PERSISTENCE AND SPREAD OF SOLUTIONS IN A TWO-SPECIES LOTKA-VOLTERRA COMPETITION-DIFFUSION MODEL WITH A SHIFTING HABITAT

FANG-DI DONG*, JIN SHANG†, WILLIAM F. FAGAN‡, AND BINGTUAN LI§

Abstract. We consider a two-species Lotka-Volterra competition-diffusion model with a shifting habitat. The growth rate of each species is nondecreasing along the x-axis, and it changes sign and shifts rightward at a speed c. We investigate the population dynamics of the model in the habitat suitable for growth of both species for two cases: (i) one species is competitively stronger and has a slower spreading speed, and (ii) both species coexist. We obtain conditions under which the outcome of competition depends critically on a number $\bar{c}(\infty)$ given by the model parameters. We show that under appropriate conditions, if $\bar{c}(\infty) > c$ then the species with the faster spreading speed spreads into the open space at its own speed and the species with the slower spreading speed spreads into its rival at speed $\bar{c}(\infty)$, and if $\bar{c}(\infty) < c$ then the species with the slower spreading speed eventually dies out in space. Our results particularly demonstrate the possibility that a competitively weaker species with a faster spreading speed can drive a competitively stronger species with a slower spreading speed to extinction. The mathematical proofs involve linear determinacy analysis, integral equations, and comparison.

Key words: Reaction-diffusion equation, shifting habitat, competition, linear determinacy, spreading speed.

AMS Subject Classification (2000): 92D25, 92D40.

1. Introduction.

Ecologists globally are focused intently on the challenges that climate change will have for species persistence. Of particular concern is the possibility that habitat shifts mediated by climate change may outpace the ability of some species to stay within tolerable zones that feature the correct temperature, rainfall, phenology, or other seasonal patterns necessary for their persistence. To date, the vast majority of ecological studies of climate change have focused on species as independent responders to the challenges imposed by climate change-mediated habitat shifts. However, there is increasing recognition that species interactions can play a central role in the ability of individual species to respond to such shifts (e.g., Gilman et al. [7], Urban et al. [23]). The idea that species that currently coexist and interact through competition, predation, or other interspecific interactions may respond quite differently to climate change processes (and thus exhibit differential matching to the resultant habitat shifts) underlies the key ecological concept of 'no-analogue' communities (Gilman et al. [7], Reu et al [21]). In this no-analogue framework, future communities whose composition has been shaped by differential spatial shifts in response to climate change will likely feature unusual combinations of species and species interactions that have no modern-day equivalent (Urban et al. [23]). Likewise, retrospective analyses suggest the widespread existence of no-analogue communities

*School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, China, Deparment of Mathematics, University of Louisville, Louisville, KY 40292, USA, and Department of Mathematics, Hangzhou Normal University, Hangzhou 310036, China (dongfd15@lzu.edu.cn). This author was supported by NSF of China (11731005, 11671180) and by the China Scholarship Council through grant (201806180083).

†Deparment of Mathematics, University of Louisville, Louisville, KY 40292, USA (jane-jjshang@gmail.com).

‡Deparment of Biology, University of Maryland, College Park, MD 20742, USA (bfagan@umd.edu). This author was partially supported by the National Science Foundation under Grant DMS-1853465.

§Deparment of Mathematics, University of Louisville, Louisville, KY 40292, USA (bing.li@louisville.edu). This author was partially supported by the National Science Foundation under Grant DMS-1515875 and Grant DMS-1951482.

thousands to millions of years in the past when climatic conditions were quite different (Huntley [15], Graham [9], Watrin et al. [25]).

Here we consider mechanisms that may contribute to the existence of no-analogue communities, specifically differential dispersal abilities between species that currently compete for resources. HilleRisLambers and colleagues [12] provide a synthetic review of empirical examples in which different types of species interactions, in concert with differential dispersal abilities, may shape the structure of future communities as climate change mediates the spatial shift of tolerable conditions. For competitive interactions, HillRisLambers et al. [12] report findings from their experimental studies of the competitive coexistence of three conifer species along an altitudinal transect at Mt. Rainier, Washington, that is expected to experience significant climate change in the coming decades. Focusing on coexistence among Pacific silver fir (Abies amabilis), western hemlock (Tsuga heterophylla), and mountain hemlock (Tsuga mertensiana), interspecific competitive interactions during early life-history stages (rather than climatic constraints) determine species' performance at lower range limits. In contrast, the direct effect of climate on performance is strongest at upper range limits, but only for adult trees and saplings (HillRisLambers et al. [12]). Spatial shifts in species distribution are expected to be slow because of low levels of adult mortality, and range shifts of the focal conifers are unlikely to keep pace with climate velocity at lower range limits, due to the interacting effects of competition and long generation times (HillRisLambers et al. [12]). Studies on competitive exclusion for species in shifting habitats are documented in HillRisLambers et al. [12]. One example is the widespread native annual legume, Chamaecrista fasciculata, interacting with competitors in its contracting habitat range and beyond [12, 22].

Understanding biological invasions within the context of climate change requires modeling approaches that incorporate dynamic landscapes. Li et al. [18] considered the following reaction-diffusion model to study the impact of climate change on invasion of species

$$u_t(t,x) = du_{xx}(t,x) + u(t,x)(r(x-ct) - u(t,x)). \tag{1.1}$$

In this model, $r(x)$ is continuous, nondecreasing and bounded with $r(-\infty) < 0 < r(\infty)$. $r(x-ct)$ thus divides the spatial domain into two parts: the region with good quality habitat suitable for growth (i.e., $r(x-ct) > 0$), and the region with poor quality habitat unsuitable for growth (i.e., $r(x-ct) < 0$). c describes the speed at which the edge of the habitat suitable for species growth shifts. It was shown that if $c > c^*(\infty) := 2\sqrt{dr(\infty)}$ then solutions with compactly supported initial values converge to zero uniformly and if $c^*(\infty) > c$ then solutions with compactly supported initial values persist in space and spread rightward at the asymptotic speed $c^*(\infty)$. The rightward and leftward spreading speeds for (1.1) with more general $r(x-ct)$ can be found in Hu et al. [13]. The problem of existence and stability of traveling waves for reaction-diffusion equations with shifting habitats related to (1.1) has been extensively studied; see, for example, Berestycki and Fang [3], Fang et al. [6], Bouhours and Giletti [5], Berestycki et al. [2], Berestycki and Rossi [4], Hamel [10], Hamel and Roques [11]. The reader is refereed to Li et al. [17], Li and Wu [19], Hu et al. [13], and Wang et al. [20, 24] for studies in spreading speeds and traveling waves in temporal-spatial models with shifting habitats in other forms including integro-difference equations and integral-differential equations.

Model (1.1) describes the persistence and spread of a single species with a shifting habitat edge, without referring to its interactions with existing species. In this paper we consider a two-species competition model, which is an extended form of the equation (1.1):

$$\begin{cases} u_t(t,x) = d_1 u_{xx}(t,x) + u(t,x)(r_1(x-ct) - u(t,x) - a_1 v(t,x)), \\ v_t(t,x) = d_2 v_{xx}(t,x) + v(t,x)(r_2(x-ct) - v(t,x) - a_2 u(t,x)). \end{cases} \tag{1.2}$$

This is a Lotka-Volterra type competition model. $u(t,x), v(t,x)$ denote the densities of two competing species, respectively, at time t and space x; $d_i > 0$ are diffusion coefficients; $a_i > 0$ represent interspecific competition coefficients; each $r_i(x-ct)$ describes a population growth rate as a function of $x-ct$, which is bounded and nondecreasing in $x-ct$, and it changes sign in $x-ct$; $c > 0$ is a speed at which the habitat shifts. Here the habitat in which two species grow and compete is shrinking in time. We investigate the population dynamics of (1.2) when two competitors consecutively or simultaneously invade the shifting habitat with compactly supported initial values. We consider $c_2^*(\infty) := 2\sqrt{d_2 r_2(\infty)} > c_1^*(\infty) := 2\sqrt{d_1 r_1(\infty)} > c$ so that in the absence of its rival, species u (v) persists and spreads rightward at speed $c_1^*(\infty)$ ($c_2^*(\infty)$), and species v spreads faster. We investigate the population dynamics of (1.2) in the habitat suitable for growth of both species for two cases: (i) species u is competitively stronger (with the slower spreading speed), and (ii) both species coexist. We obtain conditions under which the outcome of competition depends critically on a number $\bar{c}(\infty)$ given by the model parameters. We show that under appropriate conditions, if $\bar{c}(\infty) > c$ then the species with the faster spreading speed spreads into the open space at its own speed and the species with the slower spreading speed spreads into its rival at speed $\bar{c}(\infty)$, and if $\bar{c}(\infty) < c$ then the species with the slower spreading speed eventually dies out in space. Our results particularly demonstrate the possibility that a competitively weaker species with a faster spreading can drive a competitively stronger species with a slower spreading to extinction. Case (ii) was studied by Zhang el al. [28] and Yuan et al. [27] where $\bar{c}(\infty)$ was shown to be a lower bound of the speed at which u spreads into v, and the population dynamics was not explored for $\bar{c}(\infty) < c$. Our results indicate that $\bar{c}(\infty)$ serves as both an upper and a lower bound of the speed for Case (i) and Case (ii) under certain conditions. We obtain the results by extending the framework of linear determinacy, which was developed by Weinberger, Lewis and Li [16, 26] to study the invasion of a species into an equilibrium distribution of its competitor in a temporal-spatial system with constant coefficients. We particularly make use of integral equations and solutions of linearized systems to approximate the solutions of (1.2) in moving intervals.

This paper is organized as follows. The main results are given in the next section. Section 3 is about an upper bound for spreading speeds and Section 4 is about a lower bound for spreading speeds. Section 5 contains the proofs of the theorems. Section 6 is on numerical simulations. Section 7 includes some concluding remarks and discussions.

2. Main results.

We begin with making the following hypothesis:

(H) For $i = 1, 2$, $r_i(x)$ is nondecreasing, bounded, and piecewise continuously differentiable function in x for $-\infty < x < \infty$, and $r_i(-\infty)$ and $r_i(\infty)$ satisfy $-\infty < r_i(-\infty) < 0 < r_i(\infty) < \infty$.

The spatial region with $r_1(x-ct) > 0$ (or $r_2(x-ct) > 0$) is suitable for the growth of species u (or species v), and the spatial region with $r_1(x-ct) < 0$ (or $r_2(x-ct) < 0$) is unsuitable for the growth of species u (or species v). The population dynamics of two species depends on competition between two species in the region with $r_1(x-ct) > 0$ and $r_2(x-ct) > 0$. In this region, according to the standard

outcomes of two-species Lotka-Volterra type competition, at the location $x - ct$, (a) u is competitively stronger than v if $r_1(x-ct)/r_2(x-ct) > \max\{a_1, 1/a_2\}$, (b) u and v coexist if $a_1 < r_1(x-ct)/r_2(x-ct) < 1/a_2$, (c) v is competitively stronger than u if $r_2(x-ct)/r_1(x-ct) > \max\{a_2, 1/a_1\}$, and (d) u is competitively stronger than v or v is competitively stronger than u if $1/a_2 < r_1(x-ct)/r_2(x-ct) < a_1$. If $r_1(x-ct)/r_2(x-ct) > \max\{a_1, 1/a_2\}$ in the region where both $r_i(x-ct) > 0$, then $r_1(\infty)/r_2(\infty) \geq \max\{a_1, 1/a_2\}$. We consider the following slightly stronger condition
Case (i) $r_1(\infty)/r_2(\infty) > \max\{a_1, 1/a_2\}$.
This condition can be used to describe that u is competitively stronger than v in the entire habitat and more generally in an unbounded region suitable for growth of both species. Similarly
Case (ii) $a_1 < r_1(\infty)/r_2(\infty) < 1/a_2$
can be used to describe that u and v coexist in the entire habitat and more generally in an unbounded region suitable for growth of both species.

In this paper we explore the spatial dynamics of model (1.2) for Case (i) and Case (ii). The study for the scenario that v is competitively stronger than u in the habitat is similar to that for Case (i). See Yuan et al. [27] for some results obtained for the scenario that u is competitively stronger than v or v is competitively stronger than u in the habitat.

We introduce the linear determinacy condition:
(LD) $2 - d_2/d_1 > r_2(\infty)(\max\{a_1a_2, 1\} - 1)/(r_1(\infty) - a_1r_2(\infty))$.
This was first given by Lewis et al. [16] (where the inequality is not strict) for studying (1.2) with $r_i(x-ct)$ replaced by $r_i(\infty)$. The authors showed that under this condition, for cases (i)-(ii), u spreads into an equilibrium distribution of v at the speed

$$\bar{c}(\infty) = 2\sqrt{d_1(r_1(\infty) - a_1r_2(\infty))}, \tag{2.1}$$

which is the spreading speed of the linearized system about the leading edge of invasion. The linear determinacy analysis provided in [16] does not work for (1.2) as the growth rates are variables depending on $x - ct$. We extend the framework developed in [16] by using integral recursions and solutions of linearized systems to approximate the solutions of (1.2) in moving intervals.

Throughout this paper, we consider solutions of (1.2) with continuous initial values satisfying the following hypothesis:

(IV) There is a number $\tau_0 \geq 0$ such that either (i) $u(-\tau_0, x), v(0,x) \in C(\mathbb{R})$, $0 \leq u(-\tau_0, x) \leq r_1(\infty), u(-\tau_0, x) \not\equiv 0$ and $u(-\tau_0, x) \equiv 0$ outside a bounded interval, $0 \leq v(0,x) \leq r_2(\infty), v(0,x) \not\equiv 0$ and $v(0,x) \equiv 0$ outside a bounded interval, or (ii) $u(0,x), v(-\tau_0, x) \in C(\mathbb{R})$, $0 \leq v(-\tau_0, x) \leq r_2(\infty), v(-\tau_0, x) \not\equiv 0$ and $v(-\tau_0, x) \equiv 0$ outside a bounded interval, $0 \leq u(0,x) \leq r_1(\infty), u(0,x) \not\equiv 0$ and $u(0,x) \equiv 0$ outside a bounded interval.

Here the species arriving later invades in time τ_0 since the invasion of the earlier species if $\tau_0 > 0$. Both species invade simultaneously if $\tau_0 = 0$. The initial value problem of (1.2) with the initial values described by (IV) has a unique classical solution $(u(t,x), v(t,x))$ with $0 \leq u(t,x) \leq r_1(\infty), 0 \leq v(t,x) \leq r_2(\infty)$ [27, 28].

Throughout this paper, $(u_1(t,x), v_1(t,x)) \leq (\geq)(u_2(t,x), v_2(t,x))$ means that $u_1(t,x) \leq (\geq) u_2(t,x), v_1(t,x) \leq (\geq) v_2(t,x)$.

We now provide the main results.

THEOREM 2.1. *Consider Case (i). Assume that (H), (LD) and (IV) hold and $c_2^*(\infty) > c_1^*(\infty) + \bar{c}(\infty)$. Then for any small $\varepsilon > 0$,*

(i) *if* $0 \le c < \overline{c}(\infty)$,

$$\lim_{t\to\infty} \sup_{x \ge (\overline{c}(\infty)+\varepsilon)t} u(t,x) = 0,$$
$$\lim_{t\to\infty} \left[\sup_{(\overline{c}(\infty)+\varepsilon)t \le x \le (c_2^*(\infty)-\varepsilon)t} |r_2(\infty) - v(t,x)| \right] = 0,$$
$$\lim_{t\to\infty} \left[\sup_{(c+\varepsilon)t \le x \le (\overline{c}(\infty)-\varepsilon)t} \{|r_1(\infty) - u(t,x)| + v(t,x)\} \right] = 0,$$
$$\lim_{t\to\infty} \left[\sup_{x \le (c-\varepsilon)t} (u(t,x)+v(t,x)) \right] = 0, \ \textit{and} \ \lim_{t\to\infty} \sup_{x \ge (c_2^*(\infty)+\varepsilon)t} v(t,x) = 0; \ \textit{and}$$

(ii) *if* $\overline{c}(\infty) < c < c_1^*(\infty)$,

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} u(t,x) = 0,$$
$$\lim_{t\to\infty} \left[\sup_{(c+\varepsilon)t \le x \le (c_2^*(\infty)-\varepsilon)t} |r_2(\infty) - v(t,x)| \right] = 0,$$
$$\lim_{t\to\infty} \sup_{x \le (c-\varepsilon)t} v(t,x) = 0, \ \textit{and} \ \lim_{t\to\infty} \sup_{x \ge (c_2^*(\infty)+\varepsilon)t} v(t,x) = 0.$$

This theorem states that when v is a weaker species and has a much faster spreading speed and the linear determinacy condition is satisfied, (i) if $0 < c < \overline{c}(\infty)$ then v spreads rightward at its own speed $c_2^*(\infty)$ and u spreads into v at speed $\overline{c}(\infty)$, and (ii) if $c > \overline{c}(\infty)$ then v spreads rightward at its own speed $c_2^*(\infty)$ and stronger species u dies out eventually in space.

Define $u^* = (r_1(\infty) - a_1 r_2(\infty))/(1 - a_1 a_2)$ and $v^* = (r_2(\infty) - a_2 r_1(\infty))/(1 - a_1 a_2)$. For Case (ii) both u^* and v^* are positive.

Theorem 2.2. *Consider Case (ii). Assume that (H), (LD) and (IV) hold and* $c_2^*(\infty) > c_1^*(\infty) + \overline{c}(\infty)$. *Then for any small* $\varepsilon > 0$,
(i) *if* $0 \le c < \overline{c}(\infty)$,

$$\lim_{t\to\infty} \sup_{x \ge (\overline{c}(\infty)+\varepsilon)t} u(t,x) = 0,$$
$$\lim_{t\to\infty} \left[\sup_{(\overline{c}(\infty)+\varepsilon)t \le x \le (c_2^*(\infty)-\varepsilon)t} |r_2(\infty) - v(t,x)| \right] = 0,$$
$$\lim_{t\to\infty} \inf_{(c+\varepsilon)t \le x \le (\overline{c}(\infty)-\varepsilon)t} u(t,x) \ge u^*, \quad \lim_{t\to\infty} \sup_{(c+\varepsilon)t \le x \le (\overline{c}(\infty)-\varepsilon)t} v(t,x) \le v^*,$$
$$\lim_{t\to\infty} \left[\sup_{x \le (c-\varepsilon)t} (u(t,x)+v(t,x)) \right] = 0, \ \textit{and} \ \lim_{t\to\infty} \sup_{x \ge (c_2^*(\infty)+\varepsilon)t} v(t,x) = 0; \ \textit{and}$$

(ii) *if* $\overline{c}(\infty) < c < c_1^*(\infty)$,

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} u(t,x) = 0,$$
$$\lim_{t\to\infty} \left[\sup_{(c+\varepsilon)t \le x \le (c_2^*(\infty)-\varepsilon)t} |r_2(\infty) - v(t,x)| \right] = 0,$$
$$\lim_{t\to\infty} \sup_{x \le (c-\varepsilon)t} v(t,x) = 0, \ \textit{and} \ \lim_{t\to\infty} \sup_{x \ge (c_2^*(\infty)+\varepsilon)t} v(t,x) = 0.$$

This theorem states that when u and v can coexist, v has a much faster spreading speed, and the linear determinacy condition is satisfied, (i) if $0 < c < \overline{c}(\infty)$ then v spreads rightward at its own speed $c_2^*(\infty)$ and u spreads into v at speed $\overline{c}(\infty)$, and (ii) if $c > \overline{c}(\infty)$ then v spreads rightward at its own speed $c_2^*(\infty)$ and species u dies out eventually in space.

Remark 2.3. If $\overline{c}(\infty) \le 2\sqrt{d_2(r_2(\infty) - a_2 r_1(\infty))}$, then the statement

$$\lim_{t\to\infty} \inf_{(c+\varepsilon)t \le x \le (\overline{c}(\infty)-\varepsilon)t} u(t,x) \ge u^*, \quad \lim_{t\to\infty} \sup_{(c+\varepsilon)t \le x \le (\overline{c}(\infty)-\varepsilon)t} v(t,x) \le v^*$$

in (i) of Theorem 2.2 can be replaced by the following stronger statement

$$\lim_{t\to\infty} \sup_{(c+\varepsilon)t \le x \le (\overline{c}(\infty)-\varepsilon)t} \Big[|u(t,x) - u^*| + |v(t,x) - v^*| \Big] = 0.$$

The proof of this result is similar to that of Theorem 2.7 in [27] where $r_1(x - ct) \equiv r_2(x - ct)$ is assumed.

3. Upper bound for speed. In this section we provide an upper bound for the speed at which u spreads into v by establishing an upper solution for a related cooperative system (see (3.4)). For the sake of simplicity, we use $u^0(x)$ (or $v^0(x)$) to denote the actual initial value of species u (or v). That is, if $u(-\tau_0, x)$ is given, we say $u^0(x) = u(-\tau_0, x)$, and if $\tau_0 = 0$ we say $(u^0(x), v^0(x)) = (u(0,x), v(0,x))$.

3.1. Upper solutions for the case of $\overline{c}(\infty) > c \ge 0$.

LEMMA 3.1. *Assume (H) and (IV) hold and $c_2^*(\infty) > c \ge 0$. Let $c_2^*(\infty) > c_0 \ge 0$. If*

$$\lim_{t\to\infty} \sup_{x \ge c_0 t} u(t,x) = 0,$$

then for any small positive ε,

$$\lim_{t\to\infty} \left[\sup_{(\max\{c_0,c\}+\varepsilon)t \le x \le (c_2^*(\infty)-\varepsilon)t} |r_2(\infty) - v(t,x)| \right] = 0.$$

Proof. Since $\lim_{t\to\infty} \sup_{x \ge c_0 t} u(t,x) = 0$, for any small $\eta > 0$, there exists $T_0 > 0$ such that for $t > T_0$ and $x \ge c_0 t$, $u(t,x) \le \eta$. On the other hand, since $u^0(x) \equiv 0$ outside a bounded domain, there exists $x_0 > 0$ such that for $t \le T_0$ and $x \ge x_0$, $u(t,x) \le \eta$. Define

$$\sigma(x) = \begin{cases} a_2 \eta, & \text{if } x \ge 0, \\ a_2 r_1(\infty), & \text{if } x < 0. \end{cases}$$

Let $x_1 = \max\{0, x_0 - c_0 t, t \le T_0\}$. Some calculations lead to that for any $t > -\tau_0, x \in \mathbb{R}$,

$$a_2 u(t,x) \le \sigma(x - x_1 - c_0 t). \tag{3.1}$$

If $c_0 \ge c$, then $r_2(x - ct) \ge r_2(x - c_0 t)$ for all $t \ge 0$ and $x \in \mathbb{R}$. By (3.1) and the second equation of (1.2), we get

$$v_t(t,x) \ge d_2 v_{xx}(t,x) + v(t,x)(r_2(x - c_0 t) - \sigma(x - x_1 - c_0 t) - v(t,x)). \tag{3.2}$$

Note that the spreading speed for the corresponding equation is

$$c^*_{c_0}(\infty) = 2\sqrt{d_2(r_2(\infty) - a_2\eta)},$$

which is greater that $c_2^*(\infty) - \epsilon$ for small η. Let $V(t,x)$ be the solution of the equation corresponding to (3.2) with the initial value $V^0(x) = v^0(x)$. A direct application of Theorem 2.2 (iii) in [18] and comparison show that for every ϵ_0 with $0 < \epsilon_0 < (c_2^*(\infty) - c_0 - \epsilon)/2$,

$$\lim_{t\to\infty}\left[\sup_{(c_0+\epsilon_0)t\le x\le (c_2^*(\infty)-\epsilon-\epsilon_0)t} |r_2(\infty) - a_2\eta - V(t,x)|\right] = 0.$$

Because $\eta > 0$ is arbitrary and $0 \le V(t,x) \le v(t,x) \le r_2(\infty)$, for $\varepsilon = \epsilon_0 + \epsilon$,

$$\lim_{t\to\infty}\left[\sup_{(c_0+\varepsilon)t\le x\le (c_2^*(\infty)-\varepsilon)t} |r_2(\infty) - v(t,x)|\right] = 0. \tag{3.3}$$

If $c_0 \le c$, then $\sigma(x - x_1 - c_0 t) \le \sigma(x - x_1 - ct)$ for all $t \ge 0$ and all $x \in \mathbb{R}$. Hence (3.1) leads to

$$a_2 u(t,x) \le \sigma(x - x_1 - ct), \quad \forall t \ge 0, x \in \mathbb{R}.$$

This and the second equation of (1.2) indicate

$$v_t(t,x) \ge d_2 v_{xx}(t,x) + v(t,x)(r_2(x-ct) - \sigma(x - x_1 - ct) - v(t,x)).$$

Using an argument similar to that used to show (3.3), we have for any small $\varepsilon > 0$,

$$\lim_{t\to\infty}\left[\sup_{(c+\varepsilon)t\le x\le (c_2^*(\infty)-\varepsilon)t} |r_2(\infty) - v(t,x)|\right] = 0.$$

This and (3.3) lead to the desired result. The proof is complete. □

The variable change $w(t,x) = r_2(\infty) - v(t,x)$ converts (1.2) to the following cooperative system

$$\begin{cases} u_t(t,x) = d_1 u_{xx}(t,x) + u(t,x)(r_1(x-ct) - a_1 r_2(\infty) - u(t,x) + a_1 w(t,x)), \\ w_t(t,x) = d_2 w_{xx}(t,x) + (r_2(\infty) - w(t,x))(r_2(\infty) - r_2(x-ct) + a_2 u(t,x) \\ \qquad\qquad - w(t,x)). \end{cases} \tag{3.4}$$

Lemma 3.2. *Consider Case (i) and Case (ii). Assume that (H), (LD), and (IV) hold, and $c_2^*(\infty) > c_1^*(\infty) + \overline{c}(\infty)$. If $\overline{c}(\infty) > c \ge 0$, then for any small $\varepsilon > 0$,*

$$\lim_{t\to\infty} \sup_{x \ge (\overline{c}(\infty)+\varepsilon)t} u(t,x) = 0, \quad \textit{and} \ \lim_{t\to\infty}\left[\sup_{(\overline{c}(\infty)+\varepsilon)t\le x\le (c_2^*(\infty)-\varepsilon)t} w(t,x)\right] = 0. \tag{3.5}$$

Proof. For simplicity, we denote $\overline{c}(\infty), \overline{\mu}(\infty) = \sqrt{(r_1(\infty) - a_1 r_2(\infty))/d_1}, \mu_1^*(\infty) = \sqrt{r_1(\infty)/d_1}$ by $\overline{c}, \overline{\mu}, \mu_1^*$, respectively. It suffices to show that for any small $\epsilon > 0$ and $\eta > 0$, there exist $A_1, T_1, \delta > 0$ such that for $t \ge T_1 + \delta$,

$$u(t,x) \le \overline{u}(t,x), \ \forall x \in \mathbb{R}, \ \text{ and } w(t,x) \le \overline{w}(t,x), \ \forall x \le (c_2^*(\infty) - \epsilon)t, \tag{3.6}$$

where

$$\overline{u}(t,x)=A_1\xi_{\eta_1}(\overline{\mu})e^{-\overline{\mu}(x-(\overline{c}+\epsilon)(t-T_1))},\quad \overline{w}(t,x)=A_1\xi_{\eta_2}(\overline{\mu})e^{-\overline{\mu}(x-(\overline{c}+\epsilon)(t-T_1))}+\eta,$$

and $\xi_{\eta_1}(\overline{\mu}),\xi_{\eta_2}(\overline{\mu})$ are given by (3.13). Since η is arbitrary, (3.5) follows from (3.6) with $\varepsilon=2\epsilon$. To prove (3.6), we will first show that for any small $\epsilon>0$ and $\eta>0$, there exist A_1,T_1 such that (3.6) holds for $t=T_1$, we will then prove that there exist $h>\delta>0$ such that (3.6) holds for $t\in[T_1+\delta,T_1+h]$ using integral recursions, and we will finally establish (3.6) for $t\in[T_1+\delta,\infty)$.

Step 1: We first show that (3.6) holds for $t=T_1$. For any given sufficiently small $\epsilon>0$ with $\epsilon\le\min\{1/r_2(\infty),1,(c_1^*(\infty)-c)\}/2$, since

$$\lim_{t\to\infty}\sup_{x\ge(c_1^*(\infty)+\epsilon)t}u(t,x)=0,$$

by Lemma 3.1,

$$\lim_{t\to\infty}\left[\sup_{(c_1^*(\infty)+\epsilon)t\le x\le(c_2^*(\infty)-\epsilon)t}w(t,x)\right]=0.$$

Then for any small $\eta>0$ satisfying

$$(3a_1+\rho+r_1(\infty))\eta\le 2\overline{\mu}\epsilon, \tag{3.7}$$

where ρ is a constant independent on η and ϵ, and is given by (3.19), there exists $T_0>0$ such that for any $t\ge T_0$ and $(c_1^*(\infty)+\epsilon)t\le x\le(c_2^*(\infty)-\epsilon)t$,

$$0\le w(t,x)\le\eta. \tag{3.8}$$

Let L be a constant such that

$$\int_{-L}^{\infty}\frac{1}{\sqrt{\pi}}e^{-x^2}dx=\int_{-\infty}^{L}\frac{1}{\sqrt{\pi}}e^{-x^2}dx\ge\frac{1}{1+\epsilon\eta}. \tag{3.9}$$

On the other hand, since $\lim_{t\to\infty}\sup_{x\ge(c+\epsilon/2)t}\Big[r_2(\infty)-r_2(x-ct)\Big]=0$, for $\epsilon_1=\epsilon\eta/2$, choose

$$T_1\ge\max\left\{T_0,\frac{L\sqrt{4d_1}}{c_2^*(\infty)-\epsilon-c_1^*(\infty)-\overline{c}}\right\} \tag{3.10}$$

such that for $t\ge T_1$ and $x\ge(c+\epsilon/2)t$,

$$r_2(\infty)-r_2(x-ct)\le\epsilon_1, \tag{3.11}$$

and for $t\ge T_1$,

$$(c+\epsilon)t>(c+\epsilon/2)t+L\sqrt{4d_2(t-T_1)}. \tag{3.12}$$

Define

$$B_\eta(\mu):=\begin{pmatrix} d_1\mu^2+r_1(\infty)-a_1r_2(\infty)+a_1\eta, & 0\\ a_2r_2(\infty), & d_2\mu^2-r_2(\infty)+2\eta\end{pmatrix}.$$

The eigenvalues of the matrix $B_\eta(\mu)$ are

$$\lambda_{\eta_1}(\mu)=d_1\mu^2+r_1(\infty)-a_1r_2(\infty)+a_1\eta,\quad \lambda_{\eta_2}(\mu)=d_2\mu^2-r_2(\infty)+2\eta.$$

By virtue of (LD) and $\eta > 0$ sufficiently small, we have $\lambda_{\eta_1}(\overline{\mu}) > \lambda_{\eta_2}(\overline{\mu})$. An eigenvector of $B_\eta(\overline{\mu})$ corresponding to $\lambda_{\eta_1}(\overline{\mu})$ is $\boldsymbol{\xi}_\eta(\overline{\mu}) = (\xi_{\eta_1}(\overline{\mu}), \xi_{\eta_2}(\overline{\mu}))$, where

(3.13)
$$\xi_{\eta_1}(\overline{\mu}) = (d_1 - d_2)\overline{\mu}^2 + r_1(\infty) - a_1 r_2(\infty) + r_2(\infty) + \eta(a_1 - 2), \quad \xi_{\eta_2}(\overline{\mu}) = a_2 r_2(\infty).$$

Clearly $\xi_{\eta_2}(\overline{\mu}) > 0$ and $\xi_{\eta_1}(\overline{\mu}) > 0$ according to (LD), which also implies

$$\xi_{\eta_1}(\overline{\mu}) \geq \max\{a_1, 1/a_2\}\xi_{\eta_2}(\overline{\mu}). \tag{3.14}$$

Since $\lambda_{\eta_1}(\overline{\mu})/\overline{\mu} \to \overline{c}$ as $\eta \to 0$, for small $\eta > 0$,

$$\lambda_{\eta_1}(\overline{\mu}) \leq \overline{\mu}(\overline{c} + \epsilon). \tag{3.15}$$

Observe that the equation $u_t(t,x) = d_1 u_{xx}(t,x) + r_1(\infty)u(t,x)$ has a solution $\varphi(t,x) = e^{-\mu_1^*(x - c_1^*(\infty)(t - T_1))}$. Since $u^0(x)$ satisfies (IV), there exists a large $A_0 > 0$ such that $u^0(x) \leq A_0 \xi_{\eta_1}(\overline{\mu})\varphi^0(x)$ for $x \in \mathbb{R}$, where $\varphi^0(x) = \varphi(0,x)$ if $u^0(x) = u(0,x)$, otherwise $\varphi^0(x) = \varphi(-\tau_0, x)$. Comparison leads to

$$u(t,x) \leq A_0 \xi_{\eta_1}(\overline{\mu})\varphi(t,x) = A_0 \xi_{\eta_1}(\overline{\mu}) e^{-\mu_1^*(x - c_1^*(\infty)(t - T_1))}, \forall t \geq 0, x \in \mathbb{R}. \tag{3.16}$$

(3.16), $u(t,x) \leq r_1(\infty), \overline{\mu} < \mu_1^*$, (3.8) and $w(t,x) \leq r_2(\infty)$ indicate that there exists a larger constant $A_1 \geq A_0$ such that $A_1 \xi_{\eta_2}(\overline{\mu}) e^{-\overline{\mu}(\overline{c} + \epsilon)T_1} \geq r_2(\infty)$,

$$0 \leq u(T_1, x) \leq A_1 \xi_{\eta_1}(\overline{\mu}) e^{-\overline{\mu}x}, \quad \forall x \in \mathbb{R}, \tag{3.17}$$

and

$$0 \leq w(T_1, x) \leq A_1 \xi_{\eta_2}(\overline{\mu}) e^{-\overline{\mu}x} + \eta, \quad \forall x \leq (c_2^*(\infty) - \epsilon)T_1. \tag{3.18}$$

Step 2: We next show that there exist $h > \delta > 0$ such that (3.6) holds for $t \in [T_1 + \delta, T_1 + h]$ by using integral recursions. Let $(u_0(T_1, x), w_0(T_1, x)) = (u(T_1, x), w(T_1, x))$. Consider $\big(u^{(n)}(t,x), w^{(n)}(t,x)\big)$ given by

(3.19)
$$\begin{cases} u^{(n+1)}(t,x) = \int_{\mathbb{R}} K_1(t - T_1, x - y) u_0(T_1, y) dy + \int_{T_1}^t \int_{\mathbb{R}} K_1(t - \tau, x - y) u^{(n)}(\tau, y) \\ \qquad\qquad\qquad [\rho + r_1(y - c\tau) - a_1 r_2(\infty) - u^{(n)}(\tau, y) + a_1 w^{(n)}(\tau, y)] dy d\tau, \\ w^{(n+1)}(t,x) = \int_{\mathbb{R}} K_2(t - T_1, x - y) w_0(T_1, y) dy + \int_{T_1}^t \int_{\mathbb{R}} K_2(t - \tau, x - y) \\ \qquad\qquad\qquad \times \big\{[r_2(\infty) - w^{(n)}(\tau, y)][r_2(\infty) - r_2(y - c\tau) + a_2 u^{(n)}(\tau, y) \\ \qquad\qquad\qquad - w^{(n)}(\tau, y)] + \rho w^{(n)}(\tau, y)\big\} dy d\tau, \end{cases}$$

where $\big(u^{(0)}(t,x), w^{(0)}(t,x)\big) \equiv (0,0), K_i(t,x) = e^{-\rho t - \frac{x^2}{4 d_i t}} / \sqrt{4\pi d_i t}, i = 1, 2$, and ρ is a constant with $\rho > \max\{1, a_1 r_2(\infty) + 2r_1(\infty) - r_1(-\infty), a_2 r_1(\infty) + 2r_2(\infty) - r_2(-\infty)\}$. Both $u[\rho + r_1(x - ct) - a_1 r_2(\infty) - u + a_1 w]$ and $(r_2(\infty) - w)[r_2(\infty) - r_2(x - ct) + a_2 u - w] + \rho w$ are nondecreasing in u and w. Induction shows that for $t \geq T_1$ and $x \in \mathbb{R}$,

$$(0,0) \leq (u^{(n)}(t,x), w^{(n)}(t,x)) \leq (u^{(n+1)}(t,x), w^{(n+1)}(t,x)) \leq (r_1(\infty), r_2(\infty)).$$

$(u^{(n)}(t,x), w^{(n)}(t,x))$ converges to the unique solution $(u(t,x), w(t,x))$ for $t \geq T_1$. (3.17) implies for $t \geq T_1$ and $x \in \mathbb{R}$,

$$\int_{\mathbb{R}} K_1(t - T_1, x - y) u_0(T_1, y) dy$$

$$\leq \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_1(t-T_1)}} e^{-\rho(t-T_1)-\frac{y^2}{4d_1(t-T_1)}} A_1\xi_{\eta_1}(\overline{\mu}) e^{-\overline{\mu}(x-y)} dy$$

$$= A_1\xi_{\eta_1}(\overline{\mu}) e^{(d_1\overline{\mu}^2-\rho)(t-T_1)} e^{-\overline{\mu}x}. \tag{3.20}$$

So, for $t \geq T_1$ and $x \in \mathbb{R}$,

$$u^{(1)}(t,x) \leq A_1\xi_{\eta_1}(\overline{\mu}) e^{(d_1\overline{\mu}^2-\rho)(t-T_1)} e^{-\overline{\mu}x} \leq A_1\xi_{\eta_1}(\overline{\mu}) e^{-\overline{\mu}(x-(\overline{c}+\epsilon)(t-T_1))}. \tag{3.21}$$

Let $h > 0$ be a constant satisfying

$$\sqrt{h} < \min\left\{1, \frac{(c_2^*(\infty)-c-3\epsilon/2)T_1}{2L\sqrt{4d_2}+c+\epsilon/2}, \frac{(c_2^*(\infty)-c_1^*(\infty)-2\epsilon)T_1}{L\sqrt{4d_2}+c_1^*(\infty)+\epsilon}\right\}.$$

This implies that for $t \in [T_1, T_1+h]$, we have

$$\begin{cases} (c+\epsilon/2)t + L\sqrt{4d_2(t-T_1)} < (c_2^*(\infty)-\epsilon)T_1 - L\sqrt{4d_2(t-T_1)}, \\ (c_1^*(\infty)+\epsilon)t < (c_2^*(\infty)-\epsilon)T_1 - L\sqrt{4d_2(t-T_1)}. \end{cases} \tag{3.22}$$

So for $t \in [T_1, T_1+h]$ and $x \leq (c_2^*(\infty)-\epsilon)T_1 - L\sqrt{4d_2(t-T_1)}$, by (3.18),

$$\int_{\mathbb{R}} K_2(t-T_1, x-y) w_0(T_1, y) dy$$
$$\leq \int_{-\infty}^{(c_2^*(\infty)-\epsilon)T_1} K_2(t-T_1, x-y)\big[A_1\xi_{\eta_2}(\bar{\mu})e^{-\bar{\mu}y} + \eta\big] dy$$
$$+ r_2(\infty) \int_{(c_2^*(\infty)-\epsilon)T_1}^{\infty} K_2(t-T_1, x-y) dy$$
$$\leq \int_{\mathbb{R}} \frac{1}{\sqrt{\pi}} e^{-\rho(t-T_1)-z^2} \big[A_1\xi_{\eta_2}(\bar{\mu})e^{-\bar{\mu}(x-\sqrt{4d_2(t-T_1)}z)} + \eta\big] dz$$
$$+ r_2(\infty) e^{-\rho(t-T_1)} \int_{-\infty}^{-L} \frac{1}{\sqrt{\pi}} e^{-z^2} dz$$

$$= A_1\xi_{\eta_2}(\overline{\mu}) e^{(d_2\overline{\mu}^2-\rho)(t-T_1)} e^{-\overline{\mu}x} + \Big[\eta + r_2(\infty)\int_{-\infty}^{-L} \frac{1}{\sqrt{\pi}} e^{-z^2} dz\Big] e^{-\rho(t-T_1)}. \tag{3.23}$$

On the other hand, for $t \geq T_1$ and x satisfying

$$x \geq (c+\epsilon/2)t + L\sqrt{4d_2(t-T_1)}, \tag{3.24}$$

if $y \leq L\sqrt{4d_2(t-T_1)}$, then $x - y \geq (c+\epsilon/2)t$. Hence for $t \geq T_1$ and x satisfying (3.24), by (3.9), (3.11) and $\epsilon_1 = \epsilon\eta/2$,

$$\int_{T_1}^{t} \int_{\mathbb{R}} K_2(t-\tau, x-y)(r_2(\infty) - r_2(y-c\tau)) dy d\tau$$
$$= \int_0^{t-T_1} \int_{-\infty}^{L} \frac{1}{\sqrt{\pi}} e^{-\rho\tau - z^2} (r_2(\infty) - r_2(x - \sqrt{4d_2\tau}z - c(t-\tau))) dz d\tau$$
$$+ \int_0^{t-T_1} \int_{L}^{\infty} \frac{1}{\sqrt{\pi}} e^{-\rho\tau - z^2} (r_2(\infty) - r_2(x - \sqrt{4d_2\tau}z - c(t-\tau))) dz d\tau$$
$$\leq \epsilon_1 \int_0^{t-T_1} e^{-\rho\tau} \int_{-\infty}^{L} \frac{1}{\sqrt{\pi}} e^{-z^2} dz d\tau + r_2(\infty) \int_0^{t-T_1} e^{-\rho\tau} \int_{L}^{\infty} \frac{1}{\sqrt{\pi}} e^{-z^2} dz d\tau$$

$$\leq \frac{\epsilon_1}{\rho}\big(1-e^{-\rho(t-T_1)}\big) + \frac{r_2(\infty)}{\rho}\big(1-e^{-\rho(t-T_1)}\big)\frac{\epsilon\eta}{1+\epsilon\eta}$$

$$\leq\Big(\epsilon_1+\frac{\epsilon\eta}{2}\Big)\big(1-e^{-\rho(t-T_1)}\big) = \epsilon\eta\big(1-e^{-\rho(t-T_1)}\big). \tag{3.25}$$

For $t\in[T_1, T_1+h]$ and x satisfying

$$(c+\epsilon/2)t + L\sqrt{4d_2(t-T_1)} \leq x \leq (c_2^*(\infty)-\epsilon)T_1 - L\sqrt{4d_2(t-T_1)}, \tag{3.26}$$

(3.9), (3.22), (3.23) and (3.25) imply that

$$\begin{aligned} w^{(1)}(t,x) \leq & A_1\xi_{\eta_2}(\overline{\mu})e^{(d_2\overline{\mu}^2-\rho)(t-T_1)}e^{-\overline{\mu}x} + \Big[\eta + r_2(\infty)\int_{-\infty}^{-L}\frac{1}{\sqrt{\pi}}e^{-z^2}dz\Big]e^{-\rho(t-T_1)} \\ & + r_2(\infty)\epsilon\eta\big(1-e^{-\rho(t-T_1)}\big) \\ \leq & A_1\xi_{\eta_2}(\overline{\mu})e^{-\overline{\mu}(x-(\overline{c}+\epsilon)(t-T_1))} + g(t), \end{aligned}$$

where

$$g(t) = \eta e^{-\rho(t-T_1)} + r_2(\infty)\epsilon\eta\Big(1-e^{-\rho(t-T_1)}\Big) + \frac{r_2(\infty)\epsilon\eta}{1+\epsilon\eta}.$$

Note $g(T_1) = \eta + r_2(\infty)\epsilon\eta/(1+\epsilon\eta)$ and $g'(t) < 0$ for small ϵ, η. It follows that for $t\in[T_1, T_1+h]$ and x satisfying (3.26),

$$w^{(1)}(t,x) \leq A_1\xi_{\eta_2}(\overline{\mu})e^{-\overline{\mu}(x-(\overline{c}+\epsilon)(t-T_1))} + \eta + \frac{r_2(\infty)\epsilon\eta}{1+\epsilon\eta}. \tag{3.27}$$

Moreover, since $w^{(n)}(t,x) \leq w(t,x)$ for all n, t and x, (3.8) implies that for $t \in [T_1, T_1+h]$ and $(c_1^*(\infty)+\epsilon)t \leq x \leq (c_2^*(\infty)-\epsilon)t$,

$$w^{(1)}(t,x) \leq \eta. \tag{3.28}$$

For $c < \bar{c}$ and $x \leq (c+\epsilon)t, e^{-\overline{\mu}(x-(\overline{c}+\epsilon)t)} \geq 1$. By the choose of A_1, for $t \geq T_1$ and $x \leq (c+\epsilon)t$,

$$w^{(1)}(t,x) \leq r_2(\infty) \leq A_1\xi_{\eta_2}(\overline{\mu})e^{-\overline{\mu}(\overline{c}+\epsilon)T_1} \leq A_1\xi_{\eta_2}(\overline{\mu})e^{-\overline{\mu}(x-(\overline{c}+\epsilon)(t-T_1))}. \tag{3.29}$$

(3.12), (3.22), (3.27), (3.28), (3.29) and $2\epsilon \leq c_1^*(\infty)-c$ indicate that for $t\in[T_1, T_1+h]$,

$$w^{(1)}(t,x) \leq \bar{w}(t,x) + \frac{r_2(\infty)\epsilon\eta}{1+\epsilon\eta}, \quad \forall x \leq (c_2^*(\infty)-\epsilon)t.$$

This and (3.21) show that for $t\in[T_1, T_1+h]$,

$$\begin{cases} u^{(1)}(t,x) \leq \overline{u}(t,x), & \forall x\in\mathbb{R}, \\ w^{(1)}(t,x) \leq \overline{w}(t,x) + \frac{r_2(\infty)\epsilon\eta}{1+\epsilon\eta}, & \forall x \leq (c_2^*(\infty)-\epsilon)t. \end{cases} \tag{3.30}$$

Next, we assume for some positive integer $k \geq 1$ and for $t\in[T_1, T_1+h]$,

$$\begin{cases} u^{(k)}(t,x) \leq \overline{u}(t,x), & \forall x\in\mathbb{R}, \\ w^{(k)}(t,x) \leq \overline{w}(t,x) + \frac{r_2(\infty)\epsilon\eta}{1+\epsilon\eta}, & \forall x \leq (c_2^*(\infty)-\epsilon)t. \end{cases}$$

(3.14) implies for $t \geq T_1, x \in \mathbb{R}$,

$$a_1\bar{w}(t,x) - a_1\eta - \bar{u}(t,x) \leq 0, \quad \bar{w}(t,x) - \eta - a_2\bar{u}(t,x) \leq 0. \tag{3.31}$$

For $\tau \in [T_1, T_1+h], y \in \mathbb{R}$, define

$$F_1(u^{(k)}, w^{(k)}, \tau, y) = u^{(k)}(\tau,y)\big[\rho + r_1(y-c\tau) - a_1 r_2(\infty) - u^{(k)}(\tau,y) + a_1 w^{(k)}(\tau,y)\big].$$

For $\tau \in [T_1, T_1+h]$ and $y \leq (c_2^*(\infty)-\epsilon)\tau$, by $\epsilon \leq 1/(2r_2(\infty))$ and (3.31),

$$\begin{aligned}
&F_1(u^{(k)}, w^{(k)}, \tau, y)\\
\leq&\bar{u}(\tau,y)\Big[\rho + r_1(\infty) - a_1 r_2(\infty) - \bar{u}(\tau,y) + a_1\bar{w}(\tau,y) + a_1\frac{r_2(\infty)\epsilon\eta}{1+\epsilon\eta}\Big]\\
=&\bar{u}(\tau,y)\Big[\rho + r_1(\infty) - a_1 r_2(\infty) + a_1\frac{r_2(\infty)\epsilon\eta}{1+\epsilon\eta} + a_1\eta\Big]\\
&+\bar{u}(\tau,y)[a_1\bar{w}(\tau,y) - a_1\eta - \bar{u}(\tau,y)]\\
\leq&\bar{u}(\tau,y)\Big[\rho + r_1(\infty) - a_1 r_2(\infty) + 3a_1\eta/2\Big].
\end{aligned} \tag{3.32}$$

Since $F_1(u^{(k)}, w^{(k)}, \tau, y)$ is nondecreasing in $w^{(k)}$ and $w^{(k)}(t,x) \leq r_2(\infty)$, for $\tau \in [T_1, T_1+h]$ and $y \in \mathbb{R}$,

$$F_1(u^{(k)}, w^{(k)}, \tau, y) \leq \bar{u}(\tau,y)[\rho + r_1(\infty)]. \tag{3.33}$$

Note that

$$\begin{aligned}
&\int_0^{t-T_1}\int_{\mathbb{R}} \frac{1}{\sqrt{\pi}} e^{-\rho\tau - z^2}\bar{u}(t-\tau, x-\sqrt{4d_1\tau}z)dzd\tau\\
=&A_1\xi_{\eta_1}(\bar{\mu})e^{-\bar{\mu}(x-(\bar{c}+\epsilon)(t-T_1))}\frac{1-e^{-(\rho+\bar{\mu}(\bar{c}+\epsilon)-d_1\bar{\mu}^2)(t-T_1)}}{\rho+\bar{\mu}(\bar{c}+\epsilon)-d_1\bar{\mu}^2}.
\end{aligned} \tag{3.34}$$

Therefore, for $t \in [T_1, T_1+h]$ and $x \leq (c_2^*(\infty)-\epsilon)t - L\sqrt{4d_1(t-T_1)}$, by (3.7), (3.9), (3.32), (3.33), (3.34) and $\epsilon \leq 1/2$, we have

$$\begin{aligned}
&\int_{T_1}^t\int_{\mathbb{R}} K_1(t-\tau, x-y)F_1(u^{(k)}, w^{(k)}, \tau, y)dyd\tau\\
\leq&\int_{T_1}^t\int_{-\infty}^{(c_2^*(\infty)-\epsilon)\tau} K_1(t-\tau,x-y)\Big[\rho + r_1(\infty) - a_1 r_2(\infty) + 3a_1\eta/2\Big]\bar{u}(\tau,y)dyd\tau\\
&+\int_{T_1}^t\int_{(c_2^*(\infty)-\epsilon)\tau}^{\infty} K_1(t-\tau,x-y)\Big[\rho + r_1(\infty)\Big]\bar{u}(\tau,y)dyd\tau\\
\leq&\Big[\rho + r_1(\infty) - a_1 r_2(\infty) + 3a_1\eta/2\Big]\int_0^{t-T_1}\int_{\mathbb{R}}\frac{1}{\sqrt{\pi}}e^{-\rho\tau-z^2}\bar{u}(t-\tau, x-\sqrt{4d_1\tau}z)dzd\tau\\
&+[\rho + r_1(\infty)]\int_0^{t-T_1}\int_{-\infty}^{-L}\frac{1}{\sqrt{\pi}}e^{-\rho\tau-z^2}\bar{u}(t-\tau, x-\sqrt{4d_1\tau}z)dzd\tau\\
\leq&\frac{\rho + r_1(\infty) - a_1 r_2(\infty) + 3a_1\eta/2 + (\rho + r_1(\infty))\epsilon\eta}{\rho+\bar{\mu}(\bar{c}+\epsilon)-d_1\bar{\mu}^2}A_1\xi_{\eta_1}(\bar{\mu})e^{-\bar{\mu}(x-(\bar{c}+\epsilon)(t-T_1))}\\
&\times\Big[1-e^{-(\rho+\bar{\mu}(\bar{c}+\epsilon)-d_1\bar{\mu}^2)(t-T_1)}\Big]
\end{aligned}$$

(3.35)
$$\leq A_1\xi_{\eta_1}(\overline{\mu})e^{-\overline{\mu}(x-(\overline{c}+\epsilon)(t-T_1))}\Big[1-e^{-(\rho+\overline{\mu}(\overline{c}+\epsilon)-d_1\overline{\mu}^2)(t-T_1)}\Big].$$

Therefore for $t\in[T_1,T_1+h]$ and $x\leq(c_2^*(\infty)-\epsilon)t-L\sqrt{4d_1(t-T_1)}$, (3.20) and (3.35) imply

$$\begin{aligned}u^{(k+1)}(t,x)\leq &A_1\xi_{\eta_1}(\overline{\mu})e^{(d_1\overline{\mu}^2-\rho)(t-T_1)}e^{-\overline{\mu}x}+A_1\xi_{\eta_1}(\overline{\mu})e^{-\overline{\mu}(x-(\overline{c}+\epsilon)(t-T_1))}\Big[1\\ &-e^{-(\rho+\overline{\mu}(\overline{c}+\epsilon)-d_1\overline{\mu}^2)(t-T_1)}\Big]=A_1\xi_{\eta_1}(\overline{\mu})e^{-\overline{\mu}(x-(\overline{c}+\epsilon)(t-T_1))}.\end{aligned}\tag{3.36}$$

Furthermore by (3.10) and the fact $\mu_1^*\bar{c}=\bar{\mu}c_1^*(\infty)$,

$$\begin{aligned}T_1\geq&\frac{L\sqrt{4d_1}\big(\mu_1^*-\bar{\mu}\big)}{\big(c_2^*(\infty)-\epsilon-c_1^*(\infty)-\bar{c}\big)\big(\mu_1^*-\bar{\mu}\big)}\\ \geq&\frac{(\mu_1^*-\overline{\mu})L\sqrt{4d_1}}{(c_2^*(\infty)-\epsilon)(\mu_1^*-\overline{\mu})-(\mu_1^*c_1^*(\infty)-\overline{\mu}(\overline{c}+\epsilon))}.\end{aligned}$$

This implies for $t\geq T_1$,

$$\begin{aligned}(c_2^*(\infty)-\epsilon)T_1-\frac{\mu_1^*c_1^*(\infty)-\overline{\mu}(\overline{c}+\epsilon)}{\mu_1^*-\overline{\mu}}T_1-L\sqrt{4d_1}&\\ +\left((c_2^*(\infty)-\epsilon)-\frac{\mu_1^*c_1^*(\infty)-\overline{\mu}(\overline{c}+\epsilon)}{\mu_1^*-\overline{\mu}}\right)&(t-T_1)\geq 0.\end{aligned}$$

It follows that for $t\geq T_1,(c_2^*(\infty)-\epsilon)t-L\sqrt{4d_1}\geq(\mu_1^*c_1^*(\infty)t-\overline{\mu}(\overline{c}+\epsilon)t)/(\mu_1^*-\overline{\mu})$. This shows that for $t\in[T_1,T_1+h]$, if $x\geq(c_2^*(\infty)-\epsilon)t-L\sqrt{4d_1(t-T_1)}$,

$$x\geq\frac{\mu_1^*c_1^*(\infty)t-\overline{\mu}(\overline{c}+\epsilon)t}{\mu_1^*-\overline{\mu}}.$$

This leads to $\mu_1^*(x-c_1^*(\infty)t)\geq\overline{\mu}(x-(\overline{c}+\epsilon)t)$. Hence

$$\begin{aligned}\mu_1^*(x-c_1^*(\infty)(t-T_1))=&\mu_1^*(x-c_1^*(\infty)t)+\mu_1^*c_1^*(\infty)T_1\\ \geq&\overline{\mu}(x-(\overline{c}+\epsilon)t)+\overline{\mu}(\overline{c}+\epsilon)T_1=\overline{\mu}(x-(\overline{c}+\epsilon)(t-T_1)).\end{aligned}$$

Therefore, for $t\in[T_1,T_1+h]$ and $x\geq(c_2^*(\infty)-\epsilon)t-L\sqrt{4d_1(t-T_1)}$, we have

$$A_1\xi_{\eta_1}(\overline{\mu})e^{-\mu_1^*(x-c_1^*(\infty)(t-T_1))}\leq A_1\xi_{\eta_1}(\overline{\mu})e^{-\overline{\mu}(x-(\overline{c}+\epsilon)(t-T_1))}.$$

This and (3.16) imply that for $t\in[T_1,T_1+h]$ and $x\geq(c_2^*(\infty)-\epsilon)t-L\sqrt{4d_1(t-T_1)}$, $u(t,x)\leq\overline{u}(t,x)$, and thus $u^{(k+1)}(t,x)\leq\overline{u}(t,x)$. Here we used the simple fact $u^{(n)}(t,x)\leq u(t,x)$ for all n,t and x. This and (3.36) imply for $t\in[T_1,T_1+h]$,

$$u^{(k+1)}(t,x)\leq\overline{u}(t,x),\quad\forall x\in\mathbb{R}.\tag{3.37}$$

On the other hand, by $B_\eta(\overline{\mu})\boldsymbol{\xi}_\eta(\overline{\mu})=\lambda_{\eta_1}(\overline{\mu})\boldsymbol{\xi}_\eta(\overline{\mu})$ and (3.15),

$$(\rho-r_2(\infty)+3\eta/2)\xi_{\eta_2}(\overline{\mu})+a_2r_2(\infty)\xi_{\eta_1}(\overline{\mu})\leq\xi_{\eta_2}(\overline{\mu})(\rho+\overline{\mu}(\overline{c}+\epsilon)-d_2\overline{\mu}^2-\eta/2).$$

Thus for $t\geq T_1$,

$$\frac{(\rho-r_2(\infty)+3\eta/2)\xi_{\eta_2}(\overline{\mu})+a_2r_2(\infty)\xi_{\eta_1}(\overline{\mu})}{\rho+\overline{\mu}(\overline{c}+\epsilon)-d_2\overline{\mu}^2}\Big[1-e^{-(\rho+\overline{\mu}(\overline{c}+\epsilon)-d_2\overline{\mu}^2)(t-T_1)}\Big]$$

$$\le \xi_{\eta_2}(\overline{\mu}) \frac{\rho+\overline{\mu}(\overline{c}+\epsilon)-d_2\overline{\mu}^2-\eta/2}{\rho+\overline{\mu}(\overline{c}+\epsilon)-d_2\overline{\mu}^2}\Big[1-e^{-(\rho+\overline{\mu}(\overline{c}+\epsilon)-d_2\overline{\mu}^2)(t-T_1)}\Big]$$
$$\le \xi_{\eta_2}(\overline{\mu})\Big[1-e^{-(\rho+\overline{\mu}(\overline{c}+\epsilon)-d_2\overline{\mu}^2)(t-T_1)}\Big].$$

This implies

$$\begin{aligned}
&\int_0^{t-T_1}\int_{\mathbb{R}} \frac{1}{\sqrt{\pi}} e^{-\rho\tau - z^2}\Big[(\rho-r_2(\infty)+3\eta/2)\overline{w}(t-\tau, x-\sqrt{4d_2\tau}z)\\
&+a_2 r_2(\infty)\overline{u}(t-\tau, x-\sqrt{4d_2\tau}z)\Big]dzd\tau\\
=&A_1 e^{-\overline{\mu}(x-(\overline{c}+\epsilon)(t-T_1))}\big[(\rho-r_2(\infty)+3\eta/2)\xi_{\eta_2}(\overline{\mu})+a_2r_2(\infty)\xi_{\eta_1}(\overline{\mu})\big]\\
&\times\int_0^{t-T_1} e^{-(\rho+\overline{\mu}(\overline{c}+\epsilon))\tau}\int_{\mathbb{R}}\frac{1}{\sqrt{\pi}}e^{-z^2+\overline{\mu}\sqrt{4d_2\tau}z}dzd\tau+\eta(\rho-r_2(\infty)+3\eta/2)\\
&\times\int_0^{t-T_1} e^{-\rho\tau}\int_{\mathbb{R}}\frac{1}{\sqrt{\pi}}e^{-z^2}dzd\tau\\
\le& A_1\xi_{\eta_2}(\overline{\mu})e^{-\overline{\mu}(x-(\overline{c}+\epsilon)(t-T_1))}\Big[1-e^{-(\rho+\overline{\mu}(\overline{c}+\epsilon)-d_2\overline{\mu}^2)(t-T_1)}\Big]\\
&+\eta(1-e^{-\rho(t-T_1)})\Big(1-\frac{r_2(\infty)-3\eta/2}{\rho}\Big).
\end{aligned}\tag{3.38}$$

For $\tau\in[T_1,T_1+h], y\in\mathbb{R}$, define

$$F_2(u^{(k)},w^{(k)},\tau,y)=\big[r_2(\infty)-w^{(k)}(\tau,y)\big]\big[a_2u^{(k)}(\tau,y)-w^{(k)}(\tau,y)\big]+\rho w^{(k)}(\tau,y).$$

Then for $\tau\in[T_1,T_1+h]$ and $y\le(c_2^*(\infty)-\epsilon)\tau$, by $\epsilon\le 1/2r_2(\infty)$ and (3.31),

$$\begin{aligned}
&F_2(u^{(k)},w^{(k)},\tau,y)\\
\le& a_2r_2(\infty)\bar{u}(\tau,y)+\Big[\rho-r_2(\infty)+\eta+\frac{r_2(\infty)\epsilon\eta}{1+\epsilon\eta}\Big]\Big[\bar{w}(\tau,y)+\frac{r_2(\infty)\epsilon\eta}{1+\epsilon\eta}\Big]\\
&+\Big[\bar{w}(\tau,y)+\frac{r_2(\infty)\epsilon\eta}{1+\epsilon\eta}\Big]\big[\bar{w}(\tau,y)-\eta-a_2\bar{u}(\tau,y)\big]\\
\le& a_2r_2(\infty)\bar{u}(\tau,y)+\Big[\rho-r_2(\infty)+3\eta/2\Big]\Big[\bar{w}(\tau,y)+\frac{r_2(\infty)\epsilon\eta}{1+\epsilon\eta}\Big]
\end{aligned}\tag{3.39}$$

Since $F_2(u^{(k)},w^{(k)},\tau,y)$ is nondecreasing in $w^{(k)}$ and $w^{(k)}(t,x)\le r_2(\infty)$, for $\tau\in[T_1,T_1+h]$ and $y\in\mathbb{R}$,

$$F_2(u^{(k)},w^{(k)},\tau,y)\le\rho r_2(\infty).\tag{3.40}$$

Therefore for $t\in[T_1,T_1+h], x\le(c_2^*(\infty)-\epsilon)t-L\sqrt{4d_2(t-T_1)}$, by (3.38), (3.39) and (3.40), we get

$$\begin{aligned}
&\int_{T_1}^{t}\int_{\mathbb{R}}K_2(t-\tau,x-y)F_2(u^{(k)},w^{(k)},\tau,y)dyd\tau\\
\le&\int_{T_1}^{t}\int_{-\infty}^{(c_2^*(\infty)-\epsilon)\tau}K_2(t-\tau,x-y)\Big\{a_2r_2(\infty)\bar{u}(\tau,y)+\Big[\rho-r_2(\infty)+3\eta/2\Big]\\
&\times\Big[\bar{w}(\tau,y)+\frac{r_2(\infty)\epsilon\eta}{1+\epsilon\eta}\Big]\Big\}dyd\tau+\rho r_2(\infty)\int_{T_1}^{t}\int_{(c_2^*(\infty)-\epsilon)\tau}^{\infty}K_2(t-\tau,x-y)dyd\tau
\end{aligned}$$

$$
\begin{aligned}
=&\int_0^{t-T_1}\int_{\frac{x-(c_2^*(\infty)-\epsilon)(t-\tau)}{\sqrt{4d_2\tau}}}^{\infty}\frac{1}{\sqrt{\pi}}e^{-\rho\tau-z^2}\Big\{a_2r_2(\infty)\bar{u}(t-\tau,x-\sqrt{4d_2\tau}z)\\
&+(\rho-r_2(\infty)+3\eta/2)\big[\bar{w}(t-\tau,x-\sqrt{4d_2\tau}z)+r_2(\infty)\epsilon\eta/(1+\epsilon\eta)\big]\Big\}dzd\tau\\
&+\rho r_2(\infty)\int_0^{t-T_1}\int_{-\infty}^{\frac{x-(c_2^*(\infty)-\epsilon)(t-\tau)}{\sqrt{4d_2\tau}}}\frac{1}{\sqrt{\pi}}e^{-\rho\tau-z^2}dzd\tau\\
\le&\int_0^{t-T_1}\int_{\mathbb{R}}\frac{1}{\sqrt{\pi}}e^{-\rho\tau-z^2}\Big\{a_2r_2(\infty)\bar{u}(t-\tau,x-\sqrt{4d_2\tau}z)\\
&+(\rho-r_2(\infty)+3\eta/2)\big[\bar{w}(t-\tau,x-\sqrt{4d_2\tau}z)+r_2(\infty)\epsilon\eta/(1+\epsilon\eta)\big]\Big\}dzd\tau\\
&+\rho r_2(\infty)\int_0^{t-T_1}\int_{-\infty}^{-L}\frac{1}{\sqrt{\pi}}e^{-\rho\tau-z^2}dzd\tau\\
\le&A_1\xi_{\eta_2}(\overline{\mu})e^{-\overline{\mu}(x-(\overline{c}+\epsilon)(t-T_1))}\Big[1-e^{-(\rho+\overline{\mu}(\overline{c}+\epsilon)-d_2\overline{\mu}^2)(t-T_1)}\Big]+\eta\big(1-e^{-\rho(t-T_1)}\big)\\
&\times\Big(1+\frac{r_2(\infty)\epsilon}{1+\epsilon\eta}\Big)\Big(1-\frac{r_2(\infty)-3\eta/2}{\rho}\Big)+r_2(\infty)\big(1-e^{-\rho(t-T_1)}\big)\int_{-\infty}^{-L}\frac{1}{\sqrt{\pi}}e^{-z^2}dz.
\end{aligned}\tag{3.41}
$$

Then (3.9), (3.22), (3.23), (3.25) and (3.41) imply that for $t\in[T_1,T_1+h]$ and $(c+\epsilon/2)t+L\sqrt{4d_2(t-T_1)}\le x\le(c_2^*(\infty)-\epsilon)T_1-L\sqrt{4d_2(t-T_1)}$,

$$
\begin{aligned}
&w^{(k+1)}(t,x)\\
\le&A_1\xi_{\eta_2}(\overline{\mu})e^{(d_2\overline{\mu}^2-\rho)(t-T_1)}e^{-\overline{\mu}x}+\eta e^{-\rho(t-T_1)}+r_2(\infty)e^{-\rho(t-T_1)}\\
&\times\int_{-\infty}^{-L}\frac{1}{\sqrt{\pi}}e^{-z^2}dz+A_1\xi_{\eta_2}(\overline{\mu})e^{-\overline{\mu}(x-(\overline{c}+\epsilon)(t-T_1))}\Big[1-e^{-(\rho+\overline{\mu}(\overline{c}+\epsilon)-d_2\overline{\mu}^2)(t-T_1)}\Big]\\
&+\eta\big(1-e^{-\rho(t-T_1)}\big)\Big(1+\frac{r_2(\infty)\epsilon}{1+\epsilon\eta}\Big)\Big(1-\frac{r_2(\infty)-3\eta/2}{\rho}\Big)+r_2(\infty)\big(1-e^{-\rho(t-T_1)}\big)\\
&\times\int_{-\infty}^{-L}\frac{1}{\sqrt{\pi}}e^{-z^2}dz+r_2(\infty)\epsilon\eta\big(1-e^{-\rho(t-T_1)}\big)\\
\le&A_1\xi_{\eta_2}(\overline{\mu})e^{-\overline{\mu}(x-(\overline{c}+\epsilon)(t-T_1))}+G(t),
\end{aligned}
$$

where

$$
\begin{aligned}
G(t)=&\eta e^{-\rho(t-T_1)}+\eta\big(1-e^{-\rho(t-T_1)}\big)\Big(1+\frac{r_2(\infty)\epsilon}{1+\epsilon\eta}\Big)\Big(1-\frac{r_2(\infty)-3\eta/2}{\rho}\Big)\\
&+r_2(\infty)\epsilon\eta\big(1-e^{-\rho(t-T_1)}\big)+\frac{r_2(\infty)\epsilon\eta}{1+\epsilon\eta}.
\end{aligned}
$$

Note $G(T_1)=\eta+r_2(\infty)\epsilon\eta/(1+\epsilon\eta)$ and $G'(t)<0$ for small ϵ,η. It shows that for $t\in[T_1,T_1+h]$ and $(c+\epsilon/2)t+L\sqrt{4d_2(t-T_1)}\le x\le(c_2^*(\infty)-\epsilon)T_1-L\sqrt{4d_2(t-T_1)}$,

$$
w^{(k+1)}(t,x)\le A_1\xi_{\eta_2}(\overline{\mu})e^{-\overline{\mu}(x-(\overline{c}+\epsilon)(t-T_1))}+\eta+\frac{r_2(\infty)\epsilon\eta}{1+\epsilon\eta}.\tag{3.42}
$$

For $c<\overline{c}$ and $x\le(c+\epsilon)t$, $e^{-\overline{\mu}(x-(\overline{c}+\epsilon)t)}\ge1$. By the choose of A_1, for $t\ge T_1$ and $x\le(c+\epsilon)t$,

$$
w^{(k+1)}(t,x)\le r_2(\infty)\le A_1\xi_{\eta_2}(\overline{\mu})e^{-\overline{\mu}(\overline{c}+\epsilon)T_1}\le A_1\xi_{\eta_2}(\overline{\mu})e^{-\overline{\mu}(x-(\overline{c}+\epsilon)(t-T_1))}.\tag{3.43}
$$

Moreover, since $w^{(n)}(t,x) \le w(t,x)$ for all n, t and x. It then follows from (3.8), (3.12), (3.22), (3.42), (3.43) and $2\epsilon \le c_1^*(\infty) - c$ that for $t \in [T_1, T_1+h]$,

$$w^{(k+1)}(t,x) \le \bar{w}(t,x) + \frac{r_2(\infty)\epsilon\eta}{1+\epsilon\eta}, \quad \forall x \le (c_2^*(\infty)-\epsilon)t.$$

This and (3.37) show that for $t \in [T_1, T_1+h]$,

$$\begin{cases} u^{(k+1)}(t,x) \le \bar{u}(t,x), & \forall x \in \mathbb{R}, \\ w^{(k+1)}(t,x) \le \bar{w}(t,x) + \frac{r_2(\infty)\epsilon\eta}{1+\epsilon\eta}, & \forall x \le (c_2^*(\infty)-\epsilon)t. \end{cases} \tag{3.44}$$

(3.30) and induction show that (3.44) is true for all integer $k \ge 0$.

Choose δ with $0 < \delta < h$ such that for above sufficiently small ϵ and η,

$$(1-e^{-\rho\delta})\Big[\frac{r_2(\infty)}{\rho} - \frac{3\eta}{2\rho} - r_2(\infty)\epsilon - \frac{r_2(\infty)\epsilon}{1+\epsilon\eta}\Big] \ge \frac{\epsilon r_2(\infty)}{1+\epsilon\eta}.$$

Such δ can be arbitrarily small for sufficiently small ϵ and η. This implies for $t \in [T_1+\delta, T_1+h]$,

$$(1-e^{-\rho(t-T_1)})\Big[1 - r_2(\infty)\epsilon - \Big(1+\frac{\epsilon r_2(\infty)}{1+\epsilon\eta}\Big)\Big(1-\frac{r_2(\infty)-3\eta/2}{\rho}\Big)\Big] \ge \frac{\epsilon r_2(\infty)}{1+\epsilon\eta}.$$

We therefore have $G(t) \le \eta$ for $t \in [T_1+\delta, T_1+h]$. Hence for $t \in [T_1+\delta, T_1+h]$ and $(c+\epsilon/2)t + L\sqrt{4d_2(t-T_1)} \le x \le (c_2^*(\infty)-\epsilon)T_1 - L\sqrt{4d_2(t-T_1)}$,

$$w^{(k+1)}(t,x) \le A_1\xi_{\eta_2}(\overline{\mu})e^{-\overline{\mu}(x-(\overline{c}+\epsilon)(t-T_1))} + \eta. \tag{3.45}$$

Similarly, since $w^{(n)}(t,x) \le w(t,x)$ for all n, t and x. It then follows from (3.8), (3.12), (3.22), (3.43), (3.45) and $2\epsilon \le c_1^*(\infty) - c$ that for $t \in [T_1+\delta, T_1+h]$,

$$w^{(k+1)}(t,x) \le \bar{w}(t,x), \quad \forall x \le (c_2^*(\infty)-\epsilon)t.$$

Using this, (3.37) and $g(t) \le G(t)$ for $t \ge T_1$, we have that for all integer $k \ge 0$ and any $t \in [T_1+\delta, T_1+h]$,

$$\begin{cases} u^{(k+1)}(t,x) \le \bar{u}(t,x), & \forall x \in \mathbb{R}, \\ w^{(k+1)}(t,x) \le \bar{w}(t,x), & \forall x \le (c_2^*(\infty)-\epsilon)t. \end{cases}$$

Letting $k \to \infty$, for $t \in [T_1+\delta, T_1+h]$,

$$u(t,x) \le \overline{u}(t,x),\ \forall x \in \mathbb{R}, \ \text{ and } w(t,x) \le \overline{w}(t,x),\ \forall x \le (c_2^*(\infty)-\epsilon)t.$$

That is, (3.6) holds.

Step 3: We finally prove that (3.6) holds for $t \ge T_1+\delta$ using induction. For any $\overline{t} \in [T_1+\delta, T_1+h]$, repeating the above proof with T_1 replaced by $\overline{t}$, we obtain that for $t \in [T_1+h, T_1+2h]$, (3.6) is true. By induction, (3.6) is valid for $t \in [T_1+\delta, T_1+mh]$ for any positive integer m, and thus (3.6) is true for all $t \ge T_1+\delta$. Since $\eta > 0$ is arbitrary, we have

$$\lim_{t\to\infty}\Big[\sup_{x \ge (\overline{c}+2\epsilon)t} u(t,x)\Big] = 0, \quad \text{and} \lim_{t\to\infty}\Big[\sup_{(\overline{c}+2\epsilon)t \le x \le (c_2^*(\infty)-2\epsilon)t} w(t,x)\Big] = 0.$$

Let $\varepsilon = 2\epsilon$. The proof is complete. ∎

3.2. Upper solutions for the case of $c_1^*(\infty) > c > \overline{c}(\infty)$.

LEMMA 3.3. *Consider Case (i) and Case (ii). Assume that (H), (LD), and (IV) hold, and $c_2^*(\infty) > c_1^*(\infty) + \overline{c}(\infty)$. If $c_1^*(\infty) > c > \overline{c}(\infty)$, then for any small $\varepsilon > 0$,*

$$\lim_{t\to\infty}\sup_{x\in\mathbb{R}} u(t,x) = 0, \quad \textit{and} \lim_{t\to\infty}\sup_{(c+\varepsilon)t\le x\le (c_2^*(\infty)-\varepsilon)t} w(t,x) = 0.$$

Proof. We still denote $\overline{\mu}(\infty)$ by $\overline{\mu}$. For $t \ge T_1$ and $x \in \mathbb{R}$, define

$$\overline{u}(t,x) = A_1\xi_{\eta_1}(\overline{\mu})e^{-\overline{\mu}(x-(c-\epsilon)(t-T_1))}, \quad \overline{w}(t,x) = A_1\xi_{\eta_2}(\overline{\mu})e^{-\overline{\mu}(x-(c+\epsilon)(t-T_1))} + \eta,$$

where $A_1, \epsilon, \eta, T_1 > 0$ and $\xi_{\eta_1}(\overline{\mu}), \xi_{\eta_2}(\overline{\mu})$ are given by (3.13), T_1 makes (3.11) and (3.12) true. We choose A_1 sufficiently large such that $A_1\xi_{\eta_2}(\overline{\mu})e^{-\overline{\mu}(c+\epsilon)T_1} \ge r_2(\infty)$. By (LD), we can choose η satisfying $\eta(2-a_1) < (r_1(\infty) - a_1 r_2(\infty))(2 - d_2/d_1) - r_2(\infty)(\max\{a_1a_2, 1\} - 1)$ such that the inequality in (3.14) is strict, i.e.,

$$\xi_{\eta_1}(\overline{\mu}) > \max\{a_1, 1/a_2\}\xi_{\eta_2}(\overline{\mu}).$$

We can further choose a smaller ϵ such that $0 < 2\epsilon < c - \overline{c}(\infty)$ and

$$\xi_{\eta_1}(\overline{\mu}) \ge \max\{a_1, 1/a_2\}\xi_{\eta_2}(\overline{\mu})e^{2\overline{\mu}\epsilon}. \tag{3.46}$$

One can slightly modify the proof of Lemma 3.2 by replacing (3.14) with (3.46) to show that there exists $\delta_1 > 0$ such that for $t \ge T_1 + \delta_1$,

$$u(t,x) \le \overline{u}(t,x),\ \forall x \in \mathbb{R}, \ \text{ and } w(t,x) \le \overline{w}(t,x),\ \forall x \le (c_2^*(\infty) - \epsilon)t.$$

Since $\eta > 0$ is arbitrary, we have

$$\lim_{t\to\infty}\sup_{x\ge(c-\epsilon/2)t} u(t,x) = 0, \ \text{ and } \lim_{t\to\infty}\sup_{(c+2\epsilon)t\le x\le(c_2^*(\infty)-2\epsilon)t} w(t,x) = 0. \tag{3.47}$$

By virtue of Theorem 2.2 (i) in [18], for above $\epsilon > 0$, $\lim_{t\to\infty}\sup_{x\le(c-\epsilon/4)t} u(t,x) = 0$. This and (3.47) lead to the desired results by letting $\varepsilon = 2\epsilon$. The proof is complete.□

4. Lower bound for speed. In this section, we show that $\overline{c}(\infty)$ is a lower bound for the speed at which u spread into v. We have the following two lemmas whose proofs are similar to that of Theorem 2.7 in [27] and are omitted.

LEMMA 4.1. *Consider Case (i). Assume that (H) and (IV) hold. Let $w(t,x) = r_2(\infty) - v(t,x)$. If $\overline{c}(\infty) > c \ge 0$, then for any given $\varepsilon \in (0, (\overline{c}(\infty) - c)/2)$,*

$$\lim_{t\to\infty}\left[\sup_{(c+\varepsilon)t\le x\le(\overline{c}(\infty)-\varepsilon)t} |r_1(\infty) - u(t,x)| + |r_2(\infty) - w(t,x)|\right] = 0.$$

LEMMA 4.2. *Consider Case (ii). Assume that (H) and (IV) hold. Let $w(t,x) = r_2(\infty) - v(t,x)$. If $\overline{c}(\infty) > c \ge 0$, then for $w^* = r_2(\infty) - v^*$ and any given $\varepsilon \in (0, (\overline{c}(\infty) - c)/2)$,*

$$\lim_{t\to\infty}\inf_{(c+\varepsilon)t\le x\le(\overline{c}(\infty)-\varepsilon)t} u(t,x) \ge u^*, \quad \textit{and} \lim_{t\to\infty}\inf_{(c+\varepsilon)t\le x\le(\overline{c}(\infty)-\varepsilon)t} w(t,x) \ge w^*.$$

5. Proofs of theorems. In this section, we provide proofs for Theorems 2.1-2.2.

Proof of Theorem 2.1. The statement $\lim_{t\to\infty}\sup_{x\ge(c_2^*(\infty)+\varepsilon)t} v(t,x)=0$ follows from Theorem 2.2 (ii) in [18] and simple comparison. $\lim_{t\to\infty}\sup_{x\le(c-\varepsilon)t}(u(t,x)+v(t,x))=0$ comes from Theorem 2.2 (i) in [18] and simple comparison. The rest of statement (i) follows from Lemmas 3.2 and 4.1, and statement (ii) follows from Lemma 3.3. The proof is complete. ∎

Proof of Theorem 2.2. $\lim_{t\to\infty}\sup_{x\ge(c_2^*(\infty)+\varepsilon)t} v(t,x)=0$ is valid as shown in the proof of Theorem 2.1. $\lim_{t\to\infty}\sup_{x\le(c-\varepsilon)t}(u(t,x)+v(t,x))=0$ follows from Theorem 2.2 (i) in [18] and simple comparison. The rest of statement (i) follow from Lemmas 3.2 and 4.2, and statement (ii) follows from Lemma 3.3. The proof is complete. ∎

6. Simulations. In this section, we present numerical simulations with

$$r_1(x-ct)=\begin{cases}-0.5, & \text{if } x\le ct,\\ 1, & \text{elsewhere,}\end{cases}\qquad r_2(x-ct)=\begin{cases}-0.3, & \text{if } x\le ct,\\ 2, & \text{elsewhere,}\end{cases}$$

where $c>0$, the initial data $u^0(x)=0.8\sin(x-10)$ for $10\le x\le 10+\pi$ and 0 otherwise, and $v^0(x)=0.5\sin(x-20)$ for $20\le x\le 20+\pi$ and 0 otherwise. We always choose $a_1=4/9$ and $d_1=1$, and choose different values for a_2 to study competitive exclusion and competitive coexistence and different values for d_2 to consider the linear determinacy condition (LD) and the condition $c_2^*(\infty)>c_1^*(\infty)+\bar{c}(\infty)$ given in the theorems.

We first choose $a_2=9/4, d_2=49/32$. It is easily seen that

$$c_1^*(\infty)=2,\quad c_2^*(\infty)=3.5,\quad \bar{c}(\infty)=2/3,$$

and the assumptions in Theorem 2.1 are satisfied. In this case competitive exclusion occurs, and u is completely stronger and has a slower spreading speed. Figure 6.1 with $c=0.25<\bar{c}(\infty)$ displays the numerical solution supported by Theorem 2.1 (i), which shows that u spreads into v at speed $\bar{c}(\infty)$ and v spreads rightward at its own speed $c_2^*(\infty)$. Figure 6.2 with $c=1>\bar{c}(\infty)$ displays the numerical solution supported by Theorem 2.1 (ii), which indicates that v spreads rightward at its own speed $c_2^*(\infty)$ and stronger species u dies out eventually in space.

We next choose $a_2=1, d_2=49/32$. It is easily seen

$$c_1^*(\infty)=2,\quad c_2^*(\infty)=3.5,\quad \bar{c}(\infty)=2/3,$$

and the assumptions in Theorem 2.2 are satisfied. In this case, u and v can coexist and v has a faster spreading speed. Figure 6.3 with $c=0.1<\bar{c}(\infty)$ displays the numerical solution supported by Theorem 2.2 (i), which shows that v spreads rightward at its own speed $c_2^*(\infty)$ and u spreads into v at speed $\bar{c}(\infty)$. Figure 6.4 with $c=1.5>\bar{c}(\infty)$ displays the numerical solution supported by Theorem 2.2 (ii), which indicates that v spreads rightward at its own speed $c_2^*(\infty)$ and u dies out eventually in space.

We now consider $a_2=9/4$ and $d_2=441/800$, which lead to

$$c_1^*(\infty)=2,\quad c_2^*(\infty)=2.1,\quad \bar{c}(\infty)=2/3.$$

It is easily verified that all the assumptions except $c_2^*(\infty)>c_1^*(\infty)+\bar{c}(\infty)$ in Theorem 2.1 are satisfied. Figure 6.7 with $c=0.25<\bar{c}(\infty)$ shows that the speed at which the boundary between the two species moves is no longer $\bar{c}(\infty)$.

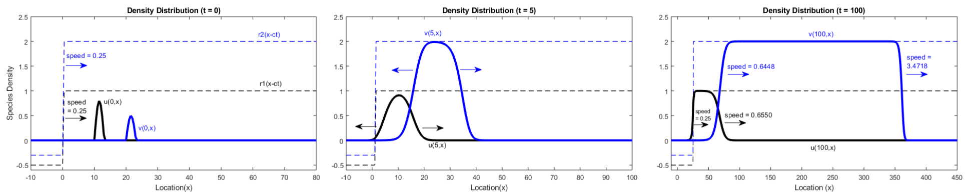

FIG. 6.1. *Simulations for* $a_1 = 4/9, a_2 = 9/4, d_1 = 1, d_2 = 49/32$. *Choose* $c = 0.25 < \bar{c}(\infty) = 2/3$. v *spreads rightward at a speed numerically close to* $c_2^*(\infty)$ *and* u *spreads into* v *at a speed numerically close to* $\bar{c}(\infty)$.

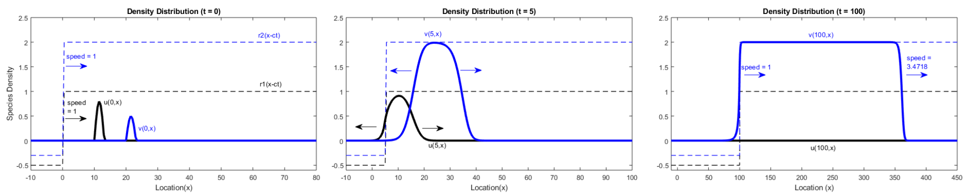

FIG. 6.2. *Simulations for* $a_1 = 4/9, a_2 = 9/4, d_1 = 1, d_2 = 49/32$. *Choose* $c = 1 > \bar{c}(\infty) = 2/3$. *The competitively stronger species* u *dies out eventually in space.*

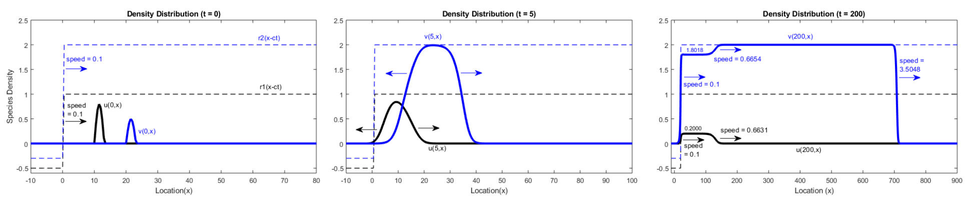

FIG. 6.3. *Simulations for* $a_1 = 4/9, a_2 = 1, d_1 = 1, d_2 = 49/32$. *Choose* $c = 0.1 < \bar{c}(\infty) = 2/3$. v *spreads rightward at a speed numerically close to* $c_2^*(\infty)$ *and* u *spreads into* v *at a speed numerically close to* $\bar{c}(\infty)$, *and both species coexist in a moving interval.*

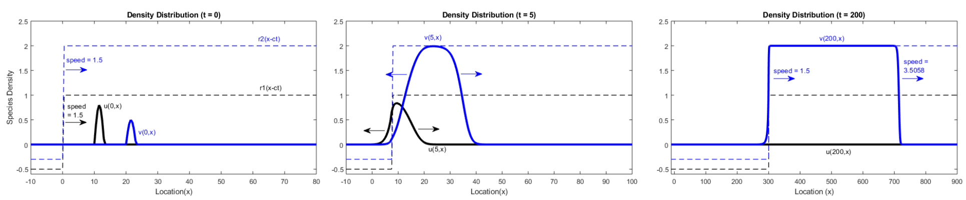

FIG. 6.4. *Simulations for* $a_1 = 4/9, a_2 = 1, d_1 = 1, d_2 = 49/32$. *Choose* $c = 1.5 > \bar{c}(\infty) = 2/3$. *Species* u *dies out eventually in space.*

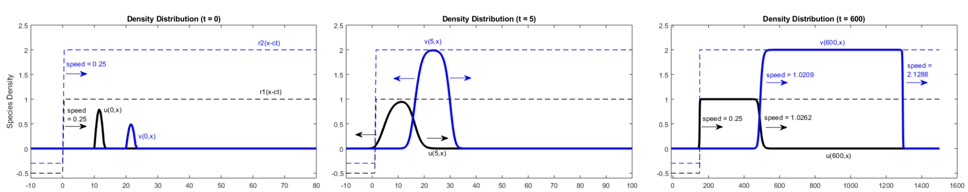

FIG. 6.5. *Simulations for* $a_1 = 4/9, a_2 = 9/4, d_1 = 1, d_2 = 441/800$. *Choose* $c = 0.25 < \bar{c}(\infty) = 2/3$. *Species* u *spreads into* v *at a speed, which is numerically very different from* $\bar{c}(\infty)$.

We finally consider $a_2 = 9/4, d_2 = 32$, which result in

$$c_1^*(\infty) = 2, \quad c_2^*(\infty) = 16, \quad \bar{c}(\infty) = 2/3,$$

and that all the assumptions except (LD) in Theorem 2.1 are satisfied. Figure 6.6 with $c = 0.25 < \bar{c}(\infty)$ shows that the speed at which the boundary between the two species moves is no longer $\bar{c}(\infty)$.

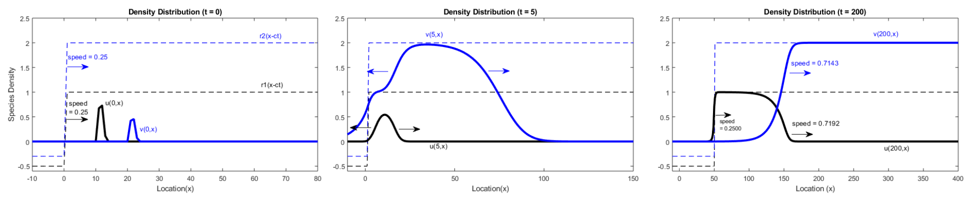

FIG. 6.6. *Simulations for $a_1 = 4/9, a_2 = 9/4, d_1 = 1, d_2 = 32$. Choose $c = 0.25 < \bar{c}(\infty) = 2/3$. Species u spreads into v at a speed, which is numerically very different from $\bar{c}(\infty)$.*

The above simulations show that the linear determinacy condition (LD) and $c_2^*(\infty) > c_1^*(\infty) + \bar{c}(\infty)$ are important in determining the population dynamics. If one of them is not satisfied, the speed at which the slower species spreads into its rival may not be $\bar{c}(\infty)$. In all the simulations above, two species initially invade the region with good quality habitat suitable for growth. Figure 6.7 shows the solution with the same parameter values and same $r_i(x - ct)$, $i = 1, 2$ as in Figure 6.1 and initial values $u^0(x) = 0.8\sin(x+10)$ for $-10 \le x \le -10+\pi$ and $v^0(x) = 0.5\sin(x+20)$ for $-20 \le x \le -20+\pi$ with compact support in the region with poor quality habitat unsuitable for growth. Figure 6.1 and Figure 6.7 indicate basically the same long term spreading dynamics. Our extensive simulations have shown that the locations of initial invasions will not affect the spreading speeds of species.

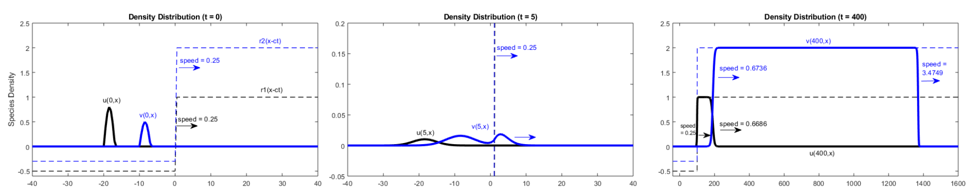

FIG. 6.7. *Simulations for $a_1 = 4/9, a_2 = 9/4, d_1 = 1, d_2 = 49/32$, $u^0(x) = 0.8\sin(x + 10)$ for $-10 \le x \le -10 + \pi$ and $v^0(x) = 0.5\sin(x + 20)$ for $-20 \le x \le -20 + \pi$. Choose $c = 0.25 < \bar{c}(\infty) = 2/3$. v spreads rightward at a speed numerically close to $c_2^*(\infty)$ and u spreads into v at a speed numerically close to $\bar{c}(\infty)$.*

7. Discussion. In this paper, we studied the two-species reaction-diffusion competition model (1.2) with a shifting habitat. It is assumed that the growth rate for each species is nondecreasing along the x-axis, and it changes sign and shifts rightward at speed $c > 0$. It is also assumed that the spreading speed of each species is greater than c so that each species can persist and spread in the absence of its rival. We determined the population dynamics of the model by examining competition between two species in the region suitable for growth of both species. We showed that under appropriate conditions, the number $\bar{c}(\infty)$ given by (2.1), plays an important role in determining long-term behavior of solutions. Specifically, (i) in the case that one species is competitively weaker and has the faster spreading speed and (LD) is satisfied, if $\bar{c}(\infty) > c$ then the weaker species spreads rightward at its own speed and the stronger species spreads into the weaker species at speed $\bar{c}(\infty)$, and if $\bar{c}(\infty) < c$ then the stronger species eventually dies out in space; and (ii) in the case that both species may coexist and (LD) is satisfied, if $\bar{c}(\infty) > c$ then the species with the faster spreading speed spreads rightward at its own speed and its competitor spreads at speed $\bar{c}(\infty)$, and if $\bar{c}(\infty) < c$ then the species with the slower spreading speed eventually becomes extinct in space. Our results particularly demonstrate that a species with a faster spreading speed can eventually win the competition. Thus mobility may be more important than competitive capability for species survival in a shifting environment.

As illustrated in Figures 6.1-6.7, a pair of competing species that differ in dispersal ability may face a range of alternative futures when their landscape is changing underneath them. We found scenarios where 1) both species may continue to exist in the same relative abundance (i.e., the density of the dominant competitor is consistently greater than that of the inferior competitor), 2) this pattern of relative abundance is reversed, and 3) one of the interacting species (surprisingly, sometimes the dominant competitor) is lost from the system. Even with only two interacting species, this set of possible outcomes includes two 'no-analogue' communities (scenarios 2 and 3) that do not match the situation that occurs in the absence of climate change. It seems logical that with a modestly larger set of competing species, vastly more alternative futures (with different combinations of species at different relative abundances) would be possible. Importantly, our results suggest that it is not just species' competitive abilities but rather their relative dispersal abilities that will shape the no-analogue communities that emerge as a result of climate. How other types of species interactions, such as predation, parasitism, and mutualism, will act together with sets of competing species to form future communities remains an open, and intriguing, question.

The method of linear determinacy was first developed by Weinberger, Lewis, and Li [26] in studying spatial-temporal models with constant coefficients, and specifically model (1.2) with $r_i(x-ct) \equiv r_i(\infty)$ [16]. We successfully applied the method to (1.2) with variable $r_i(x-ct)$ by using integral recursions and developing sequences of functions approaching real solutions in appropriate moving intervals. The condition $c_2^*(\infty) > c_1^*(\infty) + \bar{c}(\infty)$ in both Theorems 2.1-2.2 may not be simplified to $c_2^*(\infty) > c_1^*(\infty)$. The kind of condition that one speed is bigger than the other speed plus a positive number may be necessary to determine spreading dynamics of competition models where both species expand their spatial ranges even for the constant coefficient case; see Girardin and Lam [8]. Huang [14] and Alhasanat and Ou [1] obtained linear determinacy conditions that improve the results in [16] for (1.2) with $r_i(x-ct) \equiv r_i(\infty)$. It would be of interest to find linear determinacy conditions weaker than (LD) for (1.2) with a shifting habitat.

Some results obtained in this paper might be extended to n-species competition models. For example, in a habitat shifting rightward at speed c, if $n-1$ competing species have developed waves, and if the ith species among the $n-1$ invaders has the largest rightward spreading speed $c_i^*(\infty)$ with $c_i^*(\infty) > c$. The framework provided in this paper shows that the species i can establish a wave in front of all other species and spread rightward at speed $c_i^*(\infty)$. The proof of Theorem 2.1 involves useful upper solution and lower solution obtained on moving intervals on which at most one species persists. This provides a possible way to study the population dynamics when an n-th species is introduced into competition under the condition that for two species with closest spreading speeds, one species is competitively stronger than the other. We leave this problem for further investigation.

REFERENCES

[1] A. Alhasanat, C. Ou, Minimal-speed selection of traveling waves to the Lotka–Volterra competition model, J. DifferentialEquations 266 (2019) 7357–7378.
[2] H. Berestycki, O. Diekmann, C.J. Nagelkerke, P.A. Zegeling, Can a species keep pace with a shifting climate?, Bull. Math. Biol. 71 (2009) 399-429.
[3] H. Berestycki, J. Fang, Forced waves of the Fisher-KPP equation in a shifting environment, J. Differential Equations 264 (2018) 2157-2183.
[4] H. Berestycki, L. Rossi, Reaction-diffusion equations for population dynamics with forced speed. II. Cylindrical-type domains, Discrete Contin. Dyn. Syst. 25 (2009) 19-61.
[5] J. Bouhours, T. Giletti, Spreading and vanishing for a monostable reaction-diffusion equation with forced speed, J. Dynam. Differential Equations 31 (2019) 247-286.
[6] J. Fang, Y. Lou, J.H. Wu, Can pathogen spread keep pace with its host invasion?, SIAM J. Appl. Math. 76 (2016) 1633-1657.
[7] S.E. Gilman, M.C. Urban, J. Tewksbury, G.W. Gilchrist, R.D. Holt, A framework for community interactions under climate change, Trends Ecol. Evol. 25 (2010) 325-331.
[8] L. Girardin, K.Y. Lam, Invasion of open space by two competitors: spreading properties of monostable two-species competition-diffusion systems, Proc. Lond. Math. Soc. 119 (2019) 1279-1335.
[9] R.W. Graham, Quaternary mammal communities: relevance of the individualistic response and non-analogue faunas. The Paleontological Society Papers, 11 (2005) 141-158.
[10] F. Hamel, Reaction-diffusion problems in cylinders with no invariance by translation. II. Monotone perturbations, Ann. Inst. H. Poincaré Anal. Non Linéaire 14 (1997) 555-596.
[11] F. Hamel, L. Roques, Uniqueness and stability properties of monostable pulsating fronts, J. Eur. Math. Soc. 13 (2011) 345-390.
[12] J. HilleRisLambers, M.A. Harsch, A.K. Ettinger, K.R. Ford, E.J. Theobald, How will biotic interactions influence climate change-induced range shifts?. Ann. N.Y. Acad. Sci. 1297 (2013), 112-125.
[13] C. Hu, J. Shang, B. Li, Spreading speeds for reaction-diffusion equations with a shifting habitat, J. Dynam. Differential Equations 32 (2020) 1941-1964.
[14] W. Huang, Problem on minimum wave speed for a Lotka–Volterra reaction–diffusion competition model, J. Dynam. Differential Equations 22 (2010) 285–297.
[15] B. Huntley How plants respond to climate change: migration rates, individualism and the consequences for plant communities. Annals of Botany 67 (1991) 15-22.
[16] M.A. Lewis, B. Li, H.F. Weinberger, Spreading speed and linear determinacy for two-species competition models, J. Math. Biol. 45 (2002) 219-233.
[17] B. Li, S. Bewick, M.R. Barnard, W.F Fagan, Persistence and spreading speeds of integro-difference equations with an expanding or contracting habitat, Bull. Math. Biol. 78 (2016) 1337-1379.
[18] B. Li, S. Bewick, J. Shang, W.F Fagan, Persistence and spread of a species with a shifting habitat edge, SIAM J. Appl. Math. 74 (2014) 1397-1417.
[19] B. Li, J.H. Wu, Traveling waves in integro-difference equations with a shifting habitat, J. Differential Equations 268 (2020) 4059–4078.
[20] W.T. Li, J.B. Wang, X.Q. Zhao, Spatial dynamics of a nonlocal dispersal population model in a shifting environment, J. Nonlinear Sci. 28 (2018) 1189-1219.
[21] B. Reu, S. Zaehle, K. Bohn, R. Pavlick, S. Schmidtlein, J.W. Williams, A. Kleidon, Future no-analogue vegetation produced by no-analogue combinations of temperature and insolation, Global Ecol. Biogeogr. 23 (2014) 156-167.
[22] J. Stantongeddes, P. Tiffin, R.G. Shaw, Role of climate and competitors in limiting fitness across range edges of an annual plant, Ecology 93 (2012) 1604-1613.
[23] M.C. Urban, J.J. Tewksbury, K.S. Sheldon, On a collision course: competition and dispersal differences create no-analogue communities and cause extinctions during climate change, Proc. R. Soc. B. 279 (2012) 2072-2080.
[24] J.B. Wang, X.Q. Zhao, Uniqueness and global stability of forced waves in a shifting environment, Proc. Amer. Math. Soc. 147 (2019) 1467-1481.
[25] J. Watrin, A.M. Lézine, C. Hély, Plant migration and plant communities at the time of the "green Sahara", C.R. Geoscience 341 (2009) 656-670.
[26] H. Weinberger, M.A. Lewis, B. Li, Analysis of linear determinacy for spread in cooperative models, J. Math. Biol. 45 (2002) 183-218.
[27] Y.D. Yuan, Y. Wang, X.F. Zou, Spatial dynamics of a Lotka-Volterra model with a shifting habitat, Discrete Contin. Dyn. Syst. Ser. B 24 (2019) 5633-5671.
[28] Z. Zhang, W. Wang, J. Yang, Persistence versus extinction for two competing species under a climate change, Nonlinear Anal. Model. Control 22 (2017) 285-302.