

# Characterizing Trust and Resilience in Distributed Consensus for Cyberphysical Systems

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**Abstract**—This work considers the problem of resilient consensus, where stochastic values of trust between agents are available. Specifically, we derive a unified mathematical framework to characterize convergence, deviation of the consensus from the true consensus value, and expected convergence rate, when there exists additional information of trust between agents. We show that under certain conditions on the stochastic trust values and consensus protocol: First, almost sure convergence to a common limit value is possible even when malicious agents constitute more than half of the network connectivity; second, the deviation of the converged limit, from the case where there is no attack, i.e., the true consensus value, can be bounded with probability that approaches 1 exponentially; and third correct classification of malicious and legitimate agents can be attained in finite time almost surely. Furthermore, the expected convergence rate decays exponentially as a function of the quality of the trust observations between agents.

**Index Terms**—Agents' trust values, Byzantine agents, consensus systems, cyberphysical systems (CPSs), malicious agents, resilience.

## I. INTRODUCTION

IN THIS article, we address the problem of multiagent distributed consensus in the presence of malicious agents. The consensus problem is of core importance to many algorithms and coordinated behaviors in multiagent systems. It is well known, however, that these algorithms are vulnerable to malicious activity and/or noncooperating agents [1]–[5] and that several of the existing performance guarantees for the nominal case fail in the absence of cooperation [6], [7].

Many works have investigated the possibility of attaining resilient consensus or consensus in the face of noncooperating

or malicious agents. However, achieving resilience often relies on conservative assumptions of connectivity and bounds on the maximum number of malicious actors. A classical result for consensus in networks with noncooperating agents (faulty or malicious) holds from [1] and [6]:

Well-behaving agents can always agree upon a parameter if and only if the number of malicious agents is *less than 1/2 of the network connectivity*.<sup>1</sup>

This result has long been held as a fundamental requirement for achieving consensus in unreliable networks [1], [6], [8].

For purely cyber systems, where *data* are used to validate information, these limitations are difficult to circumvent. Cyberphysical systems (CPSs), however, are becoming prevalent in many fields, from robotics to power systems to traffic systems and beyond [9]. These systems provide new opportunities to additionally use *physical channels* of information for data validation. These physical channel data can be exploited to understand “trust,” or the trustworthiness of data, among the agents in the system. Examples include using observers to compare against system models [10], [11], using camera or other physical channel data [12], or using transmitted communication channels by comparing against a known carrier signal to find whether a message has been manipulated [13], [14]. Wireless channels are of particular interest since they often exist in multiagent systems as a medium for information exchange and can also be used to extract useful information such as, for example, tracking motion [15], [16], localizing robot agents [17], [18], as well as providing security to wireless systems [19].

Our recent work in [20] derives the characterization of a stochastic value of trust,  $\alpha_{ij} \in (0, 1)$ , that approaches 1 for trusted transmission and 0 for an untrusted transmission between any two agents  $i$  and  $j$  in the case of a Sybil attack. Importantly, these recent results show that information capturing the trustworthiness of communicating agents can be obtained from analyzing the communication signals themselves—alluding to the fact that the probability of trust between agents can be characterized using information that already exists in multiagent networks.

However, there remains a need for a unified mathematical framework to guide how a probabilistic understanding of trust between agents can be used to arrive at important performance guarantees for resilient coordination and consensus.

We are motivated by these recent works to derive consensus results that take trust into account. *This article provides a unified*

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<sup>1</sup>The connectivity of a graph is the maximum number of disjoint paths between any two vertices of the graph.

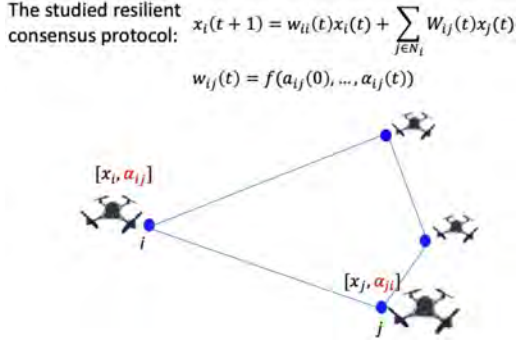


Fig. 1. Schematic of a multirobot system with a consensus protocol modified to use agent transmitted values  $x_i$  and  $x_j$  as well as observations of interagent trust  $\alpha_{ij}$ .

framework that characterizes the value of trust when it comes to achieving resilient consensus in adversarial settings.

We generalize the concept of trust between agents by considering the availability of stochastic observations  $\alpha_{ij}$  that can be thought of as a probability of trust between any two agents  $i$  and  $j$  in a system (see Fig. 1). We show that using this additional information can lead to strong results for resilience. Namely, under certain conditions on  $\alpha_{ij}$  that we characterize, we establish the following results.

1) *Convergence*: We show that convergence to a common limit value is possible almost surely, even when the number of malicious agents is larger than less than 1/2 of the network connectivity.

2) *Deviation from nominal average consensus*: Whereas without additional information of trust between agents, the presence of malicious agents comprising more than 1/2 of the network connectivity could take the consensus value to any limit, we show that by using  $\alpha_{ij}$ s, the influence of malicious agents can be *bounded* and that this bound can be characterized.

3) *Convergence rate*: We show that convergence can be attained in finite time with arbitrarily high probability, and that the expected convergence rate decays exponentially with the quality of the observations  $\alpha_{ij}$  and other problem parameters that we characterize.

A roadmap of our article is as follows. We begin by providing some background and context for the resilient consensus problem in Section I-A. In Section II, we present our consensus model, our model of trust between agents and the  $\alpha_{ij}$  observations, and we characterize the influence of malicious agents on multiagent consensus. In Section III, we provide our theoretical results and analysis of convergence, deviation, and the convergence rate for consensus with malicious agents. In Section IV, we present simulation results, followed by Section V that provides a discussion and future work directions. Finally, Section VI concludes this article.

#### A. Related Work

The problem of coordination in multiagent systems and consensus has a long history [21]–[24]. In particular, the consensus problem has enjoyed the development of many fundamental

results providing insights into conditions for convergence [25]–[29], convergence rate [30]–[32], and general analysis and implementation [33], [34]. Similarly, in the current article, we hope to start building a more general mathematical foundation for *resilient consensus* using the concept of *interagent trust*.

Resilient consensus refers to consensus in the case, where agents are noncooperative due to either faulty or malicious behavior. This is an important problem due to the prevalence of the noncooperative case. For multirobot systems for example, security is of utmost importance as these robot systems enter the world as autonomous vehicles, delivery drones, or partners in security and defense [7], [35], [36]. For CPSs, the issue of security is a highly investigated topic due to the increasing role that CPSs play in critical infrastructure such as power grids, traffic management systems, and beyond [37]–[42]. As a result, there has been increased attention on exposing the vulnerabilities of consensus in multiagent systems [1]–[4] and on developing new theory to support resilient consensus for these systems [5], [6], [8], [10], [43]. A common theme in these works is to use transmitted data to identify and thwart attacks, and as a result, they are often accompanied by important constraints on the minimum connectivity needed in the graph or requirements on the maximum number of malicious agents that can be tolerated. Take, for example, the classical result from [1], which states that well-behaving agents can always agree upon a parameter if and only if the number of malicious agents comprises less than 1/2 of the network connectivity. Additionally, the work [8] presents conditions for an asymptotic convergence to an *arbitrary* point in the convex hull of the initial values of the legitimate agents.

CPSs, including multirobot systems as a prominent example, offer more physical channels of information that can be used for data validation and for quantifying “trust” between agents (robots, sensors, autonomous vehicles, and more). Camera data, communicated wireless signals, lidar, and radar are a few examples of physical channels of information that often exist in CPSs and can be used to validate data transmitted between agents. Indeed, many works have exploited *the physics* of CPSs to validate transmitted data, which present great promise for the security of these systems [11]–[14], [44], [45].

Advances in sensing using wireless signals is especially promising in this domain as many multiagent systems transmit messages over a wireless medium [46], [47]. The wireless community has demonstrated that these wireless signals themselves contain important information that can be fruitfully exploited for security. For example, it has been shown that wireless signals can be analyzed to provide tracking and localization of agents [15], [16], [18], [48]–[50], authentication [19], [51], [52], and thwarting of spoofing attacks [53].

1) *Deriving Trust Values Using Wireless Signals*: Here, we provide some intuition on one particular example of quantifying trust between agents by exploiting physical channels of information, specifically using wireless channels. Our previous work developed a method for measuring *directional signal profiles* using channel state information from the wireless messages over each link  $(i, j)$  in the network [49], [54]. These profiles measure signal strength arriving from every direction in the 3-D plane. Directional signal profiles display two important properties: 1)

transmissions originating from the same physical agent have very similar profiles; and 2) energy can be measured coming from the direct-line path between physical agents. The paper [20] quantifies these properties, providing an analysis that shows, both analytically and experimentally, that a single scalar value  $\alpha_{ij} \in (-0.5, 0.5)$  (shifted by  $-0.5$  from [20]) can be computed for each signal profile that quantifies the likelihood that the transmission is coming from the same physical (spoofed) node or a unique (legitimate) node, a property critical for thwarting Sybil attacks. Intuitively,  $\alpha_{ij}$  was shown experimentally and theoretically to be close to  $-0.5$  if one of the agents  $j$  is a spoofed node and close to  $0.5$  if both agents are legitimate nodes in the network [20]. This is captured quantitatively by the bounds on the expectation of  $\alpha_{ij}$ . The current article takes inspiration from this previous work, which shows the existence of such  $\alpha_{ij}$  variables for stochastically characterizing trust between agents, but goes beyond to define a unified mathematical framework for using trust to obtain certain performance guarantees for consensus in multiagent systems. As such, Definition II.2 specifies the mildest characteristics of such an  $\alpha_{ij}$  term in order to achieve strong performance guarantees. Thus, the results contained herein would be compatible with stochastic variables, as derived in [20], or any other physical channels that satisfy Definition II.2. We note, however, that the focus of this article is not on the derivation of stochastic trust values. Rather, we focus on the development of the mathematical foundations for why and how such values can be key to achieving certain performance guarantees that would be difficult or impossible to obtain otherwise.

2) *Contributions in the Context of Previous Work:* This article builds most closely off of our previous work in [20] and [55], which shows that a stochastic variable  $\alpha_{ij}(t)$  can be extracted from wireless signals between agents to verify the validity of an agent's identity (in a probabilistic sense) during a Sybil attack. Furthermore, Gil *et al.* [55], [56] show the potential of using these values to arrive at resilient consensus. In the current work, we assume the availability of such probabilistic observations of trust between agents,  $\alpha_{ij}$ , and focus on the development of the mathematical machinery that can utilize these trust values to arrive at strong results for resilient consensus. Key differentiating results include the following.

- 1) We provide convergence rate results for the case where the number of malicious agents is larger than  $1/2$  of the network connectivity, showing that convergence in *finite time* is possible with arbitrarily high probability. To the best of our knowledge, this is the first time that this result has been formally proven.
- 2) We divide the consensus algorithm into two critical stages with broad consequences for the attainable performance guarantees. In the first stage, the agents *observe the network*, i.e., they collect values of  $\alpha_{ij}$  to detect malicious agents but do not start running the consensus protocol. In the second stage, agents continue to collect values of  $\alpha_{ij}$  and also adapt their data values using a modified consensus protocol that includes received data only from agents they classify as trustworthy. We show that the impact of observing the network for some time window

$T_0$  is highly consequential for deviation and convergence, where convergence rate and deviation amount decreases exponentially with increasing  $T_0$ .

- 3) We use weights that define a switching topology allowing for arbitrarily small deviation in the final consensus value.
- 4) The current work allows for the use of nonsymmetric weights, which can accommodate larger classes of multi-agent systems.

Finally, the results in this article are general to many classes of attacks so long as certain characteristics on the probability of trust between agents,  $\alpha_{ij}$ , are met. We establish these key characteristics and define their roles in convergence, deviation, and convergence rate to consensus.

## II. PROBLEM DESCRIPTION

We consider the problem of consensus for multiagent systems. The agents in the system communicate over a network, which is represented by an undirected graph,  $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ , where  $\mathbb{V} = \{1, \dots, n\}$  denotes the set of node indices for the agents and  $\mathbb{E} \subset \mathbb{V} \times \mathbb{V}$  denotes the set of undirected edges. We use  $\{i, j\}$  to denote the edge connecting agents  $i$  and  $j$ . The set of *neighbors* of node  $i$  is denoted by  $\mathcal{N}_i = \{j \in \mathbb{V} \mid \{i, j\} \in \mathbb{E}\}$ . Note that, by this definition, if  $i$  is neighbor of  $j$ , then  $j$  is also a neighbor of  $i$ . With the graph  $\mathbb{G}$ , we will associate time-varying nonnegative weights  $w_{ij}(t)$ ,  $\{i, j\} \in \mathbb{E}$ , which capture the information exchange process among the agents at a given time  $t$ . We use  $w_{ij}(t)$  to denote the  $ij$ th entry of the weight matrix  $W(t)$ . For these matrices, we have  $w_{ij}(t) \geq 0$  and  $w_{ji}(t) \geq 0$  only if  $\{i, j\} \in \mathbb{E}$ , and  $w_{ij}(t) = w_{ji}(t) = 0$  otherwise. In this way, the positive entries of the matrices  $W(t)$  are associated only with the edges in the set  $\mathbb{E}$ . Additionally, the matrices  $W(t)$  need not be symmetric. We consider the case where a subset of nodes with indices denoted by the set  $\mathcal{M}$ ,  $\mathcal{M} \subset \mathbb{V}$ , is either *malicious* or *malfunctioning* and, thus, does not reliably cooperate in the consensus protocol. The set  $\mathcal{M}$  is assumed to be unknown, with the cardinality  $|\mathcal{M}|$ . An agent that is not malicious or malfunctioning is termed *legitimate*, and the set of legitimate agents is denoted  $\mathcal{L}$ ,  $\mathcal{L} \subset \mathbb{V}$ , with the cardinality  $|\mathcal{L}|$ .

In what follows, we use  $\mathbf{0}$  to denote the zero matrix, where the dimension of  $\mathbf{0}$  is to be understood from the context. We say that a matrix  $A$  is nonnegative if its entries are nonnegative, i.e.,  $A_{ij} \geq 0$  for all  $i, j$ , and we write  $A \geq \mathbf{0}$  to denote that  $A$  is nonnegative. We write  $A > \mathbf{0}$  to denote a matrix having all entries positive, i.e.,  $A_{ij} > 0$  for all  $i, j$ . A nonnegative matrix is row-stochastic if the sum of its entries in every row is equal to 1, and it is row-substochastic if the sum of its entries in every row is equal to or less than 1. We use  $\mathbf{1}$  to denote the vector with all entries equal to 1. Given a vector  $x$ , we use  $x_i$  or  $[x]_i$  (when  $x$  has a subscript) to denote its  $i$ th entry. A similar notation is used for the entries of a matrix.

### A. Model

We are concerned with the problem of consensus, where each *legitimate* agent updates its value according to the following



update equation (see [5]) for all  $t \geq T_0 - 1$ :

$$x_i(t+1) = w_{ii}(t)x_i(t) + \sum_{j \in \mathcal{N}_i} w_{ij}(t)x_j(t), \quad (1)$$

where  $x_i(t) \in \mathbb{R}$ , while the weights  $w_{ij}(t)$  are nonnegative and sum to 1, i.e.,  $w_{ii}(t) > 0$ ,  $w_{ij}(t) \geq 0$  for  $j \in \mathcal{N}_i$  and  $w_{ii}(t) + \sum_{j \in \mathcal{N}_i} w_{ij}(t) = 1$ . The process is initiated at some time  $T_0 \geq 0$  with the agents' initial values  $x_i(T_0 - 1) = x_i(0)$  for all  $i \in \mathbb{V}$ .

**Definition II.1 (Malicious agent):** Agent  $i$  is said to be *malicious* if it does not follow the rule in (1) for updating its value  $x_i(t)$  at some time  $t \geq T_0 - 1$ .

Let  $x(t) \in \mathbb{R}^n$  denote the vector of values  $x_i(t) \in \mathbb{R}$  for all the agents at time  $t$ . Without loss of generality, we assume that the agent indices are ordered in such a way that the vector  $x(t)$  can be separated in two components: the first component representing the legitimate agent values  $x_{\mathcal{L}}(t)$  and the second component representing the malicious agent values  $x_{\mathcal{M}}(t)$ . In this way, the consensus dynamics takes the following form:

$$\begin{bmatrix} x_{\mathcal{L}}(t+1) \\ x_{\mathcal{M}}(t+1) \end{bmatrix} = \begin{bmatrix} W_{\mathcal{L}}(t) & W_{\mathcal{M}}(t) \\ \Theta(t) & \Omega(t) \end{bmatrix} \begin{bmatrix} x_{\mathcal{L}}(t) \\ x_{\mathcal{M}}(t) \end{bmatrix} \quad (2)$$

where  $W_{\mathcal{L}}(t) \in \mathbb{R}^{|\mathcal{L}| \times |\mathcal{L}|}$  is the matrix multiplying the state component corresponding to the legitimate agents' values, and  $W_{\mathcal{M}}(t) \in \mathbb{R}^{|\mathcal{L}| \times |\mathcal{M}|}$  is the matrix multiplying the state component corresponding to the malicious agents' values. The matrices  $\Theta(t)$  and  $\Omega(t)$  dictate the dynamics of the malicious agents' values and are assumed to be unknown. For the sake of simplicity, we assume that a malicious agent sends the same data value to all of its neighbors; however, our analysis holds for the case where a malicious agent can send different data values to different neighbors at each iteration. We consider the case when a parameter  $\eta > 0$  is known to all legitimate agents and it is known that the values  $|x_i(t)|$  should not exceed  $\eta$  at any time. The assumption  $|x_j(t)| \leq \eta$  for all  $t$  and  $j \in \mathcal{M}$  is reasonable, since an agent  $j$  can be classified as malicious based on its value (i.e., if  $|x_j(t)| > \eta$  at some time  $t$ ). Thus, in this article, our focus is on the update rule in (2), where  $|x_j(t)| \leq \eta$  for all  $j \in \mathcal{L} \cup \mathcal{M}$  and time instants  $t$ .

In this article, we are interested in the case where each transmission from agent  $j$  to agent  $i$  can be tagged with a random observation  $\alpha_{ij}(t) \in [0, 1]$  of a random variable  $\alpha_{ij} \in [0, 1]$ . The random variable  $\alpha_{ij}$  represents a *probability of trust* that agent  $i$  can give to its neighbor  $j \in \mathcal{N}_i$ .

**Definition II.2 ( $\alpha_{ij}$ ):** For every  $i \in \mathcal{L}$  and  $j \in \mathcal{N}_i$ , the random variable  $\alpha_{ij}$  taking values in the interval  $[0, 1]$  represents the probability that agent  $j \in \mathcal{N}_i$  is a trustworthy neighbor of agent  $i$ . We assume the availability of such observations  $\alpha_{ij}(t)$  throughout this article.

We refer to [20] for an example of such an  $\alpha_{ij}$  value. Intuitively, a random realization  $\alpha_{ij}(t)$  of  $\alpha_{ij}$  contains useful trust information if the legitimacy of the transmission can be thresholded.

In this article, we assume that a value of  $\alpha_{ij}(t) > 1/2$  indicates a legitimate transmission and  $\alpha_{ij}(t) < 1/2$  indicates a malicious transmission in a stochastic sense (misclassifications are possible). Note that  $\alpha_{ij}(t) = 1/2$  means that the observation

is completely ambiguous and contains no useful trust information for the transmission at time  $t$ .

Our ultimate goal is to understand the convergence and convergence rate properties of consensus under the influence of a malicious attack if such observations of trust are available. Toward this goal, our first objective is to construct the appropriate weight matrix  $W(t) = [W_{\mathcal{L}}(t) \ W_{\mathcal{M}}(t)]$  for legitimate agents' state dynamics in (2) by using some function of the observations  $\alpha_{ij}(t), \alpha_{ij}(t-1), \dots, \alpha_{ij}(0)$ , where  $i$  is a legitimate agent so that  $i \in \mathcal{L}$  and  $j$  is a neighbor of  $i$  so that  $j \in \mathcal{N}_i$ . In this way, the legitimate agents' ability to isolate the malicious agents is reflected in the convergence of the weight matrices  $W_{\mathcal{M}}(t)$  to the zero matrix and the convergence of the weight matrices  $W_{\mathcal{L}}(t)$  to a row-stochastic matrix, where the convergence is in a probabilistic sense (to be defined shortly). Necessary characteristics of  $\alpha_{ij}$  to allow for the convergence of the linear protocol in (2), possibly including its distribution, bounds on expectation, etc., will be derived in this work.

In line with this model, in addition to broadcasting its value  $x_i(t)$  at each time  $t$ , we assume that a message from each agent  $i \in \mathbb{V}$  is tagged with an observation  $\alpha_{ij}(t)$  for every neighboring agent  $j$ , i.e.,  $\{i, j\} \in \mathbb{E}$  (see Fig. 1). *The purpose of this work is to develop the necessary properties of  $\alpha_{ij}(t)$  such that consensus can be maintained in the presence of malicious or malfunctioning agents.*

Suppose that, in the interval of time  $[0, T_0 - 1)$ , the legitimate agents only use their measurement of the trustworthiness values  $\alpha_{ij}, j \in \mathcal{N}_i, i \in \mathcal{L}$ . We refer to this interval of time as an *observation window*, where agents can amass a history of trust values  $\alpha_{ij}$  before starting the consensus. At time  $T_0 - 1$ , the agents start the data passing phase with the weights  $w_{ij}(T_0 - 1) = f(\alpha_{ij}(0), \dots, \alpha_{ij}(T_0 - 1)) \cdot \mathbb{1}_{\{j \in \mathcal{N}_i\}}$ , where  $f$  is some function capturing the history of all observations  $\alpha_{ij}(0), \dots, \alpha_{ij}(T_0 - 1)$  and  $\mathbb{1}_{\{\cdot\}}$  denotes the indicator function. Our goal is to study the resulting consensus dynamics over time. Since we start the consensus algorithm (2) from time  $t = T_0 - 1$ , we have that  $x_{\mathcal{L}}(t) = x_{\mathcal{L}}(0)$  for all  $t \in [0, T_0 - 1]$ .

Let  $\prod_{k=p}^r H_k$  denote the backward product of the matrices  $H_p, \dots, H_r$ , i.e.,

$$\prod_{k=p}^r H_k = \begin{cases} H_r \cdots H_{p+1} H_p, & \text{if } r \geq p \\ I, & \text{if } r < p \end{cases}$$

where  $I$  denotes the identity matrix. Following (2), the value of the legitimate agents at any point in time  $t$  has two salient influence terms: the values contributed by other legitimate agents  $\tilde{x}_{\mathcal{L}}(T_0, t)$  and the values contributed by other malicious agents  $\phi_{\mathcal{M}}(T_0, t)$  so that for all  $t \geq T_0$

$$x_{\mathcal{L}}(T_0, t) = \tilde{x}_{\mathcal{L}}(T_0, t) + \phi_{\mathcal{M}}(T_0, t) \quad (3)$$

where

$$\tilde{x}_{\mathcal{L}}(T_0, t) = \left( \prod_{k=T_0-1}^{t-1} W_{\mathcal{L}}(k) \right) x_{\mathcal{L}}(0) \quad (4)$$

and

$$\phi_{\mathcal{M}}(T_0, t) = \sum_{k=T_0-1}^{t-1} \left( \prod_{l=k+1}^{t-1} W_{\mathcal{L}}(l) \right) W_{\mathcal{M}}(k) x_{\mathcal{M}}(k). \quad (5)$$

Relation (3) shows explicitly how the states of legitimate agents depend on their initial states at time  $T_0$  and on the states of malicious agents. The term  $\phi_{\mathcal{M}}(T_0, t)$  given in (5) captures the total influence of the malicious agents on the states of the legitimate agents from the initial time  $T_0$  to the current time  $t$  and will be the primary subject of study. Specifically, we will study attainable bounds on the malicious agents' influence term  $\phi_{\mathcal{M}}(T_0, t)$ .

**Remark (Arbitrary starting time  $T_0$  of the consensus)**

We emphasize that the consensus updates from (2) begin at an arbitrary time  $T_0 \geq 0$ , and that before this time, agent values  $x_i(t)$  are not updated for  $i \in \mathcal{L}$ . In other words, for  $t \in [0 : T_0]$ , only observations  $\alpha_{ij}(t)$  are collected by each agent  $i$  for its neighbors  $j \in \mathcal{N}_i$ . The choice of  $T_0$  is a user-selected parameter that affects the overall influence of malicious agents in the network and the convergence rate according to a relationship that we later characterize.

We now construct a modified weight matrix  $W(t) = [W_{\mathcal{L}}(t) \ W_{\mathcal{M}}(t)]$  that governs the state dynamics of legitimate agents [cf. (2)] and utilizes the  $\alpha$  observations. Intuitively, we wish to assign larger weights to transmissions that are most likely to originate from *legitimate* agents, and smaller or zero weights to transmissions that are most likely to originate from *malicious* agents as dictated by the observations  $\alpha_{ij}(t)$  for every neighbor  $j$  of  $i$ . Toward this end, recalling that  $\mathcal{N}_i$  is the complete set of neighbors of legitimate agent  $i$ , we define the following two quantities.

For the function  $f$  that captures the history of observations  $\alpha_{ij}(t)$ , we choose the sum and define  $\beta_{ij}(t)$  as follows:

$$\beta_{ij}(t) = \sum_{k=0}^t (\alpha_{ij}(k) - 1/2) \text{ for } t \geq 0, i \in \mathcal{L}, j \in \mathcal{N}_i. \quad (6)$$

Intuitively, following the discussion after Definition II.2 of  $\alpha_{ij}$ , the  $\beta_{ij}(t)$  will tend toward positive values for legitimate agent transmissions  $i \in \mathcal{L}, j \in \mathcal{N}_i \cap \mathcal{L}$ , and will tend toward negative values for malicious agent transmissions, where  $i \in \mathcal{L}, j \in \mathcal{N}_i \cap \mathcal{M}$ .

We also define a time-dependent *trusted neighborhood* for each agent  $i$  as

$$\mathcal{N}_i(t) = \{j \in \mathcal{N}_i : \beta_{ij}(t) \geq 0\}. \quad (7)$$

This is the set of neighbors that legitimate agent  $i$  classifies as its legitimate neighbors at time  $t$ . Denote

$$n_{w_i}(t) = \max\{\kappa, |\mathcal{N}_i(t)| + 1\} \geq 1 \quad \text{for all } i \in \mathcal{L}$$

where  $\kappa > 0$  can be thought of as a parameter limiting the maximum influence that the neighbors of agent  $i$  are allowed to have on agent  $i$ 's update values (i.e., a value of  $\kappa$  approaching infinity would preclude neighbors from influencing agent  $i$ 's updated values). Using these quantities, we define the weight matrix  $W(t)$  by choosing its entries  $w_{ij}(t)$  as follows: for every

$i \in \mathcal{L}, j \in \mathcal{N}_i$ , we have

$$w_{ij}(t) = \begin{cases} \frac{1}{n_{w_i}(t)}, & \text{if } j \in \mathcal{N}_i(t) \\ 0, & \text{if } j \notin \mathcal{N}_i(t) \cup \{i\} \\ 1 - \sum_{m \in \mathcal{N}_i} w_{im}(t), & \text{if } j = i. \end{cases} \quad (8)$$

Note that the dependence of the weights  $w_{ij}(t)$  on the trust observation history  $\beta_{ij}(t)$  comes in through the choice of time-dependent trusted neighborhood  $\mathcal{N}_i(t)$  [cf. (7)]. Note also that the matrix  $W(t)$  is (rectangular) row-stochastic, by construction. Furthermore, since  $\alpha_{ij}(t)$  are random, so are the quantities  $\beta_{ij}(t)$  and the set of neighbors of agents  $i$ ,  $\mathcal{N}_i(t)$ , that agent  $i$  classifies as legitimate at time  $t$ . Consequently, some entries of the matrix  $W(t)$  are also random, as seen from (8). We use the parameter  $\kappa$  later on, in our analysis of the dynamics in (2), to obtain an upper bound on the entries of the matrix  $W(t)$ .

We define convergence as follows.

**Definition II.3 (Convergence of the consensus protocol):** We define consensus to be achieved if:

1) *Limit values:* There exists, almost surely, a random variable  $y(T_0)$  such that

$$\lim_{t \rightarrow \infty} \tilde{x}_{\mathcal{L}}(T_0, t) = y(T_0) \mathbf{1} \quad (9)$$

where  $\mathbf{1} \in \mathbb{R}^{|\mathcal{L}| \times 1}$ , and  $y(T_0)$  is in the convex hull of the legitimate agent values  $x_i(0), i \in \mathcal{L}$ , and its distribution depends on the starting time  $T_0$  of the consensus algorithm. Furthermore, almost surely, a limit exists for the legitimate agent values, which is a random variable  $z(T_0)$  such that

$$\lim_{t \rightarrow \infty} x_{\mathcal{L}}(T_0, t) = z(T_0) \mathbf{1}. \quad (10)$$

Here,  $z(T_0)$  is in the convex hull of the legitimate and malicious agent values  $x_i(T_0), i = 1, \dots, n$ , and its deviation from the nominal consensus value (the case with no malicious agents) depends on the starting time  $T_0$  of the consensus algorithm. See Definition II.3.2.

2) *Deviation:* There is a probabilistic upper bound on the asymptotic distance between the legitimate agent values and a suitably defined weighted average of their initial values, i.e.,

$$\mathbb{P} \left( \max_{i \in \mathcal{L}} \limsup_{t \rightarrow \infty} |[x_{\mathcal{L}}(T_0, t) - \mathbf{1} v' x_{\mathcal{L}}(0)]_i| \leq \Delta(T_0, \delta) \right) \geq 1 - \delta,$$

$$\text{for some finite } \Delta(T_0, \delta) \leq \Delta_{\max}(T_0, \delta) \quad (11)$$

where  $\delta > 0$  is a user-defined error probability, while  $v$  is the Perron–Frobenius left eigenvector of the matrix  $\overline{W}_{\mathcal{L}}$  defined in (12) such that  $v' \mathbf{1} = 1$ . Fig. 2 depicts an example for the distance between the the true consensus value and the legitimate agents' values under the influence of malicious agents.

We next discuss the assumptions that we use in the following. For this, we let  $\mathbb{G}_{\mathcal{L}} = (\mathcal{L}, \mathbb{E}_{\mathcal{L}})$  be the subgraph of the graph  $\mathbb{G}$  that is induced by the legitimate agents, where  $\mathbb{E}_{\mathcal{L}} \subset \mathbb{E}$  is given by  $\mathbb{E}_{\mathcal{L}} = \{\{i, j\} \mid \{i, j\} \in \mathbb{E}, i \in \mathcal{L}, j \in \mathcal{L}\}$ . We use the following assumptions throughout this article.

**Assumption I:** Suppose that the following hold.

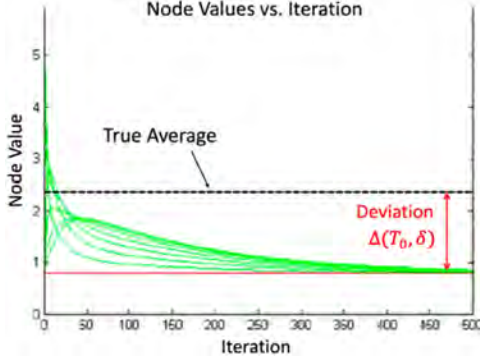


Fig. 2. Schematic showing the *deviation* from the nominal average in the case of a malicious agent attack and no resilience in the consensus protocol.

1) (*Sufficiently connected graph*) The subgraph  $\mathbb{G}_{\mathcal{L}}$  induced by the legitimate agents is connected.

2) (*Homogeneity of trust variables*) The expectation of the variables  $\alpha_{ij}$  are constant for the case of malicious transmissions and legitimate transmissions, respectively, i.e., for some scalars  $c, d$  with  $c \neq d$ , we have

$$c = E(\alpha_{ij}(t)) - 1/2 \quad \text{for all } i \in \mathcal{L}, j \in \mathcal{N}_i \cap \mathcal{M}$$

$$d = E(\alpha_{ij}(t)) - 1/2 \quad \text{for all } i \in \mathcal{L}, j \in \mathcal{N}_i \cap \mathcal{L}.$$

3) (*Independence of trust observations*) The observations  $\alpha_{ij}(t)$  are independent for all  $t$  and all pairs of agents  $i$  and  $j$ , with  $i \in \mathcal{L}$ ,  $j \in \mathcal{N}_i$ . Moreover, for any  $i \in \mathcal{L}$  and  $j \in \mathcal{N}_i$ , the observation sequence  $\{\alpha_{ij}(t)\}$  is identically distributed.

Assumption 1.1 captures the general connectivity requirement in consensus networks with a fixed topology. The assumption on the homogeneity of trust variables (cf. Assumption 1.2) is made only for the ease of exposition. The assumption can be generalized to the heterogeneous case, as discussed in Section V.

### B. Problem Definition

In the following, we summarize the three main problems that we aim to address under the assumptions in Assumption 1.

**Problem 1 (Convergence of the consensus protocol):** We aim to show that in the presence of an adversarial attack, consensus to a common limit value is attainable *even when malicious agents comprise  $> 1/2$  of the network connectivity* if observations  $\alpha_{ij}(t)$  are available. We seek conditions on  $\alpha_{ij}(t)$  and a definition of the weights  $W(t)$  that allow convergence.

**Problem 2 (Bounded deviation for average consensus):** For the case of average consensus, we aim to find a bound on deviation from the average consensus value,  $\Delta(T_0, \delta)$  that can be achieved with a probability at least  $1 - \delta$ , where the bound is given as a function of problem parameters (such as the number of legitimate and the number of malicious agents) and the characteristics of  $\alpha_{ij}$  (such as bounds on its expected value  $E(\alpha_{ij})$ ).

**Problem 3 (Characterization of convergence rate  $\tau$ ):** We aim to determine a finite convergence time for the correct

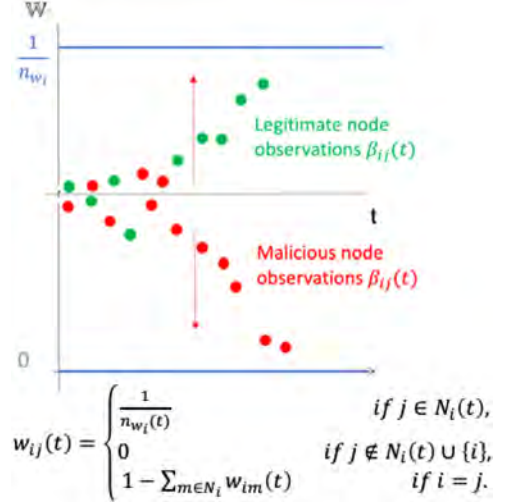


Fig. 3. Depiction of the  $\beta_{ij}(t)$  values for legitimate and malicious nodes becoming more separated and accurately classifiable with higher probability as a longer history of  $\alpha_{ij}(t)$  becomes available (i.e., as  $t$  gets larger).

classification of legitimate and malicious agents almost surely. Additionally, we aim to identify the rate of convergence as a function of the stochastic observations  $\alpha_{ij}$  and other problem parameters that we characterize.

## III. ANALYSIS

### A. Convergence of Consensus of Legitimate Agent Inputs

In this section, we analyze the state dynamics among the legitimate agents, which is governed by the weight matrix  $W_{\mathcal{L}}(k)$ , as seen from (2). Recalling that  $W(t) = [W_{\mathcal{L}}(t) \ W_{\mathcal{M}}(t)]$ , from the definition of the matrix  $W(t)$  [cf. (8)], we see that the matrix  $W_{\mathcal{L}}(t)$  is a substochastic matrix, since the sum of some rows may be strictly less than 1. This section analyzes the convergence of the (backward) product  $\prod_{k=0}^t W_{\mathcal{L}}(k)$ , as  $t \rightarrow \infty$ . We prove the strong ergodicity of this product and show that the limit matrix  $\prod_{k=0}^{\infty} W_{\mathcal{L}}(k)$  has strictly positive entries, i.e.,  $\prod_{k=0}^{\infty} W_{\mathcal{L}}(k) > \mathbf{0}$ .

Our goal here is to show convergence of the consensus system in (2) using the choice of weights from (8) that exploit the observations  $\alpha_{ij}(t)$ . Specifically, we aim to characterize the conditions on  $\alpha_{ij}(t)$  necessary to achieve consensus in the presence of malicious agents *even when the number of malicious agents is arbitrarily high*.

We will achieve this in three parts: first, we show that for a sufficiently connected network (cf. Assumption 1.1), the weight matrix in (8) reaches its limit described in (12) in finite time  $T_f$  almost surely, and that this limit is a primitive matrix (cf. Lemma 1 and Proposition 1). We depict this limiting behavior in Fig. 3. Second, we show that this, in turn, implies that the legitimate agents' influence captured in (4) approaches a finite limit almost surely (cf. Proposition 2). By the same token, the malicious agents' influence captured in (5) also approaches a finite limit almost surely (cf. Proposition 3). Finally, putting these



limits together, we show that the values of the legitimate agents,  $x_{\mathcal{L}}(T_0, t)$ , approach a finite limit, i.e., we achieve convergence.

In what follows, we use the notion of a primitive matrix, which is provided next.

**Definition III.1 (Primitive matrix):** A nonnegative square matrix  $A$  is said to be a primitive matrix if there exists an integer  $k \geq 1$  such that  $A^k > \mathbf{0}$ .

Let us define a matrix  $\bar{W}_{\mathcal{L}}$  with the entries given as: for every  $i, j \in \mathcal{L}$ , we have

$$[\bar{W}_{\mathcal{L}}]_{ij} = \begin{cases} \frac{1}{\max\{|\mathcal{N}_i \cap \mathcal{L}| + 1, \kappa\}}, & \text{if } j \in \mathcal{N}_i \cap \mathcal{L} \\ 1 - \frac{|\mathcal{N}_i \cap \mathcal{L}|}{\max\{|\mathcal{N}_i \cap \mathcal{L}| + 1, \kappa\}}, & \text{if } j = i \\ 0, & \text{otherwise.} \end{cases} \quad (12)$$

For the matrix  $\bar{W}_{\mathcal{L}}$ , we have the following result.

**Lemma 1:** The matrix  $\bar{W}_{\mathcal{L}}$  is primitive.

*Proof:* The result follows directly by the definition of the matrix  $\bar{W}_{\mathcal{L}}$  in (12) and the assumption that the graph  $\mathbb{G}_{\mathcal{L}}$  is connected (cf. Assumption 1.1).

Under Assumptions 1.2 and 1.3, we next provide a result for the random quantities  $\beta_{ij}(t)$  defining the weight matrix  $W(t)$ .

**Lemma 2:** Consider the random variables  $\beta_{ij}(t)$  as defined in (6). Then, for every  $t \geq 0$  and every  $i \in \mathcal{L}$ ,  $j \in \mathcal{N}_i \cap \mathcal{L}$ , we have

$$\mathbb{P}(\beta_{ij}(t) < 0) \leq \max\{\exp(-2(t+1)d^2), \mathbb{1}_{\{d < 0\}}\}.$$

Additionally, for every  $t \geq 0$  and every  $i \in \mathcal{L}$ ,  $j \in \mathcal{N}_i \cap \mathcal{M}$ , we have

$$\mathbb{P}(\beta_{ij}(t) \geq 0) \leq \max\{\exp(-2(t+1)c^2), \mathbb{1}_{\{c > 0\}}\}.$$

*Proof:* By the linearity of the expectation, we have  $E(\beta_{ij}(t)) = \sum_{k=0}^t (E(\alpha_{ij}(k)) - 1/2)$ . For every  $k \geq 0$ , and for all  $i, j$  with  $i \in \mathcal{L}$ ,  $j \in \mathcal{N}_i \cap \mathcal{L}$ , by Assumptions 1.2 and 1.3, we have  $d = E(\alpha_{ij}(k)) - 1/2$ . Additionally, for every  $k \geq 0$  and  $i \in \mathcal{L}$ ,  $j \in \mathcal{N}_i \cap \mathcal{M}$ , by Assumptions 1.2 and 1.3, we have  $c = E(\alpha_{ij}(k)) - 1/2$ . The result then follows directly from the Chernoff–Hoeffding inequality (see [57, Th. 1.1]).

From Lemma 2, one can observe the following.

**Corollary 1 (Bounds on expectation of  $\alpha_{ij}(t)$ ):** For the choice  $\beta_{ij}(t) = \sum_{k=0}^t [\alpha_{ij}(k) - 1/2]$ , we must have that  $c = E(\alpha_{ij}(k)) - 1/2 < 0$  for all  $i \in \mathcal{L}$ ,  $j \in \mathcal{M}$ ,  $k \geq 0$  and  $d = E(\alpha_{ij}(k)) - 1/2 > 0$ , for all  $i, j \in \mathcal{L}$ ,  $k \geq 0$  in order to have a decaying misclassification probabilities.

We provide some intuition about this corollary as it has significant implications for the characteristics of  $\alpha_{ij}(t)$  necessary for obtaining several important results. The condition that  $c$  and  $d$  be bounded away from zero in Corollary 1 intuitively means that there is information captured by the  $\alpha_{ij}(t)$  observations. The case of  $c = 0$  and  $d = 0$  is equivalent to saying that, in expectation, the observations  $\alpha_{ij}(t)$  will be the same regardless of the transmission being legitimate or malicious (the observations  $\alpha_{ij}(t)$  contain no useful information). *Hereafter, we assume that  $c < 0$  and  $d > 0$* , which means that  $\alpha_{ij}(t)$  tends to be less than  $1/2$  for malicious transmissions (has expectation value less than  $1/2$ ) and  $\alpha_{ij}(t)$  tends to be greater than  $1/2$  for legitimate transmissions (has expectation value greater than  $1/2$ ).

**Proposition 1:** There exists a finite time instant  $T_f > 0$  such that  $W_{\mathcal{L}}(k) = \bar{W}_{\mathcal{L}}$  for all  $k \geq T_f$  almost surely. Moreover, there exists a stochastic vector  $v > \mathbf{0}$  such that

$$\lim_{k \rightarrow \infty} \bar{W}_{\mathcal{L}}^k = \mathbf{1}v'.$$

In particular,  $\prod_{k=T_0-1}^{\infty} W_{\mathcal{L}}(k) > \mathbf{0}$  for every  $T_0 \geq 0$  almost surely.

*Proof:* By Lemma 2, for every  $k \geq 0$  and  $i \in \mathcal{L}$ ,  $j \in \mathcal{N}_i \cap \mathcal{L}$ , we have that

$$\mathbb{P}(\beta_{ij}(k) < 0) \leq \exp(-2(k+1)d^2). \quad (13)$$

Additionally, for every  $k \geq 0$  and  $i \in \mathcal{L}$ ,  $j \in \mathcal{N}_i \cap \mathcal{M}$ , we have

$$\mathbb{P}(\beta_{ij}(k) \geq 0) \leq \exp(-2(k+1)c^2). \quad (14)$$

Note that we have  $\sum_{k=0}^{\infty} \exp(-2(k+1)c^2) < \infty$  and  $\sum_{k=0}^{\infty} \exp(-2(k+1)d^2) < \infty$ . Thus, by the Borel–Cantelli lemma, the events  $\{\beta_{ij}(k) \geq 0 \ \forall i \in \mathcal{L}, j \in \mathcal{N}_i \cap \mathcal{M}\}$  and  $\{\beta_{ij}(k) < 0 \ \forall i \in \mathcal{L}, j \in \mathcal{N}_i \cap \mathcal{L}\}$  occur only finitely often almost surely. Therefore, there exists a (random) finite time  $T_f > 0$  such that  $W_{\mathcal{L}}(k) = \bar{W}_{\mathcal{L}}$  for all  $k \geq T_f$  almost surely.

Now, almost surely, we have

$$\begin{aligned} \prod_{k=T_0-1}^{\infty} W_{\mathcal{L}}(k) &= \prod_{k=\max\{T_f, T_0\}}^{\infty} W_{\mathcal{L}}(k) \prod_{k=T_0-1}^{\max\{T_f, T_0\}-1} W_{\mathcal{L}}(k) \\ &= \prod_{k=\max\{T_f, T_0\}}^{\infty} \bar{W}_{\mathcal{L}} \prod_{k=T_0-1}^{\max\{T_f, T_0\}-1} W_{\mathcal{L}}(k) \\ &= \lim_{k \rightarrow \infty} \bar{W}_{\mathcal{L}}^{k-\max\{T_f, T_0\}} \prod_{k=T_0-1}^{\max\{T_f, T_0\}-1} W_{\mathcal{L}}(k). \end{aligned}$$

By Lemma 1, the matrix  $\bar{W}_{\mathcal{L}}$  is primitive. Therefore, by the Perron–Frobenius theorem (see [58]), there exists a stochastic column vector  $v > \mathbf{0}$  such that  $\lim_{k \rightarrow \infty} \bar{W}_{\mathcal{L}}^k = \mathbf{1}v'$ .

Consequently, almost surely, we have

$$\prod_{k=T_0-1}^{\infty} W_{\mathcal{L}}(k) = \mathbf{1}v' \left( \prod_{k=T_0-1}^{\max\{T_f, T_0\}-1} W_{\mathcal{L}}(k) \right).$$

Since  $v > \mathbf{0}$  and the diagonal entries of the matrices  $W_{\mathcal{L}}(k)$  are positive (by construction), it follows that  $v'(\prod_{k=T_0-1}^{\max\{T_f, T_0\}-1} W_{\mathcal{L}}(k)) > \mathbf{0}$ , implying that almost surely  $\prod_{k=T_0-1}^{\infty} W_{\mathcal{L}}(k) > \mathbf{0}$ .

Our next results are aimed at proving that the legitimate agents' values reach consensus almost surely. In particular, using the decomposition  $x_{\mathcal{L}}(T_0, t) = \tilde{x}_{\mathcal{L}}(T_0, t) + \phi_{\mathcal{M}}(T_0, t)$  of the legitimate agent states [see (3)], we focus on the limiting behavior of  $\tilde{x}_{\mathcal{L}}(T_0, t)$ , as defined in (4). We show that  $\tilde{x}_{\mathcal{L}}(T_0, t)$  converges almost surely to a consensus value, as  $t \rightarrow \infty$ , as seen in the following proposition.

**Proposition 2 (Convergence of legitimate agents' values):** Consider the dynamics in (2). Let  $x_{\mathcal{L}}(0)$  be the vector of initial

values of legitimate agents, and define

$$\tilde{x}_{\mathcal{L}}^{\infty}(T_0) = \left( \prod_{k=T_0-1}^{\infty} W_{\mathcal{L}}(k) \right) x_{\mathcal{L}}(0). \quad (15)$$

Then, we have  $[\tilde{x}_{\mathcal{L}}^{\infty}(T_0)]_i = [\tilde{x}_{\mathcal{L}}^{\infty}(T_0)]_j$  for every  $i, j \in \mathcal{L}$ , almost surely.

*Proof:* By Proposition 1, there exists a finite time instant  $T_f$  such that  $W_{\mathcal{L}}(k) = \bar{W}_{\mathcal{L}}$  for all  $k \geq T_f$  almost surely, and  $\lim_{k \rightarrow \infty} \bar{W}_{\mathcal{L}}^k = \mathbf{1}v'$  for a stochastic vector  $v > \mathbf{0}$ . Hence, it follows that almost surely

$$\begin{aligned} \tilde{x}_{\mathcal{L}}^{\infty}(T_0) &= \left( \prod_{k=T_0-1}^{\infty} W_{\mathcal{L}}(k) \right) x_{\mathcal{L}}(0) \\ &= \left( \prod_{k=\max\{T_f, T_0\}}^{\infty} W_{\mathcal{L}}(k) \prod_{k=T_0-1}^{\max\{T_f, T_0\}-1} W_{\mathcal{L}}(k) \right) x_{\mathcal{L}}(0) \\ &= \left( \prod_{k=\max\{T_f, T_0\}}^{\infty} \bar{W}_{\mathcal{L}} \prod_{k=T_0-1}^{\max\{T_f, T_0\}-1} W_{\mathcal{L}}(k) \right) x_{\mathcal{L}}(0) \\ &= \mathbf{1}v' \left( \prod_{k=T_0-1}^{\max\{T_f, T_0\}-1} W_{\mathcal{L}}(k) \right) x_{\mathcal{L}}(0). \end{aligned} \quad (16)$$

By letting  $\tilde{x}_{\mathcal{L}}^{T_f}(T_0) = \left( \prod_{k=T_0-1}^{\max\{T_f, T_0\}-1} W_{\mathcal{L}}(k) \right) x_{\mathcal{L}}(0)$ , we obtain that  $\tilde{x}_{\mathcal{L}}^{\infty}(T_0) = \mathbf{1}v' \tilde{x}_{\mathcal{L}}^{T_f}(T_0)$ , almost surely.

We next consider the limiting effect that malicious agents' states have on the legitimate agents' states. Specifically, in the state decomposition relation  $x_{\mathcal{L}}(T_0, t) = \tilde{x}_{\mathcal{L}}(T_0, t) + \phi_{\mathcal{M}}(T_0, t)$  for the legitimate agents' states [see (3)], we focus on the limiting behavior of  $\phi_{\mathcal{M}}(T_0, t)$  defined in (5), as  $t \rightarrow \infty$ . We show that  $\phi_{\mathcal{M}}(T_0, t)$  converges to a consensus almost surely given our choice of weight matrices, as seen in the following proposition.

*Proposition 3:* Consider the vector  $\phi_{\mathcal{M}}^{\infty}(T_0)$  defined by

$$\phi_{\mathcal{M}}^{\infty}(T_0) = \sum_{k=T_0-1}^{\infty} \left( \prod_{l=k+1}^{\infty} W_{\mathcal{L}}(l) \right) W_{\mathcal{M}}(k) x_{\mathcal{M}}(k). \quad (17)$$

Then, the entries of this vector are all the same almost surely, i.e.,  $[\phi_{\mathcal{M}}^{\infty}(T_0)]_i = [\phi_{\mathcal{M}}^{\infty}(T_0)]_j$  almost surely for every  $i, j \in \mathcal{L}$ .

*Proof:* By Proposition 1, there exists a finite time instant  $T_f > 0$  such that almost surely

$$W_{\mathcal{L}}(k) = \bar{W}_{\mathcal{L}} \quad \text{for all } k \geq T_f. \quad (18)$$

This implies that  $W_{\mathcal{M}}(k) = \mathbf{0}$  for all  $k \geq T_f$  almost surely. Therefore, we have

$$\begin{aligned} \phi_{\mathcal{M}}^{\infty}(T_0) &= \sum_{k=T_0-1}^{\infty} \left( \prod_{l=k+1}^{\infty} W_{\mathcal{L}}(l) \right) W_{\mathcal{M}}(k) x_{\mathcal{M}}(k) \\ &= \sum_{k=T_0-1}^{T_f-1} \left( \prod_{l=k+1}^{\infty} W_{\mathcal{L}}(l) \right) W_{\mathcal{M}}(k) x_{\mathcal{M}}(k). \end{aligned} \quad (19)$$

Now, for the product  $\prod_{l=k+1}^{\infty} W_{\mathcal{L}}(l)$ , we have that almost surely

$$\begin{aligned} \prod_{l=k+1}^{\infty} W_{\mathcal{L}}(l) &= \prod_{l=\max\{T_f, k+1\}}^{\infty} W_{\mathcal{L}}(l) \prod_{l=k+1}^{\max\{T_f, k+1\}-1} W_{\mathcal{L}}(l) \\ &= \lim_{t \rightarrow \infty} \bar{W}_{\mathcal{L}}^{t-\max\{T_f, k+1\}} \prod_{l=k+1}^{\max\{T_f, k+1\}-1} W_{\mathcal{L}}(l) \end{aligned}$$

where the second relation follows from (18).

By Proposition 1, we also have for a stochastic vector  $v > \mathbf{0}$ , such that  $\lim_{k \rightarrow \infty} \bar{W}_{\mathcal{L}}^k = \mathbf{1}v'$ . Hence, almost surely, we have

$$\begin{aligned} \phi_{\mathcal{M}}^{\infty}(T_0) &= \sum_{k=T_0-1}^{T_f-1} \left( \prod_{l=k+1}^{\infty} W_{\mathcal{L}}(l) \right) W_{\mathcal{M}}(k) x_{\mathcal{M}}(k) \\ &= \mathbf{1}v' \left[ \sum_{k=T_0-1}^{T_f-1} \left( \prod_{l=k+1}^{\max\{T_f, k+1\}-1} W_{\mathcal{L}}(l) \right) W_{\mathcal{M}}(k) x_{\mathcal{M}}(k) \right] \end{aligned}$$

for some finite  $T_f$ , implying that all the coordinates of the vector  $\phi_{\mathcal{M}}^{\infty}(T_0)$  are identical almost surely.

As a direct consequence of Propositions 2 and 3, we have the following result.

*Corollary 2:* There exists, almost surely, a random variable  $y(T_0)$  such that

$$\lim_{t \rightarrow \infty} \tilde{x}_{\mathcal{L}}(T_0, t) = y(T_0) \mathbf{1} \quad (20)$$

where  $y(T_0)$  is in the convex hull of the legitimate agents' initial values  $x_i(T_0)$ ,  $i \in \mathcal{L}$ , and its distribution depends on the starting time  $T_0$  of the consensus algorithm in (2). Furthermore, almost surely, there exists a random variable  $z(T_0)$  such that

$$\lim_{t \rightarrow \infty} x_{\mathcal{L}}(T_0, t) = z(T_0) \mathbf{1} \quad (21)$$

where  $z(T_0)$  is in the convex hull of the initial values  $x_i(T_0)$ ,  $i = 1, \dots, n$ , of the legitimate and malicious agents, and its distribution depends on the starting time  $T_0$  of the consensus algorithm.

*Proof:* The results follow directly from Propositions 2 and 3, and the consensus algorithm in (2).

We note that, in the view of the fact that the legitimate agents start consensus protocol at time  $T_0$ , we have that the random variable  $y(T_0)$  is in the convex hull of the initial values  $x_i(0)$ ,  $i \in \mathcal{L}$ , of legitimate agents. The random variable  $z(T_0)$  is in the convex hull of  $x_i(0)$ ,  $i \in \mathcal{L}$ , and the values  $x_m(T_0)$ ,  $m \in \mathcal{M}$ , since the malicious agents may have deviated from their initial values at time  $t = 0$ .

We are now ready to state our main convergence result.

*Theorem 1 (Convergence of resilient consensus):* Consider the consensus system described by the dynamics in (2) with an arbitrary start time of  $T_0 \geq 0$  and weights, as described by (8). Given that 1)  $\mathbb{G}$  is a sufficiently connected network, and that 2) bounds on the expected values of the observations  $\alpha_{ij}(k)$  (see Definition II.2) satisfy  $c = E(\alpha_{ij}(k)) - 1/2 < 0$  for all  $i \in \mathcal{L}$ ,  $j \in \mathcal{N}_i \cap \mathcal{M}$ ,  $k \geq 0$  and  $d = E(\alpha_{ij}(k)) - 1/2 > 0$  for all  $i \in \mathcal{L}$ ,  $j \in \mathcal{N}_i \cap \mathcal{L}$ ,  $k \geq 0$ , the consensus algorithm converges almost surely, i.e.,

$$\lim_{t \rightarrow \infty} x_{\mathcal{L}}(T_0, t) = z(T_0) \mathbf{1}$$



almost surely, independently of the number of malicious agents in the network. The value  $z(T_0)$  is in the convex hull of the initial values  $x_i(T_0)$ ,  $i = 1, \dots, n$ , of both the legitimate and malicious agents.

*Proof:* The proof follows from Propositions 2 and 3 and Corollary 2. Specifically, we have that the legitimate agent values satisfy  $x_{\mathcal{L}}(T_0, t) = \tilde{x}_{\mathcal{L}}(T_0, t) + \phi_{\mathcal{M}}(T_0, t)$ , where the first term  $\tilde{x}_{\mathcal{L}}(T_0, t)$  converges by Proposition 2 and the second term  $\phi_{\mathcal{M}}(T_0, t)$  converges by Proposition 3.

Note that this result asserts convergence in the presence of an *arbitrary number of malicious nodes* and characterizes *necessary conditions on the observations*  $\alpha_{ij}(t)$  to attain this convergence. However, the distance of the limit value  $z(T_0)$  to the nominal average consensus value in the ideal case (when the malicious agents are known at the start time) has not been discussed. The characterization of this distance is the subject of the following section.

### B. Deviation From Nominal Consensus Value

In this section, we characterize the deviation from the nominal consensus value under our consensus model in the presence of a malicious attack. The nominal consensus value is the consensus value over the graph  $\mathbb{G}_{\mathcal{L}}$  of legitimate agents in the case of no malicious agents in the network. More specifically, by Proposition 1, we know that there exists a finite time instant  $T_f > 0$  such that  $W_{\mathcal{L}}(k) = \overline{W}_{\mathcal{L}}$  for all  $k \geq T_f$  almost surely, where  $\overline{W}_{\mathcal{L}}$  is given in (12). The nominal consensus value corresponds to consensus that would have been reached among the legitimate agents if the dynamics in (2) were governed by  $[\overline{W}_{\mathcal{L}} \mathbf{0}]$  instead of  $[W_{\mathcal{L}}(k) W_{\mathcal{M}}(k)]$ . Hence, in view of Proposition 1, *the nominal consensus value over the graph  $\mathbb{G}_{\mathcal{L}}$  of legitimate agents is  $1v'x_{\mathcal{L}}(0)$ .*

In what follows, we investigate the deviation of the consensus value  $\lim_{t \rightarrow \infty} x_{\mathcal{L}}(T_0, t)$  reached by the legitimate agents (as predicted by Theorem 1) from the nominal value  $1v'x_{\mathcal{L}}(0)$ . The user-defined start time  $T_0$ , for running the consensus dynamics in (2), plays a major role in determining the amount of the resulting deviation due to the malicious agents' inputs to the system. Intuitively, the longer  $\alpha_{ij}(t)$  values are observed for all pairs  $\{i, j\} \in \mathbb{E}$  before starting the consensus; the probability of making a misclassification error (classifying a legitimate agent as malicious and *vice versa*) decays exponentially.

Our approach in this section is to show that for larger values of  $T_0$ , the probability of our weight matrix not having reached its limit value decays exponentially (see Lemma 3). The implication of this manifests itself as a smaller overall deviation. Specifically, deviation of the convergence limit (cf. Theorem 1) from the nominal average consensus value can be bounded by the deviation introduced by legitimate agents being misclassified as malicious and by malicious agents being misclassified as legitimate. Propositions 4 and 5 proven in this section provide definitive bounds for these deviations with probability  $p(T_0)$ , which is characterized as a function of  $T_0$ .

We begin by showing that the probability of the weights from (8) not taking their ideal form  $\overline{W}_{\mathcal{L}}$  decays exponentially with  $T_0$ .

*Lemma 3:* For every  $T_0 \geq 1$ , we have

$$\begin{aligned} & \mathbb{P}(\exists k \geq T_0 - 1 : W_{\mathcal{L}}(k) \neq \overline{W}_{\mathcal{L}}) \\ & \leq |\mathcal{L}|^2 \cdot \frac{\exp(-2T_0 d^2)}{1 - \exp(-2d^2)} + |\mathcal{L}||\mathcal{M}| \cdot \frac{\exp(-2T_0 c^2)}{1 - \exp(-2c^2)}. \end{aligned}$$

*Proof:* For every  $T_0 \geq 1$ , we have

$$\begin{aligned} & \mathbb{P}(\exists k \geq T_0 - 1 : W_{\mathcal{L}}(k) \neq \overline{W}_{\mathcal{L}}) \\ & = \Pr\left(\bigcup_{k \geq T_0 - 1} \{W_{\mathcal{L}}(k) \neq \overline{W}_{\mathcal{L}}\}\right) \\ & \leq \sum_{k=T_0-1}^{\infty} \mathbb{P}(W_{\mathcal{L}}(k) \neq \overline{W}_{\mathcal{L}}) \end{aligned} \quad (22)$$

where the inequality follows from applying the union bound to the event  $\bigcup_{k \geq T_0 - 1} \{W_{\mathcal{L}}(k) \neq \overline{W}_{\mathcal{L}}\}$ . Furthermore, for every  $k$ , we have that

$$\{W_{\mathcal{L}}(k) \neq \overline{W}_{\mathcal{L}}\} = \bigcup_{\substack{i \in \mathcal{L}, \\ j \in \mathcal{N}_i \cap \mathcal{L}}} \{\beta_{ij}(k) < 0\} \bigcup_{\substack{i \in \mathcal{L}, \\ j \in \mathcal{N}_i \cap \mathcal{M}}} \{\beta_{ij}(k) \geq 0\}. \quad (23)$$

Therefore, by the union bound

$$\begin{aligned} & \mathbb{P}(W_{\mathcal{L}}(k) \neq \overline{W}_{\mathcal{L}}) \\ & \leq \sum_{i \in \mathcal{L}, j \in \mathcal{N}_i \cap \mathcal{L}} \mathbb{P}(\beta_{ij}(k) < 0) + \sum_{i \in \mathcal{L}, j \in \mathcal{N}_i \cap \mathcal{M}} \mathbb{P}(\beta_{ij}(k) \geq 0) \\ & \leq |\mathcal{L}|^2 \cdot \exp(-2(k+1)d^2) + |\mathcal{L}||\mathcal{M}| \cdot \exp(-2(k+1)c^2) \end{aligned}$$

where the last inequality follows from Lemma 2. The result then follows by combining relations (22) and (23).

Next, we consider the deviation contributed by misclassifying legitimate agents as malicious at some times and, thus, discarding their values. In the following, we use the bound of Lemma 3 and the bound we present next in Lemma 4 to ultimately bound the deviation, in Proposition 4, which is caused by its incorrect classification of the legitimate agents.

*Lemma 4:* Let  $\tilde{X}, X \in \mathbb{R}_+^{m \times m}$  be two row-substochastic matrices, and let  $\gamma > 0$  be such that  $\tilde{X}_{ii} \geq \gamma > 0$  and  $X_{ii} \geq \gamma$  for all  $i$ . Then,  $\|\tilde{X} - X\|_1 \leq 2(1 - \gamma)$ , for all  $i$ , where  $|X|$  denote the matrix with entries  $|x_{ij}|$ .

*Proof:* Let coordinate index  $i$  be arbitrary,  $1 \leq i \leq m$ . First, we prove that  $|\tilde{x}_{ii} - x_{ii}| + 2\gamma \leq \tilde{x}_{ii} + x_{ii}$ . By the triangle inequality, we obtain

$$\begin{aligned} |\tilde{x}_{ii} - x_{ii}| + 2\gamma & = |(\tilde{x}_{ii} - \gamma) - (x_{ii} - \gamma)| + 2\gamma \\ & = |\tilde{x}_{ii} - \gamma| + |x_{ii} - \gamma| + 2\gamma. \end{aligned}$$

Since  $\tilde{x}_{ii} - \gamma \geq 0$  and  $x_{ii} - \gamma \geq 0$ , we have that

$$|\tilde{x}_{ii} - \gamma| + |x_{ii} - \gamma| + 2\gamma = \tilde{x}_{ii} + x_{ii}.$$

Therefore, it follows that

$$\left[\|\tilde{X} - X\|_1\right]_i + 2\gamma = \sum_{j=1}^m |\tilde{x}_{ij} - x_{ij}| + 2\gamma$$

$$\begin{aligned}
&= \tilde{x}_{ii} + x_{ii} + \sum_{j=1, j \neq i}^m |\tilde{x}_{ij} - x_{ij}| \\
&\leq \tilde{x}_{ii} + x_{ii} + \sum_{j=1, j \neq i}^m (\tilde{x}_{ij} + x_{ij})
\end{aligned}$$

where the last inequality is obtained by using the triangle inequality and the fact that the matrices  $X$  and  $\tilde{X}$  have nonnegative entries. Since both  $\tilde{X}$  and  $X$  are stochastic matrices, we further have that  $\sum_{j=1}^m [\tilde{x}_{ij} + x_{ij}] \leq 2$ . From the preceding two relations, it follows that  $[\tilde{X} - X\mathbf{1}]_i \leq 2(1 - \gamma)$ .

Denote by  $\varphi_i(T_0, t)$  the deviation suffered by legitimate agent  $i$  that is caused by its incorrect classification of the legitimate agents, i.e., for all  $i \in \mathcal{L}$ , we have

$$\varphi_i(T_0, t) = \left| \left[ \tilde{x}_{\mathcal{L}}(T_0, t) - \left( \prod_{k=T_0}^{t-1} \overline{W}_{\mathcal{L}} \right) x_{\mathcal{L}}(0) \right]_i \right|. \quad (24)$$

Then, we have the following probabilistic bound on the deviation.

**Proposition 4:** Given an error tolerance  $\delta > 0$ , for the deviations  $\varphi_i(T_0, t)$ ,  $i \in \mathcal{L}$ , as defined in (24), we have

$$\mathbb{P} \left( \max_{i \in \mathcal{L}} \limsup_{t \rightarrow \infty} \varphi_i(T_0, t) > \tilde{g}_{\mathcal{L}}(T_0, \delta) \right) < \delta$$

where

$$\begin{aligned}
\tilde{g}_{\mathcal{L}}(T_0, \delta) &= \frac{2\eta|\mathcal{L}|^2}{\delta} \cdot \frac{\exp(-2T_0d^2)}{1 - \exp(-2d^2)} \\
&\quad + \frac{2\eta|\mathcal{L}||\mathcal{M}|}{\delta} \cdot \frac{\exp(-2T_0c^2)}{1 - \exp(-2c^2)} \quad (25)
\end{aligned}$$

and  $\eta \geq \sup_{i \in \mathcal{L} \cup \mathcal{M}, t \in \mathbb{N}} |x_i(t)|$  is a finite upper bound on the absolute input value.

*Proof:* Let  $T_f(T_0, t)$  be a random variable equal to 0 if  $W_{\mathcal{L}}(k) = \overline{W}_{\mathcal{L}}$  for all  $k \in [T_0 - 1, t - 1]$ , and equal to  $\sup\{l + 1 : W_{\mathcal{L}}(l + T_0 - 1) = \overline{W}_{\mathcal{L}}, l \in [0, t - T_0]\}$ , otherwise. In view of the evolution of  $\tilde{x}_{\mathcal{L}}(T_0, t)$  as given in (4), we have

$$\varphi_i(T_0, t) = \left| \left[ \left( \prod_{k=T_0-1}^{t-1} W_{\mathcal{L}}(k) \right) - \overline{W}_{\mathcal{L}}^{t-T_0} \right] x_{\mathcal{L}}(0) \right]_i \right|.$$

Define

$$\Delta(W_{\mathcal{L}}, T_f) = \left( \prod_{k=T_0-1}^{T_0+T_f(T_0, t)-2} W_{\mathcal{L}}(k) \right) - \left( \prod_{k=T_0-1}^{T_0+T_f(T_0, t)-2} \overline{W}_{\mathcal{L}} \right).$$

Then, we have

$$\begin{aligned}
\varphi_i(T_0, t) &= \left| \left[ \left( \prod_{k=T_0+T_f(T_0, t)-1}^{t-1} \overline{W}_{\mathcal{L}} \right) \Delta(W_{\mathcal{L}}, T_f) x_{\mathcal{L}}(0) \right]_i \right| \\
&\stackrel{(a)}{\leq} \max_{i \in \mathcal{L}} |\Delta(W_{\mathcal{L}}, T_f) x_{\mathcal{L}}(0)|_i, \quad (26)
\end{aligned}$$

where (a) follows since  $\overline{W}_{\mathcal{L}}$  is a row-stochastic matrix. Furthermore, since  $|x_{\mathcal{L}}(0)| \leq \eta$ , it follows that

$$\max_{i \in \mathcal{L}} |\Delta(W_{\mathcal{L}}, T_f) x_{\mathcal{L}}(0)|_i \leq \eta \max_{i \in \mathcal{L}} |\Delta(W_{\mathcal{L}}, T_f)| \mathbf{1}_i$$

where for a matrix  $A = [a_{ij}]$ , we use  $|A|$  to denote the matrix with entries  $|a_{ij}|$ . Therefore, we have

$$\varphi_i(T_0, t) \leq \eta \max_{i \in \mathcal{L}} |\Delta(W_{\mathcal{L}}, T_f)| \mathbf{1}_i.$$

Let  $n_w = \max\{|\mathcal{L}| + |\mathcal{M}|, \kappa\}$  and note that  $\overline{W}_{\mathcal{L}}$  is a stochastic matrix with  $[\overline{W}_{\mathcal{L}}]_{ii} \geq \frac{1}{n_w}$  [see (12)]. Additionally, for every  $k$ , the matrix  $W_{\mathcal{L}}(k)$  is substochastic with  $[W_{\mathcal{L}}(k)]_{ii} \geq \frac{1}{n_w}$  for every  $i \in \mathcal{L}$ . Therefore, by Lemma 4, we have

$$|\Delta(W_{\mathcal{L}}, T_f)| \mathbf{1}_i \leq 2 \left[ 1 - \left( \frac{1}{n_w} \right)^{T_f(T_0, t)} \right].$$

Define

$$\overline{\varphi}(T_0, t) = \left[ 1 - \left( \frac{1}{n_w} \right)^{T_f(T_0, t)} \right] 2\eta.$$

Then, we have  $\varphi_i(T_0, t) \leq \overline{\varphi}(T_0, t)$  for all  $i \in \mathcal{L}$  and  $t \geq T_0$ , thus implying that

$$\max_{i \in \mathcal{L}} \limsup_{t \rightarrow \infty} \varphi_i(T_0, t) \leq \lim_{t \rightarrow \infty} \overline{\varphi}(T_0, t).$$

By Proposition 1, for  $t \geq T_f$ , we have  $T_f(T_0, t) = t$  almost surely, implying that  $\overline{\varphi}(T_0, t)$  almost surely converges as  $t \rightarrow \infty$ . By Markov's inequality, it follows that

$$\mathbb{P} \left( \lim_{t \rightarrow \infty} \overline{\varphi}(T_0, t) > \tilde{g}_{\mathcal{L}}(T_0, \delta) \right) \leq \frac{E(\lim_{t \rightarrow \infty} \overline{\varphi}(T_0, t))}{\tilde{g}_{\mathcal{L}}(T_0, \delta)}.$$

By definition, for every  $t \geq T_0$ , we have that  $0 \leq T_f(T_0, t) \leq T_f(T_0, t+1)$ ; therefore,  $0 \leq \overline{\varphi}(T_0, t) \leq \overline{\varphi}(T_0, t+1) \leq 2$  for every  $t \geq T_0$ . Hence, by the monotone convergence theorem (see [59]), it follows that

$$\begin{aligned}
E \left( \lim_{t \rightarrow \infty} \overline{\varphi}(T_0, t) \right) &= \lim_{t \rightarrow \infty} E(\overline{\varphi}(T_0, t)) \\
&= 2\eta \left[ 1 - \lim_{t \rightarrow \infty} E \left( \left( \frac{1}{n_w} \right)^{T_f(T_0, t)} \right) \right] \\
&\leq 2\eta \left[ 1 - \lim_{t \rightarrow \infty} \mathbb{P}(T_f(T_0, t) = 0) \right] \\
&= 2\eta \lim_{t \rightarrow \infty} [1 - \mathbb{P}(T_f(T_0, t) = 0)].
\end{aligned}$$

Since  $1 - \mathbb{P}(T_f(T_0, t) = 0) = \mathbb{P}(T_f(T_0, t) > 0)$ , it further follows that

$$\begin{aligned}
E \left( \lim_{t \rightarrow \infty} \overline{\varphi}(T_0, t) \right) &= 2\eta \lim_{t \rightarrow \infty} \mathbb{P}(T_f(T_0, t) > 0) \\
&\leq 2\eta \mathbb{P}(\exists k \geq T_0 : W_{\mathcal{L}}(k) \neq \overline{W}_{\mathcal{L}}) \\
&\leq \delta \cdot \tilde{g}_{\mathcal{L}}(T_0, \delta)
\end{aligned}$$

where the last inequality follows from Lemma 3 and the definition of  $\tilde{g}_{\mathcal{L}}(T_0, \delta)$  in (25).

Now, we consider the deviation contributed by malicious agents that are misclassified as legitimate. We denote by  $\phi_i(T_0, t)$ ,  $i \in \mathcal{L}$ , the worst-case effect on legitimate agent  $i$

due to its incorrect malicious agents' classification (labeling an untrustworthy agent as trustworthy) at some time  $t \geq T_0$ . Specifically, let  $\eta = \max_{i \in \mathcal{L} \cup \mathcal{M}} |[x_{\mathcal{L}}(0)]_i|$  and define

$$\phi_i(T_0, t) = \eta \sum_{k=T_0-1}^{t-1} \sum_{\substack{j \in \mathcal{N}_i \cap \mathcal{M}: \\ \beta_{ij}(k) \geq 0}} \left[ \left( \prod_{l=k+1}^{t-1} W_{\mathcal{L}}(l) \right) W_{\mathcal{M}}(k) \right]_{ij} \quad (27)$$

for all  $i \in \mathcal{L}$ . We note that these quantities are nonnegative, since the matrices  $W_{\mathcal{L}}(k)$  and  $W_{\mathcal{M}}(k)$  are nonnegative for all  $k$ . Looking at the malicious agents' influence vector  $\phi_{\mathcal{M}}(T_0, t)$  defined in (5), we see that

$$|[\phi_{\mathcal{M}}(T_0, t)]_i| \leq \max_{j \in \mathcal{L}} \phi_j(T_0, t) \quad \forall i \in \mathcal{L}.$$

Our next result provides a probabilistic bound on the maximal influence  $\max_{j \in \mathcal{L}} \phi_j(T_0, t)$  of malicious agent inputs on the legitimate agents' values. To do so, given  $\delta > 0$ , we define

$$\tilde{g}_{\mathcal{M}}(T_0, \delta) = \frac{\eta |\mathcal{L}| |\mathcal{M}|}{\delta \cdot \kappa} \cdot \frac{\exp(-2T_0 c^2)}{1 - \exp(-2c^2)} \quad (28)$$

where  $\eta \geq \sup_{i \in \mathcal{L} \cup \mathcal{M}, t \in \mathbb{N}} |x_i(t)|$ .

**Proposition 5:** Given an error tolerance  $\delta > 0$ , for the deviations  $\phi_i(T_0, t)$ ,  $i \in \mathcal{L}$ , as defined in (27), we have

$$\mathbb{P} \left( \max_{i \in \mathcal{L}} \limsup_{t \rightarrow \infty} \phi_i(T_0, t) > \tilde{g}_{\mathcal{M}}(T_0, \delta) \right) < \delta \quad (29)$$

where  $\tilde{g}_{\mathcal{M}}(T_0, \delta)$  is given by (28).

*Proof:* First, observe that

$$\begin{aligned} & \mathbb{P} \left( \max_{i \in \mathcal{L}} \limsup_{t \rightarrow \infty} \phi_i(T_0, t) > \tilde{g}_{\mathcal{M}}(T_0, \delta) \right) \\ & \leq \sum_{i \in \mathcal{L}} \mathbb{P} \left( \limsup_{t \rightarrow \infty} \phi_i(T_0, t) > \tilde{g}_{\mathcal{M}}(T_0, \delta) \right). \end{aligned}$$

It follows by Markov's inequality and the linearity of the expectation operator that

$$\begin{aligned} & \mathbb{P} \left( \limsup_{t \rightarrow \infty} \phi_i(T_0, t) > \tilde{g}_{\mathcal{M}}(T_0, \delta) \right) \\ & \leq \frac{E(\limsup_{t \rightarrow \infty} \phi_i(T_0, t))}{\tilde{g}_{\mathcal{M}}(T_0, \delta)}. \end{aligned}$$

In view of the definition of  $\tilde{g}_{\mathcal{M}}(T_0, \delta)$ , to complete the proof, it suffices to show that

$$E \left( \limsup_{t \rightarrow \infty} \phi_i(T_0, t) \right) \leq \frac{\eta |\mathcal{M}|}{\kappa} \cdot \frac{\exp(-2T_0 c^2)}{1 - \exp(-2c^2)}. \quad (30)$$

The rest of the proof deals with establishing relation (30). By the definition of  $\phi_i(T_0, t)$ ,  $i \in \mathcal{L}$  in (27), we have that

$$\begin{aligned} \phi_i(T_0, t) &= \eta \sum_{k=T_0-1}^{t-1} \sum_{\substack{j \in \mathcal{N}_i \cap \mathcal{M}: \\ \beta_{ij}(k) \geq 0}} \left[ \left( \prod_{l=k}^{t-1} W_{\mathcal{L}}(l) \right) W_{\mathcal{M}}(k) \right]_{ij} \\ &\leq \eta \sum_{k=T_0-1}^{t-1} \sum_{\substack{j \in \mathcal{N}_i \cap \mathcal{M}: \\ \beta_{ij}(k) \geq 0}} [W_{\mathcal{M}}(k)]_{ij}, \end{aligned}$$

where the inequality is obtained by using the fact that  $W_{\mathcal{L}}(k)$  is a row-substochastic matrix. Moreover, we have that  $[W_{\mathcal{M}}(k)]_{ij} \leq 1/\kappa$  [see (8)], implying that

$$\begin{aligned} \phi_i(T_0, t) &\leq \eta \sum_{k=T_0-1}^{t-1} \sum_{\substack{j \in \mathcal{N}_i \cap \mathcal{M}: \\ \beta_{ij}(k) \geq 0}} \frac{1}{\kappa} \\ &= \eta \sum_{k=T_0-1}^{t-1} \sum_{j \in \mathcal{N}_i \cap \mathcal{M}} \mathbb{1}_{\{\beta_{ij}(k) \geq 0\}} \frac{1}{\kappa} \triangleq \bar{\phi}_i(T_0, t). \quad (31) \end{aligned}$$

Hence, we have

$$E \left( \limsup_{t \rightarrow \infty} \phi_i(T_0, t) \right) \leq E \left( \limsup_{t \rightarrow \infty} \bar{\phi}_i(T_0, t) \right). \quad (32)$$

By the definition of  $\bar{\phi}_i(T_0, t)$  in (31), the sequence  $\{\bar{\phi}_i(T_0, t)\}$  is a nonnegative nondecreasing sequence with  $t$ . Therefore, by the monotone convergence theorem [59], we have

$$\begin{aligned} E \left( \limsup_{t \rightarrow \infty} \bar{\phi}_i(T_0, t) \right) &= E \left( \lim_{t \rightarrow \infty} \bar{\phi}_i(T_0, t) \right) \\ &= \lim_{t \rightarrow \infty} E(\bar{\phi}_i(T_0, t)). \quad (33) \end{aligned}$$

We further have that

$$\begin{aligned} \lim_{t \rightarrow \infty} E(\bar{\phi}_i(T_0, t)) &= \lim_{t \rightarrow \infty} E \left( \frac{\eta}{\kappa} \sum_{k=T_0-1}^{t-1} \sum_{j \in \mathcal{N}_i \cap \mathcal{M}} \mathbb{1}_{\{\beta_{ij}(k) \geq 0\}} \right) \\ &= \frac{\eta}{\kappa} \lim_{t \rightarrow \infty} \sum_{k=T_0-1}^{t-1} \sum_{j \in \mathcal{N}_i \cap \mathcal{M}} E(\mathbb{1}_{\{\beta_{ij}(k) \geq 0\}}). \end{aligned}$$

Since  $E(\mathbb{1}_{\{\beta_{ij}(k) \geq 0\}}) = \mathbb{P}(\beta_{ij}(k) \geq 0)$ , we obtain that

$$\begin{aligned} \lim_{t \rightarrow \infty} E(\bar{\phi}_i(T_0, t)) &= \frac{\eta}{\kappa} \lim_{t \rightarrow \infty} \sum_{k=T_0-1}^{t-1} \sum_{j \in \mathcal{N}_i \cap \mathcal{M}} \mathbb{P}(\beta_{ij}(k) \geq 0) \\ &\leq \frac{\eta}{\kappa} \lim_{t \rightarrow \infty} \sum_{k=T_0-1}^{t-1} \sum_{j \in \mathcal{N}_i \cap \mathcal{M}} \exp(-2(k+1)c^2), \end{aligned}$$

where the last inequality follows by Lemma 2. Hence, we have

$$\begin{aligned} \lim_{t \rightarrow \infty} E(\bar{\phi}_i(T_0, t)) &\leq \frac{\eta |\mathcal{M}|}{\kappa} \lim_{t \rightarrow \infty} \sum_{k=T_0-1}^{t-1} \exp(-2(k+1)c^2) \\ &= \frac{\eta |\mathcal{M}|}{\kappa} \lim_{t \rightarrow \infty} \frac{\exp(-2T_0 c^2) - \exp(-tc^2)}{1 - \exp(-2c^2)} \\ &= \frac{\eta |\mathcal{M}|}{\kappa} \cdot \frac{\exp(-2T_0 c^2)}{1 - \exp(-2c^2)}. \quad (34) \end{aligned}$$

By combining estimates in (32)–(34), the desired relation in (30) follows.

We are now ready to prove our main deviation result, which states that the maximum deviation of the converged consensus value following the dynamics in (2) with the weights described in (8) can be bounded by a finite value  $\Delta_{\max}(T_0, \delta)$  that we characterize.



*Theorem 2 (Deviation from nominal average consensus):* Consider the consensus system described by the dynamics in (2) with an arbitrary start time of  $t = T_0 \geq 0$  and weights as described by (8). Given that 1)  $\mathbb{G}$  is a sufficiently connected network and that 2) bounds on the expected values of the observations  $\alpha_{ij}(k)$  (see Definition II.2) satisfy  $c = E(\alpha_{ij}(k) - 1/2) < 0$  for all  $i \in \mathcal{L}, j \in \mathcal{N}_i \cap \mathcal{M}, k \geq 0$ , and  $d = E(\alpha_{ij}(k) - 1/2) > 0$  for all  $i \in \mathcal{L}, j \in \mathcal{N}_i \cap \mathcal{L}, k \geq 0$ , for a given error probability  $\delta > 0$ , we have the following result:

$$\mathbb{P} \left( \max_{i \in \mathcal{L}} \limsup_{t \rightarrow \infty} |x_{\mathcal{L}}(T_0, t) - \mathbf{1}v'x_{\mathcal{L}}(0)|_i \leq \Delta_{\max}(T_0, \delta) \right) \geq 1 - \delta,$$

where

$$\Delta_{\max}(T_0, \delta) = 2[\tilde{g}_{\mathcal{L}}(T_0, \delta) + \tilde{g}_{\mathcal{M}}(T_0, \delta)] \quad (35)$$

with  $\tilde{g}_{\mathcal{L}}(T_0, \delta)$  and  $\tilde{g}_{\mathcal{M}}(T_0, \delta)$ , respectively, given by

$$\begin{aligned} \tilde{g}_{\mathcal{L}}(T_0, \delta) &= \frac{\eta|\mathcal{L}|^2}{\delta} \cdot \frac{\exp(-2T_0d^2)}{1 - \exp(-2d^2)} + \frac{\eta|\mathcal{L}||\mathcal{M}|}{\delta} \cdot \frac{\exp(-2T_0c^2)}{1 - \exp(-2c^2)} \end{aligned}$$

and

$$\tilde{g}_{\mathcal{M}}(T_0, \delta) = \frac{\eta|\mathcal{L}||\mathcal{M}|}{\delta \cdot \kappa} \cdot \frac{\exp(-2T_0c^2)}{1 - \exp(-2c^2)}.$$

*Proof:* First, note that by the triangle inequality, we have

$$\begin{aligned} &|x_{\mathcal{L}}(T_0, t) - \mathbf{1}v'x_{\mathcal{L}}(0)|_i \\ &\leq \left| \left[ x_{\mathcal{L}}(T_0, t) - \left( \prod_{k=T_0}^{t-1} \overline{W}_{\mathcal{L}} \right) x_{\mathcal{L}}(0) \right]_i \right| \\ &\quad + \left| \left[ \left( \prod_{k=T_0}^{t-1} \overline{W}_{\mathcal{L}} \right) x_{\mathcal{L}}(0) - \mathbf{1}v'x_{\mathcal{L}}(0) \right]_i \right|. \end{aligned}$$

Since  $\overline{W}_{\mathcal{L}}$  is a primitive stochastic matrix, by the Perron–Frobenius theorem, it follows that

$$\lim_{t \rightarrow \infty} \left| \left[ \left( \prod_{k=T_0}^{t-1} \overline{W}_{\mathcal{L}} \right) x_{\mathcal{L}}(0) - \mathbf{1}v'x_{\mathcal{L}}(0) \right]_i \right| = 0.$$

Thus, we have

$$\begin{aligned} &\mathbb{P} \left( \max_{i \in \mathcal{L}} \limsup_{t \rightarrow \infty} |x_{\mathcal{L}}(T_0, t) - \mathbf{1}v'x_{\mathcal{L}}(0)|_i \geq \Delta_{\max}(T_0, \delta) \right) \\ &= \mathbb{P} \left( \max_{i \in \mathcal{L}} \limsup_{t \rightarrow \infty} \left| \left[ x_{\mathcal{L}}(T_0, t) - \left( \prod_{k=T_0}^{t-1} \overline{W}_{\mathcal{L}} \right) x_{\mathcal{L}}(0) \right]_i \right| \geq \Delta_{\max}(T_0, \delta) \right). \end{aligned}$$

Now, since  $x_{\mathcal{L}}(T_0, t) = \tilde{x}_{\mathcal{L}}(T_0, t) + \phi_{\mathcal{M}}(T_0, t)$ , by the triangle inequality, we have

$$\begin{aligned} &\left| \left[ x_{\mathcal{L}}(T_0, t) - \left( \prod_{k=T_0}^{t-1} \overline{W}_{\mathcal{L}} \right) x_{\mathcal{L}}(0) \right]_i \right| \\ &\leq \left| \left[ \tilde{x}_{\mathcal{L}}(T_0, t) - \left( \prod_{k=T_0}^{t-1} \overline{W}_{\mathcal{L}} \right) x_{\mathcal{L}}(0) \right]_i \right| + |\phi_{\mathcal{M}}(T_0, t)|_i. \end{aligned}$$

It follows by the definition of  $\Delta_{\max}(T_0, \delta)$  in (35) and the union bound that

$$\begin{aligned} &\mathbb{P} \left( \max_{i \in \mathcal{L}} \limsup_{t \rightarrow \infty} |x_{\mathcal{L}}(T_0, t) - \mathbf{1}v'x_{\mathcal{L}}(0)|_i \geq \Delta_{\max}(T_0, \delta) \right) \\ &\leq \mathbb{P} \left( \max_{i \in \mathcal{L}} \limsup_{t \rightarrow \infty} \left| \left[ \tilde{x}_{\mathcal{L}}(T_0, t) - \left( \prod_{k=T_0}^{t-1} \overline{W}_{\mathcal{L}} \right) x_{\mathcal{L}}(0) \right]_i \right| \geq 2\tilde{g}_{\mathcal{L}}(T_0, \delta) \right) \\ &\quad + \mathbb{P} \left( \max_{i \in \mathcal{L}} \limsup_{t \rightarrow \infty} |\phi_{\mathcal{M}}(T_0, t)|_i \geq 2\tilde{g}_{\mathcal{M}}(T_0, \delta) \right). \end{aligned}$$

Since  $2\tilde{g}_{\mathcal{L}}(T_0, \delta) = \tilde{g}_{\mathcal{L}}(T_0, \delta/2)$  and  $2\tilde{g}_{\mathcal{M}}(T_0, \delta) = \tilde{g}_{\mathcal{M}}(T_0, \delta/2)$ , it follows that

$$\begin{aligned} &\mathbb{P} \left( \max_{i \in \mathcal{L}} \limsup_{t \rightarrow \infty} |x_{\mathcal{L}}(T_0, t) - \mathbf{1}v'x_{\mathcal{L}}(0)|_i \geq \Delta_{\max}(T_0, \delta) \right) \\ &\leq \mathbb{P} \left( \max_{i \in \mathcal{L}} \limsup_{t \rightarrow \infty} \left| \left[ \tilde{x}_{\mathcal{L}}(T_0, t) - \left( \prod_{k=T_0}^{t-1} \overline{W}_{\mathcal{L}} \right) x_{\mathcal{L}}(0) \right]_i \right| \geq \tilde{g}_{\mathcal{L}}(T_0, \delta/2) \right) \\ &\quad + \mathbb{P} \left( \max_{i \in \mathcal{L}} \limsup_{t \rightarrow \infty} |\phi_{\mathcal{M}}(T_0, t)|_i \geq \tilde{g}_{\mathcal{M}}(T_0, \delta/2) \right). \end{aligned}$$

Finally, by Proposition 4, we have

$$\begin{aligned} \frac{\delta}{2} &\geq \mathbb{P} \left( \max_{i \in \mathcal{L}} \limsup_{t \rightarrow \infty} \left| \left[ \tilde{x}_{\mathcal{L}}(T_0, t) - \left( \prod_{k=T_0}^{t-1} \overline{W}_{\mathcal{L}} \right) x_{\mathcal{L}}(0) \right]_i \right| \geq \tilde{g}_{\mathcal{L}}(T_0, \delta/2) \right) \end{aligned}$$

while, similarly, by Proposition 5, we have

$$\frac{\delta}{2} \geq \mathbb{P} \left( \max_{i \in \mathcal{L}} \limsup_{t \rightarrow \infty} |\phi_{\mathcal{M}}(T_0, t)|_i \geq \tilde{g}_{\mathcal{M}}(T_0, \delta/2) \right).$$

Thus, we have solved Problem 2 by showing that beyond achieving convergence of consensus in the face of a malicious attack, the deviation of the converged value from the nominal average consensus value (in the case of no malicious agents) can be characterized as a function of user-defined parameters  $T_0$  and failure probability  $\delta$ . In the next section, we characterize convergence rate of our consensus protocol.

### C. Convergence Rate of Resilient Consensus

In this section, we discuss convergence rate for the consensus protocol in (2) using the weights as defined in (8). Let

$$\|z\|_v = \sqrt{\sum_{i=1}^{n_{\mathcal{L}}} v_i z_i^2}.$$

We start by presenting a useful lemma showing a contraction property for a consensus step using a primitive matrix that we will later employ.

*Lemma 5:* Assume that  $j \in \mathcal{N}_i$  if and only if  $i \in \mathcal{N}_j$  for every  $i, j \in \mathcal{L}$  (i.e., bidirectional links between legitimate agents). Let  $\rho_2$  denote the second largest eigenvalue modulus<sup>2</sup> of  $\bar{W}_{\mathcal{L}}$  and let  $v > \mathbf{0}$  be the stochastic Perron vector satisfying  $v' \bar{W}_{\mathcal{L}} = v'$ . Then, we have

$$\|\bar{W}_{\mathcal{L}}^t x(0) - \mathbf{1} v' x(0)\|_v \leq \rho_2^t \cdot \|x(0) - \mathbf{1} v' x(0)\|_v.$$

*Proof:* It can be seen that the Perron vector  $v$  has entries  $v_i = \frac{\max\{|\mathcal{N}_i \cap \mathcal{L}|+1, \kappa\}}{\sum_{j \in \mathcal{L}} \max\{|\mathcal{N}_j \cap \mathcal{L}|+1, \kappa\}}$  for all  $i$ , implying that  $(\bar{W}_{\mathcal{L}}, v)$  is reversible Markov chain. Thus, the result follows from the convergence rate of reversible Markov chains (see [60]).

We are now ready to prove our convergence rate results. Namely, we show that the rate of convergence of legitimate agents to their limit value is governed by the second largest eigenvalue modulus of the ideal weight matrix  $\bar{W}_{\mathcal{L}}$  with a probability that approaches 1 exponentially, as  $T_0$  increases.

*Theorem 3 (Convergence rate of resilient consensus):* Consider the consensus system described by the dynamics in (2) with an arbitrary start time of  $T_0 \geq 0$  and weights, as described by (8). Assume that 1)  $\mathbb{G}$  is a sufficiently connected network and that 2) bounds on the expected values of the observations  $\alpha_{ij}$  (see Definition II.2) satisfy  $c = E(\alpha_{ij}(k)) - 1/2 < 0$  for all  $i \in \mathcal{L}, j \in \mathcal{N}_i \cap \mathcal{M}$ , for all  $k \geq 0$ , and  $d = E(\alpha_{ij}(k)) - 1/2 > 0$  for all  $i \in \mathcal{L}, j \in \mathcal{N}_i \cap \mathcal{L}$ , for all  $k \geq 0$ . Let  $\rho_2$  denote the second largest eigenvalue modulus of  $\bar{W}_{\mathcal{L}}$  and let  $v > \mathbf{0}$  be the stochastic Perron vector satisfying  $v' \bar{W}_{\mathcal{L}} = v'$ . Additionally, let  $\eta \geq \sup_{i \in \mathcal{L} \cup \mathcal{M}, t \in \mathbb{N}} |x_i(t)|$  and assume that  $j \in \mathcal{N}_i$  if and only if  $i \in \mathcal{N}_j$  for every  $i, j \in \mathcal{L}$ . For every  $T_0 > 0$ , if we start the consensus protocol from time  $T_0$ , then for every  $t \geq T_0 - 1$  and  $m = T_0 - 1, \dots, t$ , we have

$$\|x_{\mathcal{L}}(T_0, t) - z(T_0) \mathbf{1}\|_v \leq 2(m - T_0 + 1) \rho_2^{t-m} \eta \quad (36)$$

with a probability greater than

$$1 - |\mathcal{L}|^2 \cdot \frac{\exp(-2(m+1)d^2)}{1 - \exp(-2d^2)} - |\mathcal{L}| |\mathcal{M}| \cdot \frac{\exp(-2(m+1)c^2)}{1 - \exp(-2c^2)}. \quad (37)$$

*Proof:* Recall that  $\lim_{t \rightarrow \infty} x_{\mathcal{L}}(T_0, t) = z(T_0) \mathbf{1}$  almost surely. Since  $x_{\mathcal{L}}(T_0, t) = \tilde{x}_{\mathcal{L}}(T_0, t) + \phi_{\mathcal{M}}(T_0, t)$ , it follows by the triangle inequality that

$$\begin{aligned} & \|x_{\mathcal{L}}(T_0, t) - z(T_0) \mathbf{1}\|_v \\ &= \left\| x_{\mathcal{L}}(T_0, t) - \lim_{\tau \rightarrow \infty} x_{\mathcal{L}}(T_0, \tau) \right\|_v \end{aligned}$$

<sup>2</sup>The second largest absolute value of the eigenvalues.

$$\begin{aligned} & \leq \left\| \tilde{x}_{\mathcal{L}}(T_0, t) - \lim_{\tau \rightarrow \infty} \tilde{x}_{\mathcal{L}}(T_0, \tau) \right\|_v \\ & \quad + \left\| \phi_{\mathcal{M}}(T_0, t) - \lim_{\tau \rightarrow \infty} \phi_{\mathcal{M}}(T_0, \tau) \right\|_v. \end{aligned}$$

Now, assume that for all  $k \geq m$ ,  $W_{\mathcal{L}}(k) = \bar{W}_{\mathcal{L}}$  and  $W_{\mathcal{M}}(k) = \mathbf{0}$ . Then, by the Perron–Frobenius theorem and the definition of  $\eta$ , we have

$$\begin{aligned} & \left\| \tilde{x}_{\mathcal{L}}(T_0, t) - \lim_{\tau \rightarrow \infty} \tilde{x}_{\mathcal{L}}(T_0, \tau) \right\|_v \\ &= \left\| \left( \bar{W}_{\mathcal{L}}^{t-m} - \mathbf{1} v \right) \left( \prod_{k=T_0-1}^{m-1} W_{\mathcal{L}}(k) \right) x_{\mathcal{L}}(0) \right\|_v \\ & \leq \rho_2^{t-m} 2\eta. \end{aligned}$$

Additionally, we have

$$\begin{aligned} & \left\| \phi_{\mathcal{M}}(T_0, t) - \lim_{\tau \rightarrow \infty} \phi_{\mathcal{M}}(T_0, \tau) \right\|_v \\ &= \left\| \sum_{k=T_0-1}^{m-1} \left( \prod_{l=k+1}^{t-1} W_{\mathcal{L}}(l) \right) W_{\mathcal{M}}(k) x_{\mathcal{M}}(k) \right. \\ & \quad \left. - \lim_{\tau \rightarrow \infty} \sum_{k=T_0-1}^{m-1} \left( \prod_{l=k+1}^{\tau-1} W_{\mathcal{L}}(l) \right) W_{\mathcal{M}}(k) x_{\mathcal{M}}(k) \right\|_v \\ &= \left\| \sum_{k=T_0-1}^{m-1} \left( \bar{W}_{\mathcal{L}}^{t-m} - \mathbf{1} v \right) \left( \prod_{k=T_0-1}^{m-1} W_{\mathcal{L}}(k) \right) W_{\mathcal{M}}(k) x_{\mathcal{M}}(k) \right\|_v \\ & \leq (m - T_0) \rho_2^{t-m} 2\eta. \end{aligned}$$

Finally, following the proof of Lemma 3, we can lower bound the probability of the event that  $W_{\mathcal{L}}(k) = \bar{W}_{\mathcal{L}}$  and  $W_{\mathcal{M}}(k) = \mathbf{0}$  for all  $k \geq m$  by (37).

As an immediate consequence of Theorem 3, we have the following result.

*Corollary 3:* Under the conditions of Theorem 3, for every  $T_0 \geq 0$  and  $t \geq T_0$ , we have

$$\begin{aligned} & E(\|x_{\mathcal{L}}(T_0, t) - \mathbf{1} v' x_{\mathcal{L}}(0)\|_v) \\ & \leq \min_{m \in \{T_0-1, \dots, t-1\}} \left\{ 2(m - T_0 + 1) \rho_2^{t-m} \eta + 2\eta \cdot \left( \frac{|\mathcal{L}|^2 \exp(-2(m+1)d^2)}{1 - \exp(-2d^2)} + \frac{|\mathcal{L}| |\mathcal{M}| \exp(-2(m+1)c^2)}{1 - \exp(-2c^2)} \right) \right\}. \\ & \text{Note that choosing } m = \frac{t+T_0}{2} \text{ yields the bound} \\ & E(\|x_{\mathcal{L}}(T_0, t) - \mathbf{1} v' x_{\mathcal{L}}(0)\|_v) \\ & \leq 2 \left( \frac{t - T_0}{2} + 1 \right) \rho_2^{\frac{t-T_0}{2}} \eta + \left( \frac{|\mathcal{L}|^2 \exp(-(t+T_0+2)d^2)}{1 - \exp(-2d^2)} + \frac{|\mathcal{L}| |\mathcal{M}| \exp(-(t+T_0+2)c^2)}{1 - \exp(-2c^2)} \right) 2\eta \\ & = O(|\mathcal{L}| \cdot \max\{|\mathcal{L}|, |\mathcal{M}|\} t e^{-\gamma t}) \end{aligned}$$

where  $\gamma > 0$ .

#### D. Tightening the Probabilistic Bounds

Up until this point, we have placed a mild information assumption on the  $\alpha_{ij}$  (this was a consequence of Lemma 2), namely that the expected values of  $\alpha_{ij}$  are strictly bounded away from 1/2, as required in Corollary 1. In words, this means that there is some information contained in the observations  $\alpha_{ij}$  such that in expectation, these values are closer to 0 for transmissions from legitimate agents and closer to 1 for transmissions from malicious agents. An expected value of  $\alpha_{ij}$  equal to 1/2 would be the case of no information. Thus, this is the mildest assumption possible for  $\alpha_{ij}$  and is the key to the decaying probabilities from Lemma 2 that underpin the majority of our presented results in this article. However, if it is possible to obtain more information on  $\alpha_{ij}$ , for example knowing a bound on its *variance*, then the probabilistic bounds can be made tighter, and the resulting performance guarantees such as bounds on the deviation can be made stronger. This observation is substantiated with analysis in the current section.

Lemma 2 lays the foundation for deriving the convergence of consensus of legitimate agents presented in Theorem 1, the deviation probabilistic bound presented in Theorem 2, and the probabilistic upper bound on the convergence rate presented in Theorem 3. Lemma 2 considers a large family of probability measures for the random variables  $\alpha_{ij}(t)$ , where only the expectation of the variables are fixed to known values. While the bounds derived in Lemma 2 decay exponentially over time, they are somewhat loose when higher moments of the random variables  $\alpha_{ij}(t)$  are known. In the special case when the random variables  $\alpha_{ij}(t)$  belong to a family of probability measures of known variance values, we can use the concentration inequalities that consider both the first and second moments, instead of the Chernoff–Hoeffding concentration inequality used in the proof of Lemma 2. Prominent examples of such inequalities are the Bernstein inequality, Bennet’s inequality, and the improved Bennet’s inequality, providing more refined results than the Chernoff–Hoeffding inequality, are included here for completeness.

**Theorem 4 (Bernstein inequality):** Assume that  $x_1, \dots, x_n$  are independent random variables and  $E(x_i) = 0$ ,  $E(x_i^2) = \sigma_i^2$  and  $|x_i| \leq M$  almost surely. Then, we have

$$\mathbb{P} \left( \sum_{i=1}^n x_i \geq b \right) \leq \exp \left( \frac{\frac{1}{2}b^2}{\sum_{i=1}^n \sigma_i^2 + \frac{1}{3}Mb} \right).$$

**Theorem 5 (Bennet’s inequality [61]):** Assume that  $x_1, \dots, x_n$  are independent random variables and  $E(x_i) = 0$ ,  $E(x_i^2) = \sigma_i^2$  and  $|x_i| \leq M$  almost surely. Then, for any  $0 \leq t < nM$ , we have

$$\mathbb{P} \left( \sum_{i=1}^n x_i \geq b \right) \leq \exp \left( -\frac{n\sigma^2}{M^2} h \left( \frac{bM}{n\sigma^2} \right) \right)$$

where  $h(x) = (1+x) \ln(1+x) - x$  and  $n\sigma^2 = \sum_{i=1}^n \sigma_i^2$ .

**Theorem 6 (Improved Bennet’s inequality [62]):** Assume that  $x_1, \dots, x_n$  are independent random variables and  $E(x_i) = 0$ ,

$E(x_i^2) = \sigma_i^2$  and  $|x_i| \leq M$  almost surely. Additionally, let

$$A = \frac{M^2}{\sigma^2} + \frac{nM}{b} - 1 \text{ and } B = \frac{nM}{b} - 1$$

and  $\Lambda = A - W(Be^A)$ , where  $W(\cdot)$  is the Lambert  $W$  function. Let  $\sigma^2 = \frac{1}{n} \sum_{i=1}^n \sigma_i^2$ ; then, for any  $0 \leq b < nM$ , we have

$$\mathbb{P} \left( \sum_{i=1}^n x_i \geq b \right) \leq \exp \left[ -\frac{\Lambda b}{M} + n \ln \left( 1 + \frac{\sigma^2}{M^2} (e^\Lambda - 1 - \Lambda) \right) \right].$$

As stated in [63], Bennet’s inequality yields strictly tighter approximation than Bernstein’s inequality. Bennet’s inequality, in turn, is tightened by the improved Bennet’s inequality. For this reason, we present in Lemma 6 refined concentration inequalities that improve those presented in Lemma 2 using the improved Bennet’s inequality. Recall Assumptions 1.2 and 1.3 and additionally assume that

$$\text{var}(\alpha_{ij}(t) - 1/2) = \sigma_d^2 \quad \text{for all } t \geq 0, i \in \mathcal{L}, j \in \mathcal{N}_i \cap \mathcal{L}$$

and that

$$\text{var}(\alpha_{ij}(t) - 1/2) = \sigma_c^2 \quad \text{for all } t \geq 0, i \in \mathcal{L}, j \in \mathcal{N}_i \cap \mathcal{M}.$$

Applying these tighter concentration inequalities to the history of observations  $\beta_{ij}(t) = \sum_{k=0}^t (\alpha_{ij}(k) - 1/2)$ , we can obtain a faster exponential rate result for the decay of misclassification probabilities (misclassifying a legitimate agent as malicious or a malicious agent as legitimate) than that of Lemma 2, as seen in the following.

**Lemma 6:** Consider the random variables  $\beta_{ij}(t)$  defined in (6) for every legitimate node  $i \in \mathcal{L}$  and every neighbor  $j \in \mathcal{N}_i$ . Then, for every  $t \geq 0$  and every  $i \in \mathcal{L}, j \in \mathcal{N}_i \cap \mathcal{L}$ , we have

$$\mathbb{P}(\beta_{ij}(t) < 0) \leq \max \left\{ \mathbb{1}_{\{d < 0\}}, \exp \left[ -d(t+1)\Lambda_d + (t+1) \ln \left( 1 + \sigma_d^2 (e^{\Lambda_d} - 1 - \Lambda_d) \right) \right] \right\}$$

where  $d = E(\alpha_{ij}(t) - 1/2)$  for  $i \in \mathcal{L}, j \in \mathcal{N}_i \cap \mathcal{L}$ , and

$$\Lambda_d = A_d - W(B_d e^{A_d})$$

with  $W(\cdot)$  being the Lambert  $W$  function, and

$$A_d = \frac{1}{\sigma_d^2} + \frac{1}{d} - 1, \quad B_d = \frac{1}{d} - 1.$$

Moreover, for every  $t \geq 0$  and every  $i \in \mathcal{L}, j \in \mathcal{N}_i \cap \mathcal{M}$ , we have

$$\mathbb{P}(\beta_{ij}(t) \geq 0) \leq \max \left\{ \mathbb{1}_{\{c > 0\}}, \exp \left[ c(t+1)\Lambda_c + (t+1) \ln \left( 1 + \sigma_c^2 (e^{\Lambda_c} - 1 - \Lambda_c) \right) \right] \right\}$$



where  $c = E(\alpha_{ij}(t) - 1/2)$  for  $i \in \mathcal{L}, j \in \mathcal{N}_i \cap \mathcal{M}$ , and  $\Lambda_c = A_c - W(B_c e^{A_c})$ , with scalars  $A_c$  and  $B_c$  given by

$$A_c = \frac{1}{\sigma_c^2} - \frac{1}{c} - 1, \quad B_c = -\frac{1}{c} - 1.$$

*Proof:* Let  $i \in \mathcal{L}, j \in \mathcal{N}_i \cap \mathcal{L}$ , and note that the given probabilistic bound holds trivially when  $d < 0$ , so assume that  $d > 0$  and define

$$\gamma_{ij}(t) = d + 1/2 - \alpha_{ij}(t).$$

Since  $d = E(\alpha_{ij}(t) - 1/2)$ , for the random variable  $\gamma_{ij}(t)$ , we have  $E(\gamma_{ij}(t)) = 0$  and  $\text{var}(\gamma_{ij}(t)) = \sigma_d^2$ . Since  $\alpha_{ij}(t) \in [0, 1]$ , it follows that  $d \leq 1/2$  implying that  $\gamma_{ij}(t) \leq 1$ . Also, since  $d > 0$  and  $\alpha_{ij}(t) \in [0, 1]$ , it follows that  $\gamma_{ij} > -1/2$ . Thus,  $|\gamma_{ij}| \leq 1$ . Using the definition of  $\beta_{ij}(t)$  in (6), we have

$$\begin{aligned} \mathbb{P}(\beta_{ij}(t) < 0) &= \mathbb{P}\left(\sum_{k=0}^t (\alpha_{ij}(k) - 1/2) < 0\right) \\ &= \mathbb{P}\left(\sum_{k=0}^t (\alpha_{ij}(k) - 1/2 - d) < -(t+1)d\right) \\ &= \mathbb{P}\left(-\sum_{k=0}^t \gamma_{ij}(t) < -(t+1)d\right) \\ &= \mathbb{P}\left(\sum_{k=0}^t \gamma_{ij}(t) > (t+1)d\right). \end{aligned}$$

We now invoke Theorem 6 for the sequence of random variables  $\gamma_{ij}(0), \dots, \gamma_{ij}(t)$ , with  $M = 1$ ,  $n = t + 1$  and  $b = (t + 1)d$ , which yields the desired relation. Note that the condition  $b < nM$  in Theorem 6 is satisfied in our case here, since  $b = (t + 1)d$  with  $d \leq 1/2$  and  $nM = t + 1$ .

Consider now the case  $i \in \mathcal{L}, j \in \mathcal{N}_i \cap \mathcal{M}$ . If  $c > 0$ , then the result holds trivially. Thus, assume that  $c < 0$ , and consider

$$\gamma_{ij}(t) = \alpha_{ij}(t) - 1/2 - c.$$

Since  $c = E(\alpha_{ij}(t) - 1/2)$ , we have that  $E(\gamma_{ij}(t)) = 0$  and  $\text{var}(\gamma_{ij}(t)) = \sigma_c^2$ . Additionally, since  $\alpha_{ij}(t) \in [0, 1]$ , we can see (similar to the preceding case) that  $|\gamma_{ij}(t)| \leq 1$ . Hence, we have

$$\begin{aligned} \mathbb{P}(\beta_{ij}(t) \geq 0) &= \mathbb{P}\left(\sum_{k=0}^t (\alpha_{ij}(k) - 1/2) \geq 0\right) \\ &= \mathbb{P}\left(\sum_{k=0}^t (\alpha_{ij}(k) - 1/2 - c) \geq -(t+1)c\right) \\ &= \mathbb{P}\left(\sum_{k=0}^t \gamma_{ij}(t) \geq -(t+1)c\right). \end{aligned}$$

Recall that  $c < 0$ , so we can invoke Theorem 6 for the sequence of random variables  $\gamma_{ij}(0), \dots, \gamma_{ij}(t)$ , with  $\alpha = -(t + 1)c > 0$ ,  $M = 1$ , and  $n = t + 1$ . Since  $c = E(\alpha_{ij}(t) - 1/2)$  and  $\alpha_{ij}(t) \in [0, 1]$ , it follows that  $-1/2 \leq c < 0$ . Thus, in our case here, the condition  $b < nM$  in Theorem 6 reduces to  $-(t + 1)c < t + 1$ , which holds since  $-1/2 \leq c < 0$ .

Therefore, in the case that additional information on the observations  $\alpha_{ij}$  is available, in particular that a bound on the variance of the  $\alpha_{ij}$  is known, then the results of Lemma 2 can be replaced with Lemma 6, resulting in tighter deviation and convergence rate results when applied to Theorems 2 and 3. This shows the generality of our framework and the potential to improve our derived performance guarantees as more information on the  $\alpha_{ij}$  observations becomes available. Similarly, Lemma 6 can be extended to the case where  $\text{var}(\beta_{ij}(t))$  varies over time and is link dependent.

#### IV. NUMERICAL RESULTS

In this section, we evaluate the performance of the proposed scheme in this work using Monte Carlo simulations. We aim at investigating the effects, on several performance metrics, of the following system parameters: the number  $|\mathcal{M}|$  of malicious agents, the starting time  $T_0$  of the data passing stage, and the variance of the random variables  $\alpha_{ij}$ . Performance metrics that we study include deviation from the true consensus value and the convergence rate. We evaluate a system setup with 15 legitimate agents and with  $\kappa = 10$ . We consider the following values for the starting time  $T_0$ : 0, 25, 50, 100, and 150. The adjacency matrix of the subgraph  $\mathbb{G}_{\mathcal{L}}$  of links among legitimate agents is

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{pmatrix} \quad (38)$$

and we assume that every malicious agent is connected to all the legitimate agents in the system, which is the worst-case scenario. We depict the topology corresponding to the connectivity graph  $\mathbb{G}$  in Fig. 4; since every malicious agent is connected to all legitimate agents, we depict the malicious agents in Fig. 4 by a single node. The vector of initial values of legitimate agents is

$$x_{\mathcal{L}}(0) = (-2.59, -2.44, -4.23, -1.45, -1.46, 0.871, -0.51, -3.19, -0.59, -3.31, -2.25, 1.31, 1.87, 1.34, 1.76)'$$

where  $(\cdot)'$  denotes the transpose operator, and we use  $\eta = 5$  as the upper bound for the agents' initial values. Additionally, for every legitimate agent  $i$ , we have  $E(\alpha_{ij}) = 0.55$  if  $j \in \mathcal{N}_i \cap \mathcal{L}$ , and  $E(\alpha_{ij}) = 0.45$  if  $j \in \mathcal{N}_i \cap \mathcal{M}$ . The random variable  $\alpha_{ij}$  is uniformly distributed on the interval  $[E(\alpha_{ij}) - \frac{\ell}{2}, E(\alpha_{ij}) + \frac{\ell}{2}]$ . We consider the following three values for  $\ell$ : 0.2, 0.4, and 0.6, for which the standard deviation values are 0.0577, 0.1155, and 0.1732, respectively. Note that the larger the  $\ell$ , the larger

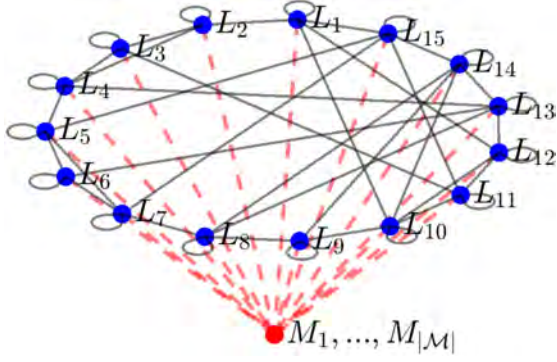


Fig. 4. Undirected subgraph  $\mathbb{G}$  of legitimate and malicious agents. Two agents are neighbors if they are connected by an edge. Legitimate agents are depicted by blue nodes and malicious agents are depicted by a red node. Edges between legitimate agents are depicted by black solid lines. Edges between legitimate and malicious agents are depicted by red dashed lines.

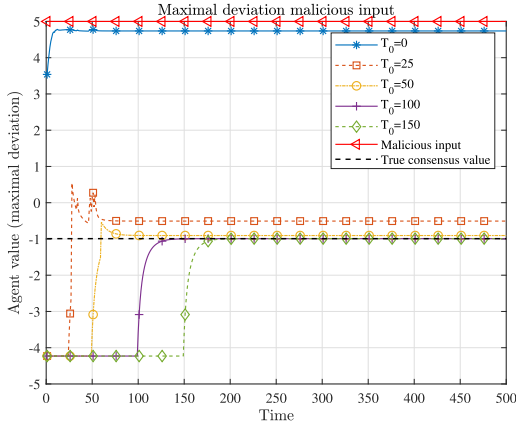


Fig. 5. Agent value with the maximal deviation from true consensus value for a single system realization for a system with 15 malicious agents,  $\ell = 0.4$ , and maximal deviation malicious inputs.

the variance of  $\alpha_{ij}$  about its mean value. We consider the following input scenarios presented in Figs. 5 and 6. Both attacks assume that the malicious agents are allowed to communicate and coordinate attacks.

#### A. Maximal Deviation Malicious Input

To measure the maximal possible deviation from true consensus value, we consider the case where the malicious agent inputs are the furthest from the true consensus value. That is, if the true consensus value is positive, then the malicious input is  $-\eta$ , and if the true consensus value is negative, then the malicious input is  $\eta$ . In our scenario, all the malicious agents choose the maximum possible input  $\eta = 5$ . Figs. 5 and 7 measure the effect of the malicious agent inputs that lead to the maximum deviation from the true consensus value. Fig. 5 depicts agent values over time

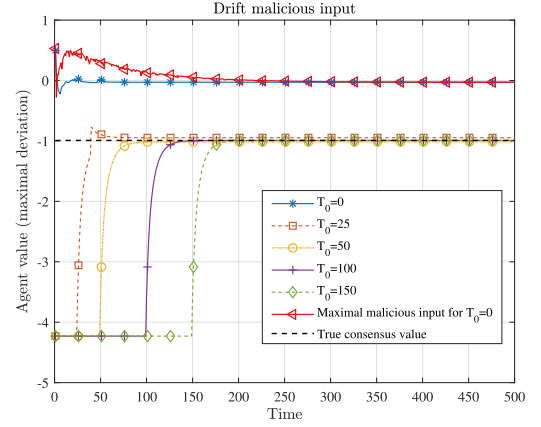


Fig. 6. Agent value with the maximal deviation from true consensus value for a single system realization for a system with 15 malicious agents,  $\ell = 0.4$ , and drift malicious inputs.

when running consensus according to the protocol from (1) and the weights defined by (8). Specifically, Fig. 5 depicts the value of the agent with the maximal deviation from the true consensus value for a single system realization with maximum deviation malicious input. Additionally, Fig. 7 depicts the absolute value of the maximum deviation from the true consensus value averaged over 500 system realizations. Figs. 5 and 7 additionally show the differences in converged values and deviation, respectively, for different consensus start times  $T_0$ .

The maximum deviation malicious input leads to an upper bound to all possible malicious attacks and depicts a worst-case scenario in terms of deviation from true consensus value. However, though this attack is useful to analyze the worst-case effect of malicious inputs, it is easy to detect using outlier rejection methods, for example [8]. For this reason, we consider the following additional attack scenario that is harder to detect.

#### B. Drift Malicious Input

Fig. 6 and 8 measure the effect of a less transparent malicious agents' attack, where every malicious agent adds a drift term to the average legitimate agent values. More specifically, let  $d_{\mathcal{L}}(m, t)$  be the average value at time  $t$  of the legitimate agents that are neighbors of the malicious agent  $m$ . Denote  $d_{\mathcal{L}}(t) = \frac{1}{|\mathcal{M}|} \sum_{m=1}^{|\mathcal{M}|} d_{\mathcal{L}}(m, t)$ . Then, at time  $t$ , the malicious agent  $m$  chooses its state value  $x_m(t)$  according to the following rule:

$$x_m(t) = d_{\mathcal{L}}(t-1) + 0.15 \times d_{\mathcal{M}}(m, t)$$

where  $x_m(t = T_0 - 1)$  is chosen randomly from the set  $[-0.15 \times \eta, 0.15 \times \eta]$ , and for every  $t \geq T_0$ , we have

$$d_{\mathcal{M}}(m, t) = -\text{sign}(d_{\mathcal{L}}(1)) \times \eta \times (0.75)^{0.05(t-T_0)} u(m, t),$$

$u(m, t) \sim U[0, 1]$ . Thus, malicious agent inputs are *time varying* and strategically close to legitimate agent values to avoid easy detection in this case. Additionally, if  $x_m(t) > \eta$ , then  $x_m(t)$  is generated randomly and uniformly from the interval  $[\eta - 0.05, \eta]$ . Similarly, if  $x_m(t) < -\eta$ , then  $x_m(t)$  is generated

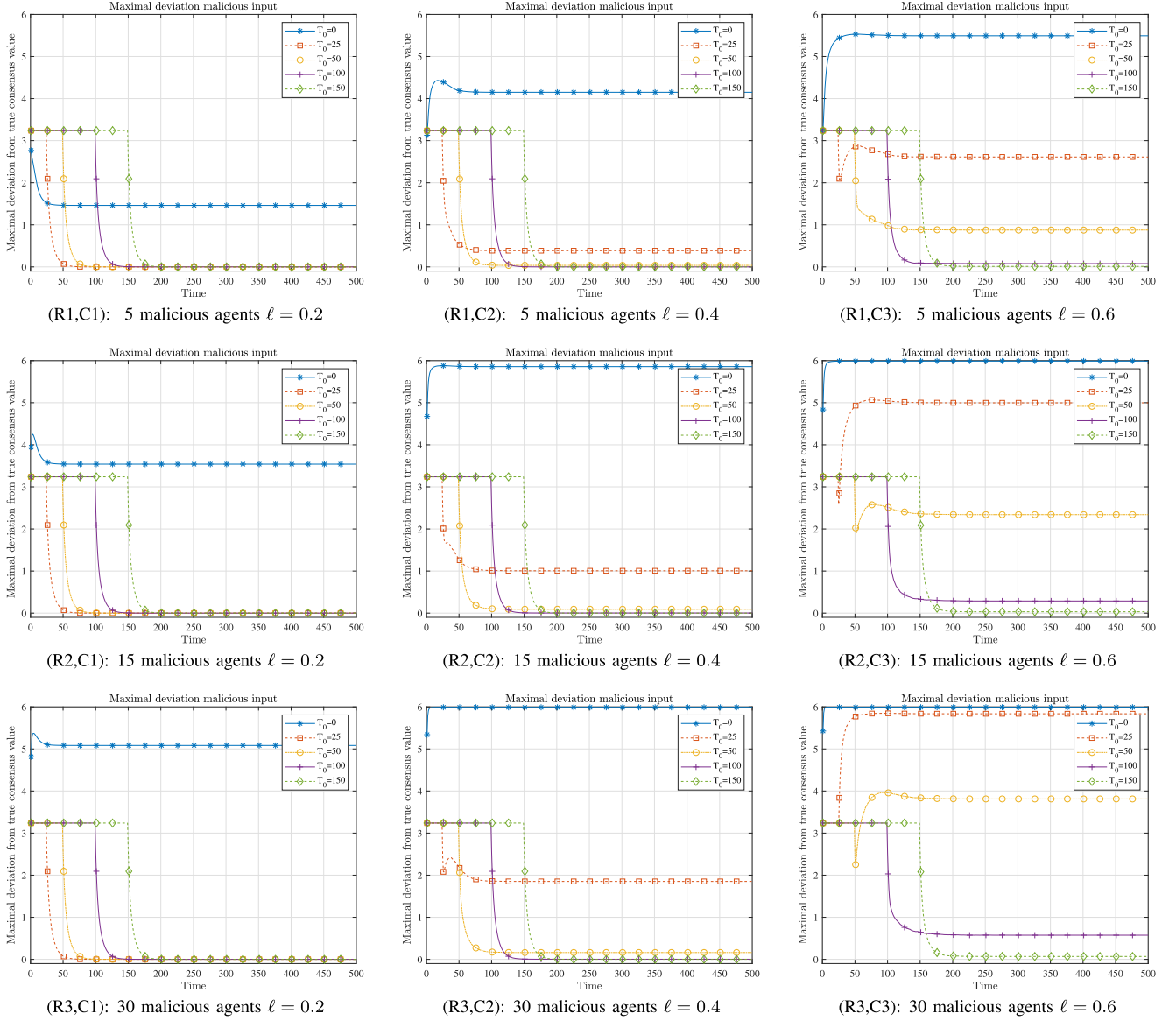


Fig. 7. Maximal deviation input.

randomly and uniformly from the interval  $[-\eta, -\eta + 0.05]$ . We note that since, in our scenario, every malicious agent is a neighbor of all the legitimate agents,  $d_{\mathcal{L}}(m_1, t) = d_{\mathcal{L}}(m_2, t)$  for all  $m_1, m_2 \in \mathcal{M}$ . Fig. 6 depicts the value of the agent with the maximal deviation from the true consensus value for a single system realization with drift malicious input. Additionally, Fig. 8 depicts the resulting absolute value of the maximum deviation from the true consensus value for a drift malicious input averaged over 500 system realizations.

Figs. 5 and 6 depict a single realization of the agent value with the maximal deviation from true consensus value for a system with 15 malicious agents and  $\ell = 0.4$ . Fig. 5 depicts the agent value with the maximal deviation from true consensus value for the maximal deviation malicious inputs, whereas Fig. 6 depicts the agent value with the maximal deviation from true consensus value for the drift malicious inputs.

In addition to Figs. 5 and 6 that depict the legitimate agent values for a *single system realization*, we depict in Figs. 7 and 8 the *average deviation* of the legitimate agent values from true consensus value. The average deviation is calculated using 500 system realizations. Figs. 7 and 8 show that the deviation from true consensus value grows both as  $\ell$  grows and as the number  $|\mathcal{M}|$  of malicious agents grows. In each figure, we label the plot by (R#,C#) where R# denotes the row number and represents the increase in the number of malicious agents, and C# denotes the column number and represents the growth of  $\ell$ . Figs. 7 and 8 show the significance of introducing the variable  $T_0$  that sets the starting time of the data passing between agents. The figures indicate that increasing  $T_0$  can significantly reduce the deviation from true consensus value. Figs. 7 and 8 also show that our scheme can combat an attack of a large number of



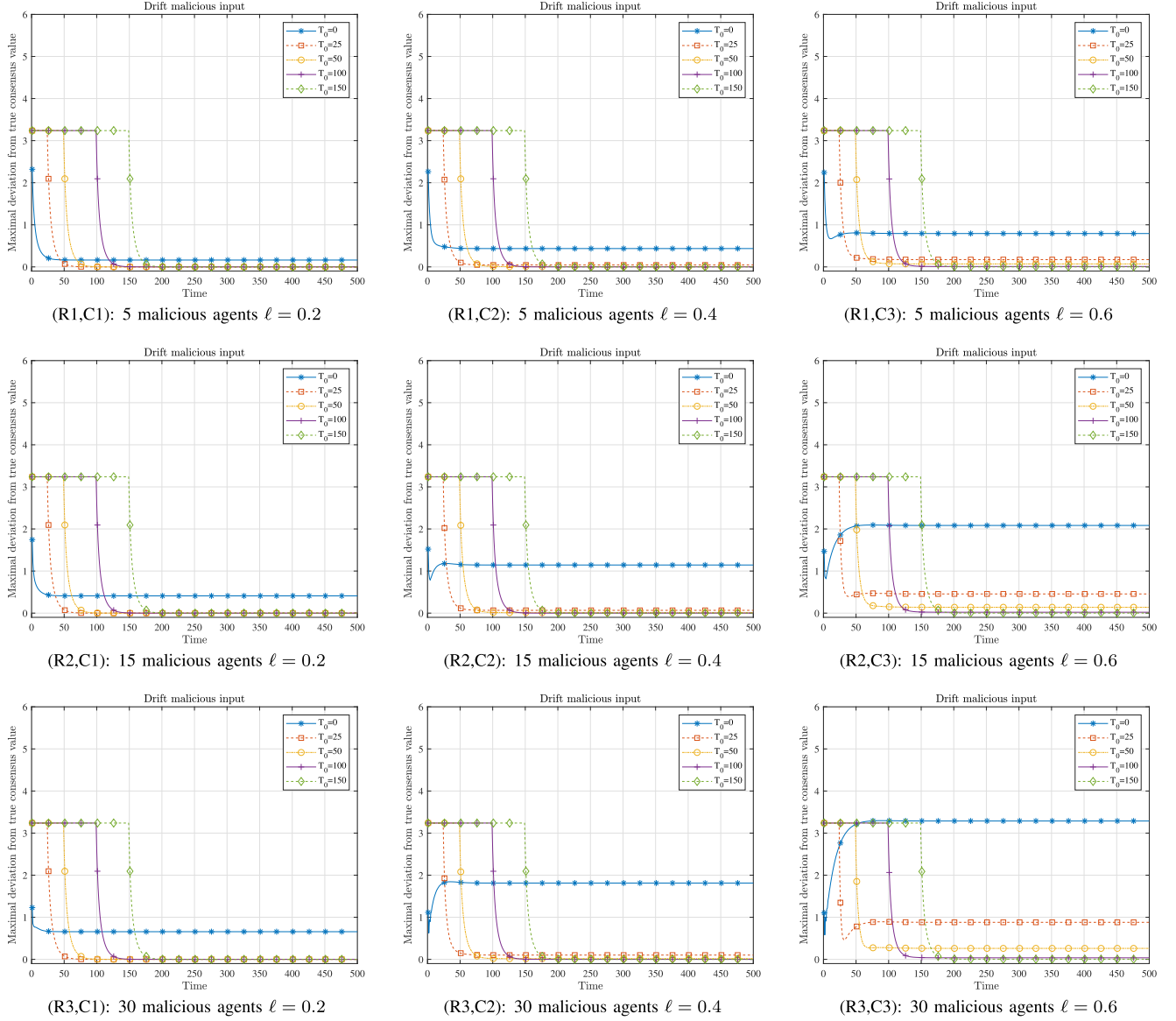


Fig. 8. Drift input.

malicious agents, even when  $|\mathcal{M}| = 30$ , that is, there is twice the number of malicious agents than legitimate agents in the system. *This is a significant improvement over classical results* that depend on the network connectivity [1], [5], [6]. Using the upper bound  $\min_{i \in \mathcal{C} \cup \mathcal{M}} |\mathcal{N}_i|$  on the network connectivity, we can conclude that classical results will fail to detect all the malicious agents when their number is larger than the number of legitimate agents; in this case, the undetected malicious agents can control the data values of the legitimate agents. More specifically, we can see from the choice of adjacency matrix (38) that for our setup, the network connectivity is at most  $3 + |\mathcal{M}|$ , and thus, classical results can tolerate less than  $\frac{3+|\mathcal{M}|}{2}$  malicious agents. Since there are  $|\mathcal{M}|$  malicious agents, the consensus algorithm can fail and be controlled by the malicious agents. Furthermore, for the connectivity graph that we consider in the numerical results, the maximal number of malicious agents leading to a

graph connectivity guaranteeing their detection and preventing them from controlling the network is 2, since we must fulfill the condition that  $|\mathcal{M}| < \frac{3+|\mathcal{M}|}{2}$ . Finally, our numerical results show that there exists a finite time  $T_0$  such that the probability to classify all the user in the network correctly for every  $t > T_0$  is sufficiently high. We can see from Figs. 7 and 8 that the minimal value of  $T_0$  that yields a negligible deviation from true consensus value depends on probability of misclassifying users, i.e., classifying a legitimate agent as malicious and *vice versa*. This probability is increased, for example, as the number of malicious agents grows and as the variance of  $\alpha_{ij}$  grows. Figs. 7 and 8 show that as we increase the number of malicious agents and the deviation  $\ell$  that governs the variance of  $\alpha_{ij}$ , we should increase  $T_0$  to limit the effect of the malicious agents on the consensus value. Additionally, Figs. 7 and 8 show that for  $T_0 = 150$ , the deviation from true consensus value for

all considered scenarios is negligible. Finally, for  $T_0 = 100$ , the deviation from true consensus value is sufficiently small, and the effect of malicious agents is limited even for  $|\mathcal{M}| = 30$ .

Fig. 7 shows that as  $\ell$  becomes larger (and therefore also the variance), the effect of malicious agents on deviation gets worse. However, even in this case, increasing  $T_0$  can recover the nominal consensus value with deviation approaching zero. The exact relationship between the deviation  $\Delta_{\max}(T_0, \delta)$  and  $T_0$  is given by Theorem 2.

In the case of a time-varying malicious agent input as with the *drift malicious input* model described in Section IV-B, nominal average consensus can also be recovered by making  $T_0$  larger, as depicted in Fig. 8. This is true even with variable variance on the  $\alpha_{ij}$  values.

## V. DISCUSSION

In this section, we discuss the significance of some of our characterizations on the trust observations  $\alpha_{ij}$  and some of our assumptions. We also discuss directions for future work. First, the results of this article assume a fixed network topology. However, we believe that it is possible to relax Assumption 1.1 to include the case where the graph over legitimate agents is not always connected. In such a case, Assumption 1.1 can be replaced with the assumption that the topology of the legitimate agents is B-connected.<sup>3</sup> We believe that the convergence results from [64, Proposition 1] can be adapted to this case along the same vein as in Theorem 3. Finally, we discuss a few interesting observations about the trust values  $\alpha_{ij}$ . Our results show that the more informed our  $\alpha_{ij}$  values are, the tighter the performance guarantees achievable become (cf. Lemma 6). This also opens a future avenue of investigation of finding  $\alpha_{ij}$  observations from physical channels in multiagent systems, which can be characterized as fully as possible. The theory in this article provides the mathematical framework for understanding *which characteristics of  $\alpha_{ij}$*  are most critical for attaining performance guarantees that are important for multiagent coordination. We hope that this can guide continued investigation into trust-based resilience, particularly for CPSs, where additional information on interagent trust can be attained.

## VI. CONCLUSION

This article presented a unified mathematical theory to treat distributed multiagent consensus in the presence of malicious agents when stochastic values of trust between agents are available. Under this model, we presented three new performance guarantees for consensus systems in the presence of malicious agents:

- 1) Convergence of consensus is possible with probability 1 even when the number of malicious agents is far larger than 1/2 of the network connectivity—in contrast to classical results in resilience;
- 2) The deviation of the converged value from average consensus can be bounded and, further, can be driven arbitrarily close to zero by using our derived edge weights and by

allowing an observation time  $T_0$  of trust values over the network; and

- 3) Convergence to the correct classification of agents can be attained in finite time with probability 1 with an exponential rate that we derive.

Additionally, we showed that our performance guarantees can be made stronger if more information on the trust observations, such as bounds on their variance, is available. Taken together, these results point to the inherent value of quantifying and exploiting trust between agents for consensus. We believe that the mathematical formulations and framework of this article hold promise for achieving a new generation of resilient multiagent systems.

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<sup>3</sup>See definition in [29].

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