HARMONIC INTRINSIC GRAPHS IN THE HEISENBERG GROUP

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ABSTRACT. Minimal surfaces in $\mathbb{R}^n$ can be locally approximated by graphs of harmonic functions, i.e., functions that are critical points of the Dirichlet energy, but no analogous theorem is known for H–minimal surfaces in the three-dimensional Heisenberg group $\mathbb{H}$, which are known to have singularities. In this paper, we introduce a definition of intrinsic Dirichlet energy for surfaces in $\mathbb{H}$ and study the critical points of this energy, which we call contact harmonic graphs. Nearly flat regions of H–minimal surfaces can often be approximated by such graphs. We give a calibration condition for an intrinsic Lipschitz graph to be energy-minimizing, construct energy-minimizing graphs with a variety of singularities, and prove a first variation formula for the energy of intrinsic Lipschitz graphs and piecewise smooth intrinsic graphs.

1. INTRODUCTION

Minimal surfaces in $\mathbb{R}^3$ are smooth, but the analogous H–minimal surfaces in the three-dimensional Heisenberg group need not be. Recall that an H–minimal surface is a stationary point of the Heisenberg area functional, which is proportional to the 3–dimensional spherical Hausdorff measure. There are many examples of H–minimal surfaces with a singularity along a curve [Pau06], as shown in Figure 1.

In fact, H–minimal surfaces can have singularities in regions that are close to flat. The surfaces in Figure 1, for example, grow closer to planes as one moves from left to right. Subtler examples come from constructions of Ritoré [Rit09] and Nicolussi Golo and Ritoré [GR21].

Theorem 1.1 ([GR21])**.** *Let S^1 be the unit circle of horizontal vectors in $\mathbb{H}$. Given a sequence (finite or countably infinite) of disjoint nonempty open arcs $A_1, A_2, \dots \subset S^1$, each of angle less than 2π, let $K = S^1 \smallsetminus \bigcup_i A_i$ be its complement. Let $a_i, b_i \in S^1$ be the endpoints of A_i and let $m_i \in S^1$ be the midpoint of A_i. Let Σ_K be the surface consisting of the union of:*

- *for each $k \in K$, a horizontal ray from the origin in the direction of k,*
- *for each i, a horizontal ray R_i from the origin in the direction of m_i, and*
- *for each $p \in R_i$, a pair of horizontal rays from p in the directions of a_i and b_i.*

Then Σ_K is a scale-invariant area-minimizing surface.

For any closed subset $K \subset S^1$ with at least two points, there are arcs A_i as above so that $K = S^1 \smallsetminus \bigcup_i A_i$. The A_i's are determined up to a permutation by K, so Σ_K is determined by K.

If there are only finitely many A_i's, then Σ_K is a C_1 surface in $\mathbb{R}^3$, and its characteristic nexus (the set of points where the Euclidean tangent plane to Σ_K is horizontal) is the

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This material is based upon work supported by the National Science Foundation under Grant Nos. 2005609 and 1926686 and research done while the author was a visiting member at the Institute of Advanced Study.

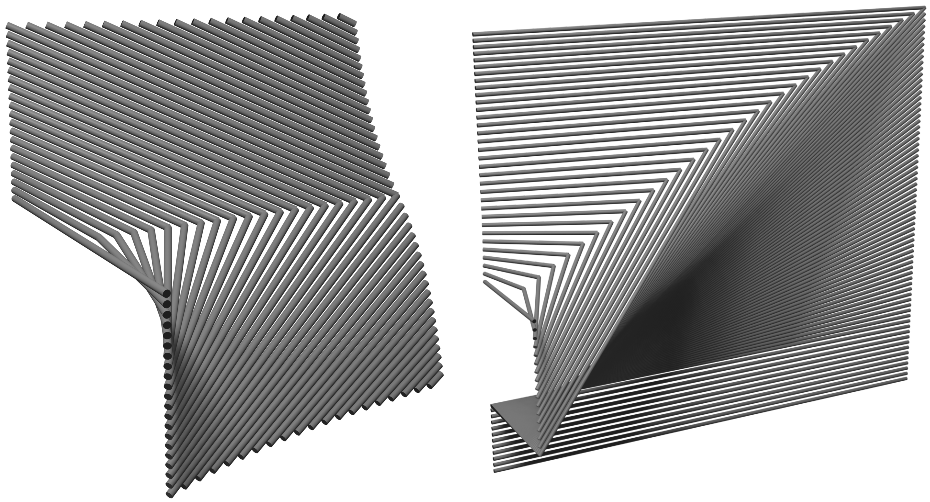

FIGURE 1. Examples of H–minimal and contact harmonic graphs with singularities. In both cases, the surfaces consist of horizontal line segments which intersect only along the characteristic nexus. 3D models can be found in the supplementary materials.

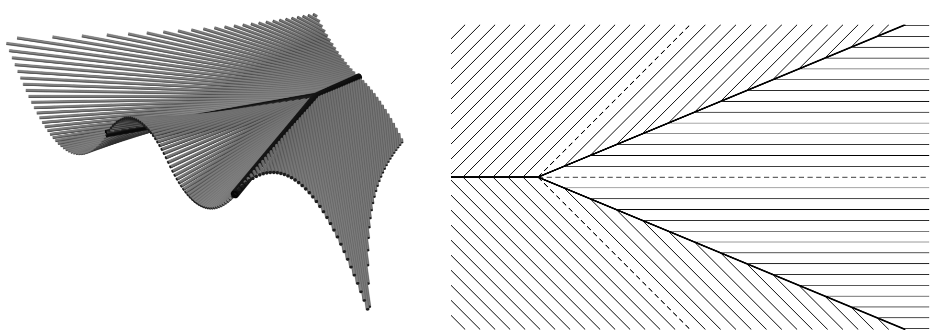

FIGURE 2. An H–minimal graph with branched characteristic nexus (left) and the foliation of Σ by horizontal curves, projected to the xy–plane (right). Dashed lines mark the boundaries between the pieces of herringbone surfaces (see Lemma 3.5) that make up Σ and thick lines mark the characteristic nexus. A 3D model of the surface can be found in the supplementary materials.

union of the R_i's. Figure 2 shows an example of such a surface; here, the rays R_i are the thick lines and K consists of the three points in S^1 corresponding to the dashed lines.

If we bring these three points close together, Σ_K converges to a vertical plane; one can show (see Proposition 1.3) that if we let $\Theta(t) = (\cos(t), \sin(t), 0)$ parametrize S^1 and let $K = \{\Theta(-\varepsilon), \Theta(0), \Theta(\varepsilon)\} \subset S^1$, then Σ_K converges to the xz–plane as $\varepsilon \to 0$. That is, Σ_K can be arbitrarily close to the xz–plane in $\mathbb{H}$ and still have a branched singularity.

In fact, there are examples with worse behavior. Suppose that K is a Cantor set with positive measure and let $\chi = \bigcup R_i$. Since K is a Cantor set, the closure $\overline{\chi}$ contains the cone over K, which has positive measure. The rectifiability of Σ_K implies that almost every point in $\overline{\chi}$ has a vertical approximate tangent plane; indeed, this holds for every point in $\overline{\chi} \setminus \chi$. If $p \in \overline{\chi} \setminus \chi$, then Σ_K is flat near p in the sense that the neighborhoods $\Sigma_K \cap B(p,\varepsilon)$ are close to a vertical plane for sufficiently small ε, but singular in the sense that $\Sigma_K \cap B(p,\varepsilon)$ is nonsmooth for all ε and the unit horizontal normal to $\Sigma_K \cap B(p,\varepsilon)$ is discontinuous.

This does not happen in $\mathbb{R}^n$. A key step in proving the regularity of codimension–1 minimal surfaces in $\mathbb{R}^n$ is to show that if E is a perimeter-minimizing subset of $\mathbb{R}^n$ and p lies in the reduced boundary of E, then ∂E can be approximated by graphs of harmonic functions on small balls around p. One can then use the regularity of harmonic functions to show that ∂E is smooth on sufficiently small balls around p; see for instance [Mag12] for an exposition.

In this paper, we propose a method to study singular points in H–minimal surfaces by using an analogue of harmonic functions that we call *contact harmonic graphs*. Many of these graphs arise as limits of H–minimal surfaces under rescaling and stretching, and we conjecture that, as in $\mathbb{R}^n$, one can use harmonic graphs to approximate sufficiently flat regions of H–minimal surfaces. A harmonic graph is a critical point of the *intrinsic Dirichlet energy* (see (1)); we describe a calibration method to prove that a graph is energy-minimizing and use calibrations to construct several examples. In addition, we prove a first variation formula for the intrinsic Dirichlet energy that lets us characterize intrinsic Lipschitz harmonic graphs.

Before we state our results, we set some notation. We define the Heisenberg group $\mathbb{H}$ to be the set $\mathbb{R}^3$ equipped with the product

$$(x,y,z)\cdot(x',y',z') = \left(x+x', y+y', z+z' + \frac{xy'-yx'}{2}\right).$$

Let $x,y,z\colon \mathbb{H}\to\mathbb{R}$ be the coordinate functions and let X,Y,Z be the coordinate vectors. The one-parameter subgroups of $\mathbb{H}$ generated by these vectors are the coordinate axes, so we write elements of these one-parameter subgroups as $X^t = (t,0,0)$, $Y^t = (0,t,0)$, and $Z^t = (0,0,t)$, for $t\in\mathbb{R}$. Let $V_0 = \{(x,y,z)\in\mathbb{H} : y=0\}$ be the xz–plane. For any $f\colon V_0\to\mathbb{R}$, we define the intrinsic graph of f to be

$$\Gamma_f = \{v\cdot Y^{f(v)} : v\in V_0\} = \left\{\left(x, f(x,z), z+\frac{1}{2}xf(x,z)\right) : x,z\in\mathbb{R}\right\}$$

and define $\Psi_f\colon V_0\to\Gamma_f$ by $\Psi_f(v) = v\cdot Y^{f(v)}$. Let μ be Lebesgue measure on V_0.

When f is an *intrinsic Lipschitz function* (see Definition 2.1), Γ_f has approximate tangent planes with respect to the Carnot–Carathéodory metric almost everywhere [FSSC11] and these planes are vertical (parallel to the z–axis). Each tangent plane projects to a line in the xy–plane with slope given by a nonlinear differential operator, the *intrinsic gradient*. When f is smooth, the intrinsic gradient of f is given by

$$\nabla_f f = (\partial_x - f\partial_z)f;$$

when f is not smooth, we define $\nabla_f f$ in the distributional sense (see Definition 2.2).

Given a bounded open subset $U\subset V_0$ and an intrinsic Lipschitz function $f\colon U\to\mathbb{R}$, we define the *intrinsic Dirichlet energy* (or simply *energy*) of f on U by

$$\widetilde{E}_U(f) = \frac{1}{2}\int_U |\nabla_f f|^2\,\mathrm{d}\mu, \tag{1}$$

where μ is Lebesgue measure on V_0. For $\Gamma = \Gamma_f$ and for any bounded open subset $W \subset \mathbb{H}$, we define the energy of Γ on W by $E_W(\Gamma) = \widetilde{E}_{\Pi(W\cap\Gamma)}(f)$, where $\Pi\colon \mathbb{H} \to V_0$, $\Pi(p) = pY^{-y(p)}$ projects $\mathbb{H}$ to V_0.

By [CMPSC14, Thm. 1.6], the spherical Hausdorff measure[1] of an intrinsic Lipschitz graph $\Gamma = \Gamma_f$ of $f\colon U \to \mathbb{R}$ can be written

$$\mathscr{S}^3(\Gamma) = \int_U \sqrt{1 + |\nabla_f f|^2}\,\mathrm{d}\mu. \tag{2}$$

In fact, the formula includes a multiplicative constant, but throughout this paper, we will normalize $\mathscr{S}^3$ so that (2) holds without a constant. When $\nabla_f f$ is small,

$$\mathscr{S}^3(\Gamma) = \int_U 1 + \frac{1}{2}|\nabla_f f|^2 + O(|\nabla_f f|^3)\,\mathrm{d}\mu = \mu(U) + \widetilde{E}_U(f) + O(\mu(U)\|\nabla_f f\|_\infty^3), \tag{3}$$

so when $\nabla_f f$ is small, the intrinsic Dirichlet energy of Γ is closely linked to its area.

This suggests that an area-minimizing surface with small $\nabla_f f$ should be close to energy-minimizing. We can formalize this notion using stretch automorphisms. For $a, b > 0$, the map $s_{a,b}(x, y, z) = (ax, by, abz)$ is an automorphism of $\mathbb{H}$ that sends intrinsic graphs to intrinsic graphs. Let $r > 0$ and let $s(x, y, z) = s_{r^{-1},r}(x, y, z) = (r^{-1}x, ry, z)$. Then r multiplies $\nabla_f f$ by $\frac{b}{a}$ and the Jacobian of $s|_{V_0}$ is r^{-1}, $E_{s(U)}(s(\Gamma)) = r^3 E_U(\Gamma)$. One can use this to construct energy-minimizing surfaces; if $\Gamma_1, \Gamma_2, \dots,$ are area-minimizing surfaces and Λ is such that $s_{i^{-1},i}(\Gamma_i) = \Lambda$ for all i, then Λ is energy-minimizing (see Section 3). This shows, for instance, that if $f(x, z) = -a\sqrt{z}\,\mathrm{sign}(z)$ for some $a \neq 0$, then Γ_f is energy-minimizing (Lemma 3.5). These surfaces were studied and generalized in [Pau06, CHY07, MSCV08, Rit09]; we call them *herringbone surfaces*, since they can be written as the union of horizontal rays that branch out from the x–axis in a herringbone pattern.

The area-minimizing cones constructed in [GR21] also have energy-minimizing analogues. As in [GR21], these surfaces can be written as a union of horizontal rays that either end at the origin or branch off of the singular set; the renderings in the figures are made up of these rays. The *slope* of a horizontal ray is defined as the slope of its projection to the xy–plane; a horizontal ray is *positive* or *negative* depending on whether it points in the $+x$– or $-x$–direction.

Theorem 1.2. *Let $\alpha > 0$. Let $I_1, I_2, \dots \subset [-\alpha, \alpha]$ be a collection of disjoint nonempty open intervals, and let $K = [-\alpha, \alpha] \setminus \bigcup I_i$. Let $I_i = (a_i, b_i)$ and let $m_i = \frac{a_i + b_i}{2}$. Let $\Lambda_K \subset \mathbb{H}$ be the surface consisting of the union of:*

1. *the negative x–axis, denoted R_0,*
2. *for each $k \in K$, a positive horizontal ray from the origin with slope k,*
3. *for each $i \geqslant 1$, a positive horizontal ray R_i from the origin with slope m_i, and*
4. *for each $p \in R_0$, a pair of positive horizontal rays from p with slopes α and $-\alpha$.*
5. *for each $p \in R_i$, $i \geqslant 1$, a pair of positive horizontal rays from p with slopes a_i and b_i.*

Then Λ_K is a scale-invariant energy-minimizing surface.

Since $\alpha = \sup K$ and $-\alpha = \inf K$, Λ_K is uniquely determined by K. Note that Λ_K is an entire Z–graph, i.e., a set of the form $\{(x, y, g(x, y)) \in \mathbb{H} : x, y \in \mathbb{R}\}$.

These surfaces are only area-minimizing when $K = [-\alpha, \alpha]$ or $K = [-\alpha, \alpha] \setminus (-\beta, \beta)$ for some $0 < \beta \leqslant \alpha$. Otherwise, there is some i such that $m_i \neq 0$. The ray R_i with slope

[1] Unless otherwise specified, distances and Hausdorff measures in $\mathbb{H}$ will be with respect to the Carnot–Carathèodory metric.

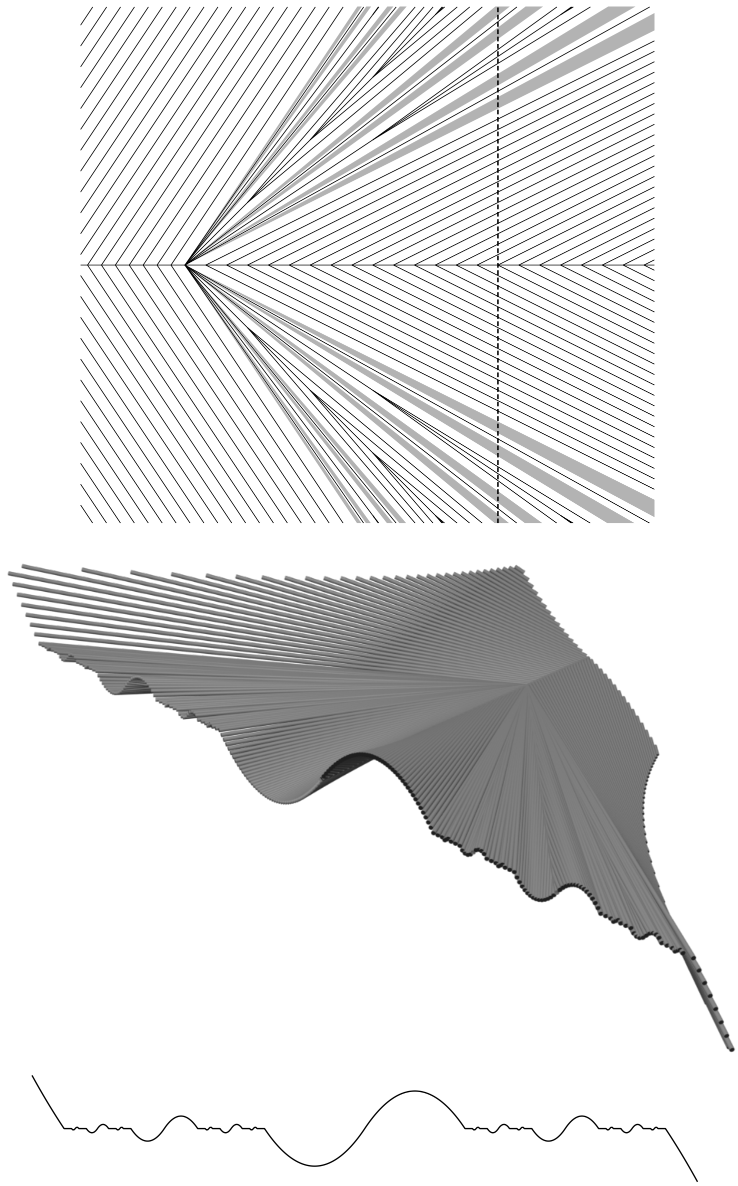

FIGURE 3. An energy-minimizing graph based on the middle-thirds Cantor set K. The top figure shows the foliation of Λ_K by horizontal curves, projected to the xy–plane. In the shaded regions, the surface coincides with the horizontal plane through $\mathbf{0}$. The bottom figure shows the cross-section of the surface in the vertical plane marked by the dashed line; the surface coincides with the xy–plane in the flat regions. A 3D model of the surface can be found in the supplementary materials.

m_i then lies in the characteristic nexus of Λ_K. By the results of [CHY07], a curve in the characteristic nexus of an H–minimal Z–graph must bisect the horizontal rays on either side. Each point on R_i, however, is the vertex of two horizontal rays of slope a_i and b_i, and since $m_i \neq 0$, the angles between R_i and these rays are not equal.

Nevertheless, any such Λ_K can be written as a limit of stretched area-minimizing surfaces.

Proposition 1.3. *Let α, $K = [-\alpha, \alpha] \setminus \bigcup I_i$ be as in Theorem 1.2. For $n > \sqrt{\alpha}$, let $K_n = \Theta(n^{-2}K) \subset S^1$ and let $S_n = s_{n^{-1},n}(\Sigma_{K_n})$, where Σ_{K_n} is constructed as in Theorem 1.1.*

The surfaces S_n and Λ_K are intrinsic graphs; let S_n^+ and Λ_K^+ be the epigraphs they bound. Then $\mathbf{1}_{S_n^+} \to \mathbf{1}_{\Lambda_K^+}$ in $L_1^{\mathrm{loc}}(\mathbb{H})$.

Conversely, one may ask when a sequence of stretched area-minimizing surfaces has a subsequence that converges to an energy-minimizing surface. We first define the *horizontal excess* of a set; the horizontal excess of the half-space bounded by a graph is related to the energy of that graph, but the horizontal excess is defined for a larger class of sets. Let $E \subset \mathbb{H}$ be a subset with finite perimeter and horizontal normal ν_E. For $p \in \partial E$, $r > 0$, and a horizontal vector $\nu \in S^1$, we define

$$\mathrm{Exc}_{B(p,r)}(E, \nu) = \frac{1}{r^3} \int_{B(p,r)} |\nu_E(p) - \nu|^2 \, \mathrm{d}|\partial E|(p)$$
$$\mathrm{Exc}_{B(p,r)}(E) = \inf_{\nu \in S^1} \mathrm{Exc}_{B(p,r)}(E, \nu),$$

where $|\partial E|$ is the perimeter measure of E (Section 2.4). Let $E \subset \mathbb{H}$ be a perimeter minimizer such that $\mathbf{0} \in \partial E$ and for $r > 0$, let $E_r = s_{r^{-1},r^{-1}}(E)$ be a scaling of E. If E_r converges to a set F as $r \to 0$, we call F the *blowup* of E at $\mathbf{0}$. If $\mathrm{Exc}_{B(\mathbf{0},r)}(E, Y) \to 0$ as $r \to 0$, then the blowup of E is a half-space bounded by the xz–plane, but one may hope that there are sequences $r_i \to 0$ and $\lambda_i \to \infty$ such that the stretched sets $s_{\lambda_i^{-1},\lambda_i}(E_{r_i})$ converge to a *anisotropic blowup* which is not bounded by a plane. More rigorously, we ask the following question.

Question 1.4. Let $c, k > 0$. Suppose that $\lambda_i \to \infty$ and $E_1, E_2, \dots \subset \mathbb{H}$ are a sequence of sets that are perimeter-minimizing in the stretched balls $W_i = s_{\lambda_i,\lambda_i^{-1}}(B(\mathbf{0}, k))$. Let $\hat{E}_i = s_{\lambda_i^{-1},\lambda_i}(E_i)$ and suppose that $\mathrm{Exc}_{B(\mathbf{0},k)}(\hat{E}_i, Y) < c$ for all i. Is it possible to choose c and k so that a subsequence of the $\hat{E}_i$'s converges to a set E such that ∂E is energy-minimizing or contact harmonic on $B(\mathbf{0}, 1)$?

(For the definition of contact harmonic, see below.)

One potential approach to this question uses intrinsic Lipschitz approximation. Monti [Mon14] showed that there is an L such that if E is perimeter-minimizing and if $\mathrm{Exc}_{B(p,r)}(E)$ is sufficiently small for some $p \in \partial E$, $r > 0$, then ∂E can be approximated by an intrinsic L–Lipschitz graph Γ near p. Consequently, if E_i satisfies the hypotheses of Question 1.4, then there is an intrinsic L–Lipschitz half-space Γ_i^+ that approximates E_i. Since stretching sends intrinsic Lipschitz graphs to intrinsic Lipschitz graphs, there is an intrinsic Lipschitz function f_i such that $\Gamma_{f_i}^+ = s_{\lambda_i^{-1},\lambda_i}(\Gamma_i^+)$, and $\Gamma_{f_i}^+$ approximates $\hat{E}_i$. One can then hope to show that a subsequence of the f_i's converges.

The problem, however, is that the intrinsic Lipschitz constants of the Γ_i's are uniformly bounded, but the intrinsic Lipschitz constants of the $\hat{\Gamma}_i$'s are not. At best, if h_i is the function such that $\hat{\Gamma}_i = \Gamma_{h_i}$, then h_i satisfies an intrinsic Sobolev bound like

$$\int_D (\nabla_{h_i} h_i)^2 \, \mathrm{d}\mu \leq C$$

on an appropriate domain $D \subset V_0$. Not much is known about such functions, and more research may be necessary to answer Question 1.4.

Another approach to Question 1.4 is to study other characterizations of energy-minimizing graphs. In an analogue to the usual notion of harmonicity, we define a *contact harmonic graph* to be a critical point of the energy with respect to a class of deformations called contact deformations. In particular, energy-minimizing graphs are contact harmonic. In Section 4, we prove some tools for working with such graphs, including first variation formulas for the energy of smooth and intrinsic Lipschitz graphs. These formulas lead to the following characterization of contact harmonicity.

Theorem 1.5. *A smooth intrinsic graph $\Gamma = \Gamma_f$ is contact harmonic on $U \subset V_0$ (a critical point of the energy with respect to contact variations supported on U) if and only if*

$$2\partial_z f \cdot \nabla_f^2 f - \nabla_f^3 f = 0 \tag{4}$$

on U. (Here, ∇_f is the vector field $\nabla_f = \partial_x - f\partial_z$. Since f is smooth, this agrees with the notation for the intrinsic gradient above, and we define $\nabla_f^2 f = \nabla_f[\nabla_f f]$ and $\nabla_f^3 f = \nabla_f[\nabla_f^2 f]$.)

For an intrinsic Lipschitz function f and for $w \in C_c^\infty(U)$, let

$$\Delta_f w = \nabla_f[\partial_x w] - \nabla_f f \cdot \partial_z w - f\nabla_f[\partial_z w].$$

(When f is smooth, $\Delta_f w = \nabla_f^2 w$.) Let

$$B_2(f,w) = -\Delta_f w \cdot \nabla_f f + \frac{1}{2}(\nabla_f f)^2 \cdot \partial_z w,$$
$$B_1(f,w) = f B_2(f,w) - \frac{3}{2}(\nabla_f f)^2 \cdot \nabla_f w,$$

Then $\Gamma = \Gamma_f$ is contact harmonic on U if and only if

$$\int_U B_1(f,w)\,\mathrm{d}\mu = \int_U B_2(f,w)\,\mathrm{d}\mu = 0$$

for every $w \in C_c^\infty(U)$.

This theorem follows from Theorem 4.5 (in the smooth case) and Theorem 4.8 (in the intrinsic Lipschitz case).

Condition (4) is weaker than the condition that the horizontal mean curvature of Γ_f vanishes (i.e., $\nabla_f^2 f = 0$). Indeed, Theorem 1.5 implies that the surface $\Gamma_f = \{(x,y,z) \in \mathbb{H} : y = x^2\}$, which has $\nabla_f^2 f = 2$, is contact harmonic. This happens because contact harmonic graphs are energy-stationary only for contact diffeomorphisms, not arbitrary diffeomorphisms. Contact diffeomorphisms preserve horizontality, so they also preserve horizontal connectivity. That is, if $\phi\colon \mathbb{H} \to \mathbb{H}$ is a contact diffeomorphism and if $p, q \in \Gamma$ are connected by a horizontal curve in Γ, then $\phi(p)$ and $\phi(q)$ are connected by a horizontal curve in $\phi(\Gamma)$. A contact harmonic graph may admit smooth deformations that reduce its energy, but these smooth deformations affect the horizontal connectivity of the graph. [2]

We use the first variation formula for smooth graphs to prove a first variation formula for piecewise smooth graphs (Theorem 4.11). This shows that if Γ is a piecewise smooth contact harmonic graph whose characteristic nexus is a curve and if the foliation by

[2] Manuel Ritoré informs us that the cylinder $\{x^2 + y^2 = r^2\}$ is an area-stationary analogue of this example, i.e. a smooth surface in $\mathbb{H}$ with nonzero horizontal mean curvature that is area-stationary under contact deformations, but as far as we are aware, the calculation has not been published.

horizontal curves is well-behaved near the characteristic nexus, then the slope of the characteristic nexus is the average of the slopes of the horizontal curves on either side of the nexus. Such singularities are similar to the singularities of H–minimal surfaces studied in [Pau06, CHY07, RR08].

Finally, we conclude with an observation. One motivation for this work was the Bernstein problem for surfaces with low regularity. Bernstein's Theorem states that an area-minimizing codimension–1 graph in $\mathbb{R}^n$ is a plane. The Bernstein problem asks for conditions under which other classes of area-minimizing surfaces are planes. Barone Adesi, Serra Cassano, and Vittone showed that if $\Gamma \subset \mathbb{H}$ is a area-minimizing intrinsic graph of an entire C^2 function, then Γ is a vertical plane [BASCV07]. The same result for C^1 functions was proved by Galli and Ritoré [GR15], and for locally Lipschitz functions by Nicolussi Golo and Serra Cassano [NSC19], but there are many examples of area-minimizing intrinsic Lipschitz graphs that are not vertical planes, such as the surfaces constructed in [Rit09] and [GR21]. All of these examples, however, are also entire Z–graphs. One may then ask the following:

Question 1.6. Is every nonplanar entire area-minimizing (or energy-minimizing) intrinsic Lipschitz graph also a Z–graph?

Remark 1.7. The 3D models used in the figures in this paper can be found in .obj format in the ancillary files on the arXiv page for this paper. These files can be opened in Preview on Macs, Paint 3D on Windows, or any 3D modeling program. They can also be found at `https://cims.nyu.edu/~ryoung/harmonic/`.

1.1. **Outline of paper.** In Section 2, we establish notation for the Heisenberg group and prove some integration formulas which will be used throughout the paper. In Section 3, we define the intrinsic Dirichlet energy, prove some basic properties of energy-minimizing surfaces, and describe a method to prove that a surface is energy-minimizing using a calibration. We use this method to prove Theorem 1.2 and Proposition 1.3. In Section 4, we define contact variations and contact harmonicity and prove first variation formulas for the energy of smooth graphs and intrinsic Lipschitz graphs, including Theorem 1.5. Many energy-minimizing and area-minimizing surfaces have singularities along curves, which we call *herringbone singularities*, and we prove a first variation formula for the energy of such a graph in Section 4.2.

The author would like to thank Roberto Monti, Sebastiano Nicolussi Golo, Manuel Ritoré, and Davide Vittone for their helpful discussions during the preparation of this paper and to thank the Institute of Advanced Study for its hospitality.

2. PRELIMINARIES

Throughout this paper, we use the following standard conventions for asymptotic notation. The notations $f \lesssim g$ and $g \gtrsim f$ mean that $f \leqslant Cg$ for some universal constant $C \in (0,\infty)$, and $f \asymp g$ means that $f \lesssim g$ and $g \lesssim f$. If C depends on some parameters, we indicate this by subscripts, i.e., $f \lesssim_t g$ implies that $f \leqslant Cg$ where $C = C(t)$ depends only on t. Finally, we use big–O notation $O(f)$ to denote an error term whose absolute value is bounded by a universal constant multiple of $|f|$.

2.1. **The Heisenberg group.** The Heisenberg group $\mathbb{H}$ is the 3–dimensional simply connected Lie group with Lie algebra

$$\mathfrak{h} := \langle X, Y, Z : [X,Y] = Z, [X,Z] = [Y,Z] = 0\rangle.$$

We identify $\mathbb{H}$ with its Lie algebra via the Baker–Campbell–Hausdorff formula, i.e., $\mathbb{H} = \langle X, Y, Z\rangle \cong \mathbb{R}^3$ and

$$(x,y,z)\cdot(x',y',z') = \left(x+x', y+y', z+z' + \frac{xy'-yx'}{2}\right).$$

Under this identification, elements of $\mathbb{H}$ correspond to left-invariant vector fields on $\mathbb{R}^3$. We write $X = (1,0,0)\in\mathbb{H}$, $Y = (0,1,0)\in\mathbb{H}$, and $Z = (0,0,1)\in\mathbb{H}$. These elements correspond to the left-invariant vector fields $X_{(x,y,z)} = \partial_x - \frac{y}{2}\partial_z$, $Y_{(x,y,z)} = \partial_y + \frac{x}{2}\partial_z$, and $Z_{(x,y,z)} = \partial_z$, where ∂_x, ∂_y, and ∂_z are the coordinate vector fields in $\mathbb{R}^3$. Every vector $v\in\mathbb{H}$ generates a one-parameter subgroup of $\mathbb{H}$; we write $\langle v\rangle = \mathbb{R}v$ for this subgroup and define $v^t = tv$ for all $t\in\mathbb{R}$.

Let $\mathsf{A} = \operatorname{span}(X,Y)$ be the left-invariant distribution spanned by X and Y; we refer to this as the *horizontal distribution*. Curves that are Lipschitz (with respect to the Euclidean metric) and almost everywhere tangent to A are called *horizontal curves*. For $h\in\mathbb{H}$ and $(a,b)\in\mathbb{R}^2\setminus(0,0)$, we call the line $L = h\cdot\langle aX+bY\rangle$ a *horizontal line*. Let $\pi\colon\mathbb{H}\to\mathbb{R}^2$ be the projection $\pi(x,y,z) = (x,y)$; we will identify $\mathbb{R}^2$ with the xy–plane in $\mathbb{H}$. The *slope* of a horizontal line L is the slope of the projection $\pi(L)$, i.e., $\frac{b}{a}$, as long as $a\neq 0$. A *vertical plane* $V\subset\mathbb{H}$ is a plane parallel to $\langle Z\rangle$; its slope is the slope of the projection $\pi(V)$.

Let d represent the *Carnot–Carathéodory metric* on $\mathbb{H}$. That is, we define a norm on A by $\|aX+bY\| = \sqrt{a^2+b^2}$, and given a horizontal curve $\gamma\colon I\to\mathbb{H}$, we define the length of γ as

$$\ell(\gamma) = \ell_{\mathbb{R}^2}(\pi\circ\gamma) = \int_I \|\gamma'(t)\|\,\mathrm{d}t.$$

For $p,q\in\mathbb{H}$, let $d(p,q) = \inf_\gamma \ell(\gamma)$, where the infimum is taken over all horizontal curves $\gamma\colon[0,1]\to\mathbb{H}$ such that $\gamma(0) = p$ and $\gamma(1) = q$. This is a left-invariant metric that satisfies the ball-box inequality

$$d(\mathbf{0},p) \asymp \max\{|x(p)|, |y(p)|, \sqrt{|z(p)|}\}. \tag{5}$$

This metric gives $\mathbb{H}$ Hausdorff dimension 4, and a smooth codimension–1 surface or intrinsic Lipschitz graph (Definition 2.1) has Hausdorff dimension 3. Let $\mathscr{S}^d$ be the spherical Hausdorff d–measure, normalized so that $\mathscr{S}^4$ is Lebesgue measure on $\mathbb{H}$ and the restriction of $\mathscr{S}^3$ to the xz–plane is Lebesgue measure on the xz–plane.

2.2. **Intrinsic graphs.** Let $V_0 = \{(x,y,z)\in\mathbb{H} : y = 0\}$ be the xz–plane. For any $U\subset V_0$ and any $f\colon U\to\mathbb{R}$, the *intrinsic graph* of f is the set

$$\Gamma_f = \{v\cdot Y^{f(v)} : v\in U\}.$$

Let $\Psi_f\colon U\to\Gamma_f$ be the map $\Psi_f(v) = v\cdot Y^{f(v)}$. Let $\Pi\colon\mathbb{H}\to V_0$,

$$\Pi(x,y,z) = (x,y,z)\cdot Y^{-y} = \left(x, 0, z - \frac{xy}{2}\right). \tag{6}$$

This is the projection to V_0 whose fibers are the cosets of $\langle Y\rangle$. The restrictions $\Pi|_{\Gamma_f}$ and $\Psi_f|_U$ are bijections (homeomorphisms when f is continuous), and $\Pi\circ\Psi_f = \mathrm{id}_U$.

Let $\gamma = (\gamma_x,\gamma_y,\gamma_z)\colon I\to\mathbb{H}$ be a horizontal curve such that γ has *unit x–speed*, i.e., $\gamma_x' = 1$. For almost every $t\in I$,

$$\gamma'(t) = \gamma_x'(t)X_{\gamma(t)} + \gamma_y'(t)Y_{\gamma(t)} = X_{\gamma(t)} + \gamma_y'(t)Y_{\gamma(t)}.$$

For any $p\in\mathbb{H}$, we have $\Pi_*(X_p) = \partial_x - y(p)\partial_z$ and $\Pi_*(Y_p) = 0$, so

$$(\Pi\circ\gamma)'(t) = \Pi_*(X_{\gamma(t)}) + \gamma_y'(t)\Pi_*(Y_{\gamma(t)}) = \partial_x - \gamma_y(t)\partial_z. \tag{7}$$

That is, one can reconstruct γ from the projection $\Pi\circ\gamma$.

Franchi, Serapioni, and Serra Cassano [FSSC06] defined the class of intrinsic Lipschitz graphs, which are analogues of graphs of Lipschitz functions in $\mathbb{R}^n$. We will give a definition which is equivalent to theirs, but with a different value for the Lipschitz constant [Rig19].

Definition 2.1. Let $\Gamma = \Gamma_f \subset \mathbb{H}$ be an intrinsic graph and let $\lambda \in (0,1)$. We say that Γ is an *intrinsic λ–Lipschitz graph* (and that f is an *intrinsic λ–Lipschitz function*) if for every $p, q \in \Gamma$,

$$|f(\Pi(q)) - f(\Pi(p))| = |y(q) - y(p)| \leqslant \lambda d(p,q).$$

If $\Pi(\Gamma_f) = V_0$, or equivalently, if the domain of f is all of V_0, we say that Γ_f is an *entire* intrinsic Lipschitz graph. Every intrinsic Lipschitz graph is a subset of an entire intrinsic Lipschitz graph; see [NY18, Thm. 27] and [Rig19].

Let $\Gamma = \Gamma_f$ be an intrinsic Lipschitz graph. Let ∇_f be the vector field on V_0 given by $\nabla_f = \partial_x - f\partial_z$. We call the corresponding operator the *intrinsic gradient.* For every point $p \in \Gamma$, there is a (possibly non-unique) unit x–speed horizontal curve $\gamma = (\gamma_x, \gamma_y, \gamma_z) \colon \mathbb{R} \to \Gamma$ such that $\gamma(0) = p$. We call the projection $\lambda = \Pi \circ \gamma$ of such a curve to V_0 a *characteristic curve* of Γ. By (7), every characteristic curve is an integral curve of ∇_f; conversely, by Theorems 1.1 and 1.2 of [BCSC15], every integral curve of ∇_f is the projection of a unit x–speed horizontal curve. Thus the characteristic curves are exactly the integral curves of ∇_f. When f is smooth, there is a unique characteristic curve passing through every point of V_0, and the intrinsic gradient is the derivative along these characteristic curves.

The intrinsic gradient $\nabla_f f$ of f is particularly important. When f is smooth and $v \in V_0$, $\nabla_f f(v)$ is the slope of the unique horizontal curve through $p = \Psi_f(v)$. If $s_{t,t}(x,y,z) = (tx, ty, t^2 z)$ is the scaling automorphism, then $s_{t,t}(p^{-1}\Gamma)$ converges to a vertical plane with slope $\nabla_f f(v)$ as $t \to \infty$. When f is intrinsic Lipschitz, the horizontal curve through p may not be unique, and the derivative $\nabla_f f$ may not be defined everywhere. We thus define $\nabla_f f$ distributionally as follows.

Definition 2.2. If $f\colon V_0 \to \mathbb{R}$ is continuous, we say that $\nabla_f f$ exists in the sense of distributions if there is a function $\theta \in L_\infty^{\mathrm{loc}}$ such that for every $\psi \in C^1_c$,

$$\int_{V_0} \theta\psi \,\mathrm{d}\mu = \int_{V_0} -f\partial_x\psi + \frac{f^2}{2}\partial_z\psi \,\mathrm{d}\mu.$$

If so, we write $\nabla_f f = \theta$. When f is C^1, this coincides with the previous definition.

The intrinsic gradient of an intrinsic λ–Lipschitz function is bounded by a function of λ. Conversely, if $U \subset V_0$ is an open set and $f\colon U \to \mathbb{R}$ is C^1 and satisfies $\|\nabla_f f\|_{L_\infty} < L$, then f is locally intrinsic Lipschitz with a constant depending on L [CMPSC14, Prop. 1.8].

Citti, Manfredini, Pinamonti, and Serra Cassano showed that intrinsic Lipschitz graphs can be approximated by smooth graphs.

Theorem 2.3 ([CMPSC14, Thm. 1.7])**.** *Let $f\colon V_0 \to \mathbb{R}$ be an intrinsic Lipschitz function and let $\omega \subset V_0$ be a bounded open set. Then there exists a sequence of functions $f_k \in C^\infty(\omega)$ such that*

(1) *$f_k \to f$ uniformly in ω,*
(2) *$|\nabla_{f_k} f_k(v)| \leqslant \|\nabla_f f\|_{L_\infty(\omega)}$ for all $v \in \omega$, and*
(3) *$\nabla_{f_k} f_k(v) \to \nabla_f f(v)$ for μ–a.e. $v \in \omega$.*

Finally, the following integration rules for ∇_f will be helpful.

Lemma 2.4. *Let $U \subset V_0$ be a closed bounded set with piecewise smooth boundary and let $f, g\colon U \to \mathbb{R}$ be functions which are smooth on the interior of U and continuous on U. Suppose that $\partial_z f, \partial_z g, \nabla_f g \in L_1(U)$. Let α parametrize ∂U in the positive direction. Then*

$$\int_U \nabla_f g \, \mathrm{d}\mu = \int_{\partial U} (fg, g) \cdot \mathrm{d}\alpha + \int_U g \cdot \partial_z f \, \mathrm{d}\mu.$$

If ∂U consists of segments on which g vanishes and characteristic curves of Γ_f, then the first term vanishes and

$$\int_U \nabla_f g \, \mathrm{d}\mu = \int_U g \cdot \partial_z f \, \mathrm{d}\mu.$$

Proof. On any subset $\omega \subset U$ with piecewise smooth boundary β, Green's Theorem and integration by parts imply that

$$\begin{aligned} \int_\omega \nabla_f g \, \mathrm{d}\mu &= \int_\omega \partial_x g \, \mathrm{d}\mu - \int_\omega f \cdot \partial_z g \, \mathrm{d}\mu \\ &= \int_\omega \partial_x g - \partial_z [fg] \, \mathrm{d}\mu + \int_\omega g \cdot \partial_z f \, \mathrm{d}\mu \\ &= \int_{\partial\omega} (fg, g) \cdot \mathrm{d}\beta + \int_\omega g \cdot \partial_z f \, \mathrm{d}\mu. \end{aligned}$$

We have $\partial_z[fg] \in L_1(U)$ and $\partial_x g = \nabla_f g + f \partial_z g \in L_1(U)$, so the lemma follows by letting $\omega \to U$. □

As a corollary, we get an analogue of the integration by parts formula.

Remark 2.5. In the corollary below and throughout this paper, we use square brackets to indicate the function acted on by a differential operator. We use $\cdot$ as a low-precedence multiplication operator, so that $\nabla_f g \cdot h$ represents $(\nabla_f g)h$, not $\nabla_f[gh]$.

Corollary 2.6. *Let U, f, and α be as above and let $g, h\colon U \to \mathbb{R}$ be smooth functions which are smooth on the interior of U and continuous on U, and satisfy $\partial_z f, \partial_z[gh], \nabla_f g, \nabla_f h \in L_1(U)$. Then*

$$\begin{aligned} \int_U \nabla_f g \cdot h \, \mathrm{d}\mu &= \int_U \nabla_f [gh] - \nabla_f h \cdot g \, \mathrm{d}\mu \\ &= \int_{\partial U} (fgh, gh) \cdot \mathrm{d}\alpha + \int_U gh \cdot \partial_z f \, \mathrm{d}\mu - \int_U \nabla_f h \cdot g \, \mathrm{d}\mu. \end{aligned}$$

If ∂U consists of segments on which gh vanishes and characteristic curves of Γ_f, then

$$\int_U \nabla_f g \cdot h \, \mathrm{d}\mu = \int_U gh \cdot \partial_z f - g \cdot \nabla_f h \, \mathrm{d}\mu.$$

2.3. **Automorphisms.** The Heisenberg group admits families of stretch and shear automorphisms. For $a, b \in \mathbb{R} \setminus \{0\}$, the *stretch automorphism* $s_{a,b}\colon \mathbb{H} \to \mathbb{H}$ is given by

$$s_{a,b}(x, y, z) = (ax, by, abz).$$

When $a = b$, this acts as a scaling on d, i.e., $d(s_{t,t}(p), s_{t,t}(q)) = td(p, q)$. The *shear automorphism* $P_b\colon \mathbb{H} \to \mathbb{H}$ is given by

$$P_b(x, y, z) = (x, y + bx, z).$$

Both of these are group automorphisms of $\mathbb{H}$ that send the horizontal distribution to itself and thus send horizontal curves to horizontal curves. In fact, they send cosets of $\langle Y \rangle$ to cosets of $\langle Y \rangle$, so the image of an intrinsic graph is an intrinsic graph. The following lemma holds.

Lemma 2.7. *Let $\Gamma = \Gamma_f$ be the intrinsic graph of a function $f\colon U \subset V_0 \to \mathbb{R}$. Let $q\colon \mathbb{H} \to \mathbb{H}$ be a stretch map, shear map, or left translation and let $\hat{q}\colon V_0 \to V_0$, $\hat{q}(v) = \Pi(q(v))$ be the map that q induces on V_0. Then there is a function $\hat{f}\colon \hat{q}(U) \to \mathbb{R}$ such that $\Gamma_{\hat{f}} = q(\Gamma)$.*

- *If $a, b \in \mathbb{R} \setminus \{0\}$ and $q = s_{a,b}$, then for all $v \in U$, $\hat{f}(\hat{q}(v)) = bf(v)$ and*
$$\nabla_{\hat{f}} \hat{f}(\hat{q}(v)) = \frac{b}{a} \nabla_f f(v).$$
- *If $b \in \mathbb{R}$ and $q = P_b$, then for all $v \in U$, $\hat{f}(\hat{q}(v)) = f(v) + bx(v)$ and $\nabla_{\hat{f}} \hat{f}(\hat{q}(v)) = \nabla_f f(v) + b$.*
- *If $h \in \mathbb{H}$ and $q(p) = hp$, then for all $v \in U$, $\hat{f}(\hat{q}(v)) = f(v) + y(h)$ and $\nabla_{\hat{f}} \hat{f}(\hat{q}(v)) = \nabla_f f(v)$.*

Proof. In all three cases, one can calculate that $q(vY^{f(v)}) = \hat{q}(v)Y^{\hat{f}(v)}$, so $\Gamma_{\hat{f}} = q(\Gamma)$.

By Theorem 2.3, it suffices to prove the formulas for $\nabla_{\hat{f}} \hat{f}$ when f is smooth. Let $v \in V_0$ and let L be the horizontal line tangent to Γ at $\Psi_f(v)$; this line has slope $S = \nabla_f f(v)$. The image $q(L)$ is tangent to $q(\Gamma)$ at $q(\Psi_f(v)) = \Psi_{\hat{f}}(\hat{q}(v))$, so $\nabla_{\hat{f}} \hat{f}(\hat{q}(v))$ is equal to the slope of $q(L)$. If $q = s_{a,b}$, then $q(L)$ has slope $\frac{b}{a}S$, if $q = P_b$, then $q(L)$ has slope $S + b$, and if $q(p) = hp$, then $q(L)$ has slope S, as desired. □

2.4. **Perimeter and divergence.** Let $C^1(\mathbb{H}; \mathsf{A})$ be the set of C^1 horizontal vector fields. For $V \in C^1(\mathbb{H}, \mathsf{A})$, $V = v_1 X + v_2 Y$, we define the *horizontal divergence* of V as $\operatorname{div}_{\mathbb{H}} V = Xv_1 + Yv_2$. For a measurable subset $E \subset \mathbb{H}$, we say that E has *locally finite perimeter* if for any bounded open set $U \subset \mathbb{H}$,

$$|\partial E|(U) \stackrel{\text{def}}{=} \sup \left\{ \int_E \operatorname{div}_{\mathbb{H}} V \, \mathrm{d}\mathscr{S}^4 : V \in C^1_c(U; \mathsf{A}), |V| \leqslant 1 \right\} < \infty.$$

We call $|\partial E|(U)$ the *perimeter* of E in U. Franchi, Serapioni, and Serra Cassano [FSSC01] showed that if E has locally finite perimeter, then $|\partial E|$ is a Radon measure, and there is a $|\partial E|$–measurable horizontal vector field ν_E (the *horizontal inward unit normal* to ∂E) such that $|\nu_E(p)| = 1$ for $|\partial E|$–a.e. p and such that for every $V \in C^1_c(\mathbb{H}; \mathsf{A})$,

$$\int_E \operatorname{div}_{\mathbb{H}} V \, \mathrm{d}\mathscr{S}^4 = \int_{\mathbb{H}} \langle \nu_E, V \rangle \, \mathrm{d}|\partial E|. \tag{8}$$

The measure $|\partial E|$ is concentrated on a subset $\partial^* E \subset \partial E$ called the *reduced boundary*. When E has locally finite perimeter, $\partial^* E$ is contained in the *measure-theoretic boundary* $\partial_{*,\mathscr{S}^4} E$, where $p \in \partial_{*,\mathscr{S}^4} E$ if and only if $\overline{\rho}_E(p) > 0$ and $\overline{\rho}_{\mathbb{H} \setminus E}(p) > 0$, where

$$\overline{\rho}_F(p) = \limsup_{r \to 0} \frac{\mathscr{S}^4(F \cap B(p,r))}{\mathscr{S}^4(B(p,r))}.$$

By [FSSC01],

$$|\partial E| = \mathscr{S}^3 \lfloor \partial^* E = \mathscr{S}^3 \lfloor \partial_{*,\mathscr{S}^4} E. \tag{9}$$

The perimeter measure can also be characterized in terms of BV functions. By [GN96], E has locally finite perimeter if and only if $\mathbf{1}_E$ is a locally BV function. In this case, the distributional gradient satisfies

$$\nabla \mathbf{1}_E = \nu_E |\partial E|. \tag{10}$$

If $E, F \subset \mathbb{H}$ have locally finite perimeter, then $E \cap F$ has locally finite perimeter, and $\nu_{E \cap F}$ agrees with ν_E or ν_F everywhere that both normals are defined.

Lemma 2.8. *Let $E,F\subset\mathbb{H}$ have locally finite perimeter. Then $E\cap F$ has locally finite perimeter. There is a $\mathscr{S}^3$–null set K such that*

$$\partial^*(E\cap F)\smallsetminus K\subset\partial^*E\cup\partial^*F. \tag{11}$$

Then $\nu_{E\cap F}(p)=\nu_F(p)$ for all $p\in\partial^*(E\cap F)\cap\partial^*F$ and $\nu_{E\cap F}(p)=\nu_E(p)$ for all $p\in\partial^*(E\cap F)\cap\partial^*E$.

(A similar lemma for general Carnot groups is proved in [AS10].)

Proof. Since $\mathbf{1}_E+\mathbf{1}_F\in\mathrm{BV}(\mathbb{H})$, the coarea formula (Theorem 5.2 of [GN96]), implies that $E\cap F=\{p\in\mathbb{H}:\mathbf{1}_E(p)+\mathbf{1}_F(p)>1\}$ has locally finite perimeter.

Suppose that $p\in\partial_{*,\mathscr{S}^4}(E\cap F)$. On one hand, $\overline{\rho}_{E\cap F}(p)>0$, $E\supset E\cap F$, and $F\supset E\cap F$, so we have $\overline{\rho}_E(p)>0$ and $\overline{\rho}_F(p)>0$. On the other hand, since $\mathbb{H}\smallsetminus(E\cap F)=(\mathbb{H}\smallsetminus E)\cup(\mathbb{H}\smallsetminus F)$, we have

$$0<\overline{\rho}_{\mathbb{H}\smallsetminus(E\cap F)}(p)\leqslant\overline{\rho}_{\mathbb{H}\smallsetminus E}(p)+\overline{\rho}_{\mathbb{H}\smallsetminus F}(p).$$

That is, $\overline{\rho}_{\mathbb{H}\smallsetminus E}(p)>0$ or $\overline{\rho}_{\mathbb{H}\smallsetminus F}(p)>0$. It follows that

$$\partial_{*,\mathscr{S}^4}(E\cap F)\subset\partial_{*,\mathscr{S}^4}E\cup\partial_{*,\mathscr{S}^4}F. \tag{12}$$

The measure-theoretic and reduced boundaries of E, F, and $E\cap F$ agree up to a null set, so (12) implies (11).

Let $p\in\partial^*(E\cap F)\cap\partial^*F$. For any $r>0$, let $(E\cap F)_r=s_{r^{-1},r^{-1}}\left(p^{-1}(E\cap F)\right)$, and let $F_r=s_{r^{-1},r^{-1}}(p^{-1}F)$. For $V\in\mathsf{A}$, let P_V^+ be the half-space

$$P_V^+=\{h\in\mathbb{H}:\langle V,\pi(h)\rangle>0\}.$$

By Theorem 4.1 of [FSSC01],

$$\lim_{r\to0}\mathbf{1}_{F_r}=\mathbf{1}_{P^+_{\nu_F(p)}}\qquad\text{and}\qquad\lim_{r\to0}\mathbf{1}_{(E\cap F)_r}=\mathbf{1}_{P^+_{\nu_{E\cap F}(p)}}$$

in $L_1^{\mathrm{loc}}(\mathbb{H})$. Since $\lim_{r\to0}\mathbf{1}_{(E\cap F)_r}\leqslant\lim_{r\to0}\mathbf{1}_{F_r}$, this implies $\nu_{E\cap F}(p)=\nu_F(p)$. Swapping E and F, we find that $\nu_{E\cap F}(p)=\nu_E(p)$ for all $p\in\partial^*(E\cap F)\cap\partial^*E$. ☐

3. Energy-minimizing surfaces and calibrations

Let $f\colon V_0\to\mathbb{R}$ be an intrinsic Lipschitz function. Let $U\subset V_0$ be a bounded measurable subset. We define the *intrinsic Dirichlet energy* of f on U by

$$\widetilde{E}_U(f)=\frac{1}{2}\int_U(\nabla_f f)^2\,\mathrm{d}\mu.$$

For $\Gamma=\Gamma_f$ and for any bounded open subset $W\subset\mathbb{H}$, we define $E_W(\Gamma)=\widetilde{E}_{\Pi(W\cap\Gamma)}(f)$.

In this section, we use calibrations to construct surfaces that minimize intrinsic energy subject to some boundary conditions. We first give a definition and some basic properties.

Definition 3.1. Let $f\colon V_0\to\mathbb{R}$ be an intrinsic Lipschitz function and let $U\subset V_0$ be an open subset. We say that f is an *energy-minimizing function* on U if

$$\widetilde{E}_{U\cap B(p,r)}(f)\leqslant\widetilde{E}_{U\cap B(p,r)}(g)$$

for every $p\in V_0$, $r>0$, and every intrinsic Lipschitz function g such that $f-g\in C_0^c(U\cap B(p,r))$.

For an entire intrinsic Lipschitz graph $\Gamma=\Gamma_f$, we define the *epigraph* of Γ to be

$$\Gamma^+=\{pY^t\in\mathbb{H}:p\in\Gamma,t\geqslant0\}.$$

We say that Γ is an *energy-minimizing graph* on an open subset $W \subset \mathbb{H}$ if

$$E_{W\cap B(p,r)}(\Gamma) \leqslant E_{W\cap B(p,r)}(\Lambda)$$

for every $p \in \mathbb{H}$, $r > 0$, and every entire intrinsic Lipschitz graph Λ such that $\Gamma^+ \triangle \Lambda^+ \Subset W \cap B(p,r)$. Here, $S \triangle T$ is the symmetric difference $S \triangle T = (S \smallsetminus T) \cup (T \smallsetminus S)$. In particular, f is energy-minimizing on U if and only if Γ_f is energy-minimizing on $\Pi^{-1}(U)$.

While perimeter minimization is preserved by scalings and rotations, a rotation of an intrinsic graph need not even be an intrinsic graph. Instead, energy minimization is preserved by stretches and shears.

Lemma 3.2. *Let $\Gamma = \Gamma_f \subset \mathbb{H}$ be a graph that minimizes energy on an open set $W \subset \mathbb{H}$ and let $h\colon \mathbb{H} \to \mathbb{H}$ be a left-translation, stretch, or shear. Then $h(\Gamma)$ minimizes energy on $h(W)$.*

We need the following calculation.

Lemma 3.3. *Let $U \subset V_0$ be a bounded open set and let $f, g\colon V_0 \to \mathbb{R}$ be intrinsic Lipschitz functions such that $f - g \in C^0_c(U)$. Then*

$$\int_U \nabla_f f \,\mathrm{d}\mu = \int_U \nabla_g g \,\mathrm{d}\mu.$$

Proof. Let $S = \operatorname{supp}(f-g) \Subset U$ and let $\psi \in C^\infty(U)$ be a function such that $\psi(s) = 1$ for all $s \in S$ and $\psi(u) = 0$ for all $u \in U \smallsetminus S$. By Definition 2.2,

$$\begin{aligned}\int_U \nabla_f f \,\mathrm{d}\mu &= \int_U \nabla_f f \cdot \psi \,\mathrm{d}\mu + \int_U \nabla_f f \cdot (1-\psi)\,\mathrm{d}\mu \\ &= \int_U -f\partial_x \psi + \frac{f^2}{2}\partial_z \psi \,\mathrm{d}\mu + \int_U \nabla_f f \cdot (1-\psi)\,\mathrm{d}\mu.\end{aligned}$$

Since ψ is constant on S and $\psi = 1$ on S, both integrals are zero on S. Therefore, since $f = g$ on $U \smallsetminus S$,

$$\int_U \nabla_f f \,\mathrm{d}\mu = \int_{U\smallsetminus S} -f\partial_x\psi + \frac{f^2}{2}\partial_z\psi\,\mathrm{d}\mu + \int_{U\smallsetminus S} \nabla_f f\cdot(1-\psi)\,\mathrm{d}\mu = \int_U \nabla_g g\,\mathrm{d}\mu,$$

as desired. □

Now we prove the lemma.

Proof of Lemma 3.2. It suffices to show that for every bounded open set $U \subset W$ and every intrinsic Lipschitz graph Λ such that $\Lambda^+ \triangle h(\Gamma^+) \Subset h(U)$, we have $E_{h(U)}(h(\Gamma)) \leqslant E_{h(U)}(\Lambda)$.

When h is a translation and $\Lambda^+ \triangle h(\Gamma^+) \Subset h(U)$, the minimality of Γ implies

$$E_{h(U)}(h(\Gamma)) = E_U(\Gamma) \leqslant E_U(h^{-1}(\Lambda)) = E_{h(U)}(\Lambda),$$

as desired.

Suppose that $h = s_{a,b}$ for some $a, b \neq 0$. By Lemma 2.7, for any intrinsic Lipschitz graph $\Sigma = \Gamma_\phi$, we have

$$E_{h(U)}(h(\Sigma)) = \frac{b^2}{a^2}\frac{\mu(h(U))}{\mu(U)}E_U(\Sigma) = \frac{b^2}{a}E_U(\Sigma). \tag{13}$$

If $\Lambda^+ \triangle h(\Gamma^+) \Subset h(U)$, the minimality of Γ implies

$$E_{h(U)}(h(\Gamma)) = \frac{b^2}{a}E_U(\Gamma) \leqslant \frac{b^2}{a}E_U(h^{-1}(\Lambda)) = E_{h(U)}(\Lambda).$$

Now suppose that h is a shear, i.e., $h = P_b$ for some $b \in \mathbb{R}$. Let $\Lambda = \Gamma_{\hat g}$ be an intrinsic Lipschitz graph such that $\Lambda^+ \triangle h(\Gamma^+) \Subset h(W)$. Let g be the intrinsic Lipschitz function such that $h^{-1}(\Lambda) = \Gamma_g$. Then $\Gamma_g^+ \triangle \Gamma^+ \Subset W$, and for any $v \in V_0$, either $f(v) = g(v)$ (in which case $\Psi_f(v) = \Psi_g(v)$) or the horizontal segment from $\Psi_f(v)$ to $\Psi_g(v)$ is contained in W. In either case, $\Psi_f(v) \in W$ if and only if $\Psi_g(v) \in W$, i.e., $\Pi(W \cap \Gamma) = \Pi(W \cap \Gamma_g)$. Let $D_0 = \Pi(W \cap \Gamma)$ and let D be a bounded open subset such that $D_0 \Subset D$ and thus $f - g \in C_c^0(D)$. The minimality of f implies that $\widetilde{E}_D(f) \leqslant \widetilde{E}_D(g)$.

Let $\hat{h}\colon V_0 \to V_0$,

$$\hat{h}(x,0,z) = \Pi(h(x,0,z)) = \left(x,0,z - \frac{1}{2}bx^2\right).$$

This is measure-preserving on V_0, and by Lemma 2.7, for any intrinsic Lipschitz graph $\Sigma = \Gamma_\phi$, there is a $\hat{\phi}$ such that $h(\Sigma) = \Gamma_{\hat{\phi}}$ and

$$\nabla_{\hat{\phi}}\hat{\phi}(v) = \nabla_\phi \phi(\hat{h}^{-1}(v)) + b$$

for all $v \in V_0$. Thus

$$\begin{aligned}
\widetilde{E}_{\hat{h}(D)}(\hat{\phi}) &= \int_{\hat{h}(D)} \left(\nabla_\phi \phi(\hat{h}^{-1}(v)) + b\right)^2 \mathrm{d}\mu(v) \\
&= \int_D \nabla_\phi \phi(v)^2 + 2b\nabla_\phi\phi(v) + b^2 \,\mathrm{d}\mu(v) \\
&= \widetilde{E}_D(\phi) + b^2\mu(D) + 2b\int_D \nabla_\phi\phi(v)\,\mathrm{d}\mu(v).
\end{aligned}$$

Since $\hat{f} = \hat{g}$ except on $\hat{h}(D_0)$,

$$E_{h(W)}(\Lambda) - E_{h(W)}(h(\Gamma)) = \widetilde{E}_{\hat{h}(D_0)}(\hat{f}) - \widetilde{E}_{\hat{h}(D_0)}(\hat{g}) = \widetilde{E}_{\hat{h}(D)}(\hat{f}) - \widetilde{E}_{\hat{h}(D)}(\hat{g}).$$

By the calculation above, Lemma 3.3, and the minimality of f,

$$\begin{aligned}
\widetilde{E}_{\hat{D}}(\hat{f}) &= \widetilde{E}_D(f) + b^2\mu(D) + 2b\int_D \nabla_f f(v)\,\mathrm{d}\mu(v) \\
&\leqslant \widetilde{E}_D(g) + b^2\mu(D) + 2b\int_D \nabla_g g(v)\,\mathrm{d}\mu(v) \\
&= \widetilde{E}_{\hat{D}}(\hat{g}),
\end{aligned}$$

so $E_{h(W)}(\Gamma) \leqslant E_{h(W)}(h(\Lambda))$, as desired. □

Furthermore, we can use stretch automorphisms to relate the energy of a surface to its perimeter.

Lemma 3.4. *Let $\Lambda = \Gamma_g$ be an intrinsic Lipschitz graph. Let $W \subset \mathbb{H}$ be a bounded open set and let $D = \Pi(W \cap \Lambda)$. Then, as $r \to \infty$,*

$$\mathscr{S}^3\left(s_{r,r^{-1}}(W \cap \Lambda)\right) = r\mu(D) + r^{-3}E_W(\Lambda) + O(r^{-7}), \tag{14}$$

where the implicit constant is bounded by a function of $\mu(D)$ and the intrinsic Lipschitz constant of Λ.

Proof. Let $g_r(x,z) = r^{-1}g(r^{-1}x,z)$, so that $s_{r,r^{-1}}(\Lambda) = \Gamma_{g_r}$. Then

$$\nabla_{g_r} g_r(v) = r^{-2}\nabla_{g_r} g_r(s_{r,r^{-1}}(v)),$$

and

$$\begin{aligned}\mathscr{S}^3\left(s_{r,r^{-1}}(W\cap\Lambda)\right) &= \int_{s_{r,r^{-1}}(D)} \sqrt{1+|\nabla_{g_r} g_r|^2}\,\mathrm{d}\mu\\ &= \int_D r\sqrt{1+r^{-4}|\nabla_g g|^2}\,\mathrm{d}\mu\\ &= r\int_D 1+\frac{r^{-4}}{2}|\nabla_g g|^2 + O(r^{-8}|\nabla_g g|^4)\,\mathrm{d}\mu\\ &= r\mu(D) + r^{-3}E_W(\Lambda) + O(r^{-7}\mu(D)\sup_{v\in D}|\nabla_g g(v)|^4),\end{aligned}$$

as desired. □

This lets us construct energy-minimizing surfaces by stretching H–minimal surfaces. We start with the H–minimal surfaces with a singularity along the x–axis constructed in [Pau06]. These surfaces were proved to be area-minimizing on any bounded set in [CHY07, Example 7.1], [MSCV08].

Lemma 3.5. *Let $a \in \mathbb{R}\smallsetminus\{0\}$ and let $f_a(x,z) = -a\sqrt{z}\,\mathrm{sign}(z)$, where $\mathrm{sign}(z) = 1, 0,$ or -1 depending on whether z is positive, zero, or negative. Let $\Gamma = \Gamma_{f_a}$. Then Γ minimizes energy on any bounded open set $W \subset \mathbb{H}$.*

We call these surfaces *herringbone surfaces*, after the arrangement of their horizontal curves. For any $a > 0$, $\Gamma = \Gamma_{f_a}$ can be written as the union of two horizontal rays coming out of each point of the x–axis with slope $\pm\frac{a^2}{2}$

Proof. Since Γ is an area-minimizing surface, if W is a bounded open set and E is a finite-perimeter set such that $E \triangle \Gamma^+ \subset W$, then $\mathrm{Per}_E(W) \geq \mathrm{Per}_{\Gamma^+}(W)$. In fact, for any $r > 0$, we have $s_{r,r^{-1}}(\Gamma) = \Gamma_{f_{r^{-1}a}}$, so $s_{r,r^{-1}}(\Gamma)$ is also area-minimizing.

Let $\Lambda = \Gamma_g$ be an intrinsic Lipschitz graph such that $\Lambda^+ \triangle \Gamma^+ \Subset W$. The minimality of $s_{r,r^{-1}}(\Gamma)$ implies that for any $r > 0$,

$$\mathscr{S}^3\left(s_{r,r^{-1}}(W\cap\Gamma)\right) \leq \mathscr{S}^3\left(s_{r,r^{-1}}(W\cap\Lambda)\right).$$

By Lemma 3.4,

$$\mathscr{S}^3\left(s_{r,r^{-1}}(W\cap\Lambda)\right) - \mathscr{S}^3\left(s_{r,r^{-1}}(W\cap\Gamma)\right) = r^{-3}\big(E_W(\Lambda) - E_W(\Gamma)\big) + O(r^{-7}).$$

This is non-negative for all $r > 0$, so as r goes to infinity, we get $E_W(\Gamma) \leq E_W(\Lambda)$, as desired. □

One can also prove the minimality of Γ using calibrations. In fact, Lemma 3.5 is a special case of Theorem 1.2; if $K = \{\pm\frac{a^2}{2}\}$, then $\Gamma_{f_a} = \Lambda_K$. In the context of minimal surfaces, a calibration is a field of unit vectors with zero divergence. We will adjust this definition for the case of harmonic graphs.

Let V be a Borel horizontal vector field. For an open set $U \subset \mathbb{H}$ and a finite-perimeter subset $E \subset \mathbb{H}$, we define the *flux* of V through ∂E as

$$\mathscr{F}_U(E,V) \stackrel{\text{def}}{=} \int_U \langle \nu_E, V\rangle\,\mathrm{d}|\partial E|.$$

We say that V is *conservative* on U if $\mathscr{F}_U(E,V) = 0$ for every finite-perimeter subset $E \Subset U$. This implies, in particular, that the distributional divergence of V is zero.

The following proposition gives a calibration condition for proving that a graph is energy minimizing.

Proposition 3.6. *Let $\Gamma = \Gamma_f$ be an intrinsic Lipschitz graph and let $W \subset \mathbb{H}$ be an open set. Suppose there is a locally bounded Borel function $\tau\colon W \to \mathbb{R}$ such that $\tau(p) = \nabla_f f(\Pi(p))$ for almost every $p \in W \cap \Gamma$. Suppose further that the vector field*

$$M(p) = -\tau(p) X_p + \left(1 - \frac{\tau(p)^2}{2}\right) Y_p \tag{15}$$

is conservative on W. Then Γ is energy-minimizing on W.

The proof of Proposition 3.6 is based on Lemma 3.4 and the following formula. Recall that for an intrinsic Lipschitz graph $\Gamma = \Gamma_f$, the *unit horizontal normal* at a point $p \in \Gamma$ is given by

$$\nu_\Gamma(p) = \frac{-\nabla_f f(\Pi(p)) X_p + Y_p}{\sqrt{1 + \nabla_f f(\Pi(p))^2}}.$$

Let

$$M_\Gamma(p) = -\nabla_f f(\Pi(p)) X_p + \left(1 - \frac{\nabla_f f(\Pi(p))^2}{2}\right) Y_p.$$

Then $|M_\Gamma(p) - \nu_\Gamma(p)| \lesssim |\nabla_f f(\Pi(p))|^3$. If M is as in Proposition 3.6, then $M_\Gamma(p) = M(p)$ almost everywhere on $W \cap \Gamma$.

Let $W \subset \mathbb{H}$ be an open set, and let $D = \Pi(W \cap \Gamma)$. Let V be a horizontal vector field V. By (2),

$$\begin{aligned}\mathscr{F}_W(\Gamma^+, M_\Gamma) = \int_W \langle M_\Gamma, \nu_\Gamma\rangle \,\mathrm{d}|\partial\Gamma^+| &= \int_D \left\langle M_\Gamma\left(\Psi_f(v)\right), -\nabla_f f(v) X_{\Psi_f(v)} + Y_{\Psi_f(v)}\right\rangle \mathrm{d}\mu(v) \\ &= \int_D \nabla_f f(v)^2 + 1 - \frac{1}{2}\nabla_f f(v)^2 \,\mathrm{d}\mu(v) = \mu(D) + E_W(\Gamma). \end{aligned} \tag{16}$$

Proof of Proposition 3.6. Without loss of generality, we may take W to be a bounded open set and τ to be bounded on W. Let $\Lambda = \Gamma_g$ be an entire intrinsic Lipschitz graph such that $\Gamma^+ \triangle \Lambda^+ \Subset W$. Let $\Gamma_r = s_{r,r^{-1}}(\Gamma)$ and $\Lambda_r = s_{r,r^{-1}}(\Lambda)$, and let $f_r, g_r\colon V_0 \to \mathbb{R}$ be the functions such that $\Gamma_{f_r} = \Gamma_r$ and $\Gamma_{g_r} = \Lambda_r$. Since $\Gamma^+ \triangle \Lambda^+ \Subset W$, we have $\Pi(W \cap \Lambda) = \Pi(W \cap \Gamma)$; let $D = \Pi(W \cap \Gamma)$ and $D_r = s_{r,r^{-1}}(D)$.

Let $W_r = s_{r,r^{-1}}(W)$. For $p \in W_r$, let $q = s^{-1}_{r,r^{-1}}(p)$, let $\tau_r(p) = r^{-2}\tau(q)$, and let $M_r(p) = -\tau_r(p) X_p + \left(1 - \frac{\tau_r(p)^2}{2}\right) Y_p$. For almost every $p \in W_r \cap \Gamma_r$,

$$\tau_r(p) = r^{-2}\nabla_f f(\Pi(q)) = \nabla_{f_r} f_r(\Pi(p)),$$

so $M_r = M_{\Gamma_r}$ almost everywhere on $W_r \cap \Gamma_r$. By (13) and (16),

$$\mathscr{F}_{W_r}(\Gamma_r^+, M_r) = \mu(D_r) + E_{W_r}(\Gamma_r) = r\mu(D) + r^{-3} E_W(\Gamma).$$

We claim that M_r is conservative. We have

$$(s_{r,r^{-1}})_*(M(q) - Y_q) = -\tau(q) r X_p + \frac{1}{2}\tau(q)^2 r^{-1} Y_p = r^3(M_r(p) - Y_p),$$

so

$$M_r = r^{-3}(s_{r,r^{-1}})_*(M - Y) + Y. \tag{17}$$

Let E be a finite-perimeter set. For any horizontal vector field V and any $|\partial E|$–measurable subset $S \subset \partial^* E$, we have

$$\int_S \langle \nu_E, V\rangle \,\mathrm{d}|\partial E| = \int_{s_{r,r^{-1}}(S)} \langle \nu_{s_{r,r^{-1}}(E)}, (s_{r,r^{-1}})_*(V)\rangle \,\mathrm{d}|\partial s_{r,r^{-1}}(E)|.$$

This follows from a calculation when S is a subset of an $\mathbb{H}$–regular hypersurface as in [FSSC01], and by the Main Theorem of [FSSC01], ∂E can be written as a union of compact subsets of $\mathbb{H}$–regular hypersurfaces, up to a $\mathscr{S}^3$–null set. Therefore,

$$\mathscr{F}_{W_r}(s_{r,r^{-1}}(E), (s_{r,r^{-1}})_*(V)) = \mathscr{F}_W(E, V),$$

and if $E \Subset W$, the conservativity of M implies

$$\begin{aligned}\mathscr{F}_{W_r}(s_{r,r^{-1}}(E), M_r) = r^{-3}\mathscr{F}_{W_r}(s_{r,r^{-1}}(E), (s_{r,r^{-1}})_*(M-Y)) + \mathscr{F}_{W_r}(s_{r,r^{-1}}(E), Y)\\ = r^{-3}\mathscr{F}_W(E, M-Y) + \mathscr{F}_W(E, rY) = 0.\end{aligned}$$

Therefore, M_r is conservative.

Now, we calculate $\mathscr{F}_{W_r}(\Lambda_r^+, M_r)$ and bound $\mathscr{S}^3(W_r \cap \Lambda_r)$ from below. Let $S = \Lambda_r^+ \smallsetminus \Gamma_r^+$ and $T = \Gamma_r^+ \smallsetminus \Lambda_r^+$. These sets have finite perimeter by Lemma 2.8, and $\mathbf{1}_{\Lambda_r^+} - \mathbf{1}_{\Gamma_r^+} = \mathbf{1}_S - \mathbf{1}_T$. Taking distributional gradients, equation (10) implies

$$\nu_{\Lambda_r^+}|\partial\Lambda_r^+| - \nu_{\Gamma_r^+}|\partial\Gamma_r^+| = \nu_S|\partial S| - \nu_T|\partial T|$$

as vector-valued Radon measures, and thus

$$\begin{aligned}\mathscr{F}_{W_r}(\Lambda_r^+, M_r) = \mathscr{F}_{W_r}(\Gamma_r^+, M_r) + \mathscr{F}_{W_r}(S, M_r) - \mathscr{F}_{W_r}(T, M_r)\\ = \mathscr{F}_{W_r}(\Gamma_r^+, M_r) = r\mu(D) + r^{-3}E_W(\Gamma).\end{aligned}$$

It follows that

$$\mathscr{S}^3(W_r \cap \Lambda_r) \geqslant \frac{\mathscr{F}_{W_r}(\Lambda_r^+, M_r)}{\sup_{q\in W_r}|M_r(q)|} = \frac{r\mu(D) + r^{-3}E_W(\Gamma)}{\sup_{q\in W_r}|M_r(q)|}. \tag{18}$$

Let $t = \sup_{p\in W}|\tau(p)|$. By (15), for $q \in W_r$,

$$|M_r(q)| = \sqrt{\tau_r(q)^2 + 1 - \tau_r(q)^2 + \frac{\tau_r(q)^4}{4}} = 1 + O(r^{-8}t^4),$$

so for sufficiently large r,

$$\mathscr{S}^3(W_r \cap \Lambda_r) \geqslant \frac{r\mu(D) + r^{-3}E_W(\Gamma)}{1 + O(r^{-8}t^4)} = r\mu(D) + r^{-3}E_W(\Gamma) + O(r^{-7}), \tag{19}$$

where the last implicit constant depends on D, t, and Γ. By Lemma 3.4,

$$\mathscr{S}^3(W_r \cap \Lambda_r) = r\mu(D) + r^{-3}E_W(\Lambda) + O(r^{-7}). \tag{20}$$

Comparing (19) and (20), we find $E_W(\Lambda) \geqslant E_W(\Gamma)$, as desired. □

Thus, we can construct energy minimizers by constructing conservative vector fields of the form (15).

Example 3.7. We say that a *ruled surface* is a surface in $\mathbb{H}$ foliated by horizontal line segments. Let Σ be a C^2 ruled Z–graph, that is, a ruled surface of the form $\Sigma = \{(x, y, g(x, y)) : x, y \in \Omega\}$ where $\Omega \subset \mathbb{R}^2$ is open and $g \in C^2(\Omega)$. The results of [CHMY05] show that Σ is an area minimizer when the horizontal lines making up Σ meet the curves in the singular set orthogonally.

Suppose that Σ has no singular set and $\Sigma = \Gamma_f$ is intrinsic Lipschitz. Then Σ is an area minimizer and the slopes of the horizontal lines making up Σ are bounded. We claim that Σ is also an energy minimizer. For every $q = (x, y) \in \Omega$, there is a unique point $p = (x, y, g(x, y)) \in \Sigma$ and a unique maximal horizontal line segment $\widetilde{L}_p$ through p that is contained in Σ. Let $L_q = \pi(\widetilde{L}_p)$ and let $\sigma(q) = \nabla_f f(\Pi(p))$ be the slope of L_q. For any

$q' \in L_q$, the uniqueness of L_q implies that $L_{q'} = L_q$, so σ is constant along lines of the form L_q.

Let $W = \pi^{-1}(\Omega)$ and for $w \in W$, let $\tau(w) = \sigma(\pi(w))$. Then $\tau(p) = \nabla_f f(\Pi(p))$ for every $p \in \Sigma$. Define M as in (15). This is a C^1 vector field. For any $p \in W$ and $q = \pi(p)$,

$$\operatorname{div}_{\mathbb{H}} M(p) = -X[\tau](p) - \tau(p)Y[\tau](p) = -(\partial_x + \sigma(q)\partial_y)[\sigma](q).$$

The vector $\partial_x + \sigma(q)\partial_y$ is tangent to L_q and σ is constant along L_q, so $\operatorname{div}_{\mathbb{H}} M(p) = 0$, i.e., M is conservative. By Proposition 3.6, Σ is energy-minimizing.

We can construct more examples by gluing together C^1 fields. Nicolussi Golo and Ritoré [GR21] gave a construction of piecewise C^1 vector fields with zero distributional divergence, and a similar construction produces conservative vector fields.

Proposition 3.8 (see [GR21, Prop. 2.5])**.** *Let $U \subset \mathbb{H}$ be an open set. Let $\{\Omega_j\}_j$ be a family of disjoint subsets of U with locally finite perimeter such that $\mathscr{S}^3(\partial\Omega_j \smallsetminus \partial^*\Omega_j) = 0$, $U = \bigsqcup_j \Omega_j$, and $\{\overline{\Omega_j}\}_j$ has locally finite multiplicity. For each j, let $V_j \in C^1(\overline{\Omega_j}; \mathsf{A})$. Let*

$$V = \sum_j V_j \mathbf{1}_{\Omega_j}.$$

Suppose that for each j, $\operatorname{div}_{\mathbb{H}} V_j = 0$ on Ω_j and for $\mathscr{S}^3$–a.e. $p \in \partial\Omega_j$,

$$\langle \nu_{\Omega_j}(p), V(p)\rangle = \langle \nu_{\Omega_j}(p), V_j(p)\rangle. \tag{21}$$

Then V is conservative.

Proof. Let $E \Subset U$ be a finite-perimeter set and for each j, let $E_j = E \cap \Omega_j$. Since $\{\overline{\Omega_j}\}_j$ has locally finite multiplicity, only finitely many of the E_j's are nonempty. By Lemma 2.8, each E_j has finite perimeter. We claim that $\mathscr{F}_U(E_j, V) = \mathscr{F}_U(E_j, V_j) = 0$. It suffices to show that

$$\langle \nu_{E_j}(p), V(p)\rangle = \langle \nu_{E_j}(p), V_j(p)\rangle \tag{22}$$

for $\mathscr{S}^3$–a.e. $p \in \partial^* E_j$.

We have $\partial^* E_j \subset \overline{E_j} \subset \overline{\Omega_j}$, so for all $p \in \partial^* E_j$ except a $\mathscr{S}^3$–null set, we have $p \in \Omega_j$ or $p \in \partial^*\Omega_j$. If $p \in \Omega_j$, then $V(p) = V_j(p)$, and (22) holds.

Otherwise, $p \in \partial^* E_j \cap \partial^*\Omega_j$. Lemma 2.8 implies $\nu_{E_j}(p) = \nu_{\Omega_j}(p)$, and (21) implies that $\langle \nu_{E_j}(p), V(p)\rangle = \langle \nu_{E_j}(p), V_j(p)\rangle$ for $\mathscr{S}^3$–a.e. $p \in \partial^* E_j \cap \partial^*\Omega_j$. Thus, by (8),

$$\mathscr{F}_U(E_j, V) = \mathscr{F}_U(E_j, V_j) = \int_{E_j} \operatorname{div}_{\mathbb{H}} V_j \, d\mathscr{S}^4 = 0.$$

By (10),

$$\nu_E |\partial E| = \nabla \mathbf{1}_E = \sum_j \nabla \mathbf{1}_{E_j} = \sum_j \nu_{E_j} |\partial E_j|$$

as vector-valued Radon measures, where the second equality uses the fact that only finitely many of the E_j's are nonempty, so there are only finitely many nonzero terms in the sum. Then

$$\mathscr{F}_U(E, V) = \int_U \langle V, \nu_E\rangle \, d|\partial E| = \sum_j \int_U \langle V, \nu_{E_j}\rangle \, d|\partial E|_j = \sum_j \mathscr{F}_U(E_j, V) = 0,$$

where we use the fact that there are only finitely many nonzero terms in the sum to pass the sum through the integral. □

When $V(p) = -\tau(p)X_p + (1 - \frac{\tau(p)^2}{2})Y_p$, as in Proposition 3.6, Condition (21) can be written in terms of τ. For example, let $\Omega^+ \subset \mathbb{H}$ be a set with locally finite perimeter and let $\Omega^- = \mathbb{H} \setminus \Omega^+$. Let $\tau^\pm \in C^1(\overline{\Omega^\pm})$,

$$V^\pm = -\tau^\pm X + \left(1 - \frac{(\tau^\pm)^2}{2}\right)Y,$$

and $V = \mathbf{1}_{\Omega^+}V^+ + \mathbf{1}_{\Omega^-}V^-$.

Let $p \in \partial^*\Omega^+$ be a point such that $\nu_{\Omega^+}(p) \neq \pm X$. Since $\Omega^+ = \mathbb{H} \setminus \Omega^-$,

$$\nu_{\Omega^+}(p) = -\nu_{\Omega^-}(p) = \frac{-\sigma X + Y}{\sqrt{1+\sigma^2}}$$

for some $\sigma \in \mathbb{R}$, and

$$\begin{aligned}\left\langle -\sigma X + Y, V^+(p) - V^-(p)\right\rangle &= \sigma(\tau^+(p) - \tau^-(p)) - \frac{1}{2}\left(\tau^+(p)^2 - \tau^-(p)^2\right)\\ &= (\tau^+(p) - \tau^-(p))\left(\sigma - \frac{\tau^+(p)^2 + \tau^-(p)^2}{2}\right).\end{aligned}$$

That is, (21) is satisfied at p if and only if $\tau^+(p) = \tau^-(p)$ or

$$\sigma = \frac{\tau^+(p) + \tau^-(p)}{2}. \tag{23}$$

This is analogous to the equal-angle condition studied in [CHY07].

We can thus construct energy-minimizing surfaces by gluing ruled surfaces along singularities. We will need a criterion that ensures that a union of horizontal curves is an intrinsic Lipschitz graph.

Lemma 3.9. *Let $c > 0$ and let*

$$C = \left\{q \in \mathbb{H} : |y(q)| > \max\left\{4c|x(q)|, \sqrt{32c|z(q)|}\right\}\right\}$$

be a scale-invariant double cone in $\mathbb{H}$. For $i = 1, 2$, let $\alpha_i : [0, t_i] \to \mathbb{H}$ be unit x–speed horizontal curves such that $\alpha_1(0) = \alpha_2(0)$ and $\mathrm{Lip}(y \circ \alpha_i) < c$. Suppose further that the projections $\pi \circ \alpha_i$ do not cross, i.e., $y(\alpha_1(t)) \leqslant y(\alpha_2(t))$ for all $t \in [0, \min(t_1, t_2)]$ or $y(\alpha_1(t)) \geqslant y(\alpha_2(t))$ for all $t \in [0, \min(t_1, t_2)]$. Let $p_i = \alpha_i(t_i)$. Then $p_1^{-1}p_2 \notin C$.

Proof. Note that $C^{-1} = C$, so $p_1^{-1}p_2 \in C$ if and only if $p_2^{-1}p_1 \notin C$. After possibly switching α_1 and α_2, we may suppose that $t_1 \leqslant t_2$. Since $s_{1,-1}(C) = C$, after possibly replacing α_i by $s_{1,-1}(\alpha_i)$, we may suppose that $y(\alpha_1(t)) \leqslant y(\alpha_2(t))$ for all $t \in [0, t_1]$.

We translate so that $p_1 = \mathbf{0}$. It suffices to show that $|y(p_2)| \leqslant 4c|x(p_2)|$ or $|y(p_2)| \leqslant \sqrt{8c|z(p_2)|}$. Suppose that $|y(p_2)| > 4c|x(p_2)|$. We claim that $z(p_2) \leqslant -\frac{y(p_2)^2}{8c}$.

Since $|y(p_2)| > 4cx(p_2)$ and $\mathrm{Lip}(y \circ \alpha_2) < c$, we have $|y(\alpha_2(t_1))| > 3cx(p_2)$. By hypothesis, $y(\alpha_1(t_1)) = 0 \leqslant y(\alpha_2(t_1))$, so

$$y(\alpha_2(0)) \geqslant y(\alpha_2(t_2)) - cx(p_2) > 3cx(p_2)$$

and $y(p_2) > 4cx(p_2)$. In particular,

$$y(\alpha_2(t)) \geqslant \frac{y(p_2)}{2} \qquad \text{for } t_1 \leqslant t \leqslant t_2. \tag{24}$$

Since α_1 and α_2 are horizontal, (7) implies $(z\circ\Pi\circ\alpha_i)' = -y\circ\alpha_i$ and

$$\begin{aligned} z(\Pi(p_1)) - z(\Pi(p_2)) = -z(\Pi(p_2)) &= \int_0^{t_1} -y(\alpha_1(t))\,\mathrm{d}t - \int_0^{t_2} -y(\alpha_2(t))\,\mathrm{d}t \\ &= \int_0^{t_1} y(\alpha_2(t)) - y(\alpha_1(t))\,\mathrm{d}t + \int_{t_1}^{t_2} y(\alpha_2(t))\,\mathrm{d}t \\ &\overset{(24)}{=} \int_0^{t_1} y(\alpha_2(t)) - y(\alpha_1(t))\,\mathrm{d}t + \frac{x(p_2)y(p_2)}{2}. \end{aligned}$$

Since $\mathrm{Lip}(y\circ\alpha_i) < c$, for $t_1 - \frac{y(p_2)}{8c} \leqslant t \leqslant t_1$,

$$y(\alpha_2(t)) - y(\alpha_1(t)) \geqslant y(\alpha_2(t_1)) - y(\alpha_1(t_1)) - 2c\frac{y(p_2)}{8c} \overset{(24)}{\geqslant} \frac{y(p_2)}{4}.$$

Thus

$$-z(\Pi(p_2)) \geqslant \int_{t_1-\frac{y(p_2)}{8c}}^{t_1} y(\alpha_2(t)) - y(\alpha_1(t))\,\mathrm{d}t + \frac{x(p_2)y(p_2)}{2} \geqslant \frac{y(p_2)^2}{32c} + \frac{x(p_2)y(p_2)}{2}.$$

By (6),

$$z(p_2) = z(\Pi(p_2)) + \frac{x(p_2)y(p_2)}{2} \geqslant \frac{y(p_2)^2}{32c},$$

so $|y(p_2)| \leqslant \sqrt{32c|z(p_2)|}$, as desired. □

Remark 3.10. We use Lemma 3.9 to show that certain Z–graphs are intrinsic Lipschitz graphs. A sufficiently regular Z–graph Γ can be written as a union of horizontal curves. In the examples we will consider, there will be a basepoint p_0 such that for every $p\in\Gamma$, there is a horizontal curve γ_p from p_0 to p. If the projections $\pi\circ\gamma_p$ are the graphs of Lipschitz functions and no two such projections cross, then for any $p,q\in\Gamma$, applying Lemma 3.9 to γ_p and γ_q implies that $p^{-1}q\notin C$. Since C is a scale-invariant open double cone containing Y, this implies that Γ is an intrinsic Lipschitz graph.

Example 3.11. Let $\gamma\colon\mathbb{R}\to\mathbb{H}$ be a smooth horizontal curve with unit x–speed. Let $\sigma(s) = (y\circ\gamma)'(s)$ be the slope of γ and let $\delta\in C^\infty(\mathbb{R})$, $\delta>0$. Let $\rho\colon\mathbb{R}^2\to\mathbb{H}$,

$$\rho(s,t) = \begin{cases} \gamma(s)\cdot\big(X + (\sigma(s)+\delta(s))\,Y\big)^{|t|} & t\geqslant 0\\ \gamma(s)\cdot\big(X + (\sigma(s)-\delta(s))\,Y\big)^{|t|} & t<0.\end{cases}$$

Let $I=(a,b)\subset\mathbb{R}$ be an open interval and choose $0<\varepsilon\leqslant\infty$ small enough that $\Sigma = \rho(I\times(-\varepsilon,\varepsilon))$ is a Z–graph. Let $p_0=\gamma(a)$. For any point $p=\rho(s,t)$, there is a horizontal curve in Σ from p_0 to p which consists of the segment $\gamma([a,s])$ concatenated with a horizontal ray of slope $\sigma(s)\pm\delta(s)$ originating at $\gamma(s)$. These curves all project to Lipschitz graphs in $\mathbb{R}^2$, and no two such graphs cross, so by Remark 3.10, Σ is an intrinsic Lipschitz graph.

We construct a calibration for Σ as follows. Let $\Omega=\pi(\Sigma)$, and let $W=\pi^{-1}(\Omega)$. For any $p\in W$, there is a unique point $(s,t)\in I\times(-\varepsilon,\varepsilon)$ such that $\pi(\rho(s,t))=\pi(p)$. Define

$$\tau(p) = \begin{cases}\sigma(s)+\delta(s) & t\geqslant 0\\ \sigma(s)-\delta(s) & t<0\end{cases}$$

so that $\tau(p) = \nabla_f f(\Pi(p))$ for every $p\in\Sigma\smallsetminus\gamma$, and let $M = -\tau X + (1-\frac{\tau^2}{2})Y$.

The surface $\gamma\langle Z\rangle$ cuts W into two halves W^+ and W^- such that τ is smooth on each half. Each half $W^\pm\cap\Sigma$ is a smooth ruled Z–graph, so by Example 3.7, $\mathrm{div}_{\mathbb{H}} M = 0$ on $W^\pm$. If we extend τ to $\tau^\pm\in C^1(\overline{W^\pm})$ by continuity, it satisfies (23) everywhere along the boundary, so by Proposition 3.8, M is conservative. By Proposition 3.6, Σ is energy-minimizing on W.

In the case that γ is a horizontal line, these examples are similar to some of the area-minimizing surfaces constructed in [Rit09].

This lets us prove Theorem 1.2.

Proof of Theorem 1.2. Let $\alpha > 0$ and let $I_1, I_2, \dots \subset [-\alpha, \alpha]$ be a collection of disjoint nonempty open intervals. Let $I_i = (a_i, b_i)$, let $m_i = \frac{a_i+b_i}{2}$, and let $\delta_i = \frac{b_i-a_i}{2}$. Let $K = [-\alpha,\alpha] \setminus \bigcup I_i$ and let $\Lambda_K \subset \mathbb{H}$ be as in the statement of the theorem. Let R_0 be the negative x–axis, and for each $i \geqslant 1$, let R_i be a positive horizontal ray from the origin with slope m_i.

We first decompose $\mathbb{R}^2$ into wedges. Let

$$W_0 = \left\{(x, y) \in \mathbb{R}^2 : |y| > \alpha x\right\},$$
$$W_i = \left\{(x, y) \in \mathbb{R}^2 : x > 0, \frac{y}{x} \in I_i\right\},$$

and

$$P_K = \left\{(x, y) \in \mathbb{R}^2 : x > 0, \left|\frac{y}{x}\right| \in K\right\} \cup \{(0,0)\}.$$

These sets are disjoint and their union is $\mathbb{R}^2$.

Every point $p \in \Lambda_K$ is the endpoint of a horizontal curve. That is, for any $p \in \Lambda_K$, there is a unique horizontal curve $\gamma_p \colon (-\infty, x(p)] \to \Lambda_K$ with unit x–speed such that $\gamma_p(x(p)) = p$. These curves are illustrated in Figure 2. The shape of the curve depends on which wedge contains $\pi(p)$. When $\pi(p) \in W_0$, γ_p travels along the negative x–axis, then along a horizontal ray of slope $\pm\alpha$. When $\pi(p) \in W_i$, γ_p travels along the negative x–axis to $\mathbf{0}$, along a horizontal ray of slope m_i, then along a horizontal ray of slope $m_i \pm \delta_i$. When $\pi(p) \in P_K$, γ_p travels along the negative x–axis to $\mathbf{0}$, then along a horizontal ray of slope k, for some $k \in K$.

For any two points $p, q \in \Lambda_K$, the curves γ_p and γ_q satisfy Lemma 3.9 with $\max\{\mathrm{Lip}(y\circ\gamma_p), \mathrm{Lip}(y\circ\gamma_q)\} \leqslant \alpha$, so there is a scale-invariant double cone such that $p \notin qC$. It follows that Λ_K is an intrinsic Lipschitz graph; in fact it is an entire intrinsic Lipschitz graph. By construction, it is also an entire Z–graph. Let $h \colon V_0 \to \mathbb{R}$ be the function such that $\Lambda_K = \Gamma_h$.

Next, we construct a function τ_K and a corresponding calibration M_K to show that Λ_K is energy-minimizing. Since Λ_K is a Z–graph, we choose a τ_K that is constant on vertical lines. For $i > 0$, let

$$W_i^+ = \{(x, y) \in W_i : y \geqslant m_i x\},$$
$$W_i^- = \{(x, y) \in W_i : y < m_i x\}.$$

Let

$$\tau_K(x,y,z) = \begin{cases} m_i \pm \delta_i & (x, y) \in W_i^{\pm} \\ \alpha & (x, y) \in W_0, y \geqslant 0 \\ -\alpha & (x, y) \in W_0, y < 0 \\ \frac{y}{x} & (x, y) \in P_K. \end{cases}$$

We claim that $M_K = -\tau_K X + (1 - \frac{\tau_K}{2})Y$ is conservative and that $\tau(p) = \nabla_h h(\Pi(p))$ almost everywhere on Λ_K. Let $K_n = [-\alpha,\alpha] \setminus (I_1 \cup I_2 \cup \dots \cup I_n)$ and define P_{K_n}, τ_{K_n} and M_{K_n} as above. Then $\mathbb{H}$ is the union of finitely many sets

$$\mathbb{H} = P_{K_n} \cup \bigcup_{i=0}^{n} W_i^+ \cup \bigcup_{i=0}^{n} W_i^-,$$

each with locally finite perimeter. On each of these sets, M_{K_n} is C^1 and $\operatorname{div}_{\mathbb{H}} M_{K_n} = 0$. Further, τ_{K_n} is continuous except along the vertical half-planes separating W_i^+ from W_i^-. These half-planes have slope m_i (taking $m_0 = 0$), and τ_{K_n} takes values $m_i \pm \delta_i$ above and below the central ray (taking $\delta_0 = \alpha$), so τ_{K_n} satisfies (23). By Proposition 3.8, M_{K_n} is conservative. Since M_K is the uniform limit of the M_{K_n}'s, M_K is also conservative.

We calculate $\nabla_h h$ by applying Theorem 1.2 of [BCSC15]. This theorem implies that there is a $\mathscr{S}^3$–null subset $R \subset \Lambda_K$ such that for any $p \in \Lambda_K \smallsetminus S$ and any unit x–speed horizontal curve $\beta = (\beta_x, \beta_y, \beta_z)\colon (-\varepsilon, \varepsilon) \to \Lambda_K$ such that $\beta(0) = p$, if β_y' is differentiable, then $\nabla_h h(\Pi(p)) = \beta_y'(0)$.[3]

Let $p \in \Lambda_K \smallsetminus S$ be a point that does not lie on the negative x–axis or on the countably many horizontal rays bisecting the W_i's. Then Λ_K contains a horizontal ray of slope $\tau_K(p)$ through p, so $\nabla_h h(\Pi(p)) = \tau_K(p)$, as desired. Proposition 3.6 then implies that Λ_K is energy-minimizing. □

Finally, we show that these energy-minimizing graphs can be written as limits of the stretched H–minimal graphs Σ_K constructed in [GR21] (see Theorem 1.1).

Proof of Proposition 1.3. Let α and $K = [-\alpha, \alpha] \smallsetminus \bigcup I_i$ be as in Theorem 1.2. Let m_i be the midpoint of I_i, and for $n > \sqrt{\alpha}$, let $\Theta(t) = (\cos(t), \sin(t), 0) \in S^1$ and let $K_n = \Theta(n^{-2}K) \subset S^1$. Let Σ_{K_n} as in Theorem 1.1, and let $S_n = s_{n^{-1},n}(\Sigma_{K_n})$. We claim that the S_n's are intrinsic Lipschitz graphs and that $\mathbf{1}_{S_n^+} \to \mathbf{1}_{\Lambda_K^+}$ in $L_1^{\mathrm{loc}}(\mathbb{H})$.

First, note that Σ_{K_n} can be written as a union of horizontal rays in the direction $\Theta(\pm n^{-2}\alpha) \in S^1$, rays in the direction of some $v \in K_n$, and rays in the direction of $\Theta(n^{-2}m_i)$. If R_n is a ray with $\angle(R, X) = n^{-2}\theta$, then $s_{n^{-1},n}(R_n)$ is a ray with slope $n^2 \tan^{-1}(n^{-2}\theta) \to \theta$ as $n \to \infty$. It follows that if n is sufficiently large, then S_n is a union of horizontal rays with slopes between -2α and 2α. In fact, for every $p \in S_n$, there is a horizontal curve $\beta_p^n\colon (-\infty, x(p)] \to S_n$ such that $x(\beta_p^n(t)) = t$ and $\beta_p^n(x(p)) = p$. Any two curves β_p^n and β_q^n satisfy Lemma 3.9, so the S_n's are intrinsic Lipschitz graphs with uniform Lipschitz constant.

The $\mathbb{R}$–trees formed by the horizontal curves in Λ_K and S_n are isomorphic. That is, there are homomorphisms $h_n\colon \Lambda_K \to S_n$ that send each horizontal curve in Λ_K to a horizontal curve in S_n, and these can be chosen so that for any $p \in \Lambda_K$, the uniform limit $\lim_{n\to\infty} \beta^n_{h_n(p)}$ is a horizontal curve in Λ_K ending at p. Consequently, $h_n \to \mathrm{id}_{\Lambda_K}$ uniformly on bounded sets, and since the S_n's are uniformly intrinsic Lipschitz, we have $\mathbf{1}_{S_n^+} \to \mathbf{1}_{\Lambda_K^+}$ in $L_1^{\mathrm{loc}}(\mathbb{H})$. □

4. Contact harmonic graphs and first variation formulas

In this section, we propose an intrinsic analogue of harmonic functions and discuss some applications and limitations. It is natural to define a harmonic intrinsic graph in $\mathbb{H}$ as a critical point of the intrinsic Dirichlet energy. It is not clear, however, how to choose an appropriate class of variations of Γ.

When Γ is smooth or piecewise smooth, we may consider smooth perturbations of Γ. Critical points of E_U with respect to smooth perturbations have vanishing horizontal mean curvature.

[3]The original theorem is stated in terms of integral curves of ∇_h. This version is obtained by applying the original theorem to $\Pi \circ \beta$, which is an integral curve of ∇_h.

Proposition 4.1 (First variation formula for smooth graphs)**.** *Let $U \subset V_0$ be a bounded open set. Let $f \in C^\infty(U)$, $h \in C_c^\infty(U)$. For $t \in (-\varepsilon, \varepsilon)$, let $f_t = f + th$ and $\Gamma_t = \Gamma_{f_t}$. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{E}_U(f_t) = -\int_U \nabla_f^2 f \cdot h \,\mathrm{d}\mu\bigg|_{t=0}.$$

Proof. First, note that

$$\begin{aligned}\partial_t \nabla_{f_t} f_t &= \partial_t \left[\partial_x [f + th] - (f + th)\partial_z [f + th]\right] \\ &= \partial_x h - h \cdot \partial_z [f + th] - (f + th)\partial_z h.\end{aligned}$$

At $t = 0$,

$$\partial_t \nabla_{f_t} f_t\big|_{t=0} = \partial_x h - h\partial_z f - f\partial_z h = \nabla_f h - h\partial_z f.$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}E(\Gamma_{f_t})\bigg|_{t=0} = \int_U \nabla_f f \cdot \left(\nabla_f h - h \cdot \partial_z f\right) \mathrm{d}\mu = \int_U \nabla_f f \cdot \nabla_f h - \nabla_f f \cdot h \cdot \partial_z f \,\mathrm{d}\mu.$$

By Corollary 2.6 applied to $\int_U \nabla_f f \cdot h \,\mathrm{d}\mu$,

$$\frac{\mathrm{d}}{\mathrm{d}t}E(\Gamma_{f_t})\bigg|_{t=0} = -\int_U \nabla_f^2 f \cdot h \,\mathrm{d}\mu.$$

□

Smooth energy-minimizing graphs are thus foliated by horizontal lines.

Proposition 4.2. *Let $U \subset V_0$ be a bounded open set and let $f : U \to \mathbb{R}$ be a smooth function which is energy-minimizing on U. Then $\nabla_f^2 f = 0$ on U, and for every $p \in \Gamma_f$, there is a horizontal line segment L_p with endpoints in $\partial\Gamma_f$ such that $p \subset L_p \subset \Gamma_f$.*

Proof. By minimality, for any $h \in C_c^\infty(U)$ and any $t \in \mathbb{R}$, we have $\widetilde{E}_U(f) \leqslant \widetilde{E}_U(f + th)$, so by Proposition 4.1,

$$\frac{\mathrm{d}}{\mathrm{d}t}E(\Gamma_{f+th})\bigg|_{t=0} = -\int_U \nabla_f^2 f \cdot h \,\mathrm{d}\mu = 0$$

and $\nabla_f^2 f = 0$ on U.

Let $p \in \Gamma$ and $u = \Pi(p)$ and suppose that $u \in U$. By the smoothness of f, there is a unique maximal integral curve of ∇_f through p, i.e., a curve $\gamma = (\gamma_x, \gamma_y, \gamma_z) : I \to U$ such that $\gamma(0) = \Pi(p)$ and $\gamma'(t) = \partial_x - f(\gamma(t))\partial_z$. Then $\lambda = \Psi_f \circ \gamma = \gamma(t) Y^{f(\gamma(t))} =: (\lambda_x, \lambda_y, \lambda_z)$ is a horizontal curve in Γ satisfying $\lambda_x' = \gamma_x' = 1$ and

$$\lambda_y''(t) = (f \circ \gamma)''(t) = \nabla_f^2 f(\gamma(t)) = 0$$

for all t. That is, $L_p = \lambda$ is a horizontal line segment containing p, as desired. □

Unfortunately, when f is intrinsic Lipschitz, the perturbation $f + th$ need not be intrinsic Lipschitz. As an alternative, one can apply contact variations, like those considered in [FMV17, Gol18]. A *contact diffeomorphism* of $\mathbb{H}$ is a diffeomorphism $\mathbb{H} \to \mathbb{H}$ that sends horizontal vectors to horizontal vectors. A *contact flow* is a one-parameter flow of such diffeomorphisms. Contact flows are generated by contact vector fields, and any contact vector field is determined by a potential function [KR95, Sec. 5]; for any smooth function $\psi \in C^\infty(\mathbb{H})$, the corresponding contact vector field is given by

$$V_\psi = (Y\psi)X - (X\psi)Y + \psi Z. \tag{25}$$

We are particularly interested in contact diffeomorphisms and flows that send intrinsic graphs to intrinsic graphs.

Definition 4.3. A *contact graph diffeomorphism* is a contact diffeomorphism that sends cosets of $\langle Y\rangle$ to cosets of $\langle Y\rangle$. Flows of contact graph diffeomorphisms are generated by contact vector fields V_ψ whose X–component is constant on each coset of $\langle Y\rangle$, i.e., fields generated by potentials satisfying $YY\psi = 0$.

Let $\phi_u(\Gamma), u \in (-\varepsilon,\varepsilon)$ be the flow generated by a contact vector field V_ψ whose potential satisfies $YY\psi = 0$. For an intrinsic graph Γ, we call the family of surfaces of the form $\phi_u(\Gamma), u \in (-\varepsilon,\varepsilon)$ a *contact graph variation*.

If Γ is the intrinsic graph of a smooth function and ψ is a smooth potential on $\mathbb{H}$, one can construct a smooth potential $\hat\psi$ that agrees with ψ up to first order on Γ and satisfies $YY\hat\psi = 0$ everywhere. Then $V_{\hat\psi}$ generates a family of contact graph diffeomorphisms and $V_\psi = V_{\hat\psi}$ on Γ, so considering only contact graph variations is not very restrictive.

Section 4 of [Gol18] describes these diffeomorphisms in terms of diffeomorphisms from V_0 to V_0. We can identify the coset space $\mathbb{H}/\langle Y\rangle$ with V_0 by the projection Π. A contact graph diffeomorphism ϕ then determines a diffeomorphism $\bar\phi\colon V_0 \to V_0$ such that $\bar\phi(v) = \Pi(\phi(v))$ for all $v \in V_0$.

When $\bar\phi$ is C^1–close to the identity, it sends characteristic curves of an intrinsic Lipschitz graph Γ to characteristic curves of $\phi(\Gamma)$. That is, let $\Gamma = \Gamma_f$ be an intrinsic Lipschitz graph and suppose that $\nabla_f[x\circ\bar\phi](v) > 0$ for all $v \in V_0$. Let $\gamma\colon \mathbb{R} \to V_0$ be a characteristic curve of Γ, parametrized with unit x–speed. Then $\lambda = \Psi_f \circ \gamma$ is a horizontal curve in Γ, and $\phi\circ\lambda$ is a horizontal curve in $\phi(\Gamma)$ such that

$$(x\circ\phi\circ\lambda)'(t) = (x\circ\bar\phi\circ\gamma)'(t) = \nabla_f[x\circ\bar\phi](\gamma(t)) > 0.$$

Thus the x–coordinate of $\Pi\circ\phi\circ\lambda = \bar\phi\circ\gamma$ is increasing, and it is a characteristic curve of $\phi(\Gamma)$.

We propose the following definition.

Definition 4.4. Let $U \subset V_0$ be an open set, let $f\colon U \to \mathbb{R}$ be an intrinsic Lipschitz function, and let $\Gamma = \Gamma_f$. We say that Γ is *contact harmonic* if f is a critical point of $\widetilde{E}_U$ among contact graph variations with potentials whose support lies in $\Pi^{-1}(U)$.

In the rest of this section, we will prove some first variation formulas for the intrinsic Dirichlet energy and use them to characterize contact harmonic graphs.

4.1. **First variation for intrinsic Lipschitz graphs.** We first prove a formula for the variation of the energy of an intrinsic Lipschitz graph under a contact flow. We start by considering the smooth case, as in [FMV17].

Theorem 4.5 (First variation formula for contact variations)**.** *Let $\Gamma = \Gamma_f$ be an intrinsic graph of a smooth function. Let $\psi \in C^\infty(\mathbb{H})$ be a potential such that $YY\psi = 0$, let $V_\psi = (Y\psi)X - (X\psi)Y + \psi Z$, and let $\phi_t\colon \mathbb{H} \to \mathbb{H}$, $t \in [-\varepsilon,\varepsilon]$ be the flow of V_ψ. Let $\Gamma_t = \phi_t(\Gamma)$ and let $f_t\colon V_0 \to \mathbb{R}$ be the function such that $\Gamma_t = \Gamma_{f_t}$.*

Let $\bar\phi_t\colon V_0 \to V_0$, $\bar\phi_t(v) = \Pi(\phi_t(v))$ be the corresponding family of diffeomorphisms of V_0. Let W be the vector field on V_0 generating the $\bar\phi_t$'s, i.e.,

$$W(v) = \Pi_*(V_\psi(v)) = (Y\psi(v), 0, \psi(v))$$

for all $v \in V_0$. Let $w_1 = x\circ W$, $w_2 = (x\circ W)\cdot f + z\circ W$, so that $W = w_1\nabla_f + w_2\partial_z$.

Let $U \subset V_0$ be a bounded open subset. There are $\varepsilon_0 = \varepsilon_0(U,\psi,\|f\|_{L_\infty(U)}) > 0$ and $C = C(U,\psi,\|f\|_{L_\infty(U)}) > 0$ such that

$$|\widetilde{E}_{\bar\phi_t(U)}(f_t) - \widetilde{E}_U(f) - (A_1 + A_2)t| \leqslant C\left(\widetilde{E}_U(f) + \mu(U)\right)t^2 \tag{26}$$

for all $t \in [-\varepsilon_0, \varepsilon_0]$*, where* A_1 *and* A_2 *depend on* w_1 *and* w_2 *respectively:*

$$A_1 = A_1(f, w_1) = \int_U w_1 \cdot \nabla_f^2 f + \frac{1}{2}(\nabla_f f)^2 \left(\partial_x w_1 - \partial_z [f w_1]\right) \mathrm{d}\mu$$

$$A_2 = A_2(f, w_2) = \int_U -\nabla_f^2 w_2 \cdot \nabla_f f + \frac{1}{2}(\nabla_f f)^2 \cdot \partial_z w_2 \, \mathrm{d}\mu.$$

If $\operatorname{supp} W \Subset U$*, then* $A_1 = 0$ *and*

$$A_2 = \int_U \left(w_2 \cdot \partial_z f + \nabla_f w_2\right) \cdot \nabla_f^2 f \, \mathrm{d}\mu = \int_U w_2 \cdot \left(2\partial_z f \cdot \nabla_f^2 f - \nabla_f^3 f\right) \mathrm{d}\mu. \tag{27}$$

Note that C only depends on the L_∞ norm of f and not f itself. This will be important later, when we prove a version of Theorem 4.5 for intrinsic Lipschitz graphs. We cannot make C completely independent of f because the existence of C is based on a compactness argument, and Γ escapes to infinity when f is large.

Note also that in the compactly supported case, the first variation is independent of w_1. Indeed, vector fields parallel to ∇_f generate flows that preserve the foliation of V_0 by characteristic curves and thus preserve Γ.

Before we prove Theorem 4.5, we make some preliminary calculations.

Lemma 4.6. *With notation as in Theorem 4.5, let* $h \in \mathbb{H}$ *and let* $u = \Pi(h)$*. Then*

$$\psi(h) = w_2(u) + (y(h) - f(u)) w_1(u) \tag{28}$$

and $Y\psi(h) = w_1(u)$*. When* $h \in \Gamma$*,*

$$X\psi(h) = \nabla_f w_2(u) - \nabla_f f(u) \cdot w_1(u). \tag{29}$$

Proof. Note that $\Pi_*(X_p) = (1, 0, -y(p))$. In particular, when $g \in \Gamma$, $\Pi_*(X_g) = (\nabla_f)_g$.

Let $h \in \mathbb{H}$ and $u = \Pi(h)$. Since ϕ_t sends cosets of $\langle Y \rangle$ to cosets of $\langle Y \rangle$, we have $\Pi(\phi_t(h)) = \Pi(\phi_t(u))$ for all $h \in \mathbb{H}$. Therefore,

$$W(u) = \Pi_*(V_\psi(u)) = \partial_t [\Pi(\phi_t(u))]\big|_{t=0} = \partial_t [\Pi(\phi_t(h))]\big|_{t=0} = \Pi_*(V_\psi(h)).$$

By (25),

$$\begin{aligned} W(u) &= \Pi_*(Y\psi(h) X_h) - \Pi_*(X\psi(h) Y_h) + \Pi_*(\psi(h) Z_h) \\ &= (Y\psi(h), 0, -Y\psi(h) y(h) + \psi(h)). \end{aligned}$$

In particular, $w_1(u) = Y\psi(h)$ and

$$w_2(u) = \psi(h) + Y\psi(h)(f(u) - y(h)) = \psi(h) + (f(u) - y(h)) w_1(u),$$

which proves (28).

Differentiating (28), we find

$$X\psi(h) = \Pi_*(X_h)[w_2 - f w_1] + y(h) \Pi_*(X_h)[w_1]. \tag{30}$$

When $h \in \Gamma$, we have $y(h) = f(u)$ and $\Pi_*(X_h) = (\nabla_f)_u$, so

$$X\psi(h) = \nabla_f [w_2 - f w_1] + f \nabla_f w_1 = \nabla_f w_2 - \nabla_f f \cdot w_1,$$

where all functions on the right are evaluated at u. □

Lemma 4.7. *With notation as in Theorem 4.5, let* $f_t \colon U \to \mathbb{R}$ *be such that* $\Gamma_t = \Gamma_{f_t}$ *and let* $F(u, t) = f_t(u)$*. Let* $u \in V_0$*. If* f *is smooth in a neighborhood of* u*, then*

$$\frac{\mathrm{d}}{\mathrm{d}t} f_t(\bar{\phi}_t(u))\bigg|_{t=0} = -\nabla_f w_2(u) + w_1(u) \cdot \nabla_f f(u)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\nabla_{f_t} f_t(\bar\phi_t(u))\Big|_{t=0} = -\nabla_f^2 w_2(u) + w_1(u)\cdot\nabla_f^2 f(u).$$

Proof. Let $p = \Psi_f(u)$ so that $f(u) = y(p)$. Then $f_t(\bar\phi_t(u)) = y(\phi_t(p))$, so

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} f_t(\bar\phi_t(u))\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} y(\phi_t(p))\Big|_{t=0} &= y(V_\psi(p)) = -X\psi(p)\\ &= -\nabla_f w_2(u) + w_1(u)\nabla_f f(u),\end{aligned}$$

as desired.

Let $\gamma\colon(-\varepsilon,\varepsilon)\to\Gamma$ be the horizontal curve in Γ such that $\gamma(0)=p$ and γ has unit x–speed (i.e., $x(\gamma(s)) = x(p)+s$). For any t, $\phi_t\circ\gamma$ parametrizes the horizontal curve in Γ_t through $\phi_t(p)$, so

$$\nabla_{f_t} f_t(\bar\phi_t(u)) = \frac{(y\circ\phi_t\circ\gamma)'(0)}{(x\circ\phi_t\circ\gamma)'(0)}.$$

Let $T_p = \gamma'(0) = X_p + \nabla_f f(u) Y_p$. Commuting derivatives with respect to s and t,

$$\frac{\mathrm{d}}{\mathrm{d}t}(x\circ\phi_t\circ\gamma)'(0)\Big|_{t=0} = \partial_t\partial_s\left[x(\phi_t(\gamma(s)))\right](0,0) = \partial_s\left[x(V_\psi(\gamma(s)))\right](0) = T_p[x\circ V_\psi]$$

and likewise $\frac{\mathrm{d}}{\mathrm{d}t}(y\circ\phi_t\circ\gamma)'(0)\big|_{t=0} = T_p[y\circ V_\psi]$.

Furthermore, $\Pi_*(T_p) = \Pi_*(X_p) = (\nabla_f)_u$ and $x(V_\psi(p)) = w_1(\Pi(p))$, so

$$T_p[x\circ V_\psi] = T_p[w_1\circ\Pi] = \Pi_*(T_p)[w_1] = \nabla_f w_1(u).$$

By (25) and Lemma 4.6, $y\circ V_\psi = -X[\psi] = -\left(\nabla_f w_2 - \nabla_f f\cdot w_1\right)\circ\Pi$, so

$$\begin{aligned}T_p[y\circ V_\psi] = -\Pi_*(T_p)\left[\nabla_f w_2 - \nabla_f f\cdot w_1\right] &= -\nabla_f\left[\nabla_f w_2 - \nabla_f f\cdot w_1\right](u)\\ &= \left(-\nabla_f^2 w_2 + \nabla_f^2 f\cdot w_1 + \nabla_f f\cdot\nabla_f w_1\right)(u)\end{aligned}$$

By our choice of parametrization, $(x\circ\gamma)'(0) = 1$ and $(y\circ\gamma)'(0) = \nabla_f f(u)$, so

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\nabla_{f_t} f_t(\bar\phi_t(u))\Big|_{t=0} &= \frac{T_p[y\circ V_\psi]\cdot(x\circ\gamma)'(0) - (y\circ\gamma)'(0)\cdot T_p[x\circ V_\psi]}{(x\circ\gamma)'(0)^2}\\ &= T_p[y\circ V_\psi] - \nabla_f f(u)\cdot T_p[x\circ V_\psi]\\ &= -\nabla_f^2 w_2(u) + \nabla_f^2 f(u)\cdot w_1(u).\end{aligned}$$

□

Theorem 4.5 follows.

Proof of Theorem 4.5. Let

$$F(t) = E_{\bar\phi_t(U)}(\Gamma_t) = \frac{1}{2}\int_U \left(\nabla_{f_t} f_t(\bar\phi_t(u))\right)^2 J_{\bar\phi_t}(u)\,\mathrm{d}\mu(u)$$

where $J_{\bar\phi_t}$ is the Jacobian determinant of $\bar\phi$. We have

$$\frac{\mathrm{d}}{\mathrm{d}t} J_{\bar\phi_t}\Big|_{t=0} = \mathrm{div}_{V_0} W = \partial_x[x\circ W] + \partial_z[z\circ W] = \partial_x w_1 + \partial_z[w_2 - f w_1]. \tag{31}$$

Exchanging derivative and integral and using Lemma 4.7,

$$F'(0) = \int_U \left(-\nabla_f^2 w_2 + w_1\cdot\nabla_f^2 f\right)\nabla_f f + \frac{1}{2}(\nabla_f f)^2\,\mathrm{div}_{V_0} W\,\mathrm{d}\mu, \tag{32}$$

Substituting (31) into (32) gives us $F' = A_1 + A_2$.

In order to prove (26), it suffices to show that there are $\varepsilon_0 = \varepsilon_0(U, \psi, \|f\|_{L_\infty(U)}) > 0$ and $C = C(U, \psi, \|f\|_{L_\infty(U)}) > 0$ such that $|F''(t)| \le C(E_U(\Gamma) + \mu(U))$ for all $t \in [-\varepsilon_0, \varepsilon_0]$. Let $K_u(t) = J_{\bar{\phi}_t}(u)$, $N_u(t) = \nabla_{f_t} f_t(\bar{\phi}_t(u))$. Then

$$F'' = \frac{1}{2}\int_U \left(N_u^2 K_u\right)'' \, d\mu = \int_U \left((N_u')^2 + N_u N_u''\right) K_u + 2N_u N_u' K_u' + \frac{N_u^2 K_u''}{2} \, d\mu. \tag{33}$$

By continuity, there is a $c_1(\psi) > 0$ such that $\max\{|K_u(t)|, |K_u'(t)|, |K_u''(t)|\} \le c_1(\psi)$ for all $u \in U$ and $t \in [-\varepsilon, \varepsilon]$.

Let $M = \{vY^r \in \mathbb{H} : v \in U, |r| \le \|f\|_{L_\infty(U)}\}$. Since $\phi_0 = \mathrm{id}_{\mathbb{H}}$, $X[x \circ \phi_0] = 1$; by continuity, we may suppose that ε_0 is sufficiently small (depending on ψ, U, and $\|f\|_{L_\infty(U)}$) that $X[x \circ \phi_t](q) > \frac{1}{2}$ for all $q \in M$. Choose c_2 such that

$$c_2 \ge \left|\partial_t^i V[v \circ \phi_t](q)\right|$$

for all $q \in M$, $i = 0, 1, 2$, $V = X, Y$, and $v = x, y$.

Let $u \in U$ and let $p = \Psi_f(u)$. Let T be the left-invariant vector field $T = X + \nabla_f f(u) Y$. As in the proof of Lemma 4.7, we have

$$N_u(t) = \frac{T[y \circ \phi_t](p)}{T[x \circ \phi_t](p)} =: \frac{\omega(t)}{\chi(t)}.$$

Since ϕ_t sends cosets of $\langle Y \rangle$ to cosets of $\langle Y \rangle$, we have $Y[x \circ \phi_t] = 0$. Therefore, $\chi(t) = X[x \circ \phi_t](p) > \frac{1}{2}$. By our choice of c_2, for $i = 0, 1, 2$ and $|t| \le \varepsilon_0$,

$$\left|\omega^{(i)}(t)\right| = \left|\partial_t^i T[y \circ \phi_t](p)\right| \le c_2(1 + \nabla_f f(u)),$$

$$\left|\omega^{(i)}(t)\right| = \left|\partial_t^i X[y \circ \phi_t](p) + \nabla_f f(u) \cdot \partial_t^i Y[y \circ \phi_t](p)\right| \le c_2(1 + \nabla_f f(u)),$$

and

$$\left|\chi^{(i)}(t)\right| = \left|\partial_t^i X[y \circ \phi_t](p)\right| \le c_2.$$

Thus for $|t| \le \varepsilon_0$,

$$\left|N_u'(t)\right| = \left|\frac{\omega'(t)\chi(t) - \omega(t)\chi'(t)}{\chi(t)^2}\right| \lesssim c_2(1 + \nabla_f f(u)),$$

and

$$\left|N_u''(t)\right| = \left|\omega(t)\left(\frac{2\chi'(t)^2}{\chi(t)^3} - \frac{\chi''(t)^2}{\chi(t)^2}\right) - 2\omega'(t)\frac{\chi'(t)}{\chi(t)^2} + \frac{\omega''(t)}{\chi(t)}\right| \lesssim c_2(1 + \nabla_f f(u)).$$

We apply these bounds to (33) to get

$$\left|F''(t)\right| \lesssim \int_U c_1 c_2^2 (1 + \nabla_f f(u))^2 \, d\mu(u) \le C(\mu(U) + E_U(\Gamma))$$

for some $C = C(\psi, U, \|f\|_{L_\infty(U)})$. Equation (26) then follows from Taylor's theorem.

It remains to consider the case that $\operatorname{supp} W \Subset U$. Let $K \subset U$ be a closed set with piecewise-smooth boundary that contains $\operatorname{supp} W$. We calculate

$$\begin{aligned} A_1 &= \frac{1}{2}\int_K \nabla_f \left[(\nabla_f f)^2\right] \cdot w_1 + (\nabla_f f)^2 \left(\partial_x w_1 - f \partial_z w_1 - w_1 \partial_z f\right) d\mu \\ &= \frac{1}{2}\int_K \nabla_f \left[(\nabla_f f)^2\right] \cdot w_1 + (\nabla_f f)^2 \cdot \nabla_f w_1 - (\nabla_f f)^2 \cdot w_1 \cdot \partial_z f \, d\mu. \end{aligned} \tag{34}$$

Corollary 2.6 with $g = (\nabla_f f)^2$ and $h = w_1$ implies that $A_1 = 0$.

We decompose A_2 as

$$A_2 = \int_K -\nabla_f^2 w_2 \cdot \nabla_f f \, d\mu + \int_K \frac{1}{2}(\nabla_f f)^2 \cdot \partial_z w_2 \, d\mu =: Q_1 + Q_2.$$

By Corollary 2.6 with $g = \nabla_f w_2$ and $h = \nabla_f f$,

$$Q_1 = -\int_K \nabla_f w_2 \cdot \nabla_f f \cdot \partial_z f \, d\mu + \int_K \nabla_f^2 f \cdot \nabla_f w_2 \, d\mu := Q_3 + Q_4.$$

Applying Corollary 2.6 again with $g = w_2$, $h = \nabla_f f \cdot \partial_z f$, we find

$$Q_3 = \int_K -w_2 \cdot \nabla_f f \cdot (\partial_z f)^2 + w_2 \cdot \nabla_f \left[\nabla_f f \cdot \partial_z f\right] d\mu$$
$$= \int_K w_2 \cdot \nabla_f f \cdot \left(-(\partial_z f)^2 + \nabla_f[\partial_z f]\right) + w_2 \cdot \partial_z f \cdot \nabla_f^2 f \, d\mu.$$

Furthermore,

$$[\nabla_f, \partial_z] = -[\partial_z, \partial_x - f\partial_z] = \partial_z f \cdot \partial_z,$$

so $-(\partial_z f)^2 + \nabla_f[\partial_z f] = \partial_z[\nabla_f f]$. Thus

$$Q_3 = \int_K w_2 \cdot \nabla_f f \cdot \partial_z[\nabla_f f] + w_2 \cdot \partial_z f \cdot \nabla_f^2 f \, d\mu. \tag{35}$$

Integrating Q_2 by parts gives

$$Q_2 = \int_K \frac{1}{2}(\nabla_f f)^2 \cdot \partial_z w_2 \, d\mu = -\int_K w_2 \cdot \nabla_f f \cdot \partial_z[\nabla_f f] \, d\mu,$$

so

$$A_2 = Q_3 + Q_4 + Q_2 = \int_{K^+} \left(w_2 \cdot \partial_z f + \nabla_f w_2\right) \cdot \nabla_f^2 f \, d\mu.$$

This is the first part of (27). Applying Corollary 2.6 once more with $g = w_2$ and $h = \nabla_f^2 f$, we get

$$A_2 = \int_U w_2 \cdot \left(2\partial_z f \cdot \nabla_f^2 f - \nabla_f^3 f\right) d\mu.$$

□

We will prove a first variation formula for intrinsic Lipschitz graphs by approximating them by smooth graphs and applying Theorem 4.5. Note, however, that when f is intrinsic Lipschitz, $\nabla_f w$ is generally not smooth, so $\nabla_f^2 w$ may be undefined. We thus introduce a new operator. For any intrinsic Lipschitz function $f\colon V_0 \to \mathbb{R}$, any smooth $w\colon V_0 \to \mathbb{R}$, and any $p \in V_0$, let $\lambda_p(t) = \Psi_f(p)(X + \nabla_f f\, Y)^t$ be the intrinsic tangent line to Γ_f at $\Psi_f(p)$ and let

$$\Delta_f w(p) = (w \circ \Pi \circ \lambda_p)''(0). \tag{36}$$

This is defined almost everywhere on V_0. When f is smooth, the first two derivatives of the characteristic curve through p agree with the first two derivatives of λ_p, so $\Delta_f w = \nabla_f^2 w$. In general, if $p = (x_0, 0, z_0)$, then

$$\Pi(\lambda_p(t)) = \left(x_0 + t, 0, z_0 - f(p)t - \frac{t^2}{2}\nabla_f f(p)\right),$$

so

$$\Delta_f w(p) = \left(\partial_x - f(p)\partial_z\right)^2[w](p) - \nabla_f f(p) \cdot \partial_z w(p)$$
$$= \nabla_f\left[\partial_x w - f(p)\partial_z w\right](p) - \nabla_f f(p) \cdot \partial_z w(p)$$
$$= \nabla_f[\partial_x w](p) - f(p)\nabla_f[\partial_z w](p) - \nabla_f f(p) \cdot \partial_z w(p). \tag{37}$$

Theorem 4.8 (First variation formula for intrinsic Lipschitz graphs)**.** *Let $\Gamma = \Gamma_f$ be the intrinsic graph of an intrinsic Lipschitz function $f\colon V_0 \to \mathbb{R}$. Let ϕ_t, ψ, $\bar{\phi}_t$, and W be as in Theorem 4.5. For each $t \in [-\varepsilon, \varepsilon]$, let f_t be the function such that $\Gamma_{f_t} = \phi_t(\Gamma)$.*

Let $U \subset V_0$ be a bounded open subset and suppose that $\operatorname{supp} W \Subset U$*. For any $w \in C^\infty(U)$ and any intrinsic Lipschitz function g, let*

$$B_2(g, w) = -\Delta_g w \cdot \nabla_g g + \frac{1}{2}(\nabla_g g)^2 \cdot \partial_z w,$$

$$B_1(g, w) = g B_2(g, w) - \frac{3}{2}(\nabla_g g)^2 \cdot \nabla_g w.$$

Then there is a $C = C(\psi, U, \|f\|_{L_\infty(U)})$ such that

$$\left| \widetilde{E}_U(f_t) - \widetilde{E}_U(f) - t \int_U B_1(f, x \circ W) + B_2(f, z \circ W)\, \mathrm{d}\mu \right| \leqslant C \left(\widetilde{E}_U(f) + \mu(U) \right) t^2 \tag{38}$$

for all $|t| \leqslant \varepsilon$.

The function B_2 above is simply the integrand in the definition of A_2, with $\nabla_f^2 w$ replaced by $\Delta_f w$. The following lemma shows that when f is smooth, Theorem 4.8 follows from Theorem 4.5.

Lemma 4.9. *Let $U \subset V_0$ be a bounded open subset. Let B_1 and B_2 be as in Theorem 4.8. When f is smooth and $w \in C_c^\infty(U)$,*

$$\int_U B_1(f, w)\, \mathrm{d}\mu = \int_U B_2(f, f w)\, \mathrm{d}\mu, \tag{39}$$

and $\int_U B_2(f, w)\, \mathrm{d}\mu = A_2(f, w)$, where A_2 is as in Theorem 4.5. Consequently, if W is a smooth vector field with $\operatorname{supp} W \Subset U$ *and if $w_2 = (x \circ W) \cdot f + z \circ W$, then*

$$A_2(f, w_2) = \int_U B_2(f, w_2)\, \mathrm{d}\mu = \int_U B_1(f, x \circ W) + B_2(f, z \circ W)\, \mathrm{d}\mu.$$

Proof. Since f is smooth, we have $\Delta_f w = \nabla_f^2 w$ and $\int_U B_2(f, w)\, \mathrm{d}\mu = A_2(f, w)$. Let $\lambda = \nabla_f f$. Then

$$\Delta_f[f w] - f \Delta_f[w] = 2\nabla_f f \cdot \nabla_f w + \nabla_f^2 f \cdot w = 2\lambda \cdot \nabla_f w + \nabla_f \lambda \cdot w,$$

so

$$B_2(f, f w) - f B_2(f, w) = -2\lambda^2 \cdot \nabla_f w - \lambda \cdot \nabla_f \lambda \cdot w + \frac{1}{2}\lambda^2 \cdot \partial_z f \cdot w.$$

By Corollary 2.6, with $f = w$ and $h = \frac{1}{2}\lambda^2$,

$$\begin{aligned}
\int_U B_2(f, f w)\, \mathrm{d}\mu &= \int_U f B_2(f, w) - 2\lambda^2 \cdot \nabla_f w - \lambda \nabla_f \lambda \cdot w + \frac{1}{2}\lambda^2 \cdot \partial_z f \cdot w\, \mathrm{d}\mu \\
&= \int_U f B_2(f, w) - \frac{3}{2}\lambda^2 \cdot \nabla_f w\, \mathrm{d}\mu \\
&= \int_U B_1(f, w)\, \mathrm{d}\mu,
\end{aligned}$$

as desired. □

Proof of Theorem 4.8. By Theorem 2.3, since U is bounded, there is a sequence $f^k \in C^\infty(U)$ of smooth functions such that $f^k \to f$ uniformly, there is a $c > 0$ such that $\|\nabla_{f^k} f^k\|_\infty < c$ for all k, and $\nabla_{f^k} f^k \to \nabla_f f$ pointwise almost everywhere in U. For $w \in C^\infty(U)$, we have

$$\lim_{k \to \infty} \nabla_{f^k} w = \lim_{k \to \infty} \partial_x w - f^k \partial_z = \nabla_f w$$

uniformly on compact subsets of U and

$$\lim_{k\to\infty} \Delta_{f^k} w \overset{(37)}{=} \lim_{k\to\infty} \nabla_{f^k}[\partial_x w] - \nabla_{f^k} f^k \cdot \partial_z w - f^k \nabla_{f^k}[\partial_z w] = \Delta_f w$$

pointwise almost everywhere in U. In particular, if $w \in C_c^\infty(U)$, then $B_1(f^k, w)$ and $B_2(f^k, w)$ are bounded by a function of c and w and converge pointwise a.e. to $B_1(f, w)$ and $B_2(f, w)$, respectively.

For each $t \in [-\varepsilon, \varepsilon]$, the image $\phi_t(\Gamma_{f^k})$ is a smooth intrinsic graph; let f_t^k be such that $\Gamma_{f_t^k} = \phi_t(\Gamma_{f^k})$. Let K be a bounded set containing $\Psi_{f^k}(U)$ for every k. For all $u \in U$, we have

$$\left|f_t^k(u) - f_t(u)\right| \le \left|f^k\left(\bar{\phi}_t^{-1}(u)\right) - f\left(\bar{\phi}_t^{-1}(u)\right)\right| \operatorname{Lip}\left(\phi_t|_K\right).$$

Since $f^k \to f$ uniformly, this implies $f_t^k \to f_t$ uniformly.

Likewise, for any $t \in [-\varepsilon, \varepsilon]$, if $\lim_{k\to\infty} \nabla_{f^k} f^k(u) = \nabla_f f(u)$, then

$$\lim_{k\to\infty} \nabla_{f_t^k} f_t^k(\bar{\phi}_t(u)) = \nabla_{f_t} f_t(\bar{\phi}_t(u)),$$

so $\nabla_{f_t^k} f_t^k \to \nabla_{f_t} f_t$ pointwise a.e. Thus, by dominated convergence,

$$\lim_{k\to\infty} \widetilde{E}_U(f_t^k) = \widetilde{E}_U(f_t). \tag{40}$$

Theorem 4.5 and Lemma 4.9 imply that there are $\varepsilon_0, C > 0$ such that for any k and any $t \in [-\varepsilon_0, \varepsilon_0]$,

$$\begin{aligned}
&\left|\widetilde{E}_U(f_t^k) - \widetilde{E}_U(f^k) - tA_2\big(f^k, (x\circ W)\cdot f + z\circ W\big)\right| \\
&\qquad = \left|\widetilde{E}_U(f_t^k) - \widetilde{E}_U(f^k) - t\int_U B_1\big(f^k, x\circ W\big) + B_2\big(f^k, z\circ W\big)\,\mathrm{d}\mu\right| \\
&\qquad \le C\big(\widetilde{E}_U(f^k) + \mu(U)\big)t^2.
\end{aligned} \tag{41}$$

Taking the limit as $k \to \infty$ and using dominated convergence to exchange the integral and the limit, we get

$$\left|\widetilde{E}_U(f_t) - \widetilde{E}_U(f) - t\int_U B_1(f, x\circ W) + B_2(f, z\circ W)\,\mathrm{d}\mu\right| \le C\left(E_U(\Gamma) + \mu(U)\right)t^2,$$

as desired. □

This gives a two-part condition for contact harmonicity for intrinsic Lipschitz functions. Namely, an intrinsic Lipschitz f is contact harmonic on U if and only if

$$\int_U B_1(f, w)\,\mathrm{d}\mu = \int_U B_2(f, w)\,\mathrm{d}\mu = 0$$

for every $w \in C_c^\infty(U)$. In contrast, when f is smooth and $\int_U B_2(f, w)\,\mathrm{d}\mu = 0$ for all $w \in C_c^\infty(U)$, (39) implies that

$$\int_U B_1(f, w)\,\mathrm{d}\mu = \int_U B_2(f, fw)\,\mathrm{d}\mu = 0$$

for all $w \in C_c^\infty(U)$, so the condition on B_2 suffices to characterize contact harmonicity. We do not know whether the condition on B_2 suffices to characterize contact harmonicity when f is not smooth.

4.2. **Vertical first variation for graphs with herringbone singularities.** One consequence of Theorem 4.5 is that when f is smooth, $W = w_1\nabla_f + w_2\partial_z$ is a compactly supported vector field on V_0, and f_t is the corresponding contact variation of f, then $\frac{\mathrm{d}}{\mathrm{d}t}[\widetilde{E}_{\tilde{\phi}_t(U)}(f_t)]$ depends only on the vertical component of W, i.e., only on w_2. It follows that a smooth intrinsic graph is contact harmonic if and only if it is a critical point of E_U with respect to *vertical contact variations*, i.e., contact variations where the corresponding vector field on V_0 can be written as $W = w_2\partial_z$. We do not know whether the same thing is true when f is nonsmooth.

In this section, we will prove a first variation formula for vertical contact variations of intrinsic graphs Γ that are smooth away from a horizontal curve. We show that if Γ is contact harmonic, the horizontal curves near the singularity must satisfy the equal-slope condition (23).

We first define the class of singularities we are interested in.

Definition 4.10. An intrinsic Lipschitz graph with a *smooth herringbone singularity* consists of two smooth intrinsic graphs Γ^+ and Γ^- meeting along a horizontal curve C and satisfying the properties below.

Let $U \subset V_0$ be an open set and let $\Gamma = \Gamma_f \subset \mathbb{H}$ be an intrinsic Lipschitz graph over U. Suppose that the characteristic nexus of Γ is a smooth horizontal curve $C \subset \Gamma$ and let $\gamma\colon I \to V_0$, $\gamma(t) = (t, 0, \gamma_z(t))$ parametrize $\Pi(C)$. Suppose that γ cuts U into two connected components, $U^+ = \{(x,0,z) \in U : z > \gamma_z(x)\}$ and $U^- = \{(x,0,z) \in U : z < \gamma_z(x)\}$. Let $\Gamma^\pm = \Psi_f(U^\pm)$.

Suppose that f is smooth on U^+ and U^- (but generally not on γ), so that Γ^+ and Γ^- are foliated by horizontal curves. We require that the foliations extend to the boundary in the following sense:

- There is a neighborhood N of C such that the projection $\pi\colon \mathbb{H} \to \mathbb{R}^2$, $\pi(x,y,z) = (x,y)$ restricts to an embedding of $N \cap \Gamma$.
- Let $H^\pm$ be the horizontal foliation of $\Gamma^\pm$. Let $M = \pi(N\cap\Gamma)$ and let $M^\pm = \pi(N\cap\Gamma^\pm)$. The projection $\pi_*(H^+)$ can be extended to a smooth foliation F^+ defined on a neighborhood of $\overline{M^+}$. Likewise, $\pi_*(H^-)$ extends to a smooth foliation F^- defined on a neighborhood of $\overline{M^-}$. These foliations are transverse to $\pi(C)$ and their tangent lines have bounded slopes.

Then C is a smooth herringbone singularity of Γ.

The behavior of Γ near C is governed by the slopes of $\pi(C)$, F^+, and F^-. Let $c = (t, c_y, c_z)\colon I \to \mathbb{H}$ be a parametrization of C. Let $\sigma^0(t) = c_y'(t)$ be the slope of $\pi \circ c$. Let $\sigma^+(t)$ (resp. $\sigma^-(t)$) be the slope of F^+ (resp. F^-) at $\pi(c(t))$. We will see in Lemma 4.12 that $\sigma^+(t) < \sigma^0(t) < \sigma^-(t)$ for all $t \in I$.

Smooth herringbone singularities are either *left-pointing* or *right-pointing*. Since π is a homeomorphism from $N \cap \Gamma$ to M, the sets M^+ and M^- are separated by the projection $\pi(C)$. If M^+ is above $\pi(C)$ (i.e., $M^+ = \{(x,y) \in \pi(N) : y > c_y(x)\}$), we say C is a right-pointing singularity (because $\pi_*(H^\pm)$ looks like $\ggg$). If M^+ is below $\pi(C)$, we say C is a left-pointing singularity (because $\pi_*(H^\pm)$ looks like $\lll$).

We now state the first variation formula for vertical contact variations.

Theorem 4.11 (First variation formula with herringbone singularities)**.** *Let $U \subset V_0$ be an open subset, let $f\colon U \to \mathbb{R}$, and let $\Gamma = \Gamma_f$ be an intrinsic Lipschitz graph with a smooth herringbone singularity C. Let σ^0, σ^+, and σ^- be as above, and let $\gamma\colon I \to V_0$, $\gamma(t) = (t, 0, \gamma_z(t))$ be a parametrization of* $\Pi(C)$.

Let $w_2 \in C_c^\infty(U)$, let $\psi = w_2 \circ \Pi$ be a potential, and let $V_\psi = -(X\psi)Y + w_2 Z$ be the corresponding field. We can write $V_\psi = \Pi^*(W)$, where $W = w_2 Z$ is a vector field on V_0. Let $\phi_t \colon \mathbb{H} \to \mathbb{H}$, $t \in (-\varepsilon, \varepsilon)$ be the flow of V_ψ, and let $\bar{\phi}_t \colon V_0 \to V_0$ be the flow of W.

Let f_t be the function such that $\Gamma_{f_t} = \phi_t(\Gamma)$. There are $\varepsilon_0 = \varepsilon_0(U, w_2, \|f\|_{L_\infty(U)}) > 0$ and $D = D(U, w_2, \|f\|_{L_\infty(U)}) > 0$ such that

$$\left|\widetilde{E}_U(f_t) - \widetilde{E}_U(f) - A_2 t\right| \leqslant D\left(\widetilde{E}_U(f) + \mu(U)\right) t^2 \tag{42}$$

for all $t \in [-\varepsilon_0, \varepsilon_0]$, where

$$\begin{aligned} A_2 &= \int_U -\nabla_f^2 w_2 \cdot \nabla_f f + \frac{1}{2}(\nabla_f f)^2 \cdot \partial_z w_2 \, \mathrm{d}\mu \\ &= \frac{1}{2}\int_I w_2(\gamma(s)) \cdot \delta(s) \, \mathrm{d}s + \int_U \left(w_2 \cdot \partial_z f + \nabla_f w_2\right) \cdot \nabla_f^2 f \, \mathrm{d}\mu, \end{aligned}$$

and $\delta(s) = (\sigma^+(s) - \sigma^0(s))^2 - (\sigma^0(s) - \sigma^-(s))^2$.

In particular, if Γ is contact harmonic on U, then $\delta(s) = 0$ and thus $\sigma^0(s) = \frac{\sigma^+(s)+\sigma^-(s)}{2}$ for all $s \in I$. That is, the slope of $\pi(C)$ is the average of the slopes of F^+ and F^-.

The condition on the slope of $\pi(C)$ is analogous to the condition in [CHY07] that a singular curve in an H–minimal Z–graph bisects the foliation lines on either side.

Before we prove Theorem 4.11, we examine the behavior of Γ near C.

Lemma 4.12. *With notation as in Theorem 4.11, for all $s \in I$, $\sigma^+(s) < \sigma^0(s) < \sigma^-(s)$. Let $\delta = 1$ if C is right-pointing and $\delta = -1$ if C is left-pointing. Let $J \subset I$ be a compact interval. Then for all $s \in J$ and all sufficiently small $v > 0$,*

$$f(\gamma(s)Z^v) - f(\gamma(s)) = \delta\sqrt{2\left(\sigma^0(s) - \sigma^+(s)\right)v} + O(v) \tag{43}$$

$$f(\gamma(s)Z^{-v}) - f(\gamma(s)) = -\delta\sqrt{2\left(\sigma^-(s) - \sigma^0(s)\right)v} + O(v) \tag{44}$$

$$\partial_z f(\gamma(s)Z^v) = \delta\sqrt{\frac{\sigma^0(s) - \sigma^+(s)}{2v}} + O(1) \tag{45}$$

$$\partial_z f(\gamma(s)Z^{-v}) = \delta\sqrt{\frac{\sigma^-(s) - \sigma^0(s)}{2v}} + O(1), \tag{46}$$

where the implicit constants depend on f and J. In particular, $\partial_z f$ is L_1 on a neighborhood of $\gamma(J)$.

Proof. Let $J' \Subset I$ be an interval such that $J \Subset J'$. Let $\varepsilon > 0$ be sufficiently small that for any $s \in J'$, there are unique unit x–speed horizontal curves $\lambda_s^\pm \colon (-\varepsilon, \varepsilon) \to \mathbb{H}$ such that $\lambda_s^\pm(0) = c(s)$ and the projection $\pi \circ \lambda_s^+$ (resp. $\pi \circ \lambda_s^-$) is a leaf of F^+ (resp. F^-), and let $\Lambda^\pm(s,t) = \lambda_s^\pm(t)$. Then Λ^+ is a smooth map defined on $D = J' \times (-\varepsilon, \varepsilon)$, and its image contains a neighborhood of $c(J)$ in Γ^+. Let $\zeta = z \circ \Pi \circ \Lambda^+$ and let $s \in J$. Since $\gamma(s) = \Pi(c(s)) = \Pi(\Lambda^+(s,0))$, we have $\zeta(s,0) = \gamma_z(s)$.

We expand ζ around $(s,0)$. Since $s \mapsto \Lambda^+(s,0) = c(s)$ is a smooth, unit x–speed horizontal curve, (7) implies $\partial_s \zeta(s,0) = -y(c(s))$ and

$$\partial_s^2 \zeta(s,0) = -(y \circ c)'(s) = -\sigma^0(s).$$

Likewise, for any s, the map $t \mapsto \Lambda^+(s,t) = \lambda_s^\pm(t)$ is a smooth, unit x–speed horizontal curve, so $\partial_t \zeta(s,t) = -y(\lambda_s^+(t))$, and

$$\partial_t^2 \zeta(s,t) = -(y \circ \lambda_s^+)'(t).$$

Setting $t = 0$, we get $\partial_t\zeta(s,0) = -y(c(s))$ and $\partial_t^2\zeta(s,0) = -\sigma^+(s)$, so $\partial_s\partial_t\zeta(s,0) = -(y\circ c)'(s) = -\sigma^0(s)$. Thus

$$\begin{aligned}\zeta(s-t,t) &= \gamma_z(s) + y(c(s))t - y(c(s))t - \frac{\sigma^0(s)}{2}t^2 + \sigma^0(s)t^2 - \frac{\sigma^+(s)}{2} + O(t^3)\\ &= \gamma_z(s) + \frac{\sigma^0(s)-\sigma^+(s)}{2}t^2 + O(t^3).\end{aligned} \tag{47}$$

For every $s \in J'$ and $0 < t < \varepsilon$, either $\Lambda^+(s-t,t)$ or $\Lambda^+(s+t,-t)$ lies in Γ^+, so $\zeta(s-t,t) > \gamma_z(s)$ or $\zeta(s+t,-t) > \gamma_z(s)$. In either case, (47) implies $\sigma^+(s) < \sigma^0(s)$. By a similar argument, $\sigma^0(s) < \sigma^-(s)$ for all $s \in J'$. Thus, if C is right-pointing, then $M^+ = \pi(N\cap\Gamma^+)$ is above $\pi(C)$, and the horizontal foliations of Γ project to lines of the form $\ggg$, while if C is left-pointing, they project to lines of the form $\lll$.

For the rest of this proof, we suppose that C is right-pointing. If C is a left-pointing singularity, we can rotate Γ by 180° around the z–axis to produce a new graph $\widehat{\Gamma} = s_{-1,-1,1}(\Gamma)$ with a right-pointing singularity. The slopes σ^0, σ^+, and σ^- stay the same under this symmetry, but the sign of f flips, so the left-pointing case follows from the right-pointing case.

Let $s \in J$. The smoothness of ζ and (47) imply that for any sufficiently small $v > 0$, there is a unique $t > 0$ such that $\zeta(s+t,-t) - \gamma_z(s) = v$. Let $p = \Lambda^+(s+t,-t) \in \Gamma$. Since C is right-pointing, $p \in \Gamma^+$. Then

$$\Pi(p) = (s+t-t, \zeta(s+t,-t)) = (s, \gamma_z(s)+v) = \gamma(s)Z^v,$$

so

$$f\big(\gamma(s)Z^v\big) = y(p) = y(\Lambda^+(s+t,-t)). \tag{48}$$

We thus consider the relationship between v and t. By (47),

$$v = \frac{\sigma^0(s)-\sigma^+(s)}{2}t^2 + O(t^3). \tag{49}$$

For $s \in J$, $\sigma^0(s) - \sigma^+(s)$ is bounded away from zero and bounded by a function of the intrinsic Lipschitz constant of f, so $v \asymp t^2$ and

$$t = \sqrt{\frac{2v}{\sigma^0(s)-\sigma^+(s)} + O(t^3)} = \sqrt{\frac{2v}{\sigma^0(s)-\sigma^+(s)}} + O\left(\frac{t^3}{\sqrt{v}}\right) = \sqrt{\frac{2v}{\sigma^0(s)-\sigma^+(s)}} + O(v).$$

Since $\partial_s[y\circ\Lambda^+](s,0) = \sigma^0(s)$ and $\partial_t[y\circ\Lambda^+](s,0) = \sigma^+(s)$,

$$\begin{aligned}f(\gamma(s)Z^v) = y(\Lambda^+(s+t,-t)) &= y(\Lambda^+(s,0)) + (\sigma^0(s)-\sigma^+(s))t + O(t^2)\\ &= f(\gamma(s)) + \sqrt{2v(\sigma^0(s)-\sigma^+(s))} + O(v),\end{aligned}$$

with implicit constants depending on f and J. This proves (43).

Differentiating (49) gives

$$\frac{\mathrm{d}v}{\mathrm{d}t} = (\sigma^0(s)-\sigma^+(s))t + O(t^2).$$

Therefore,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}v} f(\gamma(s) Z^{v}) &\overset{(48)}{=} \frac{\mathrm{d}t}{\mathrm{d}v}\frac{\mathrm{d}}{\mathrm{d}t} y(\Lambda^{+}(s+t,-t)) \\ &= \left(\frac{1}{(\sigma^0(s)-\sigma^+(s))t} + \frac{O(t^2)}{(\sigma^0(s)-\sigma^+(s))^2 t^2}\right)\left(\sigma^0(s)-\sigma^+(s)+O(t)\right) \\ &= t^{-1}+O(1) \\ &= \sqrt{\frac{\sigma^0(s)-\sigma^+(s)}{2v}}+O(1),\end{aligned}$$

proving (45). The argument for the case $v<0$ and the equations (43) and (45) is symmetric. □

Now we prove Theorem 4.11.

Proof of Theorem 4.11. By Lemma 4.12, f is smooth on $U^{\pm}=\Pi(\Gamma^{\pm})$, but not on the curve $\gamma=\Pi(C)$ dividing U^+ and U^-. Indeed, $\partial_z f\to\infty$ near γ. Nevertheless, $\nabla_f f$ is equal to the slopes of the curves in F^+ and F^-, so $\nabla_f f$ is bounded but discontinuous near γ. Let $K\subset U$ be a closed set with piecewise smooth boundary such that $\operatorname{supp} w_2 \Subset K$. Let $K^{\pm}=U^{\pm}\cap K$. We define $\lambda^+\colon \overline{K^+}\to\mathbb{R}$,

$$\lambda^+(p)=\begin{cases}\nabla_f f(p) & p\in K^+\\ \sigma^+(x(p)) & p\in\gamma\end{cases}$$

and define $\lambda^-\colon\overline{K^-}\to\mathbb{R}$ likewise.

These maps are continuous away from γ. To show continuity on γ, let N be a neighborhood of C as in Definition 4.10 and let $s^+\colon\pi(N\cap\Gamma)\to\mathbb{R}$ so that $s^+(b)$ is the slope of F^+ at b. Then for any $v=(x_v,z_v)\in\Pi(N\cap\Gamma^+)$, $\nabla_f f(v)$ is the slope of the horizontal curve through $\Psi_f(v)$, i.e.,

$$\nabla_f f(v)=s^+(\pi(\Psi_f(v)))=s^+(x_v,f(v)). \tag{50}$$

This is continuous in a neighborhood of γ, so λ^+ and λ^- are continuous.

Furthermore, we can bound $\nabla_f\lambda^{\pm}$ and $\partial_z\lambda^{\pm}$ near γ. The horizontal derivative $\nabla_f\lambda^{\pm}$ is the derivative of $s^{\pm}$ along a leaf of $F^{\pm}$, so $\nabla_f\lambda^{\pm}\in L_\infty(K^{\pm})$. By (50) and Lemma 4.12,

$$\begin{aligned}\partial_z\lambda^{\pm}(x,0,z)&=\partial_z f(x,0,z)\cdot\partial_y s^+(x,f(x,0,z))\\ &=\left(\pm\sqrt{\frac{\sigma^0(s)-\sigma^+(s)}{2(z-\gamma_z(x))}}+O(1)\right)\cdot\partial_y s^+(x,f(x,z)).\end{aligned}\tag{51}$$

Since $\partial_y s^+$ and $\sigma^0(s)-\sigma^+(s)$ are bounded, $\partial_z\lambda^{\pm}\in L_1(K^{\pm})$.

Now we turn to $\widetilde{E}_U(f_t)$. Since $\bar\phi_t(U)=U$, Theorem 4.5 implies that

$$\left|\widetilde{E}_U(f_t)-\widetilde{E}_U(f)-A_2t\right|=\left|\widetilde{E}_K(f_t)-\widetilde{E}_K(f)-A_2t\right|\le c\left(\widetilde{E}_U(f)+\mu(U)\right)t^2$$

for some $c=c(\psi,U,\|f\|_{L_\infty(U)})$, where

$$A_2=\int_K -\nabla_f^2 w_2\cdot\nabla_f f+\frac{1}{2}(\nabla_f f)^2\cdot\partial_z w_2\,\mathrm{d}\mu(v).$$

We decompose A_2 as $A_2=A_2^+ + A_2^-$, where

$$A_2^{\pm}=\int_{K^{\pm}}-\nabla_f^2 w_2\cdot\lambda^{\pm}\,\mathrm{d}\mu+\int_{K^{\pm}}\frac{1}{2}(\lambda^{\pm})^2\cdot\partial_z w_2\,\mathrm{d}\mu=:Q_1^{\pm}+Q_2^{\pm}, \tag{52}$$

and we will compute the $Q_i^{\pm}$'s as in the compactly supported case of Theorem 4.5, plus a boundary term arising from the singularity.

Consider $Q_1^+ = \int_{K^+} -\nabla_f^2 w_2 \cdot \lambda^+ \,\mathrm{d}\mu$. We have

$$\nabla_f w_2 = \nabla_f[(x\circ W)\cdot f + z\circ W] = \nabla_f[x\circ W]\cdot f + (x\circ W)\cdot\lambda^+ + \nabla_f[z\circ W]$$

on K^+, so $\nabla_f w_2$ can be extended continuously to $\overline{K^+}$. Letting $g = \nabla_f w_2$ and $h = \lambda^+$, we know that $g, h, \nabla_f g, \nabla_f h \in L_\infty(K^+)$. By Lemma 4.12 and (51), $\partial_z f, \partial_z h \in L_1(K^+)$. The smoothness of w_2 implies

$$\partial_z g = \partial_z\partial_x w_2 - \partial_z f\cdot\partial_z w_2 - f\cdot\partial_z^2 w_2 \in L_1(K^+),$$

so we can apply Corollary 2.6 to show that

$$Q_1^+ = -\int_{K^+} \nabla_f w_2\cdot\lambda^+\cdot\partial_z f\,\mathrm{d}\mu + \int_{K^+}\nabla_f\lambda^+\cdot\nabla_f w_2\,\mathrm{d}\mu := Q_3^+ + Q_4^+. \tag{53}$$

Before we integrate by parts again, we restrict to a domain that avoids γ. Let $T_i \subset K^+$ be the closed subset of K^+ bounded by $Z^{\frac{1}{i}}\gamma$. This has piecewise smooth boundary and f is smooth on T_i. Then $\nabla_f w_2, \lambda^+ \in L_\infty(K^+)$, and $\partial_z f \in L_1(K)$ by Lemma 4.12, so

$$Q_3^+ = \lim_{i\to\infty} -\int_{T_i}\nabla_f w_2\cdot\lambda^+\cdot\partial_z f\,\mathrm{d}\mu.$$

Let α_i parametrize ∂T_i in the positive direction. By Corollary 2.6 with $g = w_2$, $h = \lambda^+\cdot\partial_z f$,

$$\begin{aligned}
-\int_{T_i}\nabla_f w_2\cdot\lambda^+\cdot\partial_z f\,\mathrm{d}\mu &= \int_{T_i} -w_2\cdot\lambda^+\cdot(\partial_z f)^2 + w_2\cdot\nabla_f\left[\lambda^+\cdot\partial_z f\right]\mathrm{d}\mu + P_i\\
&= \int_{T_i} w_2\cdot\lambda^+\cdot\left(-(\partial_z f)^2 + \nabla_f[\partial_z f]\right) + w_2\cdot\partial_z f\cdot\nabla_f\lambda^+\,\mathrm{d}\mu + P_i
\end{aligned} \tag{54}$$

where P_i is the line integral $P_i = \int_{\partial T_i} -w_2\cdot\lambda^+\cdot\partial_z f\cdot(fX+Z)\cdot\mathrm{d}\alpha_i$. As in the proof of Theorem 4.5, $-(\partial_z f)^2 + \nabla_f[\partial_z f] = \partial_z\lambda^+$, so

$$Q_3^+ = \lim_{i\to\infty}\int_{T_i} w_2\cdot\lambda^+\cdot\partial_z\lambda^+ + w_2\cdot\partial_z f\cdot\nabla_f\lambda^+\,\mathrm{d}\mu + P_i.$$

This integrand is L_1, so in fact,

$$Q_3^+ = \int_{K^+} w_2\cdot\lambda^+\cdot\partial_z\lambda^+ + w_2\cdot\partial_z f\cdot\nabla_f\lambda^+\,\mathrm{d}\mu + \lim_{i\to\infty} P_i. \tag{55}$$

Now we consider $\lim_{i\to\infty} P_i$. Let $\varepsilon = i^{-1}$ and let I be the domain of γ. Since w_2 vanishes on $\partial T_i \smallsetminus Z^\varepsilon\gamma(I)$, it suffices to integrate P_i over $Z^\varepsilon\gamma(I)$. Let $p = \gamma(t)$ and $q = \gamma(t)Z^\varepsilon$. Since γ is characteristic, $\gamma'(t) = X - f(p)Z$, so

$$\begin{aligned}
P_i &= \int_I -w_2(q)\cdot\lambda^+(q)\cdot\partial_z f(q)\cdot(f(q)X+Z)\cdot\gamma'(t)\,\mathrm{d}t\\
&= \int_I -w_2(q)\cdot\lambda^+(q)\cdot\partial_z f(q)\cdot(f(q)-f(p))\,\mathrm{d}t\\
&= \int_I -w_2(q)\cdot\lambda^+(q)\cdot\left(\sigma^0(t)-\sigma^+(t)+O(\varepsilon)\right)\mathrm{d}t,
\end{aligned}$$

using Lemma 4.12 in the last step. Since $\lambda^+(q) = \sigma^+(t)$,

$$\lim_i P_i = \int_I w_2(\gamma(t))\cdot\sigma^+(t)\left(\sigma^+(t)-\sigma^0(t)\right)\mathrm{d}t. \tag{56}$$

Finally, $\partial_z \lambda^+ \in L_1(K^+)$, so integrating Q_2^+ by parts gives

$$Q_2^+ = \int_{K^+} \frac{1}{2}(\lambda^+)^2 \cdot \partial_z w_2 \,\mathrm{d}\mu = \int_{K^+} \frac{1}{2}\partial_z \left[w_2 \cdot (\lambda^+)^2 \right] \mathrm{d}\mu - \int_{K^+} w_2 \cdot \lambda^+ \cdot \partial_z \lambda^+ \,\mathrm{d}\mu.$$

Since w_2 vanishes on all of ∂T_i except for its lower boundary $\gamma(I)$,

$$\begin{aligned} Q_2^+ &= -\int_I \frac{1}{2} w_2(\gamma(s)) \cdot \lambda^+(\gamma(s))^2 \,\mathrm{d}s - \int_{K^+} w_2 \cdot \lambda^+ \cdot \partial_z \lambda^+ \,\mathrm{d}\mu \\ &= -\int_I \frac{1}{2} w_2(\gamma(s)) \cdot \sigma^+(s)^2 \,\mathrm{d}s - \int_{K^+} w_2 \cdot \lambda^+ \cdot \partial_z \lambda^+ \,\mathrm{d}\mu. \end{aligned}$$

Combining this computation with (53), (55), and (56) and canceling like terms, we find

$$\begin{aligned} A_2^+ &= Q_3^+ + Q_4^+ + Q_2^+ \\ &= \int_I w_2(\gamma(s)) \cdot \left(\frac{1}{2}\sigma^+(s)^2 - \sigma^+(s)\sigma^0(s) \right) \mathrm{d}s + \int_{K^+} (w_2 \cdot \partial_z f + \nabla_f w_2) \cdot \nabla_f^2 f \,\mathrm{d}\mu. \end{aligned}$$

A similar calculation for A_2^- gives

$$A_2^- = \int_I w_2(\gamma(s)) \cdot \left(\frac{1}{2}\sigma^-(s)^2 - \sigma^-(s)\sigma^0(s) \right) \mathrm{d}s + \int_{K^-} (w_2 \cdot \partial_z f + \nabla_f w_2) \cdot \nabla_f^2 f \,\mathrm{d}\mu$$

and

$$A_2 = A_2^+ + A_2^- = \frac{1}{2}\int_I w_2(\gamma(s)) \cdot \delta(s) \,\mathrm{d}s + \int_K (w_2 \cdot \partial_z f + \nabla_f w_2) \cdot \nabla_f^2 f \,\mathrm{d}\mu,$$

where $\delta(s) = (\sigma^+(s) - \sigma^0(s))^2 - (\sigma^0(s) - \sigma^-(s))^2$. □

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