

On the Maxey–Riley equation of motion and its extension to high Reynolds numbers

Ahmad Talaei[†], and Timothy J. Garrett

Department of Atmospheric Sciences, University of Utah, UT, USA

(Received xx; revised xx; accepted xx)

The inertial response of a particle to turbulent flows is a problem of relevance to a wide range of environmental and engineering problems. The equation most often used to describe the force balance is the Maxey–Riley equation, which includes in addition to buoyancy and steady drag forces, an unsteady Basset drag force related to past particle acceleration. Here we provide a historical review of how the Maxey–Riley equation was developed and how it is only suited for studies where the Reynolds number is less than unity. Revisiting the innovative mathematical methods employed by Basset (1888), we introduce an alternative formulation for the unsteady drag for application to a broader range of particle motions. While the Basset unsteady drag is negligible at higher Reynolds numbers, the revised unsteady drag is not.

1. Introduction

Determination of a particle’s trajectory in a turbulent flow field requires an equation that satisfies the Navier–Stokes equation and accounts for all relevant forces. The first attempt was made by Stokes for a sphere moving slowly with a uniform velocity in a viscous fluid of unlimited extent that is stationary far from the particle (Stokes 1850). Boussinesq and Basset later considered the linear inertia of flow surrounding the sphere and developed an equation for the unsteady motion of a spherical particle accelerating from rest and moving with a time-varying velocity $v_p(t)$, adding an unsteady drag force or “history term” to the equation of motion that accounts for prior particle interactions with the surrounding flow (Boussinesq 1885*a,b*; Basset 1888).

In the interests of mathematical simplicity, the derivation by Boussinesq (1885*a,b*) and Basset (1888) omitted non-linear inertia terms proportional to the squares and products of velocities of the surrounding flow relative to a moving sphere. Such an assumption can be valid in the Stokes flow regime because the particle motion can be considered to be “slow”. Fluid viscous forces dominate inertia and the Reynolds number is “small”, i.e. $Re = v_p d_p / \nu < 1$ where d_p is the sphere diameter and $\nu = \mu / \rho_f$ is the kinematic viscosity of the fluid, μ the dynamic viscosity of the fluid, and ρ_f the fluid density.

The next significant advance was introduced by Tchen (1947) who generalized the equation of motion for unsteady motion of spherical particle in a fluid at rest. He proposed an equation for the motion of a slow spherical particle in a fluid that has a velocity $v_f(t)$ independent of the sphere. To reduce the problem to that of a particle moving in a fluid at rest, Tchen assumed the particle moves with a velocity $v_p(t) - v_f(t)$. In addition, he allowed for the entire system, including both the fluid and the particle to experience a pressure gradient force due to a changing rectilinear velocity of the fluid $v_f(t)$. Corrsin & Lumley (1956) later showed that if the fluid is turbulent, and the sphere is smaller than the shortest wavelength characterizing the turbulent flow, spatial and temporal

[†] Email address for correspondence: ahmad.talaei@utah.edu

inhomogeneities in the fluid also add a torque due to spatial velocity gradients, and a force due to a static pressure gradient.

Further adaptations and extensions of the equation of motion account for the drag force due to the forced velocity curvature around the sphere, or the Faxén correction, and viscous shear stress, leading to the widely used Maxey–Riley equation (Faxén 1922; Buevich 1966; Riley 1971; Soo 1975; Gitterman & Steinberg 1980; Maxey & Riley 1983). For a particle that is at rest in a stationary fluid until the instant $t = 0$, and is sufficiently small to have a negligible effect on fluid motions far from the particle, the Maxey–Riley equation accounts for the trajectory, dispersion, and settling velocity of the particle. The force balance includes the buoyancy force, the stress gradient of the fluid flow in the absence of a particle, the force due to the virtual mass, steady Stokes drag and unsteady Basset drag

$$m_p \frac{d\mathbf{v}_p}{dt} = (m_p - \rho_f V_p) \mathbf{g} + \rho_f V_p \frac{D\mathbf{v}_f}{Dt} - k \rho_f V_p \frac{d}{dt} \left(\mathbf{v}_p - \mathbf{v}_f - \frac{1}{10} a^2 \nabla^2 \mathbf{v}_f \right) - 6\pi\mu a \left(\mathbf{v}_p - \mathbf{v}_f - \frac{1}{6} a^2 \nabla^2 \mathbf{v}_f \right) - 6\pi\mu a^2 \int_0^t \frac{\frac{d}{d\tau} (\mathbf{v}_p(\tau) - \mathbf{v}_f(\tau) - \frac{1}{6} a^2 \nabla^2 \mathbf{v}_f)}{\sqrt{\pi\nu(t-\tau)}} d\tau \quad (1.1)$$

where the index p denotes the particle and f for the fluid. $\frac{d}{dt} = \frac{\partial}{\partial t} + v_{p,j} \frac{\partial}{\partial x_j}$ is the total time derivative along the particle trajectory, and $\frac{D}{Dt} = \frac{\partial}{\partial t} + v_{f,j} \frac{\partial}{\partial x_j}$ the acceleration of the fluid along its own trajectory, m_p the particle mass, v_p the Lagrangian velocity of the particle, and v_f the Eulerian fluid velocity in the particle location. $\rho_f V_p$ is the fluid mass m_f occupied by the particle volume V_p of radius a . $k = (m'_f/m_f)$ is an added mass coefficient, and m'_f the virtual mass of the fluid, assumed to undergo the same acceleration as the particle. The coefficient k is a function of the flow regime and geometric properties of the particle. For irrotational flow around a sphere k is 0.5.

No analytical solution exists for the full expression of the Maxey–Riley equation of motion. Numerically, however, the equation provides a useful guide for exploring interactions between particles and a moving fluid flow. Its application extends to fields as wide ranging as sediment transport and waste management, combustion, particle transport, and deposition, particle clustering, atmospheric precipitation, aquatic organism behaviors, and underwater robotics (Chao 1963; Soo 1975; Murray 1970; Reeks 1977; Nir & Pismen 1979; Kubie 1980; Maxey 1987, 1990; Mei 1990; Mei *et al.* 1991a; Falkovich *et al.* 2002; Peng & Dabiri 2009; Daitche 2013; Beron-Vera *et al.* 2019).

An important point is that (1.1) assumes that the Reynolds number of the particle relative to the surrounding fluid flow satisfies $Re = |\mathbf{v}_p - \mathbf{v}_f| d_p / \nu < 1$, that is the Stokes flow regime. For larger values of Re , semi-empirical adjustments are sometimes made (Ho 1933; Hwang 1985; Tunstall & Houghton 1968; Field 1968; Murray 1970; Maxey 1990; Wang & Maxey 1993; Nielsen 1993; Stout *et al.* 1995; Good *et al.* 2014). The steady drag force $F_d = 6\pi\mu a(v_p - v_f)$ shifts from scaling linearly with relative velocity to scaling as an empirically derived steady drag coefficient C_D and the relative velocity squared $F_d = \frac{1}{2}\rho_f A_p C_D(Re)(v_p - v_f)^2$. Then the steady drag at high Reynolds numbers becomes sufficiently large that the history term in (1.1) becomes negligible and can be omitted from the equation of motion (Wang & Maxey 1993; Stout *et al.* 1995; Good *et al.* 2014).

However, it remains that the mathematical form of the history term developed by Boussinesq-Basset applies only when the Reynolds number is small ($Re < 1$). A priori, there is no mathematical justification for arguing that unsteady drag is negligible compared to steady drag when the Reynolds number is arbitrarily high. While a few

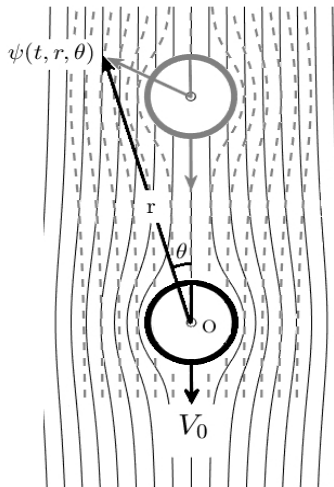


FIGURE 1. The laminar stream function around a sphere falling with constant velocity V_0 at an earlier (dashed top) and a later time t (solid bottom). The stream function changes in spatial co-ordinates r and θ with respect to a stationary observer so that it is a function of both the sphere velocity and time.

theoretical studies have considered unsteady drag on a sphere at a finite but small Reynolds number in the range $\text{Re} < 100$ (Oseen 1913; Proudman & Pearson 1957; Sano 1981; Mei *et al.* 1991b; Mei & Adrian 1992; Mei 1994; Lovalenti & Brady 1993; Michaelides 1997), as of yet, no general formulation has been presented for the unsteady drag on solid bodies moving within a viscous liquid when $\text{Re} \gg 1$. This article attempts to fill this gap by first revisiting the classical derivation of Basset's solution, and then by using a similar approach obtaining a formulation for the unsteady drag term suitable for application to higher Reynolds numbers.

2. Overview of the Stokes solution

Stokes (1850) considered a sphere of radius a falling at constant velocity V_0 under gravity along a straight axis z , considering the center of the sphere as the origin so that the motion of the fluid is symmetrical with respect to the axis of fall. Relative to the center of sphere, in a spherical coordinate system (r, θ, ϕ) where r is the radius, θ is the zenith angle, and ϕ is the azimuthal angle (Fig. 1), the u_r and u_θ components of velocities along and perpendicular to the direction of r are

$$v_r(t, r, \theta) = \frac{1}{r^2 \sin \theta} \frac{\partial \psi(t, r, \theta)}{\partial \theta} \quad (2.1)$$

$$v_\theta(t, r, \theta) = -\frac{1}{r \sin \theta} \frac{\partial \psi(t, r, \theta)}{\partial r} \quad (2.2)$$

For an incompressible fluid in an azimuthally symmetric 2D spherical coordinate system, the Navier–Stokes equations that determine the surrounding fluid flow around the moving sphere are

$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} = -\frac{1}{\rho_f} \frac{\partial p}{\partial r} + \nu \left(\nabla^2 v_r - \frac{2v_r}{r^2} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} - \frac{2v_\theta \cot \theta}{r^2} \right) \quad (2.3)$$

$$\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} = -\frac{1}{r \rho_f} \frac{\partial p}{\partial \theta} + \nu \left(\nabla^2 v_\theta + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2 \sin^2 \theta} \right) \quad (2.4)$$

Assuming the no-slip condition at the surface of sphere, at a constant fall velocity V_0 the boundary conditions are

$$v_r \Big|_{r=a} = V_0 \cos \theta, \quad v_\theta \Big|_{r=a} = -V_0 \sin \theta \quad (2.5)$$

Defining for brevity an operator

$$D = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} - \frac{\cos \theta}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \quad (2.6)$$

then using (2.1)-(2.2), the Navier–Stokes equations (2.3)-(2.4) can be rewritten in terms of the stream function as follows

$$-\frac{1}{\rho_f} \frac{\partial p}{\partial r} = \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\frac{\partial \psi}{\partial t} - \nu D \psi \right) \quad (2.7)$$

$$\frac{1}{\rho_f} \frac{\partial p}{\partial \theta} = \frac{1}{\sin \theta} \frac{\partial}{\partial r} \left(\frac{\partial \psi}{\partial t} - \nu D \psi \right) \quad (2.8)$$

Taking the derivative of (2.7) with respect to θ and the derivative of (2.8) with respect to r and eliminating the pressure term, the equation for $\psi(t, r, \theta)$ becomes

$$\underbrace{D \left(\nu D - \frac{\partial}{\partial t} \right) \psi}_{linear} + \sin \theta \underbrace{\left(\frac{\partial \psi}{\partial r} \frac{\partial}{\partial \theta} - \frac{\partial \psi}{\partial \theta} \frac{\partial}{\partial r} \right) \frac{D \psi}{r^2 \sin^2 \theta}}_{non-linear} = 0 \quad (2.9)$$

The solution to (2.9) is the stream function for a viscous and incompressible fluid surrounding a moving sphere. Note the distinction between the linear term that produces a laminar flow around a slow-moving sphere and the non-linear term that arises from retaining the velocity products and squares in (2.3)-(2.4).

In 1850, Stokes solved the linear term at steady-state, namely $D(D\psi(r, \theta)) = 0$ by switching reference frames and treating the fluid as moving with velocity V_0 relative to a stationary sphere. Therefore, by placing the origin at the center of the quiescent sphere, and supposing a solution in the form of $\psi(r, \theta) = \sin^2(\theta)f(r)$, Stokes (1850) determined the motion of a fluid for a sphere that moves slowly at a constant velocity in a fluid at rest

$$\psi(r, \theta) = \frac{1}{4} V_0 a^2 \sin^2 \theta \left(\frac{3r}{a} - \frac{a}{r} \right) \quad (2.10)$$

Stokes obtained the familiar expression $F_D = 6\pi\mu a V_0$ for the drag force of the fluid on the sphere assuming the no-slip condition at the sphere's surface. The terminal velocity of a falling sphere is then obtained from balance with the gravitational force (Appendix A).

3. Overview of Basset’s solution

Basset argued that Stokes’ formula for the terminal velocity yields values larger than those obtained by experiment. Based on his prior theoretical studies (Basset 1888), Basset (1910*a*) attributed the discrepancy to the neglect of the $\frac{\partial \psi}{\partial t}$ term in (2.9) for steady motion, suggesting that it should be replaced by $V_0 \frac{\partial \psi}{\partial z}$, and again maintaining the origin at the center of the moving sphere (Fig. 1). Stokes’ assumption that sphere starts the motion with a constant velocity V_0 also implies a discontinuity at the sphere surface. Suppose that a sphere that is set in motion with a constant velocity of V_0 . The no-slip condition requires that the fluid velocity instantly change from $\frac{1}{2}V_0 \sin\theta$ to $-V_0 \sin\theta$ (Appendix B). This discontinuity is unphysical. If instead the sphere is moving with a variable velocity $V(t)$ starting from rest then the revised linear equation to be solved is

$$\mathcal{D}\left(\nu \mathcal{D} - \frac{\partial}{\partial t}\right)\psi(t, r, \theta) = 0 \quad (3.1)$$

The solution was found first by Boussinesq (1885*b,a*) and apparently independently three years later by Basset (1888). Any more generalized analytical solution to (2.9) has yet to be determined.

Much has been written about the Basset drag force in the literature but less about how it was originally derived. Here, we revisit Basset’s solution for two reasons. His work on the problem of variable slow motion of a sphere in a viscous fluid was last published in 1888, and the innovative analytical methods he used to solve partial differential equations are not well known. Second, we extend his mathematical approach to present a revised form of the Maxey–Riley equation suitable for application to a wider range of Reynolds numbers than the Stokes regime.

The solution to (3.1) is outlined in more detail in Appendix A. Briefly, Basset’s approach to solving (3.1) for $\psi(t, r, \theta)$ was motivated by the absence of an analytical solution to the linear form of the Navier–Stokes equation for an accelerating particle. He began by first assuming that the sphere moves with constant velocity V_0 . In this case, the particular solution for the stream function around a sphere with a moving origin (3.1) is

$$\begin{aligned} \psi(t, r, \theta) = & \frac{1}{2}V_0 a^2 \sin^2\theta \left\{ \frac{3\nu t}{ra} + \frac{6\sqrt{\nu t/\pi}}{r} + \frac{a}{r} \right\} \\ & - \frac{3}{\sqrt{\pi}}V_0 a^2 \sin^2\theta \int_{\frac{r-a}{2\sqrt{\nu t}}}^{\infty} \left\{ \frac{2\xi^2 \nu t}{ra} + \frac{2\xi\sqrt{\nu t}}{r} + \frac{1}{2}\left(\frac{a}{r} - \frac{r}{a}\right) \right\} e^{-\xi^2} d\xi \end{aligned} \quad (3.2)$$

The stream function around the sphere obtained by the Basset (3.2) is laminar and its form is identical to that obtained by Stokes (2.10), as shown in Fig. 1. The difference is that the stream function is non-steady due to acceleration of the fluid around the sphere. Basset’s unsteady stream function reduces to the Stokes steady stream function at the particle surface $r = a$, and in the limit $t \rightarrow \infty$ where the integral term of Basset’s solution with $r - a/2\sqrt{\nu t}$ approaches zero. At a distance radially far from the particle surface, or for shorter times where the fluid has not yet reached a steady motion, the value of stream function calculated by Basset’s solution is greater than that found by Stokes.

By substituting (3.2) into (2.7), the solution for the fluid pressure field is

$$p(t, r, \theta) = \frac{3V_0 a \mu \cos\theta}{2r^2} \left(1 + \frac{a}{\sqrt{\pi \nu t}} \right) \quad (3.3)$$

and the fluid velocities v_r and v_θ are obtained by substituting (3.2) into (2.1) and (2.2), respectively.

The drag force owes to the upstream pressure gradient across a falling particle and the shear stress in the particle boundary layer. At the sphere surface where $r = a$, the drag force is

$$F_D = 2\pi a^2 \int_0^\pi \left\{ \left(p - 2\mu \frac{\partial v_r}{\partial r} \right) \cos\theta + \mu \left(\frac{\partial v_\theta}{\partial r} + \frac{1}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r} \right) \sin\theta \right\} \sin\theta \, d\theta$$

$$= 6\pi\mu a V_0 \left(1 + \frac{a}{\sqrt{\pi\nu t}} \right) \quad (3.4)$$

Neglecting velocity squared terms, there is a correction term to the Stokes drag. For physical insight, suppose that there is a relaxation time to the terminal velocity $\tau_p = V_0/g$ that in the Stokes flow regime is equal to $\tau_p = m_p/6\pi\mu a$, simplifying to $\tau_p = \frac{\rho_p d_p^2}{18\mu}$. The fractional addition to the Stokes drag in (3.4) varies temporally as $a/\sqrt{\pi\nu t}$, which is proportional to $\sqrt{\tau_p/t}$. Then, τ_p is the time t_{max} at which the unsteady drag is a maximum. Fluid accelerations around the particle surface exert a force on the particle that is proportional to the particle cross-section. The perturbation diffuses away from the particle as $1/\sqrt{t}$. For the case of turbulent flows, it has been suggested that the appropriate timescale to which particle relaxation time could be compared is the Kolmogorov timescale τ_η where η is the Kolmogorov length scale, in which case the fractional enhancement of unsteady drag to Stokes drag is $\sim a/\eta$ (Daitche 2015).

Effectively then, there is an extra drag force at constant V_0 that prolongs the time it takes the particle to approach its terminal velocity. For the more physical case that V is not a constant, Basset's approach was to substitute in (3.4) the time variable t with a historical time τ , and V_0 with a time-varying velocity of form $\frac{dV(t-\tau)}{dt} d\tau$, integrating the result from 0 to t . To see the justification for this substitution, consider that the transformation $\zeta = t - \tau$ leads to

$$\int_0^t \frac{dV(t-\tau)}{dt} d\tau = - \int_t^0 \frac{dV(\zeta)}{d\zeta} d\zeta$$

$$= V(t) - V(0) \quad (3.5)$$

If the sphere starts from rest, then $V(0) = 0$, and $V(\zeta)$ is finite between its limits, any integration of a time varying velocity in (3.5) will yield the current sphere velocity at time t . Basset (1888) proposed that if $V(t)$ is a solution to a partial differential equation, then the integral of $\frac{dV(t-\tau)}{dt} d\tau$ must also be a solution. The total drag force in (3.4) then becomes

$$F_D = 6\pi\mu a \left(V(t) + a \int_0^t \frac{1}{\sqrt{\pi\nu\tau}} \frac{dV(t-\tau)}{dt} d\tau \right) \quad (3.6)$$

Drag is not only a function of the current velocity but also of the particle acceleration due to prior interactions between the particle and the fluid. Basset (1910*b*) later adopted a method developed by Picciati (1907) that simplifies the procedure of first find a solution for constant velocity and then for changing velocity. Picciati's method reduces the problem to the determination of a function that satisfies Fourier's heat equation, and yields a solution equivalent to (3.6). The equation of motion for a sphere of mass m_p moving slowly with a time-varying velocity becomes

$$m_p \frac{d\mathbf{V}(t)}{dt} = (m_p - m_f)\mathbf{g} - 6\pi\mu a \left(\mathbf{V}(t) + a \int_0^t \frac{1}{\sqrt{\pi\nu(t-\tau)}} \frac{d\mathbf{V}(\tau)}{d\tau} d\tau \right) - \frac{1}{2}m_f \frac{d\mathbf{V}(t)}{dt} \quad (3.7)$$

Equation (3.7) does not consider the squares and products of flow velocities in the Navier–Stokes equation (2.3)–(2.4) and so it remains valid only for Stokes flow. It is this equation of motion that (Tchen 1947) employed to account for the effects of temporal variability in the fluid flow and that with subsequent revisions led to the Maxey–Riley equation of motion (1.1).

4. Unsteady drag at high Reynolds numbers

To determine the hydrodynamic fluid forces at higher Reynolds numbers, what is required is a particular solution to the full Navier–Stokes equations (2.3)–(2.4). This is not yet possible due to the mathematical difficulties introduced when higher order velocity terms are retained. Consequently, these terms have traditionally been either ignored or parameterized based on empirical studies. In the latter case, the steady drag on a falling sphere with velocity V_0 in a stationary and incompressible viscous fluid can then be expressed using Rayleighs formula $F_d = \frac{1}{2}\rho_f A_p C_D(Re)V_0^2$, in which case the drag force becomes

$$F_D = \frac{1}{2}\rho_f A_p C_D(Re)V_0^2 \left(1 + \frac{a}{\sqrt{\pi\nu t}} \right) \quad (4.1)$$

For example, in the Stokes flow regime, a formulation for the drag coefficient $C_D(Re) = 24/Re$ converts the Rayleigh formula to the familiar expression $F_d = 6\pi\mu a V_0$ expressed in (3.4). For higher Reynolds numbers, empirical estimates of the drag coefficient can be used.

But if a more generalized drag force is to be implemented within the context of an equation such as the Maxey–Riley equation, appropriate adjustments must be made to the equation itself. We now proceed to derive an expression for the unsteady drag at high Reynolds numbers in a manner analogous to that described in Section 3 for low Reynolds numbers. Following Basset’s approach leading to (3.6) by way of (3.5), a more general equation of motion is then

$$m_p \frac{dV(t)}{dt} = (m_p - m_f)g - \frac{1}{2}\rho_f A_p \left(C_D(Re)V^2(t) + a \int_0^t \frac{C_D(\tau)|V(\tau)|}{\sqrt{\pi\nu(t-\tau)}} \frac{dV(\tau)}{d\tau} d\tau \right) - \frac{1}{2}m_f \frac{dV(t)}{dt} \quad (4.2)$$

where the integral term expresses a more generalized unsteady drag. A possible limitation of this expression is that C_D is derived empirically for a particle moving at constant velocity, and does not account for any time variation in the drag due to acceleration. Experimental studies suggest drag coefficients that can be significantly higher (Hughes 1952; Selberg & Nicholls 1968; Igra & Takayama 1993).

What is important to note however is that within the integrand in (4.2), the particle acceleration is multiplied by the magnitude of the particle velocity, whereas with the Basset equation (3.7), it is multiplied by a constant. Therefore, for a particle falling at high velocity with a large Reynolds number, unsteady drag is not necessarily negligible

as has sometimes been assumed. Ignoring any alterations to the drag force due to forced velocity curvature around the sphere (the Faxn correction $\frac{1}{6}a^2\nabla^2 u_f$), and viscous shear stress, we propose a more general version of the Maxey–Riley equation of particle motion

$$m_p \frac{d\mathbf{v}_p}{dt} = (m_p - \rho_f V_p) \mathbf{g} + \rho_f V_p \frac{D\mathbf{v}_f}{Dt} - \frac{1}{2} \rho_f A_p \left\{ C_D(Re) |\mathbf{v}_p - \mathbf{v}_f| (\mathbf{v}_p - \mathbf{v}_f) + a \int_0^t \frac{C_D(\tau) |\mathbf{v}_p(\tau) - \mathbf{v}_f(\tau)|}{\sqrt{\pi\nu(t-\tau)}} \frac{d(\mathbf{v}_p(\tau) - \mathbf{v}_f(\tau))}{d\tau} d\tau \right\} - k\rho_f V_p \frac{d}{dt} (\mathbf{v}_p - \mathbf{v}_f) \quad (4.3)$$

where $Re = |\mathbf{v}_p(t) - \mathbf{v}_f(t)|d_p/\nu$ is the particle's relative Reynolds number.

5. Numerical analysis

The equation of motion (4.2) is now solved numerically. The particle velocity is initialized at some value close to zero, the particle's Reynolds number $Re = |\mathbf{V}(t)|d_p/\nu$ is specified, and the drag coefficient is calculated from an empirically derived relationship between the drag coefficient and the Reynolds number of a rigid sphere (White 1991)

$$C_D(Re) = \begin{cases} \frac{1}{4} + \frac{24}{Re} + \frac{6}{1+\sqrt{Re}} & \text{if } Re \leq 3,000 \\ 0.3659, & \text{otherwise} \end{cases} \quad (5.1)$$

The history term in (4.2) can be estimated except where τ approaches t , at which point the integrand becomes infinite and must be treated separately. Using the definition of an integral, the history term evolves from the previous time step ($t - \Delta t$) through

$$\int_0^t \frac{V(\tau)}{\sqrt{t-\tau}} \frac{dV(\tau)}{d\tau} d\tau = \lim_{n \rightarrow \infty} \sum_{k=1}^{n-1} \left(\frac{V(t-k\Delta t)}{\sqrt{k}} \frac{dV(t-k\Delta t)}{dt} \right) \cdot \Delta t^{1/2} \quad (5.2)$$

where t is the time of motion and Δt is the time interval employed in the simulation. The right-hand side of (5.2) is amenable to standard numerical techniques.

Equation (4.2) was solved numerically for the approach of a particle initially at rest to its terminal velocity considering both a low and high Reynolds number. Fig. 2 shows a comparison of steady and unsteady drag forces normalized by the gravity force as a function of time normalized by the particle Stokes time τ_p . For a particle with a Reynolds number of $Re = 0.2$ the generalized equation for unsteady drag, the integral term in (4.2), is equivalent to the Basset history term and is a maximum 25% of the total force when the Stokes time $t/\tau_p \simeq 1$. Its contribution to the particle acceleration is negligible as the drag turns steady and the particle approaches its terminal velocity.

For a higher Reynolds number of $Re = 1100$, the unsteady drag accounts for a maximum $\sim 15\%$ of the total force at a time much shorter than particle relaxation time τ_p while the Basset history drag plays a negligible role. Fig. 3 shows that the time at which the unsteady drag reaches its maximum value decreases logarithmically as the cube root of the particle Reynolds number $Re^{1/3}$, or that $\ln(t_{max}/\tau_p) \propto -Re^{1/3}$. Also shown is the ratio of the Basset drag to the generalized unsteady drag, which also decreases as $Re^{1/3}$, indicating a diminished relative importance of the Basset drag at higher Reynolds numbers. So while the revised unsteady drag dominates the Basset history drag, and consequently increases the drag and reduces the particle terminal velocity, relative to the Stokes time, the period over which the drag affects the particle motion is correspondingly short.

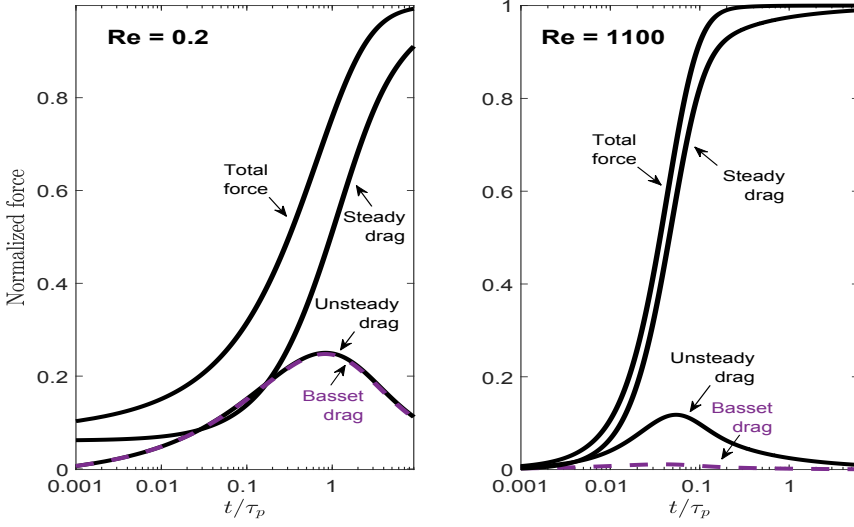


FIGURE 2. Normalized steady and unsteady drag forces acting on a particle falling into stationary air over a normalized log-time of motion, according to (4.2). The forces are normalized by gravity and the time of motion by the particle relaxation time in Stokes flow τ_p . The normalized total force is also shown. Left) particle with a density ratio of $s = 15$ and a low Reynolds number of $Re = 0.2$ Right) particle with a density ratio of $s = 830$ and a Reynolds number of $Re = 1100$. The dashed purple line shows for comparison the unsteady Basset drag calculated using the Maxey–Riley equation of motion (1.1).

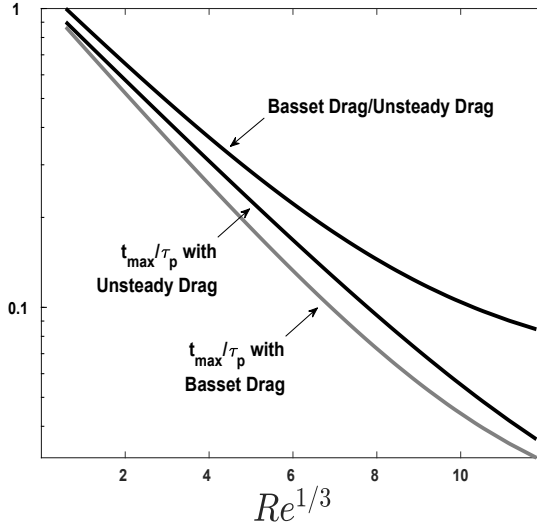


FIGURE 3. Time relative to the Stokes time τ_p at which the unsteady drag is a maximum, and the corresponding ratio of the Basset drag to the unsteady drag, as a function of the cube root of the Reynolds number.

6. Discussion

There remain some important limitations to (4.2). First, there is an implicit assumption that the particle starts from rest. While nonetheless assuming Stokes flow, Basset (1888) developed a rather more complicated equation of motion for a particle initially projected vertically with velocity V_i (for derivation, see Appendix B)

$$\frac{d\mathbf{V}(t)}{dt} = -\lambda \mathbf{V}_i e^{-\lambda t} + f \mathbf{g} e^{-\lambda t} - \lambda a \frac{d}{dt} \int_0^t \int_0^v \frac{e^{-\lambda(t-v)}}{\sqrt{\pi\nu(v-\tau)}} \frac{d\mathbf{V}(\tau)}{d\tau} d\tau dv \quad (6.1)$$

The coefficient $f = \frac{m_p - m_f}{m_p + \frac{1}{2}m_f}$ simplifies to $f = \frac{\rho_p - \rho_f}{\rho_p + \frac{1}{2}\rho_f}$, and $\lambda = \frac{6\pi\mu a}{m_p + \frac{1}{2}m_f}$ to $\lambda = \frac{9\rho_f\nu}{2a^2(\rho_p + \frac{1}{2}\rho_f)}$. Basset was unable to integrate this complicated integro-differential equation, but for the limited case of $\lambda \ll 1$, as applies to a sphere moving in a fluid whose kinematic viscosity is small, he used a method of successive approximation to obtain the acceleration and velocity to the third power in λ .

Later, Boggio (1907) successfully reduced the complexity of the problem to a solvable second order differential equation (see Appendix C). The solution employs error functions of form $\text{erf}(\sqrt{\alpha}t)$ and $\text{erf}(\sqrt{\beta}t)$ where $\alpha, \beta = \frac{\lambda}{2}\{(q-2) \pm \sqrt{q(q-4)}\}$ and $q = \lambda a^2/\nu$. Substituting this expression for λ yields $q = \frac{9\rho_f}{2\rho_p + \rho_f}$. For a particle denser than the fluid then $q < 4$, and α and β are complex numbers. For this case,

$$V(t) = \frac{fg}{\lambda} + \left(V_i - \frac{fg}{\lambda}\right)e^{\gamma t} \left\{ \cos(\delta t) - \frac{\gamma + \lambda}{\delta} \sin(\delta t) \right\} - \frac{h}{\delta} e^{\gamma t} \left\{ \cos(\delta t) \int_0^t \frac{e^{-\gamma t} \sin(\delta t)}{\sqrt{t}} dt - \sin(\delta t) \int_0^t \frac{e^{-\gamma t} \cos(\delta t)}{\sqrt{t}} dt \right\} \quad (6.2)$$

where $\gamma = \frac{\lambda}{2}(q-2) = -\frac{\lambda}{2}\left(\frac{4\rho_p - 7\rho_f}{2\rho_p + \rho_f}\right)$, $\delta = \frac{\lambda}{2}\sqrt{q(4-q)} = \frac{\lambda}{2}q^{\frac{1}{2}}\sqrt{\frac{8\rho_p - 5\rho_f}{2\rho_p + \rho_f}}$, and $h = \frac{\lambda a}{\sqrt{\pi\nu}}(fg - \lambda V_i)$. This equation is not widely known but it significantly reduces the computational expense of finding a solution for $V(t)$ by eliminating the requirement of tracking the history of the particle's motion.

A second, more troubling limitation of (6.1)-(6.2), and hence also of (3.7) and (4.2), is that for a particle starting at $t = 0$ with a finite vertical velocity, the effect of the initial velocity (or any disturbance to the flow field surrounding the sphere) on the eventual particle displacement does not decay to zero at infinite time. The end result is that the terminal velocity differs from that expected from the Stokes solution. While the effect is small, it nonetheless implies the unphysical property of infinite memory in a dissipative viscous fluid (Reeks & McKee 1984).

To resolve this issue, Sano (1981) applied a matching procedure initially developed by Bentwich & Miloh (1978) to unsteady low Reynolds number flow past a sphere to find that the drag decays faster than $t^{-1/2}$ when $t \gg \tau_p$. Thus, the temporal dependence of the Basset drag is only appropriate at times less than τ_p when inertial forces are low compared to viscous forces. A similar conclusion was reached by Mei *et al.* (1991b). Mei & Adrian (1992) applied a successive orders of approximation method to solve the Navier-Stokes equation to $O(\text{Re})$ for the case of oscillating flow over a sphere, by considering small fluctuations in velocity when the Reynolds number is not negligibly small. Mei & Adrian (1992) then proposed a modified expression for the unsteady drag that includes an integration kernel that decays as t^{-2} for $t \gg \tau_p$, limited to finite Reynolds numbers

($\text{Re} \leq 100$) and small-amplitude fluctuations in the velocity of the free stream. Mei (1994) later investigated the applicability of the kernel for other types of unsteady flows.

Mainardi (1997) went further to interpret the Basset force in terms of a fractional derivative of any order ℓ ranging in the interval $0 < \ell < 1$ as

$$F_D = \frac{9}{2}m_f \left(\frac{1}{\tau_0} + \frac{1}{\tau_0^{1-\ell}} \frac{d^\ell}{dt^\ell} \right) V(t) \quad (6.3)$$

where $\tau_0 = a^2/\nu$ represents the characteristic time to reach steady-state in a viscous fluid. $\ell = 1/2$ yields the total Basset drag in (3.6). This generalization, suggested by mathematical speculation, modifies the behaviour of the solution, changing its decay from $t^{-1/2}$ to t^ℓ for $t \gg \tau_p$. Mainardi (1997) considered three cases of $\ell = 1/4, 1/2$, and $3/4$ and compared the particle terminal velocity with a desired temporal adjustment behavior e^{-t/τ_p} expected from Stokes drag ($\ell = 0$). The results yielded improved agreement with the Stokes solution but the topic is still considered unsolved, as ideally it requires a full solution to the Navier–Stokes equations, including non-linear inertia terms involving the products of velocities.

7. Conclusions

The Maxey–Riley equation was originally developed for the study of small, slow-moving spheres but is widely used for higher Reynolds numbers under the assumption that unsteady Basset drag is insignificant relative to the steady drag. Here we have presented a historical review of the derivation of the equation of motion that leads to the Maxey–Riley equation and argue that the Basset drag can be suitably applied only when Reynolds numbers are small. Following Basset’s original approach, but considering drag proportional to the particle relative velocity squared, a revised analytical equation is developed for extension to higher Reynolds numbers. Simulations based on this equation show that the unsteady drag force contributes substantially to the total drag at timescales less than the Stokes time, even for high values of the Reynolds number.

Acknowledgments

This work is supported by the U.S. Department of Energy (DOE) Atmospheric System Research program award number DE-SC0016282 and the National Science Foundation (NSF) Physical and Dynamic Meteorology program award number 1841870.

Appendix A. Basset’s solution

Basset (1888) assumed a sphere of radius a moving slowly in a fluid with a uniform velocity V_0 , with its center at the origin moving in a straight line, surrounded by a viscous liquid that is initially at rest. As described in Section 3, the stream function must satisfy the linearized Navier–Stokes equation

$$\mathcal{D} \left(\mathcal{D} - \frac{1}{\nu} \frac{\partial}{\partial t} \right) \psi = 0 \quad (\text{A } 1)$$

where in a spherical coordinate system (r, θ, ϕ)

$$\mathcal{D} = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} - \frac{\cos \theta}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \quad (\text{A } 2)$$

Since the operators D and $(D - \frac{1}{\nu} \frac{\partial}{\partial t})$ are commutative, following Stokes (1850), Basset's solution to (A 1) was (Basset 1888):

$$\psi(t, r, \theta) = \psi_1(t, r, \theta) + \psi_2(t, r, \theta) \quad (\text{A } 3)$$

where $\psi_1(t, r, \theta)$ and $\psi_2(t, r, \theta)$ satisfy, respectively

$$\begin{cases} D\psi_1 = 0 \\ (D - \frac{1}{\nu} \frac{\partial}{\partial t})\psi_2 = 0 \end{cases} \quad (\text{A } 4)$$

Assuming the no-slip condition at the surface of a rigid falling sphere, the boundary conditions at the surface of sphere satisfying (2.1)-(2.2) are

$$\left. \frac{\partial \psi}{\partial \theta} \right|_{r=a} = V_0 a^2 \sin \theta \cos \theta, \quad \left. \frac{\partial \psi}{\partial r} \right|_{r=a} = V_0 a \sin^2 \theta \quad (\text{A } 5)$$

It is important to mention that this last boundary condition shows that θ must appear in $\psi(t, r, \theta)$ in the form of the factor $\sin^2 \theta$. Basset used separation of variables $\psi(t, r, \theta) = \psi(t, r) \sin^2 \theta$ satisfying (A 4), to obtain

$$\begin{aligned} \frac{\partial^2 \psi_1}{\partial r^2} - \frac{2\psi_1}{r^2} &= 0 \\ \frac{\partial^2 \psi_2}{\partial r^2} - \frac{2\psi_2}{r^2} &= \frac{1}{\nu} \frac{\partial \psi_2}{\partial t} \end{aligned} \quad (\text{A } 6)$$

The particular solution of $\psi_1(t, r)$ is $f(t)/r$. In an innovative approach, Basset assumed a solution of form

$$\psi_1 = \frac{1}{2r} \sqrt{\frac{\pi}{\nu t}} \int_0^\infty \kappa(\rho) e^{-\rho^2/4\nu t} d\rho \quad (\text{A } 7)$$

Provided the solution satisfies the boundary conditions, there is no restriction on how ψ_1 varies with time. For this purpose Basset chose a Gaussian distribution $\kappa(\rho) e^{-\rho^2/4\nu t}$. Note that ρ has units of length. It is zero by definition at the particle surface and increases to infinity far from the particle. $\kappa(\rho)$ is an arbitrary function to be established. To find the solution for ψ_2 , Basset used separation of variables $\psi_2(t, r) = T(t)R(r)$ so that

$$\frac{1}{R(r)} \frac{d^2 R(r)}{dr^2} - \frac{2}{r^2} = \frac{1}{\nu T(t)} \frac{d^2 T(t)}{dt^2} \quad (\text{A } 8)$$

The LHS is strictly a function of r and RHS of t , so for any range of time and space integration, both are equal and therefore can be assigned to an arbitrary constant $-m^2$ where the value of m can be specified from a particular set of boundary conditions. Note that the solution for the functionality in $R(r)$ is spherical Bessel functions of the first and second kind that can be written in terms of Rayleigh's formulas

$$R(r) = Ar \frac{d}{dr} \frac{\cos(mr)}{r} + Br \frac{d}{dr} \frac{\sin(mr)}{r} \quad (\text{A } 9)$$

With respect to the time functionality

$$T(t) = C e^{-m^2 \nu t} \quad (\text{A } 10)$$

Basset expressed a particular solution to ψ_2 in (A 6) through use of boundary conditions (A 5) at the particle surface

$$\psi_2 = Ar \frac{d}{dr} \frac{e^{-m^2 \nu t}}{r} \cos[m(r - a + \rho)] \quad (\text{A } 11)$$

Integrating (A 11) with respect to m between the limits 0 and ∞ to consider all the possible values of m , exchanging A with $A(\rho)$, and integrating the results with respect to ρ between the same limits yields:

$$\psi_2 = \frac{r}{2} \sqrt{\frac{\pi}{\nu t}} \frac{d}{dr} \int_0^\infty \frac{A(\rho)}{r} e^{-\frac{(r-a+\rho)^2}{4\nu t}} d\rho \quad (\text{A } 12)$$

First differentiating and then integrating by parts $\int u dv = uv - \int v du$ where $u = A(\rho)/r$, Basset obtained

$$\begin{aligned} \psi_2 = \frac{r}{2} \sqrt{\frac{\pi}{\nu t}} \int_0^\infty & -\frac{A(\rho)}{r^2} e^{-\frac{(r-a+\rho)^2}{4\nu t}} d\rho + \left[\frac{A(\rho)}{r} e^{-\frac{(r-a+\rho)^2}{4\nu t}} \right]_0^\infty \\ & - \frac{r}{2} \sqrt{\frac{\pi}{\nu t}} \int_0^\infty \frac{1}{r} \frac{dA(\rho)}{d\rho} e^{-\frac{(r-a+\rho)^2}{4\nu t}} d\rho \end{aligned}$$

As $A(\rho)$ is a function of arbitrary form, it is possible to define it in such a way that the term in brackets can be eliminated at both limits, namely $A(0) = 0$ and $A(\rho)e^{-\rho^2} \rightarrow 0$ when $\rho \rightarrow \infty$. Thus, the total solution for $\psi(t, r, \theta)$ becomes

$$\begin{aligned} \psi(t, r, \theta) = & \frac{\sin^2 \theta}{2r} \sqrt{\frac{\pi}{\nu t}} \int_0^\infty \kappa(\rho) e^{-\rho^2/4\nu t} d\rho \\ & - \frac{\sin^2 \theta}{2} \sqrt{\frac{\pi}{\nu t}} \int_0^\infty \left\{ \frac{A(\rho)}{r} + \frac{dA(\rho)}{d\rho} \right\} e^{-\frac{(r-a+\rho)^2}{4\nu t}} d\rho \end{aligned} \quad (\text{A } 13)$$

At the particle surface $r = a$, the exponential term in the second integral can be expressed in the form of the first integral. Thereby, the boundary condition $\left. \frac{\partial \psi}{\partial \theta} \right|_{r=a} = V_0 a^2 \sin \theta \cos \theta$ is satisfied in (A 13) if

$$\kappa(\rho) - A(\rho) - a \frac{dA(\rho)}{d\rho} = V_0 a^3 / \pi \quad (\text{A } 14)$$

Also, the boundary condition $\left. \frac{\partial \psi}{\partial r} \right|_{r=a} = V_0 a \sin^2 \theta$ requires that

$$\begin{aligned} & -\frac{1}{2} \sqrt{\frac{\pi}{\nu t}} \int_0^\infty \kappa(\rho) e^{-\rho^2/4\nu t} d\rho + \frac{1}{2} \sqrt{\frac{\pi}{\nu t}} \int_0^\infty A(\rho) e^{-\rho^2/4\nu t} d\rho \\ & - \frac{a}{2} \sqrt{\frac{\pi}{\nu t}} \int_0^\infty \left\{ A(\rho) + a \frac{dA(\rho)}{d\rho} \right\} \frac{d}{d\rho} e^{-\rho^2/4\nu t} d\rho = V_0 a^3 \end{aligned}$$

Integrating by parts the last term on the LHS, and assuming that $[A(\rho) + aA'(\rho)]e^{-\rho^2/4\nu t}$ disappears where $\rho = 0$ and $\rho \rightarrow \infty$, what is required is that $A(0) = A'(0) = 0$ and that $A(\rho)e^{-\rho^2/4\nu t}$ and $A'(\rho)e^{-\rho^2/4\nu t}$ should each vanish when $\rho \rightarrow \infty$. Then, equation (A 15) becomes

$$\frac{1}{2}\sqrt{\frac{\pi}{\nu t}} \int_0^\infty \left\{ -\kappa(\rho) + A(\rho) + a \frac{dA(\rho)}{d\rho} + a^2 \frac{d^2 A(\rho)}{d\rho^2} \right\} e^{-\rho^2/4\nu t} d\rho = V_0 a^3 \quad (\text{A } 15)$$

which is satisfied if

$$-\kappa(\rho) + A(\rho) + a \frac{dA(\rho)}{d\rho} + a^2 \frac{d^2 A(\rho)}{d\rho^2} = 2V_0 a^3 / \pi \quad (\text{A } 16)$$

Hence, we have $A(\rho) = 3V_0 a \rho^2 / 2\pi + C_1 a + C_2$, where the condition $A(0) = A'(0) = 0$ requires that $C_1 = C_2 = 0$ and $\kappa(\rho) = V_0 a \pi^{-1} (\frac{3}{2}\rho^2 + 3\rho a + a^2)$. Then, the particular solution for the stream function (A 13) around a sphere moving with a uniform velocity V_0 is:

$$\begin{aligned} \psi(t, r, \theta) = & \frac{V_0 a \sin^2 \theta}{2r\sqrt{\pi\nu t}} \int_0^\infty \left(\frac{3}{2}\rho^2 + 3\rho a + a^2 \right) e^{-\rho^2/4\nu t} d\rho \\ & - \frac{3V_0 a \sin^2 \theta}{2\sqrt{\pi\nu t}} \int_0^\infty \left\{ \frac{\rho^2}{2r} + \rho \right\} e^{-\frac{(r-a+\rho)^2}{4\nu t}} d\rho \end{aligned} \quad (\text{A } 17)$$

The first term is in the form of an exponential integral, and the second can be solved through the substitution $r - a - \rho = 2\zeta\sqrt{\nu t}$, from which Basset finally obtained

$$\begin{aligned} \psi(t, r, \theta) = & \frac{1}{2} V_0 a^2 \sin^2 \theta \left\{ \frac{3\nu t}{ra} + \frac{6\sqrt{\nu t/\pi}}{r} + \frac{a}{r} \right\} \\ & - \frac{3}{\sqrt{\pi}} V_0 a^2 \sin^2 \theta \int_{\frac{r-a}{2\sqrt{\nu t}}}^\infty \left\{ \frac{2\xi^2 \nu t}{ra} + \frac{2\xi\sqrt{\nu t}}{r} + \frac{1}{2} \left(\frac{a}{r} - \frac{r}{a} \right) \right\} e^{-\xi^2} d\xi \end{aligned} \quad (\text{A } 18)$$

This is Basset's solution to (A 1). It satisfies the Navier–Stokes equation (2.3)–(2.4) provided the advection terms involving velocity products and squares are omitted. At $t = 0$, the integral vanishes in (A 18), and the initial value of ψ becomes

$$\psi(r, \theta) = \frac{V_0 a^3 \sin^2 \theta}{2r} \quad (\text{A } 19)$$

which is the known solution of ψ for the case of a frictionless liquid. When t is very large, one may substitute $t \rightarrow \infty$ in the lower limit of the integral in (A 18) leading to

$$\psi(r, \theta) = \frac{1}{4} V_0 a^2 \sin^2 \theta \left(\frac{3r}{a} - \frac{a}{r} \right) \quad (\text{A } 20)$$

which is Stokes' solution for the motion of slowly moving rigid sphere in a viscous liquid after sufficient time has elapsed for the motion to become steady. One thing to note is that Stokes' steady-state solution for the stream function does not contain any expression of viscosity implying that the solution only applies to highly viscous liquids like water.

Appendix B. Basset's equation of motion

Zeleny & McKeehan (1910) tested Stokes' formula for the terminal velocity of small spherical spores descending in the air under gravity, and they found that the value of terminal velocity calculated by resistance expressed by Stokes's solution yields values much larger than those obtained by their experiment. As mentioned in Section 3, Basset

(1910a) proposed that with respect to a moving origin, the term $\frac{\partial \psi}{\partial t}$ should be replaced by $V_0 \frac{\partial \psi}{\partial z}$. Although, Basset was unable to obtain a complete solution for steady motion due to the difficulty of obtaining appropriate boundary conditions, his more general solution is nonetheless quite different from that given by Stokes (A 20).

Stokes' solution also ignores inertia in the disturbed fluid flow, which alters the boundary conditions very far from the sphere (Oseen 1910). For a sphere that starts from rest in a stationary fluid, the hypothesis of no-slip condition holds at the surface of the sphere because both sphere and fluid have zero initial velocities. However, for a sphere that is set in motion with a constant velocity of V_0 , there exists an initial motion of the fluid at the surface of the sphere due to an impulse needed to begin the sphere's motion. From the theory of impulsive motion, the fluid can be assumed to be frictionless at the beginning of the motion. The tangential velocity of the fluid from the equation for a frictionless fluid (A 19) becomes

$$\Theta(r, \theta) = \frac{V_0 a^2 \sin \theta}{2r^2} \quad (\text{B } 1)$$

Which at the surface of sphere is $\frac{1}{2}V_0 \sin \theta$. The velocity components of the fluid along and perpendicular to the radius vector that satisfies the boundary conditions at the surface of the sphere applying the no-slip condition (A 5) are

$$\begin{aligned} R(r, \theta) &= \frac{1}{a^2 \sin \theta} \frac{\partial \psi}{\partial \theta} = V_0 \cos \theta \\ \Theta(r, \theta) &= -\frac{1}{a \sin \theta} \frac{\partial \psi}{\partial r} = -V_0 \sin \theta \end{aligned} \quad (\text{B } 2)$$

Therefore, the consequence of the no-slip condition at the surface of sphere is that the initial velocity of fluid suddenly changes from $\frac{1}{2}V_0 \sin \theta$ to $-V_0 \sin \theta$. This discontinuity of velocity has no physical interpretation. Although the initial motion of the fluid in the neighborhood of the sphere is highly turbulent and that it gradually subsides through the action of viscosity, but the consequence of no-slip condition is that the tangential velocity of the fluid is discontinuous at the surface of sphere. Basset proposed a simple procedure to suppose that the sphere is moving with a variable velocity $V(t)$ starting from rest.

As outlined in Section 3, Basset (1888) developed the equation of motion for a sphere of mass m_p that starts the motion from rest and then moves slowly with a time-varying velocity

$$(m_p + \frac{1}{2}m_f) \frac{d\mathbf{V}(t)}{dt} = (m_p - m_f)\mathbf{g} - 6\pi\mu a \left(\mathbf{V}(t) + a \int_0^t \frac{1}{\sqrt{\pi\nu(t-\tau)}} \frac{d\mathbf{V}(\tau)}{d\tau} d\tau \right) \quad (\text{B } 3)$$

which can be simplified to

$$\frac{d\mathbf{V}(t)}{dt} = f\mathbf{g} - \lambda \mathbf{V}(t) - \frac{\lambda a}{\sqrt{\pi\nu}} \int_0^t \frac{1}{\sqrt{t-\tau}} \frac{d\mathbf{V}(\tau)}{d\tau} d\tau \quad (\text{B } 4)$$

where coefficient $\lambda = \frac{6\pi\mu a}{m_p + \frac{1}{2}m_f}$ which thereby simplifies to $\lambda = \frac{9\rho_f\nu}{2a^2(\rho_p + \frac{1}{2}\rho_f)}$, and $f = \frac{m_p - m_f}{m_p + \frac{1}{2}m_f}$.

For a sphere set in motion with an initial velocity V_0 , Basset divided the time t into two intervals ϵ and $t - \epsilon$, where ϵ is a vanishing infinitesimal. In the first interval, the

sphere starts to fall from rest due to gravity and a momentary external force that is large and constant. The value of the external force $(m_p + \frac{1}{2}m_f)X$ leads to a velocity V_i at the end of the time interval so that $V_i = X\epsilon$. X is the acceleration due to the momentary external force.

Replacing $f\mathbf{g}$ by $f\mathbf{g} + \mathbf{X}$, multiplying by $e^{\lambda t}$, and integrating between the limits t and 0 of (B 4), Basset obtained

$$\mathbf{V}(t) e^{\lambda t} = \int_0^\epsilon \mathbf{X} e^{\lambda v} dv + f \int_0^t \mathbf{g} e^{\lambda v} dv - \frac{\lambda a}{\sqrt{\pi\nu}} \int_0^t \int_0^v \frac{e^{\lambda v}}{\sqrt{v-\tau}} \frac{d\mathbf{V}(\tau)}{d\tau} d\tau dv \quad (\text{B } 5)$$

Note that integrating by parts $\int u dv = uv - \int v du$ where $u = e^{\lambda v}$ is applied to the LHS in (B 4), which cancels the second term in the RHS. $d\mathbf{V}/dt$ consists of two components; a large one that is a function of \mathbf{X} equal to \mathbf{V}_i/ϵ , and a second that is a function of $f\mathbf{g}$ and we continue to denote by $d\mathbf{V}/dt$. Hence equation (B 5) becomes

$$\mathbf{V}(t) e^{\lambda t} = \frac{\mathbf{X}}{\lambda} (e^{\lambda\epsilon} - 1) + \frac{f\mathbf{g}}{\lambda} (e^{\lambda t} - 1) - \frac{\lambda a}{\sqrt{\pi\nu}} \int_0^t \int_0^v \frac{e^{\lambda v}}{\sqrt{v-\tau}} \left(\frac{d\mathbf{V}(\tau)}{d\tau} + \frac{\mathbf{V}_i}{\epsilon} \right) d\tau dv \quad (\text{B } 6)$$

The second part of integral becomes

$$\int_0^\epsilon \frac{2\mathbf{V}_i}{\epsilon} v^{1/2} e^{\lambda v} dv = 0, \text{ when } \epsilon \rightarrow 0 \quad (\text{B } 7)$$

Also, in the limit when ϵ vanishes, the first term in RHS of (B 6) becomes

$$\lim_{\epsilon \rightarrow 0} \frac{\mathbf{V}_i}{\lambda\epsilon} (e^{\lambda\epsilon} - 1) = \mathbf{V}_i \quad (\text{B } 8)$$

so the equation of motion becomes

$$\mathbf{V}(t) = \mathbf{V}_i e^{-\lambda t} + \frac{f\mathbf{g}}{\lambda} (1 - e^{-\lambda t}) - \frac{\lambda a}{\sqrt{\pi\nu}} \int_0^t \int_0^v \frac{e^{-\lambda(t-v)}}{\sqrt{v-\tau}} \frac{d\mathbf{V}(\tau)}{d\tau} d\tau dv \quad (\text{B } 9)$$

Note that the value of the acceleration is then

$$\frac{d\mathbf{V}(t)}{dt} = -\lambda \mathbf{V}_i e^{-\lambda t} + f\mathbf{g} e^{-\lambda t} - \lambda a \frac{d}{dt} \int_0^t \int_0^v \frac{e^{-\lambda(t-v)}}{\sqrt{\pi\nu(v-\tau)}} \frac{d\mathbf{V}(\tau)}{d\tau} d\tau dv \quad (\text{B } 10)$$

Appendix C. Solution to the Basset's equation of motion

Boggio (1907) successfully integrated the equation of motion (B 4), and obtained an analytical solution to the particle velocity. The method employed by Boggio depends on the Abel integral equation. Let

$$\int_0^t \frac{1}{\sqrt{t-\tau}} \frac{dV(\tau)}{d\tau} d\tau = \phi(t) \quad (\text{C } 1)$$

for which a unique solution is

$$\int_0^t \frac{\phi(\tau)}{\sqrt{t-\tau}} d\tau = \pi[V(t) - V(0)] \quad (\text{C } 2)$$

Substituting the definite integral (C 1) in (B 4), multiplying by $(t - \tau)^{-1/2}$, integrating from 0 to t , and then eliminating the integral and differentiating with respect to t , Boggio reduced the complexity of the problem to the second order differential equation

$$\frac{d^2 V(t)}{dt^2} + \lambda \left(2 - \frac{\lambda a^2}{\nu} \right) \frac{dV(t)}{dt} + \lambda^2 V(t) + \frac{h}{\sqrt{t}} - fg\lambda = 0 \quad (\text{C } 3)$$

where the coefficient $h = \frac{\lambda a}{\sqrt{\pi\nu}}(fg - V_i\lambda)$. Basset (1910*b*) summarised Boggio's work and expressed the solution to the equation of motion (C 3) as

$$V(t) = \frac{fg}{\lambda} + \left\{ A - h\sqrt{\frac{\pi}{\alpha}} \operatorname{erf}(\sqrt{\alpha t}) \right\} \frac{e^{\alpha t}}{\alpha - \beta} + \left\{ B + h\sqrt{\frac{\pi}{\beta}} \operatorname{erf}(\sqrt{\beta t}) \right\} \frac{e^{\beta t}}{\alpha - \beta} \quad (\text{C } 4)$$

where coefficients $A = (fg - \lambda V_i)(1 + \beta/\lambda)$ and $B = -(fg - \lambda V_i)(1 + \alpha/\lambda)$ are determined by the initial conditions $V(0) = V_i$, and from (3.7), $\frac{dV(0)}{dt} = fg - \lambda V_i$. Also, the coefficient $h = \frac{\lambda a}{\sqrt{\pi\nu}}(fg - \lambda V_i)$, and $\alpha, \beta = \frac{\lambda}{2}\{(q - 2) \pm \sqrt{q(q - 4)}\}$ where $q = \lambda a^2/\nu$. When $V_i = 0$, the equation of motion simplifies to

$$V(t) = \frac{fg}{\lambda} + \frac{fg}{\alpha - \beta} \left\{ \frac{\lambda a}{\sqrt{\nu\alpha}} \operatorname{erfc}(\sqrt{\alpha t}) e^{\alpha t} - \frac{\lambda a}{\sqrt{\nu\beta}} \operatorname{erfc}(\sqrt{\beta t}) e^{\beta t} \right\} \quad (\text{C } 5)$$

This is a solution for the equation of motion (B 4) of a small sphere that is initially at rest and falls due to gravity in an infinite fluid that is also initially at rest.

REFERENCES

- BASSET, ALFRED BARNARD 1888 *A treatise on hydrodynamics: with numerous examples*, , vol. 1-2. Deighton, Bell and Company.
- BASSET, ALFRED BARNARD 1910*a* The descent of a sphere in a viscous liquid. *Nature* **83** (2122), 521–521.
- BASSET, ALFRED BARNARD 1910*b* On the descent of a sphere in a viscous liquid. *The Quarterly journal of pure and applied mathematics* **41**, 369–381.
- BENTWICH, MICHAEL & MILOH, TOUVIA 1978 The unsteady matched stokes-oseen solution for the flow past a sphere. *Journal of Fluid Mechanics* **88** (1), 17–32.
- BERON-VERA, FRANCISCO J, OLASCOAGA, MARIA J & MIRON, P 2019 Building a maxey–riley framework for surface ocean inertial particle dynamics. *Physics of Fluids* **31** (9), 096602.
- BOGGIO, TOMMASO 1907 Integrazione dell'equazione funzionale che regge la caduta di una sfera in un liquido viscoso. *Accademia dei Lincei* **16** (2), 620–620 (Nota I) and 730–737 (Nota II).
- BOUSSINESQ, JOSEPH 1885*a* *Application des potentiels l'étude de l'équilibre et du mouvement des solides lastiques*, , vol. 4. Gauthier-Villars.
- BOUSSINESQ, JOSEPH 1885*b* Sur la rsistance qu'oppose un fluide indfini en repos, sans pesanteur, au mouvement vari d'une sphre solide qu'il mouille sur toute sa surface, quand les vitesses restent bien continues et assez faibles pour que leurs carrs et produits soient ngligeables. *CR Acad. Sc. Paris* **100**, 935–937.
- BUEVICH, YU A 1966 Motion resistance of a particle suspended in a turbulent medium. *Fluid Dynamics* **1** (6), 119–119.

- CHAO, BT 1963 *Turbulent transport behavior of small particles in dilute suspension*. University of Illinois.
- CORRSIN, SE & LUMLEY, J 1956 On the equation of motion for a particle in turbulent fluid. *Applied Scientific Research, Section A* **6** (2-3), 114–116.
- DAITCHE, ANTON 2013 Advection of inertial particles in the presence of the history force: Higher order numerical schemes. *Journal of Computational Physics* **254**, 93–106.
- DAITCHE, ANTON 2015 On the role of the history force for inertial particles in turbulence. *Journal of Fluid Mechanics* **782**, 567–593.
- FALKOVICH, G, FOUXON, A & STEPANOV, MG 2002 Acceleration of rain initiation by cloud turbulence. *Nature* **419** (6903), 151–154.
- FAXÉN, HILDING 1922 Der widerstand gegen die bewegung einer starren kugel in einer zähen flüssigkeit, die zwischen zwei parallelen ebenen wänden eingeschlossen ist. *Annalen der Physik* **373** (10), 89–119.
- FIELD, WALTER G 1968 Effects of density ratio on sedimentary similitude. *Journal of the Hydraulics Division* **94** (3), 705–720.
- GITTERMAN, M & STEINBERG, V 1980 Memory effects in the motion of a suspended particle in a turbulent fluid. *The Physics of Fluids* **23** (11), 2154–2160.
- GOOD, GH, IRELAND, PJ, BEWLEY, GP, BODENSCHATZ, E, COLLINS, LR & WARHAFT, Z 2014 Settling regimes of inertial particles in isotropic turbulence. *Journal of Fluid Mechanics* **759**.
- HO, HAU-WONG 1933 Fall velocity of a sphere in a field of oscillating fluid. *PhD Dissertation, State University of Iowa*.
- HUGHES, RR 1952 Er gilliland.. *The Mechanics of Drops, Chemical Engineering Progress* **48** (10), 497–504.
- HWANG, PAUL A 1985 Fall velocity of particles in oscillating flow. *Journal of Hydraulic Engineering* **111** (3), 485–502.
- IGRA, O & TAKAYAMA, K 1993 Shock tube study of the drag coefficient of a sphere in a non-stationary flow. *Proceedings of the Royal Society of London. Series A: Mathematical and Physical Sciences* **442** (1915), 231–247.
- KUBIE, J 1980 Settling velocity of droplets in turbulent flows. *Chemical Engineering Science* **35** (8), 1787–1794.
- LOVALENTI, PHILLIP M & BRADY, JOHN F 1993 The force on a bubble, drop, or particle in arbitrary time-dependent motion at small reynolds number. *Physics of Fluids A: Fluid Dynamics* **5** (9), 2104–2116.
- MAINARDI, FRANCESCO 1997 Fractional calculus. In *Fractals and fractional calculus in continuum mechanics*, pp. 291–348. Springer.
- MAXEY, MR 1987 The gravitational settling of aerosol particles in homogeneous turbulence and random flow fields. *Journal of Fluid Mechanics* **174**, 441–465.
- MAXEY, MR 1990 On the advection of spherical and non-spherical particles in a non-uniform flow. *Phil. Trans. R. Soc. Lond. A* **333** (1631), 289–307.
- MAXEY, MARTIN R & RILEY, JAMES J 1983 Equation of motion for a small rigid sphere in a nonuniform flow. *The Physics of Fluids* **26** (4), 883–889.
- MEI, RENWEI 1990 Particle dispersion in isotropic turbulence and unsteady particle dynamics at finite reynolds number. *University of Illinois at Urbana-Champaign, Urbana, Illinois*.
- MEI, RENWEI 1994 Flow due to an oscillating sphere and an expression for unsteady drag on the sphere at finite reynolds number. *Journal of Fluid Mechanics* **270**, 133–174.
- MEI, RENWEI & ADRIAN, RONALD J 1992 Flow past a sphere with an oscillation in the free-stream velocity and unsteady drag at finite reynolds number. *Journal of Fluid Mechanics* **237**, 323–341.
- MEI, RENWEI, ADRIAN, RONALD J & HANRATTY, THOMAS J 1991a Particle dispersion in isotropic turbulence under stokes drag and basset force with gravitational settling. *Journal of Fluid Mechanics* **225**, 481–495.
- MEI, RENWEI, LAWRENCE, CHRISTOPHER J & ADRIAN, RONALD J 1991b Unsteady drag on a sphere at finite reynolds number with small fluctuations in the free-stream velocity. *Journal of Fluid Mechanics* **233**, 613–631.
- MICHAELIDES, E. E. 1997 The transient equation of motion for particles, bubbles, and droplets. *Journal of Fluids Engineering* **119** (2), 233–247.

- MURRAY, STEPHEN P 1970 Settling velocities and vertical diffusion of particles in turbulent water. *Journal of geophysical research* **75** (9), 1647–1654.
- NIELSEN, PETER 1993 Turbulence effects on the settling of suspended particles. *Journal of Sedimentary Research* **63** (5), 835–838.
- NIR, A & PISMEN, LM 1979 The effect of a steady drift on the dispersion of a particle in turbulent fluid. *Journal of Fluid Mechanics* **94** (2), 369–381.
- OSEEN, CARL WILHELM 1910 Über die stokes' sche formel und uber eine verwandte aufgabe in der hydrodynamik. *Arkiv Mat., Astron. och Fysik* **6** (29), 1.
- OSEEN, CARL WILHELM 1913 Über den gültigkeitsbereich der stokesschen widerstandsformel. *Ark. Mat. Astron. Fysik* **9** (19).
- PENG, J & DABIRI, JO 2009 Transport of inertial particles by lagrangian coherent structures: application to predator–prey interaction in jellyfish feeding. *Journal of Fluid Mechanics* **623**, 75–84.
- PICCIATI, GIUSEPPE 1907 Sul moto di una sfera in un liquido viscoso. *Rend. R. Acc. Naz. Lincei* pp. 943–951.
- PROUDMAN, IAN & PEARSON, JRA 1957 Expansions at small reynolds numbers for the flow past a sphere and a circular cylinder. *Journal of Fluid Mechanics* **2** (3), 237–262.
- REEKS, MW 1977 On the dispersion of small particles suspended in an isotropic turbulent fluid. *Journal of fluid mechanics* **83** (3), 529–546.
- REEKS, MW & MCKEE, S 1984 The dispersive effects of basset history forces on particle motion in a turbulent flow. *The Physics of fluids* **27** (7), 1573–1582.
- RILEY, JAMES J 1971 Ph.d. thesis. *The Johns Hopkins University, Baltimore, Maryland*.
- SANO, TAKAO 1981 Unsteady flow past a sphere at low reynolds number. *Journal of Fluid Mechanics* **112**, 433–441.
- SELBERG, BP & NICHOLLS, JA 1968 Drag coefficient of small spherical particles. *AIAA Journal* **6** (3), 401–408.
- SOO, SL 1975 Equation of motion of a solid particle suspended in a fluid. *The Physics of Fluids* **18** (2), 263–264.
- STOKES, GEORGE GABRIEL 1850 *On the effect of internal friction of fluids on the motion of pendulums*, , vol. IX. Trans. Camb. Phil. Soc.
- STOUT, JE, ARYA, SP & GENIKHOVICH, EL 1995 The effect of nonlinear drag on the motion and settling velocity of heavy particles. *Journal of the atmospheric sciences* **52** (22), 3836–3848.
- TCHEN, CHAN-MOU 1947 *Mean value and correlation problems connected with the motion of small particles suspended in a turbulent fluid*. Delft University of Technology.
- TUNSTALL, EB & HOUGHTON, G 1968 Retardation of falling spheres by hydrodynamic oscillations. *Chemical Engineering Science* **23** (9), 1067–1081.
- WANG, LIAN-PING & MAXEY, MARTIN R 1993 Settling velocity and concentration distribution of heavy particles in homogeneous isotropic turbulence. *Journal of fluid mechanics* **256**, 27–68.
- WHITE, FM 1991 Viscous fluid flow. *McGraw-Hill, New York*.
- ZELENY, JOHN & MCKEEHAN, LW 1910 The terminal velocity of fall of small spheres in air. *Physical Review (Series I)* **30** (5), 535.