Guaranteed State Estimation via Indirect Polytopic Set Computation for Nonlinear Discrete-Time Systems

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Abstract—This paper proposes novel set-theoretic approaches for recursive state estimation in bounded-error discrete-time nonlinear systems subject to nonlinear observations/constraints. By transforming the polytopes that are characterized as zonotope bundles (ZB) and/or constrained zonotopes (CZ), from the state space to the space of the generators of ZB/CZ, we leverage a recent result on remainder-form mixed-monotone decomposition functions to compute the propagated set, i.e., a ZB/CZ that is guaranteed to enclose the set of the state trajectories of the considered system. Further, by applying the remainder-form decomposition functions to the nonlinear observation function, we derive the updated set, i.e., an enclosing ZB/CZ of the intersection of the propagated set and the set of states that are compatible/consistent with the observations/constraints. In addition, we show that the mean value extension result in [1] for computing propagated sets can also be extended to compute the updated set when the observation function is nonlinear.

I. INTRODUCTION

State estimation is crucial in several research fields such as fault detection and isolation [2], localization [3] and state-feedback control [4]. Bayesian/stochastic estimation approaches such as particle or Kalman filtering can be applied if uncertainties/stochastic descriptions of uncertainties are known. On the other hand, in bounded-error settings where distribution-free set-valued uncertainties are considered, sets that are guaranteed to contain the true state trajectories and are compatible/consistent with constraints/observations are estimated. Obtaining the exact characterization of such sets that contain the evolution of the system states is often very complicated and computationally intractable [5], hence set-theoretic approaches that can tractably derive enclosures to such sets that are as tight as possible are of great interest.

Literature review. In the context of bounded-error settings, where dynamical systems are subject to distribution-free and bounded uncertainties, several seminal set-membership state estimation approaches for discrete-time constrained systems have been proposed to compute enclosing sets to all possible system trajectories, e.g., [1], [6], [7]. These methods commonly consider this problem in two steps, i.e., by finding an enclosing set of the image set of the dynamics vector field (propagation/prediction step), and then refining the obtained propagated set by finding an enclosure of its intersection with the set of states that are compatible/consistent with the observation/measurements (update step).

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In the case of linear systems with polytopic initial sets, it is theoretically shown that tight (exact) enclosures can be obtained [8]. However, even for linear systems, the computational complexity of polytopic propagation is extensive and grows drastically with time [9]. Hence, simpler sets such as parallelotopes [6], [10], ellipsoids [11]–[13], intervals [14]–[17] or zonotopes [7], [18] have been used to characterize the enclosures. However, structural limitations of these sets sometimes lead to conservative enclosures. To address this, [19] introduced constrained zonotopes to ease some of the limitations imposed by zonotopes, while zonotope bundles were proposed in [20] to describe the intersection of zonotopes without explicit computations, both of which were shown to be equivalent representations of polytopes.

In contrast to linear systems, obtaining efficient set-valued estimates for nonlinear systems is still very challenging. A classical approach to tackle this problem has been to use interval arithmetic-based inclusion functions [21] to propagate the current enclosing sets through the nonlinear dynamics and then to apply interval-based set inversion techniques (e.g., SIVIA) to find upper approximations for the set of compatibles states with the current measurements [3], [22]. These approaches are computationally very efficient, but unfortunately, due to the nature of interval arithmetic, the resulting bounds are often conservative.

Further, for systems with linear observation functions, zonotopic propagation methods have been developed in [23]–[25], based on first-order Taylor expansion, mean value extension or DC programming. However, significant errors are caused in the update step due to the symmetry of zonotopes, even in the case with linear observation functions [19]. More recently, an interesting approach was proposed in [1] using constrained zonotopic propagation and update algorithms for discrete-time nonlinear systems with linear observation functions, based on mean value and first-order Taylor extensions.

Contributions. This paper proposes novel methods for recursive state estimation (consisting of propagation and update steps) using indirect representations of polytopes (specifically, constrained zonotopes or zonotope bundles) for nonlinear bounded-error discrete-time systems with nonlinear observation functions by leveraging remainder-form mixed-monotone decomposition functions [26] and the standard propagation and update approach. In particular, for the propagation step, we transform the prior ZBs/CZs into the space of CZ/ZB generators, which are interval-valued, and further transform the vector field into two components,
one that is proven to attain tight image sets, as well as a linear remainder function, for which a family of remainder-form mixed-monotone decomposition functions [26] can be obtained. Each of the decomposition functions produces enclosures of the state trajectories; thus, we can intersect them to obtain the propagated ZB/CZ enclosures.

Moreover, we show that a similar idea, i.e., transformation from the state and uncertainty space to the space of generators of CZs/ZBs, can be used for the update step to find a family of enclosures of the generalized nonlinear intersection between the propagated set and the set of states that is compatible with observations, where the final enclosures are in the ZB/CZ representation of polytopes. Furthermore, we prove that the mean value extension approach used in [1] for the propagation step can also be leveraged for the update step when the observation function is nonlinear. Finally, we compare our proposed approaches with the mean value extension-based approach in [1] on two examples with a linear and a nonlinear observation function, respectively.

II. PRELIMINARIES

In this section, we briefly introduce some of the main concepts that we use throughout the paper, as well as some important existing results that will be used for deriving our main results and for comparison.

Notation. $\mathbb{N}, \mathbb{N}_n, \mathbb{R}^n$ and $\mathbb{R}^{n \times n}$ denote the set of positive integers, the first $a \in \mathbb{N}$ positive integers, the $n$-dimensional Euclidean space and the space of $m$ by $n$ real matrices, respectively. For $Z, W \subset \mathbb{R}^n, R \in \mathbb{R}^{n \times n}, Y \subset \mathbb{R}^n$, and $\mu : \mathbb{R}^n \to \mathbb{R}^m$, $Z \triangleq \{Rz \mid z \in Z\}$, $Z \ominus W \triangleq \{z + w \mid z \in Z, w \in W\}$, $Z \odot W \triangleq \{z - w \mid z \in Z, w \in W\}$, $\mu(Z) \triangleq \{\mu(z) \mid z \in Z\}$ and $Z \cup \mu Y \triangleq \{z \in Z \mid \mu(z) \in Y\}$ denote the linear mapping, Minkowski sum, Pontryagin difference, general (nonlinear) mapping and generalized (nonlinear) intersection, respectively. Further, the transpose, Moore-Penrose pseudoinverse and $(i, j)$-th element of $R$ are given by $R^T$, $R^\dagger$ and $R_{ij}$, while its row support is $r = \text{rowsupp}(R) \subset \mathbb{R}^n$, where $r_{ij} = 0$ if the $i$-th row of $R$ is zero and $r_i = 1$ otherwise, $\forall \in \mathbb{N}_m$. Moreover, $\mathbb{B}_n \triangleq \{z \in \mathbb{R}^n \mid \|z\|_\infty \leq 1\}$ and $\mathbf{0}_n$ denote an $\infty$-norm hyper-ball and a zero vector in $\mathbb{R}^n$, respectively. For $x, z \in \mathbb{R}^n$, $\text{diag}(z)$ is a diagonal matrix in $\mathbb{R}^{n \times n}$ with $z$ being its diagonal elements and $x \preceq z$ means that $z_i \leq x_i, \forall i = 1, \ldots, n$. Finally, $(\cdot, \cdot)$ denotes the inner product operator.

Definition 1 (Intervals, H-Polytopes, Constrained Zonotopes (CZ) and Zonotope Bundles (ZB)). A set $Z \subset \mathbb{R}^n$ is (i) an interval, (ii) a polytope in hyperplane representation (H-polytope), (iii) a polytope in constrained zonotope representation (CZ), or (iv) a polytope in zonotope bundle representation (ZB), if

(i) $\exists z, \tilde{z} \in \mathbb{R}^n$ such that $Z = [z, \tilde{z}] \triangleq \{z \in \mathbb{R}^n \mid z \leq \tilde{z} \leq z\}$. An interval matrix can be defined similarly, in an element-wise manner.

(ii) $\exists A_p \in \mathbb{R}^{n \times n}, b_p \in \mathbb{R}^n$ such that $Z = \{A_p, b_p\} \triangleq \{z \in \mathbb{R}^n \mid A_p z \leq b_p\}$

(iii) $\exists G \in \mathbb{R}^{n \times n}, c, \tilde{c} \in \mathbb{R}^n, A \in \mathbb{R}^{n \times n}, b \in \mathbb{R}^n$ such that $Z = \{G, c, \tilde{c}, A, b\} \triangleq \{G \xi + c \mid \xi \in \mathbb{B}^n, \tilde{A} \xi = \tilde{b}\}$

(iv) $Z$ can be represented as an intersection of $S \in \mathbb{N}$ zonotopes, i.e., $\exists \{G_s \in \mathbb{R}^{n \times n}, c_s \in \mathbb{R}^n\}_{s=1}^S$ such that $Z = \bigcap_{s=1}^S \{G_s c + c_s \mid \xi \in \mathbb{B}^n\}$, with $n_s, s = 1, \ldots, S$, being called the number of generators for each zonotope.

It is worth mentioning that a polytope $Z$ can be equivalently given in the H-polytope, CZ or ZB representations and can be exactly transformed among these representations using off-the-shelf tools, e.g., CORA 2020 [27]. This is represented throughout this paper as:

$$Z = \{A_p, b_p\} \triangleq \{G, c, \tilde{A}, \tilde{b}\} \triangleq \{G_s, c_s\}_{s=1}^S \triangleq \bigcap_{s=1}^S \{G_s c + c_s \mid \xi \in \mathbb{B}^n\}$$

Proposition 1. Consider an interval vector $IZ \triangleq \{z, \tilde{z}\} \subset \mathbb{R}^n$ and an interval matrix $J \in \mathbb{R}^{n \times n}$. Then, $IZ$ and $J$ can be equivalently represented as

$$IZ \triangleq \{z, \tilde{z}\} \preceq \text{mid}(IZ) + \frac{1}{2}\text{diag}(\text{diam}(IZ))\mathbb{B}_n,$$  

$$J \triangleq \{J_i, J_{\tilde{J}}\} \preceq \text{mid}(J) + \frac{1}{2}(\text{diam}(J))\mathbb{B}_n,$$

where for $q \in \{IZ, J\}$, $\text{mid}(q) \triangleq \frac{1}{2}(q + \tilde{q}), \text{diam}(q) \triangleq (q - \tilde{q})$, and $J_{\Delta} \in \mathbb{R}^{n \times n}$ is an interval matrix that is defined as $J_{\Delta} \triangleq \frac{1}{2}(\text{diam}(J))\text{diag}(\text{mid}(J)) \forall i \in \mathbb{N}_n, \forall j \in \mathbb{N}_m$.

Proof. To prove (1), consider $z \in IZ \Leftrightarrow z \leq \tilde{z} \leq z$ and $\text{mid}(IZ) \leq -z - \text{mid}(IZ) \Leftrightarrow -\frac{1}{2}\text{diam}(IZ) \leq -z \leq -\frac{1}{2}\text{diam}(IZ)$ and $\text{mid}(IZ) - \frac{1}{2}\text{diam}(IZ) \Leftrightarrow \exists \xi \in \mathbb{B}_n, s.t. z = \text{mid}(IZ) + \frac{1}{2}\text{diam}(IZ)) \xi \in \mathbb{B}_n$. The result in (2) is a straightforward extension of (1).

Proposition 2. [1, Theorem 1] Let $X = \{G, c, A, b\} \subset \mathbb{R}^n$ be a constrained zonotope with $n_g$ generators and $n_c$ constraints, and $J \in \mathbb{R}^{n \times n}$ be an interval matrix. Consider the set $S = JX \triangleq \{Jx \mid J \in J, x \in X\} \subset \mathbb{R}^n$. Let $\mathbf{X} = \{G, c\}$ be a zonotope satisfying $X \subseteq \mathbf{X} \subset \mathbb{R}^n$. Let $m \in \mathbb{R}^n$ be an interval vector such that $m \ominus [J \ominus \text{mid}(J)](c)$ and $\text{mid}(m) = 0_n$. Let $P \in \mathbb{R}^{n \times n}$ be a diagonal matrix defined as follows. $\forall i = 1, \ldots, n$:

$$P_{ii} = \frac{1}{2}\text{diam}(m_i) + \frac{1}{2}\sum_{j=1}^{n} \sum_{k=1}^{n} \text{diam}(J_{ik}) |G_{jk}|$$

Then, $S \subseteq \text{mid}(J)X \cap \text{PB}_\infty$.

Proposition 3 (RRSR Propagation Approach). [1, Theorem 2] Let $f : \mathbb{R}^n \times \mathbb{R}^{n_u} \to \mathbb{R}^n$ be continuously differentiable and $\nabla_x f$ denote the gradient of $f$ with respect to its first argument. Let $X = \{G_x, c_x, A_x, b_x\} \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^{n_u}$ be constrained zonotopes (CZs). Choose any $h \in \mathbb{X}$. If $Z$ is a CZ such that $f(h, W) \subseteq Z$ and $J \in \mathbb{R}^{n \times n}$ is an interval matrix satisfying $\nabla f(X, W) \subseteq J$, then

$$f(X, W) \subseteq Z \ominus \text{mid}(J)X \ominus \{h\} \cap \text{PB}_\infty,$$

where $\hat{P}$ can be computed using (3) with $J$ and an enclosing zonotope $\mathbf{X} = \{G, c\} \subset X \ominus \{h\}$.

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Definition 2 (Mixed-Monotone (One-Sided) Decomposition Functions For Discrete-Time Systems). [26, Definitions 3–4] A mapping \( f_d : Z \times Z \subset \mathbb{R}^{n_o} \to \mathbb{R}^m \) is a discrete-time mixed-monotone decomposition function with respect to \( f : Z \subset \mathbb{R}^{n_0} \to \mathbb{R}^m \) over the set \( Z \), if it satisfies the following:

(i) \( f_d(z, x) = f(z) \),
(ii) \( x \preceq y \Rightarrow f_d(z, y) \preceq f_d(z, x), \forall x, y \in Z \),
(iii) \( y \succeq y' \Rightarrow f_d(z, y') \preceq f_d(z, y), \forall x, y, x', y' \in Z \). Further, if there exists two mixed-monotone mappings \( \tilde{J}_d, \tilde{f}_d : Z \times Z \to \mathbb{R}^m \), such that for any \( z, z', \overline{z} \in Z \), the following holds: \( f(z, \overline{z}) \preceq f(z', \overline{z}) \leq \tilde{f}_d(z, \overline{z}) \leq \tilde{J}_d(z, \overline{z}) \), then \( \tilde{J}_d \) and \( \tilde{f}_d \) are called upper and lower decomposition functions for \( \tilde{f} \) over \( Z \), respectively.

It is trivial to see that \( \forall x \in [z, \overline{z}], \tilde{f}_d(z, x) = f(x) \leq \tilde{J}_d(z, \overline{z}) \).

Proposition 4 (Tight and Tractable Remainder-Form Upper and Lower Decomposition Functions). [26, Theorems 1–3] Consider a locally Lipschitz vector field \( f_i : \mathbb{Z} \to [z, \overline{z}] \subset \mathbb{R}^{n_o} \to \mathbb{R} \). Let \( N_{n_z} \triangleq \{1, \ldots, n_z\} \) and \( J^i, J^\overline{i} \in \mathbb{R}^{n_o} \) denote the upper and lower bounds for the Jacobian matrix (vector) of \( f_i \) over \( \mathbb{Z} \). Suppose that Assumption 2 in Section III holds. Then, \( f_i(\cdot) \) admits a family of mixed-monotone remainder-form decomposition functions denoted as \( \{f_d,i(z, z; m, h(\cdot))\}_{m \in M_i, h(\cdot) \in \mathcal{H}_M} \), that is parametrized by a set of supporting vectors \( M_0^i \) via

\[
m \in M_0^i \triangleq \{ m \in \mathbb{R}^{n_o} | m_j = \min(J^j, 0) \lor m_j = \max(J^\overline{j}, 0), \forall j \in N_{n_z} \},
\]

and a locally Lipschitz remainder function \( h(\cdot) \in \mathcal{H}_M \), where

\[
f_d,i(z, z; m, h(\cdot)) = h_i(c_m(z, z)) + f_i(c_m(z, z)) \equiv h_i(c_m(z, z)) \equiv \langle \hat{g}_{d,i}(\cdot) \rangle_{H_{i,j}^0 (H_{i,j}^0, \cdot)} = \hat{g}_{d,i}(\cdot), \forall \xi \in \mathcal{I}, \forall \psi \in \mathcal{H}.
\]

System Assumptions. Consider the following bounded-error nonlinear constrained discrete-time system:

\[
x_{k+1} = f(x_k, w_k, u_k) = f(x_k),
\]

\[
\mu(x_k, w_k) = \mu(x_k) \in \mathcal{Y}_k, \quad x_0 \in \mathcal{X}_0, w_k \in \mathcal{W}_k,
\]

III. PROBLEM FORMULATION

where \( z_k \triangleq (x_k, \xi_k) \), \( x_k \in \mathbb{R}^{n_x} \) is the state vector, \( w_k \in \mathcal{W}_k \subset \mathbb{R}^{n_w} \) is the augmentation of all exogenous uncertain inputs, e.g., bounded process disturbance/noise and system uncertainties such as uncertain parameters and \( u_k \in \mathcal{U}_k \subset \mathbb{R}^{n_u} \) is the known input signal. Furthermore, \( f : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x} \) (with \( n_x \triangleq n_x + n_w \)) and \( \mu : \mathbb{R}^{n_x} \to \mathbb{R}^{n_u} \) are nonlinear state vector field and observation/constraint mappings, respectively, which are well-defined, given \( f(\cdot) \) and \( \mu(\cdot) \), as well as the fact that \( u_k \) is known. Note that the mapping \( \mu(\cdot) \) along with the set \( \mathcal{Y}_k \) characterize all existing and known or even manufactured/redundant constraints over the states, observations and measurement noise signals or uncertain parameters at time step \( k \).

The unknown initial state \( x_0 \) is assumed to be in a given set \( \mathcal{X}_0 \) and moreover, we assume the following:

Assumption 1. The initial state set \( \mathcal{X}_0 \), as well as \( \mathcal{W}_k, \mathcal{Y}_k, \forall k \geq 0 \) are known polytopes, or equivalently constrained zonotopes or zonotope bundles (cf. Definition 1).

Assumption 2. The nonlinear vector fields \( f(\cdot) \) and \( \mu(\cdot) \) are locally Lipschitz on their domains. Consequently, they are differentiable and have bounded Jacobian matrix elements, almost everywhere. We further assume that given any \( Z \subset \mathbb{R}^{n_o} \) and \( X \subset \mathbb{R}^{n_x} \), some upper and lower bounds for all elements of Jacobian matrices for \( f(\cdot) \) and \( \mu(\cdot) \) over \( Z \) are available or can be computed. In other words, \( \exists J^l, J^\overline{l}, J^\mu, J^\overline{\mu} \in \mathbb{R}^{n_x \times n_z} \) such that:

\[
J^l \leq J^\mu \leq J^\overline{l}, J^\mu \leq J^\overline{l}, \forall \xi \in \mathcal{I}, \forall \psi \in \mathcal{H}.
\]

In this paper, we aim to propose novel set-membership approaches for obtaining polytopic-valued state estimates for bounded-error nonlinear systems (10) using indirect polytope representations, namely using zonotope bundles (ZBs) and constrained zonotopes (CZs). More formally, given the initial state set estimate \( \mathcal{X}_0 \), where \( x_0 \in \mathcal{X}_0 \), we consider a two-step approach for recursive state estimation by solving the following problems for the propagation and update steps, respectively, at each time step \( k \in \mathbb{N} \):
Problem 1 (Propagation). Given the 'updated set' $X^u_{k-1}$ from the previous time step and $W_{k-1}$ (with $Z_{k-1} \triangleq X^u_{k-1} \times W_{k-1}$), find the 'propagated set' $X^p_k$ that satisfies

$$f(Z_{k-1}) \triangleq \{ f(x, w, u_{k-1}) \mid x \in X^u_{k-1}, w \in W_k \} \subseteq X^p_k.$$  (11)

Problem 2 (Update). Given the 'propagated set' $X^p_k$ and the uncertain observation/constraint set $Y_k$ at time step $k$, find the 'updated set' $X^u_k$ that satisfies

$$X^p_k \cap Y_k \triangleq \{ x \in X^p_k \mid \mu(x) \in Y_k \} \subseteq X^u_k.$$  (12)

IV. INDIRECT POLYTOPIC SET COMPUTATION

We consider a recursive two-step state estimation approach consisting of state propagation (prediction) and measurement update (refinement) steps, by solving Problems 1 and 2 in Sections IV-A and IV-B, respectively. Our recursive algorithm can be either initialized at time step 0 with the initial polytopic state estimate $X_0$ as $X^u_0 = X_0$ or if $Y_0$ is available/measured, with $X^p_0 = X_0$ and the application of the update step by solving Problem 2 at time 0 to obtain $X^u_0$.

A. Decomposition-Based ZB/CZ Propagation Step

In this section, we address Problem 1, assuming that the state estimate set from the previous time step is a zonotope bundle (Lemma 1) or a constrained zonotope (Lemma 2). The main idea is to "transform" the ZBs/CZs from the $z$-space, i.e., the space of augmented state $x$ and process uncertainty $w$, to intervals in the $\xi$-space, i.e., the space of ZB/CZ generators. Then, based on our recent results in [26], we decompose the transformed vector fields in the $\xi$-space into two components, a Jacobian sign-stable (JSS) and a linear remainder mapping (cf. Corollary 1). Finally, we apply our recently developed approach to find a family of mixed-monotone remainder-form decomposition functions and to compute enclosures to the JSS components, which are proven to be tight by Corollary 1 for interval domains. Using these tight bounds and thanks to the linearity of the remainders, we show that by augmenting and intersecting all the obtained enclosures, the resulting set is a ZB/CZ. We formally summarize our proposed decomposition-based ZB/CZ approaches in the following Lemmas 1 and 2.

Lemma 1 (Decomposition-Based ZB Propagation). Suppose $f : Z \subset \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$ satisfies Assumption 2. Let $Z$ be a ZB in $\mathbb{R}^{n_x}$, i.e., $Z = \bigcap_{s=1}^{S} \{ G_s, c_s \} Z$, and $\forall s \in \mathcal{N}_S \triangleq \{ 1, \ldots, S \}$, $n_s$ be the number of generators of the corresponding zonotope. Then, the following set inclusion holds:

$$f(Z) \subseteq \bigcap_{s=1}^{S} \bigcap_{H_s \in \mathcal{H}_f} \{ G^H_s, c^H_s \} Z,$$  (13)

where $G^H_s \triangleq [H_s, \frac{1}{2} \text{diag}(g^H_s - g^H_s)], c^H_s \triangleq \frac{1}{2} (g^H_s + g^H_s)$,

$$g^H_s \triangleq g_{s, d}(1_{n_s} - 1_{n_s}^\top H_s^{\top}(i\cdot), 1_{n_s}^\top H_s^{\top}(i\cdot); \cdot),$$  (14)

$$g^H_s \triangleq g_{s, d}(1_{n_s} - 1_{n_s}^\top H_s^{\top}(i\cdot), 1_{n_s}^\top H_s^{\top}(i\cdot); \cdot),$$  (15)

while $g_{s, d}(\cdot, \cdot, \cdot)$ is the tight mixed-monotone decomposition function (cf. Proposition 4) for the JSS mapping $g_{s, i}(\xi) \triangleq f_{s, i}(\xi) - (H_{s, i}(\cdot; i\cdot), \xi) : \mathbb{B}^n_{\xi} \to \mathbb{R}^{n_s}, H_{f_s}$ is defined in Corollary 1 (with the corresponding function being $f_s$) and $f_s(\xi) \triangleq f(c_s + G_s \xi)$.

Proof. To show (13), $\forall s \in \mathcal{N}_S$, consider the zonotope $Z_s \triangleq \{ G_s c_s \} Z \triangleq \{ z = G_s \xi + c_s \mid \xi \in \mathbb{B}^n_{\xi} \}$ and let us define $f_s(\xi) : \mathbb{B}^n_{\xi} \to \mathbb{R}^{n_s} \triangleq f(G_s \xi + c_s)$, which implies that

$$f(Z_s) \subseteq f_s(\mathbb{B}^n_{\xi}), \forall s \in \mathcal{N}_S.$$  (16)

On the other hand, note that by Corollary 1, $\forall H_s \in \mathcal{H}_f, f_s(\cdot)$ can be decomposed as

$$f_s(\xi) = g^H_s(\xi) + H_s \xi, \forall s \in \mathcal{N}_S, \forall \xi \in \mathbb{B}^n_{\xi}, \forall H_s \in \mathcal{H}_f,$$  (17)

where $g^H_s(\xi)$ is a JSS function in $\mathbb{B}^n_{\xi}$ and $H_s$ can be computed from (9), with the corresponding function being $f_s$. Now (16) and (17) together imply:

$$f(Z_s) \subseteq g^H_s(\mathbb{B}^n_{\xi}) + H_s \mathbb{B}^n_{\xi}, \forall s \in \mathcal{N}_S, \forall H_s \in \mathcal{H}_f.$$  (18)

Again, it follows from Corollary 1 and the fact that $g^H_s(\xi)$ is a JSS function that in each dimension $i \in \mathbb{N}_n$, $g^H_s(\xi)$ can be tightly bounded as $g^H_{s, i} \leq g^H_s(\xi) \leq g^H_{s, i}$, $\forall \xi \in \mathbb{B}^n_{\xi}, \forall H_s \in \mathcal{H}_f$, with $g^H_{s, i}, g^H_{s, i}$ given in (14) and (15), respectively. Augmenting all these $n_x$ one-dimensional inequalities yields the following set inclusion for all $s \in \mathcal{N}_S$ and all $H_s \in \mathcal{H}_f$: $g^H_s(\mathbb{B}^n_{\xi}) \subseteq \{ g^H_s, g^H_s \} = \frac{1}{2} (g^H_s + g^H_s) \Delta \{ g^H_s, g^H_s \}$, where the last equality follows from Proposition 1. Combining this, (18) and the fact that the inclusion in (18) holds for all $s \in \mathcal{N}_S$ and all $H_s \in \mathcal{H}_f$, and hence for the intersection of all of them, we obtain (13). ■

Lemma 2 (Decomposition-Based CZ Propagation). Suppose $f : Z \subset \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$ satisfies Assumption 2 and let $Z$ be a CZ in $\mathbb{R}^{n_x}$, i.e., $Z = \{ G, c, A, b \} \subset \mathbb{R}^{n_x}$, and $n_g$ be the number of generators of $Z$. Then, the following set inclusion holds:

$$f(Z) \subseteq \bigcap_{H \in \mathcal{H}_f} \{ G^H, c^H, A, b \} \subset \mathbb{R}^{n_x}.$$  (19)

where $G^H \triangleq [H \frac{1}{2} \text{diag}(g^H - g^H)], A \triangleq [A, [0, \ldots, 0]],$

$$g^H \triangleq g_{i, d}(l_{n_g}, l_{n_g}; H_{i}, i, H_{i}, i, \cdot), c^H \triangleq \frac{1}{2} (g^H + g^H),$$  (20)

$$g^H \triangleq \min(1_{n_g}, A^2 \tilde{b} + \kappa r_{n_g}), l_{n_g} \triangleq \max(-1_{n_g}, A^2 \tilde{b} - \kappa r_{n_g}),$$  (21)

while $g_{i, d}(\cdot, \cdot, \cdot, \cdot)$ is the tight mixed-monotone decomposition function (cf. Proposition 4) for the JSS mapping $g_{i}(\xi) \triangleq f_{i}(\xi) - (H_{i}(\cdot; i\cdot), \xi) : \mathbb{B}^n_{\xi} \to \mathbb{R}^{n_s}, H_{f_i}$ is defined in Corollary 1 and $f(\xi) \triangleq f(c + G\xi), r_{n_g} \triangleq \text{rowsupp}(J_{n_g} - A^2 \tilde{A}, k$ is a very large positive real number (infinity).

Proof. To prove the inclusion in (19), consider the constrained zonotope representation of the set $Z$, i.e., $Z \triangleq \{ G, c, A, b \} \subset \{ z = G \xi + c \mid \xi \in \mathbb{B}^n_{\xi}, A \xi = b \}$. Using similar notation as in the proof of Lemma 1, let us define $f(\xi) : \mathbb{B}^n_{\xi} \to \mathbb{R}^{n_x} \triangleq f(G \xi + c)$ that consequently returns

$$f(Z) \subseteq \{ f(\xi) \mid \xi \in \mathbb{B}^n_{\xi}, A \xi = b \}.$$  (22)

Note that by [28, Theorem 2], $A \xi = b \Rightarrow \xi \in \mathbb{E} \triangleq [A]^{-b} - [A]^{-b}$.
\[ \langle b - A b \rangle, \quad \text{where } r_{nj} = \text{rowsupp}(I_{nj} - A b) \text{ and } \kappa \text{ is a very large positive real number. Combining this with the fact that } \xi \in B_{2}^{n} \text{ (cf. (22)), we find that } \xi \in \mathbb{I}_{\mathbb{E}} \subseteq \mathbb{E} \subseteq \mathbb{E}^{n} = [\xi_{n}, I_{n}], \text{ where } I_{n} \text{ is defined below (21). On the other hand, similar to the proof of Lemma 1, we conclude by Corollary 1 that } \forall H \in \mathcal{H}_{f}, f(\xi) \text{ can be decomposed as}
\]

\[ f(\xi) = g_{H}(\xi) + H \xi, \quad \forall H \in \mathcal{H}_{f}, \forall \xi \in \mathbb{I}_{\mathbb{E}},
\]

\[ \Rightarrow f(\mathbb{E}) \subseteq g_{H}(\mathbb{E}) + H \mathbb{E}, \forall H \in \mathcal{H}_{f}, \quad (23) \]

where \( g_{H}(\xi) \) is a JSS function in \( \mathbb{I}_{\mathbb{E}} \) and \( H \) is given in (9). By Corollary 1, in each dimension \( i \in N_{\mathbb{E}}, g_{H}(\xi) \) can be tightly bounded as \( g_{H}^{i} \leq g_{H}^{i}(\xi) \leq \frac{1}{2}(g_{H}^{i} + g_{H}^{j}) \), with \( g_{H}^{i}, g_{H}^{j} \) given in (20) and (21), respectively. Augmenting all these \( I_{nj} \) one-dimensional inequities as

Applying Proposition 1 yield the following set inclusion: \( \forall H \in \mathcal{H}_{f},
\]

\[ g_{H}^{i}(\mathbb{I}_{\mathbb{E}}) \subseteq \frac{1}{2}(g_{H}^{i} + g_{H}^{j}) + \text{diag}(g_{H}^{i} - g_{H}^{j}) \mathbb{B}_{2}^{n} \). Combining this, (22), and (23) and the fact that the inclusion in (23) holds for all \( H \in \mathcal{H}_{f} \) and hence for the intersection of all of them, we obtain

\[ f(\mathbb{S}) \subseteq \{ H \xi + \text{diag}(g_{H}^{i} - g_{H}^{j}) \theta + \frac{1}{2}(g_{H}^{i} + g_{H}^{j}) \xi | \xi \in \mathbb{B}_{2}^{n}, \theta \in \mathbb{B}_{2}^{n}, A \xi = b, \forall H \in \mathcal{H}_{f}, \text{ where the set on the right hand side of the inclusion is equivalent to the intersection of the CZs on the right hand side of (19).} \]

Finally, for further improvement, we can take the intersection of the resulting propagated sets in Lemmas 1 and 2. This is formally summarized in the following Theorem 1.

**Theorem 1 (Decomposition-Based ZB/ CZ Propagation).** Suppose all the assumptions in Lemmas 1 and 2 hold. Then, \( f(\mathbb{S}) \subseteq \mathcal{Z}_{B} \cap \mathcal{C}_{f} \), where \( \mathcal{Z}_{B} \) and \( \mathcal{C}_{f} \) are computed in Lemmas 1 and 2, respectively.

**Proof.** It follows from Lemmas 1 and 2 that \( f(\mathbb{S}) \subseteq \mathcal{Z}_{B} \) and \( f(\mathbb{S}) \subseteq \mathcal{C}_{f} \), and so \( f(\mathbb{S}) \subseteq \mathcal{Z}_{B} \cap \mathcal{C}_{f} \).

**B. Decomposition-Based CZZB Update Step**

In this section, we address Problem 2 for a given locally Lipschitz nonlinear vector field \( \mu(\cdot) \) and assuming that the propagated and the observation/constraint sets at each time step \( k \) are zonotope bundles (Lemma 3) or constrained zonotopes (Lemma 4). Using a similar idea as in Section IV-A, i.e., considering the space of generators, decomposing the transformed observation function into a JSS and a linear component, applying the tight remainder-form decomposition functions [26] to bound the JSS component, augmenting and intersecting, as well as taking the advantage of linear remainder functions, we obtain ZB/CZ enclosures to the nonlinear generalized intersection in (12). The results of this section are summarized in Lemmas 3 and 4 and Theorem 2.

**Lemma 3 (Decomposition-Based ZB Update).** Suppose \( \mu : \mathbb{R}^{n_{\mu}} \rightarrow \mathbb{R}^{n_{\mu}} \) satisfies Assumption 2. Let \( Z_{f} \subseteq \mathbb{R}^{n_{\mu}} \) and \( Z_{\mu} \subseteq \mathbb{R}^{n_{\mu}} \) be two ZB sets, i.e., \( Z_{f} = \mathcal{Z}_{B} = \mathcal{R} \{ G_{r}, c_{r} \} Z \) and \( Z_{\mu} = \mathcal{Z}_{B} = \mathcal{R} \{ G_{r}, c_{r} \} Z \), where \( \forall t \in \mathbb{N}_{T} = \{ 1, \ldots, T \}, \forall \nu \in \mathbb{N}_{R} \triangleq\{ 1, \ldots, R \} \), \( n_{r}, n_{t} \) be the number of generators of the corresponding zonotopes, respectively. Then, the following set inclusion holds:

\[ Z_{B} \cap_{\mu} Z_{B} \subseteq \mathcal{Z}_{B} \triangleq \mathcal{R} \{ G_{r}, c_{r}, A_{r}, b_{r} \} \mathcal{C}_{Z}, \quad (24) \]

of generators.
where \( G \equiv [\tilde{G}_f \ 0 \ 0], \ \tilde{b}_\Omega \equiv [\tilde{b}^T_f \ \tilde{b}^T_\mu \ (\tilde{c}_f - \frac{1}{2}(\tilde{\Omega} + \tilde{\Omega}^T))^T], \)
\[
A_\Omega \equiv \begin{bmatrix} A_f & 0 & 0 \\
0 & \tilde{A}_\mu & \Omega \\
-\tilde{G}_\mu & \frac{1}{2} \text{diag}(\tilde{\Omega} - \tilde{\Omega}^T) & \Omega \end{bmatrix},
\]
\[
p_\Omega^2 \equiv \nu_{id}(\tilde{I}_{n_n}, 1_{n_n}^T; \tilde{\Omega}_i; \Omega_i^T), \quad (30)
\]
\[
\tilde{p}_\Omega^2 \equiv \nu_{id}(\tilde{I}_{n_n}, 1_{n_n}^T; \tilde{\Omega}_i; \Omega_i^T),
\]
\[
\tilde{I}_{n_n} \equiv \min(1_{n_n}, \tilde{A}_f^T \tilde{b} + k_r n_n), \quad 1_{n_n} \equiv \max(-1_{n_n}, \tilde{A}_f^T \tilde{b} - k_r n_n), \quad \nu_{id}(\cdot, \cdot, \cdot; \tilde{\Omega}_i; \Omega_i^T) \text{ is the tight mixed-monotone decomposition function (cf. Proposition 4) for the JSS mapping } \nu^f(\beta) \equiv \lambda(\beta) = \langle \Omega_i^T \beta \rangle, \quad \beta : \mathbb{R}_n \rightarrow \mathbb{R}^n\), \quad \Omega_\lambda \text{ is defined similar to } H_f \text{ in Corollary 1 (with the corresponding function being } \lambda(\beta) = \mu(\tilde{c}_f + \tilde{G}_f)\), \quad n_r \equiv \text{rowsupp}(I_{n_n} - A_f^T A_f) \text{ and } k \text{ is a very large positive real number (infinity).}

Proof. Suppose \( z \in CZ_f \cap \mu \subset CZ_\mu \). Then, by the definition of the operator \( \gamma_\mu \) (cf. (12)), \( z \in CZ_f \) and \( \mu(z) \in CZ_\mu \). The former implies that \( \exists \beta \in \mathbb{R}^{n_\mu} \text{ such that } A_f^T \beta = \tilde{b}_f \) and \( z = \tilde{G}_f^T \beta + \tilde{c}_f \), while it follows from the latter that \( \mu(z) = \mu(\tilde{G}_f^T \beta + \tilde{c}_f) \equiv \lambda(\beta) \in CZ_\mu \Rightarrow \exists \gamma \in \mathbb{R}^{n_\mu} \text{ such that } A_\mu^T \gamma = \tilde{b}_\mu \) and \( \lambda(\gamma) = \tilde{c}_\mu + A_\mu^T \gamma \). Putting these two results into a set representation form, we obtain:
\[
z \in \{ \tilde{G}_f^T \beta + \tilde{c}_f \mid \lambda(\beta) = \tilde{c}_\mu + A_\mu^T \gamma, A_f^T \beta = \tilde{b}_f, \quad A_\mu^T \gamma = \tilde{b}_\mu, \beta \in \mathbb{R}^{n_\mu}, \gamma \in \mathbb{R}^{n_\mu} \}.
\]
On the other hand, using Corollary 1, \( \lambda(\cdot) \) can be decomposed into a JSS and a linear mapping as follows:
\[
\forall \Omega \in \Omega_\lambda, \forall \beta \in \mathbb{R}^{n_\mu}, \lambda(\beta) = \nu^f(\beta) + \Omega_\lambda(\beta).
\]
Further, note that by (28, Theorem 2), \( A_f^T \beta = \tilde{b}_f \Rightarrow \beta \in \mathbb{R}^{n_\mu} \equiv \{ A_f^T \tilde{b} + n_r \}, \in \mathbb{R}^{n_\mu} \equiv \{ A_f^T \tilde{b} + n_r \}, \in \mathbb{R}^{n_\mu} \equiv \{ A_f^T \tilde{b} + n_r \}, \text{ and } k \text{ is a very large positive real number. Then, since } \beta \in \mathbb{R}^{n_\mu} \text{, we have } \beta \in \mathbb{R}^{n_\mu} \equiv \{ \beta \}, \text{ and } k \text{ is the tight mixed-monotone JSS component } \nu(f(\cdot)) \text{ being tightly bounded, i.e., } \forall \beta \in \mathbb{R}^{n_\mu}, \forall \Omega \in \Omega_\lambda, \lambda(\beta) = \nu^f(\beta) + \Omega_\lambda(\beta) \Rightarrow \lambda(\beta) = \nu^f(\beta) + \Omega_\lambda(\beta), \text{ which along with (32) returns } z \in \cap_{\beta \in \mathbb{R}^{n_\mu}} \{ \tilde{G}_f^T \beta + \tilde{c}_f \mid \frac{1}{2} (\lambda_\beta + \lambda_\beta^T) + \frac{1}{2} \text{diag}(\tilde{\Omega} - \tilde{\Omega}^T) = \lambda_\beta + \Omega_\lambda(\beta) \}, \text{ where the set on the right is equivalent to the right hand side of (24).}

We conclude this subsection by combining the results in Lemmas 3 and 4 via the following theorem.

**Theorem 2** (Decomposition-Based ZB/CZ Update). Suppose all the assumptions in Lemmas 3 and 4 hold. Then
\[
Z_f \cap \mu \subset CZ_u \cap \mu 
\]
where \( Z_B \cup CZ_u \) are given in Lemmas 3 and 4, respectively.

**Proof.** By Lemmas 3 and 4: \( Z_f \cap \mu \subset Z_B \cup CZ_u \) and \( Z_f \cap \mu \subset Z_B \cup CZ_u \), and hence, \( Z_f \cap \mu \subset Z_B \cup CZ_u \).

**C. Modifications to the Approach in [1]**

The purpose of this subsection is twofold: i) We propose a potential refinement/improvement to the propagation ap-
\( A_{\mu,\gamma} = \tilde{b}_{\mu,\gamma} \), along with (37) and (38), imply that 
\[ z \in \{ \tilde{G}f_{\beta} + \tilde{c}_{\mu} | \tilde{c}_{\mu} + \tilde{G}_{\mu,\gamma} = \mu(x_0) + \text{mid}(J_{\mu})(\tilde{c}_{\mu} - x_0) \} \]
along with (37) and (38), imply that 
\[ z \in \{ \tilde{G}f_{\beta} + \tilde{c}_{\mu} | \tilde{c}_{\mu} + \tilde{G}_{\mu,\gamma} = \mu(x_0) + \text{mid}(J_{\mu})(\tilde{c}_{\mu} - x_0) \} \]
\[ \tilde{G}_{\mu,\gamma} = \tilde{b}_{\mu,\gamma} \in \mathbb{B}_n^n \}, \]
which is equivalent to the CZ on the right hand side of (35).

V. SIMULATIONS

In this section, we compare the performance of five approaches to guaranteed state estimation: i) RRSR, i.e., the mean value extension-based propagation introduced in [1] (recapped in Proposition 3) in addition to the update approach in [1] for the case when the observation function is linear (for Example I below) and its extension in Lemma 5 to nonlinear measurements (for Example II below) and ii) D-RRSR, i.e., a modification to RRSR where the bounds for Jacobian matrices are computed using the remainder decomposition functions (cf. Proposition 5), iii) D-ZB, i.e., decomposition-based propagation and update with ZBs (cf. Lemmas 1 and 3), iv) D-CZ, i.e., decomposition-based propagation and update with Czs (cf. Lemmas 2 and 4) and v) COMB, i.e., a combination of i)–iv) via intersection (based on a similar idea as Theorems 1, 2). All simulations are performed on a 1.8 GHz (8 CPU) i5-8250U, using MATLAB version 2020a and CORA 2020 [27].

A. Example I

Consider the following nonlinear discrete-time system from [1, Example 1]:
\[ x_{1,k} = 3x_{1,k-1} - \frac{x_{1,k-1}^2 - 4x_{1,k-1}^2 - 1}{4 + x_{1,k-1}} + w_{1,k-1}, \]
\[ x_{2,k} = -2x_{2,k-1} + \frac{x_{2,k-1}^2 - 4x_{2,k-1}^2 - 1}{4 + x_{2,k-1}} + w_{2,k-1} \]
\[ y_{1,k} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} x_{2,k} + v_{1,k} \]
\[ y_{2,k} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} x_{2,k} + v_{2,k} \]
with \( \|w_k\|_\infty \leq 0.1 \), an unknown initial state \( x_0 \in \mathcal{X}_0 = \{ 0.1 \ 0.2 \ -0.1 \ 0.5 \ 0.5 \} \) and \( \mathcal{Y}_k \triangleq \{ y_k - v_k | \|v_k\|_\infty \leq 0.4 \}. \)

![Fig. 1: Results for Example I from the first five time steps of set-valued state estimation, using five different approaches. Black dots are obtained from uniform sampling of the initial state and noise signals, and propagating through the system dynamics.](image)

As can be seen from Figure 1, D-ZB provides less conservative enclosures compared to the other individual approaches, and further, the COMB approach results in a significant improvement by taking advantage of all approaches via intersection. Moreover, a more systematic comparison of the average computation times and enclosure set volumes of the five approaches is given in Table I. It can be observed that D-ZB is the fastest computationally, while the combination of all approaches, i.e., COMB, took the longest, as expected. Moreover, RRSR and D-RRSR took approximately the same time on average. In terms of average set volumes, D-ZB and D-RRSR generate the least conservative (smallest) enclosures when compared to the other approaches, while a further improvement is obtained using the intersection of all approaches (COMB).

### TABLE I: Average total times (seconds) and average total volumes at each time step for five state estimators in Example I. Each average is taken over 50 simulations with uniformly sampled noise and initial state.

<table>
<thead>
<tr>
<th>Methods</th>
<th>k = 0</th>
<th>k = 1</th>
<th>k = 2</th>
<th>k = 3</th>
<th>k = 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>RRSR</td>
<td>0.0809</td>
<td>0.1926</td>
<td>0.1960</td>
<td>0.2042</td>
<td>0.2042</td>
</tr>
<tr>
<td>Vol.</td>
<td>0.012</td>
<td>0.6205</td>
<td>0.4811</td>
<td>0.3340</td>
<td>0.3340</td>
</tr>
<tr>
<td>D-RRSR</td>
<td>0.1977</td>
<td>0.4385</td>
<td>0.1472</td>
<td>0.1472</td>
<td>0.1472</td>
</tr>
<tr>
<td>D-ZB</td>
<td>0.0906</td>
<td>0.9907</td>
<td>0.1226</td>
<td>0.1226</td>
<td>0.1226</td>
</tr>
<tr>
<td>Vol.</td>
<td>0.5673</td>
<td>0.3272</td>
<td>0.3175</td>
<td>0.3175</td>
<td>0.3175</td>
</tr>
<tr>
<td>D-CZ</td>
<td>0.1926</td>
<td>0.0866</td>
<td>0.5729</td>
<td>0.3175</td>
<td>0.3175</td>
</tr>
<tr>
<td>Vol.</td>
<td>0.2042</td>
<td>0.2258</td>
<td>0.3272</td>
<td>0.3175</td>
<td>0.3175</td>
</tr>
<tr>
<td>COMB</td>
<td>0.0872</td>
<td>0.6815</td>
<td>0.2251</td>
<td>0.1926</td>
<td>0.1926</td>
</tr>
<tr>
<td>Vol.</td>
<td>0.2042</td>
<td>0.5659</td>
<td>0.2251</td>
<td>0.1926</td>
<td>0.1926</td>
</tr>
</tbody>
</table>

B. Example II (Unicycle System)

Now consider the following discretized unicycle-like mobile robot system [29]:
\[ s_{x_{i,k+1}} = s_{x_{i,k}} + T_0 \phi_{w_1} \cos(\theta_{k}) + w_{1,k}, \]
\[ s_{y_{i,k+1}} = s_{y_{i,k}} + T_0 \phi_{w_2} \sin(\theta_{k}) + w_{2,k}, \]
\[ \theta_{k+1} = \theta_{k} + T_0 \phi_{w_3} + w_{3,k}, \]
\[ y_{k} = [d_{1,k} \phi_{1,k} \ d_{2,k} \ \phi_{2,k}]^T + v_{k}, \]
where \( x_k = \begin{bmatrix} s_{x_{1,k}} & s_{x_{2,k}} & \theta_{k} \end{bmatrix}^T , \)
\[ w_k = \begin{bmatrix} w_{1,k} & w_{2,k} & w_{3,k} \end{bmatrix}^T, \]
\[ \phi_{w_1} = 0.3, \phi_{w_2} = 0.15, w_{1,k} = 0.2(0.6 \rho_{x_{1,k}} - 0.3), w_{2,k} = 0.2(0.3 \rho_{x_{2,k}} - 0.2) \] and \( \phi_{w_3} = 0.2(0.6 \rho_{x_{3,k}} - 0.4) \), with \( \rho_{x_{i,k}} \in [0,1] \) \((l = 1, 2, 3)\) and initial state \( x_0 = [0.1 \ 0.2 \ 0.1]^T \). Moreover, \( \forall i \in \{1,2\}, d_{i,k} = \sqrt{(s_{x_{i,k}} - s_{x_{i,k}})^2 + (s_{y_{i,k}} - s_{y_{i,k}})^2} \) and \( \phi_{i,k} = \theta_{k} - \arctan(s_{y_{i,k}}/s_{x_{i,k}}) \) with \( s_{x_{i,k}}, s_{y_{i,k}} \) being two known values. Furthermore, \( y_k = \begin{bmatrix} y_{k} - v_{k} \ | v_{1,k} = 0.02 \rho_{y_{1,k}} - 0.01, v_{2,k} = 0.03 \rho_{y_{2,k}} - 0.01, v_{4,k} = 0.03 \rho_{y_{4,k}} - 0.01, v_{4,k} = 0.05 \rho_{y_{4,k}} - 0.01, v_{4,k} = 0.03 \rho_{y_{4,k}} - 0.01, v_{4,k} = 0.03 \rho_{y_{4,k}} - 0.01\} \)

Applying all methods i) through v), one can observe from Figure 2 that the resulting set estimates appear comparable for all approaches. Upon closer examination, Table II shows that D-CZ takes the least average computation time followed by RRSR, D-RRSR, COMB and D-ZB, while in terms of average set volumes, the COMB approach results in the smallest volume followed by D-ZB, D-CZ, RRSR and D-RRSR. Note that the computation time for D-ZB is exceptionally large, presumably because of the specific implementation in CORA 2020 [27] for converting a polytope to its ZB representation that could result in a higher number of zonotopes than the minimal needed to exactly represent the same polytope. Thus, the reduction of the number of
Fig. 2: Results for Example II from the first five time steps of set-valued state estimation, using five different approaches. Black dots are obtained from uniform sampling of the initial state and noise signals, and propagating through the system dynamics.

TABLE II: Average total times (seconds) and average total volumes ($10^{-5}$) at each time step for five state estimators in Example II. Each average is taken over 20 simulations with uniformly sampled noise and initial state.

<table>
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<tr>
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<th>$k = 3$</th>
<th>$k = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>RRSR</td>
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<td>4.2557</td>
<td>4.1883</td>
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<td>3.6747</td>
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<td>1.6171</td>
<td>4.5573</td>
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<td></td>
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<td>D-RRSR</td>
<td>1.6690</td>
<td>42.905</td>
<td>45.571</td>
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<td></td>
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<td>5.1036</td>
<td>4.8001</td>
<td></td>
</tr>
<tr>
<td>D-ZB</td>
<td>1.3967</td>
<td>34.207</td>
<td>163.08</td>
<td>147.75</td>
<td>131.94</td>
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<tr>
<td>D-CZ</td>
<td>0.6020</td>
<td>2.1824</td>
<td>2.0195</td>
<td>2.2281</td>
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<tr>
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<td>Vol.: 4.2551</td>
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</tr>
<tr>
<td>COMB</td>
<td>0.2361</td>
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<td>65.501</td>
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<td>57.728</td>
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<td>2.7726</td>
<td>1.2220</td>
<td>3.9126</td>
<td>3.9914</td>
</tr>
</tbody>
</table>

zono-topes in the bundle could be an interesting future topic, which could significantly decrease the computation time of the D-ZB approach.

VI. CONCLUSION

Novel methods were presented in this paper for guaranteed state estimation in bounded-error discrete-time nonlinear systems subject to nonlinear observations/constraints using indirect polytopic representations, i.e., using ZBs/CZs. By considering polytopes in the space of $ZB/CZ$'s generators, our recent results on remainder-form mixed-monocone decomposition functions can be applied to compute enclosures that are guaranteed to enclose the set of all possible state trajectories. Further, the decomposition functions were leveraged to bound the nonlinear observation function to derive the updated set, i.e., to return enclosures to the intersection of the propagated set and the set of states that are consistent with noisy measurements. Finally, the mean value extension-based approach in [1] was also generalized to compute the updated set when the observation functions are nonlinear.

REFERENCES