a_1, a_2) = (2, 4)$;

and it is of wild acyclic type in all other cases.

Proof. The quiver $\mu_R Q$ is given by Fig. 2 together with the branches $s_1 \rightarrow \dots \rightarrow s_{a_1-1} \rightarrow r_{a_2}$ and $r_0 \rightarrow t_1 \rightarrow \dots \rightarrow t_{a_1-1}$, and is shown in Fig. 3. $\square$

Remark 7.3. The quiver $\mu_T \mu_R Q$ is the quiver $T_{p,q,r}$ of [4,12], with $p = q = a_1 + 1$ and $r = a_2 - 1$.

In order to study the cluster automorphism σ , we now consider the sequence of mutations μ on the level of cluster-tilting objects in the cluster category using the categorification of [7]. Each indecomposable summand T_i of the cluster-tilting object corresponds to a vertex i in the quiver, and the mutation at i is computed by an exchange triangle in the cluster category.

Lemma 7.4. *We have*

$$\mu_S(\mu_R T) = \mu_S^{-1}(\mu_R T),$$

and dually

$$\mu_T(\mu_R T) = \mu_T^{-1}(\mu_R T).$$

Proof. In $\mu_R Q$, there are no arrows between s_i and $s_{i+\ell}$ for $\ell > 1$. Therefore

$$\mu_{s_i} \mu_{s_{i+\ell}} = \mu_{s_{i+\ell}} \mu_{s_i}.$$

Thus the mutation sequence

$$\mu_{s_1} \mu_{s_2} \mu_{s_3} \cdots \mu_{r_{a_2}} \circ \cdots \circ \mu_{s_1} \mu_{s_2} \mu_{s_3} \circ \mu_{s_1} \mu_{s_2} \circ \mu_{s_1}$$

is equal to the mutation sequence

$$\mu_{s_1} \circ \mu_{s_2} \mu_{s_1} \circ \mu_{s_3} \mu_{s_2} \mu_{s_1} \cdots \circ \mu_{r_{a_2}} \cdots \mu_{s_3} \mu_{s_2} \mu_{s_1},$$

and so we have $\mu_S(\mu_R T) = \mu_S^{-1}(\mu_R T)$. $\square$

Corollary 7.5. *The mutation sequence μ is of order two, i.e.,*

$$\mu T = \mu^{-1} T.$$

Proof. There are no arrows between s_i and t_j for any $1 \leq i, j \leq a_1 - 1$, so $\mu_S \mu_T = \mu_T \mu_S$. Thus $\mu^2 = \mu_R^{-1} \mu_S \mu_T \mu_R \mu_R^{-1} \mu_S \mu_T \mu_R = \mu_R^{-1} \mu_S \mu_S \mu_T \mu_T \mu_R$, and by Lemma 7.4, this is equal to $\mu_R^{-1} \mu_R$. $\square$

Theorem 7.6. *The correspondence $T \mapsto A[1]$ induces a cluster automorphism σ . The automorphism σ has order two, and is given by the sequence of mutations μ .*

Proof. From Corollary 7.5, it is enough to show that σ is given by the sequence of mutations μ . We show that $\mu_T \mu_S \mu_R T = \mu_R A[1]$.

First, we apply the mutation sequence μ_R to T . As shown in Fig. 2, at each step prior to mutating at the vertex r_i we have exactly one arrow ending at r_i , namely $r_0 \rightarrow r_i$. Hence we replace the summand $M(r_i)$ of T with the module $X(r_i)$, given by the exchange triangle

$$M(r_0)[-1] \rightarrow X(r_i)[-1] \rightarrow M(r_i) \xrightarrow{f} M(r_0) \rightarrow X(r_i),$$

where f is an $\text{add}(T/M(r_i))$ -approximation. By Remark 4.1, we know that $M(r_0)$ is injective. Therefore the image of this triangle under the functor $\text{Hom}_{\mathcal{C}}(T, -)$ is the exact sequence in $\text{Mod } A$:

$$0 \rightarrow \tau^{-1} X(r_i) \rightarrow M(r_i) \xrightarrow{f} M(r_0).$$

So $\tau^{-1} X(r_i)$ is given by the kernel of the morphism $M(r_i) \xrightarrow{f} M(r_0)$, hence

$$\tau^{-1} X(r_i) = \begin{array}{cc} & r_0 \\ r_1 & t_1 \\ r_2 & t_2 \\ \vdots & \vdots \\ r_{i-1} & t_{a_1-1} \end{array}.$$

Now $\tau^{-1} X(r_i)$ has a projective resolution $\cdots \rightarrow P(r_i) \xrightarrow{g} P(r_0) \rightarrow \tau^{-1} X(r_i) \rightarrow 0$, so by applying the Nakayama functor ν we see that $X(r_i)$ is given by the kernel of the map $I(r_i) \xrightarrow{\nu g} I(r_0)$. Hence

$$X(r_i) = \begin{matrix} r_1 \\ r_2 \\ \vdots \\ r_i \end{matrix}, \quad \text{and} \quad \mu_R T = T \setminus \left(\bigoplus_{i=1}^{a_2-1} M(r_i) \right) \oplus \left(\bigoplus_{i=1}^{a_2-1} X(r_i) \right).$$

Second, we apply the mutation sequence μ_S . Note that the mutation sequence μ_S only mutates the quiver $\mu_R Q$ at sources. Since the quiver of T is Q^{op} , each mutation in this sequence acts on the corresponding summand of T as the shift operator. Dually, each mutation in the sequence μ_T acts on the corresponding summand of T as the inverse shift operator.

Let $s_{a_1} := r_{a_2}$ and $t_0 := r_0$. For each i , the mutation sequence μ_S mutates the vertex s_i exactly $a_1 - i + 1$ times and the mutation sequence μ_T mutates the vertex t_i exactly $i + 1$ times. We conclude that

$$\mu_T \mu_S \mu_R T = \left(\bigoplus_{i=1}^{a_2-1} X(r_i) \right) \oplus \left(\bigoplus_{i=1}^{a_1} M(s_i)[a_1 - i + 1] \right) \oplus \left(\bigoplus_{i=0}^{a_1-1} M(t_i)[-(i + 1)] \right).$$

Next, we compute $M(s_i)[a_1 - i + 1]$ and $M(t_i)[-(i + 1)]$ for all i . Given a pair (j, ℓ) such that $1 \leq j \leq \ell < a_1 - 1$, define

$$L := \begin{matrix} t_j \\ t_{j+1} \\ \vdots \\ t_\ell \end{matrix}.$$

Then L has a projective resolution $\cdots \rightarrow P(t_{\ell+1}) \xrightarrow{g} P(t_j) \rightarrow L \rightarrow 0$. Applying the Nakayama functor ν , we see that $\tau L = L[1]$ is given by the kernel of the morphism $I(t_{\ell+1}) \xrightarrow{\nu g} I(t_j)$, so

$$L[1] = \begin{matrix} t_{j+1} \\ t_{j+2} \\ \vdots \\ t_{\ell+1} \end{matrix}.$$

Recall $\tau M(s_i)$ from Lemma 4.4. Then for all i , it follows that

$$M(s_i)[a_1 - i + 1] = \tau M(s_i)[a_1 - i - 1][1],$$

$$= \begin{matrix} t_1 \\ t_2 \\ \vdots \\ t_i \end{matrix} [a_1 - i - 1][1],$$

$$= \begin{matrix} t_{a_1-i} \\ t_{a_1-i+1} \\ \vdots \\ t_{a_1-1} \end{matrix} [1],$$

$$= P(t_{a_1-i})[1].$$

Then from a dual argument, we also have

$$M(t_i)[-(i + 1)] = I(s_{a_1-i})[-1] = P(s_{a_1-i})[1].$$

Hence

$$\mu_T \mu_S \mu_R T = \left(\bigoplus_{i=1}^{a_2-1} X(r_i) \right) \oplus \left(\bigoplus_{i=0}^{a_1-1} P(s_i)[1] \right) \oplus \left(\bigoplus_{i=1}^{a_1} P(t_i)[1] \right).$$

Finally, let $A[1] = \bigoplus_{x \in Q_0} P(x)[1]$ with quiver Q^{op} . For $i = 1, 2, \dots, a_2 - 1$, let $X'(r_i)$ be the module replacing the summand $P(r_i)[1]$ in the module $\mu_R A[1]$. To complete the proof, we show that $X'(r_i) = X(r_i)$. This proves that $\mu_S \mu_T \mu_R T = \mu_R A[1]$, and thus $\mu T = A[1]$.

Recall from Fig. 2 that at each step of μ_R , prior to mutating at the vertex r_i we have exactly one arrow ending at r_i , namely $r_0 \rightarrow r_i$. So $X'(r_i)$ is given by the exchange triangle

$$P(r_i) \xrightarrow{h} P(r_0) \rightarrow X'(r_i)[-1] \rightarrow P(r_i)[1] \rightarrow P(r_0)[1] \rightarrow X'(r_i).$$

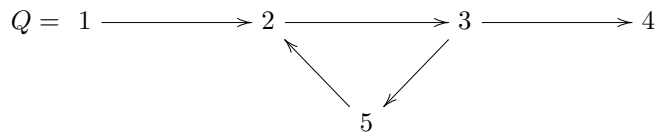
The image of this triangle under the functor $\text{Hom}_C(A[1], -)$ is the exact sequence in $\text{mod } A$:

$$P(r_i) \xrightarrow{h} P(r_0) \rightarrow \tau^{-1} X'(r_i) \rightarrow 0.$$

We have a projective resolution $\dots \rightarrow P(r_i) \xrightarrow{h} P(r_0) \rightarrow \tau^{-1} X'(r_i)$, so applying the Nakayama functor ν tells us that $X'(r_i)$ is given by the kernel of the map $I(r_i) \xrightarrow{\nu h} I(r_0)$. This is exactly how we computed $X(r_i)$, so $X'(r_i) = X(r_i)$. This completes the proof. $\square$

8. Example

We illustrate the results in the example $(a_1, a_2) = (2, 2)$ which is of type $\mathbb{A}_5$. The corresponding knot is the figure eight knot $L[2, 2]$. We have



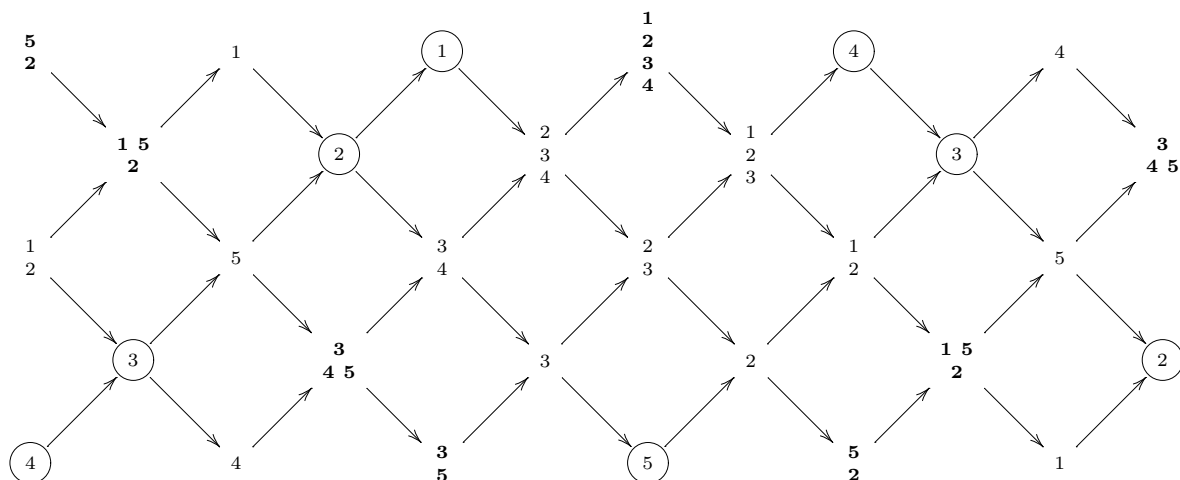
and the Jacobian algebra A is the path algebra of Q modulo the two-sided ideal generated by the subpaths of length 2 in the 3-cycle. Thus A is a cluster-tilted algebra of Dynkin type $\mathbb{A}_5$.

The A -module $T = \bigoplus_{i=1}^5 M(i)$ is given by

$$M(1) = \begin{smallmatrix} 3 \\ 5 \end{smallmatrix}, \quad M(2) = \begin{smallmatrix} 3 & \\ & 5 \end{smallmatrix}, \quad M(3) = \begin{smallmatrix} 1 & 5 \\ & 2 \end{smallmatrix}, \quad M(4) = \begin{smallmatrix} 5 \\ 2 \end{smallmatrix}, \quad M(5) = \begin{smallmatrix} 1 & \\ 2 & \\ 3 & \\ 4 \end{smallmatrix}.$$

The indecomposable summands $M(2), M(3), M(5)$ correspond to the vertices on the 3-cycle. Each of these indecomposables has precisely 5 submodules, as expected, since 5 is the number of terms in the Jones polynomial of the figure eight knot. For example, the submodules of $M(2)$ are $0, 4, 5, 4 \oplus 5$ and $M(2)$.

The Auslander-Reiten quiver of the cluster category is the following



where vertices with the same labels are identified, and the shift of the projective $P(i)$ is denoted by an i in a circle. The summands of T are set in bold face.

The mutation sequence μ of Section 7 is $\mu = \mu_5 \circ \mu_4 \circ \mu_3 \circ \mu_4 \circ \mu_1 \circ \mu_2 \circ \mu_1 \circ \mu_5$. This mutation sequence sends T to $A[1]$, more precisely

$$M(1) \mapsto P(4)[1], \quad M(2) \mapsto P(3)[1], \quad M(3) \mapsto P(2)[1], \quad M(4) \mapsto P(1)[1], \quad M(5) \mapsto P(5)[1].$$

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