ON THE RAMANUJAN CONJECTURE FOR AUTOMORPHIC FORMS OVER FUNCTION FIELDS I. GEOMETRY

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Abstract. Let G be a split semisimple group over a function field. We prove the temperedness at unramified places of automorphic representations of G, subject to a local assumption at one place, stronger than supercuspidality, and assuming the existence of cyclic base change with good properties. Our method relies on the geometry of Bun_G. It is independent of the work of Lafforgue on the global Langlands correspondence.

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1. Main result

Let F be the function field of a smooth projective curve over a finite field k. The Ramanujan conjecture that every cuspidal automorphic representation of $\mathrm{GL}(r)$ with unitary central character is tempered is established by L. Lafforgue [42]. For general reductive groups, cuspidal automorphic representations that are known to be tempered arise in the works of Lomeli [50] for generic representations of split classical groups, and of Heinloth–Ngô–Yun [33] and Yun [66, 65] for rigid representations.

For a reductive group G, it is well-known that the cuspidality condition is not sufficient to imply temperedness, which led to the formulation of Arthur's conjectures [2]. For example, there are two classical constructions of cuspidal non-tempered automorphic representations for Sp_4 by Saito–Kurokawa and Howe–Piatetskii-Shapiro [35].

Thus, if we want to prove that π is tempered, we need a condition on π stronger than cuspidality. We shall impose that π_u is supercuspidal for one place u. This is still not sufficient

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as the above examples [35] show, and Arthur's conjecture points towards the condition that π_u belongs to a supercuspidal L-packet. We shall introduce a further condition that π_u is *monomial geometric supercuspidal*, and establish the Ramanujan bound in this case. The concept will be discussed in detail below. In brief it means that π_u is compactly induced from a character on a "nice enough" open subgroup of $G(F_u)$. We also need another Condition BC from Section 5 below, on the existence of an automorphic base change for constant field extensions.

Theorem 1.1. *Assume that G is split semisimple, and that* $\mathrm{char}(F) > 2$. *Suppose that*

- *for at least one place u, the representation π_u is monomial geometric supercuspidal;*
- *π is base-changeable in the sense of Condition BC.*

Then π is tempered at every unramified place.

Langlands theorem on the analytic continuation of Eisenstein series implies that CAP representations are non-tempered at every unramified place. Combined with Theorem 1.1, it follows that π is not CAP.

Remark 1.2. Recently, V. Lafforgue [44] constructed global parameters using shtukas and excursion operators. An automorphic consequence is that π is tempered at one unramified place if and only if it is tempered at every unramified place (Theorem 11.7 below), which was [11, Conj.4(1)].

The present paper focuses on establishing a Ramanujan bound on average, see (1.1) below, and deducing Theorem 1.1. It is part of a series of two articles, and the next [55] will focus on providing examples of representations that satisfy Condition BC, and on establishing the functorial image between inner-forms which will enable us to reduce cases of the Ramanujan bounds for general reductive groups to the split semisimple case.

1.1. **Monomial geometric supercuspidal representations (mgs).** The definition of monomial geometric supercuspidal is motivated by features of the problem and our method to attack it.

We rely on studying families defined by *local prescribed behavior*, which means in our context a set of automorphic representations of $G(\mathbb{A}_F)$ that satisfy some given conditions at a fixed finite set of places and are unramified outside. If we can show temperedness for one member of the family by our method, the same argument applies to every member of the family. So we must impose strong enough local conditions. At minimum, we should avoid Eisenstein series, and, for at least one place u, requiring that π_u be supercuspidal is the easiest way to achieve this.

Our method is geometric, and requires a geometric way to check the local condition. We focus on *monomial local conditions*. These are the conditions defined by fixing a subgroup J of $G(F_u)$ and a character $\chi : J \to \mathbb{C}^\times$, and demanding that the local representation π_u of $G(F_u)$ contains a vector that transforms according to χ under the action of J. There is a natural geometric description of the set of automorphic forms satisfying a monomial local condition as long as J is the group of k-rational points of a pro-algebraic subgroup of the loop group $G[\![t]\!]$ and χ is the trace function of a character sheaf. This is certainly not the most general possible way to construct a geometric object that defines a local condition on automorphic representations — in fact the geometric Langlands program suggests that there should be geometric objects corresponding to all automorphic representations, in a suitable sense — but it is easy to work with and contains many important examples. A general formalism of monomial local conditions for automorphic representations was already used by Yun [64, §2.6.2]. Our setup (Section 6) is essentially Yun's formalism restricted to a special case for both geometric and notational simplicity (and for this reason we use somewhat different notation).

Geometric objects behave similarly over different fields. In our case, the relevant geometric objects are defined over the constant field k, and so it is possible to base change them along a constant field extension. If we use any geometric property to prove temperedness, this property will be maintained over constant field extensions, and so temperedness must hold not only for all members of the family, but also for all members of the analogous family after extension of the constant field k. In particular, these representations must not be Eisenstein series. Again, the easiest way to ensure this is to ensure that our character (J, χ) still prescribes a supercuspidal representation after a constant field extension. This yields the notion of *monomial geometric supercuspidal datum* (Definition 3.5).

Another advantage of adding the monomial and geometric modifiers to the supercuspidal local condition is that it allows us to sidestep the unipotent supercuspidal representations. The usual construction of these is not by a monomial representation but rather from representations of finite groups of Lie type. We expect that no monomial geometric construction of unipotent representations exists. For example in Deligne–Lusztig theory, irreducible representations are induced from characters on elliptic tori, but this fails to work uniformly after finite field extensions, since every torus eventually splits.

The local conditions we define are *geometric* in precisely the sense of the geometric Langlands program. However, there is one major difference in our approach. Progress in the geometric Langlands program has mainly focused on first studying automorphic forms that are everywhere unramified, and then generalizing to unipotent or tame ramification, before beginning to tackle the general case. In our problem, we find it is convenient to study highly ramified automorphic forms — in particular, including local factors with wildly ramified Langlands parameters — which necessitates working in a more general setup. We do this because when one of the local factors is supercuspidal, the Hecke kernels in the family will correspond to pure perverse sheaves (Theorem 7.36), although we also believe the more general setup is interesting on its own terms.

More formally, let G be a quasi-split reductive group over a field k. We start with the datum of a pro-algebraic subgroup H of $G[\![t]\!]$ containing the subgroup of elements congruent to 1 modulo t^m for some m, and a character sheaf $\mathcal{L}$ on H which is trivial on that subgroup. We say this datum is *geometrically supercuspidal* if for every parabolic subgroup $P \subset G_{\overline{k}}$ with radical N, and every $g \in G_{\overline{k}}[\![t]\!]$, the restriction of $\mathcal{L}_{\overline{k}}$ to the identity component of $gN_{\overline{k}}[\![t]\!]g^{-1} \cap H_{\overline{k}}$ is non-trivial. (The intersection takes place in $G_{\overline{k}}[\![t]\!]$.)

If $k = \mathbb{F}_q$ is a finite field, this occurs if and only if $\text{c-Ind}_{J_n}^{G(\mathbb{F}_{q^n}(\!(t)\!))} \chi_n$ is admissible supercuspidal for every integer $n \geq 1$, where $J_n := H(\mathbb{F}_{q^n})$ and χ_n is the trace function of $\mathcal{L}$ over $\mathbb{F}_{q^n}$ (Lemma 3.6).

1.2. **Ramanujan bound for** $\mathrm{GL}(r)$.

For the general linear group, the Ramanujan bound is the statement that a cuspidal automorphic representation of $\mathrm{GL}(r)$ with unitary central character is tempered at every place. One can distinguish two main approaches:

- Laumon [49] under a cohomological condition at one place, extending Drinfeld's first proof [19] for GL(2), using elliptic modules.
- L. Lafforgue [42] in general, extending Drinfeld's second proof [21] for GL(2), using shtukas.

Our approach is yet different, even in the case of $\mathrm{GL}(r)$, under the mgs (monomial geometric supercuspidal) condition. Rather than using moduli spaces of elliptic modules or shtukas, we study moduli spaces $\mathrm{Bun}_{\mathrm{GL}(r)}$ of vector bundles, as in the geometric Langlands program. Functions on these moduli spaces give rise to families of automorphic forms satisfying certain local prescribed conditions. We will prove temperedness using estimates for an entire family at once, rather than working with individual automorphic forms in the family.

1.3. **Outline of the proof.** We embed π in a suitable *automorphic family* $(\mathcal{V}_n)_{n\geq 1}$. We let $\mathcal{V}_1$ consist of the multi-set of automorphic representations Π of $G(\mathbb{A}_F)$, counted with multiplicities, such that Π_u has a non-zero (J,χ)-invariant vector, Π has bounded ramification at a fixed finite set of places, and Π is unramified elsewhere. The ramification bound is chosen compatibly with the original representation π in such a way that $\pi \in \mathcal{V}_1$. Since (J,χ) arises from a supercuspidal datum, all $\pi \in \mathcal{V}_1$ are supercuspidal.

For every integer $n \geq 1$, consider the constant field extension $F_n := F \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}$, assuming $k = \mathbb{F}_q$. We let $\mathcal{V}_n$ consist of automorphic representations of $G(\mathbb{A}_{F_n})$ with similar bounded ramification and with mgs prescribed behavior at the places of F_n above u, namely with a non-zero (J_n,χ_n)-invariant vector. Again all $\Pi \in \mathcal{V}_n$ are cuspidal.

Let $v \in X(k)$ be a k-rational point such that π_v is unramified. To study the temperedness of π_v, we shall consider the local components Π_v for $\Pi \in \mathcal{V}_n$. More precisely, for a coweight λ of G, we shall consider the collection of all traces of Hecke operators $\mathrm{tr}_\lambda(\Pi_v)$ for $\Pi \in \mathcal{V}_n$.

We express the kernel of this Hecke operator as the trace function of a complex of sheaves, which we will show, as consequence of our mgs local prescribed behavior, is a pure perverse sheaf (Theorem 7.36). This will imply, by standard estimates for the trace functions of perverse sheaves, a bound for the trace of a Hecke operator in the family (Theorem 10.2), which takes the form

$$\sum_{\Pi \in \mathcal{V}_n} |\,\mathrm{tr}_\lambda(\Pi_v)|^2 \lesssim C_\lambda \cdot q^{nd} \tag{1.1}$$

Here d depends on the underlying group G and the prescribed conditions, and C_λ is the dimension of some cohomology groups and it is essential for us that it is independent of n (it only depends on the underlying group G, the fixed local prescribed conditions, and the chosen unramified place v).

If we first examine the $\lambda = 0$ case, we see that the number of automorphic representations in the family is at most $C_0 \cdot q^{nd}$. This bound should be close to the truth — one expects that the sum on the geometric side of the trace formula for the number of automorphic forms in the family $\mathcal{V}_n$ is dominated by the contribution of the trivial conjugacy class, which is an adelic volume, and one can show this adelic volume $\approx C \cdot q^{nd}$ for another explicit constant C.

Furthermore, the Ramanujan bound would imply $|\,\mathrm{tr}_\lambda(\Pi_v)| \leq \dim(V_\lambda)$, so conditionally on the Ramanujan bound for all representations of $\mathcal{V}_n$, we obtain

$$\sum_{\Pi \in \mathcal{V}_n} |\,\mathrm{tr}_\lambda(\Pi_v)|^2 \lesssim C \cdot \dim(V_\lambda)^2 \cdot q^{nd}.$$

Thus, (1.1) is as strong as the Ramanujan bound on average over the family $\mathcal{V}_n$, except that the constant C_λ has unknown dependence on λ, whereas in the Ramanujan bound on average the constant $\dim(V_\lambda)^2$ has explicit, mild dependence on λ.

This suggests that we are on the right track, but that the constant C_λ is problematic.

Here comes the final step. Because C_λ is constant in n while every other term is exponential in n, the quality of the estimate (1.1) improves as n goes to infinity. To take advantage of this, we will use automorphic base change for constant field extensions F_n/F to amplify the estimate, and deduce $|\operatorname{tr}_\lambda(\pi_v)| \leq \dim(V_\lambda) \cdot q^{\frac{d}{2}}$ for our original representation π. Varying λ, we can further bootstrap this estimate to

$$|\operatorname{tr}_\lambda(\pi_v)| \leq \dim(V_\lambda),$$

which is the temperedness of the unramified representation π_v.

Remark 1.3. Recall from [11] the following conjecture: π should be tempered at every unramified place as soon as π_u is the Steinberg representation for some place u. Compared to this, our situation consists in replacing the Steinberg condition by a more ramified condition. Our method of proof doesn't extend to the case of the Steinberg representation because the Euler–Poincaré function is an alternating sum, which we do not know how to geometrize globally to a pure sheaf on Bun_G.

1.4. **Contrasting Drinfeld's modular varieties and** Bun_G**.** This subsection does not directly describe our argument, but we hope it provides some intuition that will be helpful to the reader.

The moduli spaces of shtukas and Bun_G are both stacks whose geometries carry information about automorphic forms over function fields, but they carry it in different ways and have different properties.

Each moduli space of shtukas can be related to a particular family of automorphic forms with a particular set of Hecke operators acting on it. For example, the moduli space of shtukas $\mathrm{Cht}_{D,I,W}^{(I)}$ defined in [44, Def.0.2] can be related to the family of automorphic forms of level D on $G(\mathbb{A}_F)$, with the set of Hecke operators determined by the representations W.

The geometry of the moduli space casts light on this family. More precisely, the cohomology of the moduli space $\mathrm{Cht}_{D,I,W}^{(I)}$ relative to the base is expected to be a sum over automorphic forms of level D of local systems constructed from their Langlands parameters [44, Rem.0.30]. The arithmetic structure on the moduli space carries additional information about the automorphic forms in this family. For instance, the Galois action on the cohomology of a moduli space of $\mathrm{GL}(r)$-shtukas with level structure determines the Galois action on the Langlands parameters of the cusp forms of that level [42, Lem.VI.26 and Thm.VI.27].

On the other hand, Bun_G is related to a sequence $\mathcal{V}_n$ of spaces of automorphic forms, one over each finite field extension $\mathbb{F}_{q^n}$ of the base field $\mathbb{F}_q$. In fact, the set of rational points $\mathrm{Bun}_G(\mathbb{F}_{q^n})$ is the quotient of $G(\mathbb{F}_{q^n}(X))\backslash G(\mathbb{A}_{\mathbb{F}_{q^n}(X)})$ by a maximal compact subgroup, so the space of functions on $\mathrm{Bun}_G(\mathbb{F}_{q^n})$ is the space of automorphic forms of level 1 on $G_{\mathbb{F}_{q^n}}(X)$. Thus, the space Bun_G contains information about automorphic forms of level 1 on $G_{\mathbb{F}_{q^n}}(X)$ for all n. (Variants of Bun_G with level structure hold the same relationship to spaces of automorphic forms of higher level.)

Because geometry is insensitive to base change, the geometry of Bun_G is only related to asymptotic information about these spaces of automorphic forms as $q^n \to \infty$ (or possibly other subtler sorts of information that are invariant on passing to subsequences). For instance, by the Lefschetz fixed point formula, the dimension of the space of automorphic forms of level 1 on $G_{\mathbb{F}_{q^n}}(X)$ equals the number of $\mathbb{F}_{q^n}$-points of Bun_G which equals the supertrace of Frobenius on the cohomology of Bun_G (Lemma 9.9 and Proposition 10.1), so the cohomology of Bun_G gives information about the dimension of all the spaces of automorphic forms in the sequence. (However, for any nontrivial G, there exists some n such that Bun_G will have infinitely many $\mathbb{F}_{q^n}$-points. To rigorously relate cohomology to counting automorphic forms we must make this count finite, which requires us to fix a central character, and, in addition, do something to remove Eisenstein series. In our paper the supercuspidal local prescribed conditions discussed in §1.1 are used to remove the Eisenstein series.)

This fundamental difference can explain many of the more basic differences between the geometry of the moduli space of shtukas and Bun_G — for instance, their dimensions.

The dimension of the moduli space of shtukas $\mathrm{Cht}^{(I)}_{N,I,W}$ depends on the group G and on the representations W_i of the Langlands dual group occuring at the legs $i \in I$, but does not depend on the level N — in fact, moduli spaces of shtukas of higher level are finite étale covers of moduli spaces of shtukas of lower level. On the other hand, the dimension of the moduli space $\mathrm{Bun}_{G(N)}$ of G-bundles with level N structure depends on both the group G and the level N, while the representations W do not appear in the definition.

We can explain this discrepancy between the dimensions of $\mathrm{Cht}^{(I)}_{N,I,W}$ and $\mathrm{Bun}_{G(N)}$ by looking at how the dimension is reflected in the associated spaces of automorphic forms. Recall here that the dimension of a space determines the largest possible size of Frobenius eigenvalues on its compactly supported cohomology. (Of course, in each case it is possible to calculate the dimensions much more directly than this. The point of this argument is to see why the simple concrete properties of these two spaces are necessary for their respective applications.)

We expect the cohomology of the moduli space of shtukas $\mathrm{Cht}^{(I)}_{N,I,W}$ to be a sum of contributions associated to different automorphic forms, with each contribution the tensor product over legs i of the representation W_i composed with the Langlands parameter. The size of the Frobenius eigenvalues acting on W_i depends on the weights of the representation W_i. On the other hand, there is no reason for highly ramified Langlands parameters to have different Frobenius eigenvalues from less ramified parameters. (For instance, because Langlands parameters can become more or less ramified under pullback, without changing their Frobenius weights.) Thus, it is reasonable to expect that the dimension depends on the choice of W_i, but not on the level.

On the other hand, the Frobenius eigenvalues on the cohomology of $\mathrm{Bun}_{G(N)}$ are relevant because they give a formula for the dimension of the spaces of automorphic forms of level N on $G(\mathbb{A}_{\mathbb{F}_{q^n}(X)})$. In particular, as n goes to ∞, the largest Frobenius eigenvalue should dominate, and so the largest Frobenius eigenvalue should match the asymptotic growth rate in n of the dimension of this space of automorphic forms. We can calculate the dimension of this space of automorphic forms by the trace formula, where the main term is one over the volume of the level N subgroup of $G(\mathbb{A}_{\mathbb{F}_{q^n}}(X)$. This inverse volume grows with both the degree n and level N — in fact, it is approximately $q^{n(\dim G)(g+|N|-1)}$, where $|N|$ is the degree of the divisor N. Thus, it is reasonable to expect the dimension of $\mathrm{Bun}_{G(N)}$ is $(\dim G)(g + |N| - 1)$, as indeed it is.

Similarly, the number of forms of level N on $G(\mathbb{A}_{\mathbb{F}_{q^n}(X)})$ with a nonzero (J, χ)-equivariant vector, is approximately $q^{n((\dim G)(g+|D|-1)-\dim H)}$(see §10.3).

This also suggests differences in their potential arithmetic applications. The moduli spaces of shtukas are well-suited to prove the automorphic-to-Galois direction of the Langlands correspondence because each automorphic form, and its associated Langlands parameter, appears in their cohomology. Of course this is exactly why Drinfeld [19] introduced them and how L. Lafforgue [42] and V. Lafforgue [44] used them, and it seems likely that researchers will continue to deduce information about the Langlands correspondence from study of these moduli spaces in the future. But Bun_G is not well-suited for this purpose, as with the number of automorphic forms going to infinity as $q^n \to \infty$, it is harder to pick out a single one. Though an analogue of the automorphic-to-Galois direction of the Langlands correspondence is part of the geometric Langlands program over the complex numbers, it is not clear what, if any, the finite field analogue might be.

On the other hand, Bun_G does seem well-suited to answer asymptotic questions about how analytic quantities, such as averages of Hecke operators, behave when $q^n \to \infty$, as we demonstrate in the present paper. The Ramanujan bound and Arthur's conjectures seem to lie in the intersection of these two domains — it can be attacked using Langlands parameters, but also can be viewed as a question of the $q^n \to \infty$ limit. Thus there is potential to use both approaches to prove new cases of Arthur's conjectures.

1.5. **Results on families.** Because our method to prove the main theorem relies on families of automorphic forms defined by geometric monomial local conditions, along the way we obtain some new results about these families. We expect further results can be obtained this way using our work in the future. For this reason we discuss the strengths and weaknesses of restricting to monomial representations from the point of view of families (rather than with regards to proving the Ramanujan bound for individual automorphic forms). Given a family of automorphic forms unramified away from some finite set of places, and defined by some local conditions at the remaining places, questions such as the following have been considered:

(1) Can the number of forms in the family be expressed as a finite sum of Weil numbers?
(2) What about the trace of a Hecke operator on this space of forms?
(3) Can the Weil numbers that appear in these sums be calculated explicitly?
(4) Can these sums be approximated, or can the largest Weil numbers appearing in them be estimated?

Question (1) and question (3) were answered affirmatively by Drinfeld [20] in the case of everywhere unramified automorphic forms on GL(2), by Flicker for forms on GL(2) that are Steinberg at one place and unramified everywhere else [24], by Deligne and Flicker [18] for forms on GL(r) that are Steinberg at at least two places, and unramified everywhere else, and by Yu [63] for forms on GL(r) that are unramified everywhere. Of course answering (3) is sufficient to answer question (4).

In this paper we answer question (1) in the case of monomial geometric conditions, supercuspidal at at least one place, and unramified elsewhere (Proposition 10.4). And most importantly we answer question (2), in the form that $\sum_{\Pi \in \mathcal{V}_n} q^{n\langle \lambda, \rho\rangle} |\operatorname{tr}_\lambda(\Pi_v)|^2$ is a signed sum of length C_λ of nth powers of q-Weil integers of weight $\leq 2d + \langle \lambda, 2\rho\rangle$. This is actually how we establish the main estimate (1.1). See Theorem 9.15 and §10.3 for details.

1.6. **Local prescribed behavior.** There are many different kinds of local conditions that appear in the theory of automorphic forms. As mentioned before, we work with local conditions that demand the representation contain an eigenvector of a compact open subgroup J with eigenvalue χ, where J and χ arise from geometric objects — an algebraic subgroup of $G(\kappa[t]/t^m)$ for some m and a character sheaf on that algebraic subgroup. The theory of inertial types produces many examples where this condition, for a suitable choice of (J, χ), characterizes the representation up to an unramified twist (e.g. the twist-minimal supercuspidal representations of GL(2) with conductor not congruent to 2 modulo 4). However, not all representations can be characterized up to an unramified twist this way (e.g., the twist-minimal supercuspidal representations of GL(2) with conductor congruent to 2 mod 4). But it may still be possible to characterize the representation up to a tamely ramified twist or other mild variant.

Choosing (J, χ) whose associated local condition uniquely picks out a given representation is very similar to the problem of constructing the representation as an induced representation (but slightly easier as one is allowed to produce the representation with multiplicity). Yu has shown how to construct a wide class of supercuspidal representations using Deligne–Lusztig representations of algebraic groups over finite fields and Heisenberg–Weil representations. (For instance, in the GL(2) twist-minimal case with conductor congruent to 2 mod 4, Deligne–Lusztig theory is needed for conductor 2 and Heisenberg–Weil representations are needed for higher conductor).

The matrix coefficients of the Weil representation were expressed as the trace function of a perverse sheaf in a 1982 letter of Deligne, and the same was done in [30] to the coefficients in a basis consisting of the matrices appearing in the Heisenberg representation. It is likely that much of what we do can be generalized using this geometrization. Sheaves whose trace functions are the traces of discrete series representations were constructed [51] but we do not know if there is any way to do the same for matrix coefficients (it is not clear what basis to use). It could also be possible to replicate our methods using just the trace and not all the matrix coefficients, but we are less certain of it.

Using these tools to make these representations geometric would follow the strategy of [15]. Note, however, some differences with their work. Their goal was to geometrize the trace of the automorphic representation, while our construction has the effect of geometrizing a test function, and they handled p-adic groups while we work in the equal characteristic case.

For our problem, new difficulties appear when adding Heisenberg–Weil and Deligne–Lusztig representations and their more complicated sheaves. Because restricting to one-dimensional characters, and their associated character sheaves, will simplify things at several points, we leave the full theory to a later date.

2. Preliminaries

2.1. **Unramified groups.** Let k be a finite field. We say a connected reductive group over $k(\!(t)\!)$ is *unramified* if it is quasi-split and splits over $\overline{k}(\!(t)\!)$. The following is well-known. Since we couldn't locate the result in the literature, we provide a quick proof.

Lemma 2.1. *An unramified group over $k(\!(t)\!)$ is the base change $G_{k(\!(t)\!)}$ of a reductive group G over k.*

Proof. Bruhat–Tits [7, §4.6.10], and [46, Chap.II], establish the existence of a model $\mathcal{G}$ that is a smooth affine group scheme over $k[\![t]\!]$, with reductive special fiber. Let $G := \mathcal{G}_\kappa$ be this special fiber. According to [12, Rem.7.2.4], the classification of forms of a reductive group over a Henselian local field with finite residue field is the same as the classification over the residue field. Indeed let $\mathcal{G}$, and $\mathcal{G}'$ be two connected reductive group schemes over $k[\![t]\!]$. Suppose their special fibers over k are isomorphic. The scheme of isomorphisms from $\mathcal{G}$ to $\mathcal{G}'$ is smooth, and has a point over k, so has a section over $k[\![t]\!]$. In particular if we take $\mathcal{G}'$ to be a constant group scheme G, we get that $\mathcal{G}$ is constant as well. □

Remark 2.2. The same notion of unramified group arises in mixed characteristic, that is over a finite extension K of $\mathbb{Q}_p$. In that context, it is standard that there is a smooth model $\mathcal{G}$ over the local ring $\mathfrak{o}_K$, and that $\mathcal{G}(\mathfrak{o}_K)$ is a hyperspecial maximal subgroup. This is analogous to Lemma 2.1, where the model is given by $G_{k[\![t]\!]}$, and the hyperspecial maximal subgroup by $G(k[\![t]\!])$, only that in equal characteristic the statement is simpler, and it is not necessary to introduce the group scheme $\mathcal{G}$. In mixed characteristic, the lifting argument still works, but there is no notion of constant group scheme over $\mathfrak{o}_K$ (though an analogue could likely be constructed using Witt vectors).

Lemma 2.3. *Let G be a reductive group over a finite field k. Let X be a smooth connected algebraic curve over k. Then every G-torsor on X admits a trivialization over the generic point.*

Proof. Let $F = \mathbb{F}_q(X)$. By [52, Lem.1.1], it is sufficient to check that the kernel $\ker^1(F, G)$ of the natural map from $H^1(F, G)$ to the product over all places x of $H^1(F_x, G)$ is trivial. By [60, Thm.2.6(1)], the kernel $\ker^1(F, G)$ is Pontryagin dual to $\ker^1(F, Z(\widehat{G}))$. To show that $\ker^1(F, Z(\widehat{G}))$ is trivial, it suffices to fix a nontrivial F-torsor $\mathcal{T}$ of $Z(\widehat{G})$ and show it remains nontrivial upon restriction to some place.

We can describe an F-torsor $\mathcal{T}$ by the action of $\mathrm{Gal}(F)$ on $\mathcal{T}_{\overline{F}}$, where $\mathcal{T}_{\overline{F}}$ is a $Z(\widehat{G})_{\overline{F}}$-torsor in the sense of algebraic groups. Because torsors are by definition trivial over some étale open set, this $\mathrm{Gal}(F)$-action must factor through a finite group H.

If the $\mathrm{Gal}(F)$-action on $\mathcal{T}_{\overline{F}}$ factors through $\mathrm{Gal}(k)$, then we can take H to be a finite quotient of $\mathrm{Gal}(k)$, necessarily cyclic. Thus the Frobenius element at any place of degree prime to $|H|$ generates H, and so $\mathcal{T}$ is nontrivial if and only if it is nontrivial at one of these places.

If the $\mathrm{Gal}(F)$-action on $\mathcal{T}_{\overline{F}}$ does not factor through $\mathrm{Gal}(k)$, then because the $\mathrm{Gal}(F)$-action on $Z(\widehat{G})_{\overline{F}}$ does factor through $\mathrm{Gal}(k)$, we may find some conjugacy class $\sigma \in H$ which acts trivially on $Z(\widehat{G})_{\overline{F}}$ but nontrivially on $\mathcal{T}_{\overline{F}}$. By the Chebotarev density theorem, there exists some place v such that the image of Frob_v in H is conjugate to σ. The restriction $\mathcal{T}_v$ of $\mathcal{T}$ to v must be nontrivial because, since Frob_v acts trivially on $Z(\widehat{G})_{\overline{F}}$, it acts trivially on the trivial torsor over $Z(\widehat{G})_{\overline{F}}$, so $\mathcal{T}_v$ cannot be isomorphic to the trivial torsor over $Z(\widehat{G})$ as a set with Frobenius action.

(In the case when G is split simply-connected semisimple, this result could instead be deduced from a result of Harder [31, Thm.2.4.1] that if G is split and simply-connected semisimple, then $H^1(F, G)$ is trivial.) □

2.2. **Satake isomorphism.** In this subsection, let G be a split connected reductive group over a finite field k. Let $F = k(\!(t)\!)$, $\mathfrak{o} = k[\![t]\!]$, $K = G(\mathfrak{o})$, and consider the *unramified Hecke algebra*

$$\mathcal{H}(G) = \mathcal{H}(G(F), K) = \mathcal{C}_c(K\backslash G(F)/K, \mathbb{C}).$$

We are fixing the Haar measure on $G(F)$ to give K volume one. The below results hold more generally over the base ring $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ rather than $\mathbb{C}$. Let $T \subset G$ be a maximal torus. There is an identication of the lattice $\Lambda := X_*(T)$ of coweights of G with the quotient group $T(F)/T(\mathfrak{o})$, where a cocharacter $\mu : \mathbb{G}_m \to T$ corresponds to the element $\mu(t) \in T(F)$ modulo multiplication by $T(\mathfrak{o})$. This induces an algebra isomorphism $\mathcal{C}_c(T(F)/T(\mathfrak{o})) \simeq \mathbb{C}[X_*(T)]$. The Weyl group $W = N_G(T)/Z_G(T)$ acts on both sides of this isomorphism, in particular we can form the subalgebras of W-invariant functions

$$\mathcal{C}_c(T(F)/T(\mathfrak{o}), \mathbb{C})^W \simeq \mathbb{C}[X_*(T)]^W.$$

Choose a Borel subgroup $B = TU$ and let $\Lambda^+ \subset \Lambda$ be the positive Weyl chamber. Let $\delta : B(F) \to q^{\mathbb{Z}}$ be the modulus character, where q is the size of k. Denote by $\delta^{\frac{1}{2}} : B(F) \to q^{\frac{1}{2}\mathbb{Z}}$, the positive square-root. For every $\mu \in \Lambda$, we have $\delta^{\frac{1}{2}}(\mu(t)) = q^{-\langle \rho, \mu \rangle}$, where $\rho \in X^*(T) \otimes_{\mathbb{Z}} \mathbb{C}$ is the half-sum of the positive roots. The *Satake transform* $\mathcal{S}(f)$ of a function $f \in \mathcal{H}(G) = \mathcal{H}(G(F), K)$ is defined by

$$\mathcal{S}(f)(s) := \delta^{\frac{1}{2}}(s) \int_{U(F)} f(su)du, \quad s \in T(F) \subseteq B(F),$$

where du is the Haar measure on $U(F)$ that gives $U(F) \cap K = U(\mathfrak{o})$ volume one. The value of the integral depends only on s modulo $T(\mathfrak{o})$. It induces an algebra isomorphism [29]

$$\mathcal{S} : \mathcal{H}(G) \xrightarrow{\sim} \mathcal{C}_c(T(F)/T(\mathfrak{o}), \mathbb{C})^W,$$

Denote by V_λ the irreducible representation of $\widehat{G}(\mathbb{C})$ with highest weight

$$\lambda \in \Lambda^+ \subset \Lambda = X_*(T) = X^*(\widehat{T}).$$

The trace of a finite dimensional representation V of $\widehat{G}(\mathbb{C})$ can be viewed as an element of $\mathbb{C}[X^*(\widehat{T})]^W$ by recording its weight spaces multiplicities $\dim \mathrm{Hom}_{\widehat{T}(\mathbb{C})}(\mu, V)$ for all $\mu \in X^*(\widehat{T})$, hence it corresponds to an element of the Hecke algebra $\mathcal{H}(G)$ under the Satake isomorphism. In particular, the trace of the representation V_λ is of the form

$$\mathrm{tr}(V_\lambda) = \sum_{\mu \in X^*(\widehat{T})} \dim \mathrm{Hom}_{\widehat{T}(\mathbb{C})}(\mu, V_\lambda) \cdot [\mu] = \mathcal{S}(a_\lambda) \tag{2.1}$$

for a *unique element* $a_\lambda \in \mathcal{H}(G)$. As we vary $\lambda \in \Lambda^+$, the elements a_λ form a linear basis of $\mathcal{H}(G)$ since $\mathrm{tr}(V_\lambda)$ form a linear basis of $\mathbb{C}[X^*(\widehat{T})]^W$ in view of highest weight theory.

Proposition 2.4 (Satake)**.** *There is a bijection between isomorphism classes of irreducible K-unramified representations π, algebra homomorphisms $\mathrm{tr}(\pi) : \mathcal{H}(G) \to \mathbb{C}$, W-conjugacy classes of unramified characters $\chi : T(F)/T(\mathfrak{o}) \to \mathbb{C}^\times$, and semisimple conjugacy classes t_π in $\widehat{G}(\mathbb{C})$, characterized as follows:*

(i) The bijection $\pi \mapsto t_\pi$ coincides with the composition of the two bijections $\pi \mapsto \mathrm{tr}(\pi) \mapsto t_\pi$, where $\mathrm{tr}(\pi) : \mathcal{H}(G) \to \mathbb{C}$ is the trace functional, and where t_π is characterized by the equalities

$$\mathrm{tr}(\pi)(a_\lambda) = \mathrm{tr}(t_\pi | V_\lambda),$$

for every $\lambda \in \Lambda^+$.

(ii) The bijection between χ and t_π modulo W-conjugation is via the three identifications

$$\chi \in \mathrm{Hom}(T(F)/T(\mathfrak{o}), \mathbb{C}^\times) \simeq \mathrm{Hom}(X_*(T), \mathbb{C}^\times) = \mathrm{Hom}(X^*(\widehat{T}), \mathbb{C}^\times) = \widehat{T}(\mathbb{C}) \ni t_\pi.$$

(iii) The bijection $\chi \mapsto \mathrm{tr}(\pi)$ is characterized via the Satake isomorphism by the equalities

$$\mathrm{tr}(\pi)(f) = \sum_{s \in T(F)/T(\mathfrak{o})} \mathcal{S}(f)(s)\chi(s), \quad f \in \mathcal{H}(G), \tag{2.2}$$

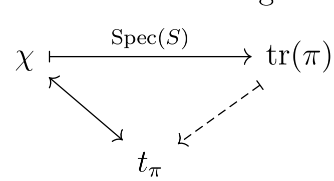
Proof. The bijection $\pi \mapsto \mathrm{tr}(\pi)$ in (i) is standard and follows from that $(G(F), K)$ is a Gelfand pair. By the second orthogonality relation of characters of the group $\widehat{G}(\mathbb{C})$, we have that the values of $\mathrm{tr}(t_\pi | V_\lambda)$ for varying $\lambda \in \Lambda^+$ characterize the element t_π up to $\widehat{G}(\mathbb{C})$-conjugation in $\widehat{G}(\mathbb{C})$, hence up to W-conjugation in $\widehat{T}(\mathbb{C})$. This shows that the identities in (i) characterise the map $\mathrm{tr}(\pi) \mapsto t_\pi$ uniquely. We shall verify below that t_π exists and the map is a bijective.

The identifications in (ii) have been given before the proposition. The Satake isomorphism induces

$$\chi \in \mathrm{Hom}(T(F)/T(\mathfrak{o}), \mathbb{C}^\times)/W \subset \mathrm{Spec}(\mathcal{C}_c(T(F)/T(\mathfrak{o}))^W) \xrightarrow{\mathrm{Spec}(\mathcal{S})} \mathrm{Spec}(\mathcal{H}(G)) \ni \mathrm{tr}(\pi),$$

where we identify $\mathrm{Hom}(T(F)/T(\mathfrak{o}), \mathbb{C}^\times)/W$ with the closed points of $\mathrm{Spec}(\mathcal{C}_c(T(F)/T(\mathfrak{o}))^W)$ which are algebra functionals $\mathcal{C}_c(T(F)/T(\mathfrak{o}))^W \to \mathbb{C}$. This yields the bijection in (iii) between χ and $\mathrm{tr}(\pi)$ via (2.2).

The rest of the proposition amounts to the following commuting triangle of bijections:



Indeed, we have verified above that the middle map, and the lower-left map are bijections. As a final step, it remains to show that the element t_π obtained by following the inverse bijections $\mathrm{tr}(\pi) \leftarrow\!\shortmid \chi \leftarrow\!\shortmid t_\pi$ so as to make the triangle commute satisfies the equalities $\mathrm{tr}(\pi)(a_\lambda) = \mathrm{tr}(t_\pi|V_\lambda)$ in (i).

In view of (2.2), we have for every $\lambda \in \Lambda^+$,

$$\mathrm{tr}(\pi)(a_\lambda) = \sum_{s\in T(F)/T(\mathfrak{o})} \mathcal{S}(a_\lambda)(s)\chi(s).$$

Under the identifications

$$s \in T(F)/T(\mathfrak{o}) \simeq X_*(T) = X^*(T) \ni \mu,$$

which are dual to those of (ii), we have the equality $\chi(s) = \mu(t_\pi)$. Moreover,

$$\dim \mathrm{Hom}_{\widehat{T}(\mathbb{C})}(\mu, V_\lambda) = \mathcal{S}(a_\lambda)(s)$$

by the definition (2.1) of a_λ. We obtain that the latter integral is equal to

$$\sum_{\mu\in X_*(T)=X^*(\widehat{T})} \dim \mathrm{Hom}_{\widehat{T}(\mathbb{C})}(\mu, V_\lambda)\cdot \mu(t_\pi) = \mathrm{tr}(t_\pi|V_\lambda),$$

which concludes the proof of the claim. □

Definition 2.5. For $\lambda \in \Lambda^+$, and an irreducible K-unramified representation π, define

$$\mathrm{tr}_\lambda(\pi) := \mathrm{tr}(\pi)(a_\lambda) = \mathrm{tr}(t_\pi|V_\lambda).$$

The *unramified principal series* $\mathrm{Ind}_{B(F)}^{G(F)}(\delta^{\frac{1}{2}}\chi)$ contains a unique non-zero K-fixed vector v° given in the induced model

$$\{v : G(F) \to \mathbb{C}, \quad v(tug) = \delta^{\frac{1}{2}}(t)\chi(t)v(g), \quad t \in T(F),\ u \in U(F),\ g \in G(F)\}$$

by the formula

$$v^\circ(tuk) := \delta^{\frac{1}{2}}(t)\chi(t), \quad t \in T(F),\ u \in U(F),\ k \in K,$$

which is justified by the Iwasawa decomposition $G = B(F)K$ because $\delta^{\frac{1}{2}}\chi$ is trivial on $T(\mathfrak{o}) = T(F)\cap K = B(F)\cap K$. Every $f \in \mathcal{H}(G)$ acts on the vector v° by the scalar

$$\sum_{s\in T(F)/T(\mathfrak{o})} \mathcal{S}(f)(s)\chi(s) = \int_{T(F)} \mathcal{S}(f)(s)\chi(s)ds,$$

as can be seen from the following calculation. For every $t \in T(F)$,

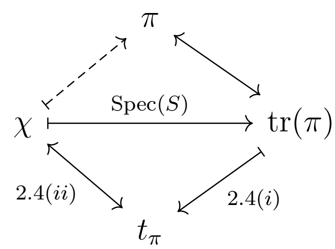
$$\int_{G(F)} v^\circ(tg)f(g)dg = \int_{B(F)} v^\circ(tb)f(b)d_{\mathrm{left}}b$$

$$= \delta^{\frac{1}{2}}(t)\chi(t)\int_{T(F)} \delta^{\frac{1}{2}}(s)\chi(s)\int_{U(F)} f(su)duds = v^\circ(t)\int_{T(F)} \mathcal{S}(f)(s)\chi(s)ds, \tag{2.3}$$

where $d_{\mathrm{left}}(su) = dsdu$ is the left Haar measure on $B(F)$ that gives $B(\mathfrak{o})$ volume one, and $d(bk) = d_{\mathrm{left}}bdk$ is the Haar measure on $G(F)$ that gives K volume one.

The functor of K-fixed vectors $(\pi, V) \rightsquigarrow V^K$ is exact from the category of admissible $G(F)$-representations to the category of finite-dimensional $\mathcal{H}(G)$-modules as follows from the existence of the projection $\int_K \pi(k)dk : V \twoheadrightarrow V^K$. Since the unramified principal series $\mathrm{Ind}_{B(F)}^{G(F)}(\delta^{\frac{1}{2}}\chi)$ has finite length, it has a unique irreducible K-unramified $G(F)$-subquotient (in its Jordan–Hölder decomposition).

Proposition 2.6. *For every unramified character $\chi : T(F)/T(\mathfrak{o}) \to \mathbb{C}$, the irreducible K-unramified $G(F)$-subquotient of the unramified principal series $\mathrm{Ind}_{B(F)}^{G(F)}(\delta^{\frac{1}{2}}\chi)$ corresponds with χ under the bijection of Proposition 2.4. This completes the following commutative diagram of bijections:*



Proof. Let (π, V) be the irreducible K-unramified $G(F)$-subquotient of $\mathrm{Ind}_{B(F)}^{G(F)}(\delta^{\frac{1}{2}}\chi)$, and write $V_2 \hookrightarrow V_1 \twoheadrightarrow V$ for two $G(F)$-subrepresentations V_1, V_2 of $\mathrm{Ind}_{B(F)}^{G(F)}(\delta^{\frac{1}{2}}\chi)$. We verify that the equalities (2.2) which characterize the middle arrow of the diagram are satisfied. Since $V_1^K \to V^K$ is surjective, $\dim V_1^K \leq 1$ and $\dim V^K = 1$, we deduce that it is a bijection (in fact an $\mathcal{H}(G)$-isomorphism). In particular $\dim V_1^K = 1$ and the K-fixed vector $v^\circ \in \mathrm{Ind}_{B(F)}^{G(F)}(\delta^{\frac{1}{2}}\chi)^K$ necessarily belongs to V_1^K. In the above calculation (2.3), we have found the action of $f \in \mathcal{H}(G)$ on $V_1^K = \mathbb{C} \cdot v^\circ$ is given by the right-hand side of (2.2). On the other hand, the action of $f \in \mathcal{H}(G)$ on V^K is via the scalar $\mathrm{tr}(\pi)(f)$. Since $V_1^K \xrightarrow{\sim} V^K$, this verifies (2.2). □

A smooth representation of $G(F)$ is said to be *tempered* if it is unitary and weakly contained in the regular representation by translation on $L^2(G(F))$. An irreducible smooth unitary representation of $G(F)$ is tempered if and only if its matrix coefficients belong to $L^{2+\epsilon}(G^{\mathrm{der}}(F))$ for every $\epsilon > 0$ (this follows from [14]). Denote by $\Xi(g) := \int_K \delta^{\frac{1}{2}}(kg)dk$ the Harish-Chandra function, where δ is inflated to $G(F) = B(F)K$ using the Iwasawa decomposition.

Proposition 2.7. *Let π be an irreducible K-unramified representation of $G(F)$, and let χ, t_π, $\mathrm{tr}(\pi)$ be as in Proposition 2.4. The following six properties are equivalent:*

(i) π *is tempered,*
(ii) π *is unitary and* $|\mathrm{tr}(\pi)(f)| \leq \int_G |f(g)|\Xi(g)dg$ *for every* $f \in \mathcal{H}(G)$*,*
(iii) χ *is unitary,*
(iv) t_π *is a compact element, i.e.,* t_π *belongs to the maximal compact subgroup of* $\widehat{T}(\mathbb{C})$*,*
(v) $|\mathrm{tr}_\lambda(\pi)| \leq \dim V_\lambda$ *for every* $\lambda \in \Lambda^+$*,*
(vi) there exists $C > 0$ *such that* $|\mathrm{tr}_\lambda(\pi)| \leq C \cdot \dim V_\lambda$ *for every* $\lambda \in \Lambda^+$*.*

Proof. The result is implicit in early work of Langlands. We couldn't locate a proof in the literature, hence we provide one.

(i) ⇔ (ii). Let v° be a K-fixed vector of π with $\langle v^\circ, v^\circ \rangle = 1$. Then

$$\mathrm{tr}(\pi)(f) = \int_{G(F)} f(g)\langle \pi(g)v^\circ, v^\circ \rangle dg, \quad f \in \mathcal{H}(G).$$

If π is tempered, then [14, Thm.2] says that the matrix coefficient is bounded by $|\langle \pi(g)v^\circ, v^\circ \rangle| \leq \Xi(g)$, which implies (ii). Conversely, (ii) implies the inequality $|\langle \pi(g)v^\circ, v^\circ \rangle| \leq \Xi(g)$, and since $\Xi \in L^{2+\epsilon}(G^{\mathrm{der}}(F))$ for every $\epsilon > 0$, we have that [14, Thm.1] implies that π is tempered.

(ii) ⇔ (iii). Since Proposition 2.6 says that π is a $G(F)$-subquotient of the unramified principal series $\mathrm{Ind}_{B(F)}^{G(F)}(\delta^{\frac{1}{2}}\chi)$, we have that (iii) implies that $\mathrm{Ind}_{B(F)}^{G(F)}(\delta^{\frac{1}{2}}\chi)$ is unitary which in turn implies that π is unitary. Applying (2.2), we find

$$\mathrm{tr}(\pi)(f) = \int_{T(F)} \chi(s)\mathcal{S}(f)(s)ds = \int_{T(F)} \chi(s)\delta^{\frac{1}{2}}(s) \int_{U(F)} f(su)duds$$

$$= \int_{B(F)} \chi(b)f(b)\delta^{\frac{1}{2}}(b)d_{\mathrm{left}}b = \int_{B(F)} \chi(b) \int_{K} f(bk)\delta^{\frac{1}{2}}(b)dkd_{\mathrm{left}}b$$

This shows that (iiii) implies the inequalities in (ii). The converse follows from Macdonald's formula for the spherical function.

(iii) $\Leftrightarrow$ (iv). We may identify the maximal compact subgroup of $\widehat{T}(\mathbb{C})$ with $\mathrm{Hom}(X^*(\widehat{T}), S^1)$, which in turn can be identified following Proposition 2.4(ii) with $\mathrm{Hom}(T(F)/T(\mathfrak{o}), S^1)$, the group of unitary unramified characters of $T(F)/T(\mathfrak{o})$.

The equivalences (iv) $\Leftrightarrow$ (v) and (iv) $\Leftrightarrow$ (vi) follow from $\mathrm{tr}_\lambda(\pi) = \mathrm{tr}(t_\pi|V_\lambda)$ in Definition 2.5. □

Remark 2.8. Property (ii) is related to Harish-Chandra's definition of temperedness of an admissible representation as having the property that its trace character extends to a continuous distribution on the Schwartz space.

We have been using consistently the *unitary normalization* of the character χ, of the Satake transform, and of the Satake parameter t_π. There is also an algebraic normalization, which is that $q^{\langle\lambda,\rho\rangle}a_\lambda$ corresponds to the trace function of the IC-sheaf of the closure of the cell of the affine Grassmannian associated to λ.

Example 2.9. For the trivial representation $\mathbf{1}$, we have $\mathrm{tr}_\lambda(\mathbf{1}) = \mathrm{tr}(\mathbf{1})(a_\lambda)$. The Satake parameter $t_{\mathbf{1}}$ is equal to the principal semisimple element $\rho(q) \in \widehat{T}(\mathbb{C})$, where ρ is seen as a cocharacter $X_*(\widehat{T})_{\mathbb{C}}$. In particular, we obtain

$$\sum_{x\in K\backslash G/K} a_\lambda(x) = \mathrm{tr}(\mathbf{1})(a_\lambda) = \mathrm{tr}(\rho(q)|V_\lambda) = q^{\langle\lambda,\rho\rangle}\left(1 + O(q^{-\frac{1}{2}})\right).$$

We conclude that $\mathrm{tr}_\lambda(\mathbf{1}) = q^{\frac{d(\lambda)}{2}}\left(1 + O(q^{-\frac{1}{2}})\right)$, where

$$d(\lambda) := \langle\lambda, 2\rho\rangle = \dim \mathrm{Gr}_\lambda \in \mathbb{Z}_{\geq 0},$$

which we interpret as the *degree* of the Hecke operator of coweight $\lambda \in \Lambda^+$.

2.3. **Base change.** Notation is as in the previous subsection, and we consider the degree n extension $k' = \mathbb{F}_{q^n}$ of $k = \mathbb{F}_q$. There is a base change algebra homomorphism $b : \mathcal{H}(G_{k'}) \to \mathcal{H}(G)$, see e.g. [40]. For any K-unramified irreducible representation π of $G(k((t)))$, there corresponds a unique K'-unramified irreducible representation Π of $G(k'((t)))$ such that $\mathrm{tr}(\Pi)(f) = \mathrm{tr}(\pi)(b(f))$ for every $f \in \mathcal{H}(G_{k'})$. Indeed the corresponding Satake parameters satisfy the relation $t_\Pi = t_\pi^n$. In particular the representation π is tempered if and only if the base change representation Π is tempered. We can identify the positive Weyl chamber $\Lambda^+ \subset X_*(T)$ for the groups G and $G_{k'}$. We have then the relation,

$$\mathrm{tr}_\lambda(\Pi) = \mathrm{tr}(t_\pi^n|V_\lambda),$$

which will be used often in relation to taking the limit as $n \to \infty$.

2.4. Character sheaves.

Definition 2.10. For a connected algebraic group H, say a *character sheaf* on H is a rank one lisse sheaf $\mathcal{L}$ with an isomorphism $\mathcal{L} \boxtimes \mathcal{L} \cong m^*\mathcal{L}$ for $m : H \times H \to H$ the multiplication map.

Remark 2.11. Given a character sheaf $\mathcal{L}$, we have an isomorphism $\mathcal{L}_e = \mathcal{L}_e \otimes \mathcal{L}_e$, hence an isomorphism $\overline{\mathbb{Q}}_\ell = \mathcal{L}_e$.

Let $m_3 : H \times H \times H \to H$ be the multiplication of three elements. Because $m_3 = m \circ (m \times id) = m \circ (id \times m)$, the isomorphism $\mathcal{L} \boxtimes \mathcal{L} \cong m^*\mathcal{L}$ induces two different isomorphism $\mathcal{L} \boxtimes \mathcal{L} \boxtimes \mathcal{L} \cong m_3^*\mathcal{L}$. These two isomorphisms are necessarily equal, because they are maps between lisse sheaves on a connected scheme and are equal on the identity point.

For convenience, we give here many important facts about character sheaves, almost all of which are surely well-known.

Lemma 2.12. *Let H be an algebraic group over a finite field $\mathbb{F}_q$. The trace function of a character sheaf is a one-dimensional character of $H(\mathbb{F}_q)$.*

Throughout this paper, the *trace function* of a sheaf $\mathcal{F}$ will be the function that takes a point x to the trace of the *geometric* Frobenius on the stalk of $\mathcal{F}$ at x.

Proof. Let $\mathcal{L}$ be a character sheaf and let χ be the trace function of $\mathcal{L}$ on $H(\mathbb{F}_q)$. Then by the definition of a character sheaf, for $x, y \in H(\mathbb{F}_q)$, $\chi(xy) = \chi(x)\chi(y)$. Moreover because $\mathcal{L}$ is a rank one lisse sheaf, χ is nonzero. Hence it is an homomorphism to $\overline{\mathbb{Q}}_\ell^\times$ and thus a character. □

Remark 2.13. Not every character of $H(\mathbb{F}_q)$ necessarily arises from a character sheaf. Consider the group of matrices of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & a^p \\ 0 & 0 & 1 \end{pmatrix}$$

under matrix multiplication. Any character sheaf, restricted to the subgroup H' defined by $a = 0$, is a lisse character sheaf on $H' \cong \mathbb{A}^1$. By evaluating the character sheaf on a commutator, one can see that this sheaf is necessarily trivial when pulled back along the map $(x, y) \to (x^p y - xy^p)$ whose generic fiber is geometrically irreducible, and hence the sheaf is trivial when restricted to $H'(\mathbb{F}_p)$. However, not all characters of $H(\mathbb{F}_p)$ are trivial on $H'(\mathbb{F}_p)$.

Let σ be the arithmetic Frobenius automorphism of $H(\overline{\mathbb{F}}_q)$. The Lang isogeny is the covering $H \to H$ sending g to $\sigma(g)g^{-1}$, which is finite étale Galois with automorphism group $H(\mathbb{F}_q)$.

Lemma 2.14. *Let H be an algebraic group over a finite field $\mathbb{F}_q$, $\mathcal{L}$ a character sheaf on H, and χ its trace function. Then the pullback of $\mathcal{L}$ along the Lang isogeny is trivial, and as a representation of the fundamental group, $\mathcal{L}$ is equal to the composition of the map $\pi_1(H_{\mathbb{F}_q}) \to H(\mathbb{F}_q)$ with the character $\chi^{-1} : H(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^\times$.*

Proof. For the first fact, observe that the pullback of $\mathcal{L}$ along the Lang isogeny is $\sigma^*\mathcal{L} \otimes \mathcal{L}^{-1} = \overline{\mathbb{Q}}_\ell$ as $\mathcal{L}$ is defined over $\mathbb{F}_q$ and hence invariant under σ. It follows that the monodromy representation of $\mathcal{L}$ factors through $H(\mathbb{F}_q)$. By examining the Frobenius elements at points of $H(\mathbb{F}_q)$, we obtain χ — the inverse is obtained because of the difference between arithmetic and geometric Frobenius. □

Lemma 2.15. *Let H be an algebraic group over a finite field $\mathbb{F}_q$. Every one-dimensional character of $H(\mathbb{F}_q)$ arises from at most one character sheaf.*

The orders of the arithmetic monodromy group of the character sheaf, the geometric monodromy group of the character sheaf, and the character all agree.

Proof. These statements follow immediately from Lemma 2.14. For the second, it is sufficient to observe that the image of the geometric fundamental group inside $H(\mathbb{F}_q)$ is also $H(\mathbb{F}_q)$, because the total space H of the Lang isogeny is geometrically connected. □

To check that a character arises from a character sheaf, we will mainly use the following lemma:

Lemma 2.16. *(i) Let H be an abelian algebraic group over $\mathbb{F}_q$. Every one-dimensional character of $H(\mathbb{F}_q)$ arises from a unique character sheaf. The trace function on $H(\mathbb{F}_{q^n})$ of this sheaf is the composition of the original character with the norm map.*

(ii) Let $f : H_1 \to H_2$ be an algebraic group homomorphism and let $\mathcal{L}$ be a character sheaf on H_2. Then $f^*\mathcal{L}$ is a character sheaf on H_1 whose trace function is the composition of the trace function of $\mathcal{L}$ with h.

Hence every character of the $\mathbb{F}_q$-points of an algebraic group that factors through a homomorphism to an abelian algebraic group arises from a unique character sheaf.

Proof. For assertion (i), one uses the construction of Lemma 2.14 to construct a sheaf from a character, and then checks immediately the necessary isomorphism to make it a character sheaf.

Assertion (ii) is a direct calculation. □

When performing harmonic analysis calculations with character sheaves, it is helpful to have a description of character sheaves directly in terms of points. This is provided, based on central extensions, by the following lemmas:

Lemma 2.17. *Let $\tilde{H}$ be a central extension $1 \to \overline{\mathbb{Q}}_\ell^\times \to \tilde{H} \to H(\overline{\mathbb{F}}_q) \to 1$ with an action of σ such that both maps involved are equivariant.*

Then there exists a unique character sheaf $\mathcal{L}$ on H whose trace function over $\mathbb{F}_{q^n}$ is given by $g \mapsto \sigma^n(\tilde{g})\tilde{g}^{-1}$ for $\tilde{g}$ any lift of g from $H(\overline{\mathbb{F}}_q)$ to $\tilde{H}$.

Furthermore, every character sheaf arises from a central extension in this way.

Proof. For the purposes of this proof, it is simpler to define the trace function using the arithmetic Frobenius, and then we invert to get the true trace function.

Given a central extension $\tilde{H}$, we form the associated character $\chi : H(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^\times$, $g \mapsto \sigma(\tilde{g})\tilde{g}^{-1}$. It is easy to check that this is actually a group homomorphism. We compose the Lang isogeny homomorphism $\pi_1(H_{\mathbb{F}_q}) \to H(\mathbb{F}_q)$ with χ to produce a homomorphism $\pi_1(H) \to \overline{\mathbb{Q}}_\ell^\times$ and hence a rank one sheaf $\mathcal{L}_\chi$, as in Lemma 2.14.

Let us check that the trace function of $\mathcal{L}_\chi$ over $\mathbb{F}_{q^n}$ is given by $g \mapsto \sigma^n(\tilde{g})\tilde{g}^{-1}$. Let g be an element of $H(\mathbb{F}_{q^n})$ and let $\sigma(h)h^{-1} = g$. By definition, the trace function of $\mathcal{L}_\chi$ at g is defined as $\chi(a)$ for the unique $a \in H(\mathbb{F}_q)$ such that $\sigma^n(h) = ha$. (Such a exists because $\sigma(\sigma^n(h))(\sigma^n(h))^{-1} = \sigma^n(g) = g$.) In other words, the trace of $\mathcal{L}_\chi$ at g is $\chi(h^{-1}\sigma^n(h))$. Choose $\tilde{h}$ a lift of h and let $\tilde{g} = \sigma(\tilde{h})\tilde{h}^{-1}$, so that

$$\chi(h^{-1}\sigma^n(h)) = \sigma(\tilde{h}^{-1}\sigma^n(\tilde{h}))\left(\tilde{h}^{-1}\sigma^n(\tilde{h})\right)^{-1} = \sigma(\tilde{h})^{-1}\sigma^{n+1}(\tilde{h})\sigma^n(\tilde{h})^{-1}\tilde{h}$$
$$= \sigma^{n+1}(\tilde{h})\sigma^n(\tilde{h})^{-1}\tilde{h}\sigma(\tilde{h})^{-1} = \sigma^n(\tilde{g})\tilde{g}^{-1},$$

where we use the fact that we are working with an element of the center and hence may freely conjugate it by any element.

Second, let us check that the trace function of $\mathcal{L}_\chi$ over $\mathbb{F}_{q^n}$ is actually a character. This follows because

$$\sigma^n(\tilde{g}_1\tilde{g}_2)(\tilde{g}_1\tilde{g}_2)^{-1} = \sigma^n(\tilde{g}_1)\sigma^n(\tilde{g}_2)\tilde{g}_2^{-1}\tilde{g}_1^{-1} = \sigma^n(\tilde{g}_1)\tilde{g}_1^{-1}\sigma^n(\tilde{g}_2)\tilde{g}_2^{-1}$$

where we use that $\sigma^n(\tilde{g}_2)\tilde{g}_2^{-1}$ is central.

It now follows by the Chebotarev density theorem that $\mathcal{L}_\chi$ admits an isomorphism $\mathcal{L}_\chi \boxtimes \mathcal{L}_\chi \cong m^*\mathcal{L}_\chi$ because these two sheaves have the same trace function over every finite field. The uniqueness follows from Lemma 2.15.

Conversely, given a character sheaf $\mathcal{L}$, define $\tilde{H}$ to be the set of pairs of a point $x \in H(\overline{\mathbb{F}}_q)$ and a nonzero section of $\mathcal{L}_x$. Multiplication is given by $(x, s_x)(y, s_y) = (xy, s_x \otimes s_y)$ where we use the isomorphism $\mathcal{L}_x \otimes \mathcal{L}_y = \mathcal{L}_{xy}$ induced by taking stalks in the isomorphism $\mathcal{L} \boxtimes \mathcal{L} = m^*\mathcal{L}$ that is part of the definition of a character sheaf. Associativity for this multiplication follows from associativity for the isomorphism. To find units and inverses, it is sufficient to find them in the stalk over the identity of H, where they are obvious.

By definition, the trace function of $\mathcal{L}$ at x is the trace of Frobenius on $\mathcal{L}_x$, which because $\mathcal{L}_x$ is one-dimensional is the eigenvalue of Frobenius on the $\mathcal{L}_x$, which can be calculated as $\sigma^n(s_x)s_x^{-1}$ for s_x a section of $\mathcal{L}_x$, which is equal to $\sigma^n(x, s_x)(x, s_x)^{-1}$ for (x, s_x) a lift of x. $\square$

Lemma 2.18. *(1) For H_1, H_2 two algebraic groups, any character sheaf on $H_1 \times H_2$ is $\mathcal{L}_1 \boxtimes \mathcal{L}_2$ for $\mathcal{L}_1$ and $\mathcal{L}_2$ character sheaves on H_1 and H_2.*

(2) For H an algebraic group over $\mathbb{F}_{q^n}$, any character sheaf on $\operatorname{Res}_{\mathbb{F}_{q^n}}^{\mathbb{F}_q} H$ is the Weil restriction of a character sheaf on H.

Proof. (1) Let $\mathcal{L}$ be the character sheaf, let $\mathcal{L}_1$ be its pullback to H_1, and let $\mathcal{L}_2$ be its pullback to H_2. Then $\mathcal{L}_1 \boxtimes \mathcal{L}_2$ and $\mathcal{L}$ have the same trace function, hence are equal.

(2) Let $\mathcal{L}$ be the character sheaf, let $\mathcal{L}'$ be its pullback to $(\operatorname{Res}_{\mathbb{F}_{q^n}}^{\mathbb{F}_q} H)_{\mathbb{F}_{q^n}}$ and then to H, embedded diagonally. Then $\mathcal{L}$ and $\operatorname{Res}_{\mathbb{F}_{q^n}}^{\mathbb{F}_q} \mathcal{L}'$ have the same trace function and thus are equal. $\square$

2.5. **Weil Restrictions.**

Notation 2.19. We work with the convention that, for an algebraic group G over k and a finite-dimensional ring R over k, $G\langle R\rangle$ is the algebraic group whose S-points for a ring S over k are the $R \otimes_k S$ points of G. Equivalently, $G\langle R\rangle$ is the Weil restriction $\operatorname{Res}_k G_R$ from R to k of the base-change G_R.

Example 2.20. If we view k^n as a ring by pointwise multiplication, then $G\langle k^n\rangle = G^n$. More generally, $G\langle R_1 \times R_2\rangle = G\langle R_1\rangle \times G\langle R_2\rangle$. For another generalization, if k' is a separable k-algebra of degree n, and $\overline{k}$ is the algebraic closure of k, then $(G\langle k'\rangle)_{\overline{k}} = G_{\overline{k}}^n$.

Example 2.21. $G\langle k[t]/t^2\rangle$ is an extension of G by the Lie algebra $\mathfrak{g}$ of G, where $\mathfrak{g}$ is viewed as an additive group scheme. More generally, $G\langle k[t]/t^n\rangle$ is an $n-1$-fold iterated extension of G by $\mathfrak{g}$.

By definition, we have $G\langle R\rangle(k) = G(R)$, which we will use several times. This method of constructing a scheme whose k-points are $G(R)$ has many good properties. For us, the most important is that it is stable under base change, i.e., for any field k' over k, $G_{k'}\langle R \otimes_k k'\rangle = (G\langle R\rangle)_{k'}$.

2.6. **Sheaves on Stacks.** We always denote Verdier duality by D.

Lemma 2.22. *Let Y be a stack of finite type over an algebraically closed field and let K_1 and K_2 be bounded complexes of ℓ-adic sheaves on Y.*

(1) $H^i_c(Y, DK_1 \otimes K_2)$ is naturally dual to $\operatorname{Ext}^{-i}_Y(K_2, K_1)$;
(2) If K_1 and K_2 are perverse, then $H^i_c(Y, DK_1 \otimes K_2)$ vanishes for $i > 0$;
(3) If K_1 and K_2 are perverse and semisimple, then $H^0_c(Y, DK_1 \otimes K_2) = \operatorname{Hom}(K_1, K_2)$.

Proof. For part 1, by the definition of cohomology with compact supports [47, §9.1],

$$H^i_c(Y, DK_1 \otimes K_2) = \left(H^{-i}(Y, D(DK_1 \otimes K_2))\right)^\vee.$$

By [47, Prop.6.0.12 and Thm.7.3.1],

$$H^{-i}(Y, D(DK_1 \otimes K_2)) = H^{-i}(Y, \mathcal{H}om(K_2, K_1)),$$

which in turn is equal to $\operatorname{Ext}^{-i}_Y(K_2, K_1)$, by definition of Ext, see [47, Rem.5.0.11].

Part 2 follows because perverse sheaves are the heart of a t-structure by [48, Thm.5.1] and so their Ext^{-i} vanishes for $i > 0$.

Part 3 follows because for semisimple perverse sheaves $\operatorname{Ext}^0(K_2, K_1) = \operatorname{Hom}(K_2, K_1)$ is dual to $\operatorname{Hom}(K_1, K_2)$. ☐

Lemma 2.23. *Let $\iota : \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ be an embedding. Let Y be an Artin stack of finite type over $\mathbb{F}_q$ with affine stabilizers and let K_1 and K_2 be bounded complexes of ℓ-adic sheaves on Y, ι-pure of weights w_1 and w_2. Then for any $j \in \mathbb{Z}$,*

$$\sum_{i=-\infty}^{j} (-1)^i \operatorname{tr}\left(\operatorname{Frob}_{q^e}, \iota(H^i_c(Y_{\overline{\mathbb{F}}_q}, DK_1 \otimes K_2))\right) = O\left((q^e)^{\frac{j+w_2-w_1}{2}}\right),$$

where the constant in the big O is independent of e but may depend on (Y, K_1, K_2).

Proof. The tensor product $DK_1 \otimes K_2$ is necessarily mixed of weight $\le w_2 - w_1$ so by [59, Thm.1.4], $H^i_c(Y_{\overline{\mathbb{F}}_q}, DK_1 \otimes K_2)$ is mixed of weight $\le i + w_2 - w_1$.

Let $|\operatorname{Frob}_q|$ be the operator that acts on generalized eigenspaces of Frob_q with eigenvalue the absolute value of the corresponding eigenvalue of Frob_q. Then we have

$$\left|\sum_{i=-\infty}^{j-1} (-1)^i \operatorname{tr}\left(\operatorname{Frob}_{q^e}, \iota(H^i_c(Y_{\overline{\mathbb{F}}_q}, DK_1 \otimes K_2))\right)\right| \le \sum_{i=-\infty}^{j-1} \operatorname{tr}\left(|\operatorname{Frob}_q|^e, \iota(H^i_c(Y_{\overline{\mathbb{F}}_q}, DK_1 \otimes K_2))\right).$$

Then because all eigenvalues of Frob_q are $\le q^{\frac{j+w_2-w_1-1}{2}}$, for any $0 < s \le e$, we have

$$\sum_{i=-\infty}^{j-1} \operatorname{tr}(|\operatorname{Frob}_q|^e, \iota(H^i_c(Y_{\overline{\mathbb{F}}_q}, DK_1 \otimes K_2))) \le q^{(e-s)\frac{j+w_2-w_1-1}{2}} \left(\sum_{i=-\infty}^{j-1} \operatorname{tr}(|\operatorname{Frob}_q|^s, \iota(H^i_c(Y_{\overline{\mathbb{F}}_q}, DK_1 \otimes K_2)))\right)$$

(by [59, Thm.4.2(i)])

$$\le q^{(e-s)\frac{j+w_2-w_1-1}{2}} O_s(1).$$

Note that for s sufficiently small we have

$$q^{(e-s)\frac{j+w_2-w_1-1}{2}} < q^{e\frac{j+w_2-w_1}{2}} = (q^e)^{\frac{j+w_2-w_1}{2}}$$

because if $j+w_2-w_1-1 \geq 0$ this holds for all nonnegative s and if $j+w_2-w_1-1 < 0$ this holds for all $s < \frac{e}{-(j+w_2-w_1-1)}$.

Thus we have

$$\left| \sum_{i=-\infty}^{j-1} (-1)^i \operatorname{tr}\left(\mathrm{Frob}_{q^e}, \iota(H^i_c(Y_{\overline{\mathbb{F}}_q}, DK_1 \otimes K_2))\right) \right| = O\left((q^e)^{\frac{j+w_2-w_1}{2}} \right).$$

The remaining term satisfies

$$(-1)^j \operatorname{tr}(\mathrm{Frob}_{q^e}, \iota(H^j_c(Y_{\overline{\mathbb{F}}_q}, DK_1 \otimes K_2))) = O\left((q^e)^{\frac{j+w_2-w_1}{2}} \right)$$

where the constant in the big O is the dimension of $H^j_c(Y_{\overline{\mathbb{F}}_q}, DK_1 \otimes K_2)$. Thus, the desired bound holds for both terms. □

2.7. **Linear recursive sequences and tensor power trick.** The following is a variant of Gelfand's formula $\lim_{n\to\infty} ||t^n||^{\frac{1}{n}}$ for the spectral radius of an endomorphism t.

Lemma 2.24 ([16, §3], [6]). *Let V be a finite-dimensional complex vector space, and $t \in \mathrm{End}(V)$. Then*

$$\rho := \limsup_{n\to\infty} |\operatorname{tr}(t^n|V)|^{\frac{1}{n}}$$

is the spectral radius of t, and

$$|\operatorname{tr}(t^n|V)| \leq \dim V \cdot \rho^n, \quad \textit{for every } n \geq 0.$$

Proof. Let $\lambda_1, \ldots, \lambda_{\dim(V)}$ denote the eigenvalues of t, so that $\operatorname{tr}(t^n|V) = \sum_i \lambda_i^n$. The power series

$$\sum_{n=1}^{\infty} \operatorname{tr}(t^n|V)\frac{z^n}{n} = -\log\det(1-zt|V) = -\sum_i \log(1-\lambda_i z)$$

has radius of convergence equal to ρ^{-1} by the Cauchy–Hadamard theorem (note that $n^{\frac{1}{n}} \to 1$ as $n \to \infty$). Since it cannot be extended to an holomorphic function past the singularities at $z = \lambda_i^{-1}$, we deduce that ρ is equal to $\max_i |\lambda_i|$, the spectral radius of t. This establishes the first assertion, and then the inequality of the second assertion follows. □

3. Compactly induced representations

This section is concerned with first developing some preliminary material, leading up to the key definition of mgs representations, followed by giving some basic properties of the definition, then providing some examples and non-examples, and finally describing some additional useful properties.

We begin, in §3.1, with some purely representation-theoretic computations. In particular, we give in Corollary 3.3 a concrete criterion on a subgroup $J \subset G(F)$ and a character χ such that every irreducible smooth representation of $G(F)$ containing a nonzero (J,χ)-invariant vector is supercuspidal.

In §3.2, we define a "monomial datum" as a geometric version of (J,χ), and say a datum is "geometrically supercuspidal" if it satisfies a geometric version of this concrete criterion. These geometric analogues contain the classical versions in the sense that we can extract from a monomial datum a subgroup J and character χ, and they do satisfy the concrete condition if the original monomial datum is geometrically supercuspidal.

In §3.4, we define the notion of a mgs representation as an irreducible smooth representation containing a (J,χ)-invariant vector, where (J,χ) arise from a geometrically supercuspidal monomial datum in this way. (In particular, mgs representations are always supercuspidal.)

The calculations in §3.1 involve the compact induction $\text{c-Ind}_J^{G(F)}\chi$, but they do not require us to show that $\text{c-Ind}_J^{G(F)}\chi$ is itself an irreducible supercuspidal representation. Thus they have a different approach than works which aim to construct supercuspidal representations as inductions, where showing that the induced representation is irreducible is of the highest importance. On the other hand, in §3.3, we give a way to check that a monomial datum is geometrically supercuspidal, which does involve showing that $\text{c-Ind}_J^{G(F)}\chi$ is irreducible and supercuspidal, and in §3.9, we show that $\text{c-Ind}_J^{G(F)}\chi$ is at worst a finite direct sum of supercuspidal representations under mild additional assumptions.

Our examples of mgs representations come in §3.5 and §3.6. These examples arise from existing constructions of supercuspidal representations, such as epipelagic representations and toral supercuspidals.

In §3.7, we give examples of monomial data that are not geometrically supercuspidal.

In §3.8, we check that mgs representations are preserved under certain natural operations.

3.1. **Vanishing of Jacquet modules.** Let $G(F)$ be a reductive group over a non-archimedean local field F. Let P be a parabolic subgroup with Levi decomposition $P = MN$. The *Jacquet module* (π_N, V_N) of a smooth representation (π, V) of $G(F)$ is the N-coinvariants of V, regarded as an M-module. This is an exact functor.

Lemma 3.1. *Let χ be a character of an open-compact subgroup J and $P = MN$ a parabolic subgroup. The following properties are equivalent:*

(i) The Jacquet module of N-coinvariants of the induced representation $\text{c-Ind}_J^{G(F)}\chi$ *vanishes;*
(ii) for every $g \in G(F)$, the restriction of χ to $gNg^{-1} \cap J$ is non-trivial;
(iii) for every $g_1, g_2 \in G(F)$, $\int_N f_\chi(g_1 n g_2)dn = 0$, where

$$f_\chi(g) := \begin{cases} \chi(g), & \text{if } g \in J, \\ 0, & \text{if } g \notin J. \end{cases}$$

Proof. We first show the direction (i) $\implies$ (ii). We view $\text{c-Ind}_J^{G(F)}\chi$ as the space of smooth compactly supported functions f on $G(F)$ satisfying $f(gh) = f(g)\chi(h)$ for $h \in J$. Since the functional $f \mapsto \int_{n\in N} f(ng^{-1})$ factors through the Jacquet module of N-coinvariants of $\text{c-Ind}_J^{G(F)}\chi$, it vanishes. Take f in this space to be the function supported on the left coset $g^{-1}J$ such that $f(g^{-1}h) = \chi(h)$ for $h \in J$. Then

$$0 = \int_{n\in N} f(ng^{-1}) = \int_{n\in N\cap g^{-1}Jg} \chi(gng^{-1}) = \int_{h\in gNg^{-1}\cap J} \chi(h),$$

where the integrations are with respect to Haar measures. This implies that the restriction of χ to the subgroup $gNg^{-1} \cap J$ is non-trivial.

For the direction (ii) $\implies$ (i), observe that a linear basis of $\text{c-Ind}_J^{G(F)}\chi$ consists of the above functions $f_g : gh \mapsto \chi(h)$ supported on the left cosets gJ of J in $G(F)$ for varying $g \in G(F)/J$. For $h \in gNg^{-1} \cap J$, the right translation of f_g by h is equal to $\chi(h)f_g$, and the right translation of f_g by h is equal to the left translation of f_g by an element of N, which implies that the images of f_g and $\chi(h)f_g$ in the Jacquet module of N-coinvariants are equal. Since property (ii) says that χ is nontrivial when restricted to $gNg^{-1} \cap J$, this implies that the image of f_g in the Jacquet module is zero. Since $\text{c-Ind}_J^{G(F)} \chi$ projects onto the Jacquet module, and the f_g's form a linear basis, we deduce property (i), i.e, that the Jacquet module vanishes.

For the implication (ii) ⇒ (iii), suppose that $g_1 n_0 g_2 \in J$ for some $n_0 \in N$. Then the condition $g_1 n g_2 \in J$ is equivalent to $g_2^{-1} n_0^{-1} n g_2 \in J$. Therefore

$$\int_{n \in N} f_\chi(g_1 n g_2) = \chi(g_1 n_0 g_2) \int_{h \in g_2^{-1} N g_2 \cap J} \chi(h) = 0.$$

The implication (iii) ⇒ (ii) follows by taking $g_1 = g$ and $g_2 = g^{-1}$. □

Lemma 3.2. *The following properties of a smooth irreducible representation (π, V) of $G(F)$ are equivalent:*

(i) it has a non-zero (J, χ)-invariant vector;
(ii) it is a quotient of $\text{c-Ind}_J^{G(F)} \chi$.

If one of these conditions holds and the Jacquet module of N-coinvariants of $\text{c-Ind}_J^{G(F)} \chi$ vanishes, then the Jacquet module of N-coinvariants of π also vanishes: $V_N = 0$.

Proof. The equivalence of $\text{Hom}_J(\chi, \pi) = 0$ and $\text{Hom}(\text{c-Ind}_J^{G(F)} \chi, \pi) = 0$ is a form of Frobenius reciprocity [10, Thm.3.2.4]. The second assertion is consequence of the exactness of the Jacquet functor. □

Recall that an admissible representation (π, V) is *supercuspidal* if $V_N = 0$ for every proper parabolic subgroup $P = MN$ of $G(F)$. It is equivalent [10, Thm.5.3.1] to that all the matrix coefficients of (π, V) have compact support mod center. If (π, V) is irreducible, then it is sufficient to verify that one nonzero matrix coefficient has compact support mod center. We deduce from Lemma 3.1 and Lemma 3.2 the following which will be used often.

Corollary 3.3. *Let χ be a character of an open-compact subgroup J of $G(F)$. The following four properties are equivalent:*

(i) $\text{c-Ind}_J^{G(F)} \chi$ has vanishing Jacquet module of N-coinvariants for every proper parabolic subgroup $P = MN$;
(ii) the restriction of χ to $N \cap J$ is non-trivial for every proper parabolic subgroup $P = MN$ of $G(F)$;
(iii) f_χ is a cuspidal function on $G(F)$;
(iv) every irreducible smooth representation of $G(F)$ with a non-zero (J, χ)-invariant vector is supercuspidal.

Remark 3.4. It is proved in [8] that the following properties on the induced representation $\text{c-Ind}_J^{G(F)} \chi$ are equivalent:

(i') it is admissible;
(ii') it is supercuspidal;
(iii') it is a finite direct sum of irreducible supercuspidals.

These properties (i')-(iii') are stronger that the properties (i)-(iv) of Corollary 3.3, because (ii') $\implies$ (i), or because (iii') $\implies$ (iv).

3.2. **Geometric version.** Let G be a reductive group over a finite field κ, m a natural number, H a connected subgroup of $G\langle\kappa[t]/t^m\rangle$, and $\mathcal{L}$ a character sheaf on H. We call the quadruple $(G, m, H, \mathcal{L})$ a *monomial datum.*

Let J be the inverse image of $H(\kappa)$ in $G(\kappa[\![t]\!])$ and let χ be the character induced by $\mathcal{L}$ on $H(\kappa)$ (see Lemma 2.12), pulled back to J. The situation is described by the diagram

$$\begin{array}{ccccc} U_m(G(\kappa[\![t]\!])) & \hookrightarrow & J & \twoheadrightarrow & H(\kappa) \\ & & \downarrow & & \downarrow \\ & & G(\kappa[\![t]\!]) & \twoheadrightarrow & G(\kappa[t]/t^m) \end{array} \tag{3.1}$$

where $U_m(G(\kappa[\![t]\!]))$ is the subgroup of $G(\kappa[\![t]\!])$ consisting of elements congruent to 1 modulo t^m. In this diagram, the square is Cartesian and the sequence $U_m(G(\kappa[\![t]\!])) \to J \to H(\kappa)$ is short exact.

This datum defines a monomial representation $\text{c-ind}_J^{G(\kappa(\!(t)\!))} \chi$. The following definition gives the geometric version of the property that all of the Jacquet modules of $\text{c-Ind}_J^{G(\kappa(\!(t)\!))} \chi$ vanish:

Definition 3.5. We say that the monomial datum $(G, m, H, \mathcal{L})$ is *geometrically supercuspidal* if for every proper parabolic subgroup $P = MN$ of $G_{\overline{\kappa}}$, and every $g \in G(\overline{\kappa}[t]/t^m)$, the restriction of $\mathcal{L}_{\overline{\kappa}}$ to the identity component of the intersection $gN\langle\overline{\kappa}[t]/t^m\rangle g^{-1} \cap H_{\overline{\kappa}}$ is non-trivial.

The next Lemma 3.6 will imply a close relationship between this geometric property and the previous vanishing property of the Jacquet modules. For any finite field extension κ' of κ, the datum $(G, m, H, \mathcal{L})$ is geometrically supercuspidal if and only $(G_{\kappa'}, m, H_{\kappa'}, \mathcal{L}_{\kappa'})$ is geometrically supercuspidal. Let $J_{\kappa'}$ be the inverse image of $H(\kappa')$ in $G(\kappa'[\![t]\!])$ as in the diagram (3.1). Let $\chi_{\kappa'}$ be the character induced by $\mathcal{L}$ on $H(\kappa')$, pulled back to $J_{\kappa'}$.

Lemma 3.6. *The following properties are equivalent:*

(i) for every finite extension κ' of κ, the induction $\text{c-Ind}_{J_{\kappa'}}^{G(\kappa'(\!(t)\!))} \chi_{\kappa'}$ *has vanishing Jacquet modules;*
(ii) for every finite extension κ' of κ, every proper parabolic subgroup $P = MN$ of $G_{\kappa'(\!(t)\!)}$, the restriction of $\chi_{\kappa'}$ to $N(\kappa'(\!(t)\!)) \cap J_{\kappa'}$ is non-trivial;
(iii) $(G, m, H, \mathcal{L})$ is geometrically supercuspidal;
(iv) for every field extension κ' of κ, any proper parabolic subgroup $P = MN$ of $G_{\kappa'}$, and any $g \in G(\kappa'[t]/t^m)$, the restriction of $\mathcal{L}_{\kappa'}$ to the intersection of $gN\langle\kappa'[t]/t^m\rangle g^{-1}$ with $H_{\kappa'}$ is not geometrically isomorphic to a constant sheaf.

Proof. The equivalence between (i) and (ii) follows from Lemma 3.1. The implication (iv) $\implies$ (iii) follows by taking $\kappa' = \overline{\kappa}$.

The direction (iii) $\implies$ (ii) is straightforward. It uses the fact that any quasi-split reductive group with a Borel subgroup, the Galois group of the base field acts on its Dynkin diagram, and parabolic subgroups are classified up to conjugacy by Galois-invariant subsets of the roots. Let B be a Borel of $G_{\kappa'}$ and $B_{\kappa'(\!(t)\!)} \subseteq G_{\kappa'(\!(t)\!)}$ its pullback. Let P be a parabolic subgroup of $G_{\kappa'(\!(t)\!)}$. Then P is conjugate to a parabolic P' containing $B_{\kappa'(\!(t)\!)}$. Let S be the set of simple roots of $B_{\kappa'(\!(t)\!)}$ contained in the Levi of P'. Then S is invariant under $\text{Gal}(\kappa'(\!(t)\!))$. Because the action of $\text{Gal}(\kappa'(\!(t)\!))$ on the simple roots of $B_{\kappa'(\!(t)\!)}$ factors through the action of $\text{Gal}(\kappa')$ on the simple roots of B', S is also invariant under $\text{Gal}(\kappa')$, so it corresponds to a parabolic subgroup P_0 of $G_{\kappa''}$ containing B. Because $P_{0,\kappa'(\!(t)\!)}$ contains $B_{\kappa'(\!(t)\!)}$, and has the same set S of simple roots in its Levi, we have $P_{0,\kappa'(\!(t)\!)} = P'$ and thus $P_{0,\kappa'(\!(t)\!)}$ is conjugate to P. See [12, Ex.7.2.3].

Because of the Iwasawa decomposition $G(\kappa'(\!(t)\!)) = P_0(\kappa'(\!(t)\!))G(\kappa'[\![t]\!])$, we have that P is $G(\kappa'[\![t]\!])$-conjugate to $P_{0,\kappa'(\!(t)\!)}$. Thus $P = gP_0g^{-1}$ for some $g \in G(\kappa'[\![t]\!])$, and so we have $N = gN_{0,\kappa'(\!(t)\!)}g^{-1}$ for N_0 the unipotent radical of P_0. Hence $N(\kappa'(\!(t)\!)) = gN_0(\kappa'(\!(t)\!))g^{-1}$. Finally, note that the restriction of $\chi_{\kappa'}$ to $gN_0(\kappa'(\!(t)\!))g^{-1} \cap J_{\kappa'}$ is the trace function over κ' of the restriction of $\mathcal{L}$ to $gN_0\langle\kappa'[t]/t^m\rangle g^{-1} \cap H$. Since this restriction is non-trivial, Lemma 2.15 implies that its trace function is a non-trivial character.

So it remains to prove the converse (ii) $\implies$ (iii) $\implies$ (iv).

To verify (iii) $\implies$ (iv), let us first check that, given a morphism $f : Y \to X$ of schemes of finite type over a field and a lisse sheaf $\mathcal{F}$ on Y, the property that $\mathcal{F}$ restricted to a fiber of f is constant defines a constructible subset of X. By Noetherian induction, it is sufficient to solve the problem after restricting to any open subset of X. By [17, Thm. Finitude, Théorème 1.9(2)], there exists an open subset of X such that for each point x in that subset,

$$(f_*\mathcal{F})_x = H^0(Y_x, \mathcal{F}).$$

Restrict to that open subset. Because the image of each irreducible component of Y is constructible, we can choose a smaller open subset of X which is contained in the image of each irreducible component of Y with dense image and does not intersect the image of any irreducible component of Y without dense image. After base-changing to this open subset, each irreducible component of Y maps surjectively onto X (because the irreducible components without dense image no longer exist). Now we prove the result in this case. At any point x, if there is a section of $H^0(Y_x, \mathcal{F})$ that gives an isomorphism between $\mathcal{F}$ and the constant sheaf, then the corresponding section of $f_*\mathcal{F}$ extends to some neighborhood, which gives an extension of the section of $\mathcal{F}$ to the inverse image of that neighborhood, where because $\mathcal{F}$ is lisse it must be an isomorphism on every connected component of Y that intersects that fiber. By construction, every connected component of Y intersects the fiber over x, so the map is an isomorphism on the inverse image of the neighborhood of x. Hence the set where $\mathcal{F}$ is isomorphic to the constant sheaf is open, thus constructible, verifying the claim.

Consider the family of schemes $gN\langle\kappa[t]/t^m\rangle g^{-1} \cap H$ parameterized by $g \in G\langle\kappa[t]/t^m\rangle$. Let $\mathcal{F}$ be the pullback of $\mathcal{L}$ to this family. The set in $G\langle\kappa[t]/t^m\rangle$ where $\mathcal{F}$ is geometrically trivial on the fiber is constructible. Property (iv) is equivalent to the claim that this set does not contain any point defined over any field extension of κ. Because this set is constructible, it is sufficient to check this for every point defined over $\overline{\kappa}$, which is exactly geometric supercuspidality. This establishes the direction (iii) $\implies$ (iv).

We now establish (ii) $\implies$ (iii). Fix a point $g \in G(\overline{\kappa}[t]/t^m)$. There exist some finite field extension κ^* of κ such that g is defined over κ^* and every connected component of $gN\langle\kappa[t]/t^m\rangle g^{-1} \cap H$ is defined over κ^*. If the character sheaf $\mathcal{L}$ is geometrically trivial on $gN\langle\kappa[t]/t^m\rangle g^{-1} \cap H$, then its trace function is necessarily constant on each connected component of $gN\langle\kappa^*[t]/t^m\rangle g^{-1} \cap H$, and hence it corresponds to a character of the component group $\pi_0\left(gN\langle\kappa^*[t]/t^m\rangle g^{-1} \cap H\right)$. Thus the eigenvalue of Frobenius at each point is a root of unity of order dividing the order of the component group. We can pass to a further finite field extension κ'/κ^* that trivializes the eigenvalues of Frobenius at each point. Over this field extension, the corresponding character $\chi_{\kappa'}$ must be trivial when restricted to

$$\left(gN\langle\kappa[t]/t^m\rangle g^{-1} \cap H\right)(\kappa') = gN(\kappa'[t]/t^m)g^{-1} \cap H(\kappa'). \qquad \square$$

We deduce from Lemma 3.6 that, in order to establish that a monomial datum $(G, m, H, \mathcal{L})$ is geometrically supercuspidal, it suffices to verify the vanishing of all the Jacquet modules of all the representations compactly induced from (the inflation of) the characters $\chi_{\kappa'}$ of $H(\kappa')$, for all finite field extension κ'/κ. This will enable us to apply standard techniques from representation theory of reductive groups over local fields to verify geometric supercuspidality. Indeed, we will see examples of $(G, m, H, \mathcal{L})$ satisfying these properties later in this section.

The following lemma shows that H always lies in a maximal unipotent subgroup. This is useful because a maximal unipotent subgroup of $G(\kappa(\!(t)\!))$ sometimes lies in two subgroups, both isomorphic to $G(\kappa[\![t]\!])$, but not conjugate to each other. The lemma allows us to transfer geometrically supercuspidal monomial data between the two subgroups.

Lemma 3.7. *If $(G, m, H, \mathcal{L})$ is geometrically supercuspidal, then H is unipotent mod center.*

Proof. We will prove the contrapositive. Assume that H is not unipotent modulo the center of $G\langle\kappa[t]/t^m\rangle$; we will show that $(G, m, H, \mathcal{L})$ is not geometrically supercuspidal. A smooth connected algebraic group fails to be unipotent if and only if it admits a nontrivial homomorphism from $\mathbb{G}_m$ (possibly after extending the base field κ, which we may freely do). Thus H admits a homomorphism $\alpha : \mathbb{G}_m \to H$ that doesn't factor through the center of $G\langle\kappa[t]/t^m\rangle$.

The natural map $\kappa \to \kappa[t]/t^m$ defines a map $G \to G\langle\kappa[t]/t^m\rangle$. Let us check that G is a maximal reductive subgroup of $G\langle\kappa[t]/t^m\rangle$. To do this, observe that the projection $G\langle\kappa[t]/t^m\rangle \twoheadrightarrow G$ has reductive image and unipotent kernel, and because the composition $G \to G\langle\kappa[t]/t^m\rangle \twoheadrightarrow G$ is an isomorphism, G is a maximal reductive subgroup.

It follows that every reductive subgroup of $G\langle\kappa[t]/t^m\rangle$ is conjugate to a subgroup of G. In particular, the image of α is conjugate to a a subgroup of G. Because the definition of geometrically supercuspidal is invariant under conjugacy, we may assume without loss of generality that the image of α is a subgroup of G. In other words, we may assume that the composition $\mathbb{G}_m \xrightarrow{\alpha} H \subset G\langle\kappa[t]/t^m\rangle$ factors through a nontrivial homomorphism $\mathbb{G}_m \xrightarrow{\beta} G \to G\langle\kappa[t]/t^m\rangle$.

Let T be a maximal κ-split torus of G containing the image of β. Let P be the parabolic subgroup of G containing T and every root subgroup of G on which β acts by conjugating with eigenvalue a nonnegative power of the parameter $\mathbb{G}_m \xrightarrow{\mathrm{id}} \mathbb{G}_m$. Let N be the maximal unipotent subgroup of P. Then β acts on each root subgroup of N with eigenvalue a positive power of id. Let $H' := H \cap N\langle\kappa[t]/t^m\rangle$. Then H' is an iterated extension of copies of $\mathbb{G}_a$, on each of which β acts by conjugation by a nonzero power of id. In other words, H' admits a β-invariant filtration $\{1\} = H'_0 \subseteq H'_1 \subseteq \cdots \subseteq H'_m = H'$. Let i be the largest natural number such that $\mathcal{L}$ is geometrically trivial on H'_i. Then $\mathcal{L}$ descends to H'/H'_i and is nontrivial on H'_{i+1}/H'_i. Because $\mathcal{L}$ is a character sheaf on H, it is conjugacy-invariant. Hence it is invariant by the conjugacy action of β. Thus its restriction to H'_{i+1} followed by descent to H'_{i+1}/H'_i is invariant under the action of β, which is scaling by some nonzero power of id. But there is no nontrivial lisse sheaf on $\mathbb{G}_a$ which is invariant by scaling by a nonzero power. So in fact $i = m$, and $\mathcal{L}$ is trivial on H', so $(G, m, H, \mathcal{L})$ is not geometrically supercuspidal. □

3.3. **Intertwining.** Let G be a reductive group over a finite field κ, m a natural number, H a connected subgroup of $G\langle\kappa[t]/t^m\rangle$ containing the center, and $\mathcal{L}$ a character sheaf on H. We can check that $(G, m, H, \mathcal{L})$ is geometrically supercuspidal using a geometric analogue of the standard method, based on intertwining sets.

The *intertwining set* of $\mathcal{L}$ is the set of $g \in G(\overline{\kappa}(\!(t)\!))$ such that $\mathcal{L} \simeq \mathcal{L}^g$ on $H_{\overline{\kappa}} \cap H^g_{\overline{\kappa}}$.

Lemma 3.8. *If the intertwining set is equal to $J_{\overline{\kappa}}$, then $(G, m, H, \mathcal{L})$ is geometrically supercuspidal.*

Proof. Applying Lemma 3.6, it suffices to verify the vanishing of all the Jacquet modules of the induced representation $\text{c-Ind}_{J_{\kappa'}}^{G(\kappa'(\!(t)\!))} \chi_{\kappa'}$ for every finite extension κ'/κ. By assumption, an element $g \in G(\kappa'((t)))$ interwines $\chi_{\kappa'}$ in the sense that $\chi_{\kappa'}(h) = \chi_{\kappa'}(ghg^{-1})$ for every $h \in J_{\kappa'} \cap g^{-1} J_{\kappa'} g$ if and only if $g \in J_{\kappa'}$. The vanishing of all the Jacquet modules of $\text{c-Ind}_{J_{\kappa'}}^{G(\kappa'((t)))} \chi_{\kappa'}$ then follows, in the stronger form of the irreducibility and cuspidality of $\text{c-Ind}_{J_{\kappa'}}^{G(\kappa'((t)))} \chi_{\kappa'}$, from the argument of [9, §3.11.4]. See also [8, Prop.2.4]. □

Lemma 3.9. *Suppose that there is another subgroup K containing H as a normal subgroup, and that the intertwining set is equal to the set of g whose reduction modulo t^m is in $K(\overline{\kappa})$ and such that $\mathcal{L} \simeq \mathcal{L}^g$. Then $(G, m, H, \mathcal{L})$ is geometrically supercuspidal.*

Proof. We again apply Lemma 3.6, and verify the vanishing of the Jacquet module for every finite extension κ'. This follows from [53, Lem.2.2]. □

3.4. **Monomial geometric supercuspidal representations.** Let G be a reductive group over a finite field κ.

Definition 3.10. We say that an irreducible smooth representation of $G(\kappa((t)))$ is *mgs* if there exists a natural number m, a connected subgroup H of $G\langle\kappa[t]/t^m\rangle$, and a character sheaf $\mathcal{L}$ on H, such that

(1) $(G, m, H, \mathcal{L})$ is geometrically supercuspidal;
(2) π is a quotient of $\text{c-Ind}_J^{G(\kappa((t)))} \chi$ where J is the inverse image of $H(\kappa)$ in $G(\kappa[[t]])$ and χ is the trace function of $\mathcal{L}$ on $H(\kappa)$, pulled back to J.

Furthermore, in this setting, we say that $(G, m, H, \mathcal{L})$ is an *mgs datum* for π. By Lemma 3.2, condition (2) is equivalent to that the representation has a non-zero (J, χ)-invariant vector.

Lemma 3.11. *Let π be an mgs representation of $G(\kappa((t)))$. Then the pullback of π by any automorphism of the field $\kappa((t))$ is an mgs representation.*

Proof. Any such automorphism is a composition of an automorphism of κ with a change of variables that sends t to a power series with leading term a constant multiple of t. Automorphisms of κ act in a natural way on the mgs datum $(G, m, H, \mathcal{L})$. Changes of variables in t act in a natural way on $G\langle\kappa[t]/t^m\rangle$ and hence act in a natural way on H and $\mathcal{L}$. Both of these automorphisms agree with the action of the field automorphism on the induced representation, hence preserve the vanishing property of Jacquet modules vanishing, and also therefore agree with the pullback of χ. □

Lemma 3.12. *Let π be an mgs representation of $G(\kappa((t)))$. Then the pullback of π by any automorphism of the group G defined over $\kappa((t))$ is an mgs representation.*

Proof. Let $(G, m, H, \mathcal{L})$ be an mgs datum for π. By Lemma 3.7, H is unipotent. In particular, its image inside G is solvable, and hence contained in a Borel subgroup B. The inverse image of B in $G(\kappa[[t]])$ is a minimal parahoric subgroup I of $G(\kappa((t)))$. (This follows from the explicit description of the parahoric subgroup in terms of roots. If we take an apartment corresponding to the inverse image of a torus of G and perturb the hyperspecial point associated to $G(\kappa[[t]])$ in a generic direction, producing a point in the interior of a chamber whose associated subgroup is a minimal parahoric, we see that the parahoric subgroup is the inverse image of some Borel, and because all Borels are conjugate all such subgroups are minimal parahoric.) Because all minimal parahoric subgroups are conjugate [46, §9], every automorphism of $G_{\kappa((t))}$ can be expressed as an inner automorphism composed with an automorphism σ such that $\sigma(I) = I$. Conjugation by an element of $G(\kappa((t)))$ produces a representation isomorphic to π, so it suffices to show that the pullback of π by σ is mgs.

Expressing σ in the coordinates of G, let δ be the highest power of t^{-1} that appears. Then for any $g \in I$, $\sigma(g) \in I$ and $\sigma(g)$ modulo t^m depends only on g modulo $t^{m+\delta}$. Hence σ defines a map $\overline{\sigma}$ from the subset of $G\langle\kappa[t]/t^{m+\delta}\rangle$ congruent to B modulo t to the subset of $G\langle\kappa[t]/t^m\rangle$ congruent to B modulo t. Because σ acts as an automorphism of I, $\overline{\sigma}$ is surjective.

Consider the datum $(G, m+\delta, \overline{\sigma}^{-1}(H), \overline{\sigma}^*\mathcal{L})$. Let J' be the inverse image of $\overline{\sigma}^{-1}(H)(\kappa)$ in $G(\kappa[\![t]\!])$ and let χ' be the pullback of the trace function of $\overline{\sigma}^*\mathcal{L}$ to J'. Then $J' = \sigma^{-1}(J)$ and $\chi' = \chi \circ \sigma$, so $\text{c-Ind}_{J'}^{G(\kappa(\!(t)\!))} \chi'$ is the pullback of $\text{c-Ind}_{J}^{G(\kappa(\!(t)\!))} \chi$ by σ and hence $\pi \circ \sigma$ is a quotient of it.

Similarly, over any finite field extension κ' of κ, $\text{c-Ind}_{J'_{\kappa'}}^{G(\kappa'(\!(t)\!))} \chi'_{\kappa'}$ is the pullback of $\text{c-Ind}_{J'_{\kappa'}}^{G(\kappa'(\!(t)\!))} \chi_{\kappa'}$, hence has vanishing Jacquet module for every parabolic subgroup, so by Lemma 3.6 it is geometrically supercuspidal.

Hence $\pi \circ \sigma$ is an mgs representation. □

Any unramified reductive group over an equal characteristic local field F necessarily descends to a reductive group G over the residue field κ (Lemma 2.1). Combined with the previous two lemmas, that allows us to give an intrinsic definition of mgs representations of an unramified group over F. Namely π is mgs if for some (equivalently any) uniformizer t of F, and for reductive group G over κ and some (equivalently any) isomorphism with $G_{\kappa(\!(t)\!)}$, the representation π is a quotient of $\text{c-Ind}_J^{G(F)} \chi$ for some geometrically supercuspidal datum $(G, m, H, \mathcal{L})$.

Remark 3.13. We can make a similar definition over a mixed characteristic local field F, and for a general reductive group G over F as follows. Let $\mathfrak{o}_F$ be its ring of integers, ϖ an uniformizer, and κ its residue field. Let $\mathcal{G}$ be a smooth group scheme over $\mathfrak{o}_F$ whose generic fiber is isomorphic to G. Let $\mathcal{G}_m$ be the algebraic group over κ whose R-points for a ring R over κ are the $W_m(R) \otimes_{W(\kappa)} \mathfrak{o}_F$-points of $\mathcal{G}$, where W is the Witt vectors functor and $W_m(R)$ is the ring of truncated Witt vectors modulo p^m. (Here the Witt vectors are defined using universal polynomials over an imperfect ring). A *monomial datum* consists of a connected closed subgroup H of $\mathcal{G}_m$ and a character sheaf $\mathcal{L}$ on H. The datum is *geometrically supercuspidal* if for every proper parabolic subgroup $P \subset G$ with maximal unipotent N, closure $\mathcal{N}$ in $\mathcal{G}$, and associated κ-group $\mathcal{N}_m$, and every $g \in \mathcal{G}_m(\overline{\kappa})$, the restriction of $\mathcal{L}_{\overline{\kappa}}$ to the identity component of $H_{\overline{\kappa}} \cap g\mathcal{N}_{m,\overline{\kappa}} g^{-1}$ is nontrivial. Let J be the inverse image of $H(\kappa) \subseteq \mathcal{G}_m(\kappa) = \mathcal{G}(\mathfrak{o}_F/\varpi^m)$ in $\mathcal{G}(\mathfrak{o}_F)$ and let χ be the pullback of the trace function of $\mathcal{L}$ from $H(\kappa)$ to J. We say that an irreducible smooth representation of $G(F)$ is *mgs* if is has a non-zero (J,χ)-invariant vector, or equivalently if it appears as a quotient of $\text{c-Ind}_J^{G(F)} \chi$.

3.5. **Moy–Prasad types and epipelagic representations.** Let G be a quasi-split reductive group over κ, and $F = \kappa(\!(t)\!)$. Let x be a point in the Bruhat–Tits building of $G(F)$, and let $r > 0$ be a positive real. Let $G(F)_{x,r}$ and $G(F)_{x,r+}$ refer as usual to Moy–Prasad subgroups of $G(F)_x$. Then $G(F)_{x,r}/G(F)_{x,r+}$ is a vector space over κ. Let χ be a character of $G(F)_{x,r}$ that factors through this vector space.

Lemma 3.14. *$G(F)_{x,r}$ is conjugate to a subgroup of $G(\kappa[\![t]\!])$.*

Proof. It is contained in a minimal parahoric subgroup (e.g. the one associated to any adjacent chamber of the Bruhat–Tits building), and we may conjugate it to a minimal parahoric subgroup inside $G(\kappa[\![t]\!])$ [46, §9]. □

Lemma 3.15. *Suppose that $G(F)_{x,r} \subseteq G(\kappa[\![t]\!])$. Then there exists a natural number m, algebraic subgroups $H \subseteq G\langle\kappa[t]/t^m\rangle$, $H' \subseteq H$ such that the inverse image of $H(\kappa)$ in $G(\kappa[\![t]\!])$ is $G(F)_{x,r}$*

and the inverse image of $H'(\kappa)$ is $G(F)_{x,r+}$. Furthermore, for any finite field extension κ' of κ, the inverse image of $H(\kappa')$ is $G(\kappa'((t)))_{x,r}$ and the inverse image of $H'(\kappa')$ is $G(\kappa'((t)))_{x,r+}$.

Finally, H/H' is isomorphic to a vector space as an algebraic group.

The conditions on the rational points uniquely characterize the groups H and H'.

Proof. For some m, $G(F)_{x,r+}$ contains the subgroup of elements congruent to the identity modulo t^m, so that $G(F)_{x,r}$ and $G(F)_{x,r+}$ are the inverse images of their projections to $G(\kappa[t]/t^m)$.

It is clear from the definition of $G(F)_{x,r}$ and $G(F)_{x,r+}$ that these projections are algebraic subgroups — the Moy–Prasad subgroups are defined as the subgroups generated by certain additive and multiplicative groups, and we can simply take the algebraic subgroup generated by these groups.

Furthermore, because $r > 0$, all the involved subgroups are additive, and their commutators lie in $G(F)_{x,r+}$, so the H/H' is a vector space. ☐

Any character χ of $G(F)_{x,r}$ trivial on $G(F)_{x,r+}$ defines a character of $G(F)_{x,r}/G(F)_{x,r+} = H(\kappa)/H'(\kappa) = H/H'(\kappa)$ and hence, by Lemma 2.16, a character sheaf $\mathcal{L}$ on H. By construction, this datum $(G, m, H, \mathcal{L})$ satisfies $J = G(F)_{x,r}$ and $\chi = \chi$. Hence if $(G, m, H, \mathcal{L})$ is geometrically supercuspidal, any irreducible representation containing a vector on which $G(F)_{x,r}$ acts through the character χ is mgs.

A concrete description of when this occurs is provided by Lemma 3.6.

We give here a different condition, inspired by the construction of epipelagic representations of Reeder and Yu [53].

Lemma 3.16. *Let H and H' be the subgroups of Lemma 3.15. Let $\lambda : H/H' \to \mathbb{G}_a$ be a linear map, let $pr : H \to H/H'$ be the projection, let ψ be an additive character of κ, and let $\chi = \psi \circ \lambda \circ pr$ be the trace function of the character sheaf $pr^*\lambda^*\mathcal{L}_\psi$.*

Then $(G, m, H, pr^*\lambda^*\mathcal{L}_\psi)$ is geometrically supercuspidal if and only if λ is GIT-stable for the action of $G(F)_{x,0}/G(F)_{x,0+}$ on $(H/H')^\vee$.

Proof. By conjugation, we may assume that $G(F)_{x,0}$ contains the standard minimal parahoric subgroup I (the inverse image in $G(\kappa[[t]])$ of a fixed Borel subgroup of the quasi-split group $G(\kappa)$) and hence that x lies in the apartment of the standard maximal torus T. Let P be a standard parabolic, and consider a conjugate gPg^{-1}. Because G/P is proper and $H^1(\operatorname{Spec} \kappa[[t]], P)$ is trivial, the natural map $G(\kappa((t)))/P(\kappa[[t]]) \to G(\kappa((t)))/P(\kappa((t)))$ is a bijection, and so we may assume $g \in G(\kappa[[t]])$. By further multiplying on the right by an element of P, we may assume that g mod t is an element of the Borel times an element of the Weyl group, so $g = g_0 w$ where $g_0 \in I$ and w lies in the Weyl group.

Because P is a standard parabolic subgroup, there is some cocharacter $\alpha : \mathbb{G}_m \to T$ of the standard maximal torus T such that the unipotent subgroup N of P consists of those roots which have positive eigenvalue under α. Then wNw^{-1} consists of those roots which have a positive eigenvalue under $w\alpha w^{-1}$. Hence $wNw^{-1} \cap G(F)_{x,r}/(wNw^{-1} \cap G(F)_{x,r+}) = wNw^{-1} \cap H/(wNw^{-1} \cap H')$ is generated by the elements of H/H' which have a positive eigenvalue under $w\alpha w^{-1}$, as H/H' has a basis consisting of roots. Thus the projection onto H/H' of gNg^{-1} is generated by the elements which have a positive eigenvalue under $g_0 w\alpha w^{-1} g_0^{-1}$, which is a cocharacter of $G(F)_{x,0}$. Therefore the set of linear forms on H/H' that vanish on that projection is the subspace generated by the elements which have a nonnegative eigenvalue under $g\alpha g^{-1}$. If λ is GIT-stable, then by the Hilbert–Mumford criterion it does not lie in this space, so it is nontrivial on the image, hence the pullback of $\mathcal{L}_\psi$ under λ is nontrivial on this image, as desired.

For the converse, if λ is not stable, we have a cocharacter of $G(F)_{x,0}$ such that λ is a sum of linear forms on H/H' that are eigenvectors of this cocharacter with nonnegative eigenvalue. Hence λ vanishes on all elements of H/H' that have positive eigenvalue under the cocharacter. Now let P be the parabolic subgroup of G generated by the maximal torus and all the roots that have nonnegative eigenvalue under this character. Then all elements of N have positive eigenvalue, so λ vanishes on $H \cap N$ and therefore $(G, m, H, pr^* \lambda^* \mathcal{L}_\psi)$ is not mgs. □

Corollary 3.17. *If G is unramifed semisimple, then the epipelagic supercuspidal representations constructed in* [53] *are mgs.*

Proof. They are by definition summands of $\text{c-Ind}_{G(F)_{x,r}}^{G(F)} \chi$ for r the minimum positive value and χ a GIT-stable character of $G(F)_{x,r}/G(F)_{x,r}^+$. □

Example 3.18. We review the simplest example of an epipelagic representation, which is also the simplest example of an mgs representation. Let $G = SL_2$ and let x be the midpoint of an edge between two vertices of the Bruhat–Tits tree. Let $\mathfrak{o} = \kappa[\![t]\!]$ and $\mathfrak{p} = t\kappa[\![t]\!]$. Then $G(F)_{x,0+} = G(F)_{x,1/2}$ is the subgroup of matrices of the form $\begin{pmatrix} 1+\mathfrak{p} & \mathfrak{o} \\ \mathfrak{p} & 1+\mathfrak{p} \end{pmatrix}$ and $G(F)_{x,1/2+}$ is the subgroup of matrices of the form $\begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & 1+\mathfrak{p} \end{pmatrix}$, so the quotient is isomorphic to κ^2, given by extracting the leading terms of the top-right and bottom-left matrix entries.

Furthermore $G(F)_{x,0}$ is the subgroup of matrices of the form $\begin{pmatrix} \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o} \end{pmatrix}$, and so $G(F)_{x,0}/G(F)_{x,0+}$ consists of the cosets $\begin{pmatrix} a+\mathfrak{p} & \mathfrak{o} \\ \mathfrak{p} & a^{-1}+\mathfrak{p} \end{pmatrix} \in G(F)_{x,0}/G(F)_{x,0+}$ for $a \in \kappa^\times$. The action of such a coset is by multiplication by a^2 on the top-right entry and a^{-2} on the bottom-left entry, so the stable characters are exactly the characters nontrivial on the top-right and bottom-left entries.

The associated mgs datum has $m = 2$, H the four-dimensional subgroup of matrices in $SL_2\langle \kappa[t]/t^2 \rangle$ congruent mod t to an upper-triangular unipotent matrix, and $\mathcal{L}$ the unique character sheaf on H whose trace function is any fixed character nontrivial on the top-right and bottom-left entries.

3.6. **Adler datum and toral representations.** We now describe a special case of the construction of [1] that produces mgs representations. To that end, we borrow some notation from [1]. Let G be an unramified semisimple group over $F = \kappa(\!(t)\!)$ satisfying [1, Hypothesis 2.1.1]. This allows us to take a G-equivariant symmetric bilinear form on the Lie algebra $\mathfrak{g}$ of G, so that there is an induced isomorphism between $\mathfrak{g}$ and its dual.

Let T be a maximal F-torus of G that splits over a tamely ramified extension E of F but such that $T/Z(G)$ has no nontrivial map to $\mathbb{G}_m$ defined over any unramified extension of F. Let X be an element of the Lie algebra of T. Assume that there is a positive rational number r such that the valuation of $d\alpha(X)$ for every root α of T defined over E is equal to r.

Let x be the unique point of the Bruhat–Tits building of G that belongs to the apartment of T inside the Bruhat–Tits building of $G(E)$. Let $G(F)_{x,r}, G(F)_{x,r+}, \mathfrak{g}_{x,r}, \mathfrak{g}_{x,r+}$ be the corresponding Moy–Prasad subgroups of G and $\mathfrak{g}$. Then because $r > 0$, we may identify $G(F)_{x,r}/G(F)_{x,r+} = \mathfrak{g}_{x,r}/\mathfrak{g}_{x,r+}$ [1, (1.5.2)]. Using the bilinear form, we may view X as a character of $\mathfrak{g}_{x,r}/\mathfrak{g}_{x,r+}$, defining a character χ of $G(F)_{x,r}$.

Proposition 3.19. *Any irreducible representation π of $G(F)$ that contains $(G(F)_{x,r}, \chi)$ is mgs.*

Proof. We use the datum (G, m, H, ψ) constructed in the previous section. It remains to check that this datum is geometrically supercuspidal, which we do using Lemma 3.6.

It is sufficient to show that, after base-changing to a finite extension of κ, the Jacquet modules of this induced representation vanish. Because all our assumptions are stable under base change of κ, it in fact suffices to show that, for all F, G, T, X satisfying these assumptions, the Jacquet module of $\text{c-Ind}_{G(F)_{x,r}}^{G(F)} \chi$ vanishes.

Adler defines M to be the centralizer of X in G. In our case, that is simply $T(F)$, because our assumptions imply that $d\alpha(X) \neq 0$ for every root α of T. Because T is anisotropic, M is compact. Adler defines J as $\phi_x(\mathfrak{m}_{x,r} \oplus \mathfrak{m}^{\perp}_{x,(r/2)})$, where $\mathfrak{m}_{x,r}$ and $\mathfrak{m}^{\perp}_{x,(r/2)}$ are the Moy–Prasad subspaces of the Lie algebra of T and its orthogonal complement in the Lie algebra of G respectively, and ϕ_x is an approximate exponential map. For our purposes, it is most significant that J is compact and contains $G(F)_{x,r}$, and is normalized by M, so MJ is compact and contains $G(F)_{x,r}$ as an open subgroup.

Thus $\text{c-Ind}_{G(F)_{x,r}}^{MJ} \chi$ is a sum of irreducible representations σ of MJ, each containing $(G(F)_{x,r}, \chi)$ by Frobenius reciprocity. The induced representations $\text{c-Ind}_{G(F)_{x,r}}^{G(F)} \chi = \text{c-Ind}_{MJ}^{G(F)} \text{c-Ind}_{G(F)_{x,r}}^{MJ} \chi$ is the sum of $\text{c-Ind}_{MJ}^{G(F)} \sigma$, and by the discussion at the beginning of [1, §2.5], these are supercuspidal, so it is a sum of supercuspidal representations, hence has vanishing Jacquet modules. ☐

Proposition 3.19 shows that all the representations produced by the construction of Adler in the case where G is unramified and semisimple and the centralizer M of X is not just anisotropic over the base field but over all unramified extensions are mgs. (To see this, we must observe that M anisotropic over unramified extensions implies that M is a torus, as all groups become quasi-split over some unramified extension, and hence equals T. If $M = T$, then $d\alpha(X) \neq 0$ for any root α of T. This condition, plus the stronger anisotropic condition for T, are our only points of departure from the setup of [1].)

3.7. **Non-examples.** We discuss some examples of data $(G, m, H, \mathcal{L})$ that are not geometrically supercuspidal and so do not lead to mgs representations.

Example 3.20. If $\mathcal{L}$ is trivial then $(G, m, H, \mathcal{L})$ cannot be mgs unless G is a torus, as there will always be at least one proper parabolic subgroup. In particular, we can simply take H to be trivial.

Example 3.21. If the order of the monodromy group of $\mathcal{L}$, which, by Lemma 2.15, is equal to the order of the associated character, is prime to p, then its pullback to the intersection with any unipotent subgroup will have order prime to p, but the order of the unipotent subgroup is a power of p, so the character sheaf is trivial on that intersection. Thus $(G, m, H, \mathcal{L})$ is not mgs unless G is a torus.

For instance, we can take $m = 1$, H a Borel subgroup of G, and $\mathcal{L}$ the pullback of a character sheaf on the maximal torus. It is possible in this case for $\text{c-Ind}_J^{G(\kappa[\![t]\!])} \chi$ to be irreducible (the inflation of an irreducible principle series representation of $G(\kappa)$) but the Jacquet module of the induced representation is nonvanishing.

Example 3.22. We provide an example of a $(G, m, H, \mathcal{L})$ which is not mgs even though the Jacquet modules of the induced representation are trivial. Let $G = \mathrm{GL}(2)$, $m = 2$, and H be the subgroup of elements congruent to 1 mod t, which is isomorphic to the Lie algebra of G, i.e., the vector space of 2×2 matrices. Consider the linear function $A \mapsto \mathrm{tr}(AB)$ on the Lie algebra of G, where B is a non-scalar element of a non-split Cartan of $M_2(\kappa)$. View H as the Lie algebra of G and pull back an Artin–Schreier sheaf $\mathcal{L}_\psi$ to H under this linear function. Then for any parabolic subgroup P, $gNg^{-1} \cap H$ is a one-dimensional vector space of nilpotent matrices, so the character is trivial when pulled back to that subgroup if and only if the trace of B times the nilpotent matrix vanishes, which happens if and only if B is contained in the associated Borel. Over κ, this is impossible, so the Jacquet module vanishes, and the induced representation is a sum of supercuspidals. However, over a quadratic extension of κ, there are two Borels containing B, so $(G, m, H, \mathcal{L})$ is not geometrically supercuspidal.

3.8. **Preservation of mgs.** We show that mgs representations are preserved under some natural operations on algebraic groups. For this subsection and §4.1 only, we denote groups over the local field F by the roman letter G, and groups over the residue field κ by the bold letter $\mathbf{G}$.

Lemma 3.23. *Let $f : G_1 \to G_2$ be a homomorphism of unramified reductive groups over an equal characteristic local field F whose kernel is a torus and whose image is a normal subgroup with quotient a torus. Let π_2 be an mgs representation of $G_2(F)$. Then any irreducible quotient π_1 of $\pi_2 \circ f$ is mgs.*

Proof. Let $F = \kappa((t))$. We may choose descents $\mathbf{G}_1$ and $\mathbf{G}_2$ of G_1 and G_2 to κ such that f is defined over κ, because G_1 and G_2 have the same Bruhat–Tits buildings and the same hyperspecial points.

Let $(\mathbf{G}_2, m, H, \mathcal{L})$ be mgs datum for π_2. Let J_2 be the subgroup defined by this datum and χ the character. Let v be a vector in π_2 which transforms under the subgroup J_2 by the character χ_2. Because π_2 is irreducible, there must be some $g \in G_2(F)$ such that gv remains nonzero in the quotient π_1. This vector gv transforms under the subgroup $f^{-1}(gJ_2g^{-1})$ by $\chi_2 \circ f$. Because conjugation by g is an outer automorphism of G_1, and geometric supercuspidality is preserved by automorphisms (Lemma 3.12), we may assume π_1 contains a vector that transforms under the subgroup $f^{-1}(J_2)$ by the character $\chi_2 \circ f$.

We have a map $f : \mathbf{G}_1\langle \kappa[t]/t^m \rangle \to \mathbf{G}_2\langle \kappa[t]/t^m \rangle$. It suffices to show that $(G_1, m, f^{-1}(H), f^*\mathcal{L})$ is mgs datum for π_1. Let J_1 be the subgroup defined by this datum and let χ_1 be the character. We have $J_1 = f^{-1}(J)$ and $\chi_1 = \chi_2 \circ f$, so it remains to show that $(\mathbf{G}_1, m, f^{-1}(H), f^*\mathcal{L})$ is geometrically supercuspidal. Let P_1 be a parabolic subgroup of $\mathbf{G}_1$. Then P_1 is the inverse image under f of a parabolic subgroup P_2 of $\mathbf{G}_2$. Moreover, for N_1 and N_2 the maximal unipotent subgroups of P_1 and P_2, $f : N_1\langle \kappa[t]/t^m \rangle \to N_2\langle \kappa[t]/t^m \rangle$ is an isomorphism, because the kernel of f is a torus and does not intersect the unipotent subgroups, while the cokernel of f is a torus and so the image of the unipotent subgroup in it is trivial. So for any g in $\mathbf{G}_1\langle \kappa[t]/t^m \rangle$, $f : gN_1g^{-1} \cap f^{-1}(H) \to f(g)N_2f(g^{-1}) \cap H$ is an isomorphism, and since the pullback of $\mathcal{L}$ to $f(g)N_2f(g^{-1}) \cap H$ is nontrivial, the pullback of $\mathcal{L}$ to $gN_1g^{-1} \cap f^{-1}(H)$ is nontrivial. ☐

Lemma 3.24. *Let G_1 and G_2 be unramified reductive groups over an equal characteristic local field F. Let $\pi = \pi_1 \boxtimes \pi_2$ be a mgs representation of $G_1(F) \times G_2(F)$, where π_1 is a representation of $G_1(F)$ and π_2 is a representation of $G_2(F)$. Then π_1 and π_2 are mgs representations of $G_1(F)$ and $G_2(F)$ respectively.*

Proof. Let $F = \kappa((t))$. Choose descents $\mathbf{G}_1$ and $\mathbf{G}_2$ and isomorphisms $\mathbf{G}_{1,F} = G_1, \mathbf{G}_{2,F} = G_2$. Let $G = G_1 \times G_2$, and $\mathbf{G} = \mathbf{G}_1 \times \mathbf{G}_2$.

Choose an mgs datum $(\mathbf{G}, m, H, \mathcal{L})$ for π. Let $H_1 = H \cap \mathbf{G}_1\langle \kappa[t]/t^m \rangle$ and let $H_2 = H \cap \mathbf{G}_2\langle \kappa[t]/t^m \rangle$. Let $\mathcal{L}_1$ be the pullback of $\mathcal{L}$ to H_1 and let $\mathcal{L}_2$ be the pullback of $\mathcal{L}$ to H_2.

To show that $(\mathbf{G}_1, m, H_1, \mathcal{L}_1)$ and $(\mathbf{G}_2, m, H_2, \mathcal{L}_2)$ are geometrically supercuspidal, observe that for any parabolic subgroup P_1 of $\mathbf{G}_1$ with maximal unipotent subgroup N_1, $P_1 \times \mathbf{G}_2$ is a parabolic subgroup of $\mathbf{G} = \mathbf{G}_1 \times \mathbf{G}_2$ with maximal unipotent subgroup $N_1 \times e$, and for any $(g_1, g_2) \in \mathbf{G}_1\langle\kappa[t]/t^m\rangle \times \mathbf{G}_2\langle\kappa[t]/t^m\rangle$,

$$H \cap (g_1, g_2)(N_1 \times e)(g_1, g_2)^{-1} = H_1 \cap g_1 N_1 g_1^{-1}$$

so the pullback of $\mathcal{L}_1$ to $g_1 N_1 g_1^{-1}$ is geometrically nontrivial. The same argument works symmetrically for $\mathbf{G}_2$.

Letting $J, J_1, J_2, \chi, \chi_1, \chi_2$ be the subgroups and characters associated to the various data, we have $J_1 \times J_2 \subseteq J$ and $\chi_1 \times \chi_2$ is the restriction of χ to $J_1 \times J_2$, so there is a surjection

$$\text{c-Ind}_{J_1}^{G_1(F)} \chi_1 \boxtimes \text{c-Ind}_{J_2}^{G_2(F)} \chi_2 = \text{c-Ind}_{J_1 \times J_2}^{G_1(F)\times G_2(F)} (\chi_1 \times \chi_2) \to \text{c-Ind}_{J}^{G_1(F)\times G_2(F)} \chi \to \pi = \pi_1 \boxtimes \pi_2$$

and thus surjections $\text{c-Ind}_{J_1}^{G_1(F)} \to \pi_1$ and $\text{c-Ind}_{J_2}^{G_2(F)} \chi_2 \to \pi_2$, as desired. ∎

Lemma 3.25. *Let E/F be an unramified extension of local fields. Let G be an unramified reductive group over E. Let π be a representation of $G(E)$. Then π is an mgs representation of G over E if π is an mgs representation of the F-points of the Weil restriction of G from E to F.*

Proof. We may take $F = \kappa((t))$ and let $E = \kappa'((t))$. Let $\mathbf{G}'$ be a group over κ' with $\mathbf{G}'_E = G$. Let $\mathbf{G}$ be the Weil restriction of $\mathbf{G}$ from κ' to κ. Then $\mathbf{G}_F$ is the Weil restriction of G from E to F. Let $(\mathbf{G}, m, H, \mathcal{L})$ be an mgs datum for π.

For R a ring over κ', by definition

$$\mathbf{G}_{\kappa'}(R) = \mathbf{G}'(R \otimes_\kappa \kappa') = \mathbf{G}'(\prod_{\sigma \in \mathrm{Gal}(\kappa'/\kappa)} R) = \prod_{\sigma \in \mathrm{Gal}(\kappa'/\kappa)} \mathbf{G}'(R).$$

This defines an isomorphism $\mathbf{G}_{\kappa'} \equiv \mathbf{G}'^{[\kappa':\kappa]}$. Let i be the map $\mathbf{G}' \to \mathbf{G}_{\kappa'}$ defined as the inclusion of the factor corresponding to the identity element of the Galois group under this isomorphism.

Let $H' = i^{-1}(H)$ and let $\mathcal{L}'$ be the restriction of $\mathcal{L}$ to H. Then to check that $(\mathbf{G}', m, H', \mathcal{L}')$ is geometrically supercuspidal, fix P' a parabolic subgroup of $\mathbf{G}'$, and let P be the product of P' on the factor corresponding to the identity element with $\mathbf{G}'_\kappa$ on all the other factors, so that $N = i(N')$, and thus for any $g \in \mathbf{G}'\langle\kappa'[t]/t^m\rangle$, $i(g)Ni(g)^{-1} = i(gN'g^{-1})$. Hence because $\mathcal{L}$ is nontrivial on $i(g)Ni(g)^{-1} \cap H$, the restriction of $\mathcal{L}$ is nontrivial on $gN'g^{-1} \cap H'$.

Next observe that for $g' \in \mathbf{G}'(\kappa'[t]/t^m)$, since $\mathbf{G}'(\kappa'[t]/t^m) = \mathbf{G}(\kappa[t]/t^m)$ by the definition of Weil restriction, there is a corresponding element g in $\mathbf{G}(\kappa[t]/t^m)$, which we may pull back to $\mathbf{G}(\kappa'[t]/t^m)$. We can calculate

$$g = \prod_{\sigma \in \mathrm{Gal}(\kappa'/\kappa)} \sigma(i(g'))$$

where σ acts on $G(\kappa'[t]/t^m)$ by its action on κ', not on the group scheme G over κ. If $g' \in H'(\kappa)$ then $i(g) \in H(\kappa')$. Because H is defined over κ, this means $\sigma(i(g)) \in H(\kappa')$ for all automorphisms σ. Hence $g \in H(\kappa')$, and then because $g \in \mathbf{G}(\kappa[t]/t^m)$, we finally have $g \in H(\kappa)$.

It follows that J' is a subgroup of J when both are viewed as subgroups of $G(E)$. Furthermore χ' is the restriction of χ to J', so since π contains a vector transforming under the character χ of the subgroup J, it contains a vector transforming under the character χ' of the subgroup J'. ∎

3.9. **Admissibility.** In §3.1, we discussed the vanishing of Jacquet modules of certain induced representations $\text{c-Ind}_J^{G(F)} \chi$, but did not otherwise describe their structure. We now present a lemma giving a condition for such an induced representation (and some more general ones) to be admissible, in which case it follows that it is a finite direct sums of supercuspidals. The lemma has some similarities with [32, §III.2].

Lemma 3.26. *Let $F = \kappa((t))$, G a semisimple group over κ, J a compact open subgroup of $G(F)$, and σ a smooth finite-dimensional representation of J. Suppose for any proper parabolic subgroup P of $G(F)$, with unipotent radical N, the restricted representation $\sigma|_{J\cap N}$ does not contain the trivial representation. Then $\text{c-Ind}_J^{G(F)}(\sigma)$ is a finite direct sum of supercuspidal representations.*

Remark 3.27. The semisimplicity condition is necessary because $\text{c-Ind}_J^{G(F)} \sigma$ is never admissible if the center of $G(F)$ is non-compact. For example, all unramified characters of $\mathbb{F}_q((t))^\times$ appear as quotient of $\text{c-Ind}_{\mathbb{F}_q[[t]]^\times}^{\mathbb{F}_q((t))^\times} 1$.

Proof. By [8, Thm.1, (ii) $\implies$ (iv)], the assertion follows if we prove that $\text{c-Ind}_J^G(\sigma)$ is admissible. (The same proof works for an unramified semisimple group G over a local field F of characteristic zero.)

Let U_m be the principal congruence subgroup of $G(\kappa[[t]])$ consisting of elements congruent to 1 mod t^m. To prove that $\text{c-Ind}_J^G(\sigma)$ is admissible, it is sufficient to prove that the subspace of U_m-invariant vectors is finite-dimensional for every integer m. It suffices to prove that there are only finitely many double cosets $U_m gJ$ such that

(C) σ restricted to $g^{-1}U_m g \cap J$ contains the trivial representation.

By the Cartan decomposition, we write $g = k'\mu(t)k$ with $k, k' \in G(\kappa[[t]])$ with μ a dominant cocharacter of G. We have $g^{-1}U_m g = k^{-1}\mu^{-1}(t)U_m\mu(t)k$ (because U_m is normalized by k').

It is sufficient to prove that there are only finitely many possibilities for μ such that there is $g \in G(\kappa[[t]])\mu G(\kappa[[t]])$ satisfying the condition (C), as U_m and J are finite index in $G(\kappa[[t]])$.

We shall show that (C) implies that $\langle \mu, \alpha\rangle < m$ for any simple root α. This defines a finite subset of the cocharacter lattice.

Suppose for contradiction that $\langle \mu, \alpha\rangle > m$ for some simple root α. Let N be the maximal unipotent of the maximal parabolic associated to α.

Then $\mu^{-1}(t)U_m\mu(t)$ contains $N \cap G(\kappa[[t]])$. To check this, it is sufficient to check that for any element $u \in N \cap G(\kappa[[t]])$, the matrix coefficients of $\mu(t)u\mu(t)^{-1}$ in any representation are congruent to the matrix coefficients of the identity matrix mod t^m. We fix a representation, and choose a basis for that representation consisting of eigenvectors for the maximal torus T. For any i, j in the index set of this basis, the function $u \mapsto u_{ij}$ that sends an element of N to its ij matrix coefficient is a polynomial function on N which is equivariant according to some character χ_{ij} of T.

If $u \mapsto u_{ij}$ is the constant function on N, then

$$(\mu(t)u\mu(t)^{-1})_{ij} = e_{ij}$$

where $e \in N$ is the identity matrix, and so certainly $(\mu(t)u\mu(t)^{-1})_{ij}$ is congruent mod t^m to e_{ij}.

If $u \mapsto u_{ij}$ is nonconstant, then the character χ_{ij} is a nonempty product of characters $\chi_{\alpha'}$ associated to roots α' of N.

Each root α' of N is a positive root, hence a sum of simple roots. Because all the simple roots of G other than α are roots of the Levi of P, any sum of them lies in the root lattice of the Levi, and hence is not a root of N, so α' is a sum of simple roots, at least one of which is α. Because

μ is dominant and thus its pairings with the simple roots other than α are nonnegative, we have $\langle\alpha', \mu\rangle \geq \langle\alpha, \mu\rangle \geq m$ by assumption. It follows that $\chi_{\alpha'}(\mu(t)) = t^{\langle\alpha',\mu\rangle}$ is divisible by t^m.

Because χ_{ij} is a nonempty product of characters $\chi_{\alpha'}$, $\chi_{ij}(\mu(t))$ is divisible by t^m. Because $u \mapsto u_{ij}$ is χ_{ij}-equivariant, we have

$$(\mu(t)u\mu(t)^{-1})_{ij} = \chi_{ij}(\mu(t))u_{ij}.$$

Because $u \in G(\kappa[\![t]\!])$, $u_{ij} \in \kappa[\![t]\!]$, so $(\mu(t)u\mu(t)^{-1})_{ij} \in t^m\kappa[\![t]\!]$. Because the identity matrix e commutes with T, we have $e_{ij} = 0$, so $(\mu(t)u\mu(t)^{-1})_{ij}$ is congruent to e_{ij} mod t^m.

As we have checked both cases, it follows that $\mu(t)u\mu(t)^{-1}$ is congruent as a matrix to the identity matrix mod t^m. So after conjugation by k, we obtain that $g^{-1}U_m g$ contains $k^{-1}Nk \cap G(\kappa[\![t]\!])$, thus $k^{-1}Nk \cap J$. By assumption, the restriction of σ to $k^{-1}Nk \cap J$ does not contain the trivial representation. A fortiori, the restriction of σ to $g^{-1}U_m g \cap J$ does not contain the trivial representation, hence (C) is not satisfied. □

4. The base change transfer for mgs matrix coefficients

In [40], Kottwitz proves the base change fundamental lemma for unramified extensions at not just the unit elements of Hecke algebras but the characteristic functions of quite general compact open subgroups. In this section, we prove the analogous statement for one-dimensional characters of these compact open subgroups. This result should be useful in any attempt to describe how mgs representations behave under base change using the trace formula — in particular, in a proof of the conjecture we make in Section 5 — and may have other applications.

There is no direct way to base change an arbitrary compact open subgroup J and a one-dimensional character χ of it from a field to an unramified extension. On the other hand, it is easy to base change the monomial datum $(G, m, H, \mathcal{L})$ mentioned earlier, and this datum can be used to define a subgroup J and a character χ. The fact that the fundamental lemma holds in this setting can be motivated by the geometric Langlands philosophy: because the induced representations defined over two different fields from the datum $(G, m, H, \mathcal{L})$ correspond to the same geometric object, i.e., the category of $(H, \mathcal{L})$-equivariant sheaves on the loop group $G(\!(t)\!)$, they should have the same geometric Langlands parameter, so automorphic base change should take one to the other, which suggests that the fundamental lemma should hold.

However, in the proof of the fundamental lemma, the geometric description is not necessary. We have isolated the datum needed for a compact open subgroup of a group over a local field and a character to both have well-defined base changes to an arbitrary unramified extension. Our results hold in this setting, and work equally well over equal characteristic and mixed characteristic local fields. They may be of general interest.

4.1. Character datum.

Let F be a non-archimedean local field, let L be the completion of its maximal unramified extension, let σ be the Frobenius of F acting on L, and let G be a connected reductive group over F.

Definition 4.1. A *character datum* on $G(F)$ consists of a bounded open σ-invariant subgroup J_L of $G(L)$ and a central extension of topological groups with an action of σ

$$1 \to \mathbb{C}^\times \to \tilde{J}_L \to J_L \to 1.$$

We take the discrete topology and the trivial σ action on $\mathbb{C}^\times$.

For $E \subset L$ a degree l unramified extension of F, the subgroup $G(E)$ consists in the σ^l-invariant elements of of $G(L)$. Given a character datum on $G(F)$, define the subgroup J_E to be the σ^l-invariant subset of J_L and define $\chi_E : J_E \to \mathbb{C}^\times$ to take a σ^l-invariant element g to $\sigma^l(\tilde{g})\tilde{g}^{-1}$, where $\tilde{g}$ is a lift of g from J_L to $\tilde{J}_L$. In particular, in the $l = 1$ case, the character χ_F sends $g \in J_F$ to $\sigma(\tilde{g})\tilde{g}^{-1}$. Note that J_E and χ_E are invariant under σ and hence independent of the choice of isomorphism of E with the σ^l-invariant subfield of L.

Definition 4.2. Given an integer $l \geq 1$, we say that the character datum satisfies the axiom Lang_l if the map $g \mapsto \sigma^l(g)g^{-1}$ from J_L to itself is surjective.

Let $(\mathbf{G}, m, H, \mathcal{L})$ be a monomial datum, that is a group $\mathbf{G}$ over a finite field κ, a natural number m, a connected algebraic subgroup H of $\mathbf{G}\langle \kappa[t]/t^m \rangle$, and a character sheaf $\mathcal{L}$ on H, we can define a character datum on $G(\kappa((t)))$. Take J_L to be the elements of $\mathbf{G}(\overline{\kappa}[[t]])$ congruent mod t^m to elements of $H(\overline{\kappa})$. Lemma 2.17 defines a central extension of $H(\overline{\kappa})$ by $\overline{\mathbb{Q}}_\ell^\times$ with an action of σ associated to $\mathcal{L}$. By applying an embedding ι of $\overline{\mathbb{Q}}_\ell$ into $\mathbb{C}$, and pulling back from $H(\overline{\kappa})$ to J_L, we obtain a central extension $1 \to \mathbb{C}^\times \to \tilde{J}_L \to J_L \to 1$.

Lemma 4.3. *When we obtain a character datum from $(\mathbf{G}, m, H, \mathcal{L})$ in this way, the following holds:*

1. *The axiom Lang_l is satisfied for every integer $l \geq 1$.*
2. *For κ' a finite extension of κ, and $E = \kappa'((t))$, the character χ_E is equal to $\iota \circ \chi_{\kappa'}$, the trace function of $\mathcal{L}_{\kappa'}$, pulled-back from $H(\kappa')$ to $J_E = J_{\kappa'}$.*

Proof. (1) By Lang's theorem [57, §4.4.17], the map $g \to \sigma^l(g)g^{-1}$ from $H(\overline{\kappa})$ to itself is surjective for all l, and by iteratively lifting solutions to the equation $\sigma^l(g)g^{-1} = h$, the same map is surjective on J_L, so the axiom Lang_l is satisfied for all l.

(2) This follows by comparing the definition with Lemma 2.17. □

Remark 4.4. Character data have many of the nice geometric properties of monomial data, in particular those needed to prove the base change fundamental lemma below, without bringing any geometry into the definition. A character datum does not necessarily come from an algebraic subgroup, even if one assumes the axioms Lang_1 and Lang_l. For instance, consider the group of all matrices in $\mathrm{SL}_2(\mathbb{F}_q[[t]])$ that are unipotent upper triangular $\pmod t$ and whose upper-right entry $\pmod t$ lies in an extension of $\mathbb{F}_{q^l}$ of degree a power of p, where p is the characteristic of $\mathbb{F}_q$. Then for any a in $\mathbb{F}_{q^{lp^r}}$, the action of $\mathrm{Frob}_{q^{lp^r}}$ on solutions in $\overline{\mathbb{F}_q}$ of $x^q - x = a$ and $x^{q^l} - x = a$ is by translation, hence has order at most p, so both these equations have solutions in $\mathbb{F}_{q^{lp^{r+1}}}$. Because of this, the axioms Lang_1 and Lang_l are satisfied for this group, by taking an upper unipotent solution mod t and lifting to a t-adic solution.

4.2. **Matching of orbital integrals.** Assume that G_{der} is simply connected. Let $l \geq 1$, and $\tilde{J}_L \to J_L$ be character datum on $G(F)$ satisfying Lang_1 and Lang_l. We keep the other notation from the definition of character datum.

Let E be an unramified extension of F of degree l, embedded as the fixed points of σ^l in L. Let θ be an automorphism of E, with $E^\theta = F$.

Let f on $G(F)$ be equal to χ on J_F and 0 elsewhere. Let f_E on $G(E)$ equal χ_E on J_E and 0 elsewhere. We have the orbital integral

$$O_\gamma(f) = \int_{G_\gamma(F)\backslash G(F)} f(g^{-1}\gamma g) dg/dt$$

for G_γ the centralizer of γ in G, dg the Haar measure on g that gives J_F measure one, and dt any fixed Haar measure on $G_\gamma(F)$.

Similarly, we define

$$O_{\delta\theta}(f_E) = \int_{I_{\delta\theta}(F)\backslash G(E)} f_E(g^{-1}\delta\theta(g)) dg_E/d\mu$$

where $I_{\delta\theta}$ is the algebraic subgroup of $\mathrm{Res}_F^E G$ consisting of h satisfying the equation $h = \delta\theta(h)\delta^{-1}$, g_E is the Haar measure on G such that J_E has total mass one, and $d\mu$ is a Haar measure on $I_{\delta\theta}(F)$. We shall assume that these integrals converge absolutely.

Let j be an integer such that $\theta = \sigma^j$ as automorphisms of E, and let a, b be integers with $bl - aj = 1$.

Kottwitz's argument [40] relies on the system in (γ, δ, c) of two equations

$$\begin{cases} c\gamma^a\sigma^l c^{-1} = \sigma^l, \\ c\gamma^b\sigma^j c^{-1} = \delta\sigma^j, \end{cases} \tag{4.1}$$

valued in the semidirect product of $G(L)$ with the free abelian group on σ.

Lemma 4.5. *Suppose that $\gamma \in J_F$, $\delta \in J_E$, $c \in J_L$ satisfy the system* (4.1). *Then $\chi(\gamma) = \chi_E(\delta)$.*

Proof. Choose lifts $\tilde{\gamma}$ and $\tilde{\delta}$ to $\tilde{G}_L$. We will perform calculations in the semidirect product of $\tilde{J}_L$ with the free abelian group on σ. We have

$$\chi(\gamma) = \sigma(\tilde{\gamma})\tilde{\gamma}^{-1} = [\sigma, \tilde{\gamma}].$$

Because γ and σ commute, $\tilde{\gamma}$ and σ commute modulo center, so because $bl - aj = 1$,

$$[\sigma, \tilde{\gamma}] = [\tilde{\gamma}^a\sigma^l, \tilde{\gamma}^b\sigma^j].$$

Then because this commutator is central, it commutes with c, and thus

$$[\tilde{\gamma}^a\sigma^l, \tilde{\gamma}^b\sigma^j] = c[\tilde{\gamma}^a\sigma^l, \tilde{\gamma}^b\sigma^j]c^{-1} = [c\tilde{\gamma}^a\sigma^l c^{-1}, c\tilde{\gamma}^b\sigma^j c^{-1}].$$

Finally, because this commutator is independent of the choice of lift to a central extension,

$$[c\tilde{\gamma}^a\sigma^l c^{-1}, c\tilde{\gamma}^b\sigma^j c^{-1}] = [\sigma^l, \tilde{\delta}\sigma^j] = [\sigma^l, \tilde{\delta}] = \sigma^l(\tilde{\delta})\tilde{\delta}^{-1} = \chi_E(\delta).$$ □

Lemma 4.6. *Suppose that $\gamma \in G(F)$, $\delta \in G(E)$, $c \in G(L)$ satisfy* (4.1), *and also satisfy $x^{-1}\gamma x \in J_F$, $y^{-1}\delta\theta(y) \in J_E$, $y^{-1}cx \in J_L$.*

Then $\chi_E(y^{-1}\delta\theta(y)) = \chi(x^{-1}\gamma x)$.

Proof. This follows by applying Lemma 4.5 to $x^{-1}\gamma x$, $y^{-1}\delta\theta(y)$, $y^{-1}cx$, which can be immediately seen to satisfy the system of equations (4.1). □

The remainder of the argument closely follows [40]. We repeat the arguments in our setting for clarity, and because Kottwitz works in mixed characteristic only and we need equal characteristic.

Lemma 4.7. *Suppose that $\gamma \in G(F)$, $\delta \in G(E)$, $c \in G(L)$ satisfy* (4.1). *Conjugation by c defines an isomorphism from G_γ to $I_{\delta\theta}$, and we have*

$$O_{\delta\theta}(f_E) = O_\gamma(f),$$

where we use this isomorphism to match the Haar measures on G_γ and $I_{\delta\theta}$.

Proof. We break the integral $\int_{I_{\delta\theta}(F)\backslash G(E)} f_E(g^{-1}\delta\theta(g))dg_E/d\mu$ into a sum over double cosets $y \in I_{\delta\theta}(F)\backslash G(E)/J_E$. For each double coset, we claim that f_E is constant. This is because f_E vanishes outside J_E, a set which is invariant under twisted J_E-conjugation, and is a θ-invariant character on J_E which is also invariant under twisted J_E conjugation. This follows from the fact that for $\tilde{k}$ a lift of k, and $g \in J_E$, $\tilde{g}^{-1}\tilde{k}\sigma^j(\tilde{g})$ is a lift of $g^{-1}k\theta(g)$, and we have

$$\sigma^l(\tilde{g}^{-1}\tilde{k}\sigma^j(\tilde{g})) = \sigma^l(\tilde{g})^{-1}\sigma^l(\tilde{k})\sigma^j(\sigma^l(\tilde{g})) = \tilde{g}^{-1}\chi_E(g)^{-1}\tilde{k}\chi_E(k)\sigma^j(\tilde{g}\chi_E(g)) = \tilde{g}^{-1}\tilde{k}\sigma^j(\tilde{g})\chi_E(k).$$

Hence we can express the integral as a sum over $y \in I_{\delta\theta}(F)\backslash G(E)/J_E$ such that $y^{-1}\delta\theta(y) \in J_E$ of $\chi_E(y^{-1}\delta\theta(y))$ times the measure of $I_{\delta\theta}(F)\backslash I_{\delta\theta}(F)yJ_E$.

Similarly, in the $l = 1$ case, the integral is the sum over $x \in G_\gamma(F)\backslash G(F)/J_F$ such that $x^{-1}\gamma x \in J_F$ of $\chi(x^{-1}\gamma x)$ times the measure of $G_\gamma(F)\backslash G_\gamma(F)x$.

Using the axiom Lang_l, one can view $G(E)/J_E$ as the σ^l-fixed points in $G(L)/J_L$, and the set with $y^{-1}\delta\theta(y) \in J_E$ as the $\delta\sigma^j$ -fixed points among those. Similarly, by Lang_1, $G(F)/J_F$ is the set of σ-fixed points in $G(L)/J_L$, and the subset of x with $x^{-1}\gamma x \in J_F$ is the γ-fixed points. Now (4.1) implies precisely that the map that sends x to $y = cx$ gives a bijection between the points fixed by γ and σ and the points fixed by σ^l and $\delta\sigma^j$. Furthermore, the points of $G(L)$ fixed by conjugation by γ and σ are precisely $G_\gamma(F)$, and the points fixed by $\delta\sigma^j$ and σ^l are precisely $I_{\delta\theta}(F)$, so this gives a bijection between the double cosets $I_{\delta\theta}(F)\backslash G(E)/J_E$ and $G_\gamma(F)\backslash G(F)/J_F$.

By construction, for x and y paired by this bijection, we have $y = cx \in G(L)/J_L$, so $y^{-1}cx \in J_L$, thus by Lemma 4.6, $\chi_E(y^{-1}\delta\theta(y)) = \chi(x^{-1}\gamma x)$.

It remains to check that, for x and y paired by this bijection, the measure of $I_{\delta\theta}(F)\backslash I_{\delta\theta}(F)yJ_E$ equals the measure of $G_\gamma(F)\backslash G_\gamma(F)xJ_F$. To do this, observe that we have fixed measures so that J_E and J_F have measure 1, so that the measure of $I_{\delta\theta}(F)\backslash I_{\delta\theta}(F)yJ_E$ is equal to the inverse of the measure of the stabilizer of yJ_E in $I_{\delta\theta}$, and $G_\gamma(F)\backslash G_\gamma(F)xJ_F$ is equal to the inverse of the measure of the stabilizer of xJ_F in $G_\gamma(F)$. We can equivalently view these stabilizers as the stabilizers of the points x and y in $G(L)/J_L$. Thus, because $y = cx$, these stabilizers are sent to each other by the isomorphism between $G_\gamma(F)$ and $I_{\delta\theta}(F)$ defined by conjugation by c, which by assumption is a measure-preserving isomorphism, so these measures are equal.

Hence the sums are equal and the orbital integrals are equal. □

4.3. **Stable orbital integrals.** We say $\gamma, \gamma' \in G(F)$ are *stably conjugate* if they are conjugate as elements of $G(\overline{F})$.

An *inner twisting* between two algebraic groups is an isomorphism defined over the separable closure of the base field, which is Galois-invariant up to compositions with inner automorphisms, and where we take two inner twistings to be equivalent if they are equal up to composition with an inner automorphism [49, p.68]. Given an inner twisting between two groups, there is a natural transfer, explained in *loc. cit*, of Haar measures from one group to Haar measures on the other via the Lie algebras.

In particular, if γ and γ' are stably conjugate, then there is a canonical inner twisting (i.e., canonical isomorphism over the separable closure of the base field, up to conjugacy) between their centralizers G_γ and $G_{\gamma'}$. This enables us to define, after fixing a Haar measure on G_γ, the stable orbital integral

$$SO_\gamma(f) = \sum_{\gamma'} e(G_{\gamma'})O_{\gamma'}(f)$$

where γ' traverses a system of conjugacy classes of elements stably conjugate to γ, and $e(G_{\gamma'})$ is the sign defined by Kottwitz.

Less obviously, for $\delta \in G_E$, let $\mathcal{N}\delta = \delta\theta(\delta)\theta^2(\delta)\dots\theta^{l-1}(\delta)$ be the norm of δ. If $\mathcal{N}\delta$ is stably conjugate to γ then there is a canonical inner twisting $I_{\delta\theta} \to G_\gamma$. Indeed,

Lemma 4.8. *Let $I = \operatorname{Res}_F^E G$ be the Weil restriction, and I_E its base change to E.*

Let p be the projection $I_E \to G_E$ defined, using the fact that R-points of $\operatorname{Res}_F^E G$ are $R \otimes_F E$-points of G for any ring R, by the map $G(R \otimes_F E) \to G(R)$ for an E-algebra R induced by the multiplication map $R \otimes_F E \to R$.

For $d \in G(F^s)$ such that $d^{-1}\mathcal{N}\delta d = \gamma$, the map $g \mapsto d^{-1}p(g)d$ from $I_{\delta\theta,F^s} \to G_{\gamma,F^s}$ is an isomorphism.

This defines an isomorphism $I_{\delta\theta} \to G_\gamma$ which depends only on γ, δ, up to conjugation by G_γ.

The proof is the same as [39, Lem.5.8] and [49, §I, p.115], though neither reference is in the exact context we work in.

Proof. We use the fact that $I_E \cong G^{\mathrm{Gal}(E/F)}$. Under this isomorphism, the action of δ is by translation, and the map p is projection onto one of the factors. (This follows from the fact that $E \otimes_F E = E^{\mathrm{Gal}(E/F)}$, with the action of θ by translation, and the multiplication map to E is projection onto one of the factors).

Thus the action of $\delta\theta$ on I is by translation by $\theta \in \mathrm{Gal}(E/F)$ and then conjugation by δ. So a fixed point of this action is determined by a tuple of l elements of G, each of which when conjugated by δ becomes equal to the next one. Such a tuple is determined by its value in one copy of G, and an element of G extends to a tuple if and only if it returns to itself when conjugated and translated l times, which is equivalent to commuting with $\mathcal{N}\delta$. This shows that the projection p defines an isomorphism $I_{\delta\theta} \cong G_{\mathcal{N}\delta}$ over L, and then conjugating by d gives a further isomorphism onto G_γ.

Because any d' satisfying the same equation as d, for instance a Galois conjugate of d, is equal to d times an element of G_γ, this map depends only on δ, γ up to conjugation by G_γ. ☐

Using this isomorphism $I_{\delta\theta} \to G_\gamma$ to transfer a fixed Haar measure on G_γ, we can define the stable twisted orbital integral

$$SO_{\delta\theta}(f_E) = \sum_{\delta'} e(I_{\delta'\theta})O_{\delta'\theta}(f_E)$$

where δ' traverse a system of representatives for the twisted conjugacy classes inside the stable twisted conjugacy class of δ. (The transfer of Haar measure depends only on γ, δ because the Haar measure on G_γ is invariant under conjugation.)

We will now show an identity of stable twisted orbital integrals, continuing to follow [40].

Lemma 4.9. *For each $\delta \in G(E)$, there is at most one $\gamma \in G(F)$ up to conjugacy satisfying* (4.1)*, and always at least one if $O_{\delta\theta}(f_E) \neq 0$. Similarly, for each $\gamma \in G(F)$, there is at most one $\delta \in G(E)$ up to θ-conjugacy satisfying* (4.1)*, and always at least one if $O_\gamma(f) \neq 0$.*

Finally, δ and γ satisfying (4.1) *have $\mathcal{N}\delta = c\gamma c^{-1}$.*

Proof. Fix γ. The identity $c\gamma^a\sigma^l c^{-1} = \sigma^l$ implies

$$c^{-1}\sigma^l(c) = \gamma^a,$$

which uniquely determines c up to left multiplication by something σ^l-invariant. In other words, this determines c up to left-multiplication by an element of $G(E)$. For any choice of c, the identity $c\gamma^b\sigma^j c^{-1} = \delta\sigma^j$ determines δ, and multiplying c on the left by $G(F)$ is equivalent to conjugating $\delta\sigma^j$ by an element of $G(E)$ and thus is equivalent to θ-conjugating δ by an element of $G(E)$. So for each γ, there is at most one δ up to θ-conjugacy.

For there to exist at least one δ satisfying (4.1), it suffices that the equation $c^{-1}\sigma^l(c) = \gamma^a$ has a solution, for which by the axiom Lang_l it suffices that γ is conjugate to an element of J_F, which is implied by the nonvanishing of $O_\gamma(f)$. Moreover, any δ satisfying (4.1) lies in $G(E)$ because the two equations together imply that δ commutes with σ^l.

For the opposite direction, we change the equations slightly. Because γ and σ commute with each other, and σ^l and $\delta\sigma^j$ commute with each other, we can invert the two-by-two matrix to obtain the equivalent equations

$$(\delta\sigma^j)^l\sigma^{-jl} = c\gamma c^{-1}$$
$$(\delta\sigma^j)^{-a}\sigma^{bl} = c\sigma c^{-1}$$

Fixing δ, the second equation determines $c\sigma c^{-1}$, hence determines c up to right multiplication by an element of $G(F)$. Examining the first equation, we see it determines γ after fixing δ, c, and right multiplying c by an element of $G(F)$ has the effect of conjugating γ by an element of $G(F)$.

For γ to exist, it suffices that there exists a c with $c\sigma(c)^{-1} = (\delta\sigma^j)^{-a}\sigma^{bl-1}$, for which by the axiom Lang_1 it suffices that δ is θ-conjugate to an element of J_E, which is implied by the nonvanishing of $O_{\delta\theta}(f_E)$. (Indeed, if $O_{\delta\theta}(f_E) \neq 0$ then there exists $g \in G(E)$ with $u = g^{-1}\delta\theta(g) \in J_E$, so that $(\delta\sigma^j)^{-a}\sigma^{bl-1} = g^{-1}(u\sigma^j)^a g\sigma^{bl-1} = g^{-1}u\theta(u)\dots\theta^{a-1}(u)\sigma^{1-bl}(g)$ and then applying Lang_1 to $u\theta(u)\dots\theta^{a-1}(u)$ and using $\sigma^{1-bl}(g) = \sigma(g)$ we obtain c.) Furthermore this implies $\gamma \in G(F)$, because it implies γ commutes with σ.

Finally, observe that

$$c\gamma c^{-1} = (\delta\sigma^j)^l\sigma^{-jl} = \delta\theta(\delta)\theta^2(\delta)\dots\theta^{l-1}(\delta) = \mathcal{N}\delta. \qquad \square$$

Lemma 4.10. *For any δ, γ, c satisfying (4.1) with γ semisimple, the map from $I_{\delta\theta}(F)$ to $G_\gamma(F)$ defined by conjugation by c in fact arises from an isomorphism of group schemes over F, which is equivalent to the isomorphism of Lemma 4.8 in the case $d = c$.*

In particular, the transfer of the Haar measure from $G_\gamma(F)$ to $I_{\delta\theta}(F)$ under this map matches the transfer via the isomorphism of Lemma 4.8.

Proof. The isomorphism $g \mapsto c^{-1}p(g)c$ of Lemma 4.8 is, by construction, defined over L.

To show it descends from L to F, we use the fact that G_γ is reductive and thus, by Lemma 4.8, $I_{\delta\theta}$ is reductive, so there exists a scheme parameterizing isomorphisms between G_γ and $I_{\delta\theta}$. To check that an L-point of this scheme is defined over F, it suffices to check that it is stable under the Frobenius σ. In other words we must check that it commutes with σ. It suffices to check it commutes with σ^l and σ^j.

Observe that σ^l commutes with p, and that

$$\sigma^l(c^{-1}gc) = \sigma^l(c)^{-1}\sigma^l(g)\sigma^l(c) = \gamma^{-a}c^{-1}\sigma^l(g)c\gamma^a = c^{-1}\sigma^l(g)c$$

using (4.1) and the fact that γ commutes with $c^{-1}\sigma^l(g)c \in G_\gamma$.

Next observe that

$$\sigma^j(c^{-1}p(g)c) = \sigma^j c^{-1}p(g)c\sigma^{-j} = \gamma^{-b}c^{-1}\delta\sigma^j p(g)\sigma^{-j}\delta^{-1}c\gamma^b$$
$$= \gamma^{-b}c^{-1}p(\delta\theta\sigma^j g\sigma^{-j}\theta\delta^{-1})c\gamma^b = \gamma^{-b}c^{-1}p(\sigma^j(c))c^{-1} = c^{-1}p(\sigma^j(c))c^{-1}$$

using (4.1), the fact that $\sigma^j(g) \in I_{\delta\theta}$ commutes with $\delta\theta$, and the fact that $c^{-1}p(\sigma^j(c))c^{-1} \in G_\gamma$ commutes with γ. $\square$

Theorem 4.11. *For every semisimple $\gamma \in G(F)$, the stable orbital integral $SO_\gamma(f)$ vanishes unless the stable conjugacy class of γ is equal to the norm $\mathcal{N}\delta$ for some $\delta \in G(E)$, in which case it is given by $SO_\gamma(f) = SO_{\delta\theta}(f_E)$.*

Here we define both stable orbital integrals using the same Haar measure on G_γ.

Proof. For each stable conjugate γ' of γ, if the associated orbital integral is nonvanishing, then γ' is conjugate to an element of J_F. Hence by Lemma 4.9 there exists a δ' satisfying Kottwitz's equations, and the norm of δ' is stably conjugate to γ.

So we may assume that γ is stably conjugate to the norm of δ. Now for each γ' for which the orbital integral is nonvanishing there exists a unique δ' up to θ-conjugacy satisfying (4.1) by Lemma 4.9, and because the norm of δ' is stably conjugate to the norm of δ, δ' is stably θ-conjugate to δ. (To see, this, base change to E, so that $I = G^l$ and θ acts by permutation. Then if two elements of G^l have conjugate norms, we can θ-conjugate one to the other by adjusting each element of the l-tuple step-by-step.) By Lemma 4.7 and Lemma 4.10, the orbital integrals and signs of γ' and δ' agree. (The signs agree because they depend only on the isomorphism class, and we have an isomorphism between the two groups.) Because each γ' corresponds to a unique δ' up to stable θ-conjugation, and by Lemma 4.9 each δ' with nonvanishing orbital integral corresponds to a unique γ' up to conjugation, the signed sums of orbital integrals over conjugacy classes and θ-conjugacy classes agree, so the orbital integrals agree. □

The analogue for κ-orbital integrals should also be possible, by an argument analogous to that in [40].

5. Automorphic base change

For every place y of every constant field extension F_n of F of degree $n \geq 1$, we will always take the standard hyperspecial maximal compact $G(\mathfrak{o}_y)$ defined by the globally split structure of G. We say that a representation is unramified when it is $G(\mathfrak{o}_y)$-unramified. Let π be an automorphic representation of $G(\mathbb{A}_F)$, and $u \in |X|$ a place such that π_u is mgs. In this context, we say that π is base-changeable if the following holds.

Condition (BC). *There exists a finite set of mgs data at u, such that for every constant field extension F_n of F, there exists a base change representation Π_n of $G(\mathbb{A}_{F_n})$, which at places lying over u is mgs with one of the given mgs data, over the unramified places of π is unramified and compatible under the Satake isomorphism, and at all other places has depth bounded independently of n.*

We make the following conjecture.

Conjecture 5.1. *Every automorphic representation of $G(\mathbb{A}_F)$ that is mgs at a place u satisfies Condition BC.*

This is a standard conjecture on the existence of cyclic base change, analogous to results that have been proved over number fields by Labesse [41, Thm.4.6.2], except for the compatibility condition at places lying over u, and for the boundedness of depth [27]. Our main evidence that a cyclic base change compatible at u should exist is Theorem 4.11, which gives the local transfer identities needed to compare twisted orbital integrals involving a test function which detects the mgs condition with usual orbital integrals for an analogous test function. Hence the conjecture is amenable to endoscopically stabilizing the trace formula and twisted trace formula and proving a comparison result between them. Special cases are accessible either by establishing stability of a finite set of mgs data at u, or by inserting stabilizing test functions at an additional place, we hope to do this in the sequel [55].

6. Geometric setup

We now discuss geometric models for a family of automorphic forms with prescribed local behavior. Afterwards, we will use these geometric models to bound the traces of Hecke operators on this family.

Let k be a field, let X be a curve over k, and let $F = k(X)$. When we connect to analysis we will assume k finite, but for the purely geometric parts we will not need that assumption. Let G be a split semisimple algebraic group over k. Let D be an effective divisor on X, which we will often view as a closed subscheme in X. We write $D = \sum_{x \in D} m_x[x]$ where m_x is the multiplicity of x in D.

Definition 6.1. Let $\mathrm{Bun}_{G(D)}$ be the moduli space of G-bundles on X with a trivialization along D (notation is in analogy with that of principal congruence subgroups).

We write $|X|$ for the set of closed points of X and $|X - D|$ for the points outside the support of D. For $x \in |X|$, let κ_x be the residue field at x. We fix a local coordinate t of X at each closed point x, so that $\mathfrak{o}_x = \kappa_x[\![t]\!]$ is the complete local ring at x, but our constructions will be independent of the choice of coordinate and so this is really just a notational convenience. With this convention, $F_x = \kappa_x(\!(t)\!)$. The adèle ring $\mathbb{A}_F$ is the restricted product $\prod'_{x \in |X|} F_x$.

Notation 6.2. Let

$$\mathbf{K}(D) = \prod_{x \in |X-D|} G(\mathfrak{o}_x) \times \prod_{x \in D} U_{m_x}(G(\mathfrak{o}_x)),$$

where $U_{m_x}(G(\kappa_x[\![t]\!]))$ is the subgroup of $G(\kappa_x[\![t]\!])$ consisting of elements congruent to 1 modulo t^{m_x}. Then Weil's parameterization lets us write $\mathrm{Bun}_{G(D)}(k)$ as the adelic double quotient $G(F)\backslash G(\mathbb{A}_F)/\mathbf{K}(D)$, see Lemma 9.1 below.

Let $\mathcal{O}_D$ be the ring of global sections of the structure sheaf on the scheme D, so that $G\langle \mathcal{O}_D \rangle$ is the group of automorphisms of the trivial G-bundle on D.

Lemma 6.3. *We have isomorphisms*

$$\mathcal{O}_D \simeq \prod_{x \in D} \kappa_x[t]/t^{m_x}, \quad G\langle \mathcal{O}_D \rangle \simeq \prod_{x \in D} G\langle \kappa_x[t]/t^{m_x} \rangle.$$

Proof. The first isomorphism follows from viewing D as a disjoint union of schemes $m_x[x]$, and choosing local coordinates for each x, and the second isomorphism follows from the first. □

Definition 6.4. We say that an algebraic subgroup $H \subseteq G\langle \mathcal{O}_D \rangle$ is *factorizable* if it is equal to a product $\prod_{x \in D} \mathrm{Res}^k_{\kappa_x} H_x$ where H_x is an algebraic subgroup of $G_{\kappa_x}\langle \kappa_x[t]/t^{m_x} \rangle$ and $\mathrm{Res}^k_{\kappa_x} H_x$ is its Weil restriction from κ_x to k, making it a subgroup of $G\langle \kappa_x[t]/t^{m_x} \rangle$.

Lemma 6.5. *If $H \subseteq G\langle \mathcal{O}_D \rangle$ is factorizable, then for any separable field extension k' of k, the base change $H_{k'}$ of H from k to k' remains factorizable as a subgroup of $G_{k'}\langle \mathcal{O}_D \otimes k' \rangle$.*

Proof. Write $H = \prod_{x \in D} \mathrm{Res}^k_{\kappa_x} H_x$. Let us check that

$$(\mathrm{Res}^k_{\kappa_x} H_x)_{k'} = \mathrm{Res}^{k'}_{\kappa_x \otimes k'} (H_x)_{\kappa_x \otimes k'} = \prod_{x' | x} \mathrm{Res}^{k'}_{\kappa_{x'}} (H_x)_{\kappa_{x'}}. \tag{6.1}$$

The first equality follows from taking the definition of the Weil restriction and base changing everything from k to k'. The second follows from the fact that k'/k is separable and thus $\kappa_x \otimes k' = \prod_{x'|x} \kappa_{x'}$ is a product of fields.

Taking the product of (6.1) over $x \in D$, the resulting subgroup $H_{k'}$ is factorizable. □

Fix a smooth connected factorizable subgroup $H \subseteq G\langle \mathcal{O}_D \rangle$ and a character sheaf $\mathcal{L}$ on H. By Lemma 2.18, $\mathcal{L}$ splits as a product $\boxtimes_{x\in D} \operatorname{Res}^k_{\kappa_x} \mathcal{L}_x$ for character sheaves $\mathcal{L}_x$ on H_x. A datum $(G, D, H, \mathcal{L})$ will give rise to a set of *monomial local conditions* on an automorphic representation of $G(\mathbb{A}_F)$ as follows.

Notation 6.6. Let J_x be the inverse image of $H_x(\kappa_x)$ in $G(\kappa_x[\![t]\!])$, which maps to $G(\kappa_x[t]/t^{m_x}) = G\langle \kappa_x[t]/t^{m_x}\rangle(\kappa_x)$ by the natural projection.

Definition 6.7. Let χ_x be the character of $H_x(\kappa_x)$, and thus of J_x, induced by $\mathcal{L}_x$ and let χ be the character of $H(k)$ induced by $\mathcal{L}$.

Under these definitions, we have a commutative diagram

$$\begin{array}{ccccc} \mathbf{K}(D) & \hookrightarrow & \prod_{x\in|X-D|} G(\mathfrak{o}_x) \times \prod_{x\in D} J_x & \twoheadrightarrow & H(\kappa) \\ & & \big\downarrow & & \big\downarrow \\ & & \prod_{x\in|X|} G(\mathfrak{o}_x) & \twoheadrightarrow & G(\mathcal{O}_D) \end{array}$$

where the square is a Cartesian and the top row is a short exact sequence.

For clarity and concreteness, we explicate the datum $(G, m_x, H_x, \mathcal{L}_x)$ that will appear in the proof of the main theorem of the paper. At each place, we will either take H_x the trivial group and $\mathcal{L}$ the trivial sheaf, or we will take $(G, m_x, H_x, \mathcal{L}_x)$ to be geometrically supercuspidal. Examples of the second kind of data were provided in Lemma 3.5.

Remark 6.8. Assume that k is a finite field. Consider the space of L^2-functions on $\operatorname{Bun}_{G(D)}(k) = G(F)\backslash G(\mathbb{A}_F)/\mathbf{K}(D)$ that are χ-equivariant for the natural right action of

$$H(k) \subseteq G\langle \mathcal{O}_D\rangle(k) = \prod_{x\in D} G(\kappa_x[t]/t^{m_x}) = \prod_{x\in D} G(\mathfrak{o}_x)/U_{m_x}(G(\mathfrak{o}_x)) = G(\mathfrak{o}_F)/\mathbf{K}(D)$$

on $\operatorname{Bun}_{G(D)}(k)$, where $\mathfrak{o}_F = \prod_{x\in|X|} \mathfrak{o}_x = \prod_{x\in|X|} \kappa_x[\![t]\!]$. We view this as a space of automorphic forms.

We break this space into eigenspaces under Hecke operators, with irreducible subconstituents given by automorphic representations of $G(\mathbb{A}_F)$. All automorphic representations that appear as subquotients are unramified away from D, and at every point $x \in D$ admit a nontrivial map from the compact induction $\text{c-Ind}_{J_x}^{G(\kappa_x(\!(t)\!))} \chi_x$.

The dimension of the space associated to an automorphic representation π of $G(\mathbb{A}_F)$ is equal to its global multiplicity in $L^2(G(F)\backslash G(\mathbb{A}_F))$ times the product over x of the dimension of the (J_x, χ_x)-eigenspace in π_x.

Remark 6.9. We compare our datum $(G, D, H, \mathcal{L})$ defining a space of automorphic forms to the "geometric automorphic datum" defined by Yun in [64, §2.6.2]. Both are geometric versions of the notion of an automorphic representation defined by local conditions, but Yun's is somewhat more general, as we have made various restrictions for technical and notational simplicity.

We work with semisimple groups, while Yun fixes a central character. The group "$\mathbf{K}_S$" in [64] carries the same information as our H. The group "$\mathbf{K}_S$" is a pro-algebraic subgroup of $\prod_{x\in S} G\langle \kappa_x[\![t]\!]\rangle$, whereas H is an algebraic subgroup of $G\langle \mathcal{O}_D\rangle$. This is only a technical difference: by truncating, we avoid working with pro-algebraic groups. More significantly, Yun allows the local subgroups to be contained in any parahoric subgroup, while we allow only the standard hyperspecial subgroup, and he allows them to be arbitrary subgroups of $G\langle \kappa_x[\![t]\!]\rangle$ and not just Weil restrictions from $G_{\kappa_x[[t]]}$, which means that his definition is not stable under base field extension (this can be repaired by either specializing to subgroups that are Weil restrictions or generalizing to subgroups of the product of local groups at all places, rather than products of local subgroups). The notation "$\mathcal{K}_S$" in [64] is our $(\mathcal{L}_x)_{x\in D}$.

Remark 6.10. Most of our methods apply over an arbitrary base field k, and it would not be surprising if they could be generalized to the derived category of D-modules. For instance, Theorem 7.36 could possibly be established for D-modules, in which case Lemma 8.3 would be the statement that a D-module pushforward is supported in a single degree. Similarly, the Ramanujan bound in a particular case established in [33] has been used in [45] to prove that certain character D-modules were concentrated in a single degree.

If this were done, it might have relevance to the characteristic zero geometric Langlands program. However, it is easy to see that the geometric supercuspidality condition cannot be satisfied by any tamely ramified character sheaf, and thus cannot be satisfied at all for sheaves or D-modules with regular singularities in characteristic zero. Hence using this technique requires dealing with irregular singularities.

Remark 6.11. We note that this geometric setup can also be used to motivate Condition BC. Let π be an automorphic representation generated by some automorphic function on $G(F)\backslash G(\mathbb{A}_F)/\mathbf{K}(D)$ which is χ-equivariant for the right action of $H(k)$. Suppose that it is the trace function of a Hecke eigensheaf on $\mathrm{Bun}_{G(D)}$ that is $\mathcal{L}$-equivariant for the right action of H. Then π satisfies Condition BC, except possibly for finitely many extensions. Indeed, over each finite field extension k' of k, we can take the trace function of the Hecke eigensheaf over k', which is a Hecke eigenfunction (assuming it is non-zero), and generates one or more automorphic representations with the same Satake parameters at unramified places. Because the Hecke eigenvalues come from the same geometric Langlands parameter as the Hecke eigensheaf, they have matching Satake parameters with π. Because the automorphic function lies on $\mathrm{Bun}_{G(D)}(k')$, the generated representations have bounded depth, and because it is $(H(k'), \chi_{k'})$-equivariant, the generated representations are compatible with the same mgs data at every mgs place. The only potential problem is if the trace function is identically zero, which can only happen for finitely many field extensions.

6.1. **Moduli Spaces.**

As in §2.2, let Λ^+ be a Weyl cone in the cocharacter lattice of G (which is naturally in bijection with a Weyl cone in the character lattice of $\widehat{G}$).

Let x be a point in X and let $U \subseteq X$ be a neighborhood of x. Let α_1 and α_2 be two G-bundles defined over U, and let $f : \alpha_1 \to \alpha_2$ be an isomorphism over $U - \{x\}$. If we choose trivializations of α_1 and α_2 in a formal neighborhood of x, we can represent the restriction of f to the punctured formal neighborhood of x as an element of $G(\kappa_x((t)))$. Changing the trivializations corresponds to the left and right action of $G(\kappa_x[[t]])$ on this element, so the isomorphism f defines a double coset in $G(\kappa_x[[t]])\backslash G(\kappa_x((t)))/G(\kappa_x[[t]])$. These double cosets are naturally in bijection, under the Cartan decomposition, with Λ^+. We can view this decomposition as coming from the affine Grassmannian $G((t))/G[[t]]$, because each double coset in $G(\kappa_x[[t]])\backslash G(\kappa_x((t)))/G(\kappa_x[[t]])$ is a $G(\kappa_x[[t]])$-orbit in the κ_x-points $G(\kappa_x((t)))/G(\kappa_x[[t]])$ of the affine Grassmannian. These orbits are the Schubert cells of the affine Grassmannian, which again are in bijection with Λ^+. This geometric description makes clear that, in any algebraic family of G-bundles α_1, α_2 and maps f between them, the set of points where the double coset associated to f is in a particular cell of the affine Grassmannian is locally closed and, moreover, the set of points where f is in the closure of a particular Schubert cell of the affine Grassmannian is closed. Using these closed cells, we will define a Hecke correspondence.

Let W be a function from $|X|$ to Λ^+, that sends all but finitely many points to the trivial cocharacter and sends all the points of D to the trivial cocharacter. Define the support of W to be the set of points that W sends to a nontrivial cocharacter (i.e., the usual definition of the support of a function, if we view the trivial cocharacter as the zero element of Λ^+).

Definition 6.12. Let $\mathcal{H}k_{G(D),W}$ be the moduli space of pairs α_1, α_2 of G-bundles with an isomorphism $f : \alpha_1 \to \alpha_2$ away from the support of W, and with a trivialization

$$t_1 : \alpha_1|_D \xrightarrow{\sim} G \times \operatorname{Spec}(\mathcal{O}_D)$$

of the first bundle along D, such that near each point x of the support of W, when f is viewed as a point in the $G(\kappa_x((t)))$ as above, it projects to a point in the affine Grassmannian that lies in the closed cell corresponding to W_x.

Definition 6.13. We define a map $\Delta^W : \mathcal{H}k_{G(D),W} \times H \to \mathrm{Bun}_{G(D)} \times \mathrm{Bun}_{G(D)}$ that sends $(\alpha_1, \alpha_2, f, t_1)$ to $((\alpha_1, t_1), (\alpha_2, h \circ t_1 \circ f|_D^{-1}))$. In other words, the left projection is taking the first G-bundle with trivialization over D, and the right projection is taking the second G-bundle α_2, using f to carry over the trivialization t_1, and then twisting the trivialization by the element $h \in H$.

We will work with the intersection cohomology complex $IC_{\mathcal{H}k_{G(D),W}}$ on $\mathcal{H}k_{G(D),W}$, which by definition is the unique irreducible perverse sheaf isomorphic to $\overline{\mathbb{Q}}_\ell[\dim \mathcal{H}k_{G(D),W}]$ on the open set where $\mathcal{H}k_{G(D),W}$ is smooth.

Remark 6.14. The trace function of $\Delta^W_!(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L})$, which is a function on $\mathrm{Bun}_{G(D)}(k) \times \mathrm{Bun}_{G(D)}(k)$, is the kernel for the composition of the Hecke operator associated to W by the Satake isomorphism with the averaging operator of the $(H(k), \chi)$-action (Lemma 9.9). Thus it acts as a Hecke operator on the space of automorphic forms described in Remark 6.8.

The aim of Section 7 will be to prove the following cleanness property of Δ^W.

Theorem 6.15 (=Theorem 7.36)**.** *Assume that $(G, m_u, H_u, \mathcal{L}_u)$ is geometrically supercuspidal for some $u \in D$ and* $\mathrm{char}(k) > 2$. *Then the natural map*

$$\Delta^W_! \left(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L} \right) \to \Delta^W_* \left(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L} \right)$$

is an isomorphism.

Using this, in Section 8, we will prove that $\Delta^W_!(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L})$ is a pure perverse sheaf, which we will use in Section 9 to derive numerical consequences.

Remark 6.16. Let us explain some of the motivation for Theorem 7.36. As we mentioned before, the trace function of $R\Delta^W_!(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L})$ is a Hecke kernel on a particular space of automorphic forms. In particular, in the case when W is trivial, it is simply the idempotent projector onto this space of automorphic forms.

In the case where $G = SL_2$, D is empty, and W is trivial, the trace function of $R\Delta^W_*(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L})$ was calculated by Schieder [56, Prop.8.15]. Viewing the trace function as a kernel, the induced operator on the space of automorphic forms was calculated by Drinfeld and Wang, who found that it acts as the identity on cusp forms [22, Prop.3.2.2(i), Theorem 1.3.4, and Equation 3.2]. A similar calculation was done by Wang for general groups in [62, Thm.C.7.2 and Thm.1.4.3]. If this fact is true for the families of automorphic forms with more general local conditions, then the trace function of $R\Delta_*^W(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L})$ should equal the trace function of $R\Delta_!^W(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L})$ as soon as one of the local prescribed conditions ensures that the automorphic forms in the family are cuspidal by mandating that one of the local factors is supercuspidal. If we believe this, then we might conjecture that they should agree as sheaves and not just trace functions as long as the local condition also forces cuspidality over finite field extensions.

7. Cleanness of the Hecke complex

As before, let X be a smooth projective curve over a finite field k, G a split semisimple algebraic group over k, D an effective divisor on X, H a smooth factorizable subgroup of $G\langle\mathcal{O}_D\rangle$, and $\mathcal{L}$ a character sheaf on H.

We begin, in §7.1, by constructing a compactification of $\mathcal{H}k_{G(D),W} \times H$ over $\mathrm{Bun}_{G(D)} \times \mathrm{Bun}_{G(D)}$. The advantage of having a compactification is that it reduces the cleanness property of Δ_W that we are trying to prove (in Theorem 7.36) to the corresponding cleanness statement for the open immersion j of $\mathcal{H}k_{G(D),W} \times H$ into its compactification (Theorem 7.35). We can prove this cleanness statement by working locally with individual points of the compactification. This compactification will also help in proving (in Lemma 7.9) that Δ_W is schematic and affine. These facts (Lemma 7.9 and Theorem 7.36), will be the main results from this section that are relevant to subsequent sections, as they together imply very strong properties (Lemma 8.3) of $\Delta_!^W\left(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L}\right)$.

We define this compactification by giving explicit coordinates for a map of G-bundles. We do this by using a faithful representation V of G. We then allow these coordinates to go to infinity. What this means in practice is described in §7.2, which is devoted to describing the projective closure of the affine variety $G \subset \operatorname{End} V$. We give (in Lemma 7.12) a classification of points on this affine closure, and then describe how G acts on them. This will eventually allow us to classify the points of the compactification.

For an open immersion j, the cleanness can be interpreted as a vanishing of stalks. We begin the proof, in §7.3, by proving the vanishing of stalks for a special set of points in the compactification, those "near the cusps", which arise from highly unstable G-bundles (Lemma 7.28). These G-bundles have extra symmetries, and we use these symmetries to obtain the vanishing. Roughly, we show that these symmetries act trivially on the stalk, and, if the stalk is nontrivial, they act nontrivially on it.

We continue the proof in §7.4 by showing how to relate the stalks at different points in the compactification. We show that if a stalk vanishes at one point, it vanishes at certain related points. This will enable, in §7.5, an inductive proof that the stalk vanishes everywhere in the compactification outside the original $\mathcal{H}k_{G(D),W} \times H$. We do this by defining a Hecke correspondence between the compactification and itself. Just as, in the classical setting of modular curves, the graph of a Hecke correspondence is itself a modular curve, and therefore admits Hecke correspondences at coprime places, we have a notion of Hecke correspondence for $\mathcal{H}k_{G(D),W} \times H$, and even its compactification. The most technically difficult part is checking that these Hecke correspondences are smooth (Lemma 7.32). This then enables us to relate the stalks at two corresponding points by smooth base change (Lemma 7.33).

We conclude in §7.5 with an induction on the "height" of a point, which we think of as a generalization of the y-coordinate of a point on the upper-half plane (Definition 7.25). The larger this height is, the more a point is near the cusp. (The points "near the cusp" are exactly the points with height over some threshold.) The key lemma for this induction is that every point is related by a Hecke correspondence to some point of greater height (Lemma 7.34).

7.1. **A compactification of $\mathcal{H}k_{G(D),W} \times H$.** Let V be a faithful representation of G, which we also view as a functor $\alpha \mapsto V(\alpha)$ from G-bundles to vector bundles. Throughout this section, we will be working geometrically and so we can and will assume that k is algebraically closed. We ssume that V lifts to the Witt vectors of k and the pairing of any root of G with any weight of V is less than the characteristic p of k (this technical condition is used in Lemma 7.14, and the existence of a suitable V is checked in Lemma 7.17). We fix a maximal torus and a Borel $T \subset B$ inside G. As in the previous section, let W be a function from $|X|$ to Λ^+ with finite support disjoint from the effective divisor $D = \sum_{x\in|X|} m_x[x]$.

Definition 7.1. For each point $x \in |X|$, consider the composition $\mathbb{G}_m \xrightarrow{W_x} G \to \mathrm{GL}(V)$ of the representation V with the cocharacter $W_x \in \Lambda^+$. This is a representation of $\mathbb{G}_m$, hence is a sum of one-dimensional representations, which we can express as $\lambda \mapsto \lambda^{e_1}, \dots, \lambda^{e_{\dim V}}$ for a tuple of integer weights $e_1, \dots, e_{\dim V}$. Let $\{W\}_x = -\min(e_1, \dots, e_{\dim V})$.

Let $\{W\} : |X| \to \mathbb{Z}$ be the divisor, whose multiplicity at each point $x \in |X|$ is $\{W\}_x$.

The support of $\{W\}$ is less than the support of W. In particular we have that $\{W\}$ is disjoint from D.

Example 7.2. (i) If $G = \mathrm{Sp}_{2n}$, V is the standard representation, and W_x is the cocharacter with eigenvalues $\lambda^{w_1}, \dots, \lambda^{w_n}, \lambda^{-w_n}, \dots, \lambda^{-w_1}$ where $w_1, \dots, w_n$ are integers with $w_1 \geq \dots \geq w_n \geq 0$ then $\{W\}_x = w_1$.

(ii) If $G = \mathrm{SL}_n$, V is the adjoint representation, and W_x is the cocharacter whose eigenvalues on the standard representation are $\lambda^{w_1}, \dots, \lambda^{w_n}$ for $w_1, \dots, w_n$ integers with $w_1 \geq \dots \geq w_n$ and $\sum_{i=1}^n w_i = 0$, then its eigenvalues on the adjoint representation have the form $\lambda^{w_i - w_j}$, so $\{W\}_x = w_1 - w_n$.

Before compactifying $\mathcal{H}k_{G(D),W} \times H$, we compactify G by considering the projective completion of $\mathrm{End}(V)$:

Notation 7.3. Let $\overline{G}$ be the closure of $G \subseteq \mathrm{End}\, V \subseteq \mathbb{P}(\mathrm{End}\, V \oplus k)$, where we embed $\mathrm{End}\, V$ into the projective space $\mathbb{P}(\mathrm{End}\, V \oplus k)$ by $x \mapsto [x : 1]$. (The map $G \to \mathrm{GL}(V) \to \mathrm{End}\, V$ is an immersion because V is a faithful representation).

Given two pairs $(\alpha_1, t_1), (\alpha_2, t_2)$ of a G-bundle and a trivialization over D and a projective section $\varphi \in \mathbb{P}(\mathrm{Hom}(V(\alpha_1), V(\alpha_2) \otimes \mathcal{O}_X(\{W\})) \oplus k)$, because $\mathrm{Hom}(V(\alpha_1), V(\alpha_2) \otimes \mathcal{O}_X(\{W\})) \oplus k$ is the vector space of global sections of

$$\mathcal{H}om(V(\alpha_1), V(\alpha_2) \otimes \mathcal{O}_X(\{W\})) \oplus \mathcal{O}_X,$$

we can view φ as a nonzero global section of $\mathcal{H}om(V(\alpha_1), V(\alpha_2) \otimes \mathcal{O}_X(\{W\})) \oplus \mathcal{O}_X$, well-defined up to scaling. Locally over any open set, closed set, or punctured formal neighborhood, that does not intersect the support of W and where we have a trivialization of α_1 and α_2, we obtain a section of $(\mathrm{End}\, V \oplus k) \otimes \mathcal{O}_X$ up to scaling.

Definition 7.4. Let $\overline{\mathcal{H}k}_{G(D),H,W,V}$ be the moduli space of five-tuples consisting of $\alpha_1, t_1, \alpha_2, t_2, \varphi$ where $(\alpha_1, t_1), (\alpha_2, t_2)$ are two pairs of a G-bundle and a trivialization over D and

$$\varphi \in \mathbb{P}(\mathrm{Hom}(V(\alpha_1), V(\alpha_2) \otimes \mathcal{O}_X(\{W\})) \oplus k)$$

such that

(1) Over any point x in the complement of the support of W, for any trivialization of α_1 and α_2 over x, the induced point of $(\operatorname{End} V \oplus k) \otimes \kappa_x$ lies in the affine cone of $\overline{G}$. (Note that $\overline{G}$ is invariant under the left and right action of G, so this does not depend on the choice of trivialization.)
(2) In a punctured formal neighborhood of any point x in the support of W, for any trivialization of α_1 and α_2 over that punctured formal neighborhood, the induced section of $\operatorname{End} V \oplus \mathcal{O}_X$, when viewed as a point in the formal loop space $(\operatorname{End} V \oplus k)((t))$, is in the closure of the set of pairs $(\lambda V(g), \lambda)$ where $\lambda \in \mathbb{G}_m$ and $g \in G((t))$ is in the Schubert cell associated to W_x.
(3) Over D, using the trivializations t_1 and t_2, the induced element of $(\operatorname{End} V \oplus k)\langle \mathcal{O}_D \rangle$ lies in the closure of the set of pairs $(\lambda h, \lambda)$ where $\lambda \in \mathbb{G}_m$ and $h \in H \subseteq G\langle \mathcal{O}_D \rangle \subseteq \operatorname{End} V\langle \mathcal{O}_D \rangle$. Equivalently, using an arbitrary trivialization over D, $V(t_2) \circ \varphi|_D \circ V(t_1)^{-1}$ lies in this closure, where $V(t_i) : V(\alpha_i)|_D \xrightarrow{\sim} V\langle \mathcal{O}_D \rangle$ are the associated trivialization.

For interpreting the last two conditions, remember that a global section of $\mathcal{O}_X$ is always constant over X, so forcing the last coordinate to be locally constant over X is not any additional restriction. Recall from Definition 6.12 that $\mathcal{H}k_{G(D),W}$ is the moduli space of four-tuples $(\alpha_1, \alpha_2, f, t_1)$ consisting of a pair of G-bundles α_1, α_2, an isomorphism $f : \alpha_1 \to \alpha_2$ away from the support of W, that near each point in the support of W is in the closure of the cell of the affine Grassmannian associated to the corresponding representation, and a trivialization t_1 of α_1.

To understand $\overline{\mathcal{H}k}_{G(D),H,W,V}$ geometrically, it helps to first describe the analogous moduli space without the conditions (1), (2), (3). We can describe this as a projective bundle.

Lemma 7.5. *The moduli space of five-tuples* $((\alpha_1, t_1), (\alpha_2, t_2), \varphi)$ *where* $(\alpha_1, t_1), (\alpha_2, t_2) \in \operatorname{Bun}_{G(D)}$ *and* $\varphi \in \mathbb{P}(\operatorname{Hom}(V(\alpha_1), V(\alpha_2) \otimes \mathcal{O}_X(\{W\})) \oplus k)$ *is a projective bundle over* $\operatorname{Bun}_{G(D)} \times \operatorname{Bun}_{G(D)}$*, in the sense of* Proj *of the symmetric algebra of a coherent sheaf on* $\operatorname{Bun}_{G(D)} \times \operatorname{Bun}_{G(D)}$.

Proof. The projectivization of a vector space is Proj of the symmetric algebra of the dual vector space. So it suffices to check that there is a coherent sheaf on $\operatorname{Bun}_{G(D)} \times \operatorname{Bun}_{G(D)}$ whose fiber at each point is the dual of $\operatorname{Hom}(V(\alpha_1), V(\alpha_2) \otimes \mathcal{O}_X(\{W\})) \oplus k$. By Serre duality, this dual is $H^1(X, V(\alpha_2)^\vee \otimes \mathcal{O}_X(-\{W\}) \otimes V(\alpha_1) \otimes K_X) \oplus k$. Because H^1 is the top cohomology group, its value at each point is the fiber of the coherent sheaf $R^1\pi_*(V(\alpha_2)^\vee \otimes \mathcal{O}_X(-\{W\}) \otimes V(\alpha_1) \otimes K_X)$, for π the projection $X \times \operatorname{Bun}_{G(D)} \times \operatorname{Bun}_{G(D)} \to \operatorname{Bun}_{G(D)} \times \operatorname{Bun}_{G(D)}$.

Finally, the sum of H^1 with k is the fiber of the sum of this coherent sheaf with $\mathcal{O}_{\operatorname{Bun}_{G(D)} \times \operatorname{Bun}_{G(D)}}$. □

Lemma 7.6. *There is a well-defined map* $j : \mathcal{H}k_{G(D),W} \times H \to \overline{\mathcal{H}k}_{G(D),H,W,V}$ *that sends* $(\alpha_1, t_1, \alpha_2, f, h)$ *to* $((\alpha_1, t_1), (\alpha_2, h \circ t_1 \circ f|_D^{-1}), \varphi)$ *where*

$$\varphi \in \operatorname{Hom}(V(\alpha_1), V(\alpha_2) \otimes \mathcal{O}_X(\{W\})) \subseteq \mathbb{P}(\operatorname{Hom}(V(\alpha_1), V(\alpha_2) \otimes \mathcal{O}_X(\{W\})) \oplus k)$$

is $V(f) : V(\alpha_1) \to V(\alpha_2)$ *tensored with the natural map* $\mathcal{O}_X \to \mathcal{O}_X(\{W\})$.

Proof. First we show that φ is in fact a homomorphism from $V(\alpha_1)$ to $V(\alpha_2) \otimes \mathcal{O}_X(\{W\})$ defined everywhere on X. This is clear away from the support of W, where f is an isomorphism. In a formal neighborhood of each point x in the support of W, for f whose associated point of $G((t))$ is in the Schubert cell corresponding to W_x, the order of the pole of $V(f)$ is at most $\{W\}_x$, by definition of $\{W\}$. For f whose associated point of $G((t))$ is in the closure of the Schubert cell, because the pole order is a lower semicontinuous function, the order of the pole is also at most $\{W\}_x$, and so it becomes a homomorphism after we tensor with $\mathcal{O}(\{W\})$.

Next we show that φ satisfies the local conditions (1), (2), and (3) of the definition of $\mathrm{Hom}(V(\alpha_1), V(\alpha_2) \otimes \mathcal{O}_X(\{W\}))$. It satisfies condition (1) because f is an isomorphism away from the support of W, condition (2) because f is in the closure of the correct cell of the affine Grassmannian near points in the support of W, and condition (3) because over D, we have $t_2 \circ f|_D \circ t_1^{-1} = h \in H$. ☐

Let $\overline{\Delta}^W : \overline{\mathcal{H}k}_{G(D),H,W,V} \to \mathrm{Bun}_{G(D)} \times \mathrm{Bun}_{G(D)}$ send $(\alpha_1, t_1, \alpha_2, t_2, \varphi)$ to $((\alpha_1, t_1), (\alpha_2, t_2))$.

Lemma 7.7. *The map $\overline{\Delta}^W$ is projective and $\overline{\Delta}^W \circ j = \Delta^W$.*

Proof. The first claim follows immediately from Lemma 7.5 because the graph of $\overline{\Delta}^W$ is defined as a subset of a projective bundle consisting of triples satisfying three closed conditions, and thus is a closed subset, hence projective. The second claim follows because $\overline{\Delta}^W \circ j$ sends $(\alpha_1, t_1, \alpha_2, f, h)$ to $((\alpha_1, t_1), (\alpha_2, h \circ t_1 \circ f|D^{-1}))$ which is precisely the definition of Δ^W. ☐

Lemma 7.8. *j is an open immersion, and its image is the locus in $\overline{\mathcal{H}k}_{G(D),H,W,V}$ where $\varphi \in \mathrm{Hom}(V(\alpha_1), V(\alpha_2) \otimes \mathcal{O}_X(\{W\})) \subseteq \mathbb{P}(\mathrm{Hom}(V(\alpha_1), V(\alpha_2) \otimes \mathcal{O}_X(\{W\})) \oplus k)$.*

Proof. By construction, a point in the image of j has φ contained in $\mathrm{Hom}(V(\alpha_1), V(\alpha_2) \otimes \mathcal{O}_X(\{W\}))$. The subset where $\varphi \in \mathrm{Hom}(V(\alpha_1), V(\alpha_2) \otimes \mathcal{O}_X(\{W\}))$ is the inverse image of a standard affine open chart of projective space, and thus is an open subset U of $\overline{\mathcal{H}k}_{G(D),H,W,V}$. Hence, to prove that j is an open immersion whose image is U, it suffices to find an inverse of j over this U.

Fix a point $(\alpha_1, t_1, \alpha_2, t_2, \varphi) \in U$ and an open set away from the support of W where α_1 and α_2 can be trivialized, so that $\mathrm{Hom}(V(\alpha_1), V(\alpha_2) \otimes \mathcal{O}_X(\{W\})) = \mathrm{End}\, V \otimes \mathcal{O}_X$. Using this isomorphism, we can view the section φ as a map from the curve X to the vector space $\mathrm{End}\, V$. By definition, its image must lie in $\mathrm{End}\, V \cap \overline{G}$. Because $\mathrm{End}\, V \cap \overline{G} = G$, we can view φ as a map from X to G. Remembering the trivialization, φ defines an isomorphism of G-bundles $\alpha_1 \to \alpha_2$.

Because changing the two trivializations acts on $\mathrm{End}\, V$ by left and right multiplication by G, this isomorphism does not depend on the choice of trivialization. Thus it glues to a global isomorphism away from W. Hence we obtain an isomorphism $f : \alpha_1 \to \alpha_2$ as G-bundles away from W. By assumption we know that φ, when viewed as a point in the formal loop space $(\mathrm{End}\, V \oplus k)(\!(t)\!)$, is in the closure of the set of pairs $(\lambda V(g), \lambda)$ where $\lambda \in \mathbb{G}_m$ and $g \in G(\!(t)\!)$ is in the Schubert cell associated to W_x. Because $\varphi = (V(f), 1)$ and V is faithful, this implies that f, when viewed as a point in $G(\!(t)\!)$, it is in the closure of the Schubert cell associated to W_x, hence modulo $G[\![t]\!]$, it is in the closure of the cell of the affine Grassmannian associated to W_x.

Over D, $t_2 \circ f \circ t^{-1}$ lies in the closure of the set of points $(h\lambda, \lambda)$ for $h \in H$. Because the last coordinate is nonzero, we may fix it to equal 1, and thus take $\lambda = 1$, so it lies in the closure of H inside $\mathrm{End}\, V\langle \mathcal{O}_D \rangle$. Because H is a closed subgroup of $G\langle \mathcal{O}_D \rangle$, which is closed in $\mathrm{End}\, V\langle \mathcal{O}_D \rangle$, in fact $t_2 \circ f \circ t_1^{-1}$ lies in H, so we may take h to be $t_2 \circ f \circ t_1^{-1}$.

Verifying that this is an inverse is a routine calculation. ☐

Lemma 7.9. *The map Δ^W is schematic and affine.*

Proof. By Lemma 7.7 and 7.8, this map is the composition of the open immersion j with the projective morphism $\overline{\Delta}^W$. Moreover, this open immersion is the complement of the hyperplane $\mathbb{P}(\mathrm{Hom}(V(\alpha_1), V(\alpha_2) \otimes \mathcal{O}_X(\{W\})))$ inside $\mathbb{P}(\mathrm{Hom}(V(\alpha_1), V(\alpha_2) \otimes \mathcal{O}_X(\{W\})) \oplus k)$. Thus, by Lemma 7.5, Δ^W is a hyperplane complement in a projective morphism, so it is affine. ☐

Remark 7.10. Throughout Section 7, we do not need the full formalism of étale cohomology on stacks. This is because the relevant morphisms are schematic morphisms between Artin stacks, so we can define $\Delta_!^W$ and Δ_*^W smooth-locally as derived pushforwards with respect to morphisms of schemes.

If $G = \mathrm{SL}_n$ and V is the standard representation, we can classify the points of $\overline{\mathcal{H}k}_{G(D),H,W,V}$ according to the generic rank of φ. For each rank, we can consider the maximal parabolic subgroup that preserves the kernel of φ, and its unipotent radical, elements of which fix φ when acting by composition on the right. Sections of this unipotent radical act as local automorphisms of $\overline{\mathcal{H}k}_{G,H,W,V}$. These automorphisms can be used to show the vanishing of $j_*(IC_{\mathcal{H}k_{G(D),H}} \boxtimes \mathcal{L})$ at these points. In the general group case, we will replace the study of the rank with the orbits in $\overline{G}$ of the joint left and right action of $G \times G$. We describe these orbits using the standard theory of reductive groups in the next subsection.

Remark 7.11. If G is adjoint and V is an irreducible representation whose highest weight is regular, i.e., not fixed by any nontrivial element of the Weyl group, then $\overline{G}$ is isomorphic to "wonderful compactification" of G. We expect in this case that $\overline{\mathcal{H}k}_{G(0),H,1,V}$ is very close to the Drinfeld–Lafforgue–Vinberg compactification of Bun_G as defined by Schieder [56], which is closely connected to the wonderful compactification. Our proof uses heavily the explicit representation V as a form of coordinates, but it seems plausible that a "coordinate-free" proof of the same result can be obtained using the abstract theory of the wonderful and Drinfeld–Lafforgue–Vinberg compactifications.

However, for our proof, there is no reason to choose V to be the representation associated to a regular weight. If we instead choose a representation like the standard representation (for G a classical group), the compactification we use, and other concepts involved like the height, admit particularly simple descriptions. The reader may wish to follow along with the case $G = \mathrm{Sp}_{2g}$ in mind, say.

7.2. **Lemmas on semisimple groups.** Let G be a split semisimple group, V a faithful representation over k, and fix a split maximal torus T of G.

Lemma 7.12. *Any point in $\overline{G} - G \subseteq \mathbb{P}(\mathrm{End}\, V \oplus k)$ can be expressed as $(g_1 e g_2, 0)$ where $g_1, g_2 \in G$ and e is the idempotent projector onto the sum of eigenspaces of T whose weights lie in some proper face of the convex hull of the weights of V.*

Example 7.13. Let us provide some examples of what these idempotent projectors look like:

(i) Let $G = \mathrm{SL}_n$ and let V be the standard representation. Then the weights of V are n linearly independent vectors, forming the vertices of an $(n-1)$-simplex. Hence any nonempty proper subset of the weights is the set of weights lying in some proper face of the convex hull. Thus any diagonal matrix with all diagonal entries 0 and 1, not all 1 and not all 0, is such an e.

(ii) Let $G = \mathrm{Sp}_{2g}$ and let V be the standard representation. Then the weights of V are the vectors with one entry ± 1 and the rest 0 in $\mathbb{Z}^g$. The convex polytope this forms is a cross-polytope, whose proper faces are all simplices. The weights lying in a face form a subset S of these vectors, such that for any $v \in S$, $-v \notin S$. Thus e is an idempotent projector onto an isotropic subspace, whose kernel contains a maximal isotropic subspace.

(iii) Let $G = G_2$ and let V be the unique seven-dimensional irreducible representation. Then the weights of V form the six vertices and center of a hexagon. The proper faces consist of either one vertex or two adjacent vertices, so the sum of the eigenspaces is a subspace of dimension one or two. These subspaces are isotropic under the G_2-invariant quadratic form on V and the two-dimensional subspaces are sent to zero by the unique G_2-equivariant map $\wedge^2 V \to V$ (as the product of their eigenvalues under T is not a weight of V).

In all the above examples, the stabilizer of the sum of the eigenspaces of T whose weights lie in a proper face is a maximal parabolic subgroup of G. We will later prove that it is always a parabolic subgroup, but it need not be maximal — for instance when $G = \mathrm{SL}_n$ and V is the adjoint representation, it need not be maximal for $n \geq 3$.

Proof of Lemma 7.12. First note that any point x of the closure of G is the limit as t goes to 0 of a $k'((t))$-valued point of G for some field k'. To see this, choose a generic linear subspace L of dimension $1 + \dim \operatorname{End} V - \dim G$ containing x. By genericity, $L \cap \overline{G}$ has dimension 1 and $L \cap (\overline{G} \setminus G)$ has dimension 0. The normalization of $L \cap \overline{G}$ is a smooth curve mapping to $\overline{G}$ whose image contains x but all but finitely many points of which map to G, and choosing a local coordinate at some point mapping to x gives the desired $k'[[t]]$-valued point.

By the Bruhat decomposition, any such point can be written as $g_1(t)\chi(t)g_2(t)$ where g_1, g_2 are $k'[[t]]$-valued points of G and χ is a cocharacter of T. Now $\chi(t)$ converges as t goes to 0 to a point $\chi(0) \in \mathbb{P}(\operatorname{End} V \oplus k)$, and because the left and right group actions are continuous, $g_1(t)\chi(t)g_2(t)$ converges as t goes to 0 to $g_1(0)\chi(0)g_2(0)$.

If χ is trivial, then $\chi(0)$ is the identity element and this limit is in G.

Otherwise, in an eigenbasis, $\chi(t)$ is a diagonal matrix whose entries are integer powers of t, where the integer power appearing is a linear function of the weight. The projective coordinates for $\chi(t)$ are the entries of this matrix plus an additional 1. Because χ is nontrivial, not all these exponents are 0, and because G is semisimple, the sum of the exponents vanishes, so some are negative and some are positive. To calculate the limit in projective space as t goes to 0, we first divide each coordinate by the minimal power of t that appears, making each coordinate a nonnegative power of t, and then set $t = 0$, making each coordinate 1 or 0. The 1s occur exactly on the diagonal entries corresponding to eigenspaces with minimal exponent. Thus $\chi(0)$ is the idempotent projector e onto the sum of eigenspaces of T with minimal exponent. These are the eigenspaces where some nontrivial linear function of the weights is minimized, i.e. some proper face of the convex hull of the weights.

The last coefficient of $\chi(0)$ is 0, so multiplying on the left by $g_1(0)$ and the right by $g_2(0)$ we obtain $(g_1 e g_2, 0)$. ☐

Fix a proper face of the convex hull of the weights of V, and take the idempotent projector e, so that $\operatorname{Im}(e)$ is the sum of the T-eigenspaces whose weights lie on that face and $\ker(e)$ is the sum of the T-eigenspaces whose weights do not lie on that face.

Associated to a point in the $\overline{G} - G$ is a natural parabolic subgroup, the stabilizer of $\ker(e)$ (as we will see below, in Lemma 7.14). A key useful property is that its unipotent radical acts trivially on e (Lemma 7.15). In §7.3, we will define a height function so that, at points of large height, there are many global automorphisms of the G-bundle α_1 that lie in the unipotent radical. We will then exploit these extra symmetries. Thus, we will define our height function using this particular parabolic subgroup.

Lemma 7.14. *The stabilizer of* $\ker(e)$ *is a parabolic subgroup of* G*, and this stabilizer remains smooth after lifting* G *and* V *to the Witt vectors of* k*.*

Proof. Let S be the stabilizer of $\ker(e)$, viewed as a group scheme over the Witt vectors $W(k)$. We first check that $S_{W(k)[1/p]}$ is a parabolic subgroup. It suffices to check that $S_{W(k)[1/p]}$ is proper and contains a Borel subgroup.

By Lemma 7.12, there exists a linear form ω on the weight space such that e is the idempotent projector onto the eigenspaces of weights that maximize ω.

The stabilizer $S_{W(k)[1/p]}$ is proper because the weights of $\mathrm{Im}(e)$ are the weights maximzing ω, so the sum of ω over the weights of $\mathrm{Im}(e)$ is positive, and thus the sum of ω over the weights of $\ker(e)$ is negative, which is impossible if $\ker(e)$ is a representation of G. Thus $\ker(e)$ is not G-stable and so its stabilizer $S_{W(k)[1/p]}$ is proper.

To show that $S_{W(k)[1/p]}$ contains a Borel, note that the linear form ω is in some Weyl chamber of the dual to weight space. With regards to the ordering induced by that Weyl chamber, ω takes nonnegative values on all the simple roots, hence takes nonnegative values on all the positive roots. Hence the set of weights of V where ω takes its maximal value is closed under addition of positive roots, and the complement of this set is closed under addition of negative roots. Therefore $\ker(e)$ is closed under the lowering operators and thus stable under the opposite Borel.

To show that S is smooth over $W(k)$, and thus remains parabolic in characteristic p, it suffices to check that the cotangent space of S at the identity is p-torsion free, in other words that every element of the Lie algebra of S in characteristic p is the reduction mod p of an element in the Lie algebra of S in characteristic zero. Because $\ker(e)$ is T-invariant, the Lie algebra of S is a sum of T-eigenspaces, and so it is sufficient to check this for raising operators associated to roots. Let J^+ be the raising operator associated to a root and let J^- be the lowering operator associated to the opposite root.

Suppose that J^+ does not stabilize $\ker(e)$ in characteristic zero but does in characteristic p. Because J^+ does not stabilize, it raises ω, so $J^+ \mathrm{Im}(e) = 0$, and J^- lowers ω so $J^- \mathrm{Im}(e) \subseteq \ker(e)$. Thus in characteristic p, $J^+J^- \mathrm{Im}(e) \subseteq J^+ \ker(e) \subseteq \ker(e)$, and $J^-J^+ \mathrm{Im}(e) \subseteq J^- 0 = 0$, so $[J^+, J^-] \mathrm{Im}(e) \subseteq \ker(e)$. Now $[J^+, J^-]$ is an element of the Lie algebra of the maximal torus. More precisely, $[J^+, J^-]$ is the coroot corresponding to J^+, so $\mathrm{Im}(e)$ is a sum of eigenspaces of this coroot, and thus all the eigenvalues must be 0 mod p. Because the eigenvalues are pairings of the coroot corresponding to J^+ with weights of V, and hence are integers at most p, they must be zero, by our assumption from the beginning of this section. Because $J^+ \mathrm{Im}(e) = 0$, all eigenvalues of $[J^+, J^-]$ on $\mathrm{Im}(e)$ are highest weights of their corresponding representations, so all irreducible representations of the $\mathfrak{sl}_2$ generated by J^+, J^-, and $[J^+, J^-]$ other than those contained in $\ker(e)$ have highest weight zero, hence are trivial, hence have J^+ vanish on them, which contradicts the assumption that J^+ does not stabilize $\ker(e)$ in characteristic zero. □

Lemma 7.15. *Let P be the stabilizer of* $\ker(e)$ *and let M be its Levi subgroup. The action of P on $V/\ker(e)$ factors through the projection $P \to M$.*

Proof. The set of weights in a proper face is the locus where some linear form ω on the weight lattice takes its maximal value among the weights of V.

Because the subspace $\ker(e)$ is stable under the maximal torus, P contains, and hence is normalized by, the maximal torus, so the Lie algebra of P is generated by some subset of the raising and lowering operators corresponding to roots. The maximal unipotent subgroup of P is generated by the operators corresponding to some further subset of the roots.

If the raising or lowering operator corresponding to some root α acts nontrivially on $V/\ker(e)$, then there must be two weights maximizing ω that differ by α, so we must have $\omega(\alpha) = 0$, and thus the operator corresponding to $-\alpha$ is also in the stabilizer P, and hence the unipotent element corresponding to α is in some SL_2-triple and thus is not in the maximal unipotent subgroup of P.

Because no generator of the maximal unipotent subgroup of P acts nontrivially on $V/\ker(e)$, the whole unipotent subgroup acts trivially, and so the action factors through M. □

This statement will be useful to prove the smoothness of the Hecke correspondence later:

Lemma 7.16. *Let e be the idempotent projector onto the T-eigenspaces in some proper face of the convex hull of the weights of V. Let P be the parabolic subgroup of G consisting of elements stabilizing $\ker(e)$. Then*

(i) The natural map $\pi : G \to P\backslash G$ extends to a map π' from an open subset U of $\overline{G}$ to $P\backslash G$, such that $(e, 0) \in U$ and $\pi'(e) = P \in P\backslash G$.

(ii) Let $\overline{P}$ be the projective closure of P inside $\mathbb{P}(\operatorname{End} V \oplus k)$. Any element of U sent to the identity under π' lies in $\overline{P}$.

Proof. (i) Let $U \subseteq \overline{G}$ be the open subset consisting of $(x, \lambda) \in \overline{G}$ where $\operatorname{rank}(ex) = \operatorname{rank}(e)$. There is a map k from U to the Grassmannian $\operatorname{Gr}(\dim \ker(e), \dim V)$ that sends x to $\ker(ex)$. Such a map is invariant under the left action of P, which by definition preserves $\ker(e)$, so we have a commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\quad\pi\quad} & P\backslash G \\ \downarrow & & \downarrow{\scriptstyle i} \\ U & \xrightarrow{\ k\ } & \operatorname{Gr}(\dim \ker(e), \dim V) \end{array}$$

Because P is the schematic stabilizer of the kernel of e, i is an embedding, and because P is parabolic, $P\backslash G$ is proper, and so i is a closed immersion. In particular, the image of i is closed. Because G is dense in U, the image of k is contained in the image of i, so we can factor $k = i \circ \pi'$ for a unique map $\pi' : U \to P\backslash G$. By commutativity, this extends π.

Because e is idempotent, $\operatorname{rank}(e^2) = \operatorname{rank}(e)$, so by the definition of U, $(e, 0) \in U$. Furthermore $i \circ \pi'(e) = \ker(e^2) = \ker(e) = i(1)$, so because i is injective, $\pi'(e) = 1$.

(ii) There is a map $m : \overline{P} \times G \to \overline{G}$ defined by the embedding $\overline{P} \subseteq \overline{G}$ and the right action of G on $\overline{G}$. Because m is stable under the action of $p \in P$ on $\overline{P} \times G$ that sends (x, g) to $(xp, p^{-1}g)$, m descends to a map $\gamma : P\backslash(\overline{P} \times G) \to \overline{G}$. Now $P\backslash(\overline{P} \times G)$ is an $\overline{P}$-bundle on $P\backslash G$ and both of these are proper, so $P\backslash(\overline{P} \times G)$ is proper. Because the map to $\overline{G}$ is proper and has dense image, it is surjective. Let U be the open subset of $\overline{G}$ on which the map $\pi' : U \to P\backslash G$ is defined. Then $\gamma^{-1}(U)$ admits two maps to $P\backslash G$, the first given by $\pi' \circ \gamma$ and the second by projection to the second factor, which agree on the dense subset $P\backslash(P \times G) = G$ and hence are equal as $P\backslash G$ is separated. Hence every point that is sent to the identity must be an element of $(\overline{P} \times G)/P$ with the second factor in P, in other words an element of $\overline{P}$, as desired. □

Lemma 7.17. *Let G be a split semisimple algebraic group over a field k of characteristic p. If $p > 2$, then there exists a faithful representation V of G defined over $\mathbb{Z}$ such that the pairing of any weight of V with any coroot of G is less than p.*

If $p = 2$, there exists such a representation if each nontrivial normal subgroup of G has nontrivial center.

Proof. In fact, we will construct V where all the pairings are at most 2. We will construct it over $\mathbb{Z}$ as a sum of highest weight representations, and reduce modulo p. It is sufficient to show that, for each character χ of the center $Z(G)$ of G, there exists such a representation V_χ whose central character is χ and whose kernel is contained in $Z(G)$. Then taking $V = \sum_\chi V_\chi$, the kernel of V will be the intersection inside $Z(G)$ of the kernels of all characters of $Z(G)$, and hence be trivial.

For this statement, we can assume that G is simply-connected, as any representation of the universal cover of G with a central character pulled back from G is in fact a representation of G with the same central character.

For $G = G_1 \otimes G_2$, V_1 a representation of G_1 satisfying the condition on weights, and V_2 a representation of G_2 satisfying the condition on weights, $V_1 \otimes V_2$ satisifies the condition on weights. The same is true for the condition on kernels. Thus, expressing G as a product of simple groups, we may assume that G is simple.

We will now check that, for each simply-connected simple G, and each character χ, there exists such a representation V_χ.

For the trivial character, the adjoint representation satisfies the pairing condition if and only if the Dynkin diagram has no edges of multiplicity greater than 2, so we can use the adjoint representation for any group except G_2. Because the center of G_2 is trivial, we can use the seven-dimensional standard representation for G_2.

It remains to handle the nontrivial characters. For any simple group, there exists a unique minuscule representation for each central character, and for any nontrivial character, the minuscule representation satisfies both conditions. Indeed, because it is not the trivial representation, its kernel is contained in the center, and because the Weyl group acts transitively on the weights (the definition of minuscule) the weights lie on a sphere, and so no three are collinear. But any weight whose pairing with a coroot is k lies in a $k+1$-dimensional representation of the SL_2 containing the dual root, hence lies in a series of $k+1$ weights in a line, so we must have $k \leq 1$.

In the $p = 2$ case, we can take V to be the sum of all minuscule representations of G, which necessarily have all pairings ≤ 1. Because, for each character of the center of G, there exists a minuscule representation with that central character, V is a faithful representation of the center of G. If V were not a faithful representation of G, then some nontrivial normal subgroup of G would act trivially on V. By assumption, this subgroup has nontrivial center, which also acts trivially, contradicting the faithfulness restricted to the center. Thus V is faithful.

□

7.3. **Vanishing near the cusp.** Fix a point $(\alpha_1, t_1, \alpha_2, t_2, \varphi)$ of $\overline{\mathcal{H}k}_{G(D),H,W,V}$ not in the image of j. We will define, using this data, a parabolic subgroup P of G and a group scheme $\mathcal{P}_{\alpha_1,\varphi}$ over X, locally isomorphic to P.

Let U be an open subset of X on which α_1 and α_2 are trivialization. Over U, we have $\operatorname{Hom}(V(\alpha_1), V(\alpha_2) \otimes \mathcal{O}_X(\{W\})) \oplus \mathcal{O}_X \cong (\operatorname{End} V \oplus k) \otimes \mathcal{O}_X$. Restricting φ, which is a section of $\operatorname{Hom}(V(\alpha_1), V(\alpha_2) \otimes \mathcal{O}_X(\{W\})) \oplus \mathcal{O}_X$ well-defined up to scaling, to U, we obtain a map $U \to (\operatorname{End} V \oplus k)$ up to scaling, and hence a map $U \to \mathbb{P}(\operatorname{End} V \oplus k)$. By the definition of $\overline{\mathcal{H}k}_{G(D),H,W,V}$, this map has image in $\overline{G}$, and so we obtain a map $U \to \overline{G}$. By Lemma 7.8, the last coordinate of φ vanishes, so this map has image contained in $\overline{G} \setminus G$.

By Lemma 7.12, $\overline{G} \setminus G$ is a finite union of locally closed $G \times G$-orbits of the form $G(e,0)G$. Because this union is finite, one must contain the image of an open subset $X_0 \subseteq U$. Let e be this idempotent projector and let P be the stabilizer of $\ker e$, which by Lemma 7.14 is a parabolic subgroup. Let N be the unipotent radical of P.

We note that e, and thus P, is independent of the choice of trivialization, since changing the trivializations would have the effect of multiplying on the left and right by maps $U \to G$, which preserves all $G \times G$ orbits.

Definition 7.18. The Grassmannian of $\dim(\ker e)$-dimensional subspaces of $V(\alpha_1)$ forms a fiber bundle over X. Over U', $\ker\varphi$ defines a section of this bundle. Because the Grassmannian is proper, this extends to a section s_φ over X. Let $\mathcal{P}_{\alpha_1,\varphi} \subseteq \mathrm{Aut}(\alpha_1)$ be the group scheme over X of automorphisms preserving s_φ.

We can check that $\mathcal{P}_{\alpha_1,\varphi}$ is locally conjugate to P. The G-orbit of $\ker e$ inside the Grassmannian of $\dim(\ker e)$-dimensional subspaces of V is isomorphic to $P\backslash G$, thus proper, hence closed. For any subset U^* of X on which α_1 is trivialized, the image of $U^* \cap X_0$ under the section $\ker\varphi$ lies in $G\cdot(\ker e)$, so because $G\cdot(\ker e)$ is closed, the image of s_φ is contained in $G\cdot(\ker e)$, and thus the stabilizer of s_φ at any point is conjugate to the stabilizer P of $\ker e$.

Definition 7.19. Let $\mathcal{N}_{\alpha_1,\varphi}$ be the unipotent radical of $\mathcal{P}_{\alpha_1,\varphi}$.

We can observe that the closed subset of the Grassmannian of subspaces of $V(\alpha_1)$ which locally under a trivialization is $G\cdot(\ker e)$ is

$$((G\cdot(\ker e))\times\alpha_1)/G = (P\backslash G\times\alpha_1)/G = P\backslash\alpha_1$$

and so the section s_φ defines a reduction of α_1 from a G-bundle to a P-bundle. Then $\mathcal{P}_{\alpha_1,\varphi}$ is the twist of P by this P-bundle under the conjugation action of P, and similarly $\mathcal{N}_{\alpha_1,\varphi}$ is the twist of N.

Lemma 7.20. *Let σ be a section of $\mathcal{N}_{\alpha_1,\varphi}$, viewed as an automorphism of $V(\alpha_1)$. Then*

$$\varphi\circ\sigma = \varphi.$$

Proof. Because this equation is a closed condition, it suffices to check this over X_0, and to work locally. In particular, we may trivialize α_1 and α_2. Using that trivialization, from Lemma 7.12, φ can be expressed as $g_1 e g_2$. From the definition of $\mathcal{P}_{\alpha_1,\varphi}$ and $\mathcal{N}_{\alpha_1,\varphi}$, we see that $\mathcal{P}_{\alpha_1,\varphi} = g_2^{-1}Pg_2$ and $\mathcal{N}_{\alpha_1,\varphi} = g_2^{-1}Ng_2$. So it suffices to check that for $\sigma\in N$, $e\sigma = e$. Elements of N certainly lie in P and thus preserve the kernel of e, so to check $e\sigma = e$ it suffices to check that they act trivially on the quotient by this kernel, which is done in Lemma 7.15. □

The global sections of $\mathcal{N}_{\alpha_1,\varphi}$ will be crucial in our vanishing argument. We next give a reasonable criterion for there to be sufficiently many global sections of $\mathcal{N}_{\alpha_1,\varphi}$, by writing it as an iterated extension of vector bundles, and then assuming those vector bundles have no low-degree quotient line bundle. The Riemann–Roch theorem then implies that there are enough sections in a precise sense — see Lemma 7.27.

Because G is split, we may assume that P is defined over $\mathbb{Z}$.

Definition 7.21. Let $N_{\mathbb{Q}} = N_{0,\mathbb{Q}} \supseteq N_{1,\mathbb{Q}} \supseteq N_{2,\mathbb{Q}} \supseteq \cdots \supseteq N_{r,\mathbb{Q}} = 1$ be the derived series of $N_{\mathbb{Q}}$. Let $N_{0,\mathbb{Z}} \supseteq N_{1,\mathbb{Z}} \supseteq N_{2,\mathbb{Z}} \supseteq \cdots \supseteq N_{r,\mathbb{Z}} = 1$ be their schematic closure in $N_{\mathbb{Z}}$, and let $N_0 \supseteq N_1 \supseteq N_2 \supseteq \cdots \supseteq N_r = 1$ be their reductions mod p.

Lemma 7.22. *For all i, N_i is a smooth connected P-invariant subgroup of N, and N_i/N_{i+1} is isomorphic to a vector space (i.e., a power of $\mathbb{G}_a$), where the action of P on N_i/N_{i+1} is by vector space automorphisms.*

Proof. We can verify all these facts by the theory of root groups.

Let U be a maximal unipotent subgroup of G, defined over $\mathbb{Z}$, containing N. For each root α of U, there is a root group U_α, a subgroup isomorphic to $\mathbb{G}_a$ over $\mathbb{Z}$, which in characteristic zero is the exponential of that root [12, Thm.4.1.4 and Def.4.2.3]. (In general the root group may be a line bundle, but over $\mathbb{Z}$ the only line bundle is $\mathbb{G}_a$.) Moreover, U is isomorphic as a scheme to the product of these root groups, with the isomorphism given by multiplication in the group law, for any fixed ordering of the roots [12, Thm.5.1.13].

Choose an ordering where the roots not in $N_{\mathbb{Q}}$ are first, then the roots in $N_{0,\mathbb{Q}}$ but not in $N_{1,\mathbb{Q}}$, and so on, and use the induced isomorphism to a product of copies of $\mathbb{G}_a$ as coordinates on U. In this ordering, each of the closed subsets $N_{i,\mathbb{Q}}$ is defined by the vanishing of an initial segment of the coordinates. Hence their schematic closures, and the reductions mod p, are defined by the same equations. In particular, they are smooth and connected. The fact that these closed subsets are P-invariant, and are subgroups, can be expressed by algebraic equations and hence holds in the reduction mod p because it holds over $\mathbb{Q}$.

Because the commutator of two roots in $N_{i,\mathbb{Q}}$ necessarily lies in $N_{i+1,\mathbb{Q}}$, the group law on $N_{i,\mathbb{Q}}/N_{i+1,\mathbb{Q}}$ is simply given by addition in our fixed coordinates, and thus the action of P is linear in these coordinates. Because these are both closed conditions, they also hold modulo p. ☐

Definition 7.23. Let $\mathcal{N}_{\alpha_1,\varphi,i}$ be the subgroup of $\mathcal{N}_{\alpha_1,\varphi}$ that is locally P-conjugate to N_i, which is well-defined since N_i is a P-invariant subgroup of N.

Lemma 7.24. *The quotient $\mathcal{N}_{\alpha_1,\varphi,i}/\mathcal{N}_{\alpha_1,\varphi,i+1}$ is a vector bundle on X.*

Proof. This follows from the fact that N_i/N_{i+1} is a vector space and P acts by vector space automorphisms. ☐

Definition 7.25. Let the *height* of $(\alpha_1, t_1, \alpha_2, t_2, \varphi)$ be minus the smallest degree of a line bundle which occurs as a quotient of any of the vector bundles $\mathcal{N}_{\alpha_1,\varphi}$.

We will first, in Lemma 7.26, see how the stalk of $j_*(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L})$ changes upon composing t_1 by an element $h \in H$. In Lemma 7.28, we will contrast this with the fact that the stalk is invariant under composing t_1 by a global section of $\mathcal{N}_{\alpha_1,\varphi}$, restricted to H, to show that the stalk vanishes.

Lemma 7.26. *Fix $(\alpha_1, \alpha_2, t_1, t_2, \varphi)$ in $\overline{\mathcal{H}k}_{G(D),H,W,V}$. Consider the map $c : H \to \overline{\mathcal{H}k}_{G(D),H,W,V}$ that sends $h \in H$ to $(V_1, V_2, h \circ t_1, t_2, \varphi)$. The pullback $c^* j_*(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L})$ is isomorphic to the tensor product of the stalk of $j_*(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L})$ at $(\alpha_1, \alpha_2, t_1, t_2, \varphi)$ with $\mathcal{L}^{-1}$.*

Proof. Consider the commutative diagram

$$\begin{array}{ccccc}
\mathcal{H}k_{G(D),W} \times H & \xrightarrow{\ j\ } & \overline{\mathcal{H}k}_{G(D),H,W,V} & & \\
{\scriptstyle a}\uparrow & & {\scriptstyle b}\uparrow & \overset{c}{\nwarrow} & \\
\mathcal{H}k_{G(D),W} \times H \times H & \xrightarrow{\ j\times id\ } & \overline{\mathcal{H}k}_{G(D),H,W,V} \times H & \xleftarrow[d]{} & H
\end{array}$$

where the vertical map b sends $((\alpha_1, \alpha_2, t_1, t_2, \varphi), h)$ to $(\alpha_1, \alpha_2, h \circ t_1, t_2, \varphi)$, the vertical map a sends $((\alpha_1, \alpha_2, f, t_1), (h_1, h_2))$ to $((\alpha_1, \alpha_2, f, h_2 \circ t_1), h_1 h_2^{-1})$, the arrow d sends $h \in H$ to $((V_1, V_2, t_1, t_2, \varphi), h)$, and so c sends $h \in H$ to $(V_1, V_2, t_1, h \circ t_2, \varphi)$.

We have

$$c^* j_*(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L}) = d^* b^* j_*(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L}).$$

We have

$$b^* j_*(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L}) = (j \times id)_* a^*(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L}) = (j \times id)_*(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L} \boxtimes \mathcal{L}^{-1}) = j_*(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L}) \boxtimes \mathcal{L}^{-1}$$

with the first identity by smooth base change, because the left square is Cartesian, the third identity is by the Künneth formula, and the second identity requires some thought: By the character sheaf property of $\mathcal{L}$, the pullback of $\mathcal{L}$ along the map $(h_1, h_2) \to (h_1 h_2^{-1})$ is $\mathcal{L} \boxtimes \mathcal{L}^{-1}$. The pullbacks of $IC_{\mathcal{H}k_{G(D),W}}$ along the morphisms sending $((\alpha_1, \alpha_2, f, t_1), (h_1, h_2))$ to $(\alpha_1, \alpha_2, f, h_2 \circ t_1)$ and $(\alpha_1, \alpha_2, f, t_1)$ are equal since these are both smooth morphisms of the same relative dimension. Thus $a^*(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L}) = IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L} \boxtimes \mathcal{L}^{-1}$.

Finally, $d^*(j_*(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L}) \boxtimes \mathcal{L}^{-1})$ is the tensor product of the stalk of $j_*(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L})$ at $(\alpha_1, \alpha_2, t_1, t_2, \varphi)$ with $\mathcal{L}^{-1}$. □

Lemma 7.27. *Let β be a P-bundle on X. Let $\mathcal{P}_\beta$ be the associated twisted form of P and $\mathcal{N}_\beta$ its unipotent radical. Assume that all vector bundles in the canonical filtration of $\mathcal{N}_\beta$ have no nontrivial quotients of degree at most $2g - 2 + |D|$. Then there is a section of $\mathcal{N}_\beta$ over $\operatorname{Res}_k^D(\mathcal{N}_\beta|_D) \times X$, whose restriction to $\operatorname{Res}_k^D(\mathcal{N}_\beta|_D) \times D$ is the canonical section.*

Proof. Let $i : D \to X$ be the immersion, so that $\Gamma(D, i^*\mathcal{N}_\beta) = \Gamma(X, i_* i^* \mathcal{N}_\beta)$. First we will show that the map $\Gamma(X, \mathcal{N}_\beta) \to \Gamma(X, i_* i^* \mathcal{N}_\beta)$ is surjective. The cokernel is contained in the H^1 of X with coefficients in the kernel of the natural map $\mathcal{N}_\beta \to i_* i^* \mathcal{N}_\beta$. The kernel of the natural map $\mathcal{N}_\beta \to i_* i^* \mathcal{N}_\beta$ has a filtration, induced by pulling back the filtration of $\mathcal{N}_\beta$, whose associated graded objects are $(\mathcal{N}_{i,\beta}/\mathcal{N}_{i+1,\beta}) \otimes \mathcal{O}(-D)$. By the assumption on height, $(\mathcal{N}_{i,\beta}/\mathcal{N}_{i+1,\beta}) \otimes \mathcal{O}(-D)$ has no line bundle quotients of degree $2g - 2$, thus admits no nontrivial maps to the canonical bundle, hence has vanishing H^1, so the kernel has vanishing H^1 as well, and the map is surjective.

Moreover, the H^1 of the kernel will still vanish when base changed by any affine scheme, as these are flat over the base field, and so the natural map $\Gamma(X \times Y, \mathcal{N}_\beta) \to \Gamma(X \times Y, i_* i^* \mathcal{N}_\beta)$ is surjective for any affine Y. We take Y to be the Weil restriction $\operatorname{Res}_k^D(\mathcal{N}_\beta|_D)$ of $\mathcal{N}_\beta$ from D to k, over which there is a canonical element of $\Gamma(D, \mathcal{N}_\beta)$. This gives a section of $\mathcal{N}_\beta$ over $\operatorname{Res}_k^D(\mathcal{N}_\beta|_D) \times X$, whose restriction to $\operatorname{Res}_k^D(\mathcal{N}_\beta|_D) \times D$ is the canonical section. □

Lemma 7.28. *Assume that some $(G, m_u, H_u, \mathcal{L}_u)$ is geometrically supercuspidal. Then the stalk of $j_*(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L})$ vanishes at points whose height is greater than $2g - 2 + |D|$.*

Proof. Consider a point $(\alpha_1, \alpha_2, t_1, t_2, \varphi)$ in $\overline{\mathcal{H}k}_{G(D),H,W,V}$ of height greater than $2g-2+|D|$. Let β be the associated P-bundle, so that $\mathcal{P}_\beta = \mathcal{P}_{\alpha_1,\varphi}$. By Lemma 7.27, there is a section s of $\mathcal{N}_{\alpha_1,\varphi}$ over $\operatorname{Res}_k^D(\mathcal{N}_{\alpha_1,\varphi}|_D) \times X$, whose restriction to $\operatorname{Res}_k^D(\mathcal{N}_{\alpha_1,\varphi}|_D) \times D$ is the canonical section.

Now consider the map τ from $\operatorname{Res}_k^D(\mathcal{N}_{\alpha_1,\varphi}|_D)$ to $\overline{\mathcal{H}k}_{G(D),H,W,V}$ that sends $g \in \operatorname{Res}_k^D(\mathcal{N}_{\alpha_1,\varphi}|_D)$ to $(\alpha_1, \alpha_2, t_1 \circ s(g)|_D, t_2, \varphi)$. Because the restriction of s to D is the canonical section, $s(g)|_D$ is the section of $\mathcal{N}_{\alpha_1,\varphi}$ over D induced by g. This map is actually equal to the constant map by a diagram

$$\begin{array}{ccc} \alpha_1 & \xrightarrow{\varphi} & \alpha_2 \\ \downarrow{\scriptstyle s(g)} & & \downarrow{\scriptstyle id} \\ \alpha_1 & \xrightarrow{\varphi} & \alpha_2 \end{array}$$

which commutes by Lemma 7.20 because $s(g) \in \mathcal{N}_{\alpha_1,\varphi}$.

Because τ is the constant map,

$$\tau^* j_*(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L}) = \mathbb{Q}_\ell \boxtimes \Big(j_*(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L}) \Big)_{(\alpha_1,\alpha_2,t_1,t_2,\varphi)}$$

where ${}_{(\alpha_1,\alpha_2,t_1,t_2,\varphi)}$denotes the stalk at $(\alpha_1, \alpha_2, t_1, t_2, \varphi)$.

Now $t_1 \circ s(g)|_D = (t_1 s(g)|_D t_1^{-1}) \circ t_1$. Because α_1 admits a trivialization over N, $\mathcal{N}_{\alpha_1,\varphi}$ is conjugate over D to N, and so $\mathrm{Res}_k^D(\mathcal{N}_{\alpha_1,\varphi})$ is isomorphic to $N\langle \mathcal{O}_D \rangle$, in such a way that the embedding $g \mapsto (t_1 g t_1^{-1})$ into $G\langle \mathcal{O}_D\rangle$ is conjugate to the standard embedding.

Now consider the pullback of $j_*(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L})$ along the map that sends h to $(\alpha_1, \alpha_2, t_1 \circ h, t_2, \varphi)$ for h in the intersection of H with this conjugate copy of $N\langle \mathcal{O}_D\rangle$. This pullback is $\mathbb{Q}_\ell \otimes \left(j_*(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L})\right)_{(\alpha_1,\alpha_2,t_1,t_2,\varphi)}$. On the other hand, from Lemma 7.26, we know that this same pullback is $\mathcal{L}^{-1} \otimes \left(j_*(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L})\right)_{(\alpha_1,\alpha_2,t_1,t_2,\varphi)}$. From the definition of geometric supercuspidal, we know that even restricting to a further intersection with H_x, the pullback of $\mathcal{L}^{-1}$ is not a geometrically constant sheaf, and so its tensor product with no nonzero vector space is geometrically constant, and hence the stalk $\left(j_*(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L})\right)_{(\alpha_1,\alpha_2,t_1,t_2,\varphi)}$ vanishes, as desired. □

7.4. Hecke Correspondences.

We will use the following space to compare the stalks of $j_*(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L})$ at different points:

Definition 7.29. Fix a geometric point $Q \in X$ that is neither in D nor the support of W and a cocharacter μ in the Weyl cone of G. Let $\mathcal{H}k_{Q,\mu}\left(\overline{\mathcal{H}k}_{G(D),H,W,V}\right)$ be the moduli space of quadruples consisting of two points $(\alpha_1, \alpha_2, t_1, t_2, \varphi)$ and $(\alpha_3, \alpha_4, t_3, t_4, \varphi')$ in $\overline{\mathcal{H}k}_{G(D),H,W,V}$ and isomorphisms $m_1 : \alpha_3 \to \alpha_1$ and $m_2 : \alpha_4 \to \alpha_2$ away from Q, such that $t_1 \circ m_1|_D = t_3$, $t_2 \circ m_2|_D = t_4$, $\varphi \circ V(m_1) = V(m_2) \circ \varphi'$, and such that m_1 and m_2, expressed as points in $G((t))$ via local coordinates at Q, are in $G[[t]]\mu(t)G[[t]]$. (Note that here we use a Schubert cell and not its closure.)

Let pr_{12} and $pr_{34} : \mathcal{H}k_{Q,\mu}\left(\overline{\mathcal{H}k}_{G(D),H,W,V}\right) \to \overline{\mathcal{H}k}_{G(D),H,W,V}$ be the maps induced by $(\alpha_1, \alpha_2, t_1, t_2, \varphi)$ and $(\alpha_3, \alpha_4, t_1, t_2, \varphi')$ respectively.

Let $(\alpha_1, t_1, \alpha_2, t_2, \varphi)$ be a point of $\overline{\mathcal{H}k}_{G(D),H,W,V}$ not in the image of j. As we did at the beginning of the previous subsection, we can choose some open set X_0 where φ locally takes the form $g_1 e g_2$ for the idempotent projector e onto the space of T-eigenvalues of some proper face of the convex hull of the weights of V. Equivalently, we can trivialize α_1 and α_2 over X_0, using Lemma 2.3, so that φ in the induced coordinates is an idempotent projector e. Fix such trivializations.

Let Q be a point in X_0 that does not lie in D. Let P be the stabilizer of the kernel of e. Let $\mu : \mathbb{G}_m \to T$ be a cocharacter such that the eigenvalue of $g \to \mu(\lambda)^{-1} g \mu(\lambda)$ is a nonnegative power of λ on roots in P and is negative on roots not in P (which exists by [13, Prop.2.2.9]).

In this subsection, we will show how to choose a point of $\mathcal{H}k_{Q,\mu}\left(\overline{\mathcal{H}k}_{G(D),H,W,V}\right)$ whose image under pr_{12} is $(\alpha_1, \alpha_2, t_1, t_2, \varphi)$, whose image under pr_{34} has greater height than $(\alpha_1, \alpha_2, t_1, t_2, \varphi)$, and such that the stalks of the pullbacks of $j_*(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L})$ on its image under pr_{12} and its image under pr_{34} are isomorphic. This is precisely what we will need to inductively show that the stalk vanishes in the proof of Theorem 7.35 in the next subsection.

The key step in comparing the stalks is to show that the maps pr_{12} and pr_{34} are smooth, as it allows us to use the smooth base change theorem. This can be checked by comparing sections of the relevant stalks over the local ring, which can be reduced by a Beauville–Laszlo argument to a purely algebraic calculation, which we handle first:

Lemma 7.30. *Let R be a Henselian local ring, with maximal ideal $\mathfrak{m}$. Let $M \in \operatorname{End} V(R[[t]])$ be a matrix and $s \in R[[t]]$ an element such that (M, s) are the projective coordinates of an $R[[t]]$-point* *of $\overline{G}$. Assume that (M,s) is congruent to $(e,0)$ modulo $\mathfrak{m}$. Let g_a and g_b be elements of $G(R[\![t]\!])$ such that $g_a\mu(t)^{-1}g_b$ is congruent to $\mu(t)$ mod $\mathfrak{m}$, where the cocharacter μ is as above.*

Then there exist elements g_c and g_d in $G(R[\![t]\!])$, such that $g_c\mu(t)g_d$ is congruent to $\mu(t)$ mod $\mathfrak{m}$, and such that

$$\left(g_a\mu(t)^{-1}g_b\right) M \left(g_c\mu(t)g_d\right)$$

belongs to $\operatorname{End} V(R[\![t]\!]) \subseteq \operatorname{End} V(R(\!(t)\!))$.

Moreover the products $g_c\mu(t)g_d$ for all g_c and g_d satisfying these two conditions lie in a single orbit under the right action of $G(R[\![t]\!])$.

Proof. We make a series of reductions.

First note that we may assume R is Noetherian. This is because the problem only depends on finitely many entries of M, g_a, g_b — those entries that are nonvanishing mod a power of t equal to the sum of the highest negative power of t appearing in entries of $\mu(t^{-1})$ and $\mu(t)$. Hence the problem is defined over a Henselization of a finitely generated subring of R, which is Noetherian. For the uniqueness statement, because

$$\left(G(R[\![t]\!])\mu(t)G(R[\![t]\!])\right)/G(R[\![t]\!])$$

is represented by a scheme of finite type — more specifically, a Schubert cell of the affine Grassmannian — we may check uniqueness in the Henselization of another finitely generated subring of R, that generated by the finitely many entries of M, g_a, g_b plus the coordinates in this Schubert cell of two different possible values of g_c, g_d.

Next we will show that, by possibly changing g_a, we may assume that g_b is congruent to 1 mod $\mathfrak{m}$. This is because the map

$$G[\![t]\!] \to G[\![t]\!]\backslash\left(G[\![t]\!]\mu(t)^{-1}G[\![t]\!]\right)$$

that sends g to $G[\![t]\!]\mu(t)^{-1}g$ (equivalently to $G[\![t]\!]g_a\mu(t)^{-1}g$) is smooth at the identity, and so we can lift $G[\![t]\!]g_a\mu(t)^{-1}g_b$, which is congruent to $\mu(t)^{-1}$ mod $\mathfrak{m}$, to an R-point of $G[\![t]\!]$ congruent to 1 mod $\mathfrak{m}$.

Now because $\overline{G}$ is stable under left-multiplication by G, we may replace M by g_bM and so assume $g_b = 1$. Because left-multiplication by g_a does not affect integrality, we may assume $g_a = 1$.

Now applying Lemma 7.16(1), from $(M,s) \in \overline{G}(R[\![t]\!])$ we obtain a point $\pi'(M,s) \in (P\backslash G)(R[\![t]\!])$ congruent to P mod $\mathfrak{m}$. Since the map $G \to P\backslash G$ is smooth, and the point $\pi'(M,s) \in (P\backslash G)(R[\![t]\!])$ lifts mod $\mathfrak{m}$ to the point $1 \in G(R[\![t]\!]/\mathfrak{m})$, it follows that $\pi'(M,s)$ lifts to some $\sigma \in G(R[\![t]\!])$. We can multiply M on the right by σ^{-1} without affecting the existence of g_c, g_d or their uniqueness, because we can always multiply g_c on the left by σ to cancel it. So we may assume that $\pi'(M,s)$ is the identity, and hence by Lemma 7.16(2) that (M,s) lies in $\overline{P}(R[\![t]\!])$.

This implies the existence of a solution. In fact we can take $g_c = g_d = 1$, so it suffices to check that $\mu(t)^{-1}M\mu(t)$ is integral. By construction, all the nonzero entries of elements of the Lie algebra of P are multiplied by a nonnegative power of t when conjugated by $\mu(t)$. In characteristic zero, this implies that all the nonzero entries of elements of P are multiplied by a nonnegative power of t when conjugated by $\mu(t)$, as these are exponentials of the Lie algebra elements. Because V lifts to characteristic zero, the same thing is true for the nonzero entries in the characteristic p representation, and thus the same thing is true for elements of the closure $\overline{P}$ of P, including (M,s). So indeed $\mu(t)^{-1}M\mu(t)$ is integral, as desired.

The argument for uniqueness is more subtle. It suffices to show that, for M, g_a, g_b in this special form, all solutions $g_c\mu(t)g_d$ map to the point $\mu(t)G(R[\![t]\!])$ of the Schubert cell

$$(G(R[\![t]\!])\mu(t)G(R[\![t]\!]))\,/G(R[\![t]\!]).$$

By induction, it is sufficient to assume that the solution maps to this point modulo $\mathfrak{m}^n$ for some $n \geq 1$ and show that it also maps to this point modulo $\mathfrak{m}^{n+1}$. Here, to ensure that the map $R \to \hat{R}$ is injective, we use the Noetherian hypothesis. Because the map $G(R[\![t]\!]) \to (G(R[\![t]\!])\mu(t)G(R[\![t]\!]))\,/G(R[\![t]\!])$ sending g to $g\mu(t)G(R[\![t]\!])$ is smooth, and because $g_c\mu(t)g_dG(R[\![t]\!])$ is congruent to $\mu(t)G(R[\![t]\!])$ modulo $\mathfrak{m}^n$, we may assume g_c is congruent to 1 modulo $\mathfrak{m}^n$. Then modulo $\mathfrak{m}^{n+1}$, g_c is $1+\tau$ for some $\tau \in \mathfrak{m}^n\mathfrak{g}(R[\![t]\!])$, where $\mathfrak{g}$ is the Lie algebra of G. Then we can write

$$\mu(t^{-1})Mg_c\mu(t)g_d = \mu(t^{-1})M(1+\tau)\mu(t)g_d = \mu(t^{-1})M\mu(t)g_d + \mu(t^{-1})M\tau\mu(t)g_d.$$

We know that $\mu(t^{-1})M\mu(t)g_d$ is integral, so this implies that $\mu(t^{-1})M\tau\mu(t)g_d$ is integral, which, inverting g_d, implies that $\mu(t)^{-1}M\tau\mu(t)$ is integral. Because τ is divisible by $\mathfrak{m}^n$ and M is congruent to e modulo $\mathfrak{m}$, modulo $\mathfrak{m}^{n+1}$ we have

$$\mu(t)^{-1}M\tau\mu(t) = \mu(t)^{-1}e\tau\mu(t) = e\mu(t)^{-1}\tau\mu(t).$$

Thus $e\mu(t)^{-1}\tau\mu(t)$ is integral. If we write $\mu(t)^{-1}\tau\mu(t)$ as $\sum_i v_it^i$ for $i \in \mathbb{Z}$, then we have $ev_i = 0$ for $i < 0$. For $i < 0$, since $ev_i = 0$, $\mathrm{Im}(v_i) \subseteq \ker(e)$, so $v_i(\ker(e)) \subseteq \ker(e)$, thus v_i lies in the Lie algebra of P because P is by definition the stabilizer of $\ker(e)$.

Fix a basis of $\mathfrak{g}$ consisting of roots of the maximal torus and an arbitrary basis for the Lie algebra of the maximal torus. In such a basis, the Lie algebra $\mathrm{Lie}(P)$ is the span of a subset of the basis vectors, consisting of the roots in P and the maximal torus. Thus, because $v_i \in \mathrm{Lie}(P)$ for $i < 0$, if we express $\mu(t)^{-1}\tau\mu(t)$ as a $R(\!(t)\!)$-linear combination of the basis vectors, the coefficients of every basis vector not in $\mathrm{Lie}(P)$ will be integral. However, because the eigenvalues of conjugation by $\mu(t)$ on $\mathrm{Lie}(P)$ are nonnegative powers of t, the coefficient of every basis vector in in P of $\mu(t)^{-1}\tau\mu(t)$ will be integral. So all coefficients are integral, and thus $\mu(t)^{-1}\tau\mu(t)$ is integral. Finally, because $g_c\mu(t)g_d \equiv 1\mu(t)(1+\mu(t)^{-1}\tau\mu(t))g_d \mod m^{n+1}$, this shows that $g_c\mu(t)g_d$ maps to the point $\mu(t)G(R[\![t]\!])$ of the Schubert cell $(G(R[\![t]\!])\mu(t)G(R[\![t]\!]))\,/G(R[\![t]\!])$ modulo $\mathfrak{m}^{n+1}$, as desired. □

We will define a special point of $\mathcal{H}k_{Q,\mu}\left(\overline{\mathcal{H}k}_{G(D),H,W,V}\right)$ where the smoothness of pr_{12} and pr_{34} is as easy as possible to check. Recall that we have already fixed trivalizations of α_1 and α_2 on the open set X_0, and thus on a formal neighborhood of Q. Let $m_1 : \alpha_3 \to \alpha_1$ and $m_2 : \alpha_4 \to \alpha_2$ be the unique modifications of α_1 and α_2 respectively that are isomorphisms away from Q and that in a formal neighborhood of Q are locally isomorphic to the map $\mu(t)$. (This uniquely characterizes them by Beauville–Laszlo.) Let $t_3 = t_1 \circ m_1$ and $t_4 = t_2 \circ m_2$ be the trivializations. Let $\varphi' : V(\alpha_3) \to V(\alpha_4)$ be the map that, away from Q, is φ, and in a formal neighborhood of Q, is e. Let $y = ((\alpha_1, \alpha_2, t_1, t_2, \varphi), (\alpha_3, \alpha_4, t_1, t_2, \varphi'), m_1, m_2)$. Because e commutes with $\mu(t)$, $\varphi \circ V(m_1) = V(m_2) \circ \varphi'$ and so y is a point of $\mathcal{H}k_{Q,\mu}\left(\overline{\mathcal{H}k}_{G(D),H,W,V}\right)$.

We can translate Lemma 7.30 into a geometric lifting lemma:

Lemma 7.31. *Let R be a Henselian local ring with maximal ideal $\mathfrak{m}$. Let $(\alpha_1^*, \alpha_2^*, t_1^*, t_2^*, \varphi^*)$ be an R-point of of $\overline{\mathcal{H}k}_{G(D),H,W,V}$ that modulo the maximal ideal of R is $(\alpha_1, \alpha_2, t_1, t_2, \varphi)$. Let α_4^* be a G-bundle on X_R and let m_2^* be an isomorphism: $m_2^* : \alpha_4^* \to \alpha_2^*$ away from Q that expressed in local coordinates over a formal neighborhood of Q lies in $G[\![t]\!]\mu(t)G[\![t]\!]$ and such that (α_4^*, m_2^*) mod $\mathfrak{m}$ is isomorphic to (α_4, m_2).*

Then there exists a unique triple of a G-bundle α_3^* on X_R, isomorphism $m_1^* : \alpha_3^* \to \alpha_1^*$ away from Q that in a formal neighborhood of Q lies in $G[[t]]\mu(t)G[[t]]$, and $\varphi'^* \in \mathbb{P}(\mathrm{Hom}_X(V(\alpha_3), V(\alpha_4)+k)$ such that $\varphi^* \circ V(m_1^*) = V(m_2^*) \circ \varphi'^*$, that is congruent to $(\alpha_3, m_1, \varphi')$ modulo $\mathfrak{m}$ up to isomorphism.

Proof. Fix trivializations of $\alpha_1^*, \alpha_2^*, \alpha_4^*$ over the formal neighborhood of Q that agree modulo $\mathfrak{m}$ with the trivializations of α_1 and α_2 we have chosen and with the trivialization of α_4 in which m_2 is $\mu(t)$.

By Beauville–Laszlo, the data of α_3^* is equivalent to the data of a G-bundle over a formal neighborhood of Q, a G-bundle over the complement of Q, and an isomorphism between the two over the punctured formal neighborhood. Because m_1^* is an isomorphism over the complement of Q, we can take the G-bundle over the complement of Q to be α_1^*, so the data of (α_3^*, m_1^*) is simply a G-bundle over a formal neighborhood of Q with an isomorphism to α_1^* over the punctured formal neighborhood. Because we have a trivialization of α_1^*, this data is equivalent to an element of $G(R((t)))$ modulo the right action of $G(R[[t]])$. We can view this element as m_1^* because it is the isomorphism from α_3^* to α_1^* in formal coordinates.

The map $\varphi^{'*}$ is uniquely determined by the other data, as we must have $V(m_2^*)^{-1} \circ \varphi^* \circ V(m_1^*) = \varphi^{'*}$. However, this formula may not define any $\varphi^{'*}$, as it defines a section of $\mathcal{H}om(V(\alpha_3), V(\alpha_4)) + \mathcal{O}_X$ away from Q that may have a pole of Q.

If we express φ^* in our trivialization over the punctured formal neighborhood as (M, s), then by assumption (M, s) are the projective coordinates of an $R[[t]]$-point of $\overline{G}$ and are congruent to $(e, 0)$ mod $\mathfrak{m}$.

If we view m_2^* over the punctured formal neighborhood of Q as an element of $G(R((t)))$, by assumption on m_2, it can be expressed as $g_b^{-1}\mu(t)g_a^{-1}$ for $g_a, g_b \in G(R[[t]])$ and it is congruent to $\mu(t)$ modulo $\mathfrak{m}$.

Then the possible values of (α_1^*, m_1^*) are parameterized by those elements of $G(R((t)))$ that are of the form $g_c\mu(t)g_d$ for $g_c, g_d \in G(R[[t]])$, that are congruent to $\mu(t)$ modulo $\mathfrak{m}$, and such that $g_a\mu(t)^{-1}g_bMg_c\mu(t)g_d$ is integral, up to the right action of elements of $G(R[[t]])$ that are congruent to 1 modulo $\mathfrak{m}$. By Lemma 7.30, there is a unique such element up to equivalence. ☐

We can now prove the desired smoothness statement:

Lemma 7.32. *Both pr_{12} and pr_{34} are smooth at y.*

Proof. We can factor pr_{12} as the composition of first, the map p' that projects onto a point $(\alpha_1, \alpha_2, t_1, t_2, \varphi)$ of $\overline{\mathcal{H}k}_{G(D),H,W,V}$ with a G-bundle α_4 and isomorphism $m_2 : \alpha_4 \to \alpha_2$ such that m_2 near Q is in the cell of the affine Grassmannian corresponding to μ, with, second, the map that forgets α_4 and m_2. This second map is a locally trivial fibration by the cell of the affine Grassmannian associated to μ and hence is smooth.

Thus it is sufficient to show that the first projection p' is étale at y. To do this we may ignore the trivializations t_3, t_4 as these are uniquely determined by the other data. The projection p' is then defined by adding α_3, m_1, φ'. Then p' is schematic of finite type, since the data of the pair (α_3, m_1) is equivalent to a section of a locally trivial fibration by the cell of the affine Grassmannian associated to μ, and then φ' is a section of a projective bundle satisfying a closed condition, so p' is represented by a closed subset of a projective bundle on a fibration by a variety. To check that p' is étale at the point y, we use the fact that each R-point of the base for a Henselian local ring R congruent mod $\mathfrak{m}$ to the image of y has a unique lift to an R-point of the total space congruent mod $\mathfrak{m}$ to y, which is Lemma 7.31. This implies that there is a section of p' over the étale local ring at the $p'(y)$, and that this section is equal over the étale local ring at y to the identity, which implies the natural map from the étale local ring at $p'(y)$ to the étale local ring at y is an isomorphism and so the map is étale.

Finally, we can deduce the pr_{34} case from the pr_{12} case by symmetry, taking the dual of V and so reversing all the arrows. Note that the assumption on the weights of V is preserved by duality. $\square$

Using smoothness, we can prove an isomorphism of stalks, which will be a key ingredient in our induction argument:

Lemma 7.33. *The stalks of* $pr_{12}^* j_*(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L})$ *and* $pr_{34}^* j_*(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L})$ *at* y *are isomorphic.*

Proof. By Lemma 7.8, the image of j inside $\overline{\mathcal{H}k}_{G(D),H,W,V}$ consists of those $(\alpha_1, \alpha_2, t_1, t_2, \varphi)$ where the last coordinate of φ is nonzero. For a point of $\mathcal{H}k_{Q,\mu}\left(\overline{\mathcal{H}k}_{G(D),H,W,V}\right)$, the equation $\varphi \circ V(m_1) = V(m_2) \circ \varphi'$ ensures that the last coordinate of φ is nonzero if and only if the last coordinate of φ' is nonzero. Let $\mathcal{H}k_{Q,\mu}\left(\mathcal{H}k_{G(D),W} \times H\right)$ be the open subset where the last coordinates of φ and φ' are nonzero, j' its inclusion into $\mathcal{H}k_{Q,\mu}\left(\overline{\mathcal{H}k}_{G(D),H,W,V}\right)$, and pr'_{12} and pr'_{34} the projections onto $\mathcal{H}k_{G(D),W} \times H$. This gives a commutative diagram:

$$\begin{array}{ccccc}
\overline{\mathcal{H}k}_{G(D),H,W,V} & \xleftarrow[pr_{12}]{} & \mathcal{H}k_{Q,\mu}\left(\overline{\mathcal{H}k}_{G(D),H,W,V}\right) & \xrightarrow{pr_{34}} & \overline{\mathcal{H}k}_{G(D),H,W,V} \\
{\scriptstyle j}\uparrow & & {\scriptstyle j'}\uparrow & & {\scriptstyle j}\uparrow \\
\mathcal{H}k_{G(D),W} \times H & \xleftarrow[pr'_{12}]{} & \mathcal{H}k_{Q,\mu}\left(\mathcal{H}k_{G(D),W} \times H\right) & \xrightarrow{pr'_{34}} & \mathcal{H}k_{G(D),W} \times H
\end{array}$$

To show the isomorphism, observe that in a neighborhood of y, $pr_{12}^* j_*(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L}) = j'_* pr'^*_{12}(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L})$ by smooth base change and Lemma 7.32. So it suffices to show that $pr'^*_{12}(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L}) = pr'^*_{34}(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L})$. Let $p_c : \mathcal{H}k_{G(D),W} \times H \to \mathcal{H}k_{G(D),W}$ and $p_h : \mathcal{H}k_{G(D),W \times H} \to H$ be the projections. We have $IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L} = p_c^* IC_{\mathcal{H}k_{G(D),W}} \otimes p_h^* \mathcal{L}$ so

$$pr'^*_{12}(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L}) = pr'^*_{12} p_c^* IC_{\mathcal{H}k_{G(D),W}} \otimes pr^{*'}_{12} p_h^* \mathcal{L},$$

and similarly for pr_{34}. Hence it suffices to show that

$$pr'^*_{12} p_c^* IC_{\mathcal{H}k_{G(D),W}} = pr'^*_{34} p_c^* IC_{\mathcal{H}k_{G(D),W}}$$

and

$$pr'^*_{12} p_h^* \mathcal{L} = pr'^*_{34} p_h^* \mathcal{L}.$$

The map $pr_{12'}$ is smooth by Lemma 7.32, and p_c is smooth because H is. Thus, $pr'^*_{12} p_c^* IC_{\mathcal{H}k_{G(D),W}}$ is simply a shift of $IC_{\mathcal{H}k_{Q,\mu}\left(\mathcal{H}k_{G(D),W} \times H\right)}$. The same argument works for for pr_{34}, which gives the first desired identity.

The second desired identity follows from $p_h \circ pr'_{12} = p_h \circ pr'_{34}$, which can be expressed also as the commutativity of the extended diagram

$$\begin{array}{ccccc}
\overline{\mathcal{H}k}_{G(D),H,W,V} & \xleftarrow{pr_{12}} & \mathcal{H}k_{Q,\mu}\left(\overline{\mathcal{H}k}_{G(D),H,W,V}\right) & \xrightarrow{pr_{34}} & \overline{\mathcal{H}k}_{G(D),H,W,V} \\
\uparrow j & & \uparrow j' & & \uparrow j \\
\mathcal{H}k_{G(D),W} \times H & \xleftarrow{pr'_{12}} & \mathcal{H}k_{Q,\mu}\left(\mathcal{H}k_{G(D),W} \times H\right) & \xrightarrow{pr'_{34}} & \mathcal{H}k_{G(D),W} \times H \\
& \searrow^{p_h} & & {}_{p_h}\swarrow & \\
& & H & &
\end{array}$$

If $(\alpha_1, \alpha_2, t_1, t_2, \varphi)$ is in the image under j of some point $((\alpha_1, t_1, \alpha_2, f), h) \in \mathcal{H}k_{G(D),W\times H}$, then $\varphi = V(f)$ for some isomorphism f of G-bundles $\alpha_1 \to \alpha_2$, and $t_2 = h \circ t_1 \circ f^{-1}|_D$, so $h = t_2 \circ f|_D \circ t_1^{-1}$. Similarly if $\varphi' = V(f')$ then we have $h' = t_4 \circ f'|_D \circ t_3^{-1}$. To check that the diagram commutes, we must check $h = h'$. Because V is faithful, the identity $V(m_1) \circ \varphi' = \varphi \circ V(m_2)$ implies $m_2 \circ f' = f \circ m_1$. Thus we have

$$t_2 \circ f|_D \circ t_1^{-1} = t_2 \circ f|_D \circ m_1|_D \circ t_3^{-1} = t_2 \circ m_2|_D \circ f'|_D \circ t_3^{-1} = t_4 \circ f'|_D \circ t_3^{-1}$$

showing that the diagram commutes and completing the proof. □

The final ingredient in our induction is a lemma that checks that the height grows:

Lemma 7.34. *For $y = ((\alpha_1, \alpha_2, t_1, t_2, \varphi), (\alpha_3, \alpha_4, t_3, t_4, \varphi'), m_1, m_2)$ defined as before, the height of $(\alpha_3, \alpha_4, t_1, t_2, \varphi')$ is strictly greater than the height of $(\alpha_1, \alpha_2, t_1, t_2, \varphi)$.*

Proof. Consider the natural isomorphism $\mathcal{N}_{\alpha_1,\varphi} \to \mathcal{N}_{\alpha_3,\varphi'}$ away from Q that is induced by the isomorphism m_1. This isomorphism respects the canonical filtration of N by vector spaces. Hence it defines an isomorphism from the associated graded vector bundles of $\mathcal{N}_{\alpha_1,\varphi}$ to the associated graded vector bundles of $\mathcal{N}_{\alpha_3,\varphi'}$. We will show that each map of vector bundles appearing this way extends to a map of vector bundles over all of X that vanishes over the fiber of Q.

To do this, it is sufficient to calculate in a neighborhood of Q. Over that neighborhood, we can assume that φ and φ' are both simply the map e, so that $\mathcal{N}_{\alpha_1,\varphi}$ and $\mathcal{N}_{\alpha_3,\varphi'}$ are each N, and the induced map is the homomorphism $g \to m_1^{-1} \circ g \circ m_1 = \mu(t)^{-1} g \mu(t)$. So it is sufficient to show that the eigenvalues of $\mu(t)$ acting by conjugation on the associated graded module of the canonical filtration of N are all positive powers of t. Because the associated graded is also the associated graded of the Lie algebra of a filtration on the Lie algebra of N, it is sufficient to show that all the eigenvalues of $\mu(t)$ on the Lie algebra of N are positive powers of t. To do this, observe that for any root in the Lie algebra of N, its dual root is not in the Lie algebra of P, so the eigenvalue of $\mu(t)$ on it is a negative power of t.

Given a map $V_1 \to V_2$ that is an isomorphism away from a point Q and vanishes Q , any line bundle L that appears as a quotient of V_2 admits a nontrivial map from V_1 which vanishes at a point, and so $L_1(-Q)$ admits a nontrivial map from V_1, and thus some line bundle which maps to $L_1(-Q)$ and thus has degree $< \deg L_1$ must appear as a quotient of V_1. It follows that the height of $(\alpha_1, \alpha_2, t_1, t_2, \varphi)$ is less than the height of $(\alpha_3, \alpha_4, t_1, t_2, \varphi')$. □

7.5. Conclusion.

Theorem 7.35. *Assume that V lifts to the Witt vectors of k and that the pairing of any weight of V with any coroot of G is less than p.*

Assume that $(G, m_u, H_u, \mathcal{L}_u)$ is geometrically supercuspidal for some $u \in D$ and $\operatorname{char}(k) > 2$. *Then the natural map*

$$j_!(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L}) \to j_*(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L})$$

is an isomorphism.

Proof. We check the isomorphism on stalks at each point. By Lemma 7.8, j is an open immersion, and thus the isomorphism holds for points in the image of j. At points outside the image of j, it is sufficient to prove that the stalk of $j_*(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L})$ vanishes. We do this by induction on the height. The base case when the height is greater than $2g-2+|D|$ is handled by Lemma 7.28.

For the induction step, we assume it is true for height $> h$ and prove it for height h. Given a point $(\alpha_1, \alpha_2, t_1, t_2, \varphi)$ of height h, we have defined a point y of $\mathcal{H}k_{Q,\mu}\left(\overline{\mathcal{H}k}_{G(D),H,W,V}\right)$. By Lemma 7.33, the stalk at $pr_{12}(y)$ is equal to the stalk at $pr_{34}(y)$. By Lemma 7.34, the height of $p_{34}(y)$ is greater than h, so by our induction hypothesis the stalk vanishes, and then the stalk at $(\alpha_1, \alpha_2, t_1, t_2, \varphi)$ vanishes, completing the induction step. □

Theorem 7.36. *Assume that $(G, m_u, H_u, \mathcal{L}_u)$ is geometrically supercuspidal for some $u \in D$ and* $\operatorname{char}(k) > 2$. *Then the natural map*

$$\Delta^W_! \left(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L}\right) \to \Delta^W_* \left(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L}\right)$$

is an isomorphism.

Proof. By Lemma 7.17, there exists a representation V satisfying the condition of Theorem 7.35.

We have observed that $\overline{\Delta}^W \circ j = \Delta^W$ and that $\overline{\Delta}^W$ is proper. We thus have

$$\begin{aligned}\Delta^W_! \left(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L}\right) = \overline{\Delta}^W_* j_! \left(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L}\right) &= \overline{\Delta}^W_* j_* \left(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L}\right) \\ &= \Delta^W_* \left(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L}\right).\end{aligned}$$

□

In fact, this result also holds in characteristic 2 if G has no nontrivial normal subgroup with trivial center by a similar proof, using the second part of Lemma 7.17.

8. Properties of the Hecke complex

Let X be a smooth projective curve over k, G a split semisimple group over k, D an effective divisor on X, H a smooth connected factorizable subgroup of $G\langle\mathcal{O}_D\rangle$, and $\mathcal{L}$ a character sheaf on H.

For $W : |X| \to \Lambda^+$ a function with finite support, supported away from D, let

$$K_W := \Delta^W_! \left(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L}\right)[\dim H].$$

We will use Theorem 7.36, and other tools, to show important properties of K_W. In §8.1 we will show it is a pure perverse sheaf. In §8.2 we will describe its support. In §9.1 we will calculate its trace function.

8.1. **Purity and Perversity.**

Notation 8.1. Let $d(W) := \sum_{x\in|X|} 2(\deg x)\langle W_x, \rho\rangle$ where ρ is half the sum of the positive roots of the maximal torus of G.

Lemma 8.2. *(i) The dimension of* $\mathrm{Bun}_{G(D)}$ *is* $(\dim G)(g + |D| - 1)$.
(ii) The dimension of $\mathcal{H}k_{G(D),W}$ *is* $(\dim G)(g + |D| - 1) + d(W)$

Proof. (i) $\mathrm{Bun}_{G(D)}$ is a $G\langle\mathcal{O}_D\rangle$-torsor on Bun_G. The dimension of Bun_G is $(\dim G)(g-1)$ and the dimension of $G\langle\mathcal{O}_D\rangle$ is $(\dim G)|D|$.
(ii) $\mathcal{H}k_{G(D),W}$ is a fiber bundle over $\mathrm{Bun}_{G(D)}$ in the étale topology. The fiber over each point of $\mathrm{Bun}_{G(D)}$ is equal to the product over x in the support of W of the Weil restriction from κ_x to k of the closure of the Schubert cell of the affine Grassmannian associate with W_x. The dimension of this fiber is the sum over x of $\deg x$ times the dimension of this cell. The dimension of the cell is $2\langle W_x, \rho\rangle$ so the sum is $d(W)$. □

We refer the reader to [5, 38] for the foundations of the theory of perverse sheaves in characteristic p and [48] for the generalization to stacks.

Lemma 8.3. *Assume that* $(G, m_u, H_u, \mathcal{L}_u)$ *is geometrically supercuspidal for some* $u \in D$ *and* $\mathrm{char}(k) > 2$*. Then the complex* K_W *is perverse, pure of weight* $(\dim G)(g + |D| - 1) + d(W) + \dim H$*, and geometrically semisimple.*

Proof. The intersection cohomology complex $IC_{\mathcal{H}k_{G(D),W}}$ is defined as the intermediate extension of $\mathbb{Q}_\ell[\dim \mathcal{H}k_{G(D),W}]$ from the smooth locus of $\mathcal{H}k_{G(D),W}$ to the whole space, and thus is perverse by [5, Thm.4.3(ii)]. Because $\mathbb{Q}_\ell[\dim IC_{\mathcal{H}k_{G(D),W}}]$ is pure of weight $\dim \mathcal{H}k_{G(D),W}$ on the smooth locus [5, §5.1.8], and the intermediate extension preserves purity [5, Cor.5.3.2], $IC_{\mathcal{H}k_{G(D),W}}$ is pure of weight $\dim \mathcal{H}k_{G(D),W} = (\dim G)(g + |D| - 1) + d(W)$ (by Lemma 8.2).

Because $\mathcal{L}$ is lisse on a smooth variety of dimension $\dim H$, $\mathcal{L}[\dim H]$ is perverse. By Lemma 7.9, Δ^W is schematic and affine. Thus by Artin's theorem, $\Delta^W_*\left(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L}[\dim H]\right)$ is semiperverse [5, Thm.4.1.1] and $K_W = \Delta^W_!\left(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L}[\dim H]\right)$ is cosemiperverse [5, Cor.4.1.2]. Because they are equal by Theorem 7.36, they are each perverse. (We can apply these results for schemes because perversity is a smooth-local condition, so we may check it locally, and Artin stacks are smooth-locally modeled by schemes.)

By Lemma 2.15, $\mathcal{L}$ has arithmetic monodromy of finite order, so every Frobenius eigenvalue of $\mathcal{L}$ has finite order, and hence has absolute value 1, so $\mathcal{L}$ is pure of weight 0. Thus its shift $\mathcal{L}[\dim H]$ is pure of weight $\dim H$, so the exterior product $IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L}[\dim H]$ is pure of weight $(\dim G)(g + |D| - 1) + d(W) + \dim H$. Hence by Deligne's theorem (which we may apply because Δ^W is schematic), $K_W = \Delta^W_!\left(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L}[\dim H]\right)$ is mixed of weight $\leq (\dim G)(g + |D| - 1) + d(W) + \dim H$ and $\Delta^W_*\left(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L}[\dim H]\right)$ is mixed of weight $\geq (\dim G)(g + |D| - 1) + d(W) + \dim H$ [5, Stabilities 5.1.14(i,i*)]. Because they are equal by Theorem 7.36, they are each pure of weight $(\dim G)(g + |D| - 1) + d(W) + \dim H$.

The geometric semisimplicity of a pure perverse sheaf on an Artin stack with affine stabilizers follows from [58, Thm.1.2] □

Lemma 8.4. *Assume that* $(G, m_u, H_u, \mathcal{L}_u)$ *is geometrically supercuspidal for some* $u \in D$ *and* $\mathrm{char}(k) > 2$*. Then the Verdier dual of* K_W *is the analogue of* K_W *defined with the dual character sheaf* $\mathcal{L}^\vee$*, twisted by* $\overline{\mathbb{Q}}_\ell((\dim G)(g + |D| - 1) + d(W) + \dim H)$*.*

Proof. We have

$$DK_W = D\Delta_!^W \left(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L}[\dim H] \right) = \Delta_*^W D \left(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L}[\dim H] \right)$$

$$= \Delta_*^W \left(DIC_{\mathcal{H}k_{G(D),W}} \boxtimes D(\mathcal{L}[\dim H]) \right).$$

Now $D(\mathcal{L}[\dim H]) = \mathcal{L}^\vee(\dim H)[\dim H]$ and $DIC_{\mathcal{H}k_{G(D),W}} = IC_{\mathcal{H}k_{G(D),W}}(\dim \mathcal{H}k_{G(D),W}) = IC_{\mathcal{H}k_{G(D),W}}(\dim G(g + |D| - 1) + d(W))$ so

$$DK_W = \Delta_*^W \left(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L} \right) ((\dim G)(g + |D| - 1) + d(W) + \dim H)[\dim H]$$

$$= \Delta_!^W \left(IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L} \right) ((\dim G)(g + |D| - 1) + d(W) + \dim H))[\dim H]. \qquad \square$$

8.2. **Vanishing Properties.** The following definition is one way of generalizing to the ramified case the very unstable bundles of Frenkel–Gaitsgory–Vilonen [25, §3.2].

Definition 8.5. Let P be a parabolic subgroup of G with maximal unipotent subgroup N. To a P-bundle on X, we attach a form of N twisted by the conjugation action of P on N, which admits a natural filtration into vector bundles (see Lemma 7.24). We say that a *P-bundle is very unstable* if none of these vector bundles admit a nontrivial map to $K_X(D)$. We say that a *G-bundle is very unstable* if it admits a reduction to a very unstable P-bundle for some maximal parabolic subgroup P of G.

This definition makes sense for G-bundles on X defined over any field, and in particular an algebraically closed field. The utility of this definition is that it allows us to prove that the stalk of K_W vanishes:

Lemma 8.6. *Assume that $(G, m_u, H_u, \mathcal{L}_u)$ is geometrically supercuspidal for some $u \in D$ and $\mathrm{char}(k) > 2$. Then the stalk of K_W at a geometric point $((\alpha_1, t_1), (\alpha_2, t_2))$ of $\mathrm{Bun}_{G(D)} \times \mathrm{Bun}_{G(D)}$ vanishes if V_1 or V_2 is very unstable, as does the stalk of its Verdier dual.*

Proof. By Lemma 8.4, and because geometric supercuspidality is preserved by duality, we can reduce to the case of K_W. By switching α_1 and α_2 and replacing W by the conjugate of $-W$ under the longest element of the Weyl group, we can reduce to the case where α_1 is very unstable.

By proper base change, the stalk of K_W at $((\alpha_1, t_1), (\alpha_2, t_2))$ is the cohomology with compact supports of the fiber of Δ^W over $(\alpha_1, t_1), (\alpha_2, t_2)$ with coefficients in $IC_{\mathcal{H}kG(D)} \boxtimes \mathcal{L}$. This fiber consists of isomorphisms $\varphi : \alpha_1 \to \alpha_2$ away from the support of W, satisfying local conditions at points in the support of W, such that $t_2 \circ \varphi|_D \circ t_1 \in H$.

Let β be a reduction of α_1 to a very unstable P-bundle. By Lemma 7.27, there is a section over $\mathrm{res}_k^D(\mathcal{N}_\beta|D) \times X$ of $\mathcal{N}_\beta$, and therefore a section s of the automorphism group of α_1, that restricted to D is the canonical section. Let S be the subgroup of $\sigma \in \mathrm{res}_k^D(\mathcal{N}_\beta|D) \times X$ such that $t_1^{-1} \circ \sigma \circ t_1 \in H$. Then S acts on this fiber by sending φ to $\varphi \circ s(\sigma)$, which satisfies

$$t_2^{-1} \circ \varphi|_D \circ s(\sigma)|_D \circ t_1 = t_2^{-1} \circ \varphi|_D \circ \sigma \circ t_1 \in H$$

by assumption. This action preserves $IC_{\mathcal{H}kG(D)}$, because it is canonical, but acts on $\mathcal{L}$ by tensoring with $\mathcal{L}(t_1^{-1}\sigma t_1)$. Hence the action of the automorphism on the cohomology is by tensoring with $\mathcal{L}(t_1^{-1}\sigma t_1)$, which is nontrivial by the geometrically supercuspidal assumption, so the cohomology is equal to itself tensored with a nontrivial local system, hence the cohomology vanishes, as desired. $\square$

Next, we will describe an explicit open set of $\mathrm{Bun}_{G(D)}$ whose complement consists entirely of very unstable G-bundles. We will be able to restrict attention to this open subset, which has many useful properties (most crucially, it is quasicompact), for most calculations.

First, it is necessary to prove a version of the main theorem of reduction theory that is uniform in q. In the work [25], the role of this lemma is played by some calculations with the Harder–Narasimhan filtration. See also [31]. Recall that G is split.

Lemma 8.7. *Every G-bundle on X admits a reduction to a B-bundle whose induced T-bundle, composed with the character associated to any simple positive root to produce a line bundle, has degree $\geq -2g$.*

We use the convention that in the SL_2 triple where the upper-right nilpotent is the given positive root, the associated cocharacter is $t \mapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$.

Proof. Fix a G-bundle β. First we check β admits a reduction to a B-bundle. To prove this, note that β admits a trivialization over the generic point, hence a B-reduction over the generic point, which extends to the whole curve because the associated G/B-bundle is proper.

Next we define a height on the set of B-reductions of β. Observe that the associated G/B-bundle (i.e., β modulo the right action of B) is a projective scheme Y over X. Given a character χ_0 of T, which induces a character of B, we can form the associated line bundle L_{χ_0} on Y by composing the universal B-bundle with the inverse character $B \to \mathbb{G}_m$. Fix a character χ_0 of T that is in the interior of the Weyl chamber of B, so that it is positive on all the positive coroots. Then the associated line bundle L_{χ_0} is ample. (We use the inverse character so that dominant weights will correspond to ample line bundles.)

Any B-reduction, consisting of a B-bundle $\alpha \subseteq \beta$, defines a section $s : X \to Y$. The Weil height of s according to L_{χ_0} is defined to be the degree of $s^* L_{\chi_0}$. This is manifestly an integer and is bounded below. Hence it takes a minimum value. For s the section associated to a B-bundle α, $s^* L_{\chi_0}$ is the inverse of the composition of α with χ_0 , so the height of s is minus the degree of $\chi_0(\alpha)$. Choose a B-reduction α_1 whose height attains the minimum value. We will show that the composition of α_1 with every simple root character has degree $\geq -2g$, giving the desired conclusion.

Fix a simple root. Let χ be the associated character of B and let P be the associated parabolic. Then the quotient of the Levi subgroup of P by its center is a split adjoint-form group of rank one, hence is isomorphic to PGL_2. We have a commutative diagram with Cartesian square.

$$\begin{array}{ccccc} \mathbb{G}_m & \xleftarrow{\lambda_1/\lambda_2} & B(\mathrm{PGL}_2) & \longrightarrow & \mathrm{PGL}_2 \\ & \nwarrow^{\chi} & \uparrow & & \uparrow \\ & & B & \longrightarrow & P \end{array}$$

By functoriality, α_1 defines a P-bundle $P(\alpha_1)$ and hence a PGL_2-bundle $\mathrm{PGL}_2(\alpha_1)$, which we can view as a rank two vector bundle V on X, up to a twist by a line bundle. After twisting, we may assume that V has degree $2g-1$ or $2g$. By Riemann–Roch, $H^0(X, V)$ has dimension $\geq (2g-1)+2-2g = 1$, so it has a global section, and hence V can be written as the extension by a line bundle L_1 of degree ≥ 0 of another line bundle L_2, which necessarily has degree $\leq 2g$.

This gives a reduction E of $\mathrm{PGL}_2(\alpha_1)$ to $B(\mathrm{PGL}_2)$. Let α_2 be the fiber product

$$P(\alpha_1) \times_{\mathrm{PGL}_2(\alpha_1)} E.$$

Then because

$$B = P \times_{\mathrm{PGL}_2} B(\mathrm{PGL}_2),$$

α_2 is a B-bundle. Furthermore, α_2 agrees with α_1 when projected to P, and hence α_2 is another B-reduction of β. Finally, α_2 agrees with E when projected to $B(\mathrm{PGL}_2)$.

We can express χ_0 as a sum of some character that factors through P with a positive multiple of χ. This is because the characters that factor through P form a wall of the Weyl chamber, to which χ is perpendicular, and pointing towards the interior of the Weyl cone. Observe that the degree of $\chi(\alpha_2)$ is equal to the degree of L_1 minus the degree of L_2, which is at least $-2g$ by construction. So if $\chi(\alpha_1) < -2g$, then $\chi(\alpha_2) > \chi(\alpha_1)$ and thus $\chi_0(\alpha_2) > \chi_0(\alpha_1)$, which contradicts the assumption that the height $-\chi_0(\alpha)$ is minimized by α_1. □

We are now ready to define our key open subset U: Let V be a faithful representation of G. Let r be the maximum number of simple roots that can be added to form a positive root of G and let k be the maximum ℓ^1-norm of any weight of V, measured in a basis of simple roots of G. Let ϵ be 1 if $r = 1$ and D is empty and 0 otherwise. Let L be a line bundle on X of degree at least $k(2rg + \deg D + \epsilon) + 2g - 1$.

Definition 8.8. Let U consist of $(\alpha, t) \in \mathrm{Bun}_{G(D)}$ such that $H^1(X, V(\alpha) \otimes L(-Q))$ vanishes for each point Q in X.

Lemma 8.9. *(i) U is an open subset of $\mathrm{Bun}_{G(D)}$.*
(ii) U is quasicompact.
(iii) U is the quotient of a smooth scheme of finite type by a reductive algebraic group of finite type.
(iv) Every vector bundle in the complement of U inside $\mathrm{Bun}_{G(D)}$ is very unstable.
(v) The stalk of K_W vanishes on $\mathrm{Bun}_{G(D)} \times \mathrm{Bun}_{G(D)}$ outside $U \times U$.

Proof. To prove assertion (i), observe that U is the complement of the projection from $\mathrm{Bun}_{G(D)} \times X$ to $\mathrm{Bun}_{G(D)}$ of the locus where $H^1(X, V(\alpha) \otimes L(-Q)) \neq 0$. By the semicontinuity theorem, this locus is closed, and X is proper, hence universally closed, so the projection is closed as well.

Assertion (ii) follows from assertion (iii). To prove assertion (iii), observe that a G-bundle α satisfies the condition from Definition 8.8 if and only if $V(\alpha) \otimes L$ is globally generated and satisfies $H^1(X, V(\alpha) \otimes L) = 0$. In this case, $H^0(X, V(\alpha) \otimes L)$ is a $(\dim V)(\deg L + 1 - g)$-dimensional vector space.

Thus, let $\mathcal{M}_3$ be the moduli space of triples of a G-bundle α satisfying the condition of Definition 8.8, a trivialization of α over D, and a basis for $H^0(X, V(\alpha) \otimes L)$. Then U is the quotient of $\mathcal{M}_3$ by $\mathrm{GL}_{(\dim V)(\deg L+1-g)}$. (In particular, $\mathcal{M}_3$ is a $\mathrm{GL}_{(\dim V)(\deg L+1-g)}$-torsor over U, hence $\mathcal{M}_3$ is smooth.) Thus, to prove (iii), it suffices to check that $\mathcal{M}_3$ is a scheme of finite type.

Let $\mathcal{M}_2$ be the moduli space of pairs of a G-bundle α satisfying the condition of Definition 8.8 and a basis for $H^0(X, V(\alpha) \otimes L)$. Then $\mathcal{M}_3$ is a $G\langle \mathcal{O}_D \rangle$-torsor over $\mathcal{M}_2$, so it suffices to show that $\mathcal{M}_2$ is a scheme of finite type.

Given a point of $\mathcal{M}_2$, and in particular a basis for $H^0(X, V(\alpha) \otimes L)$, we obtain a map from X to the Grassmannian Gr of rank $\dim V$ quotients of a fixed $(\dim V)(\deg L + 1 - g)$-dimensional vector space, where the map has degree $(\dim V)(\deg L)$. Let $\mathcal{M}_1$ be the moduli space of maps f

from X to Gr with degree $(\dim V)(\deg L)$. Let V_{taut} be the tautological bundle on Gr. Then $\mathcal{M}_1$ is a scheme of finite type, and $\mathcal{M}_2$ maps to $\mathcal{M}_1$. The image of this map is contained in the open subset $\mathcal{M}_1'$ of $\mathcal{M}_1$ where $H^1(X, f^*V_{\mathrm{taut}}) = 0$ and the natural map $H^0(\mathrm{Gr}, V_{\mathrm{taut}}) \to H^0(X, f^*V_{\mathrm{taut}})$ is an isomorphism. The fiber of the map $\mathcal{M}_2 \to \mathcal{M}_1'$ parameterizes reductions of the structure group of $f^*V_{\mathrm{taut}} \otimes L^{-1}$ to G. Thus $\mathcal{M}_2 \to \mathcal{M}_1$ is a schematic morphism of finite type [61, Cor.3.2.4], and so $\mathcal{M}_2$, and finally $\mathcal{M}_3$, are schematic of finite type.

To prove assertion (iv), let α be a G-bundle outside U. Then for some point Q, we have $H^1(X, V(\alpha) \otimes L(-Q)) \neq 0$. Hence by Serre duality we have $H^0(X, K_X \otimes V(\alpha)^\vee \otimes L^\vee(Q)) \neq 0$, so $V(\alpha)$ admits a nontrivial map to the line bundle $K_X \otimes L^\vee(Q)$, which has degree at most $-k(2rg + \deg D + \epsilon)$. Let $\gamma_1, \dots, \gamma_n$ be the simple roots of B. Choose a B-reduction of α as in Lemma 8.7, and let β be the induced T-bundle, where T is the maximal torus of T. Using Lemma 8.7, we have chosen β so that

$$\deg(\gamma_1(\beta)), \dots, \deg(\gamma_n(\beta)) \geq -2g. \tag{8.1}$$

As a representation of B, V admits a filtration by one-dimensional characters. The filtration of $V(\alpha)$ induced by this B-reduction is a filtration by line bundles, each arising by β from a one-dimensional character of T. Because $V(\alpha)$ admits a nontrivial map to a line bundle of degree $\leq -k(2rg + \deg D + \epsilon)$, at least one of these line bundles has degree $\leq -k(2rg + \deg D + \epsilon)$. The degree of the line bundle associated to a character of T is a linear form ω on the weight space. If we had

$$|\deg(\gamma_i(\beta))| < 2rg + \deg D + \epsilon$$

for each root γ_i, then the linear form ω would have absolute value $< 2rg + \deg D + \epsilon$ on each basis vector, hence have absolute value $< k(2rg + \deg D + \epsilon)$ on each vector with ℓ^1 norm at most k, so by the definition of k have absolute value $< k(2rg + \deg D + \epsilon)$ on each weight of T, giving a contradiction. Thus, for some i, we must have

$$|\deg(\gamma_i(\beta))| \geq 2rg + \deg D + \epsilon > 2g.$$

Combined with (8.1), this implies that

$$\deg(\gamma_i(\beta)) > 2rg + \deg D + \epsilon.$$

Let P be the parabolic subgroup defined by the set of all the roots other than γ_i. Let N be the unipotent radical of P. Then N is an iterated extension, as an algebraic group, of one-dimensional representations of B, each a character of B corresponding to a positive root in the unipotent radical of P and thus to the sum of at most r positive roots, at least one of which is γ_i. Because each of the other roots has degree $\geq -2g$ and γ_i has degree $\geq 2rg + \deg D$, the product has degree at least $2g + \deg D$ and so does not admit a nontrivial map to $K_X(D)$. Hence none of the N_i's do either, and the bundle is very unstable.

Assertion (v) follows from assertion (iv) and Lemma 8.6. □

9. The trace function of the Hecke complex

We maintain the assumptions and notation of Section 8.

9.1. Calculation of the trace function. To describe the trace function of K_W explicitly, we will first give an explicit description of the points of $\mathrm{Bun}_{G(D)}(\mathbb{F}_q)$, that the trace function is a function on, in terms of adelic double cosets. This is a variant of the classical Weil parameterization. It will be helpful for later to give an adelic description of the automorphisms of a point of $\mathrm{Bun}_{G(D)}$, which we do in Lemma 9.3. The trace function of K_W can be calculated as a sum. We will describe the set to be summed over in Lemma 9.5, and define the function to be summed in Definition 9.6, culminating in a description of the trace function in Lemma 9.7.

Recall some of our earlier notation: $\mathbf{K}(D) = \prod_{x\in|X-D|} G(\mathfrak{o}_x) \times \prod_{x\in D} U_{m_x}(G(\mathfrak{o}_x))$, where $\mathfrak{o}_x = \kappa_x[[t]]$ and $U_{m_x}(G(\mathfrak{o}_x))$ is the subgroup of $G(\mathfrak{o}_x)$ consisting of elements congruent to 1 modulo t^{m_x}.

Lemma 9.1. *There is a bijection between $G(F)\backslash G(\mathbb{A}_F)/\mathbf{K}(D)$ and $\mathrm{Bun}_{G(D)}(k)$.*

Moreover, this bijection arises from a bijection between $G(\mathbb{A}_F)$ and the set of tuples $(\alpha, z_\eta, (z_x)_{x\in|X|})$ of a G-bundle α and a trivialization $z_\eta : \alpha|_\eta \xrightarrow{\sim} G_\eta$ of α over the generic point and a trivialization $z_x : \alpha|_{\mathfrak{o}_x} \xrightarrow{\sim} G_{\mathfrak{o}_x}$ for each closed point $x \in |X|$. Explicitly, the bijection sends $(\alpha, z_\eta, (z_x)_{x\in|X|})$ to the tuple

$$(z_\eta|_{\kappa_x((t))} \circ z_x^{-1}|_{\kappa_x((t))})_{x\in|X|} \in {\prod_{x\in|X|}}' G(\kappa_x((t))) = G(\mathbb{A}_F)$$

of transition maps defined over the punctured formal neighborhood of x. Forgetting z_η corresponds to quotienting out by $G(F)$ on the left, and keeping from $(z_x)_{x\in|X|}$ only the trivialization z_x modulo t^{m_x} for $x \in D$ corresponds to quotienting by $\mathbf{K}(D)$ on the right. Here, the trivialization z_x modulo t^{m_x} for $x \in D$ matches the trivialization of α over D that comes with a point of $\mathrm{Bun}_{G(D)}(k)$.

Proof. This is the standard definition of the Weil parameterization. By Lemma 2.3, for any G-bundle there in fact exists a trivialization over the generic point, and because there are no nontrivial torsors of connected algebraic groups over finite fields, there exists a trivialization over a formal neighborhood of every closed point.

One then checks that this map sends the set of all possible trivializations to a double coset in $G(F)\backslash G(\mathbb{A}_F)/\mathbf{K}(D)$ and that each double coset arises from a unique isomorphism class of G-bundles. ☐

Recall that J_x is the inverse image of $H_x(\kappa_x)$ under the map $G(\mathfrak{o}_x) \twoheadrightarrow G(\kappa_x)$.

Definition 9.2. For $g \in G(\mathbb{A}_F)$, let $\mathrm{Aut}_D(g)$ be the subgroup of $\gamma \in G(F)$ such that $g^{-1}\gamma g \in \mathbf{K}(D)$. Let $\mathrm{Aut}_{D,H}(g)$ be the subgroup of $\gamma \in G(F)$ such that

$$g^{-1}\gamma g \in \prod_{x\in|X-D|} G(\mathfrak{o}_x) \times \prod_{x\in D} J_x.$$

We have that $\mathrm{Aut}_D(g)$ is a normal subgroup of $\mathrm{Aut}_{D,H}(g)$.

There is an action of $H(k)$ on $\mathrm{Bun}_{G(D)}(k)$ where $h \in H(k)$ acts by fixing the G-bundle α and composing the trivialization t_D of α over D with H.

For the action of $H(k)$ on $\mathrm{Bun}_{G(D)}(k)$, we say that the stabilizer in $H(k)$ of a point (α, t_D) consists of all elements $H(k)$ that send (α, t_D) to a point isomorphic to (α, t_D). Equivalently, this is the stabilizer of the isomorphism class of (α, t_D) for the induced action of $H(k)$ on the set of isomorphism classes. The analogous definition works for any group action of a groupoid.

Lemma 9.3. *Let g be an element of $G(\mathbb{A}_F)$, and (α, t) be the point of $\mathrm{Bun}_{G(D)}(k)$ corresponding to the double coset of g. Then*

(i) The automorphism group of (α, t_D) is $\mathrm{Aut}_D(g)$.
(ii) Under the identification $H(k) = \prod_{x\in D} H_x(\kappa_x) = \prod_{x\in D} J_x / \prod_{x\in D} U_{m_x}(G(\mathfrak{o}_x))$, the action of $H(k)$ on $\mathrm{Bun}_{G(D)}(k)$ is intertwined with the action of $\prod_{x\in D} J_x$ by right multiplication on $G(F)\backslash G(\mathbb{A}_F)/\mathbf{K}(D)$.
(iii) The stabilizer in $H(k)$ of a point (α, t_D) is $\mathrm{Aut}_{D,H}(g)/\mathrm{Aut}_D(g)$.

Proof. (i) Any automorphism of (α, t_D), when restricted to the generic point by the trivialization t_η, defines an element $\gamma \in G(F)$. Conversely, any element $\gamma \in G(F)$ defines an automorphism of α over the generic point. The condition that the automorphism extends to a place x is precisely the condition that $g_x^{-1}\gamma g_x$ is in $G(\mathfrak{o}_x)$. For $x \in D$, the condition that the automorphism commute with the trivialization t is the condition that $g_x^{-1}\gamma g_x \in U_{m_x}(G(\mathfrak{o}_x))$.

(ii) For $h \in H(k)$ write $h = (h_x)_{x \in D}$ under the identification $H(k) = \prod_{x \in D} H_x(\kappa_x)$. The double coset corresponding to a G-bundle with a trivialization over D arises, by Lemma 9.1, from all choices of a trivialization z_η over the generic point and z_x in a formal neighborhood over each point x, such that for $x \in D$, z_x is congruent mod t^{m_x} to the trivialization over D. Thus, the action of h on the trivialization over D is equivalent to composing z_x with an element of J_x in the inverse image of h_x. This is equivalent to multiplying g_x by an element of J_x in the inverse image of h_x.

(iii) $\mathrm{Aut}_D(g)$ is the kernel of the natural map from $\mathrm{Aut}_{D,H}(g)$ to $H(k)$ given by projection $\gamma \mapsto g_x^{-1}\gamma g_x$ from J_x to $H_x(\kappa_x)$. The elements in the image are exactly those elements of $H(k)$ that can be lifted to elements in $\prod_{x \in D} J_x$ whose action by right multiplication fixes the double coset of g, i.e., the stabilizer in $H(k)$ of (α, t_D). $\square$

From now on, let $k = \mathbb{F}_q$. We need a lemma about the compatibility of the geometric and classical Satake isomorphisms, which is well-known. This is implicit in the 1982 combinatorial formulas of Lusztig and Kato, whose relationship to the IC sheaf is the generalization to the affine Grassmannians of the calculations by Kazhdan–Lusztig of the trace of Frobenius on the IC-sheaves of the closure of Schubert cells in a complete flag variety. Our proof is an elaboration of a sketch by Richarz and Zhu [54, p.449], and we provide some details since we were not able to find a more detailed exposition in the literature.

Lemma 9.4. *Let $\lambda \in \Lambda^+$ be a coweight of G. Let IC_λ be the IC-sheaf of the closure of the cell of the affine Grassmannian $\mathrm{Gr}_G = G(\!(t)\!)/G[\![t]\!]$ associated to λ. The trace of Frobenius on the stalk $IC_{\lambda,x}$ of IC_λ at a point $x \in \mathrm{Gr}_G(\mathbb{F}_q) = G(\mathbb{F}_q(\!(t)\!))/G(\mathbb{F}_q[\![t]\!])$ is equal to the value at $G(\mathbb{F}_q[\![t]\!])xG(\mathbb{F}_q[\![t]\!])$ of the function $a_\lambda \in \mathcal{H}(G)$ associated to the representation of $\widehat{G}$ with highest weight λ by the Satake isomorphism, times $q^{\langle \lambda, \rho \rangle}$.*

Proof. Consider the function $f_\lambda \colon \mathrm{Gr}_G(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell \cong \mathbb{C}$ defined by the stalks of IC_λ times $q^{-\langle \lambda, \rho \rangle}$, i.e., the stalks of the twist $IC_\lambda(\langle \lambda, \rho \rangle)$. Because the Schubert cell is left $G(\mathbb{F}_q[\![t]\!])$-invariant, IC_λ is left $G(\mathbb{F}_q[\![t]\!])$-invariant, and so f_λ descends to a function on $G(\mathbb{F}_q[\![t]\!])\backslash G(\mathbb{F}_q(\!(t)\!))/G(\mathbb{F}_q[\![t]\!])$. Because the Satake transform is an isomorphism, in order to verify that it coincides with a_λ, it suffices to check that the Satake transform of f_λ is the character of the representation of $\widehat{G}$ with highest weight λ.

For $\mu : \mathbb{G}_m \to T$ a cocharacter, let $[\mu] \in X^*(\widehat{T})$ be the associated character of the dual torus. Then by definition, the Satake transform of f_λ is given by

$$\sum_{\mu : \mathbb{G}_m \to T} [\mu] \cdot q^{-\langle \mu, \rho \rangle} \int_{h \in N(\mathbb{F}_q(\!(t)\!))} f_\lambda(h\mu(t))dh$$

where N is the unipotent radical of a Borel, and we take the Haar measure dh on $N(\mathbb{F}_q(\!(t)\!))$ where $N(\mathbb{F}_q[\![t]\!])$ has measure one. For

$$g \in N(\mathbb{F}_q[\![t]\!])\mu(t)G(\mathbb{F}_q[\![t]\!]),$$

consider the total measure assigned by dh to

$$gG(\mathbb{F}_q[\![t]\!])\mu(t)^{-1} \cap N(\mathbb{F}_q(\!(t)\!)).$$

Because the Haar measure dh is left $N(\mathbb{F}_q[\![t]\!])$-invariant, this equals the measure of

$$\mu(t)G(\mathbb{F}_q[\![t]\!])\mu(t)^{-1} \cap N(\mathbb{F}_q(\!(t)\!)) = \mu(t)\left(N(\mathbb{F}_q(\!(t)\!)) \cap G(\mathbb{F}_q[\![t]\!])\right)\mu(t)^{-1},$$

which by definition is the index of $N(\mathbb{F}_q(\!(t)\!)) \cap G(\mathbb{F}_q[\![t]\!])$ inside $\mu(t)\left(N(\mathbb{F}_q(\!(t)\!)) \cap G(\mathbb{F}_q[\![t]\!])\right)\mu(t)^{-1}$. By viewing N as an iterated extension of root groups, and observing that the action of $\mu(t)$ on the root group associated to α is scaling by $q^{\langle \mu, \alpha\rangle}$, we can see that this index is $q^{2\langle \mu,\rho\rangle}$, where ρ as usual is half the sum of the positive roots.

On the other hand, if

$$g \notin N(\mathbb{F}_q[\![t]\!])\mu(t)G(\mathbb{F}_q[\![t]\!])$$

then the total measure assigned to $gG(\mathbb{F}_q[\![t]\!])$ by dh is zero.

Thus,

$$\int_{h\in N(\mathbb{F}_q(\!(t)\!))} f_\lambda(h\mu(t))dh = \sum_{g\in N(\mathbb{F}_q(\!(t)\!))\mu(t)G(\mathbb{F}_q[\![t]\!])/G(\mathbb{F}_q[\![t]\!])} f_\lambda(g)q^{2\langle\mu,\rho\rangle}$$

so the Satake transform of f_λ is

$$\sum_{\mu:\mathbb{G}_m\to T} [\mu]\cdot q^{\langle\mu,\rho\rangle} \sum_{g\in N(\mathbb{F}_q(\!(t)\!))\mu(t)G(\mathbb{F}_q[\![t]\!])/G(\mathbb{F}_q[\![t]\!])} f_\lambda(g).$$

The subset $N(\mathbb{F}_q[\![t]\!])\mu(t)G(\mathbb{F}_q[\![t]\!])/G(\mathbb{F}_q[\![t]\!]) \subseteq \mathrm{Gr}_G(\mathbb{F}_q)$ is the set of $\mathbb{F}_q$-points of the locally closed subscheme S_μ of the affine Grassmannian defined by Mirković-Vilonen [67, §5.3.5], see also [4, §3.2]. Hence the sum of the trace function f_λ of $IC_\lambda(\langle\lambda,\rho\rangle)$ over this set is the trace of Frobenius on $H^*_c(S_{\mu,\overline{\mathbb{F}}_q}, IC_\lambda(\langle\lambda,\rho\rangle))$. By [67, Thm.5.3.9(2)] and [4, Prop.10.1], all eigenvalues of Frobenius on this cohomology group are equal to $q^{-\langle\mu,\rho\rangle}$ and occur in degree $\langle 2\rho,\mu\rangle$, so the trace of Frobenius is $q^{-\langle\mu,\rho\rangle}\dim H_c^{\langle 2\rho,\mu\rangle}(S_{\mu,\overline{\mathbb{F}}_q}, IC_\lambda(\langle\lambda,\rho\rangle))$. Thus, the Satake transform of f_λ is

$$\sum_{\mu:\mathbb{G}_m\to T} [\mu]\cdot \dim H_c^{\langle 2\rho,\mu\rangle}(S_{\mu,\overline{\mathbb{F}}_q}, IC_\lambda(\langle\lambda,\rho\rangle)). \tag{9.1}$$

By [67, Thm.5.3.9(3) and Lem.5.3.17], this cohomology group is isomorphic to the $\widehat{T}$-eigenspace with character $[\mu]$ in the representation of $\widehat{G}$ with highest weight λ. This means the multiplicity of $[\mu]$ in the sum (9.1) is the multiplicity of $[\mu]$ in the representation V_λ of $\widehat{G}$ with highest weight λ, so (9.1) is the character $\mathrm{tr}(V_\lambda)$ of that representation, as desired. □

To state and prove a bijection between isomorphisms φ satisfying a list of conditions and $\gamma\in G(F)$ satisfying a different list of conditions, it is helpful to name these conditions. These will be used only in the following Lemma 9.5.

Let g_1, g_2 be two elements of $G(\mathbb{A}_F)$, and let (α_1, t_1), (α_2, t_2) be the corresponding points of $\mathrm{Bun}_{G(D)}(k)$.

We say an isomorphism $\varphi : \alpha_1 \to \alpha_2$ away from the support of W satisfies condition (C-φ) if φ, expressed as an element of $G(\!(t)\!)$ by local coordinates near each point x in the support of W, is in the closed cell of the affine Grassmannian associated to W_x, and if $t_2\circ\varphi|_D\circ t_1^{-1}$ is contained in H.

In other words, φ satisfies (C-φ) if and only if $((\alpha_1,t_1),(\alpha_2,t_2),\varphi)$ is a point of $\mathcal{H}k_{G(D),W}\times H$.

We say $\gamma\in G(F)$ satisfies condition (C-γ) if $g_2^{-1}\gamma g_1$ is in $G(\mathfrak{o}_x)$ at all points outside the support of W and the support of D, is in the closure of the cell of the Bruhat decomposition of $G(F_x)$ associated to W_x for each point x in the support of W, and lies in J_x for each point $x \in D$.

Lemma 9.5. *Let g_1, g_2 be two elements of $G(\mathbb{A}_F)$, and let (α_1, t_1), (α_2, t_2) be the corresponding points of $\mathrm{Bun}_{G(D)}(k)$.*

There is a bijection between the set of isomorphisms $\varphi : \alpha_1 \to \alpha_2$ away from the support of W satisfying the above condition (C-φ) and $\gamma \in G(F)$ satisfying the above condition (C-γ) such that, if φ and γ correspond under this bijection, we have the two identities:

(1) $t_2 \circ \varphi|_D \circ t_1^{-1} \in H(k)$ equals the product over $x \in D$ of the projection of the local component of $g_2 \gamma g_1^{-1}$ from J_x to $H_x(\kappa_x)$.

(2) The trace function at $IC_{\mathcal{H}k_{G(D),W}}$ at $(\alpha_1, \alpha_2, \varphi, t_1) \in \mathcal{H}k_{G(D),W}$ equals $\prod_{x \in W} f_x^W(g_2 \gamma g_1^{-1})$, where f_x^W is the function on $G(F_x)$ associated by the Satake isomorphism to the character of the representation of $\widehat{G}$ whose highest weight corresponds to W_x.

For interpreting the identities (1) and (2) above, it is helpful to note that the projection $\mathcal{H}k_{G(D),W} \times H \to H$ sends $((\alpha_1, t_1), (\alpha_2, t_2), \varphi)$ to $t_2 \circ \varphi|_D \circ t_1^{-1}$ and the projection $\mathcal{H}k_{G(D),W} \times H \to \mathcal{H}k_{G(D),W}$ sends $((\alpha_1, t_1), (\alpha_2, t_2), \varphi)$ to $(\alpha_1, \alpha_2, \varphi, t_1)$.

Proof. Let $t_{\eta,1}, t_{x,1}, t_{\eta,2}, t_{x,2}$ be the trivializations of α_1 and α_2 at the generic point and in formal neigborhoods respectively. Then because $t_{\eta,1}$ and $t_{\eta,2}$ are isomorphisms, there is a bijection between isomorphisms $\varphi_\eta : \alpha_1 \to \alpha_2$ over the generic points and the elements $t_{\eta,2} \circ \varphi_\eta \circ t_{\eta,1}^{-1}$ of $G(F)$. Let $\gamma = t_{\eta,2} \circ \varphi_\eta \circ t_{\eta,1}^{-1}$.

We define our bijection to send φ to γ. The inverse map defines φ_η over the generic point as $t_{\eta,2}^{-1} \circ \gamma \circ t_{\eta,1}$ and then extends φ_η uniquely to an isomorphism φ away from the support of W.

To show this gives a bijection, it suffices to check that the extension φ of φ_η exists and satisfies condition (C-φ) if and only if γ satisfies condition (C-γ). To check this, first note that, restricted to the punctured formal neighborhood of x,

$$t_{x,2} \circ \varphi \circ t_{x,1}^{-1} = t_{x,2} \circ t_{\eta,2}^{-1} \circ \gamma \circ t_{\eta,1} \circ t_{x,1}^{-1} = g_{2,x}^{-1} \gamma g_{1,x}$$

is the local component of $g_2 \gamma g_1^{-1}$ at x.

Now we check that the restriction (C-φ) places on φ at a point x is equivalent to a corresponding restriction (C-γ) places on the local component of $g_2 \gamma g_1^{-1}$ at the same point x:

- For x not in the support of W, the condition that φ_η extends to an an isomorphism in a neighborhood of x is equivalent to the condition that $g_2^{-1} \gamma g_1$ lies in $G(\mathfrak{o}_x)$. (If $x \in D$, this is implied by the stronger condition that $g_2^{-1} \gamma g_1$ lies in J_x).
- Let x be in the support of W. The condition that, expressed in local coordinates at x, φ is in the closure of the cell in the affine Grassmannian associated to W_x is equivalent to the condition that $g_2^{-1} \gamma g_1$ lies in the closure of the cell of the Bruhat decomposition of $G(F_x)$ associated to W_x.
- The fact that $t_2 \circ \varphi|_D \circ t_1^{-1}$ lies in H is equivalent to the condition that $g_2^{-1} \gamma g_1$ is in H modulo D, or equivalently modulo t^{m_x} for each x in D, which is precisely the definition of J_x.

Combining these equivalences at all points x, we see that (C-γ) is equivalent to (C-φ), together with the claim that the extension φ of φ_η exists, and so the map that sends φ to γ is a bijection.

The identity (1) follows from the fact that, for $x \in D$, $g_{2,x}^{-1} \gamma g_{1,x} = t_{x,2} \circ \varphi \circ t_{x,1}^{-1}$ and thus is congruent to $t_2 \circ \varphi \circ t_1^{-1}$ modulo t^{m_x}.

The identity (2) follows from Lemma 9.4. ∎

Definition 9.6. For x a closed point of X, let f_x^W on $G(F_x)$ equal:

- If x is not contained in D or the support of W, the characteristic function of $G(\mathfrak{o}_x)$.
- If x is contained in the support of W, the function associated by the Satake isomorphism to the character of the representation of $\widehat{G}$ whose highest weight corresponds to W_x, times $q^{\deg x \langle W_x, \rho \rangle}$.
- If x is contained in D, the function that vanishes outside of J_x and is equal to χ_x on J_x.

Lemma 9.7. *Let g_1, g_2 be two elements of $G(\mathbb{A}_F)$. Let (α_1, t_1) and (α_2, t_2) be the points of $\mathrm{Bun}_{G(D)}(k)$ corresponding to the double cosets of g_1 and g_2 respectively. Then the trace of Frob_k on the stalk of K_W at $((\alpha_1, t_1), (\alpha_2, t_2))$ is*

$$\sum_{\gamma \in G(F)} \prod_{x \in |X|} f_x^W (g_2^{-1} \gamma g_1).$$

Proof. By the Lefschetz formula, the trace is the sum of the trace function of $IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L}$ over $(\Delta^W)^{-1}((\alpha_1, t_1), (\alpha_2, t_2))$, where $(\Delta_W)^{-1}$ denotes the inverse image. (The fiber $(\Delta^W)^{-1}((\alpha_1, t_1), (\alpha_2, t_2))$ is an affine scheme of finite type, so we do not need to apply the Lefschetz formula for stacks.)

By Definition 6.12, $(\Delta^W)^{-1}((\alpha_1, t_1), (\alpha_2, t_2))$ consists of isomorphisms $\varphi : \alpha_1 \to \alpha_2$ away from the support of W, that expressed as elements of $G(\!(t)\!)$ by local coordinates near each point x in the support of W are in the closed cell of the affine Grassmannian associated to W_x, such that $t_2 \circ \varphi|_D \circ t_1^{-1}$ is contained in H.

By Lemma 9.5, such maps φ are in bijection with γ in $G(F)$ such that $g_2^{-1} \gamma g_1$ is in $G(\mathcal{O}_{F_v})$ at all places outside the support of W and the support of D, is in the closure of the cell of the Bruhat decomposition of $G(F_x)$ for each place x associated to W_x for each point x in the support of W, and lies in J_x for each point x of D.

Furthermore, the trace function of $IC_{\mathcal{H}k_{G(D),W}} \boxtimes \mathcal{L}$ is equal to the product of the trace function of $IC_{\mathcal{H}k_{G(D),W}}$ and the trace function of $\mathcal{L}$. The trace function of $IC_{\mathcal{H}k_{G(D),W}}$ is the product over the places lying in the support of W of the function associated to the corresponding representation of $\widehat{G}$ in the Satake isomorphism times $q^{\langle W_x, \rho \rangle}$ by Lemma 9.5. The trace function of $\mathcal{L}$ is a character of $H(k)$, which by definition is $\prod_{x \in D} \chi_x$.

Examining, we see that the trace of the point associated to an element γ is precisely $\prod_{x \in |X|} f_x^W (g_2^{-1} \gamma g_1)$. Summing over γ, we obtain the stated sum. $\square$

Definition 9.8. For $g_1, g_2 \in G(\mathbb{A}_F)$, let

$$\mathsf{K}_W(g_1, g_2) = \sum_{\gamma \in G(F)} \prod_{x \in |X|} f_x^W (g_2^{-1} \gamma g_1)$$

be the trace function of K_W.

9.2. **Cohomological interpretation of the trace.** We can interpret the inner product of two functions K_{W_1}, K_{W_2} cohomologically. Using this cohomological interpretation, we will get a very strong bound, in Theorem 9.10. We will later express this inner product as the trace of a Hecke operator on our space of automorphic forms (in Proposition 10.1), and therefore obtain bounds for traces of Hecke operators.

Lemma 9.9. *Assume that $p > 2$ and some $(G, m_u, H_u, \mathcal{L}_u)$ is geometrically supercuspidal. Then we have*

$$\sum_{g_1,g_2\in G(F)\backslash G(\mathbb{A}_F)/\mathbf{K}(D)} \frac{\overline{\mathsf{K}_{W_1}(g_1,g_2)}\mathsf{K}_{W_2}(g_1,g_2)}{|\operatorname{Aut}_D(g_1)||\operatorname{Aut}_D(g_2)|}$$

$$= q^{(\dim G)(g+|D|-1)+d(W_1)+\dim H} \sum_{i\in\mathbb{Z}} (-1)^i \operatorname{tr}(\operatorname{Frob}_q, H^i_c(U_{\overline{k}} \times U_{\overline{k}}, DK_{W_1} \otimes K_{W_2}))$$

where the sum on the left is finitely supported and the sum on the right is absolutely convergent.

Proof. By [59, Thm.4.2(i)], $\sum_{i\in\mathbb{Z}}(-1)^i \operatorname{tr}(\operatorname{Frob}_q, H^i_c(U_{\overline{k}} \times U_{\overline{k}}, DK_{W_1} \otimes K_{W_2}))$ is absolutely convergent and the Lefschetz formula for algebraic stacks [59, Thm.4.2(ii)] reads

$$\sum_{i\in\mathbb{Z}}(-1)^i \operatorname{tr}(\operatorname{Frob}_q, H^i_c(U_{\overline{k}}\times U_{\overline{k}}, DK_{W_1}\otimes K_{W_2}) = \sum_{((\alpha_1,t_1),(\alpha_2,t_2))\in U(k)\times U(k)} \frac{\operatorname{tr}(\operatorname{Frob}_q, (DK_{W_1} \otimes K_{W_2})_{((\alpha_1,t_1),(\alpha_2,t_2))}}{|\operatorname{Aut}((\alpha_1,t_1),(\alpha_2,t_2))|}.$$

By Lemma 8.9, $(DK_{W_1} \otimes K_{W_2})$ vanishes outside $U(k) \times U(k)$ and so the above is equal to

$$\sum_{((\alpha_1,t_1),(\alpha_2,t_2))\in \operatorname{Bun}_{G(D)}(k)\times \operatorname{Bun}_{G(D)}(k)} \frac{\operatorname{tr}(\operatorname{Frob}_q, (DK_{W_1} \otimes K_{W_2})_{((\alpha_1,t_1),(\alpha_2,t_2))}}{|\operatorname{Aut}((\alpha_1,t_1),(\alpha_2,t_2))|}.$$

Furthermore, this sum is finitely supported.

We have

$$\operatorname{tr}(\operatorname{Frob}_q, (DK_{W_1}\otimes K_{W_2})_{((\alpha_1,t_1),(\alpha_2,t_2))} = \operatorname{tr}(\operatorname{Frob}_q, (DK_{W_1})_{((\alpha_1,t_1),(\alpha_2,t_2))}) \operatorname{tr}(\operatorname{Frob}_q, (K_{W_2})_{((\alpha_1,t_1),(\alpha_2,t_2))})$$

$$= q^{-(\dim G)(g+|D|-1)+d(W_1)-\dim H} \overline{\operatorname{tr}(\operatorname{Frob}_q, (K_{W_1})_{((\alpha_1,t_1),(\alpha_2,t_2))})} \operatorname{tr}(\operatorname{Frob}_q, (K_{W_2})_{((\alpha_1,t_1),(\alpha_2,t_2))})$$

$$= q^{-(\dim G)(g+|D|-1)+d(W_1)-\dim H} K_{W_1}(g_1,g_2)K_{W_2}(g_1,g_2),$$

where g_1 corresponds to α_1, t_1 and g_2 corresponds to α_2, t_2 under the bijection of Lemma 9.5. The first identitiy is straightforward, the second identity uses an application due to Katz of a result of Gabber [36, Lem.1.8.1(1)] and the fact that K_{W_1} is pure and perverse of weight $(\dim G)(g + |D| - 1) + d(W_1) + \dim H$, and the third identity uses Lemma 9.7.

Finally, we have $|\operatorname{Aut}((\alpha_1,t_1),(\alpha_2,t_2))| = |\operatorname{Aut}_D(g_1)||\operatorname{Aut}_D(g_2)|$ by Lemma 9.3. Plugging these in, we get the stated formula. □

Let n be a natural number. Define $F_n = \mathbb{F}_{q^n}(X)$ and $X_n = X_{\mathbb{F}_{q^n}}$. We can base change the datum $(G, D, H, \mathcal{L}, W_1, W_2)$ from $\mathbb{F}_q$ to $\mathbb{F}_{q^n}$ in the following way: We pull back G from $\mathbb{F}_q$ to $\mathbb{F}_{q^n}$, we pull back D from X to X_n, we compose W_1 and W_2 with the projection $|X_n| \to |X|$, and we base change H and $\mathcal{L}$ from $G\langle\mathcal{O}_D\rangle$ to $(G\langle\mathcal{O}_D\rangle)_{\mathbb{F}_{q^n}}$. Let $f_x^{W_i,n}$ be the local factors defined by this new datum and $\mathsf{K}_{W_i,n}(g_1,g_2) = \sum_{\gamma\in G(F_n)} \prod_{x\in|X|} f_x^{W_i,n}(g_2^{-1}\gamma g_1)$ for $g_1, g_2 \in G(\mathbb{A}_{F_n})$. Let $\mathbf{K}(D)_n$ be defined also in terms of this base-changed datum.

Theorem 9.10. *Assume that $p > 2$ and some $(G, m_u, H_u, \mathcal{L}_u)$ is geometrically supercuspidal. Then*

$$\sum_{g_1,g_2\in G(F_n)\backslash G(\mathbb{A}_{F_n})/\mathbf{K}(D)_n} \frac{\overline{\mathsf{K}_{W_1,n}(g_1,g_2)}\mathsf{K}_{W_2,n}(g_1,g_2)}{|\operatorname{Aut}_D(g_1)||\operatorname{Aut}_D(g_2)|} = O((q^n)^{(\dim G)(g+|D|-1)+\frac{d(W_1)}{2}+\frac{d(W_2)}{2}+\dim H}).$$

Proof. By Lemma 8.3, K_{W_2} is pure of weight $w_2 = (\dim G)(g + |D| - 1) + d(W_2) + \dim H$ and K_{W_1} is pure of weight $w_1 = (\dim G)(g+|D|-1)+d(W_1)+\dim H$, so $w_2 - w_1 = d(W_2)-d(W_1)$.

By Lemma 2.23, taking $j = 0$, it follows that

$$\sum_{i=-\infty}^{0} (-1)^i \operatorname{tr}(\mathrm{Frob}_{q^n}, H^i_c(U_{\overline{k}} \times U_{\overline{k}}, DK_{W_1} \otimes K_{W_2})) = O\left((q^n)^{\frac{d(W_2)-d(W_1)}{2}}\right).$$

Applying Lemma 2.22(2), this cohomology group vanishes for $i > 0$, so

$$\sum_{i\in\mathbb{Z}}(-1)^i \operatorname{tr}(\mathrm{Frob}_{q^n}, H^i_c(U_{\overline{k}} \times U_{\overline{k}}, DK_{W_1} \otimes K_{W_2})) = O\left((q^n)^{\frac{d(W_2)-d(W_1)}{2}}\right).$$

Then we apply Lemma 9.9 over $\mathbb{F}_{q^n}$. It is clear that base changing all the data in this way is equivalent to base-changing $\mathcal{H}k_{G(D),W} \times H$ and thus to base-changing K_{W_1}, K_{W_2}, so we obtain

$$\sum_{g_1,g_2\in G(F_n)\backslash G(\mathbb{A}_{F_n})/\mathbf{K}(D)_n} \overline{\mathsf{K}_{W_1,n}(g_1, g_n)}\mathsf{K}_{W_2,n}(g_1, g_2)$$
$$= (q^n)^{(\dim G)(g+|D|-1)+d(W_1)+\dim H} \sum_{i\in\mathbb{Z}}(-1)^i \operatorname{tr}(\mathrm{Frob}_{q^n}, H^i_c(U_{\overline{k}} \times U_{\overline{k}}, DK_{W_1} \otimes K_{W_2}))$$
$$= O\left((q^n)^{(\dim G)(g+|D|-1)+d(W_1)+\dim H} (q^n)^{\frac{d(W_2)-d(W_1)}{2}}\right) = O\left((q^n)^{(\dim G)(g+|D|-1)+\frac{d(W_1)}{2}+\frac{d(W_2)}{2}+\dim H}\right).$$

□

9.3. **Integrality and Weil numbers.** Dimensions of spaces of automorphic forms over function fields can often be expressed naturally as sums of Weil numbers. The same is true for the traces of Hecke operators that we study here. In fact, these Weil numbers are algebraic integers. We prove this indirectly, by first proving, in Lemmas 9.11, 9.12, and 9.13, that the traces themselves are algebraic integers, then proving in Lemma 9.14 that the eigenvalues of Frobenius acting on the relevant cohomology groups are Weil numbers, but not necessarily integers, and finally combining these, in Lemma 9.15, to express the trace in terms of integral Weil numbers.

Let m be the order of the arithmetic monodromy group of $\mathcal{L}$, which is equal to the order of the character χ by Lemma 2.15. It is also stable under finite field extension by Lemma 2.15, as the arithmetic and geometric monodromy groups are equal.

Lemma 9.11. *For all $x \in |X|$ and all $W : |X| \to \Lambda^+$, the function f^W_x takes values in $\mathbb{Z}[\mu_m]$.*

Proof. If x lies in D, this follows from the fact that χ is an eigenvalue of Frobenius on $\mathcal{L}$ and hence is a root of unity in the monodromy group. If x does not lie in D or the support of W, then f_x takes the values zero and one, both integers. If x lies in the support of W, then the value is a polynomial in q by the Kazhdan–Lusztig purity theorem. □

Lemma 9.12. *For all $g_1, g_2 \in G(\mathbb{A}_F)$, $\mathsf{K}_W(g_1, g_2)$ is divisible in $\mathbb{Z}[\mu_m]$ by $|\operatorname{Aut}_{D,H}(g_1)|$ and by $|\operatorname{Aut}_{D,H}(g_2)|$.*

Proof. Let γ' be an element of $\operatorname{Aut}_{D,H}(g_1)$. Then for all $x \in |X - D|$, $g_1^{-1}\gamma' g_1 \in G(\kappa_x[\![t]\!])$ and so $f^W_x(g_2^{-1}\gamma\gamma' g_1) = f^W_x(g_2^{-1}\gamma g_1)$. For $x \in D$, $g_1^{-1}\gamma' g_1 \in J_x$ and so

$$f^W_x(g_2^{-1}\gamma\gamma' g_1) = f^W_x(g_2^{-1}\gamma g_1)\chi_x(g_1^{-1}\gamma' g_1).$$

Hence right multiplication by γ' multiplies $\prod_{x\in|X|} f^W_x(g_2^{-1}\gamma g_1)$ by $\prod_{x\in D} \chi_x(g_1^{-1}\gamma' g_1)$. It follows that $\mathsf{K}_W(g_1, g_2) = \mathsf{K}_W(g_1, g_2)\prod_{x\in D} \chi_x(g_1^{-1}\gamma' g_1)$ and hence $\mathsf{K}(g_1, g_2) = 0$, and we are done, unless $\prod_{x\in D} \chi_x(g_1^{-1}\gamma' g_1) = 1$. So we may assume that $\prod_{x\in D} \chi_x(g_1^{-1}\gamma' g_1) = 1$ for all $\gamma' \in \operatorname{Aut}_{D,H}(g_1)$

This implies that $\prod_{x\in|X|} f_x^W(g_2^{-1}\gamma g_1)$ is invariant under right multiplication of γ by elements of $\operatorname{Aut}_{D,H}(g_1)$. We can write $\sum_{\gamma\in G(F)}\prod_{x\in|X|} f_x^W(g_2^{-1}\gamma g_1)$ as a sum over orbits of this right multiplication action. Because the action is by multiplication in a group, its orbits are cosets of $\operatorname{Aut}_{D,H}(g_1)$, and so the size of each orbit is $|\operatorname{Aut}_{D,H}(g_1)|$, and by Lemma 9.11, the sum over each orbit is an element of $\mathbb{Z}[\mu_m]$ times $|\operatorname{Aut}_{D,H}(g_1)|$, so the final (finite) sum is divisible by $|\operatorname{Aut}_{D,H}(g_1)|$.

A symmetrical argument works for $\operatorname{Aut}_{D,H}(g_2)$, using left multiplication instead. □

Lemma 9.13. *The sum*

$$\frac{1}{|H(k)|^2}\sum_{g_1,g_2\in G(F)\backslash G(\mathbb{A}_F)/\mathbf{K}(D)}\frac{\overline{\mathsf{K}_{W_1}(g_1,g_2)}\mathsf{K}_{W_2}(g_1,g_2)}{|\operatorname{Aut}_D(g_1)||\operatorname{Aut}_D(g_2)|}$$

is an element of $\mathbb{Z}[\mu_m]$.

Proof. Break the sum into a sum over orbits under the action of $H(k)\times H(k)$ on $G(F)\backslash G(\mathbb{A}_F)/\mathbf{K}(D)\times G(F)\backslash G(\mathbb{A}_F)/\mathbf{K}(D)$ by right multiplication. It suffices to show that the sum over each orbit, divided by $|H(k)|^2$, lies in $\mathbb{Z}[\mu_m]$.

Because this action corresponds to right multiplication by $\prod_{x\in D} J_x$, it multiplies $\prod_{x\in|X|} f_x^W(g_1^{-1}\gamma g_2)$ by $\prod_{x\in D}\chi_x(h)$, so it multiplies $\mathsf{K}_{W_i}(g_1,g_2)$ by $\prod_{x\in D}\chi_x(h)$, which is a root of unity, so it fixes $\overline{\mathsf{K}_{W_1}(g_1,g_2)}\mathsf{K}_{W_2}(g_1,g_2)$. Hence the sum over each orbit is the size of that orbit times $\frac{\overline{\mathsf{K}_{W_1}(g_1,g_2)}\mathsf{K}_{W_2}(g_1,g_2)}{|\operatorname{Aut}_D(g_1)||\operatorname{Aut}_D(g_2)|}$ for some g_1,g_2 in that orbit. By the orbit-stabilizer theorem and Lemma 9.3, the size of the orbit is $\frac{|H(k)|^2|\operatorname{Aut}_D(g_1)||\operatorname{Aut}_D(g_2)|}{|\operatorname{Aut}_{D,H}(g_1)||\operatorname{Aut}_{D,H}(g_2)|}$. Hence the sum over the orbit, divided by $|H(k)|^2$, is $\frac{\overline{\mathsf{K}_{W_1}(g_1,g_2)}\mathsf{K}_{W_2}(g_1,g_2)}{|\operatorname{Aut}_{D,H}(g_1)||\operatorname{Aut}_{D,H}(g_2)|}$, which is an algebraic integer by Lemma 9.12. □

We use the convention (following [59, Def.10.1]) that Weil q-numbers are algebraic numbers whose absolute values are a power of $\sqrt{q}$ independent of the choice of complex embedding, while Weil q-integers are algebraic integers with the same property.

Lemma 9.14. *All the eigenvalues of* Frob_q *on* $H_c^i(U_{\overline{k}}\times U_{\overline{k}}, DK_{W_1}\otimes K_{W_2})$ *are Weil q-numbers.*

Proof. By Lemma 8.4 have

$$DK_{W_1}=\Delta_!^{W_1}\left(IC_{\mathcal{H}k_{G(D),W_1}}\boxtimes\mathcal{L}^{-1}\right)[\dim H]((\dim G)(g+|D|-1)+d(W_1)+\dim H).$$

Then if we form a Cartesian square

$$\begin{array}{ccc} Y & \xrightarrow{p_1} & \mathcal{H}k_{G(D),W_1}\times H \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle \Delta^{W_1}} \\ \mathcal{H}k_{G(D),W_2}\times H & \xrightarrow{\Delta^{W_2}} & U\times U \end{array}$$

By the projection formula, proper base change, and the projection formula again

$$\begin{aligned} &H_c^i(U_{\overline{k}}\times U_{\overline{k}}, DK_{W_1}\otimes K_{W_2})\\ &=H_c^{i+\dim H}(\mathcal{H}k_{G(D),W_2}\times H, \Delta^{W_2*}DK_{W_1}\otimes(IC_{\mathcal{H}k_{G(D),W_2}}\boxtimes\mathcal{L}))\\ &=H_c^{i+2\dim H}(\mathcal{H}k_{G(D),W_2}\times H, p_{2!}p_1^*(IC_{\mathcal{H}k_{G(D),W_1}}\boxtimes\mathcal{L}^{-1})\otimes(IC_{\mathcal{H}k_{G(D),W_2}}\boxtimes\mathcal{L}))\\ &=H_c^{i+2\dim H}(Y_{\overline{k}}, p_1^*(IC_{\mathcal{H}k_{G(D),W_1}}\boxtimes\mathcal{L}^{-1})\otimes p_2^*(IC_{\mathcal{H}k_{G(D),W_2}}\boxtimes\mathcal{L})). \end{aligned}$$

We can stratify $\mathcal{H}k_{G(D)}^{W_i}$ into strata, the inverse images of Schubert cells, on which the Kazhdan–Lusztig purity theorem implies that $IC_{\mathcal{H}k_{G(D),W_1}}$ is a shift of a Tate twist of a constant sheaf. By excision, it suffices to prove that all eigenvalues of Frobenius on the cohomology of the inverse images of these strata are q-Weil numbers. We can remove the $\mathcal{L}$ and $\mathcal{L}^{-1}$ terms by noting that these are summands of the pushforward of the constant sheaf along the Lang isogeny (Lemma 2.14), so the whole cohomology group is a summand of the cohomology of the inverse image of one of these strata under the Lang isogeny of $H \times H$. Because this is an algebraic stack, it follows from [59, Lem.10.2] that all eigenvalues of Frobenius on its cohomology are q-Weil numbers. ☐

Theorem 9.15. *There exists a natural number N, q-Weil integers $\alpha_1, \dots, \alpha_N$ of weight $\leq 2(\dim G)(g + |D| - 1) + d(W_1) + d(W_2) - 2\dim H$, and signs $\epsilon_1, \dots, \epsilon_N \in \{\pm 1\}$, such that for all n,*

$$\frac{1}{|H(\mathbb{F}_{q^n})|^2} \sum_{g_1, g_2 \in G(F_n)\backslash G(\mathbb{A}_{F_n})/\mathbf{K}(D)_n} \frac{\overline{\mathsf{K}_{W_1,n}(g_1,g_2)}\mathsf{K}_{W_2,n}(g_1,g_2)}{|\operatorname{Aut}_D(g_1)||\operatorname{Aut}_D(g_2)|} = \sum_{i=1}^{N} \epsilon_i \alpha_i^n .$$

Furthermore, we may arrange such that

- *$\alpha_1, \dots, \alpha_{\dim \operatorname{Hom}_{\overline{\mathbb{F}}_q}(K_{W_1}, K_{W_2})}$ are $q^{(\dim G)(g+|D|-1)+d(W_1)-\dim H}$ times the eigenvalues of* Frob_q *on* $\operatorname{Hom}_{\overline{\mathbb{F}}_q}(K_{W_1}, K_{W_2})$*, which are of weight $d(W_2) - d(W_1)$,*
- *$\epsilon_1, \dots, \epsilon_{\dim \operatorname{Hom}_{\overline{\mathbb{F}}_q}(K_{W_1}, K_{W_2})}$ are all equal to* 1*,*
- *α_i has weight $< 2(\dim G)(g+|D|-1)+d(W_1)+d(W_2)-2\dim H$ for $i > \dim \operatorname{Hom}_{\overline{\mathbb{F}}_q}(K_{W_1}, K_{W_2})$.*

Proof. Let S_n be the left-hand side of the formula. We apply Lemma 9.9 over $\mathbb{F}_{q^n}$. It is clear that base-changing all the data in this way is equivalent to base-changing $\mathcal{H}k_{G(D),W} \times H$ and thus to base-changing K_{W_1}, K_{W_2}, so we obtain

$$|H(\mathbb{F}_{q^n})|^2 \cdot S_n = (q^n)^{(\dim G)(g+|D|-1)+d(W_1)+\dim H} \sum_{i \in \mathbb{Z}} (-1)^i \operatorname{tr}(\operatorname{Frob}_{q^n}, H_c^i(U_{\overline{k}} \times U_{\overline{k}}, DK_{W_1} \otimes K_{W_2}))$$

By Lemma 8.3, K_{W_2} is pure of weight $w_1 = (\dim G)(g+|D|-1) + d(W_2) + \dim H$ and K_{W_1} is pure of weight $w_2 = (\dim G)(g+|D|-1)+d(W_1)+\dim H$, so $w_2 - w_1 = d(W_2) - d(W_1)$. Hence the eigenvalues of Frob_q on $H_c^i(U_{\overline{k}} \times U_{\overline{k}}, DK_{W_1} \otimes K_{W_2})$ are Weil numbers of weight $\leq d(W_2) - d(W_1) + i$.

By Lemma 2.22(2), this cohomology group vanishes for $i > 0$. Hence we can write $|H(\mathbb{F}_{q^n})|^2 \cdot S_n$ as a convergent signed sum of nth powers of Weil numbers, with the largest possible weight being

$$\begin{aligned} &2(\dim G)(g + |D| - 1) + 2d(W_1) + 2\dim H + d(W_2) - d(W_1) \\ &= 2(\dim G)(g + |D| - 1) + 2\dim H + d(W_1) + d(W_2), \end{aligned}$$

and appearing in H^0.

Now $|H(\mathbb{F}_{q^n})|$ is a finite signed sum of nth powers of Weil numbers, with the largest weight $2\dim H$ appearing with multiplicity 1 and sign 1, because H is smooth and geometrically connected. Hence $\frac{1}{|H(\mathbb{F}_{q^n})|^2}$ is a convergent signed sum of nth powers of Weil numbers, with the largest weight $-4\dim H$ appearing with multiplicity 1 and sign 1.

Hence the product $S_n = \frac{1}{|H(\mathbb{F}_{q^n})|^2} \cdot (|H(\mathbb{F}_{q^n})|^2 \cdot S_n)$ is also a convergent signed sum of nth powers of Weil numbers. By Lemma 2.23 this convergence is uniform in n. Thus the generating function $\sum_{n=1}^{\infty} u^n S_n$ is a signed sum of terms of the form $\frac{\alpha_i u}{1-\alpha_i u}$, with the α_i Weil numbers. In particular, it is a meromorphic function with poles of order 1 at inverses of Weil numbers α_i and with residues integer multiples of $1/\alpha_i$.

However, it is also a power series with coefficients in the ring of integers of a number field $\mathbb{Q}(\mu_m)$. A variant due to Dwork of a result of E. Borel implies that it is a rational function [23, Thm.3, p.645], so all but finitely many of the α_i occur with zero multiplicity, and we have the stated claim, except with q-Weil numbers rather than q-Weil integers. To check they are algebraic integers, it is sufficient to check that they are ℓ-adic integers for each prime ℓ. The ℓ-adic radius of convergence of this rational function is at least one, because all its coefficients are algebraic integers, so all its poles have ℓ-adic norm at least one, and the α_i are the inverses of its poles.

The maximum weight of the Weil numbers occurring is

$$\begin{aligned}&-4\dim H+2(\dim G)(g+|D|-1)+2\dim H+d(W_1)+d(W_2)\\&=2(\dim G)(g+|D|-1)-2\dim H+d(W_1)+d(W_2).\end{aligned}$$

A Weil number meets that bound only if it a Weil number from H^0 multiplied by the constant $q^{(\dim G)(g+|D|-1)+d(W_1)+\dim H}$ and then multiplied by $q^{-2\dim H}$. By Lemma 2.22(3), H^0 is isomorphic to $\mathrm{Hom}(K_1,K_2)$. Because K_1 and K_2 are pure, all eigenvalues on $\mathrm{Hom}(K_1,K_2)$ actually have size $q^{\frac{d(W_1)-d(W_2)}{2}}$, so a Weil number meets that bound if and only if it comes from H^0 in this way. Bringing these numbers to the front of the line we obtain the stated claim. ☐

10. q-aspect families

We continue with the set-up of Sections 8 and 9, that is G is a split semisimple group over k, D is an effective divisor on X, and $\mathcal{L}$ is a character sheaf on the factorizable subgroup H of $G\langle\mathcal{O}_D\rangle$. Suppose $k=\mathbb{F}_q$. We shall define the q-aspect family $\mathcal{V}=\mathcal{V}(G,X,D,H,\mathcal{L})$. This q-aspect family will be crucial in turning our bound for the trace of a Hecke operator into a bound for the individual Hecke eigenvalues.

For every $n\geq 1$, let $F_n:=F\otimes_{\mathbb{F}_q}\mathbb{F}_{q^n}$. We define $\mathcal{V}_n$ as consisting of automorphic representations Π of $G(\mathbb{A}_{F_n})$, that are $G(\mathfrak{o}_y)$-unramified for every $y\in|(X-D)_n|$, and at each place $y\in D_n$ lying over a place $x\in D$, and with residue field κ_y/κ_x, admit a vector on which the preimage $J_y\subset G(\kappa_y[\![t]\!])$ of $H(\kappa_y)$ acts by the character χ_y associated to the sheaf $\mathcal{L}$. The automorphic representations are counted with multiplicity, more precisely it is the product of the automorphic multiplicity of Π with the dimension of the space of (J_y,χ_y)-invariant vectors in Π_y for every $y\in D_n$.

10.1. **Spectral expansion of the trace.** For any $n\geq 1$, $\Pi\in\mathcal{V}_n$, and $y\in|(X-D)_n|$, the representation Π_y is $G(\mathfrak{o}_y)$-spherical. Recall from §2.2 that to every $G(\mathfrak{o}_y)$-unramified irreducible representation Π_y is attached a Satake parameter $t_{\Pi_y}\in\widehat{T}(\mathbb{C})/W$. For a dominant weight $\lambda\in\Lambda^+$, we have defined

$$\mathrm{tr}_\lambda(\Pi_y):=\mathrm{tr}(\Pi_y)(a_\lambda)=\mathrm{tr}(t_{\Pi_y}|V_\lambda).$$

For a function $W:|X|\to\Lambda^+$ of finite support $\mathrm{supp}(W)$ disjoint from D, let

$$\mathrm{tr}_W(\Pi):=\prod_{y\in\mathrm{supp}(W)_n}\mathrm{tr}_{W_y}(\Pi_y).$$

Proposition 10.1. *For every $W_1,W_2:|X|\to\Lambda^+$ with finite support disjoint from D, and for every $n\geq 1$,*

$$|H(\mathbb{F}_{q^n})|^2\sum_{\Pi\in\mathcal{V}_n}\mathrm{tr}_{W_1}(\Pi)\overline{\mathrm{tr}_{W_2}(\Pi)}=\frac{1}{(q^n)^{\frac{d(W_1)+d(W_2)}{2}}}\sum_{g_1,g_2\in G(F_n)\backslash G(\mathbb{A}_{F_n})/\mathbf{K}(D)}\frac{\mathsf{K}_{W_1}(g_1,g_2)\overline{\mathsf{K}_{W_2}(g_1,g_2)}}{|\mathrm{Aut}_D(g_1)||\mathrm{Aut}_D(g_2)|}$$

Proof. Recall Definition 9.6 of the test functions f_y^W for $y \in |X_n|$, and let $f := \prod_{y \in |X_n|} f_y^W$. By Definition 9.8, $\mathsf{K}_W(g_1, g_2)$ is the kernel of the convolution operator $*f$ on the vector space of all forms. Here, and below, we shall work with the counting measure on $G(\mathbb{A}_{F_n})/\mathbf{K}(D)$ when forming convolutions.

Since f_u^W is a cuspidal function for the place u, the operator $*f$ has image inside the space of cusp forms. More precisely, consider an orthonormal Hecke basis $\mathcal{B}_n = \{\varphi\}$ of the space of cuspidal automorphic forms on $G(F_n)\backslash G(\mathbb{A}_{F_n})/\mathbf{K}(D)$, where the inner product is

$$\frac{1}{|H(\mathbb{F}_{q^n})|} \sum_{g \in G(F_n)\backslash G(\mathbb{A}_{F_n})/\mathbf{K}(D)} \frac{|\varphi(g)|^2}{|\operatorname{Aut}_D(g)|}.$$

Since automorphic representations in $\mathcal{V}_n$ are counted with multiplicity, this implies that we can arrange the basis so that there is an injection $\mathcal{V}_n \hookrightarrow \mathcal{B}_n$, which we shall denote by $\Pi \mapsto \varphi_\Pi$. In other words, $\{\varphi_\Pi\}$ is a basis of the subspace of automorphic functions on $\operatorname{Bun}_{G(D)}(\mathbb{F}_q)$ which are $(\prod_{y \in D} J_y, \prod_{y \in D} \chi_y)$-equivariant. We can also arrange so that $\varphi * f = 0$ if $\varphi \in \mathcal{B}_n - \mathcal{V}_n$ (for this consider the case $W = 0$, in which case the operator $*f$ is idempotent, and its kernel forms an orthogonal complementary subspace).

The convolution operator $*f$ is an integral operator with kernel

$$\sum_{\varphi \in \mathcal{B}_n} (\varphi * f)(g_1)\overline{\varphi(g_2)} = \sum_{\Pi \in \mathcal{V}_n} (\varphi_\Pi * f)(g_1)\overline{\varphi_\Pi(g_2)}$$

We can show that we have $\varphi_\Pi * f = |H(\mathbb{F}_{q^n})|\, q^{\frac{d(W)}{2}} \operatorname{tr}_W(\Pi)\varphi_\Pi$. To do this, observe that for every $y \in \operatorname{supp}(W)_n$, $*f_y^W$ acts on the representation Π_y by scalar multiplication by $\operatorname{tr}_{W_y}(\Pi_y)\,(q^n)^{\langle W_y, \rho\rangle}$, and that for $y \in D$, $*f_y^W$ acts on φ_Π by a volume factor. Precisely,

$$\begin{aligned}
\sum_{g_2 \in G(\mathbb{A}_{F_n})/\mathbf{K}(D)} \prod_{y \in |X_n|} f_y^W(g_2^{-1}g_1)\varphi_\Pi(g_2) &= \frac{1}{\operatorname{vol}(\mathbf{K}(D))} \int_{g_2 \in G(\mathbb{A}_{F_n})} \prod_{y \in |X_n|} f_y^W(g_2^{-1}g_1)\varphi_\Pi(g_2) \\
&= \frac{1}{\operatorname{vol}(\mathbf{K}(D))} \prod_{y \in |X_n|} \int_{h \in G(F_y)} f_y^W(h)\varphi_\Pi(g_1 h^{-1}) \\
&= \frac{\prod_{y \in D} \operatorname{vol}(J_y)}{\operatorname{vol}(\mathbf{K}(D))} \cdot \prod_{y \in \operatorname{supp}(W)_n} \operatorname{tr}_{W_x}(\Pi_y)\,(q^n)^{\langle W_y, \rho\rangle} \cdot \varphi_\Pi(g_1),
\end{aligned}$$

and the ratio of volumes is equal to $|H(\mathbb{F}_{q^n})|$ by Definition 6.6. We deduce the identity

$$K_W(g_1, g_2) = |H(\mathbb{F}_{q^n})|\,(q^n)^{d(W)/2} \sum_{\Pi \in \mathcal{V}_n} \operatorname{tr}_W(\Pi)\varphi_\Pi(g_1)\overline{\varphi_\Pi(g_2)}.$$

The proposition now follows from orthogonality relations for the orthonormal basis $\mathcal{B}_n$. □

10.2. **Average Ramanujan bound.** We fix a place $v \in |X - D|$.

Theorem 10.2. *Let $\lambda \in \Lambda^+$ be a dominant weight. For every integer $n \geq 1$,*

$$\sum_{\Pi \in \mathcal{V}_n} \prod_{w|v} |\operatorname{tr}_\lambda(\Pi_w)|^2 \ll q^{n(\dim G(g+|D|-1) - \dim H)}.$$

The multiplicative constant is independent of n, it depends only on $X, v, \lambda, (G, D, H, \mathcal{L})$.

Proof. Let $W : |X| \to \Lambda^+$ be defined by

$$W_x := \begin{cases} \lambda, & \text{if } x = v, \\ 0, & \text{if } x \neq v. \end{cases}$$

Let K_W be the function defined in Definition 9.8. Then by Proposition 10.1 and Theorem 9.10

$$\sum_{\Pi \in \mathcal{V}_n} |\operatorname{tr}_W(\Pi)|^2 = \frac{1}{(q^n)^{d(W)} |H(\mathbb{F}_{q^n})|^2} \sum_{g_1, g_2 \in G(F_n) \backslash G(\mathbb{A}_{F_n}) / \mathbf{K}(D)} \frac{\mathsf{K}_W(g_1, g_2) \overline{\mathsf{K}_W(g_1, g_2)}}{|\operatorname{Aut}_D(g_1)||\operatorname{Aut}_D(g_2)|}$$

$$= \frac{O\left((q^n)^{(\dim G)(g+|D|-1)+d(W)+\dim H}\right)}{(q^n)^{d(W)} |H(\mathbb{F}_{q^n})|^2} = O\left((q^n)^{(\dim G)(g+|D|-1)-\dim H}\right). \qquad \square$$

Corollary 10.3. *Let $n \geq 1$, $\Pi \in \mathcal{V}_n$, and let λ be a dominant weight of G. Then*

$$\prod_{w|v} |\operatorname{tr}_\lambda(\Pi_w)|^2 \ll q^{n(\dim G(g+|D|-1)-\dim H)}$$

with the constant independent of n.

Proof. This follows from Theorem 10.2 because the left side is a sum of squares and hence any term is bounded by the whole. □

10.3. **Sums of Weil numbers.** In the course of the proof above we have shown that several spectral quantities are sums of Weil numbers. Such results are of independent interest, and we spell them out in more detail in this subsection.

Proposition 10.4. *There exist q-Weil integers α_i of weight $\leq 2(\dim G)(g + |D| - 1) - 2\dim H$, such that*

$$|\mathcal{V}_n| = \sum\nolimits_i \alpha_i^n, \quad \text{for every } n \geq 1.$$

Proof. This follows from Theorem 9.15 and Proposition 10.1, taking $W_1 = W_2 = 0$. In this case $d(W_1) = d(W_2) = 0$ so the factor of $q^{d(W_1)/2+d(W_2)/2}$ may be ignored. □

Proposition 10.5. *For every $W : |X| \to \Lambda^+$ of finite support disjoint from D, there exist q-Weil integers β_j of weight $\leq 2(\dim G)(g + |D| - 1) - 2\dim H + d(W)$, such that*

$$q^{n\frac{d(W)}{2}} \sum_{\Pi \in \mathcal{V}_n} \operatorname{tr}_W(\Pi) = \sum\nolimits_j \beta_j^n, \quad \text{for every } n \geq 1.$$

Proof. This follows from Theorem 9.15 and Proposition 10.1, taking $W_1 = W$ and $W_2 = 0$. □

10.4. **The main theorem.** To prove the main theorem, we shall embed the automorphic representation π of $G(\mathbb{A}_F)$ in a suitable automorphic family $(\mathcal{V}_n)_{n \geq 1}$ in the q-aspect: at the place u, we shall use the mgs datum, and at the other ramified places, we shall choose a datum with trivial character, and with sufficient depth that π and its base changes Π_n have a nonzero invariant vector.

Lemma 10.6. *Let G be a reductive group over a local field. There is a constant c such that for any two points x, y in the Bruhat–Tits building, for all depths r, the Moy–Prasad subgroup $G_{x,r}$ contains a conjugate of $G_{y,r+c}$. If G is split, we can take c to depend only on the root data of G and not on the base field.*

Proof. After conjugation, we may assume that x and y are contained in the same apartment. Define a metric on this apartment where the distance $d(x,y)$ is the max over all roots of the absolute value of the difference between the evaluations of the linear function associated to this root on x and y. Then by construction, it is clear that $G_{x,r}$ contains $G_{y,r+d(x,y)}$. Take c to be the supremum over pairs x,y of the minimum distance between x and any conjugate of y under the affine Weyl group action. Because this action is cocompact, a finite supremum in fact exists. Because the metric on the apartment and the affine Weyl group can be defined combinatorially, c depends only on the underlying root data. □

Lemma 10.7. *Let G be a split semisimple algebraic group. Let $F = \mathbb{F}_q(X)$. Let π be an automorphic representation of $G(\mathbb{A}_F)$, mgs at a place u, with Condition BC. Then there exists an effective divisor D on X, a subgroup $H \subseteq G\langle \mathcal{O}_D \rangle$, and a character sheaf $\mathcal{L}$ on H, that is geometrically supercuspidal on U, and such that for all n, the base change Π_n of π to F_n is contained in the associated family $\mathcal{V}_n$.*

Proof. By the definition of Condition BC, there exists mgs datum $(G_{\kappa_u}, m_u, H_u, \mathcal{L}_u)$ such that for all n, for all places u' of F_n lying over u with local field $E_{u'}$, $\Pi_{n,u'}$ is a quotient of $\text{c-Ind}_{J_{u,E}}^{G(E_{u'})} \chi_{u,E}$.

Let S be the set of ramified places of π other than u. Again by the definition of Condition BC, Π_n is unramified outside $S \cup \{u\}$, with a bound on the depth inside S. Let m be some integer greater than this bound on the depth plus the constant of Lemma 10.6. It follows that for all places x lying over a place in S, Π_n contains a vector invariant under the depth m subgroup of the standard hyperspecial maximal compact, which is the subgroup of elements of $G(\kappa_x[[t]])$ congruent to 1 mod t^m.

It follows that if we let D be the divisor of multiplicity m at each point of S and multiplicity m_u at u, $H = H_u$, and $\mathcal{L} = \mathcal{L}_u$, then $\Pi_n \in \mathcal{V}_n$ for all n.

Finally, $(G, D, H, \mathcal{L})$ is geometrically supercuspidal at u because $(G_{\kappa_u}, m_u, H_u, \mathcal{L}_u)$ is geometrically supercuspidal. □

To improve the bound of Corollary 10.3 for this family, and obtain the main theorem, we use a variant of the tensor power trick, where bounds for large n will imply stronger bounds for small n.

Theorem 10.8. *Let G be a split semisimple algebraic group. Assume the characteristic of F is not 2. Let π be an automorphic representation of $G(\mathbb{A}_F)$, mgs at a place u, and satisfying Condition BC. Let v be a place at which π is unramified for the standard hyperspecial maximal compact subgroup $G(\mathfrak{o}_v)$. Then π is tempered at v.*

Proof. Let $\lambda \in \Lambda^+$ be a dominant weight. We apply Corollary 10.3 to the family produced by Lemma 10.7 to obtain that

$$\prod_{w|v} |\operatorname{tr}_\lambda(\Pi_{n,w})|^2 \ll (q^n)^{(\dim G)(g+|D|-1)-\dim H}.$$

Let $n_0 := \gcd(n, [\kappa_v : k])$, and $n_1 := n/n_0$. All the places $w|v$ have isomorphic residue field κ_w, with $[\kappa_w : \kappa_v] = n_1$, and by the definition of base change, they have the same Satake parameter. So all of the n_0 terms in the above product are equal to each other, and we deduce

$$|\operatorname{tr}_\lambda(\Pi_{n,w})| \ll (q^{n_1})^{((\dim G)(g+|D|-1)-\dim H)/2}.$$

Let t_{π_v} be the Satake parameter of π_v. Then the Satake parameter of $\Pi_{n,w}$ is equal to $t_{\pi_v}^{n_1}$, hence $\operatorname{tr}_\lambda(\Pi_{n,w}) = \operatorname{tr}(t_{\pi_v}^{n_1}|V_\lambda)$. Because all $n_1 \geq 1$ arise for some n (specifically for $n = [\kappa_v : k]n_1$), Lemma 2.24 implies that we have the improved inequalities

$$|\operatorname{tr}(t_{\pi_v}^{n_1}|V_\lambda)| \leq \dim V_\lambda \cdot (q^{n_1})^{((\dim G)(g+|D|-1)-\dim H)/2}.$$

In particular for $n_1 = 1$,

$$|\operatorname{tr}_\lambda(\pi_v)| \leq \dim V_\lambda \cdot q^{((\dim G)(g+|D|-1)-\dim H)/2}.$$

Since the inequality holds for every $\lambda \in \Lambda^+$, we deduce by Proposition 2.7 that in fact $|\operatorname{tr}_\lambda(\pi_v)| \leq \dim V_\lambda$, and π_v is tempered. □

Remark 10.9. A close analogue of the argument may be found in the Bombieri–Stepanov proof of the Riemann hypothesis for curves over finite fields. Weil's proof for a curve C of genus g over $\mathbb{F}_q$ immediately proves in one stroke the Riemann bound $|\#C(\mathbb{F}_q) - q - 1| \leq 2g\sqrt{q}$. The proof of Bombieri–Stepanov, say in the special case of a Galois cover of $\mathbb{P}^1$, involves more steps. One first deduces an estimate $\#C(\mathbb{F}_q) \leq 1 + q + (2g+1)\sqrt{q}$, then by applying this bound to twists of C, obtains $\#C(\mathbb{F}_q) - q - 1 \geq 1 + q - O((2g+1))\sqrt{q})$, with a constant depending on the order of the Galois group. To improve the constant from $O(2g+1)$ to the correct value $2g$, it is necessary to use the rationality of the zeta function. From the estimate for $\#C(\mathbb{F}_{q^n})$ for n large, one deduces the sharp bound for the zeroes of the zeta function and thus a sharp bound for the number of points.

Our method closely follows the strategy of the last deduction. Instead of the zeroes of the zeta function, we are attempting to bound the eigenvalues of the Satake parameter. Instead of using the rationality of the zeta function, we use cyclic base change to compare the Satake eigenvalues for the base changed automorphic form to the Satake eigenvalues of the original form. The main difference is that, while the bound $(2g+1)\sqrt{q}$ is sufficient for most practical purposes, the constant factor which we amplify away is ineffective, and would render the estimate useless in the λ aspect if not dealt with.

Remark 10.10. We compare our use of the tensor power trick to Rankin's trick. In both cases, some special case of functoriality is used to amplify a weaker bound into a stronger one. The needed functoriality is rather weak in our case, where it is cyclic base change. However, our argument and Rankin's trick are different in one crucial respect, other than the different versions of functoriality applied. Rankin's trick produces an improvement in the dependence on q in the bound. Speaking geometrically, we may refer to it as an improvement of the weight. In our method, however, the weight is fixed as q varies (unsurprising as it arises geometrically as the weight of a cohomology group), and is not improved directly. Instead, we pass to the large q^n limit to handle a constant term independent of q.

10.5. **Hecke eigenvalues are Weil numbers.**

We establish the following strengthening of the previous Theorem 10.8. Assumptions are as before.

Theorem 10.11. *For every $\lambda \in \Lambda^+$, the trace $q^{\langle\lambda,\rho\rangle}\operatorname{tr}_\lambda(\pi_v)$ of the λ-Hecke operator is a sum of length $\dim(V_\lambda)$ of q-Weil integers of weight $\langle\lambda, 2\rho\rangle$.*

Proof. Hecke eigenvalues are algebraic numbers because of the finiteness of the support of cuspidal automorphic functions with prescribed local conditions. Next we will prove that the Hecke eigenvalues have size $q^{\langle\lambda,\rho\rangle}$ for every embedding of the coefficient field into $\mathbb{C}$. Every embedding comes from another automorphic form satisfying the same assumptions, possibly with a different mgs datum. Indeed the local mgs condition at u is preserved under $\operatorname{Aut}(\mathbb{C})$, and also the global Condition BC. Thus the previous Theorem 10.8 applies. Finally the integrality follows either from [43, Prop.2.1], or from Lemma 9.13 by varying $\lambda \in \Lambda^+$. □

Example 10.12. Consider the rigid automorphic sheaves constructed in [33, 66]. The Condition BC is satisfied because the trace function over each finite extension $\mathbb{F}_{q^n}$ defines an automorphic function that generates a corresponding automorphic representation (see Remark 6.11). We have seen in Section 3.5 that epipelagic representations are mgs. Thus Theorem 10.8 applies, and the temperedness is consistent with the results of *loc. cit.*, indeed the construction of ℓ-adic sheaves on $\mathbb{P}^1_{\setminus\{0,\infty\}}$ that generalize Kloosterman sums. The conclusion of Theorem 10.11 on integrality is also consistent with *loc. cit.*, precisely, it follows from [33, (5.8)], which explicates Kl^{V_λ} as an exponential sum, and because each of the Kummer, Artin–Schreier, and IC sheaves is integral. This is analogous to Lemma 9.13.

11. Relationship with Lafforgue–Langlands parameters and Arthur parameters

In this section, we will describe a potential approach to provide a different proof of the main theorem of this paper, using V. Lafforgue's Langlands parameterization, the Lafforgue–Genestier semisimplified local Langlands parameterization, and some conjectural explicit calculations with that parameterization. We will then express the same strategy, or a very similar strategy, in the language of Arthur parameters, and again without direct reference to parameters of any kind, using only the notion of two representations being in the same L-packet.

The starting point of all three approaches will be a guess about the Langlands parameters of mgs representations. We can verify this conjecture in the GL_r case, where the local Langlands correspondence is known by results of Laumon–Rapoport–Stuhler, and Henniart–Lemaire [34].

Proposition 11.1. *Let F_u be a non-archimedean local field and let π_u be a mgs representation of $\mathrm{GL}_r(F_u)$. Then its local Langlands parameter $\sigma_u : W_{F_u} \to \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell)$ is irreducible when restricted to the inertia group of F_u.*

Proof. For each unramified extension F'_u of F_u, let π'_u be the base change representation of π_u. It follows from [34, Prop.II.2.9], [34, Prop.II.5.15.2], and the orbital integral identity in Theorem 4.11 that π'_u is an mgs representation, with datum compatible with that of π_u. In particular π'_u is supercuspidal.

It is established in [34, Thm.IV.1.5] that the Langlands parameter of π'_u is the restriction of the Langlands parameter σ_u to $W_{F'_u}$. Since π'_u is supercuspidal, we have that σ_u restricts to an irreducible $W_{F'_u}$ representation.

Because $\sigma_u(I_{F_u})$ is a finite group, the action of $\sigma_u(\mathrm{Frob}_u)$ on it by conjugation has finite order m. Let F'_u be an unramified extension of F_u of degree m. Then $\sigma_u(W_{F'_u})$ is generated by $\sigma_u(I_{F_u})$ and the mth power of Frob_u, which commutes with it. Hence $\sigma_u(\mathrm{Frob}_u^m)$ lies in the center of $\sigma_u(W_{F'_u})$, which acts irreducibly, so $\sigma_u(\mathrm{Frob}_u^m)$ is a scalar, and hence $\sigma_u(I_{F_u})$ acts irreducibly, as desired. $\square$

To conjecturally apply this to general groups, and use it to verify Ramanujan, we use the work of V. Lafforgue and Genestier–Lafforgue on the Langlands correspondence over function fields, which we now review: Recall that D is an effective divisor on X, and $\mathbf{K}(D)$ is the compact subgroup of the adelic points $G(\mathbb{A}_F)$ of the split semisimple G consisting at each place of local sections of the group scheme congruent to the identity modulo D.

Lafforgue [44] defines a $\mathcal{C}_c(\mathbf{K}(D)\backslash G(\mathbb{A}_F)/\mathbf{K}(D), \overline{\mathbb{Q}}_\ell)$-module decomposition of $\mathcal{C}_c^{\mathrm{cusp}}(\mathrm{Bun}_{G(D)}(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell)$ indexed by continuous semisimple representations $\sigma : \mathrm{Gal}(\overline{F}/F) \to \widehat{G}(\overline{\mathbb{Q}}_\ell)$, unramified away from D. Since $\pi^{\mathbf{K}(D)}$ is irreducible and nonzero, it appears inside a module of this decomposition.

Letting ι be an embedding $\overline{\mathbb{Q}}_\ell \to \mathbb{C}$, we say a continuous representation of $\mathrm{Gal}(\overline{F}/F)$ is *ι-pure* of weight w if for each unramified place v, the image by ι of the eigenvalues of Frob_v on the representation are complex numbers of norm $|\kappa_v|^{\frac{w}{2}}$. We say that a representation is *ι-mixed* if it has a filtration whose associated graded components are ι-pure of increasing weights. All representations σ appearing in the above decomposition, composed with any representation of $\widehat{G}$, are ι-mixed. (In fact by [42] this is known for any representation, but it has a direct proof in this case.)

Genestier–Lafforgue [28] define for each local representation π_u a semisimple representation $\sigma_{\pi_u} : \mathrm{Gal}(\overline{F_u}/F_u) \to \widehat{G}(\overline{\mathbb{Q}}_\ell)$, which satisfies the following compatibility condition: Whenever $\pi^{\mathbf{K}(D)}$ appears as an irreducible $\mathcal{C}_c(\mathbf{K}(D)\backslash G(\mathbb{A}_F)/\mathbf{K}(D), \overline{\mathbb{Q}}_\ell)$-module inside the summand of $\mathcal{C}_c^{\mathrm{cusp}}(\mathrm{Bun}_{G(D)}(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell)$ indexed by a representation $\sigma : \mathrm{Gal}(\overline{F}/F) \to \widehat{G}(\overline{\mathbb{Q}}_\ell)$, the semisimplification of the restriction of σ to $\mathrm{Gal}(\overline{F_u}/F_u)$ is equal to σ_{π_u}.

The key conjecture, which is expected to generalize Proposition 11.1, is as follows. In the case of an epipelagic representation π_u, it is consistent with the conjectures of [53, §7.1], in which the assertion is expressed in the form $\widehat{\mathfrak{g}}^{\sigma_{\pi_u}(I_{F_u})} = 0$.

Conjecture 11.2. *For π_u a mgs representation, the image of the inertia subgroup I_{F_u} of $\mathrm{Gal}(\overline{F_u}/F_u)$ under the parameter σ_{π_u} is not contained in any proper parabolic subgroup of $\widehat{G}(\overline{\mathbb{Q}}_\ell)$.*

It follows from this conjecture that, if π is mgs at one place, then π is tempered at all unramified places. This follows from the below chain of reasoning, which depends on the Lemmas 11.3,11.4, and 11.5 immediately afterwards.

(1) Assume that π_u is mgs.

Then, under Conjecture 11.2:

(2) The image of $\mathrm{Gal}(\overline{F_u}/F_u)$ under σ_{π_u} is not contained in any proper parabolic subgroup of $\widehat{G}(\overline{\mathbb{Q}}_\ell)$.

Thus we deduce:

(3) The image of $\mathrm{Gal}(\overline{F}/F)$ under the Lafforgue–Langlands parameter σ of π is not contained in any proper parabolic subgroup of $\widehat{G}(\overline{\mathbb{Q}}_\ell)$.
(4) The composition of the Lafforgue–Langlands parameter σ of π with every representation of $\widehat{G}(\overline{\mathbb{Q}}_\ell)$ is pure of weight 0.
(5) π is tempered at every unramified place.

Indeed the implication (2) $\implies$ (3) is Lemma 11.3, then (3) $\implies$ (4) is Lemma 11.4, and Lemma 11.5 gives (4) $\implies$ (5).

Lemma 11.3. *Let $\sigma : \mathrm{Gal}(\overline{F}/F) \to \widehat{G}(\overline{\mathbb{Q}}_\ell)$ be a representation with image contained in a proper parabolic subgroup. For any place u of F, the image of the semisimplification of the restriction of σ to $\mathrm{Gal}(\overline{F_u}/F_u)$ is contained in a proper parabolic subgroup.*

Proof. That the property of being contained in a parabolic subgroup is stable under restriction is obvious. That it is preserved under semisimplification is immediate from the definition of semisimplification — we take a minimal parabolic subgroup containing the image of the representation, if any, and then project onto the Levi of that parabolic. Furthermore, the semisimplification is independent of which minimal parabolic we take. Thus, as long as some proper parabolic subgroup contains the image, some proper Levi subgroup contains the image of the semisimplification. □

Lemma 11.4. *Let $\sigma : \mathrm{Gal}(\overline{F}/F) \to \widehat{G}(\overline{\mathbb{Q}}_\ell)$ be a ι-mixed representation whose image is not contained in a proper parabolic subgroup. Then for every representation V of $\widehat{G}$, the composite $V(\sigma)$ is pure of weight zero.*

Proof. Because $V(\sigma)$ is ι-mixed, it has a canonical filtration into pure representations. The image of σ is contained in the stabilizer of this filtration inside $\widehat{G}$. We will show that either this stabilizer is a proper parabolic subgroup of $\widehat{G}$ or $V(\sigma)$ is pure of weight zero.

Let v be a place at which σ is unramified and let T be a torus containing the semisimplication Frob_v^{ss} of Frob_v. Then the generalized eigenspaces of Frob_v are sums of eigenspaces of T. For χ a character of T, let $\omega(\chi) = \log|\iota(\chi(\mathrm{Frob}_v^{ss}))|$. Then ω is a linear function on the weight lattice of T. Because each associated graded of the weight filtration is pure of increasing weight, the eigenvalues of Frob_v on each associated graded all have the same absolute value, so each associated graded of the weight filtration is a sum of eigenspaces of T where ω takes a fixed value, and this value of ω is increasing in the filtration. Thus an element preserves the weight filtration if and only if it sends eigenspaces of T to eigenspaces of T where ω takes equal or lower values on their weights.

This is exactly the subgroup of $\widehat{G}$ generated by all roots where ω takes a nonnegative value on their weights. This subgroup is parabolic unless it contains every root, in which case ω is zero on all roots, which because $\widehat{G}$ is semisimple implies it is zero on all characters of T, so the representation is pure of weight zero. ☐

Lemma 11.5. *Let π be a representation of $G(\mathbb{A}_F)$ such that $\pi^{\mathbf{K}(D)}$ is nonzero and appears inside the summand of $\mathcal{C}_c^{\mathrm{cusp}}(\mathrm{Bun}_{G(D)}(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell)$ indexed by a parameter σ such that $V(\sigma)$ is ι-pure of weight zero for every representation V of $\widehat{G}$. Then π is tempered at all unramified places.*

Proof. This follows from Proposition 2.7 and the compatibility between the action of $\mathcal{H}(G(F_v), G(\mathfrak{o}_v))$ on the summand of $\mathcal{C}_c^{\mathrm{cusp}}(\mathrm{Bun}_{G(D)}(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell)$ indexed by σ and the conjugacy class of $\sigma(\mathrm{Frob}_v)$. ☐

We now sketch two, more conjectural, analogues of this argument.

The first is based on Arthur parameters, and explains how we expect our main theorem can be related to Arthur's conjectures. We can, conditionally on different conjectures, prove that all representations π mgs at one place are tempered at every unramified place by a modified chain of deductions (1) $\implies$ (2) $\implies$ (3) $\implies$ (4') $\implies$ (5), where (4') is as follows.

(4') The image of SL_2 in every global Arthur parameter of π is trivial.

The implication (3) $\implies$ (4') depends on the conjectural existence of Arthur parameterizations compatible with Lafforgue's Langlands parameterization. Using this, the proof is similar to the proof of Lemma 11.4, but with a diagonal element in SL_2 replacing the Frobenius element. The implication (4') $\implies$ (5) is part of Arthur's conjectures on Arthur parameters. It is clear that if the conjectural relationship of Lafforgue–Langlands parameters with Arthur parameters could be proved, then this argument would be essentially the same as the previous argument.

The second analogue avoids mentioning parameters of any kind, except through their L-packets, and relies on conjectures only in terms of automorphic representations. Conditionally on conjectures, we can prove (1) $\implies$ (5) via a chain of implications (1) $\implies$ (2") $\implies$ (3") $\implies$ (4") $\implies$ (5), where (2"), (3"), (4") are as follows.

(2") All representations of $G(F_u)$ in the L-packet containing π_u are supercuspidal.
(3") All automorphic representations π' such that π_v and π'_v are in the same L-packet for every place v of F, are cuspidal.

(4") All automorphic representations π' such that $\pi_v \simeq \pi'_v$ for all but finitely many places v of F, are cuspidal.

The implication (4") $\implies$ (5) is consequence of the conjecture [26] that non-tempered cuspidal automorphic representations are CAP. The implications (2") $\implies$ (3") $\implies$ (4") are trivial, and the implication (1) $\implies$ (2") is a variant of Conjecture 11.2.

Our method of proof of the main result is also purely automorphic, and in some respects follows this last strategy. Indeed property (4") is necessary to construct a spectral set $\mathcal{V}$, prescribed by local behavior containing π_u, which is obtained by projection from an automorphic kernel $K(x,y)$ of compact support. See the related discussion in §1.1. Properties (2") and (3") appear implicitly in Condition BC, since the theory of base change and stabilization of trace formulas is related to the notion L-packet.

Remark 11.6. Many of the reverse implications are known or conjectured. In the Arthur parameter setting, (4') implies (3), since discrete series representations should have elliptic Arthur parameters, meaning that the Weil group and SL_2 aren't both contained in the same parabolic subgroup. The same statement is true in the Lafforgue–Langlands parameter setting, conditional on conjectural relationship with Arthur parameters. In every setting, (5) is known to imply (4) (resp. (4'), (4")). However (3) never implies (2) as cuspidality of an automorphic representation cannot imply local supercuspidality of its constituents. Hence it is not possible to prove the conjecture that (1) implies (2) as a corollary of our main result.

Finally, we include for comparison a proof of a part of a conjecture of Clozel [11, Conj.4(1)] in the function field case, obtainable unconditionally from the work of V. Lafforgue [44], which we mentioned in Remark 1.2 of the introduction.

Theorem 11.7. *Let G be a split semisimple group over a function field F and π a cuspidal automorphic representation of $G(\mathbb{A}_F)$. If π is tempered at one unramified place, then π is tempered at all unramified places.*

Proof. Choose some compact open subgroup $\mathbf{K}(D)$ which fixes a nonzero vector $f \in \pi^{\mathbf{K}(D)}$, where D is an effective divisor containing the ramified places of π. Viewing π as a subrepresentation of $L^2(G(F)\backslash G(\mathbb{A}_F))$, this vector defines a locally constant function f on $G(F)\backslash G(\mathbb{A}_F)/\mathbf{K}(D)$. Because π is cupsidal, f is compactly supported. Fix an isomorphism $\iota\colon \overline{\mathbb{Q}}_\ell \cong \mathbb{C}$. Lafforgue's theorem [44] gives a decomposition of $\mathcal{C}_c^{\mathrm{cusp}}(\mathrm{Bun}_{G(D)}(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell)$ indexed by continuous semisimple representations $\sigma\colon \mathrm{Gal}(\overline{F}/F) \to \widehat{G}(\overline{\mathbb{Q}}_\ell)$. Because f is nonzero, there must exist a parameter σ such that the projection of f onto the module indexed by σ is nonzero.

Similarly, we can choose a parameter σ' such that the projection of the complex conjugate $\overline{f}$ to the space indexed by σ' is nonzero.

Now for v an unramified place of π, and V any representation of $\widehat{G}$, because f is a $G(\mathfrak{o}_v)$-invariant vector in the representation space of π, it is an eigenfunction of the corresponding V-Hecke operator, with eigenvalue $\mathrm{tr}(t_{\pi_v}, V)$, where t_{π_v} is the Satake parameter of π_v. By [44], this eigenvalue coincides with $\mathrm{tr}(\mathrm{Frob}_v, V(\sigma))$. So we must have

$$\iota(\mathrm{tr}(\mathrm{Frob}_v, V(\sigma))) = \mathrm{tr}(t_{\pi_v}, V).$$

Similarly, we have $\iota(\mathrm{tr}(\mathrm{Frob}_v, V(\sigma'))) = \overline{\mathrm{tr}(t_{\pi_v}, V)}$. More strongly, the characteristic polynomials of Frob_v acting on $V(\sigma)$ is sent by ι to the characteristic polynomial of t_{π_v} acting on V, while the characteristic polynomial of Frob_v acting on $V(\sigma')$ is sent by ι to the complex conjugate polynomial. Thus $V(\sigma) \oplus V(\sigma')$ is ι-real in the sense that its characteristic polynomial of Frobenius has real coefficients (at every unramified place, under ι).

Now assume π_v is tempered for the given unramified place v. Then the Satake paramater t_{π_v} is unitary by Proposition 2.7, so by this previous identity of characteristic polynomials, all the eigenvalues of Frob_v on $V(\sigma)$ are sent by ι to complex numbers of norm 1. The same is true for their complex conjugates, the images under ι of the eigenvalues of Frob_v on $V(\sigma')$. We can now apply [37, Thm.4.1] to $V(\sigma) \oplus V(\sigma')$ — because it is ι-real and its eigenvalues of Frobenius at one place are complex numbers of norm 1, it follows that its eigenvalues of Frobenius at every place are complex numbers of norm 1. It follows at every other unramifed place w that the eigenvalues of the Satake parameter t_{π_w} on V have norm 1. □

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