2021 American Control Conference (ACC)
New Orleans, USA, May 25-28, 2021

Interval Observers for Simultaneous State and Model Estimation of Partially Known Nonlinear Systems

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Abstract— **We consider the problem of designing interval observers for partially unknown nonlinear systems with bounded noise signals that simultaneously estimate the system states and learn a model of the unknown dynamics. Leveraging affine abstraction methods and nonlinear decomposition functions, as well as a data-driven function over-approximation/abstraction approach to over-estimate the unknown dynamic model, our proposed observer recursively computes the maximal and minimal elements of the interval estimates that are proven to frame the true augmented states. Then, using observed output/measurement signals, the observer iteratively shrinks the intervals by eliminating estimates that are not compatible with the measurements. Moreover, given new interval estimates, the observer updates the over-approximation model of the unknown dynamics. Finally, we provide sufficient conditions for uniform boundedness of the sequence of interval estimate widths, i.e., for the stability of the designed observer.**

I. Introduction

Motivated by the need to ensure safe and smooth operation under various forms of uncertainties in many safety-critical engineering applications such as fault detection, urban transportation, attack (unknown input) detection and mitigation in cyber-physical systems and aircraft tracking [1]–[3], many robust set-valued algorithms have been recently developed for state and input estimation of these systems. On the other hand, dynamic models of many practical systems are often only partially known. Thus, the development of algorithms that can combine model learning and set membership estimation approaches is an interesting and important problem.

Literature review. Various approaches have been proposed in the literature to design model-based set-based or interval observers for several classes of systems [3]–[13], including linear time-invariant (LTI) [7], linear parameter-varying (LPV) [11], Metzler and/or partial linearizable [6], [8], cooperative [6], Lipschitz nonlinear [9], monotone nonlinear [5] and uncertain nonlinear [10] systems. However, these approaches often do not consider the presence of unknown inputs (e.g., other agent's input, disturbance, attack or simply unobserved signals) nor unknown dynamics. Some more recent works considered the set-valued observer design problem for simultaneously estimating states and unknown inputs for LTI [3], LPV [12], switched linear [13] and nonlinear [14] systems with bounded-norm noise.

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This work is partially supported by NSF grant CNS-1943545 and an Early Career Faculty grant 80NSSC21K0071 from NASA's Space Technology Research Grants Program.

On the other hand, when the system model is not exactly known, set-valued data-driven approaches that use input-output data to *abstract* or over-approximate unknown dynamics or functions have gained increased popularity over the last few years [15]–[18], where the objective is to find *a set of dynamics* that frame/bracket the unknown system dynamics [15], under the assumption that the unknown dynamics is univariate Lipschitz continuous [16], multivariate Lipschitz continuous [17] or Hölder continuous [18]. Nonetheless, to our knowledge, set-valued or interval observers for such data-driven models have not been considered in the literature.

Contributions. The goal of this paper is to bridge the gap between model-based set membership observer design approaches, e.g., [3]–[13], and data-driven function approximation methods (i.e., model learning methods), e.g., [15]–[18], to design interval observers for partially known nonlinear dynamical systems with bounded noise, where the state and observation vector fields belong to a fairly general class of nonlinear functions and the vector field of the unknown (input) dynamics is an *unknown function*. Our approach builds upon and extends the observer design approach in [14] by including a crucial *update step*, where starting from the intervals from the propagation step, the framers are iteratively updated by computing their intersection with the augmented state intervals that are compatible with the observations, resulting in tighter intervals (i.e., with decreased interval width) for the updated framers.

In addition, our design incorporates a data-driven function approximation/abstraction approach based on [19] to recursively over-approximate the unknown dynamics function from noisy observation data and interval estimates from the update step. Furthermore, by leveraging the combination of nonlinear decomposition/bounding functions [14], [20], [21] and affine abstractions [22], we prove that our observer is correct, i.e., the framer property [8] holds and our estimation/abstraction of the unknown dynamics model becomes more precise and tighter over time. More importantly, we provide sufficient conditions, in the form of a finite number of constraint satisfaction checks, for the stability of our observer (i.e., for the uniform boundedness of the sequence of interval estimate widths), and compute the upper bounds for the interval widths of the sequence of estimates and derive their steady-state values.

II. Preliminaries

Notation. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space and $\mathbb{R}_{++}$ positive real numbers. For vectors $v, w \in \mathbb{R}^n$ and a matrix $M \in \mathbb{R}^{p\times q}$, $\|v\| \triangleq \sqrt{v^\top v}$ and $\|M\|$ denote their (induced) 2-norm, and $v \leq w$ is an element-wise inequality. The transpose, Moore-Penrose pseudoinverse, (i,j)-th element and rank of M are given by $M^\top$, $M^\dagger$, $M_{i,j}$ and $\mathrm{rk}(M)$, while $M_{(r:s)}$ is a sub-matrix of M, consisting of its r-th through s-th rows, and its row support is $r = \mathrm{rowsupp}(M) \in \mathbb{R}^p$, where $r_i = 0$ if the i-th row of M is zero and $r_i = 1$ otherwise, $\forall i \in \{1 \dots p\}$. Also, $M^+ \triangleq \max(M, 0_{p\times q})$, $M^{++} \triangleq M^+ - M$ and $|M| \triangleq M^+ + M^{++}$. M is a non-negative matrix, if $M_{i,j} \geq 0, \forall (i,j) \in \{1 \dots p\} \times \{1 \dots q\}$.

Next, we introduce some definitions and results that will be useful throughout the paper.

Definition 1 (Interval, Maximal and Minimal Elements, Interval Width)**.** *An (multi-dimensional) interval $\mathcal{I} \subset \mathbb{R}^n$ is the set of all real vectors $x \in \mathbb{R}^n$ that satisfies $\underline{s} \leq x \leq \overline{s}$, where $\underline{s}$, $\overline{s}$ and $\|\overline{s} - \underline{s}\|$ are called minimal vector, maximal vector and width of $\mathcal{I}$, respectively.*

Definition 2 (Mixed-Monotone Mappings and Decomposition Functions)**.** *[20, Definition 4] A mapping $f : \mathcal{X} \subseteq \mathbb{R}^n \to \mathcal{T} \subseteq \mathbb{R}^m$ is mixed-monotone if there exists a decomposition function $f_d : \mathcal{X} \times \mathcal{X} \to \mathcal{T}$ satisfying: i) $f_d(x,x) = f(x)$, ii) $x_1 \geq x_2 \Rightarrow f_d(x_1, y) \geq f_d(x_2, y)$ and iii) $y_1 \geq y_2 \Rightarrow f_d(x, y_1) \leq f_d(x, y_2)$.*

Proposition 1. *[21, Theorem 1] Let $f : \mathcal{X} \subseteq \mathbb{R}^n \to \mathcal{T} \subseteq \mathbb{R}^m$ be a mixed-monotone mapping with decomposition function $f_d : \mathcal{X}\times\mathcal{X} \to \mathcal{T}$ and $\underline{x} \leq x \leq \overline{x}$, where $\underline{x}, x, \overline{x} \in \mathcal{X}$. Then $f_d(\underline{x}, \overline{x}) \leq f(x) \leq f_d(\overline{x}, \underline{x})$.*

Note that the decomposition function of a vector field is not unique and a specific one is given in [20, Theorem 2]: If a vector field $q = \begin{bmatrix} q_1^\top & \dots & q_n^\top \end{bmatrix}^\top : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is differentiable and its partial derivatives are bounded with known bounds, i.e., $\frac{\partial q_i}{\partial x_j} \in (a^q_{i,j}, b^q_{i,j}), \forall x \in X \in \mathbb{R}^n$, where $a^q_{i,j}, b^q_{i,j} \in \mathbb{R}$, then q is mixed-monotone with a decomposition function $q_d = \begin{bmatrix} q_{d1}^\top & \dots & q_{di}^\top & \dots q_{dn}^\top \end{bmatrix}^\top$, where $q_{di}(x,y) = q_i(z) + (\alpha^q_i - \beta^q_i)^\top (x - y), \forall i \in \{1, \dots, n\}$, and $z, \alpha^q_i, \beta^q_i \in \mathbb{R}^n$ can be computed in terms of $x, y, a^q_{i,j}, b^q_{i,j}$ as given in [20, (10)–(13)]. Consequently, for $x = [x_1 \dots x_j \dots x_n]^\top$, $y = [y_1 \dots y_j \dots y_n]^\top$, we have

$$q_d(x,y) = q(z) + C^q (x - y), \tag{1}$$

where $C^q \triangleq \left[[\alpha^q_1 - \beta^q_1] \dots [\alpha^q_i - \beta^q_i] \ \dots [\alpha^q_m - \beta^q_m]\right]^\top \in \mathbb{R}^{m\times n}$, with α^q_i, β^q_i given in [20, (10)–(13)], $z = [z_1 \dots z_j \dots z_m]^\top$ and $z_j = x_j$ or y_j (dependent on the case, cf. [20, Theorem 1 and (10)–(13)] for details). On the other hand, when the precise lower and upper bounds, $a_{i,j}, b_{i,j}$, of the partial derivatives are not known or are hard to compute, we can obtain upper and lower approximations of the bounds by using Proposition 2 with the slopes set to zero, or by leveraging interval arithmetics [4].

III. Problem Formulation

System Assumptions. Consider a partially unknown nonlinear discrete-time system with bounded noise

$$\begin{aligned} x_{k+1} &= f(x_k, d_k, u_k, w_k), \\ y_k &= g(x_k, d_k, u_k, v_k), \end{aligned} \tag{2}$$

where $x_k \in \mathcal{X} \subset \mathbb{R}^n$ is the state vector at time $k \in \mathbb{N}$, $u_k \in \mathcal{U} \subset \mathbb{R}^m$ is a known input vector, $y_k \in \mathbb{R}^l$ is the measurement vector and $d_k \in \mathcal{D} \subset \mathbb{R}^p$ is an unknown (dynamic) input vector whose dynamics is governed by an *unknown*[a] *vector field* $h(\cdot)$:

$$d_{k+1} = h(x_k, d_k, u_k, w_k). \tag{3}$$

Moreover, we refer to $z_k \triangleq \begin{bmatrix} x_k^\top & d_k^\top \end{bmatrix}^\top$ as the augmented state. The process noise $w_k \in \mathbb{R}^{n_w}$ and the measurement noise $v_k \in \mathbb{R}^l$ are assumed to be bounded, with $\underline{w} \leq w_k \leq \overline{w}$ and $\underline{v} \leq v_k \leq \overline{v}$, where $\underline{w}$, $\overline{w}$ and $\underline{v}$, $\overline{v}$ are the known lower and upper bounds of the process and measurement noise signals, respectively. We also assume that lower and upper bounds, $\underline{z}_0$ and $\overline{z}_0$, for the initial augmented state $z_0 \triangleq \begin{bmatrix} x_0^\top & d_0^\top \end{bmatrix}^\top$ are available, i.e., $\underline{z}_0 \leq z_0 \leq \overline{z}_0$.

The vector fields $f(\cdot) : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R}^{n_w} \to \mathbb{R}^n$ and $g(\cdot) : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R}^l \to \mathbb{R}^l$ are known, while the vector field $h(\cdot) = \begin{bmatrix} h_1^\top(\cdot) \dots h_p^\top(\cdot) \end{bmatrix}^\top : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R}^{n_w} \to \mathbb{R}^p$ is *unknown*, but each of its arguments $h_j(\cdot) : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R}^{n_w} \to \mathbb{R}$, $\forall j \in \{1 \dots p\}$ is known to be Lipschitz continuous. For simplicity and without loss of generality, we assume that the Lipschitz constant L^h_j is known; otherwise, we can estimate the Lipschitz constants with any desired precision using the approach in [19, Equation (12) and Proposition 3]. Moreover, we assume the following:

Assumption 1. *The vector field $f(\cdot)$ is mixed-monotone.*

Assumption 2. *The entire space $\mathbb{X} \triangleq \mathcal{Z} \times \mathcal{U}$ is bounded, where $\mathcal{Z} \triangleq \mathcal{X} \times \mathcal{D}$ and $\mathcal{U}$ are the spaces of the augmented states $z_k \triangleq \begin{bmatrix} x_k^\top & d_k^\top \end{bmatrix}^\top$ and the known inputs u_k, $\forall k \in \{0 \dots \infty\}$, respectively.*

Note that Assumption 1 is satisfied for a broad range of nonlinear functions [20], while Assumption 2 is reasonable for most practical systems.

The observer design problem can be stated as follows:

Problem 1. *Given a partially known nonlinear discrete-time system* (2) *with bounded noise signals and unknown dynamics* (3)*, design a stable observer that simultaneously finds bounded intervals of compatible augmented states and learns an unknown dynamics model for* (3)*.*

IV. State and Model Interval Observers (SMIO)

A. Recursive Interval Observer

In this section, we introduce a three-step recursive interval observer that combines model-based estimation and data-driven model learning approaches. The observer structure

[a]Note that if the vector field $h(\cdot)$ is partially known (i.e., consists of the sum of a known component $\hat{h}(\cdot)$ and an unknown component $\tilde{h}(\cdot)$), we can simply consider $d_{k+1} - \hat{h}(\cdot)$ as the output data for the model learning procedure to learn a model of the (completely) unknown function $\tilde{h}(\cdot)$.

is composed of a State Propagation (SP), a Measurement Update (MU) step and a Model Learning (ML) step. In the state propagation step, the interval estimate for the augmented states (consisting of the state and the unknown input) is propagated for one time step through the nonlinear state equation and the estimated model of the unknown dynamics function obtained in previous time step. In the update step, compatible intervals of the augmented states are iteratively updated given new measurements and the nonlinear observation function, and finally, the model learning step estimates the upper and lower *framer* functions (abstractions) for the unknown dynamics function. More formally, the three observer steps have the following form (with $z_k \triangleq [x_k^\top \; d_k^\top]^\top$, $z_k^p \triangleq [x_k^{p\top} \; d_k^{p\top}]^\top$):

$$\begin{aligned}
&\textit{SP:} \quad \mathcal{I}_k^{z^p} = \mathcal{F}^p(\mathcal{I}_{k-1}^z, y_{k-1}, u_{k-1}, \overline{h}_{k-1}(.), \underline{h}_{k-1}(.)),\\
&\textit{MU:} \quad \mathcal{I}_k^{z} = \mathcal{F}^u(\mathcal{I}_{k}^{z^p}, y_{k}, u_{k}),\\
&\textit{ML:} \quad [\underline{h}_k^\top(.) \; \overline{h}_k^\top(.)]^\top = \mathcal{F}^l(\{\mathcal{I}_{k-t}^z, u_{k-t}\}_{t=0}^k),
\end{aligned}$$

with $\mathcal{F}^p$ and $\mathcal{F}^u$ being to-be-designed interval-valued mappings and $\mathcal{F}^l$ a to-be-constructed function over-approximation procedure (abstraction model), while $\mathcal{I}_k^{z^p}$ and $\mathcal{I}_k^z$ are the intervals of compatible propagated and estimated augmented states, respectively, and $\{\overline{h}_k(\cdot), \underline{h}_k(\cdot)\}$ is a *data-driven abstraction/over-approximation model* for the unknown function $h(\cdot)$, at time step k, i.e.,

$$\forall \zeta_k \in \mathcal{D}_h : \underline{h}_k(\zeta_k) \le h(\zeta_k) \le \overline{h}_k(\zeta_k),$$

where $\mathcal{D}_h$ is the domain of $h(\cdot)$ and $\zeta_k \triangleq [z_k^\top \; u_k^\top \; w_k^\top]^\top$.

To leverage the properties of intervals [11] while avoiding the computational complexity of optimal observers [23], we consider the following form of interval estimates in the propagation and update steps:

$$\begin{aligned}
\mathcal{I}_k^{z^p} &= \{z \in \mathbb{R}^{n+p} : \underline{z}_k^p \le z \le \overline{z}_k^p\},\\
\mathcal{I}_k^{z} &= \{z \in \mathbb{R}^{n+p} : \underline{z}_k \le z \le \overline{z}_k\},
\end{aligned}$$

where the estimation boils down to finding the maximal and minimal values of $\mathcal{I}_k^{z^p}$ and $\mathcal{I}_k^z$, i.e., $\overline{z}_k^p, \underline{z}_k^p, \overline{z}_k, \underline{z}_k$. Further, at the model learning step, given the sequence of interval estimates up to the current time, we plan to leverage the data-driven function abstraction/over-approximation approach developed in our previous work [19] to update and refine the learned/estimated model of the unknown dynamics function $h(\cdot)$ at the current time step.

Specifically, our interval observer at each time step $k \ge 1$ is given as follows (with the augmented state $z_k \triangleq [x_k^\top \; d_k^\top]^\top$, $\zeta_k \triangleq [z_k^\top \; u_k^\top \; w_k^\top]^\top$ and known $\underline{x}_0$ and $\overline{x}_0$ such that $\underline{x}_0 \le x_0 \le \overline{x}_0$):

State Propagation (SP):

$$\begin{bmatrix} \overline{x}_k^p \\ \underline{x}_k^p \end{bmatrix} = \begin{bmatrix} \min(f_d(\overline{z}_{k-1}, u_{k-1}, \overline{w}, \underline{z}_{k-1}, u_{k-1}, \underline{w}), \overline{x}_k^{a,p}) \\ \max(f_d(\underline{z}_{k-1}, u_{k-1}, \underline{w}, \overline{z}_{k-1}, u_{k-1}, \overline{w}), \underline{x}_k^{a,p}) \end{bmatrix}, \quad (4a)$$

$$\begin{bmatrix} \overline{d}_k^p \\ \underline{d}_k^p \end{bmatrix} = \mathbb{A}_k^h \begin{bmatrix} \overline{z}_{k-1}^p \\ \underline{z}_{k-1}^p \end{bmatrix} + \mathbb{B}_k^h u_{k-1} + \mathbb{W}_k^h \begin{bmatrix} \overline{w} \\ \underline{w} \end{bmatrix} + \tilde{e}_k^h, \quad (4b)$$

$$\overline{z}_k^p = \begin{bmatrix} \overline{x}_k^{p^\top} & \overline{d}_k^{p^\top} \end{bmatrix}^\top, \underline{z}_k^p = \begin{bmatrix} \underline{x}_k^{p^\top} & \underline{d}_k^{p^\top} \end{bmatrix}^\top; \quad (4c)$$

Measurement Update (MU):

$$\begin{bmatrix} \overline{z}_k & \underline{z}_k \end{bmatrix} = \lim_{i \to \infty} \begin{bmatrix} \overline{z}_{i,k}^u & \underline{z}_{i,k}^u \end{bmatrix}, \quad (5a)$$

$$\begin{bmatrix} \overline{x}_k & \underline{x}_k \\ \overline{d}_k & \underline{d}_k \end{bmatrix} = \begin{bmatrix} \overline{z}_{k,(1:n)} & \underline{z}_{k,(1:n)} \\ \overline{z}_{k,(n+1:n+p)} & \underline{z}_{k,(n+1:n+p)} \end{bmatrix}; \quad (5b)$$

Model Learning (ML):

$$\overline{h}_{k,j}(\zeta_k) = \min_{t \in \{0,\dots,T-1\}} (\overline{d}_{k-t,j} + L_j^h \|\zeta_k - \tilde{\zeta}_{k-t}\|) + \varepsilon_{k-t}^j, \quad (6a)$$

$$\underline{h}_{k,j}(\zeta_k) = \max_{t \in \{0,\dots,T-1\}} (\underline{d}_{k-t,j} - L_j^h \|\zeta_k - \tilde{\zeta}_{k-t}\|) + \varepsilon_{k-t}^j, \quad (6b)$$

where $j \in \{1 \dots p\}$, $\{\tilde{\zeta}_{k-t} = \frac{1}{2}(\overline{\zeta}_{k-t} + \underline{\zeta}_{k-t})\}_{t=0}^k$ and $\{\overline{d}_{k-t}, \underline{d}_{k-t}\}_{t=0}^k$ are the *augmented* input-output data set. At each time step k, the augmented data set constructed from the estimated framers gathered from the initial to the current time step, is used in the model learning step to recursively derive over-approximations of the unknown function $h(\cdot)$, i.e., $\{\overline{h}_k(.), \underline{h}_k(.)\}$ by applying [19, Theorem 1]. In addition,

$$\begin{bmatrix} \overline{x}_k^{a,p} \\ \underline{x}_k^{a,p} \end{bmatrix} = \mathbb{A}_k^f \begin{bmatrix} \overline{z}_{k-1}^p \\ \underline{z}_{k-1}^p \end{bmatrix} + \mathbb{B}_k^f u_{k-1} + \mathbb{W}_k^f \begin{bmatrix} \overline{w} \\ \underline{w} \end{bmatrix} + \tilde{e}_k^f, \quad (7)$$

with $\mathbb{J}_k^q = \begin{bmatrix} J_k^{q+} & -J_k^{q++} \\ -J_k^{q++} & J_k^{q+} \end{bmatrix}$, $\mathbb{B}_k^q = \begin{bmatrix} B_k^{q\top} & B_k^{q\top} \end{bmatrix}^\top$, $\tilde{e}_k^q = \begin{bmatrix} \overline{e}_k^{q\top} & \underline{e}_k^{q\top} \end{bmatrix}^\top$, $\varepsilon_{k-t}^j = 2L_j^h \|\overline{\zeta}_{k-t} - \underline{\zeta}_{k-t}\|$, $\forall \mathbb{J} \in \{\mathbb{A}, \mathbb{W}\}, q \in \{f, h\}, J \in \{A, W\}$. Moreover, the *sequences of updated framers* $\{\overline{z}_{i,k}^u, \underline{z}_{i,k}^u\}_{i=1}^\infty$ are iteratively computed as follows:

$$\begin{bmatrix} \overline{z}_{0,k}^u & \underline{z}_{0,k}^u \end{bmatrix} = \begin{bmatrix} \overline{z}_k^p & \underline{z}_k^p \end{bmatrix}, \quad \forall i \in \{1 \dots \infty\} : \quad (8)$$

$$\begin{bmatrix} \overline{z}_{i,k}^u \\ \underline{z}_{i,k}^u \end{bmatrix} = \begin{bmatrix} \min(A_{i,k}^{g\dagger+} \overline{\alpha}_{i,k} - A_{i,k}^{g\dagger++} \underline{\alpha}_{i,k} + \omega_{i,k}, \overline{z}_{i-1,k}^u) \\ \max(A_{i,k}^{g\dagger+} \underline{\alpha}_{i,k} - A_{i,k}^{g\dagger++} \overline{\alpha}_{i,k} - \omega_{i,k}, \underline{z}_{i-1,k}^u) \end{bmatrix}, \quad (9)$$

where

$$\begin{bmatrix} \overline{t}_{i,k} \\ \underline{t}_{i,k} \end{bmatrix} = \begin{bmatrix} y_k - B_{i,k}^g u_k \\ y_k - B_{i,k}^g u_k \end{bmatrix} + \begin{bmatrix} W_{i,k}^{g++} & -W_{i,k}^{g+} \\ -W_{i,k}^{g+} & W_{i,k}^{g++} \end{bmatrix} \begin{bmatrix} \overline{v} \\ \underline{v} \end{bmatrix} - \begin{bmatrix} \underline{e}_{i,k}^g \\ \overline{e}_{i,k}^g \end{bmatrix}, \quad (10)$$

$$\begin{bmatrix} \overline{\alpha}_{i,k} \\ \underline{\alpha}_{i,k} \end{bmatrix} = \begin{bmatrix} \min(\overline{t}_{i,k}, A_{i,k}^{g+} \overline{z}_{i-1,k}^u - A_{i,k}^{g++} \underline{z}_{i-1,k}^u) \\ \max(\underline{t}_{i,k}, A_{i,k}^{g+} \underline{z}_{i-1,k}^u - A_{i,k}^{g++} \overline{z}_{i-1,k}^u) \end{bmatrix}, \quad (11)$$

and $\omega_{i,k} = \kappa \text{rowsupp}(I - A_{i,k}^{g\dagger} A_{i,k}^g)$, $\forall i \in \{1 \dots \infty\}$. In addition, $(A_k^q, B_k^q, W_k^q, \overline{e}_k^q, \underline{e}_k^q)$ for $q \in \{f, h\}$ and $(A_{i,k}^g, B_{i,k}^g, W_{i,k}^g, \overline{e}_{i,k}^g, \underline{e}_{i,k}^g)$ are solutions to the problem (12a) for the corresponding functions $\{\underline{g}(\cdot) = \overline{g}(\cdot) = g(\cdot)\}$, $\{\underline{f}(\cdot) = \overline{f}(\cdot) = f(\cdot)\}$ and $\{\overline{h}_k(\cdot), \underline{h}_k(\cdot)\}$, on the intervals $[[\underline{z}_{i-1,k}^{u\top} \; u_{k-1}^\top \; \underline{v}^\top]^\top, [\overline{z}_{i-1,k}^{u\top} \; u_{k-1}^\top \; \overline{v}^\top]^\top]$ for g and $[[\underline{z}_{k-1}^\top \; u_{k-1}^\top \; \underline{w}^\top], [\overline{z}_{k-1}^\top \; u_{k-1}^\top \; \overline{w}^\top]^\top]$ for f, $\overline{h}_k$, $\underline{h}_k$, respectively, at time k and iteration i, while κ is a very large positive real number (infinity), while f_d is the bounding function based on (1).

Note that since the tightness of the upper and lower bounding functions for the observation function g (cf. Propositions 1 and 2) depends on the *a priori* interval $\mathcal{B}$, the measurement update step is done iteratively (see proof of Theorem 2 for more explanation). Hence, if tighter updated intervals are obtained starting from the compatible intervals from the propagation step, we can use them as the new $\mathcal{B}$ to obtain better abstraction/bounding functions for g, which in turn may lead to even tighter updated intervals.

Repeating this process results in a sequence of monotonically tighter updated intervals, that is convergent by the monotone convergence theorem, and its limit is chosen as the final interval estimate at time k. Note that when implementing the observer, a desired user-specified stopping criterion/threshold can be used so that the observer can be implemented in finite time. The choice of the stopping criterion may have an impact on the observer performance but does not affect the correctness and stability guarantees provided in this paper.

Further, building upon our previous result in [19, Theorem 1], in the model learning step with the history of obtained compatible intervals up to the current time, $\{[\underline{z}_s, \overline{z}_s]\}_{s=0}^k$ as the noisy input data and the compatible interval of unknown inputs, $[\underline{d}_k, \overline{d}_k]$, as the noisy output data, we recursively construct a sequence of *abstraction/over-approximation models* $\{\overline{h}_k(\cdot), \underline{h}_k(\cdot)\}_{k=1}^\infty$ for the unknown input function $h(\cdot)$, that by construction satisfy (16), i.e., our model estimation is correct (i.e., is guaranteed to frame/bracket the true function) and becomes more precise with time (cf. Lemma 1).

B. Correctness of the Observer

The objective of this section is to design the SMIO observer gains such that the *framer property* [8] holds, i.e., we desire to guarantee that the observer returns correct interval estimates, in the sense that starting from the initial interval $\underline{z}_0 \leq z_0 \leq \underline{z}_0$, the true augmented states of the dynamic system (2) are guaranteed to be within the estimated intervals, given by (4a)-(6b). If the observer is correct, $\{\overline{z}_k, \underline{z}_k\}_{k=0}^\infty$ is a *framer sequence* for system (2).

Before deriving our main first result on correctness of the observer, we state a modified version of our previous result in [22, Theorem 1], in a unified manner that enables us to derive parallel global and local affine bounding functions for our known $f(\cdot), g(\cdot)$ and unknown $h(\cdot)$ vector fields. For brevity, some of the more straightforward proofs of lemmas and propositions are omitted. Interested readers are referred to an extended version of this paper [24] for more details.

Proposition 2 (Parallel Affine Abstractions)**.** *Let the entire space be defined as $\mathbb{X}$ and suppose that Assumption 2 holds. Consider the vector fields $\overline{q}(.), \underline{q}(.) : \mathbb{X} \subset \mathbb{R}^{n'} \to \mathbb{R}^{m'}$ satisfying $\underline{q}(\zeta) \leq \overline{q}(\zeta), \forall \zeta \in \mathbb{X}$ and the following Linear Program (LP):*

$$\min_{\theta^q_{\mathcal{B}}, A^q_{\mathcal{B}}, \overline{e}^q_{\mathcal{B}}, \underline{e}^q_{\mathcal{B}}} \theta^q_{\mathcal{B}} \tag{12a}$$

$$\begin{aligned} s.t \ \ & A^q_{\mathcal{B}}\zeta_s + \underline{e}^q_{\mathcal{B}} + \sigma^q \leq \underline{q}(\zeta_s) \leq \overline{q}(\zeta_s) \leq A^q_{\mathcal{B}}\zeta_s + \overline{e}^q_{\mathcal{B}} - \sigma^q, \\ & \overline{e}^q_{\mathcal{B}} - \underline{e}^q_{\mathcal{B}} - 2\sigma^q \leq \theta^q \mathbf{1}_{m'}, \\ & \underline{e}^q - \underline{e}^q_{\mathcal{B}} \leq (A^q_{\mathcal{B}} - \mathbb{A}^q)\zeta_s \leq \overline{e}^q - \overline{e}^q_{\mathcal{B}}, \forall \zeta_s \in \mathcal{V}_{\mathcal{B}}, \end{aligned} \tag{12b}$$

where $\mathcal{B}$ is an interval with $\overline{\zeta}, \underline{\zeta}$ and $\mathcal{V}_{\mathcal{B}}$ being its maximal, minimal and set of vertices, respectively, $\mathbf{1}_m \in \mathbb{R}^m$ is a vector of ones, σ^q is given in [22, Proposition 1 and (8)] for different classes of vector fields and $(\mathbb{A}^q, \overline{e}^q, \underline{e}^q)$ are the global parallel affine abstraction matrices for the pair of functions $\overline{q}(.), \underline{q}(.)$ on the entire space $\mathbb{X}$, i.e.,

$$\mathbb{A}^q\zeta + \underline{e}^q \leq \underline{q}(\zeta) \leq \overline{q}(\zeta) \leq \mathbb{A}^q\zeta + \overline{e}^q, \forall \zeta \in \mathbb{X}. \tag{13}$$

Using the above proposition, we first solve (12a) on the entire space $\mathbb{X}$, i.e., with $\mathcal{B} = \mathbb{X}$ (where the constraint (12b) is trivially satisfied and is thus redundant) and obtain a tuple of $(\theta^q, \mathbb{A}^q, \overline{e}^q, \underline{e}^q)$ that satisfies (13), i.e., we construct a global affine abstraction model for the pair of functions $\overline{q}(.), \underline{q}(.)$ on the entire space $\mathbb{X}$.

Next, given the (global) tuple $(\mathbb{A}^q, \overline{e}^q, \underline{e}^q)$ computed as described above, we solve (12a) on $\mathcal{B}$ subject to (12b) to obtain a tuple of local parallel affine abstraction matrices for the pair of functions $\{\underline{q}(\cdot), \overline{q}(\cdot)\}$ on the interval $\mathcal{B}$, satisfying the following: $\forall \zeta \in \mathcal{B}$,

$$\mathbb{A}^q\zeta + \underline{e}^q \leq A^q_{\mathcal{B}}\zeta + \underline{e}^q_{\mathcal{B}} \leq \underline{q}(\zeta) \leq \overline{q}(\zeta) \leq A^q_{\mathcal{B}}\zeta + \overline{e}^q_{\mathcal{B}} \leq \mathbb{A}^q\zeta + \overline{e}^q. \tag{14}$$

Now, equipped with all the required tools, we state our first main result on the framer property of the SMIO observer.

Theorem 1 (Correctness of the Observer)**.** *Consider the system* (2) *with its augmented state defined as $z \triangleq \begin{bmatrix} x^\top & d^\top \end{bmatrix}^\top$, along with the SMIO observer in* (4a)–(6b)*. Suppose that Assumptions 1–2 hold and $f_d(\cdot)$ is a decomposition function of $f(\cdot)$. Then, the SMIO observer estimates are correct, i.e., the sequences of intervals $\{\overline{z}_k, \underline{z}_k\}_{k=0}^\infty$ are framers of the augmented state sequence of system* (2) *that satisfy $\underline{z}_k \leq z_k \leq \overline{z}_k$ for all k.*

Proof. We will prove this by induction. For the base case, by assumption, $\underline{z}_0 \leq z_0 \leq \overline{z}_0$ holds. Now, for the induction step, suppose that $\underline{z}_{k-1} \leq z_{k-1} \leq \overline{z}_{k-1}$. Then, by [9, Lemma 1], Propositions 1–2, (2),(7)–(4c) and [19, Theorem 1], we have $\underline{z}^p_k \leq z_k \leq \overline{z}^p_k$. Given this, by iteratively obtaining upper and lower abstraction matrices for the observation function $g(.)$ based on Proposition 2 and applying [9, Lemma 1]:

$$\underline{\alpha}_{i,k} \leq A^g_{i,k} z_k \leq \overline{\alpha}_{i,k}, \tag{15}$$

where $\underline{\alpha}_{i,k}, \overline{\alpha}_{i,k}$ are given in (11) and $A^g_{i,k}$ is a solution of the LP in (12a), i.e., the parallel abstraction slope for function $g(.)$ at iteration i in the corresponding compatible interval $[\underline{z}^u_{i-1,k}, \overline{z}^u_{i-1,k}]$. Then, multiplying (15) by $A^{g\dagger}_{i,k}$ and using the fact that $\underline{z}^u_{i-1,k}, \overline{z}^u_{i-1,k}$ are framers for the augmented state z_k at time k, [9, Lemma 1] and [25], we obtain $\underline{z}^u_{i,k} \leq z_k \leq \overline{z}^u_{i,k}$, with $\underline{z}^u_{i,k}, \overline{z}^u_{i,k}$ given in (9). Now, note that by construction, the sequences of updated upper and lower framers, $\{\overline{z}^u_{i,k}\}_{i=0}^\infty$ and $\{\underline{z}^u_{i,k}\}_{i=0}^\infty$ with $\overline{z}^u_{0,k} = \overline{z}^p_k$ and $\overline{z}^u_{0,k} = \overline{z}^p_k$, are monotonically decreasing and increasing, respectively, and hence are convergent by the *monotone convergence theorem*. Consequently, their limits $\overline{z}_k, \underline{z}_k$ are the tightest possible framers, i.e., $\forall i \in \{1 \dots \infty\}$:

$$\begin{aligned} \underline{z}^u_{0,k} \leq \dots \leq \underline{z}^u_{i,k} \leq \dots \leq \lim_{i\to\infty} \underline{z}^u_{i,k} \triangleq \underline{z}_k, \\ \overline{z}_k \triangleq \lim_{i\to\infty} \overline{z}^u_{i,k} \leq \dots \leq \overline{z}^u_{i,k} \leq \dots \leq \overline{z}^u_{0,k}, \end{aligned}$$

where $\overline{z}_k, \underline{z}_k$ are the returned updated augmented state framers by the observer. This completes the proof. ■

Next, we show that given correct interval estimates, the abstraction model of the unknown dynamics function becomes tighter (i.e., more precise) over time, so our model estimate of the unknown dynamics becomes more accurate over time.

Lemma 1. *Consider the system* (2) *and the SMIO observer* *in* (4a)–(6b) *and suppose that all the assumptions in Theorem 1 hold. Then, the following holds:*

$$\begin{array}{l}\underline{h}_0(\zeta_0)\leq\ldots\leq\underline{h}_k(\zeta_k)\leq\ldots\leq\lim_{k\to\infty}\underline{h}_k(\zeta_k)\leq h(\zeta_k)\\ h(\zeta_k)\leq\lim_{k\to\infty}\overline{h}_k(\zeta_k)\leq\ldots\leq\overline{h}_k(\zeta_k)\leq\ldots\leq\overline{h}_0(\zeta_0),\end{array} \tag{16}$$

i.e, the unknown input model estimations/abstractions are correct and become more precise or tighter with time.

C. Observer Stability

In this section, we investigate the stability of the designed observer in the following sense:

Definition 3 (Stability)**.** *The observer SMIO* (4a)–(6b) *is stable, if the sequence of interval widths* $\{\|\Delta^z_{k-1}\| \triangleq \|\overline{z}_{k-1}-\underline{z}_{k-1}\|\}_{k=1}^{\infty}$ *is uniformly bounded, and consequently, the sequence of estimation errors* $\{\|\tilde{z}_{k-1}\|\triangleq\max(\|z_{k-1}-\underline{z}_{k-1}\|,\|\overline{z}_{k-1}-z_{k-1}\|)$ *is also uniformly bounded.*

Next, we derive a property for the decomposition function given in (1), which will be helpful in deriving sufficient conditions for the observer stability.

Lemma 2. *Let* $q(\zeta):\mathbb{X}\subset\mathbb{R}^n\to\mathbb{R}^m$ *be a mixed-monotone vector-field with a corresponding decomposition function* $q_d(.,.)$ *constructed using* (1)*. Suppose that Assumption 2 holds and let* $(\mathbb{A}^q,\overline{e}^q,\underline{e}^q)$ *be the parallel affine abstraction matrices for function* $q(\cdot)$ *on its entire domain* $\mathbb{X}$ *(can be computed via Proposition 2). Consider any ordered pair* $\underline{\zeta}\leq\overline{\zeta}\in\mathbb{X}$*. Then,* $\Delta q_\zeta\leq(|\mathbb{A}^q|+2C^q)\Delta\zeta+\Delta e^q$*, with* $\Delta q_\zeta\triangleq q_d(\overline{\zeta},\underline{\zeta})-q_d(\underline{\zeta},\overline{\zeta})$*,* $\Delta\zeta\triangleq\overline{\zeta}-\underline{\zeta}$ *and* C^q *given in* (1)*.*

We are now ready to state our next main result on the SMIO observer stability in the following theorem.

Theorem 2 (Observer Stability)**.** *Consider the system* (2) *along with the SMIO observer in* (4a)–(6b)*. Let* $\mathbb{D}_m$ *be the set of all diagonal matrices in* $\mathbb{R}^{m\times m}$ *with their diagonal arguments being* 0 *or* 1*. Suppose that all the assumptions in Theorem 1 hold and the decomposition function* f_d *is constructed using* (1)*. Then, the observer is stable if there exist* $D_1\in\mathbb{D}_{n+p},D_2\in\mathbb{D}_l,D_3\in\mathbb{D}_n$ *that satisfy* $D_{1,i,i}=0$ *if* $r(i)=1$*, i.e., if there exist* $(D_1,D_2,D_3)\in\mathbb{D}^*\triangleq\{(D_1,D_2,D_3)\in\mathbb{D}_{n+p}\times\mathbb{D}_l\times\mathbb{D}_n|D_{1,ii}r(i)=0\}$ *such that*

$$\mathcal{L}^*(D_1,D_2,D_3)\triangleq\|\mathcal{A}^g(D_1,D_2)\mathcal{A}^{f,h}(D_3)\|\leq 1, \tag{17}$$

with $\mathcal{A}^g(D_1,D_2)\triangleq(I-D_1)+D_1|A^{g\dagger}|(I-D_2)|A^g|$*,* $\mathcal{A}^{f,h}(D_3)\triangleq\left[(|A^f|+2(I-D_3)C^f_z)^\top\ |A^h|^\top\right]^\top$*,* $\{A^q\triangleq\mathbb{A}^q_{(1:n+p)}\}_{q\in\{f,g,h\}}$*,* $\mathbb{A}^q$ *given in Proposition 1,* $r\triangleq\mathrm{rowsupp}(I-A^{g\dagger}A^g)$*, and* $C^f\triangleq\left[C^f_z\ \ C^f_u\ \ C^f_w\right]$ *from* (1)*.*

Proof. Note that our goal is to obtain sufficient stability conditions that can be checked *a priori* instead of for each time step k. On the other hand, for the implementation of the update step, we iteratively find new *local* parallel abstraction slopes $A^g_{i,k}$ by iteratively solving the LP (12a) for g on the intervals obtained in the previous iteration, $\mathcal{B}^u_{i,k}=[\underline{z}^u_{i-1,k},\overline{z}^u_{i-1,k}]$, to find *local* framers $\overline{z}^u_{i,k},\underline{z}^u_{i,k}$ (cf. (8)–(11)), with additional constraints given in (12b) in the optimization problems, which guarantees that the iteratively updated *local* intervals obtained using the local abstraction slopes are inside the global interval, i.e.,

$$\begin{array}{l}\underline{z}^u_k\leq\underline{z}^u_{0,k}\leq\cdots\leq\underline{z}^u_{i,k}\leq\cdots\leq\lim_{i\to\infty}\underline{z}^u_{i,k}\triangleq\underline{z}_k,\\ \overline{z}_k\triangleq\lim_{i\to\infty}\overline{z}^u_{i,k}\leq\cdots\leq\overline{z}^u_{i,k}\leq\cdots\leq\overline{z}^u_{0,k}\leq\overline{z}^u_k,\end{array}$$

where we apply (9) for just one iteration (dropping index i) with $\overline{z}^u_{k,0}=\overline{z}^p_k,\underline{z}^u_{k,0}=\underline{z}^p_k$ to obtain:

$$\begin{bmatrix}\overline{z}^u_k\\ \underline{z}^u_k\end{bmatrix}=\begin{bmatrix}\min(A^{g\dagger+}\overline{\alpha}_k-A^{g\dagger++}\underline{\alpha}_k+\omega,\overline{z}^p_k)\\ \max(A^{g\dagger+}\underline{\alpha}_k-A^{g\dagger++}\overline{\alpha}_k-\omega,\underline{z}^p_k)\end{bmatrix}. \tag{18}$$

This allows us to use the *global* parallel affine abstraction slope A^g for the stability analysis as follows. Dropping index i in (10)–(11) and defining $\Delta^z_k\triangleq\overline{z}_k-\underline{z}_k$ (and similarly for $\Delta^{z^p}_k,\Delta^g_e,\Delta^f_e,\Delta^h_e,\Delta^\alpha_k,\Delta^t_k$), (9) implies that $\forall D_1\in\mathbb{D}_{n+p}$

$$\begin{aligned}\Delta^z_k&\leq\min(|A^{g\dagger}|\Delta^\alpha_k+2\kappa\mathbf{r},\Delta^{z^p}_k)\\ &\leq D_1(|A^{g\dagger}|\Delta^\alpha_k+2\kappa\mathbf{r})+(I-D_1)\Delta^{z^p}_k,\end{aligned} \tag{19}$$

where the second inequality follows from generalization of the fact that $\min(a,b)\leq\lambda a+(1-\lambda)b,\forall a,b\in\mathbb{R},\lambda\in[0,1]$. Moreover, from (10)–(11) and a similar reasoning, we observe that $\forall D_2\in\mathbb{D}_l$:

$$\begin{aligned}\Delta^\alpha_k&\leq\min(|W^g|\Delta v+\Delta^g_e,|A^g|\Delta^{z^p}_k)\\ &\leq D_2(|W^g|\Delta v+\Delta^g_e)+(I-D_2)|A^g|\Delta^{z^p}_k.\end{aligned} \tag{20}$$

On the other hand, by similar arguments, it follows from (4a)–(4c) that $\forall D_3\in\mathbb{D}_n$,

$$\Delta^{z^p}_k\leq\begin{bmatrix}D_3(|A^f|\Delta^z_{k-1}+|W^f|\Delta w+\Delta^f_e)+(I-D_3)\Delta^f_{k-1}\\ |A^h|\Delta^z_{k-1}+|W^h|\Delta w+\Delta^h_e\end{bmatrix}, \tag{21}$$

where $\Delta^f_{k-1}\triangleq f_d(\overline{\zeta}_{k-1},\underline{\zeta}_{k-1})-f_d(\underline{\zeta}_{k-1},\overline{\zeta}_{k-1})$. Furthermore, by Lemma 2, $\Delta^f_{k-1}\leq(|A^f|+2C^f_z)\Delta^z_{k-1}+(|W^f|+2C^f_w)\Delta w+\Delta^f_e$, with $C^f=\left[C^f_z\ \ C^f_u\ \ C^f_w\right]$ given in (1). This, in addition to (19)–(21), [9, Lemma 1] and non-negativity of both sides of all the inequalities, lead to: $\forall(D_1,D_2,D_3)\in\mathbb{D}_{n+p}\times\mathbb{D}_l\times\mathbb{D}_n$:

$$\begin{aligned}\Delta^z_k\leq\ &\mathcal{A}^g(D_1,D_2)\mathcal{A}^{f,h}(D_3)\Delta^z_{k-1}\\ &+\Delta^g(D_1,D_2)+\mathcal{A}^g(D_1,D_2)\Delta^{f,h}(D_3)+2\kappa D_1\mathbf{r},\end{aligned} \tag{22}$$

where $\mathcal{A}^g(D_1,D_2)\triangleq D_1|A^{g\dagger}|D_2|A^g|+(I-D_1)$, $\mathcal{A}^{f,h}(D_3)\triangleq\left[(|A^f|+2(I-D_3)C^f_z)^\top\ |A^h|^\top\right]^\top$, $\Delta^g(D_1,D_2)\triangleq D_1|A^{g\dagger}|D_2(|W^g|\Delta v+\Delta^g_e)$ and $\Delta^{f,h}(D_3)\triangleq\left[((|W^f|+2(I-D_3)C^f_w)\Delta w+\Delta^f_e)^\top\ (|W^h|\Delta w+\Delta^h_e)^\top\right]^\top$. Since κ can be infinitely large, to make the right hand side of (22) finite in finite time, we choose $D_1\in\mathbb{D}_{n+p}$ such that $D_1\mathbf{r}=0$, i.e., $D_{1,i,i}=0$ if $r(i)=1,i=1,\ldots,n+p$. Then, by the *Comparison Lemma* [26], it suffices for uniform boundedness of $\{\Delta^z_k\}_{k=0}^\infty$ that the following system:

$$\Delta^z_k=\mathcal{A}^g(D_1,D_2)\mathcal{A}^{f,h}(D_3)\Delta^z_{k-1}+\tilde{\Delta}(D_1,D_2), \tag{23}$$

be stable, where $\tilde{\Delta}(D_1,D_2)\triangleq\Delta^g(D_1,D_2)+\mathcal{A}^g(D_1,D_2)\Delta^{f,h}(D_3)$ is a bounded disturbance. This implies that the system (23) is stable (in the sense of uniform stability of the interval sequnces) if and only if the matrix $\mathcal{A}(D_1,D_2,D_3)\triangleq\mathcal{A}^g(D_1,D_2)\mathcal{A}^{f,h}(D_3)$ is (non-strictly) stable for at least one choice of (D_1,D_2,D_3), equivalently (17) should hold. ∎

Remark 1. *The sufficient condition in Theorem 2 has a* *finitely countable feasible set ($|\mathbb{D}^*| \le 2^{2n+p+l}$); hence, the condition can be easily checked by enumerating all possible cases and checking the satisfaction of* (17).

Finally, we conclude this section by providing upper bounds for the interval widths and their steady-state values.

Proposition 3 (Upper Bounds of the Interval Widths and their Convergence)**.** *Consider the system* (2) *and the observer* (4a)–(6b) *and suppose all the assumptions in Theorem 2 hold. Then, the sequence of* $\{\Delta^z_k \triangleq \overline{z}_k - \underline{z}_k\}_{k=0}^{\infty}$ *is uniformly upper bounded by a convergent sequence, as follows:*

$$\Delta^z_k \le \overline{\mathcal{A}}^k \Delta^z_0 + \sum_{j=0}^{k-1} \overline{\mathcal{A}}^j \overline{\Delta} \xrightarrow{k\to\infty} e^{\overline{\mathcal{A}}} \overline{\Delta}, \tag{24}$$

where

$\overline{\mathcal{A}} = \mathcal{A}(D_1^*, D_2^*, D_3^*) \triangleq \mathcal{A}^g(D_1^*, D_2^*)\mathcal{A}^{f,h}(D_3^*)$,
$\overline{\Delta} = \Delta^g(D_1^*, D_2^*) + \mathcal{A}^g(D_1^*, D_2^*)\Delta^{f,h}(D_3^*)$,
$\mathcal{A}^g(D_1, D_2) \triangleq D_1|A^{g\dagger}|D_2|A^g| + (I - D_1)$,
$\mathcal{A}^{f,h}(D_3) \triangleq \begin{bmatrix}(|A^f| + 2(I - D_3)C^f_z)^\top & |A^h|^\top\end{bmatrix}^\top$,
$\Delta^g(D_1, D_2) \triangleq D_1|A^{g\dagger}|D_2(|W^g|\Delta v + \Delta e^g)$, $\Delta^{f,h}(D_3) \triangleq$
$\begin{bmatrix}((|W^f|+2(I-D_3)C^f_w)\Delta w+\Delta^f_e)^\top (|W^h|\Delta w+\Delta^h_e)^\top\end{bmatrix}^\top$,
and (D_1^*, D_2^*, D_3^*) *is a solution of the following problem:*

$$\min_{D_1,D_2,D_3} \|e^{\mathcal{A}(D_1,D_2,D_3)}(\Delta^g(D_1,D_2)+\mathcal{A}^g(D_1,D_2)\Delta^{f,h}(D_3))\|$$
$$s.t. (D_1, D_2, D_3) \in \{(D_1, D_2, D_3) \in \mathbb{D}^* | \mathcal{L}^*(D_1, D_2, D_3) < 1\}.$$

Consequently, the sequence of interval widths $\{\|\Delta^z_k\|\}_{k=1}^{\infty}$ *is uniformly upper bounded by a convergent sequence, i.e.,*

$$\|\Delta^z_k\| \le \delta^z_k \triangleq \|\overline{\mathcal{A}}^k \Delta^z_0 + \sum_{j=0}^{k-1} \overline{\mathcal{A}}^j \overline{\Delta}\| \xrightarrow{k\to\infty} \|e^{\overline{\mathcal{A}}}\overline{\Delta}\|. \tag{25}$$

Proof. The proof is straightforward by applying [9, Lemma 1], computing (22) iteratively, using the fact that by Theorem 2, $\mathcal{A}(D_1, D_2, D_3)$ is a stable matrix, where (D_1, D_2, D_3) is a solution of (17), and from triangle inequality. ∎

V. Illustrative Example

We consider a slightly modified version of the continuous-time predator-prey system in [27]: $\dot{x}_1 = -x_1x_2 - x_2 + u + d + w_1, \dot{x}_2 = x_1x_2 + x_1 + w_2, \dot{d} = 0.1(\cos(x_1) - \sin(x_2)) + w_d$, where the (unknown input) dynamics $\dot{d}$ is an unknown function, and the output equations are given by:

$$y_1 = x_1 + v_1, y_2 = x_2 + v_2, y_3 = \sin(d) + v_3,$$

We use the forward Euler method to discretize the system and the system can be described in the form (2)–(3) with the following parameters: $n = 2$, $l = 3$, $p = m = 1$, $f(.) = \begin{bmatrix}f_1(.) & f_2(.)\end{bmatrix}^\top$, $g(.) = \begin{bmatrix}g_1(.) & g_2(.) & g_3(.)\end{bmatrix}^\top$, $u_k = 0$, $w_k = \begin{bmatrix}w_{1,k} & w_{2,k} & w_{d,k}\end{bmatrix}^\top$, $v_k = \begin{bmatrix}v_{1,k} & v_{2,k} & v_{3,k}\end{bmatrix}^\top$, $\overline{v} = -\underline{v} = \overline{w} = -\underline{w} = \begin{bmatrix}0.1 & 0.1 & 0.1\end{bmatrix}^\top$, $\overline{x}_0 = \begin{bmatrix}0 & 0.6\end{bmatrix}^\top$, $\underline{x}_0 = \begin{bmatrix}-0.35 & -0.1\end{bmatrix}^\top$, where $g_3(\cdot) = \sin(d_k) + v_{3,k}$,

$$\begin{aligned} f_1(\cdot) &= x_{1,k} + \delta_t(-x_{1,k}x_{2,k} - x_{2,k} + u_k + d_k + w_{1,k}),\\ f_2(\cdot) &= x_{2,k} + \delta_t(x_{1,k}x_{2,k} + x_{1,k} + w_{2,k}),\\ h(\cdot) &= d_k + \delta_t(0.1(\cos(x_{1,k}) - \sin(x_{2,k})) + w_{d,k})\\ g_1(\cdot) &= x_{1,k} + v_{1,k},\ g_2(\cdot) = x_{2,k} + v_{2,k}, \end{aligned}$$

with sampling time $\delta_t = 0.01s$. Moreover, using Proposition 2 with abstraction slopes set to zero, we can obtain finite-valued upper and lower bounds (horizontal abstractions) for the partial derivatives of $f(\cdot)$ as: $\begin{bmatrix}a^f_{11} & a^f_{12} & a^f_{12}\\ a^f_{21} & a^f_{22} & a^f_{23}\end{bmatrix} = \begin{bmatrix}0.994 & -0.01 & 1-\epsilon\\ 0.009 & 0.9965 & -\epsilon\end{bmatrix}$, $\begin{bmatrix}b^f_{11} & b^f_{12} & b^f_{13}\\ b^f_{21} & b^f_{22} & b^f_{23}\end{bmatrix} = \begin{bmatrix}1.006 & -0.0065 & 1+\epsilon\\ 0.016 & 1 & \epsilon\end{bmatrix}$, where ϵ is a very small positive value, ensuring that the partial derivatives are in open intervals (cf. [20, Theorem 1]). Therefore, Assumption 1 holds by [20, Theorem 1]). Hence, we expect that the true states and unknown inputs are within the interval estimates by Theorem 1, i.e., the interval estimates are correct. This can be observed from Figures 1 and 2, where the true states and unknown inputs as well as interval estimates are depicted. Furthermore, solving the optimization problem in Proposition 2 for the global abstraction matrices, we obtained $A^f = \begin{bmatrix}0.6975 & -0.0083 & 0.01\\ 0.0125 & 0.9982 & 0\end{bmatrix}$, $A^g = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0.995\end{bmatrix}$, $A^h = \begin{bmatrix}0 & -0.0015 & .6\end{bmatrix}$ and from [20, (10)–(13)]), we obtained $C_f = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$ when using (1). Consequently, (17) is satisfied and so, the sufficient condition in Theorem 2 holds. Moreover, as can be seen in Figure 3, we obtain uniformly bounded and convergent interval estimate errors when applying our observer design procedure, where at each time step, the actual error sequence is upper bounded by the interval widths, which converge to steady-state values. Further, Figure 4 shows the framer intervals of the learned/estimated unknown dynamics model (depicted by the "kinky" red and blue meshes) that frame the actual unknown dynamics function $h(\cdot)$, as well as the global abstraction that is computed via Proposition 2 at the initial step.

Note that as discussed in the proof of Theorem 2, since

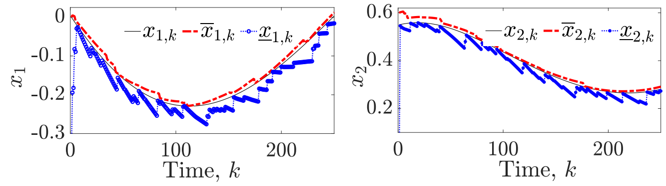

Fig. 1: Actual states, $x_{1,k}$, $x_{2,k}$, as well as their estimated maximal and minimal values, $\overline{x}_{1,k}$, $\underline{x}_{1,k}$, $\overline{x}_{2,k}$, $\underline{x}_{1,k}$.

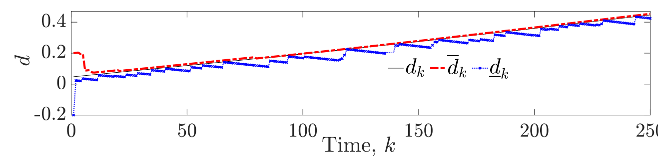

Fig. 2: Actual unknown input, d_k, as well as its estimated (learned) maximal and minimal values, $\overline{d}_k$, $\underline{d}_k$.

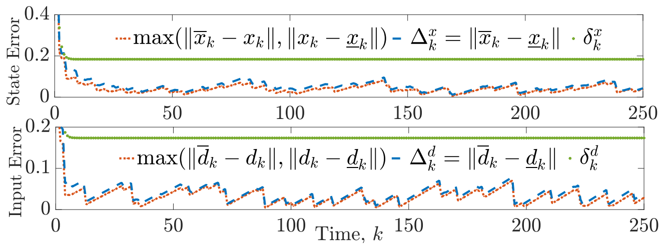

Fig. 3: Actual estimation errors $\max(\|\overline{x}_k - x_k\|, \|x_k - \underline{x}_k)\|)$, interval estimate widths $\|\overline{x}_k - \underline{x}_k\|$ and their upper bounds for the interval estimates of states, $\|\tilde{x}_{k|k}\|$, $\|\Delta^x_k\|$, δ^x_k, and unknown inputs, $\|\tilde{d}_k\|$, $\|\Delta^d_k\|$, δ^d_k.

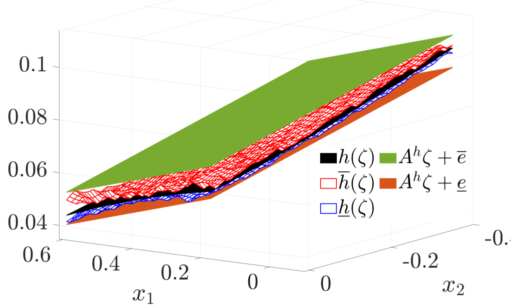

Fig. 4: Actual unknown dynamics function $h(\zeta)$, its upper and lower framer intervals $\overline{h}_k(\zeta), \underline{h}_k(\zeta)$ at time step $k = 250$, and its global abstraction $A^h\zeta + \overline{e}^h, A^h\zeta + \underline{e}^h$ at $k = 0$.

we need to check an *a priori* condition (i.e., offline or before starting to implement the observer) for observer stability, we use global abstraction slopes for stability analysis. However, for the implementation, we iteratively update the framers and consequently, obtain the updated local abstractions, which, in turn, lead to updated local intervals that by construction are tighter than the global ones, as shown in the proof of Theorem 2. Hence, it might be the case that the (relatively conservative) global abstraction-based sufficient conditions for the observer stability given in Theorem 2 do not hold, while the implemented local-abstraction-based intervals are still uniformly bounded. This is the main benefit of using iterative local affine abstractions, but at the cost of higher computational effort.

VI. Conclusion

This paper proposed an interval observer for partially unknown nonlinear systems with bounded noise that simultaneously estimates the augmented states and learns the unknown dynamics. By leveraging a combination of nonlinear bounding/decomposition functions, affine abstractions and a data-driven function abstraction method, we introduced a recursive interval observer design whose interval estimates' maximal and minimal elements are guaranteed to frame/bracket the true augmented states. Moreover, using observed output/measurement signals at run time, the observer also iteratively shrinks the intervals by eliminating estimates that are not compatible with the measurements. Further, tractable sufficient conditions for uniform boundedness of the sequence of interval estimate widths, i.e., for stability of the designed observer were provided.

References

[1] W. Liu and I. Hwang. Robust estimation and fault detection and isolation algorithms for stochastic linear hybrid systems with unknown fault input. *IET control theory & applications*, 5(12):1353–1368, 2011.

[2] S.Z. Yong, M. Zhu, and E. Frazzoli. Switching and data injection attacks on stochastic cyber-physical systems: Modeling, resilient estimation and attack mitigation. *ACM Transactions on Cyber-Physical Systems*, 2(2):9, 2018.

[3] S.Z. Yong. Simultaneous input and state set-valued observers with applications to attack-resilient estimation. In *2018 Annual American Control Conference (ACC)*, pages 5167–5174. IEEE, 2018.

[4] L. Jaulin. Nonlinear bounded-error state estimation of continuous-time systems. *Automatica*, 38(6):1079–1082, 2002.

[5] O. Bernard and J-L. Gouzé. Closed loop observers bundle for uncertain biotechnological models. *J. of Process Control*, 14(7):765–774, 2004.

[6] T. Raïssi, D. Efimov, and A. Zolghadri. Interval state estimation for a class of nonlinear systems. *IEEE Transactions on Automatic Control*, 57(1):260–265, 2011.

[7] F. Mazenc and O. Bernard. Interval observers for linear time-invariant systems with disturbances. *Automatica*, 47(1):140–147, 2011.

[8] F. Mazenc, T-N. Dinh, and S-I. Niculescu. Robust interval observers and stabilization design for discrete-time systems with input and output. *Automatica*, 49(11):3490–3497, 2013.

[9] D. Efimov, T. Raïssi, S. Chebotarev, and A. Zolghadri. Interval state observer for nonlinear time varying systems. *Automatica*, 49(1):200–205, 2013.

[10] G. Zheng, D. Efimov, and W. Perruquetti. Design of interval observer for a class of uncertain unobservable nonlinear systems. *Automatica*, 63:167–174, 2016.

[11] N. Ellero, D. Gucik-Derigny, and D. Henry. An unknown input interval observer for LPV systems under L_2-gain and L_∞-gain criteria. *Automatica*, 103:294–301, 2019.

[12] M. Khajenejad and S.Z. Yong. Simultaneous input and state set-valued $\mathcal{H}_\infty$-observers for linear parameter-varying systems. In *American Control Conference (ACC)*, pages 4521–4526. IEEE, 2019.

[13] M. Khajenejad and S.Z. Yong. Simultaneous mode, input and state set-valued observers with applications to resilient estimation against sparse attacks. In *IEEE Conference on Decision and Control (CDC)*, pages 1544–1550, 2019.

[14] M. Khajenejad and S.Z. Yong. Simultaneous input and state interval observers for nonlinear systems with full-rank direct feedthrough. In *IEEE Conference on Decision and Control*, pages 5443–5448, 2020.

[15] M. Milanese and C. Novara. Set membership identification of nonlinear systems. *Automatica*, 40:957–975, 2004.

[16] Z.B Zabinsky, R.L Smith, and B.P Kristinsdottir. Optimal estimation of univariate black-box Lipschitz functions with upper and lower error bounds. *Computers & Operations Res.*, 30(10):1539–1553, 2003.

[17] G. Beliakov. Interpolation of Lipschitz functions. *Journal of computational and applied mathematics*, 196(1):20–44, 2006.

[18] J.P. Calliess. *Conservative decision-making and inference in uncertain dynamical systems*. PhD thesis, University of Oxford, 2014.

[19] Z. Jin, M. Khajenejad, and S.Z. Yong. Data-driven model invalidation for unknown lipschitz continuous systems via abstraction. In *American Control Conference (ACC)*, pages 2975–2980. IEEE, 2020.

[20] L. Yang, O. Mickelin, and N. Ozay. On sufficient conditions for mixed monotonicity. *IEEE Transactions on Automatic Control*, 64(12):5080–5085, 2019.

[21] S. Coogan and M. Arcak. Efficient finite abstraction of mixed monotone systems. In *Hybrid Systems: Computation and Control*, pages 58–67. ACM, 2015.

[22] K.R. Singh, Q. Shen, and S.Z. Yong. Mesh-based affine abstraction of nonlinear systems with tighter bounds. In *Conference on Decision and Control (CDC)*, pages 3056–3061. IEEE, 2018.

[23] M. Milanese and A. Vicino. Optimal estimation theory for dynamic systems with set membership uncertainty: An overview. *Automatica*, 27(6):997–1009, 1991.

[24] M. Khajenejad, Z. Jin, and S.Z. Yong. Interval observers for simultaneous state and model estimation of partially known nonlinear systems (extended version). In *arXiv preprint arXiv:2004.03665*, 2021.

[25] M. James. The generalised inverse. *The Mathematical Gazette*, 62(420):109–114, 1978.

[26] H.K. Khalil. Nonlinear systems. *Upper Saddle River*, 2002.

[27] D. Pylorof, E. Bakolas, and K.S. Chan. Design of robust Lyapunov-based observers for nonlinear systems with sum-of-squares programming. *IEEE Control Systems Letters*, 4(2):283–288, 2019.