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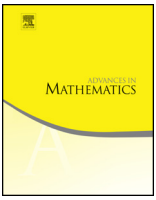

The Minkowski problem in Gaussian probability space ☆

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ABSTRACT

The Minkowski problem in Gaussian probability space is studied in this paper. In addition to providing an existence result on a Gaussian-volume-normalized version of this problem, the main goal of the current work is to provide uniqueness and existence results on the Gaussian Minkowski problem (with no normalization required).

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1. Introduction

The seminal work [66] by Minkowski in 1903 can be viewed as the starting point of the now vibrant Brunn-Minkowski theory in convex geometry. Minkowski's work relied on a remarkable result by Steiner in 1840 that combines the power of the usual operations in the Euclidean space with the usual Lebesgue measure in the space. The so-called *Steiner formula* states that in $\mathbb{R}^n$, the Lebesgue measure of $K + t \cdot B = \{x + ty : x \in K, y \in B\}$ where K is a convex body (compact convex set), B is the Euclidean unit ball and $t > 0$ is a polynomial of degree n in t with coefficients carrying essential geometric information regarding K. These coefficients are known as *quermassintegrals* and include volume, surface area, mean width and many more geometric invariants. Inequalities and their equality conditions involving these invariants can then be used to identify geometric shapes—perhaps the most well-known one is the isoperimetric inequality that identifies balls. These invariants, when being "differentiated", generate geometric measures that arguably carry more geometric information and at times all information as they can be used to uniquely recover the geometric shape. The celebrated Minkowski problem is one such example (perhaps the most well-known one). Minkowski asked if a given Borel measure μ on S^{n-1} can be used to reconstruct a convex body whose *surface area measure* is precisely the given measure μ and if the reconstruction is unique. Here, the surface area measure of K, denoted by S_K, is uniquely determined by the Aleksandrov's variational formula

$$\lim_{t\to 0} \frac{\mathcal{H}^n(K + tL) - \mathcal{H}^n(K)}{t} = \int_{S^{n-1}} h_L(v) dS_K(v) \tag{1.1}$$

where $h_L : S^{n-1} \to \mathbb{R}$ is the support function of L (see (2.1)). The influence of the Minkowski problem is widespread. In differential geometry, this is the problem of the prescription of Gauss curvature; in nonlinear PDE, it has the appearance of Monge-Ampère equation. For the last three decades, there have been many Minkowski-type problems, each involving a certain geometric measures generated by "differentiating" an invariant in Steiner's formula in a way such as in (1.1). These Minkowski problems can be understood as the problems of reconstructing convex bodies in manners specified by the geometric measures in question and each of them, when asked in the smooth category, reduces to a certain fully nonlinear elliptic PDE of varying natures. Some of the most prominent Minkowski-type problems include the L_p Minkowski problem (see [20,41,62]), the logarithmic Minkowski problem (see [13]), and the dual Minkowski problem (see [40]). We shall provide a short review of these problems shortly.

Perhaps of equal significance as the Lebesgue measure in $\mathbb{R}^n$ is the Gaussian probability measure γ_n given by

$$\gamma_n(E) = \frac{1}{(\sqrt{2\pi})^n} \int_E e^{-\frac{|x|^2}{2}} dx.$$

Unlike Lebesgue measure, Gaussian probability measure is neither translation invariant nor homogeneous. Moreover, the density decays exponentially fast as $|x| \to \infty$. The "surface area measure" in the Gaussian probability space is known as the *Gaussian surface area measure*, which was studied in, for example, Ball [3] and Nazarov [68]. In this paper, we will retrace the steps of Minkowski, Aleksandrov among many others and study the corresponding Minkowski problem in Gaussian probability space. As we will see shortly, the missing features such as translation invariance and homogeneity, along with exponential decay, causes the behavior of the *Gaussian Minkowski problem* to be quite mysterious and differs significantly from that of the Minkowski problem.

The following variational formula allows us to "differentiate" the Gaussian volume $\gamma_n(\cdot)$ on the set of convex bodies:

$$\lim_{t\to 0} \frac{\gamma_n(K+tL) - \gamma_n(K)}{t} = \int_{S^{n-1}} h_L dS_{\gamma_n,K},$$

for any convex bodies K and L containing the origin in their interiors. The proof will be given in Theorem 3.3. The uniquely determined Borel measure $S_{\gamma_n,K}$ is defined, in an equivalent way, in (3.1) and will be referred to as the *Gaussian surface area measure* of K for its corresponding role in Gauss probability space when compared to surface area measure in the Lebesgue measure space. When K is sufficiently smooth, its Gaussian surface area measure is absolutely continuous with respect to spherical Lebesgue measure:

$$dS_{\gamma_n,K}(v) = \frac{1}{(\sqrt{2\pi})^n} e^{-\frac{|\nabla h_K|^2 + h_K^2}{2}} \det(\nabla^2 h_K + hI), \tag{1.2}$$

where $h_K : S^{n-1} \to \mathbb{R}$ is the support function of K and ∇, ∇^2 are gradient and Hessian operators on S^{n-1} with respect to the standard metric.

It is therefore natural to wonder what measures can be used to reconstruct convex bodies in Gaussian probability space based on their Gaussian surface area measure and whether Gaussian surface area measure uniquely identifies the body.

The Gaussian Minkowski problem. Given a finite Borel measure μ, what are the necessary and sufficient conditions on μ so that there exists a convex body K with $o \in \operatorname{int} K$ such that

$$\mu = S_{\gamma_n,K}? \tag{1.3}$$

If K exists, to what extent is it unique?

Because of (1.2), when the given measure μ has a density $d\mu = f dv$, the Gaussian Minkowski problem reduces to solving the following Monge-Ampère type equation on S^{n-1},

$$\frac{1}{(\sqrt{2\pi})^n} e^{-\frac{|\nabla h|^2+h^2}{2}} \det(\nabla^2 h + hI) = f.$$

By the works of Ball [3] and Nazarov [68], it is simple to notice that the allowable μ in the Gaussian Minkowski problem cannot have an arbitrarily big total mass. In fact, the Gaussian surface area of any convex set in $\mathbb{R}^n$ is up to a constant bounded from above by $n^{\frac{1}{4}}$.

We will briefly discuss several other features that distinguish the Gaussian Minkowski problem from the Minkowski problem in Lebesgue measure space, which are what makes the problem more interesting and simultaneously more challenging.

To start, notice that when K is a centered ball of radius r, according to (1.2), the density of its Gaussian surface area measure is given by $f_r \equiv \frac{1}{(\sqrt{2\pi})^n} e^{-r^2/2} r^{n-1}$. Notice that $e^{-r^2/2} r^{n-1} \to 0$ both when r approaches 0 and ∞. Thus, in full generality, even when $\mu = cdv$ for some constant $c > 0$, the solutions to the Gaussian Minkowski problem are not unique. This is a result of the fact that the Gaussian probability space "thins" out exponentially as you travel away from the origin and therefore both larger and smaller convex bodies in $\mathbb{R}^n$ can have relatively small Gaussian surface area. However, as we will show in Section 4, when restricted to convex bodies with larger than 1/2 Gaussian volume, uniqueness part of the Gaussian Minkowski problem can be established.

Theorem 1.1. *Suppose K, L are two convex bodies in $\mathbb{R}^n$ that contain the origin in their interiors and K, L both solve the Gaussian Minkowski problem; i.e.,*

$$S_{\gamma_n, K} = S_{\gamma_n, L} = \mu.$$

If $\gamma_n(K), \gamma_n(L) \geq 1/2$, then $K = L$.

Our uniqueness result utilizes the Ehrhard's inequality with its equality condition, along with several of its consequences. Ehrhard's inequality is an isoperimetric inequality in the Gaussian probability space and implies that half-spaces, among all other sets of the same Gaussian volume, attain the least Gaussian surface area. As mentioned before, the Gaussian probability measure does not enjoy any homogeneity. As a result, there are many isoperimetric inequalities in the Gaussian probability space. Of particular interest is the dimensional Brunn-Minkowski inequality (for o-symmetric convex bodies) conjectured by Gardner-Zvavitch [27], with important contribution by Kolesnikov-Livshyts [47] followed by a recent confirmation by Eskenazis-Moschidis [23]. Gardner-Zvavitch [27] observed that this inequality neither implies nor is implied by Ehrhard's inequality. The dimensional Brunn-Minkowski inequality is also linked with the conjectured log-Brunn-Minkowski inequality (planar case established in [12])—an inequality in the Lebesgue measure space but with different addition—following a result by Livshyts-Marsiglietti-Nayar-Zvavitch [56], which was very recently extended in [39]. We also would like to mention the work of Borell [5].

Since Gaussian probability measure is not translation-invariant, the position of a convex body with respect to the origin is of critical importance. In terms of the existence part of the Gaussian Minkowski problem, we will restrict ourselves to the o-symmetric case; that is, when the given measure μ is an even measure and the solution set is the set of all o-symmetric convex bodies. As Example 7.1 in Appendix shows, even in this restricted case, the characterization of permissible measures μ is quite complicated. The situation is made worse by the lack of homogeneity in the Gaussian probability space (and therefore the Gaussian volume and surface area measure). Minkowski-type problems in which one deals with non-homogeneous geometric measures are typically known as of Orlicz type, which has their origin in the work [33] by Haberl-Lutwak-Yang-Zhang for the Orlicz Minkowski problem that generalizes both the classical Minkowski problem and the L_p Minkowski problem. See also [24,38,42,51,75,82] for additional results and contributions in the Orlicz extension of the classical Brunn-Minkowski theory. Very recently, Gardner-Hug-Weil-Xing-Ye extended the recently posed dual Minkowski problem to its Orlicz counterpart [25,26]. Inspired by their work, particularly the work [33] by Haberl-Lutwak-Yang-Zhang, we obtain the following normalized solution to the even Gaussian Minkowski problem.

Theorem 1.2. *Suppose μ is an finite even Borel measure not concentrated in any closed hemisphere. Then for each $0 < \alpha < \frac{1}{n}$, there exists an o-symmetric convex body K such that*

$$\mu = cS_{\gamma_n,K},$$

where

$$c = \frac{1}{\gamma_n(K)^{1-\alpha}}.$$

The proof of Theorem 1.2 is contained in Section 3.1, which is of variational nature. If the Gaussian surface area measure $S_{\gamma_n,K}$ was homogeneous in K, one would then be able to get rid of the constant c. This, however, is *far* from a simple procedure and as a matter of fact, to the best knowledge of the authors', solutions to Orlicz-Minkowski-type problem in the works mentioned above are all normalized solutions (meaning that there exists a constant c in the solutions). It is unclear whether the reconstruction process is unique by allowing such a constant c in the solution. Therefore, it is much desired to obtain a solution to the non-normalized version of the Gaussian Minkowski problem as stated in (1.3) where uniqueness to a certain extent is guaranteed by one of our results (Theorem 1.1). One of our main results in the current paper is a progress in this direction by obtaining an existence result of the Gaussian Minkowski problem (non-normalized, restricted to o-symmetric case) via a degree theory approach. In particular, we will show

Theorem 1.3 *(Existence of smooth solutions). Let* $0 < \alpha < 1$ *and* $f \in C^{2,\alpha}(S^{n-1})$ *be a positive even function with* $|f|_{L_1} < \frac{1}{\sqrt{2\pi}}$. *Then there exists a unique* $C^{4,\alpha}$ *o-symmetric* K *with* $\gamma_n(K) > 1/2$ *such that*

$$\frac{1}{(\sqrt{2\pi})^n} e^{-\frac{|\nabla h_K|^2 + h_K^2}{2}} \det(\nabla^2 h_K + h_K I) = f. \tag{1.4}$$

An approximation argument is then used to obtain an existence result (weak solution) for the Gaussian Minkowski problem.

Theorem 1.4. *Let* μ *be an even measure on* S^{n-1} *that is not concentrated in any subspace and* $|\mu| < \frac{1}{\sqrt{2\pi}}$. *Then there exists a unique origin-symmetric* K *with* $\gamma_n(K) > 1/2$ *such that*

$$S_{\gamma_n, K} = \mu.$$

Notice that Theorem 1.4 *trivially* implies that if μ is an even measure on S^{n-1} that is not concentrated in any subspace, there are infinitely many pairs of c and K such that $\mu = cS_{\gamma_n, K}$.

The constant $\frac{1}{\sqrt{2\pi}}$ in both Theorems 1.3 and 1.4 comes from the fact that convex bodies with their Gaussian volume equal to $1/2$ must have their Gaussian surface area larger than or equal to $\frac{1}{\sqrt{2\pi}}$. See Corollary 4.6 for detail. It is also worth mentioning that the supremum of the Gaussian surface area of any convex body in $\mathbb{R}^n$ is asymptotically on the order of $n^{\frac{1}{4}}$. This fact is due to Ball [3] and Nazarov [68]. See Livshyts [54] for further generalizations.

The author would like to point out that *a differently normalized version* of the Gaussian Minkowski problem is a special case of the general dual Orlicz-Minkowski problem considered in [25,26]. However, it is important to note that none of the main theorems in the current paper overlap with those presented there. One should also note that the Minkowski problem in measurable spaces whose densities possess certain homogeneity and concavity has been previously considered in Livshyts [55].

Before ending this section, a short review of the aforementioned Minkowski-type problems in $\mathbb{R}^n$ with Lebesgue measure will be provided given their relevance to the current work and their importance in convex geometry. However, the more eager readers should feel free to skip it and jump to Section 2.

Nine decades after Minkowski's seminal work [66], in the early 1990s, Lutwak [62,63] laid the foundation to the now fruitful L_p Brunn-Minkowski theory. Due to limit of space, we mention only a selection of beautiful results in this area, [10,12,35,36,58,60, 64,65,67,69,73,74,76] and refer the interested readers to Schneider's book [70] for more details. In [62], Lutwak introduced the L_p surface area measure which is the counterpart of the classical surface area measure in the L_p theory and posed the corresponding L_p Minkowski problem. When $p = 1$, the L_p Minkowski problem is precisely the classical Minkowski problem Lutwak himself solved the problem when $p > 1$ in the *o*-symmetric

case whereas the more general case (non-symmetric) was settled by [20,41]. The L_p Minkowski problem when $p < 1$ is much more complicated and contains *challenging* problems such as the logarithmic Minkowski problem ($p = 0$) and the centro-affine Minkowski problem ($p = -n$).

The logarithmic Minkowski problem characterizes cone volume measure which has been the central topic in a number of recent works. When the given data is even, the existence of solutions to the logarithmic Minkowski problem was completely solved in Böröczky-Lutwak-Yang-Zhang [13]. In the general case (non-even case), important contributions were made by Zhu [80], and later by Böröczky-Hegedűs-Zhu [9] and Chen-Li-Zhu [18]. The logarithmic Minkowski problem has strong connections with isotropic measures (Böröczky-Lutwak-Yang-Zhang [14]) and curvature flows (Andrews [1,2]).

The centro-affine Minkowski problem characterizes the centro-affine surface area measure whose density in the smooth case is the centro-affine Gauss curvature. The characterization problem, in this case, is the centro-affine Minkowski problem posed in Chou-Wang [20]. See also Jian-Lu-Zhu [43], Lu-Wang [57], Zhu [81], *etc.*, on this problem.

The readers are also referred to [4,17] for some recent development of the L_p Minkowski problem when $p < 1$.

Recently, there has been some crucial progress towards the uniqueness part of the L_p Minkowski problem when $0 < p < 1$ within the class of smooth o-symmetric convex bodies. In particular, a local uniqueness result was given in [48]. Using PDE methods, this was later extended to a global uniqueness result in [15].

The Minkowski problem and the L_p Minkowski problem are within the framework introduced by Brunn and Minkowski. A parallel theory, which is known as the dual Brunn-Minkowski theory, was introduced by Lutwak (see Schneider [70]) in the 1970s. The dual Brunn-Minkowski theory has been most effective in answering questions related to intersections. One major triumph of the dual Brunn-Minkowski theory is tackling the famous Busemann-Petty problem, see Gardner [28], Gardner-Koldobsky-Schlumprecht [29], Koldobsky [44–46], Lutwak [61], and Zhang [77]. The dual theory makes extensive use of techniques from harmonic analysis. Recently, the dual Brunn-Minkowski theory took a huge step forward when Huang-Lutwak-Yang-Zhang [40] discovered the family of fundamental geometric measures—called dual curvature measures—in the dual theory. The dual Minkowski problem is the problem of prescribing dual curvature measures. The dual Minkowski problem introduces intrinsic PDEs—something long missing—to the dual Brunn-Minkowski theory. The dual Minkowski problem, while still largely open, has been solved in the o-symmetric case when the associated index q satisfies $q \in [0, n]$, see, for example [8,11,16,37,40,52,78,79].

The current work is along the lines of the classical problem raised by Minkowski, but now considered in the Gaussian probability space rather than the Lebesgue measure space. The lack of translation-invariance and homogeneity in the Gaussian probability space creates many challenges not encountered in the classical Minkowski problem.

2. Preliminaries

Some basics, as well as notations, regarding convex bodies will be provided in this section. For a general reference on the theory of convex bodies, the readers are referred to the book [70] by Schneider.

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space. The unit sphere in $\mathbb{R}^n$ is denoted by S^{n-1}. We will write $C(S^{n-1})$ for the space of continuous functions on S^{n-1}. We will use the subscript e for even functions and the superscript $+$ for positive function so that $C_e^+(S^{n-1})$ is used to denote the set of all even positive functions on S^{n-1}. For a Borel measure μ in a measure space, we will use $|\mu|$ for its total measure.

A convex body in $\mathbb{R}^n$ is a compact convex set with nonempty interior. The boundary of K is written as ∂K. Denote by $\mathcal{K}_0^n$ the class of convex bodies that contain the origin in their interiors in $\mathbb{R}^n$ and by $\mathcal{K}_e^n$ the class of origin-symmetric convex bodies in $\mathbb{R}^n$.

Let K be a compact convex subset of $\mathbb{R}^n$. The support function h_K of K is defined by

$$h_K(y) = \max\{x \cdot y : x \in K\}, \quad y \in \mathbb{R}^n. \tag{2.1}$$

The support function h_K is a continuous function homogeneous of degree 1. Suppose K contains the origin in its interior. The radial function ρ_K is defined by

$$\rho_K(x) = \max\{\lambda : \lambda x \in K\}, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

The radial function ρ_K is a continuous function homogeneous of degree -1. It is not hard to see that $\rho_K(u)u \in \partial K$ for all $u \in S^{n-1}$.

For each $f \in C^+(S^{n-1})$, the Wulff shape $[f]$ generated by f is the convex body defined by

$$[f] = \{x \in \mathbb{R}^n : x \cdot v \le f(v), \text{ for all } v \in S^{n-1}\}.$$

It is apparent that $h_{[f]} \le f$ and $[h_K] = K$ for each $K \in \mathcal{K}_0^n$.

Suppose K_i is a sequence of convex bodies in $\mathbb{R}^n$. We say K_i converges to a compact convex subset $K \subset \mathbb{R}^n$ in Hausdorff metric if

$$\max\{|h_{K_i}(v) - h_K(v)| : v \in S^{n-1}\} \to 0, \tag{2.2}$$

as $i \to \infty$. If K contains the origin in its interior, equation (2.2) implies

$$\max\{|\rho_{K_i}(u) - \rho_K(u)| : u \in S^{n-1}\} \to 0,$$

as $i \to \infty$.

For a compact convex subset K in $\mathbb{R}^n$ and $v \in S^{n-1}$, the supporting hyperplane $H(K, v)$ of K at v is given by

$$H(K,v) = \{x \in K : x \cdot v = h_K(v)\}.$$

By its definition, the supporting hyperplane $H(K,v)$ is non-empty and contains only boundary points of K. For $x \in H(K,v)$, we say v is an outer unit normal of K at $x \in \partial K$.

Since K is convex, for $\mathcal{H}^{n-1}$ almost all $x \in \partial K$, the outer unit normal of K at x is unique. In this case, we use ν_K to denote the Gauss map that takes $x \in \partial K$ to its unique outer unit normal. Therefore, the map ν_K is almost everywhere defined on ∂K. We use ν_K^{-1} to denote the inverse Gauss map. Since K is not assumed to be strictly convex, the map ν_K^{-1} is set-valued map and for each set $\eta \subset S^{n-1}$, we have

$$\nu_K^{-1}(\eta) = \{x \in \partial K : \text{there exists } v \in \eta \text{ such that } v \text{ is an outer unit normal at } x\}.$$

Occasionally, for simplicity, we will sometimes use the following renormalization of the Gauss and inverse Gauss map.

For those $u \in S^{n-1}$ such that ν_K is well-defined at $\rho_K(u)u \in \partial K$, we write $\alpha_K(u)$ for $\nu_K(\rho_K(u)u)$.

Let $\eta \subset S^{n-1}$ be a Borel set. The reverse radial Gauss image of K, denoted by $\alpha_K^*(\eta)$, is defined to be the set of all radial directions such that the corresponding boundary points have at least one outer unit normal in η, i.e.,

$$\alpha_K^*(\eta) = \{u \in S^{n-1} : v \cdot u\rho_K(u) = h_K(v) \text{ for some } v \in \eta\}.$$

When $\eta = \{v\}$ is a singleton, we usually write $\alpha_K^*(v)$ instead of the more cumbersome notation $\alpha_K^*(\{v\})$. It follows from Theorem 2.2.11 in [70] that for $\mathcal{H}^{n-1}$ almost all $v \in S^{n-1}$, the set $\alpha_K^*(\eta)$ contains only a singleton. Thus, we will sometimes treat α_K^* as an almost everywhere defined map on S^{n-1} when no confusion arises.

We recall that by Lemma 2.2 in [40] that if K_i converges to $K_0 \in \mathcal{K}_o^n$ in Hausdorff metric, then α_{K_i} converges to α_{K_0} almost everywhere on S^{n-1} with respect spherical Lebesgue measure.

3. Gaussian surface area measure and the Gaussian Minkowski problem

The purpose of this section is to introduce Gaussian surface area measure and the Gaussian Minkowski problem, and prove some basic properties as well as basic statements made in the Introduction but were not proved there. The authors would like to point out that these definitions and properties have already appeared in previous literatures (for example, [25,26]) and are only included in this paper for the sake of completeness. With the exception of Theorem 3.5, we take no credit for the other results presented in this section.

We define the following Borel measure on S^{n-1} and refer to it as *Gaussian surface area measure*.

Definition 3.1. Let $K \in \mathcal{K}_o^n$. The Gaussian surface area measure of K, denoted by $S_{\gamma_n,K}$, is a Borel measure on S^{n-1} given by

$$S_{\gamma_n,K}(\eta) = \frac{1}{(\sqrt{2\pi})^n} \int\limits_{\nu_K^{-1}(\eta)} e^{-\frac{|x|^2}{2}} d\mathcal{H}^{n-1}(x), \tag{3.1}$$

for each Borel measurable $\eta \subset S^{n-1}$.

We will need the following lemma.

Lemma 3.2. *Let $K \in \mathcal{K}_o^n$ and $f \in C(S^{n-1})$. Suppose $\delta > 0$ is sufficiently small so that for each $t \in (-\delta, \delta)$, we have*

$$h_t = h_K + tf > 0.$$

Then,

$$\lim_{t\to 0} \frac{\rho_{[h_t]}(u) - \rho_K(u)}{t} = \frac{f(\alpha_K(u))}{h_K(\alpha_K(u))} \rho_K(u)$$

for almost all $u \in S^{n-1}$ with respect to spherical Lebesgue measure. Moreover, there exists $M > 0$, such that

$$|\rho_{[h_t]}(u) - \rho_K(u)| < M|t|,$$

for all $u \in S^{n-1}$ and $t \in (-\delta, \delta)$.

Proof. The desired result follows immediately from Lemmas 2.8 and 4.1 in [40] and also the fact that

$$\log h_t = \log h_K + t\frac{f}{h_K} + o(t). \quad \square$$

The following variational formula gives rise to the corresponding surface area measure in the Gaussian probability space and therefore justifies the name *Gaussian surface area measure*. The proof is an adaptation of the variational formula obtained in [40].

Theorem 3.3. *Let $K \in \mathcal{K}_o^n$ and $f \in C(S^{n-1})$. Then,*

$$\lim_{t\to 0} \frac{\gamma_n([h_K + tf]) - \gamma_n(K)}{t} = \int\limits_{S^{n-1}} f dS_{\gamma_n,K}. \tag{3.2}$$

Proof. Write $h_t = h_K + tf$. Using polar coordinates, we have

$$\gamma_n([h_t]) = \frac{1}{(\sqrt{2\pi})^n} \int_{S^{n-1}} \int_0^{\rho_{[h_t]}(u)} e^{-\frac{r^2}{2}} r^{n-1} dr du.$$

Since $K \in \mathcal{K}_o^n$ and $f \in C(S^{n-1})$, for t close to 0, there exists $M_1 > 0$ such that $[h_t] \subset M_1 B$. Denote $F(s) = \int_0^s e^{-\frac{r^2}{2}} r^{n-1} dr$. By mean value theorem,

$$|F(\rho_{[h_t]}(u)) - F(\rho_K(u))| \leq |F'(\theta)||\rho_{[h_t]}(u) - \rho_K(u)| < M|F'(\theta)||t|,$$

where M comes from Lemma 3.2 and θ is between $\rho_{[h_t]}(u)$ and $\rho_K(u)$. Since $[h_t] \subset M_1 B$, we have $\theta \in (0, M_1]$. Therefore, by definition of F, we have $|F'(\theta)|$ is bounded from above by some constant that depends on M_1. Therefore, there exists $M_2 > 0$ such that

$$|F(\rho_{[h_t]}(u)) - F(\rho_K(u))| \leq M_2|t|.$$

Using dominated convergence theorem, together with Lemma 3.2, we have

$$\begin{aligned}
\lim_{t\to 0} \frac{\gamma_n([h_K + tf]) - \gamma_n(K)}{t} &= \frac{1}{(\sqrt{2\pi})^n} \int_{S^{n-1}} f(\alpha_K(u)) e^{-\frac{\rho_K(u)^2}{2}} \frac{\rho_K(u)^n}{h_K(\alpha_K(u))} du \\
&= \frac{1}{(\sqrt{2\pi})^n} \int_{\partial K} f(\nu_K(x)) e^{-\frac{|x|^2}{2}} d\mathcal{H}^{n-1}(x) \\
&= \frac{1}{(\sqrt{2\pi})^n} \int_{S^{n-1}} f dS_{\gamma_n, K}. \quad \square
\end{aligned}$$

The Gaussian surface area measure is weakly convergent with respect to Hausdorff metric.

Theorem 3.4. *Let $K_i \in \mathcal{K}_o^n$ such that K_i converges to $K_0 \in \mathcal{K}_o^n$ in Hausdorff metric. Then S_{γ_n, K_i} converges to S_{γ_n, K_0} weakly.*

Proof. Note that since $K_0 \in \mathcal{K}_o^n$, there exists $C > 0$ such that $\frac{1}{C} B \subset K_i \subset CB$ for sufficiently large i. Therefore, we have

$$e^{-\frac{\rho_{K_i}(u)^2}{2}} \rho_{K_i}^{n-1}(u) \to e^{-\frac{\rho_{K_0}(u)^2}{2}} \rho_{K_0}^{n-1}(u) \qquad \text{uniformly on } S^{n-1}. \tag{3.3}$$

Let $g \in C(S^{n-1})$. Since $\alpha_{K_i} \to \alpha_{K_0}$ almost everywhere on S^{n-1} with respect to spherical Lebesgue measure, we also have

$$\frac{g(\alpha_{K_i}(u))}{u \cdot \alpha_{K_i}(u)} \to \frac{g(\alpha_{K_0}(u))}{u \cdot \alpha_{K_0}(u)} \qquad \text{almost everywhere on } S^{n-1}. \tag{3.4}$$

By definition of $S_{\gamma_n,K}$, we have

$$\begin{aligned}\int_{S^{n-1}} g dS_{\gamma_n,K_i} &= \frac{1}{(\sqrt{2\pi})^n}\int_{\partial K_i} g(\nu_{K_i}(x))e^{-\frac{|x|^2}{2}}d\mathcal{H}^{n-1}(x)\\ &= \frac{1}{(\sqrt{2\pi})^n}\int_{S^{n-1}} g(\alpha_{K_i}(u))e^{-\frac{\rho_{K_i}(u)^2}{2}}\frac{\rho_{K_i}(u)^n}{h_{K_i}(\alpha_{K_i}(u))}du\\ &= \frac{1}{(\sqrt{2\pi})^n}\int_{S^{n-1}} \frac{g(\alpha_{K_i}(u))}{u\cdot\alpha_{K_i}(u)}e^{-\frac{\rho_{K_i}(u)^2}{2}}\rho_{K_i}(u)^{n-1}du\\ &\to \frac{1}{(\sqrt{2\pi})^n}\int_{S^{n-1}} \frac{g(\alpha_{K_0}(u))}{u\cdot\alpha_{K_0}(u)}e^{-\frac{\rho_{K_0}(u)^2}{2}}\rho_{K_0}(u)^{n-1}du\\ &= \int_{S^{n-1}} g dS_{\gamma_n,K_0},\end{aligned}$$

where the limit is due to (3.3) and (3.4). □

By a simple calculation, it follows from the definition of Gaussian surface area measure that if $K\in\mathcal{K}_o^n$ is convex, then $S_{\gamma_n,K}$ is absolutely continuous with respect to surface area measure and

$$dS_{\gamma_n,K} = \frac{1}{(\sqrt{2\pi})^n}e^{-\frac{|\nabla h_K|^2+h_K^2}{2}}dS_K.$$

If, in addition, the body K is C^2, then $S_{\gamma_n,K}$ is absolutely continuous with respect to spherical Lebesgue measure and

$$dS_{\gamma_n,K}(v) = \frac{1}{(\sqrt{2\pi})^n}e^{-\frac{|\nabla h_K|^2+h_K^2}{2}}\det(\nabla^2 h_K + h_K I)dv. \tag{3.5}$$

When $P\in\mathcal{K}_o^n$ is a polytope with unit normal vectors v_i with the corresponding faces F_i, the Gaussian surface area measure $S_{\gamma_n,P}$ is a discrete measure given by

$$S_{\gamma_n,P}(\cdot) = \sum_{i=1}^N c_i\delta_{v_i}(\cdot),$$

where c_i is given by

$$c_i = \frac{1}{(\sqrt{2\pi})^n}\int_{F_i} e^{-\frac{|x|^2}{2}}d\mathcal{H}^{n-1}(x).$$

The classical Minkowski problem asks for the existence, uniqueness and regularity of a convex body K whose surface area measure is prescribed. It has played a fundamental

role, not only in convex geometric analysis, but also in PDE, differential geometry, functional analysis. Given this, it is natural to study the corresponding problem for Gaussian surface area measure, which we refer to as *the Gaussian Minkowski problem.*

The Gaussian Minkowski problem. Given a finite Borel measure μ, what are the necessary and sufficient conditions on μ so that there exists a convex body K with $o \in \operatorname{int} K$ such that

$$\mu = S_{\lambda_n, K}?$$

If K exists, to what extent is it unique?

It follows from (3.5) that if $\mu = f dv$, then the Gaussian Minkowski problem is equivalent to the study of the following Monge-Ampère type equation on S^{n-1}:

$$\frac{1}{(\sqrt{2\pi})^n} e^{-\frac{|\nabla h|^2 + h^2}{2}} \det(\nabla^2 h + hI) = f.$$

It is well-known that the classical surface area measure S_K when viewed as a map from the set of convex bodies to the set of Borel measures on S^{n-1} is a valuation. In fact, Haberl-Parapatits [34] gave a valuation characterization of surface area measure. Valuation theory plays an important role in convex geometry, see, *e.g.*, [32,59,60,71]. Similar to the proof for that of surface area measure, it is not hard to see that Gaussian surface area measure is also a valuation; that is, if K and L are two convex bodies such that $K \cup L$ is also a convex body, then

$$S_{\gamma_n, K \cup L} + S_{\gamma_n, K \cap L} = S_{\gamma_n, K} + S_{\gamma_n, L}.$$

It is of great interest to see if there is a valuation characterization of Gaussian surface area measure.

3.1. The normalized problem and its solution

Motivated by the work of Haberl-Lutwak-Yang-Zhang [33], we derive the solution to the following normalized version of the even Gaussian Minkowski problem.

Theorem 3.5. *Suppose μ is an finite even Borel measure not concentrated in any closed hemisphere. Then, for each $0 < \alpha < \frac{1}{n}$, there exists an o-symmetric convex body K such that*

$$\mu = \frac{S_{\gamma_n, K}}{\gamma_n(K)^{1-\alpha}}. \tag{3.6}$$

Our approach to the normalized problem is variational and involves the following optimization problem:

$$\sup\{\Gamma(f) : f \in C_e^+(S^{n-1})\}, \tag{3.7}$$

where $\Gamma : C^+(S^{n-1}) \to \mathbb{R}$ is given by

$$\Gamma(f) = \frac{1}{\alpha}\gamma_n([f])^\alpha - \int_{S^{n-1}} f d\mu.$$

Lemma 3.6. *Let $0 < \alpha < \frac{1}{n}$. If an even function f_0 is a maximizer to the optimization problem (3.7), then f_0 must be the support function of an o-symmetric convex body; that is, there exists an o-symmetric convex body K_0 such that $f_0 = h_{K_0}$. Moreover, K_0 satisfies (3.6).*

Proof. Note that for each $f \in C_e^+(S^{n-1})$, by the definition of Wulff shape, we have

$$\Gamma(f) \geq \Gamma(h_{[f]}).$$

Therefore, the maximizer f_0 must be the support function of some o-symmetric convex body K_0.

We now use the variational formula (3.2) to establish that K_0 satisfies (3.6).

Towards this end, for each $g \in C_e^+(S^{n-1})$, consider the one-parameter family

$$K_t = [h_{K_0} + tg].$$

Since $f_0 = h_{K_0}$ is a maximizer, we have

$$0 = \frac{d}{dt}\Big|_{t=0} \Gamma(h_{K_t}) = \gamma_n(K_0)^{\alpha-1} \int_{S^{n-1}} g dS_{\gamma_n, K_0} - \int_{S^{n-1}} g d\mu,$$

where in the second equality, we used the variational formula (3.2). Note that the above equation holds for every $g \in C_e^+(S^{n-1})$. Therefore, we conclude that K_0 satisfies (3.6). □

For simplicity, when no confusion arises, we will write $\Gamma(K)$ in place of $\Gamma(h_K)$.

Lemma 3.7. *Let $0 < \alpha < \frac{1}{n}$. For sufficiently small $r > 0$, we have $\Gamma(rB) > 0$.*

Proof. By definition of Γ, we have

$$\begin{aligned}\Gamma(rB) &= \frac{1}{\alpha}\left(\frac{1}{(2\pi)^{\frac{n}{2}}}\int\limits_{rB} e^{-\frac{|x|^2}{2}}dx\right)^{\alpha} - \int\limits_{S^{n-1}} r d\mu \\ &= \frac{1}{\alpha}\left(\frac{n\omega_n}{(2\pi)^{\frac{n}{2}}}\int\limits_0^r e^{-\frac{t^2}{2}}t^{n-1}dt\right)^{\alpha} - r|\mu| \\ &= r\left[\frac{1}{\alpha}\left(\frac{n\omega_n}{(2\pi)^{\frac{n}{2}}}\right)^{\alpha}\left(\frac{\int_0^r e^{-\frac{t^2}{2}}t^{n-1}dt}{r^{\frac{1}{\alpha}}}\right)^{\alpha} - |\mu|\right].\end{aligned}$$

It follows from simple computation that when $0 < \alpha < \frac{1}{n}$, we have

$$\lim_{r\to 0^+}\frac{\int_0^r e^{-\frac{t^2}{2}}t^{n-1}dt}{r^{\frac{1}{\alpha}}} = \lim_{r\to 0^+}\frac{\alpha e^{-\frac{r^2}{2}}r^{n-1}}{r^{\frac{1}{\alpha}-1}} = \alpha\lim_{r\to 0^+} r^{n-\frac{1}{\alpha}} = \infty.$$

The desired result is therefore established. □

We are now ready to give a proof to Theorem 3.5.

Proof of Theorem 3.5. By Lemma 3.6, it suffices to show that a maximizer to the optimization problem (3.7) exists. We assume that K_i is a sequence of o-symmetric convex bodies and

$$\lim_{i\to\infty}\Gamma(K_i) = \sup\{\Gamma(f) : f \in C_e^+(S^{n-1})\} > 0, \tag{3.8}$$

where the last inequality follows from Lemma 3.7.

Choose $r_i > 0$ and $u_i \in S^{n-1}$ such that $r_i u_i \in K_i$ and

$$r_i = \max_{u\in S^{n-1}}\rho_{K_i}(u).$$

It is simple to notice that $K_i \subset r_i B$. We claim that r_i is a bounded sequence. Otherwise, by taking a subsequence, we may assume that $\lim_{i\to\infty} r_i = \infty$. Since $K_i \subset r_i B$, we have

$$\Gamma(K_i) \le \frac{1}{\alpha}\gamma_n(r_i B)^{\alpha} - \int\limits_{S^{n-1}} h_{K_i} d\mu.$$

Since $r_i u_i \in K_i$, we have by the definition of support function that

$$h_{K_i}(v) \ge r_i|v\cdot u_i|.$$

Therefore, we have

$$\Gamma(K_i) \leq \frac{1}{\alpha}\gamma_n(r_iB)^\alpha - r_i \int\limits_{S^{n-1}} |v \cdot u_i| d\mu.$$

By the fact that μ is not concentrated in any closed hemisphere, we may find $c_0 > 0$ such that

$$\int\limits_{S^{n-1}} |v \cdot u_i| d\mu \geq c_0.$$

Therefore,

$$\begin{aligned}\Gamma(K_i) &\leq \frac{1}{\alpha}\gamma_n(r_iB)^\alpha - r_ic_0 \\ &= \frac{1}{\alpha}\left(\frac{n\omega_n}{(2\pi)^{\frac{n}{2}}}\int_0^{r_i} e^{-\frac{t^2}{2}}t^{n-1}dt\right)^\alpha - r_ic_0 \\ &\to -\infty,\end{aligned}$$

as $i \to \infty$, since the integral $\int_0^\infty e^{-\frac{t^2}{2}}t^{n-1}dt$ is convergent. But this is a contradiction to K_i being a maximizing sequence and (3.8). Therefore, the sequence of convex bodies K_i is uniformly bounded. We may therefore use Blaschke selection theorem and assume (by taking a subsequence) that K_i converges in Hausdorff metric to a compact convex o-symmetric set K_0. Note that by the continuity of the Gaussian volume with respect to the Hausdorff metric, definition of Γ, and (3.8), we have

$$\frac{1}{\alpha}\gamma_n(K_0)^\alpha \geq \Gamma(K_0) = \lim_{i\to\infty}\Gamma(K_i) > 0.$$

This, when combined with the fact that K_0 is o-symmetric, implies that K_0 contains the origin as its interior point. Therefore, the convex body K_0 (or, its support function h_{K_0}) is a maximizer to the optimization (3.7). □

It is of great interest to ask whether the convex body satisfying (3.6) is uniquely determined.

4. Isoperimetric inequalities in Gaussian probability space

In this section, we recall the Ehrhard inequality and several of its consequences.

The Ehrhard inequality was shown by Ehrhard [21] when both Borel sets involved are convex, by Latała [49] when only one of the sets is assumed to be convex, and more recently by Borell [7] when neither set is required to be convex. However, for the purpose of this paper, only Ehrhard's original version is required.

Theorem 4.1 *(Ehrhard inequality). Let K, L be two convex bodies in $\mathbb{R}^n$. For $0 < t < 1$, we have*

$$\Phi^{-1}(\gamma_n((1-t)K + tL)) \geq (1-t)\Phi^{-1}(\gamma_n(K)) + t\Phi^{-1}(\gamma_n(L)).$$

Here,

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt. \tag{4.1}$$

Moreover, equality holds if and only if $K = L$.

The equality condition in the above lemma was shown by Ehrhard [22]. More recently, for more general versions of Ehrhard inequality, the equality condition has been settled by Shenfeld and van Handel [72].

The following lemma is a direct consequence of Ehrhard inequality. See also Borell [6] for a characterization of log-concave measures.

Lemma 4.2. *Let K and L be two convex bodies in $\mathbb{R}^n$. For $0 < t < 1$, we have*

$$\gamma_n((1-t)K + tL) \geq \gamma_n(K)^{1-t}\gamma_n(L)^t, \tag{4.2}$$

with equality if and only if $K = L$.

Using the variational formula (3.2), we obtain the following Minkowski-type inequality.

Lemma 4.3. *Let K and L be two convex bodies in $\mathbb{R}^n$. We have*

$$\int_{S^{n-1}} h_L - h_K dS_{\gamma_n, K} \geq \gamma_n(K) \log \frac{\gamma_n(L)}{\gamma_n(K)},$$

with equality if and only if $K = L$.

Proof. Lemma 4.2 implies that the function $g : [0, 1] \to \mathbb{R}$ given by

$$g(t) = \log \gamma_n((1-t)K + tL)$$

is concave. Therefore, the slope of the tangent line at $t = 0$ is no smaller than the slope of the secant line joining $(0, g(0))$ and $(1, g(1))$. Using (3.2) to compute $g'(0)$, we immediately arrive at the desired inequality.

If equality holds, then $g(t)$ is a linear function. Thus, equality holds in (4.2), which then implies that $K = L$. □

An immediate consequence is

Lemma 4.4. *Let K and L be two convex bodies in $\mathbb{R}^n$. If $\gamma_n(K) = \gamma_n(L)$, then*

$$\int_{S^{n-1}} h_L dS_{\gamma_n,K} \geq \int_{S^{n-1}} h_K dS_{\gamma_n,K}, \tag{4.3}$$

with equality if and only if $K = L$.

The Ehrhard inequality also implies the following isoperimetric inequality in Gaussian probability space. See, for example, [50].

Theorem 4.5 *(Gaussian isoperimetric inequality). Let K be a convex body in $\mathbb{R}^n$. Then,*

$$|S_{\gamma_n,K}| \geq \varphi(\Phi^{-1}(\gamma_n(K))),$$

where $\varphi(t) = (\sqrt{2\pi})^{-1} \exp(-t^2/2)$ and Φ is as given in (4.1).

The Guassian isoperimetric inequality is a consequence of Ehrhard inequality. A direct consequence of the Gaussian isoperimetric inequality is the following.

Corollary 4.6. *If K is a convex body in $\mathbb{R}^n$ such that $\gamma_n(K) = 1/2$, then $|S_{\gamma_n,K}| \geq \frac{1}{\sqrt{2\pi}}$.*

5. Uniqueness of solution

In this section, we will show that the solution to the Gaussian Minkowski problem is unique if one restricts the solution set to bodies with sufficiently big Gaussian volume.

Lemma 5.1. *Suppose $K, L \in \mathcal{K}_o^n$ and K, L both solve the Gaussian Minkowski problem; i.e.,*

$$S_{\gamma_n,K} = S_{\gamma_n,L} = \mu. \tag{5.1}$$

If $\gamma_n(K), \gamma_n(L) \geq 1/2$, then,

$$\gamma_n(K) = \gamma_n(L).$$

Proof. For simplicity, we write $\Psi = \Phi^{-1}$ where Φ is given in (4.1). Then, Ehrhard inequality says

$$\Psi(\gamma_n((1-t)K + tL)) \geq (1-t)\Psi(\gamma_n(K)) + t\Psi(\gamma_n(L)), \tag{5.2}$$

with equality if and only if $K = L$. Notice that Ψ is C^∞ and strictly monotonically increasing. Utilizing Theorem 3.3, we take the first derivative of (5.2) at $t = 0$ and get

$$\Psi'(\gamma_n(K)) \int_{S^{n-1}} h_L - h_K dS_{\gamma_n,K} \geq \Psi(\gamma_n(L)) - \Psi(\gamma_n(K)). \tag{5.3}$$

Switching the role of K and L, we get

$$\Psi'(\gamma_n(L)) \int_{S^{n-1}} h_K - h_L dS_{\gamma_n,L} \geq \Psi(\gamma_n(K)) - \Psi(\gamma_n(L)). \tag{5.4}$$

By (5.1), we have

$$\Psi'(\gamma_n(L)) \int_{S^{n-1}} h_L - h_K dS_{\gamma_n,K} \leq \Psi(\gamma_n(L)) - \Psi(\gamma_n(K)), \tag{5.5}$$

from (5.4). Since $\Psi' > 0$, (5.3) and (5.5) imply that

$$\frac{\Psi(\gamma_n(L)) - \Psi(\gamma_n(K))}{\Psi'(\gamma_n(L))} \geq \frac{\Psi(\gamma_n(L)) - \Psi(\gamma_n(K))}{\Psi'(\gamma_n(K))},$$

or, equivalently,

$$(\Psi'(\gamma_n(K)) - \Psi'(\gamma_n(L)))\,(\Psi(\gamma_n(K)) - \Psi(\gamma_n(L))) \leq 0. \tag{5.6}$$

By the definition of Ψ and chain rule, we can compute

$$\Psi'(x) = \sqrt{2\pi} e^{\frac{\Psi(x)^2}{2}}.$$

Since Ψ is strictly increasing on $[1/2, 1]$, this implies that Ψ' is also strictly increasing. This, when combined with the fact that Ψ is strictly increasing, shows that

$$(\Psi'(a) - \Psi'(b))(\Psi(a) - \Psi(b)) \geq 0,$$

with equality if and only if $a = b$. Equation (5.6) now gives us the desired result. □

We are now ready to prove the uniqueness part of the Gaussian Minkowski problem when we restrict to the set of convex bodies whose Gaussian measure is no smaller than $1/2$.

Theorem 5.2. *Suppose $K, L \in \mathcal{K}_o^n$ and K, L both solve the Gaussian Minkowski problem; i.e.,*

$$S_{\gamma_n,K} = S_{\gamma_n,L} = \mu.$$

If $\gamma_n(K), \gamma_n(L) \geq 1/2$, then $K = L$.

Proof. By Lemma 5.1, we have $\gamma_n(K) = \gamma_n(L)$. By Corollary 4.4 and the fact that $S_{\gamma_n,K} = S_{\gamma_n,L}$, we conclude that equality holds in (4.3) and therefore by the equality condition, we have $K = L$. □

6. Existence of *o*-symmetric solutions

For the rest of the paper, we are going to prove existence results regarding the Gaussian Minkowski problem. For this purpose, we shall restrict ourselves to the *o*-symmetric case; that is, when the given data (μ or in the smooth case, its density f) is even and the potential solution set is restricted to $\mathcal{K}_e^n$.

In this section, we will first prove the existence result when the given data is sufficiently smooth and everywhere positive. To do that, some *a-priori* estimates are required. At the end of the section, an approximation argument will be deployed to get the solution when the given data is a measure.

6.1. C^0 estimate

Lemma 6.1. *There exists a constant $c > 0$ (that depends only on n) such that if $K \in \mathcal{K}_e^n$ and $\gamma_n(K) \geq \frac{1}{2}$, then its support function h_K is bounded from below by c on S^{n-1}.*

Proof. We argue by contradiction and assume that there exists $K_i \in \mathcal{K}_e^n$ with $\gamma_n(K_i) \geq \frac{1}{2}$ and $v_i \in S^{n-1}$ such that $h_i := h_{K_i}(v_i) \to 0$. Then, by definition of support function, we have

$$K_i \subset \{x \in \mathbb{R}^n : |x \cdot v_i| \leq h_i\}.$$

Therefore,

$$\gamma_n(K_i) \leq \gamma_n(\{x \in \mathbb{R}^n : |x \cdot v_i| \leq h_i\}) \to 0,$$

which contradicts with the given condition that $\gamma_n(K_i) \geq \frac{1}{2}$. □

Lemma 6.2. *Let K_i be a sequence of convex bodies in $\mathcal{K}_e^n$ and $r_i = ||h_{K_i}||_\infty$. If $\lim_{i\to\infty} r_i = \infty$, then for each $0 < c \leq 1$ and $v \in S^{n-1}$, we have*

$$\lim_{i\to\infty} \int_{\omega_i} e^{-\frac{|x|^2}{2}} d\mathcal{H}^{n-1}(x) = 0,$$

where

$$\omega_i = \{x \in \partial K_i : x \cdot v > cr_i\}.$$

Proof. We first recall that based on Cauchy's surface area formula, if $K \subset L$, then $\mathcal{H}^{n-1}(\partial K) \leq \mathcal{H}^{n-1}(\partial L)$.

For simplicity, we write B_i for the centered ball of radius r_i. For each $j = 1, 2, \ldots$, write

$$L_{i,j} = B_i \cap \{x \in \mathbb{R}^n : j \leq x \cdot v \leq j+1\}.$$

Note that $L_{i,j}$ can be contained in a cylinder with the base of an $(n-1)$-dimensional ball of radius r_i and unit height. Thus,

$$\mathcal{H}^{n-1}(\partial L_{i,j}) \leq \mathcal{H}^{n-1}(\partial(B_i \cap \mathbb{R}^{n-1}) \times [0,1]) \leq c(n) r_i^{n-1}. \tag{6.1}$$

Here and in the rest of the proof, we frequently use symbols such as $c(n)$ to denote nonessential constants that only depend on the dimension.

Let $\Gamma_{i,j} = K_i \cap \{x \in \mathbb{R}^n : j \leq x \cdot v \leq j+1\}$. Note that

$$\omega_i \subset \bigcup_{j=\lfloor cr_i \rfloor}^{\infty} \partial\Gamma_{i,j}. \tag{6.2}$$

For each individual $\partial\Gamma_{i,j}$, we have

$$\int\limits_{\partial\Gamma_{i,j}} e^{-\frac{|x|^2}{2}} d\mathcal{H}^{n-1}(x) \leq e^{-\frac{j^2}{2}} \mathcal{H}^{n-1}(\partial\Gamma_{i,j}) \leq e^{-\frac{j^2}{2}} \mathcal{H}^{n-1}(\partial L_{i,j}),$$

where in the last inequality, we used the fact that $\Gamma_{i,j} \subset L_{i,j}$. Combining with (6.1) and (6.2), we have

$$\begin{aligned}
\int\limits_{\omega_i} e^{-\frac{|x|^2}{2}} d\mathcal{H}^{n-1}(x) &\leq \sum_{j=\lfloor cr_i \rfloor}^{\infty} \int\limits_{\partial\Gamma_{i,j}} e^{-\frac{|x|^2}{2}} d\mathcal{H}^{n-1}(x) \\
&\leq c(n) r_i^{n-1} \sum_{j=\lfloor cr_i \rfloor}^{\infty} e^{-\frac{j^2}{2}} \\
&\leq c(n) r_i^{n-1} \sum_{j=\lfloor cr_i \rfloor}^{\infty} e^{-j} \\
&\leq c(n) r_i^{n-1} e^{-\lfloor cr_i \rfloor} \sum_{j=0}^{\infty} e^{-j} \\
&\to 0,
\end{aligned}$$

as $i \to \infty$, since $r_i \to \infty$. $\square$

Lemma 6.3. *Suppose K_i is a sequence of convex bodies in $\mathcal{K}^n_e$. For simplicity, write $\mu_i = S_{\gamma_n,K_i}$. If $||h_{K_i}||_\infty \to \infty$ and $|\mu_i| > \frac{1}{C}$ for some $C > 0$, then for every $\delta > 0$, there exists $v \in S^{n-1}$ and $N > 0$ such that*

$$|\mu_i| < \mu_i(\overline{\xi_{v,\delta}}) + \delta,$$

for each $i > N$. Here $\xi_{v,\delta}$ is given by

$$\xi_{v,\delta} = \{u \in S^{n-1} : |u \cdot v| < \delta\}.$$

Proof. By John's theorem, there exists o-symmetric ellipsoid E_i:

$$E_i = \left\{ x \in \mathbb{R}^n : \frac{|x \cdot e_{i,1}|^2}{r_{i,1}^2} + \cdots + \frac{|x \cdot e_{i,n}|^2}{r_{i,n}^2} \leq 1 \right\}$$

with $r_{i,1} \geq r_{i,2} \geq \cdots \geq r_{i,n}$ such that $E_i \subset K_i \subset \sqrt{n}E_i$. Since $||h_i||_\infty \to \infty$, we have $r_{i,1} \to \infty$.

By taking a subsequence, we may assume

$$0 \leq \lim_{i\to\infty} \frac{r_{i,j}}{r_{i,j-1}} = a_j \leq 1, \text{ for } j = 2, \dots n.$$

Define $a_1 = 1$. We argue that there exists $j \in \{1, \dots, n\}$ such that $a_j = 0$. If not, then there exists $0 < c \leq 1$ such that $cr_{i,1} < r_{i,n} \leq r_{i,1}$. Choose an orthonormal basis $v_1, \dots, v_n$. By definition of $r_{i,n}$, we see that

$$(\sqrt{2\pi})^n |S_{\gamma_n,K_i}| = \int_{\partial K_i} e^{-\frac{|x|^2}{2}} d\mathcal{H}^{n-1}(x) \leq \sum_{j=1}^{n} \int_{\omega_{i,j}} e^{-\frac{|x|^2}{2}} d\mathcal{H}^{n-1}(x), \tag{6.3}$$

where

$$\omega_{i,j} = \{x \in \partial K_i : |x \cdot v_j| > cr_{i,1}\}.$$

Lemma 6.2 and (6.3) now implies that $\lim_{i\to\infty} |S_{\gamma_n,K_i}| = 0$, which is a contradiction to the uniform lower bound of $|\mu_i|$.

Now, let $s = \min\{i - 1 : a_i = 0\}$. Then there exists $0 < c \leq 1$ such that

$$r_{i,1} \geq r_{i,2} \geq \cdots \geq r_{i,s} \geq cr_{i,1}, \tag{6.4}$$

and

$$\lim_{i\to\infty} \frac{r_{i,j}}{r_{i,s}} = 0, \text{ for each } j > s. \tag{6.5}$$

By possibly taking another subsequence, we may assume

$$\lim_{i\to\infty} e_{i,j} = e_j, \text{ for } j = 1, 2, \dots, n. \tag{6.6}$$

Here e_j is an orthonormal basis in $\mathbb{R}^n$.

Choose $\tau_n, \varepsilon_1 \in (0,1)$ such that

$$\frac{n}{c^2}(2\tau_n + \sqrt{n}\varepsilon_1)^2 \le 1,$$

where $c > 0$ is from (6.4).

Let

$$\Omega_i = \{x \in \partial K_i : |x \cdot e_j| \le \tau_n r_{i,s}, j = 1, 2, \dots, s\}$$
$$\eta_{i,j} = \{x \in \partial K_i : |x \cdot e_j| > \tau_n r_{i,s}\}.$$

Then, it is simple to see that

$$\partial K_i = \Omega_i \cup \left(\cup_{j=1}^{s} \eta_{i,j}\right).$$

Recall that

$$(\sqrt{2\pi})^n |S_{K_i,\gamma_n}| = \int_{\partial K_i} e^{-\frac{|x|^2}{2}} d\mathcal{H}^{n-1}(x) = \int_{\Omega_i} e^{-\frac{|x|^2}{2}} d\mathcal{H}^{n-1}(x) + \sum_{j=1}^{s} \int_{\eta_{i,j}} e^{-\frac{|x|^2}{2}} d\mathcal{H}^{n-1}(x). \tag{6.7}$$

By Lemma 6.2 and (6.4),

$$\lim_{i\to\infty} \sum_{j=1}^{s} \int_{\eta_{i,j}} e^{-\frac{|x|^2}{2}} d\mathcal{H}^{n-1}(x) = 0. \tag{6.8}$$

It remains to estimate

$$\int_{\Omega_i} e^{-\frac{|x|^2}{2}} d\mathcal{H}^{n-1}(x).$$

Take $x \in \Omega_i$. Set $z = (x \cdot e_{i,1} \dots, x \cdot e_{i,s-1}, x \cdot e_{i,s} + \tau_n r_{i,s}, 0, \dots, 0)$, where the coordinates are under $e_{i,1}, \dots e_{i,n}$. By (6.6), there exists $N_1 > 0$ such that for each $i > N_1$, we have

$$|e_{i,j} - e_j| < \varepsilon_1, \text{ for each } j = 1, \dots, n. \tag{6.9}$$

We claim that for $i > N_1$, we have $z \in E_i$. Indeed, by the triangle inequality, the definition of Ω_i, (6.9), the fact that $K_i \subset \sqrt{n}E_i$, (6.4), and the choice of τ_n and ε_1,

$$\sum_{j=1}^{n} \frac{|z \cdot e_{i,j}|^2}{r_{i,j}^2} = \sum_{j=1}^{s-1} \frac{|x \cdot e_{i,j}|^2}{r_{i,j}^2} + \frac{|x \cdot e_{i,s} + \tau_n r_{i,s}|^2}{r_{i,s}^2}$$

$$\leq \sum_{j=1}^{s-1} \frac{(|x\cdot e_j| + |x\cdot(e_{i,j}-e_j)|)^2}{r_{i,j}^2} + \frac{(|x\cdot e_s + \tau_n r_{i,s}| + |x\cdot(e_{i,s}-e_s)|)^2}{r_{i,s}^2}$$

$$\leq \sum_{j=1}^{s-1} \frac{(\tau_n r_{i,s} + \sqrt{n} r_{i,1}\varepsilon_1)^2}{r_{i,j}^2} + \frac{(2\tau_n r_{i,s} + \sqrt{n} r_{i,1}\varepsilon_1)^2}{r_{i,s}^2}$$

$$\leq \sum_{j=1}^{s-1} (\tau_n + \sqrt{n}\varepsilon_1)^2 \frac{r_{i,1}^2}{r_{i,j}^2} + (2\tau_n + \sqrt{n}\varepsilon_1)^2 \frac{r_{i,1}^2}{r_{i,s}^2}$$

$$\leq \frac{1}{c^2} n(2\tau_n + \varepsilon_1)^2 \leq 1.$$

Therefore, $z \in E_i \subset K_i$.

Note that by definition of z and the fact that $K_i \subset \sqrt{n}E_i$,

$$d(x + \tau_n r_{i,s} e_{i,s}, z) \leq \left(\sum_{j=s+1}^{n} |x\cdot e_{i,j}|^2 \right)^{\frac{1}{2}} \leq \left(\sum_{j=s+1}^{n} n r_{i,j}^2 \right)^{\frac{1}{2}}.$$

According to (6.5), we have

$$\lim_{i\to\infty} \frac{d(x + \tau_n r_{i,s} e_{i,s}, z)}{r_{i,s}} = 0. \tag{6.10}$$

Notice also that by definition of the Gauss map $\nu_{K_i}(x)$, we have

$$\nu_{K_i}(x)\cdot(x-z) \geq 0,$$

which is equivalent to

$$\nu_{K_i}(x)\cdot\left(\frac{x - z + \tau_n r_{i,s} e_{i,s}}{\tau_n r_{i,s}} - e_{i,s}\right) \geq 0.$$

By (6.10), for each $\delta > 0$, there exists $N_2 > N_1$ such that for every $i > N_2$, we have

$$\nu_{K_i}(x)\cdot e_{i,s} < \delta.$$

By symmetry, we yield

$$|\nu_{K_i}(x)\cdot e_{i,s}| < \delta.$$

Since $e_{i,s} \to e_s$, there exists $N_3 > N_2$ such that for every $i > N_3$, we have

$$|\nu_{K_i}(x)\cdot e_s| < 2\delta.$$

This, implies that

$$\frac{1}{(\sqrt{2\pi})^n}\int_{\Omega_i} e^{-\frac{|x|^2}{2}} d\mathcal{H}^{n-1}(x) \leq \mu_i(\xi_{e_s,2\delta}),$$

This, (6.7) and (6.8) imply that for each $\delta > 0$, there exists $N > 0$ such that for each $i > N$, we have

$$|\mu_i| \leq \mu_i(\overline{\xi_{e_s,\delta}}) + \delta. \quad \square$$

The following lemma contains the desired C^0 estimate.

Lemma 6.4. *Suppose the support function of $K \in \mathcal{K}^n_e$ is C^2 and satisfies*

$$\frac{1}{(\sqrt{2\pi})^n} e^{-\frac{|\nabla h|^2+h^2}{2}} \det(\nabla^2 h + h\delta_{ij}) = f,$$

and $\gamma_n(K) > 1/2$. If there exists $C > 0$ such that $\frac{1}{C} < f < C$, then there exists $C' > 0$ such that $1/C' < h_K < C'$.

Proof. The lower bound for h_K comes from the assumption that $\gamma_n(K) > \frac{1}{2}$ and Lemma 6.1.

For the upper bound, assume for the sake of contradiction that there exists a sequence of C^2 o-symmetric convex bodies K_i with $\gamma_n(K_i) > 1/2$ and

$$1/C < f_i = \frac{1}{(\sqrt{2\pi})^n} e^{-\frac{|\nabla h_i(v)|^2+h_i^2(v)}{2}} \det(\nabla^2 h_i(v) + h_i(v)I) < C,$$

but $||h_i||_\infty \to \infty$. Here we abbreviated h_{K_i} by h_i.

Write $\mu_i = f_i dv$.

By Lemma 6.3, for every $\delta > 0$, there exists $v \in S^{n-1}$ and $N > 0$ such that for each $i > N$, we have

$$|\mu_i| < \mu_i(\overline{\xi_{v,\delta}}) + \delta. \tag{6.11}$$

Note that there exists $c > 0$ such that $\mathcal{H}^{n-1}(\overline{\xi_{v,\delta}}) < c\delta$, which when combined with (6.11) and the uniform upper bound of f_i, shows

$$|\mu_i| < cC\delta + \delta.$$

When δ is small enough, this is a contradiction to the uniform lower bound of f_i. $\square$

6.2. Higher order a-priori *estimate*

Lemma 6.5 *(a-priori estimate). Let $0 < \alpha < 1$. Suppose $f \in C^{2,\alpha}(S^{n-1})$ and there exists $C > 0$ such that $\frac{1}{C} < f < C$ and $|f|_{C^{2,\alpha}} < C$. If the support function of $K \in \mathcal{K}^n_e$ is $C^{4,\alpha}$ and satisfies*

$$\frac{1}{(\sqrt{2\pi})^n} e^{-\frac{|\nabla h|^2+h^2}{2}} \det(\nabla^2 h + hI) = f \tag{6.12}$$

and $\gamma_n(K) > \frac{1}{2}$*, then there exists* $C' > 0$ *that only depends on* C *such that*

(1) $\frac{1}{C'} < \sqrt{|\nabla h_K|^2 + h_K^2} < C'$
(2) $\frac{1}{C'} I < (\nabla^2 h_K + h_K I) < C' I$
(3) $|h_K|_{C^{4,\alpha}} < C'$.

Proof. (1) Note that by Lemma 6.4, there exists $C' > 0$ such that $1/C' < h_K < C$, or, in another word, $1/C'B \subset K \subset C'B$. Recall that by the definition of support function, we have for each $v \in S^{n-1}$,

$$\nabla h_K(v) + h_K(v)v = \nu_K^{-1}(v) \in \partial K.$$

Therefore the upper and lower bound on $|\nabla h_K|^2 + h_K^2$ follows from the bounds on K.

(2) Our strategy to prove this statement is to show that

(a) The trace of the matrix $(\nabla^2 h_K + h_K I)$, or the sum of the matrix's eigenvalues, is bounded from above.
(b) The determinant of the matrix $(\nabla^2 h_K + h_K I)$, or the product of the matrix's eigenvalues is bounded both from above and from below (by a positive constant).

Note that (a) and (b), when combined together, immediately imply that all eigenvalues of $(\nabla^2 h_K + h_K I)$ have positive upper and lower bounds—a consequence of the fact that $(\nabla^2 h_K + h_K I)$ is positive definite.

Claim (b) permits a quick proof based on equation (6.12). Indeed,

$$\det(\nabla^2 h_K + h_K I) = (\sqrt{2\pi})^n f e^{\frac{|\nabla h_K|^2+h_K^2}{2}},$$

where the right side has positive upper and lower bounds based on the bounds of f and statement (1).

To prove Claim (a), let us denote

$$H = \operatorname{trace}(\nabla^2 h_K + h_K I) = \Delta h_K + (n-1)h_K.$$

Since H is continuous on S^{n-1}, there exists $v_0 \in S^{n-1}$ such that $H(v_0) = \max_{v \in S^{n-1}} H$. Then, at v_0, we have $\nabla H = 0$ and the matrix $\nabla^2 H$ is negative semi-definite. We choose a local orthonormal frame $e_1, \ldots, e_{n-1}$, such that the Hessian of h_K, $(h_K)_{ij}$, is diagonal. Recall the commutator identity [31, p. 1361]:

$$H_{ii} = \Delta w_{ii} - (n-1)w_{ii} + H. \tag{6.13}$$

We use w^{ij} to denote the inverse of the matrix $w_{ij} = ((h_K)_{ij} + h_K\delta_{ij})$. Equation (6.13), the fact that (w^{ij}) is positive definite and $\nabla^2 H$ is negative semi-definite, and that (w^{ij}) is diagonal, imply that at v_0, we have

$$0 \geq w^{ii}H_{ii} = w^{ii}\Delta w_{ii} + H\sum_i w^{ii} - (n-1)^2 \geq w^{ii}\Delta w_{ii} - (n-1)^2. \tag{6.14}$$

Taking the logarithm of (6.12), we have

$$\log\det(\nabla^2 h_K + h_K I) = \log f + \frac{|\nabla h_K|^2 + h_K^2}{2}.$$

We take the spherical Laplacian of the above equation and get

$$\begin{aligned}&\sum_\alpha (w^{ij})_\alpha (w_{ij})_\alpha + w^{ij}\Delta w_{ij}\\ =&\Delta\log f + \sum_{i,j}(h_K)_{ij}^2 + \sum_i (h_K)_i(\Delta(h_K)_i) + |\nabla h_K|^2 + h_K\Delta h_K\end{aligned} \tag{6.15}$$

By definition of H, we have

$$\sum_{i,j}(h_K)_{ij}^2 = H^2 - 2h_K H + (n-1)h_K^2. \tag{6.16}$$

By definition of H and equation (4.11) in Cheng-Yau [19], we have

$$\sum_i (h_K)_i(\Delta(h_K)_i) = \sum_i (h_K)_i(\Delta h_K)_i = \nabla h_K\cdot\nabla H - (n-1)|\nabla h_K|^2. \tag{6.17}$$

We also note that

$$h_K\Delta h_K = h_K H - (n-1)h_K^2. \tag{6.18}$$

Finally, using the fact that (w^{ij}) is the inverse matrix of (w_{ij}), we get

$$(w^{ij})_\alpha (w_{jk})_\alpha = -w^{im}(w_{ml})_\alpha w^{lj}(w_{jk})_\alpha,$$

which implies its trace is non-positive; that is

$$(w^{ij})_\alpha (w_{ij})_\alpha \leq 0. \tag{6.19}$$

Combining (6.15), (6.16), (6.17), (6.18), and (6.19), we get

$$w^{ij}\Delta w_{ij} \geq \Delta\log f + H^2 - h_K H + \nabla h_K\cdot\nabla H - (n-2)|\nabla h_K|^2.$$

When evaluated at v_0 where H reaches its maximum, we have

$$w^{ii}\Delta w_{ii} \geq H^2 - h_K H + (\Delta \log f - (n-2)|\nabla h_K|^2). \tag{6.20}$$

Equations (6.14) and (6.20) now imply that

$$0 \geq H^2 - h_K H + (\Delta \log f - (n-2)|\nabla h_K|^2 - (n-1)^2).$$

The right side of the above inequality is a quadratic polynomial in H. Note that by the bounds on f and $|f|_{C^{2,\alpha}}$, statements (i) and (ii), the coefficients have bounds that only depend on C. Therefore, H is bounded from above by a positive constant that only depends on C.

(3) By statement (2), the Monge-Ampère equation (6.12) is uniformly elliptic. Thus, the standard Evans-Krylov-Safonov theory [30] implies the higher estimates in statement (3). □

6.3. Existence of smooth solutions via degree theory

Theorem 6.6 *(Existence of smooth solutions). Let $0 < \alpha < 1$ and $f \in C^{2,\alpha}(S^{n-1})$ be a positive even function with $|f|_{L_1} < \frac{1}{\sqrt{2\pi}}$. Then there exists a unique $C^{4,\alpha}$ o-symmetric K with $\gamma_n(K) > 1/2$ such that*

$$\frac{1}{(\sqrt{2\pi})^n} e^{-\frac{|\nabla h_K|^2 + h_K^2}{2}} \det(\nabla^2 h_K + h_K I) = f. \tag{6.21}$$

Proof. The uniqueness part follows from Theorem 5.2.

We use the degree theory for second-order nonlinear elliptic operators developed in Li [53] for the existence part.

It follows by simple application of intermediate value theorem and Theorem 5.2 that (6.21) admits a unique constant solution $h_K \equiv r_0 > 0$ such that $\gamma_n(K) > 1/2$ if $f \equiv c_0 > 0$ is small enough. We also require that $c_0 > 0$ is small enough so that $|c_0|_{L_1} < \frac{1}{\sqrt{2\pi}}$. Our final requirement for $c_0 > 0$ is that the operator $L\phi = \Delta_{S^{n-1}}\phi + ((n-1) - r_0^2)\phi$ is invertible. This is possible since spherical Laplacian has a discrete spectrum.

Let $F(\cdot; t) : C^{4,\alpha}(S^{n-1}) \to C^{2,\alpha}(S^{n-1})$ be defined as

$$F(h; t) = \det(\nabla^2 h + hI) - (\sqrt{2\pi})^n e^{\frac{|\nabla h|^2 + h^2}{2}} f_t,$$

for $t \in [0, 1]$. Here

$$f_t = (1-t)c_0 + tf.$$

Since $f \in C^{2,\alpha}(S^{n-1})$ and $f > 0$, there exists $C > 0$ such that $\frac{1}{C} < f, c_0 < C$ and $|f|_{C^{2,\alpha}} < C$. We let $C' > 0$ be the constant extracted from Lemma 6.5. Note that for each $t \in [0, 1]$, the function f_t has the same bound as f; namely, $\frac{1}{C} < f_t < C$, $|f_t|_{L_1} < \frac{1}{\sqrt{2\pi}}$, and $|f_t|_{C^{2,\alpha}} < C$.

Define $O \subset C^{4,\alpha}(S^{n-1})$ by

$$O = \left\{ h \in C^{4,\alpha}(S^{n-1}) : \frac{1}{C'} < h < C', \frac{1}{C'} I < (\nabla^2 h + hI) < C' I, |h|_{C^{4,\alpha}} < C', \gamma_n(h) > \frac{1}{2} \right\}.$$

Here $\gamma_n(h) = \gamma_n([h])$ is well-defined and moreover since h is strictly convex, h is precisely a support function. We note that it is simple to see that O is an open bounded set under the norm $|\cdot|_{C^{4,\alpha}}$.

We also note that since every $h \in O$ has uniform upper and lower bounds for the eigenvalues of its Hessian, the operator $F(\cdot; t)$ is uniformly elliptic on O for any $t \in [0, 1]$.

We claim that for each $t \in [0, 1]$, if $h \in \partial O$, then

$$F(h; t) \neq 0.$$

Indeed, if $F(h; t) = 0$, then h solves

$$\frac{1}{(\sqrt{2\pi})^n} e^{-\frac{|Dh|^2}{2}} \det(\nabla^2 h + hI) = f_t. \tag{6.22}$$

Since $h \in \partial O$, we also have that $\gamma_n(h) \geq \frac{1}{2}$. If $\gamma_n(h) > 1/2$, by Lemma 6.5, we would have

$$\frac{1}{C'} < h < C', \frac{1}{C'} I < (\nabla^2 h + hI) < C' I, |h|_{C^{4,\alpha}} < C'.$$

Thus, by the definition of O, the only way for $h \in \partial O$ is that

$$\gamma_n(h) = \frac{1}{2}.$$

By Corollary 4.6, we have

$$|S_{\gamma_n,[h]}| \geq \frac{1}{\sqrt{2\pi}}.$$

But this contradicts with the fact that h solves (6.22) and that $|f_t|_1 < \frac{1}{\sqrt{2\pi}}$.

Using Proposition 2.2 in Li [53], we conclude that

$$\deg(F(\cdot; 0), O, 0) = \deg(F(\cdot; 1), O, 0).$$

If we can show that $\deg(F(\cdot; 0), O, 0) \neq 0$, then it follows immediately that

$$\deg(F(\cdot; 1), O, 0) \neq 0,$$

which then implies the existence of $h \in O$ such that $F(h; 1) = 0$.

The rest of the proof focuses on showing $\deg(F(\cdot;0), O, 0) \neq 0$. For simplicity, we will simply write $F(\cdot) = F(\cdot;0)$.

Recall that $r_0 > 0$ is so that $h_K \equiv r_0$ is the unique solution in O to (6.21) when $f \equiv c_0$. We denote by $L_{r_0} : C^{4,\alpha}(S^{n-1}) \to C^{2,\alpha}(S^{n-1})$ the linearized operator of F at the constant function r_0. It is simple to compute that

$$\begin{aligned} L_{r_0}(\phi) &= r_0^{n-2}\Delta_{S^{n-1}}\phi + ((n-1)r_0^{n-2} - r_0^n)\phi \\ &= r_0^{n-2}\left(\Delta_{S^{n-1}}\phi + ((n-1) - r_0^2)\phi\right). \end{aligned}$$

Recall that we have specifically chosen a $c_0 > 0$ so that L_{r_0} is invertible. By Proposition 2.3 in Li [53] and the fact that $h \equiv r_0$ is the unique solution for $F(h) = 0$ in O, we have

$$\deg(F, O, 0) = \deg(L_{r_0}, O, 0) \neq 0,$$

where the last inequality follows from Proposition 2.4[1] in Li [53]. □

6.4. Existence of general o-symmetric solutions

In this section, we solve the general measure case by using an approximation argument. The key step is to establish uniform C^0 estimate.

Theorem 6.7. *Let μ be an even measure on S^{n-1} that is not concentrated in any subspace and $|\mu| < \frac{1}{\sqrt{2\pi}}$. Then there exists a unique origin-symmetric K with $\gamma_n(K) > 1/2$ such that*

$$S_{\gamma_n, K} = \mu.$$

Proof. We approximate μ weakly by a sequence of measures $\mu_i = f_i dv$ where $f_i \in C^{2,\alpha}$, $f_i > 0$ with $0 < |f_i|_{L_1} < \frac{1}{\sqrt{2\pi}}$. By Theorem 6.6, there are $C^{4,\alpha}$ o-symmetric K_i with $\gamma_n(K_i) > \frac{1}{2}$ such that

$$\frac{1}{(\sqrt{2\pi})^n} e^{-\frac{|\nabla h_{K_i}|^2 + h_{K_i}^2}{2}} \det(\nabla^2 h_{K_i} + h_{K_i} I) = f_i.$$

Since f_i converges weakly to μ, by discarding the first finitely many terms, we may assume that $|f_i|_{L_1} > \varepsilon_0$ for some positive absolute constant ε_0.

We argue that K_i is uniformly bounded. Otherwise, by taking a subsequence, we may assume that $||h_{K_i}||_\infty \to \infty$. Now, Lemma 6.3 tells us that for every $\delta > 0$, there exists $v \in S^{n-1}$ and $N > 0$ such that

$$|\mu_i| < \mu_i(\overline{\xi_{v,\delta}}) + \delta,$$

[1] Proposition 2.4 in Li [53] contain some typos, which were corrected by Li on his personal webpage.

for each $i > N$. Let $i \to \infty$. Since μ_i converges to μ weakly, we have

$$|\mu| \leq \mu(\overline{\xi_{v,\delta}}) + \delta.$$

Notice that the above inequality holds for all $\delta > 0$. Therefore,

$$|\mu| \leq \mu(v^{\perp} \cap S^{n-1}),$$

which is a contradiction to the fact that μ is not concentrated in any great subsphere. □

It is of great interest to characterize Gaussian surface area measures for convex bodies that do not necessarily have large Gaussian volume. Example 7.1 in Appendix shows that some essential condition must be found and that how complicated this can be even in the most simple rectangular case on the plane. However, one should keep in mind that the task, even in Example 7.1, is not to find the relation between the weights of the measure, but rather to give a characterization of the permissible measures.

7. Appendix

This appendix consists of two examples.

The first one shows that the Gaussian Minkowski problem even when restricted to the o-symmetric case contains some complications that were masked by our assumption that $\gamma_n(K) > \frac{1}{2}$.

Example 7.1. Consider the even discrete measure on S^1:

$$\mu = \mu_1\delta_{e_1} + \mu_2\delta_{e_2} + \mu_1\delta_{-e_1} + \mu_2\delta_{-e_2}.$$

In order to have an o-symmetric convex body K in $\mathbb{R}^2$—in this case, a centered rectangle—so that $\mu = S_{\gamma_2,K}$, the weights μ_1 and μ_2 cannot be chosen independently of each other.

Proof. Note that potential solutions for $\mu = S_{\gamma_2,K}$ consist of o-symmetric rectangles with sides parallel to the coordinate axes. Let us assume that K is such a rectangle generated by its vertices $(\pm a_1, \pm a_2)$ where $a_1, a_2 > 0$.

It is simple to see that it has to be the case that $0 < \mu_1, \mu_2 < \frac{1}{\sqrt{2\pi}}$ since, for example,

$$\mu_1 = \frac{1}{2\pi}\int_{-a_2}^{a_2} e^{-\frac{a_1^2+y^2}{2}}\,dy < \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{y^2}{2}}\,dy = \frac{1}{\sqrt{2\pi}}.$$

The claimed codependence between μ_1 and μ_2 is best observed when one of the weights is close enough to $\frac{1}{\sqrt{2\pi}}$. For this purpose, let $\varepsilon_0 > 0$ be small enough and $\mu_1 = \frac{1}{\sqrt{2\pi}} - \varepsilon_0$.

It follows from basic computation that

$$\begin{aligned} \mu_1 &= \frac{1}{\pi} e^{-\frac{a_1^2}{2}} \int_0^{a_2} e^{-\frac{t^2}{2}} dt \\ \mu_2 &= \frac{1}{\pi} e^{-\frac{a_2^2}{2}} \int_0^{a_1} e^{-\frac{t^2}{2}} dt. \end{aligned} \tag{7.1}$$

According to (7.1) and the value of μ_1, we have

$$\frac{1}{\sqrt{2\pi}} - \varepsilon_0 < \frac{1}{\pi} e^{-\frac{a_1^2}{2}} \int_0^{\infty} e^{-\frac{t^2}{2}} dt = \frac{1}{\sqrt{2\pi}} e^{-\frac{a_1^2}{2}}.$$

This implies the existence of $\delta_0 = \sqrt{-2\ln(1-\sqrt{2\pi}\varepsilon_0)} > 0$ such that $0 < a_1 < \delta_0$.

Now, using the second equation in (7.1), we have

$$\mu_2 < \frac{1}{\pi} \int_0^{\delta_0} e^{-\frac{t^2}{2}} dt.$$

This crude analysis shows that when one of the weights is close enough to its maximally allowable value, then the other weight must be correspondingly close enough to 0. Moreover, the dependence takes a complicated nonlinear form involving Gaussian distribution function. A similar, but arguably much more complicated computation can show that this phenomenon happens to any o-symmetric polygons on the plane.

Note that in the above computation, $|\mu| > 2(\frac{1}{\sqrt{2\pi}} - \varepsilon_0)$, which is excluded by the hypotheses in Theorem 6.7. □

It follows from a trivial calculation that the Gaussian surface area measure $S_{\gamma_n, rB}$ of a centered ball of radius r is given by

$$dS_{\gamma_n, rB}(v) = \frac{1}{(\sqrt{2\pi})^n} e^{-\frac{r^2}{2}} r^{n-1} dv.$$

Because of the behavior of $e^{-\frac{r^2}{2}} r^{n-1}$ on $(0, \infty)$, it is simple to conclude that for every $c > 0$ that is small enough, there are exactly two balls $r_1 B$ and $r_2 B$ whose Gaussian surface area measure is given by cdv. When this is combined with our uniqueness result (Theorem 5.2), it is tempting to think that uniqueness also holds when restricted to convex bodies whose Gaussian volume is sufficiently small. This turned out to be false, as illustrated by the following example.

Example 7.2. As in Example 7.1, consider an o-symmetric rectangle K in $\mathbb{R}^2$ generated by its vertices $(\pm a_1, \pm a_2)$. Suppose its Gaussian surface area measure is given by

$$dS_{\gamma_n,K} = \mu_1\delta_{e_1} + \mu_2\delta_{e_2} + \mu_1\delta_{-e_1} + \mu_2\delta_{-e_2}.$$

As has already been discussed in Example 7.1, when $\varepsilon_0 > 0$ is sufficiently small, the weight $\mu_1 = \frac{1}{\sqrt{2\pi}} - \varepsilon_0$ is an allowable choice. In particular, we just need to choose sufficiently small $0 < a_1 < \delta_0$ and correspondingly a big enough a_2 so that

$$\frac{1}{\sqrt{2\pi}} - \varepsilon_0 = \mu_1 = \frac{1}{\pi} e^{-\frac{a_1^2}{2}} \int_0^{a_2} e^{-\frac{t^2}{2}}\, dt \tag{7.2}$$

holds.

It is simple to note that by

$$\mu_2 = \frac{1}{\pi} e^{-\frac{a_2^2}{2}} \int_0^{a_1} e^{-\frac{t^2}{2}}\, dt,$$

when $a_1 \to 0$, we have $\mu_2 \to 0$.

Note also that by (7.2), when $a_1 \to \delta_0$, we have $a_2 \to \infty$. Therefore $\mu_2 \to 0$ as well in this scenario.

By intermediate value theorem, we can conclude that when $\mu_1 = \frac{1}{\sqrt{2\pi}} - \varepsilon_0$ and μ_2 is sufficiently small, there are two rectangles—one whose a_1 is close to 0, the other whose a_1 is close to δ_0—such that they have the same Gaussian surface area measure.

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