2021 60th IEEE Conference on Decision and Control (CDC)
December 13-15, 2021. Austin, Texas

Guaranteed State Estimation via Indirect Polytopic Set Computation for Nonlinear Discrete-Time Systems

Mohammad Khajenejad, Fatima Shoaib, Sze Zheng Yong

2021 60th IEEE Conference on Decision and Control (CDC) | 978-1-6654-3659-5/21/$31.00 ©2021 IEEE | DOI: 10.1109/CDC45484.2021.9683626

Abstract— **This paper proposes novel set-theoretic approaches for recursive state estimation in bounded-error discrete-time nonlinear systems subject to nonlinear observations/constraints. By transforming the polytopes that are characterized as zonotope bundles (ZB) and/or constrained zonotopes (CZ), from the state space to the space of the generators of ZB/CZ, we leverage a recent result on remainder-form mixed-monotone decomposition functions to compute the propagated set, i.e., a ZB/CZ that is guaranteed to enclose the set of the state trajectories of the considered system. Further, by applying the remainder-form decomposition functions to the nonlinear observation function, we derive the updated set, i.e., an enclosing ZB/CZ of the intersection of the propagated set and the set of states that are compatible/consistent with the observations/constraints. In addition, we show that the mean value extension result in [1] for computing propagated sets can also be extended to compute the updated set when the observation function is nonlinear.**

I. Introduction

State estimation is crucial in several research fields such as fault detection and isolation [2], localization [3] and state-feedback control [4]. Bayesian/stochastic estimation approaches such as particle or Kalman filtering can be applied if distributions/stochastic descriptions of uncertainties are known. On the other hand, in bounded-error settings where distribution-free set-valued uncertainties are considered, sets that are guaranteed to contain the true state trajectories and are compatible/consistent with constraints/observations are estimated. Obtaining the exact characterization of such sets that contain the evolution of the system states is often very complicated and computationally intractable [5], hence set-theoretic approaches that can tractably derive enclosures to such sets that are as tight as possible are of great interest.

Literature review. In the context of bounded-error settings, where dynamical systems are subject to distribution-free and bounded uncertainties, several seminal set-membership state estimation approaches for discrete-time constrained systems have been proposed to compute enclosing sets to all possible system trajectories, e.g., [1], [6], [7]. These methods commonly consider this problem in two steps, i.e., by finding an enclosing set of the image set of the dynamics vector field (*propagation/prediction* step), and then refining the obtained propagated set by finding an enclosure of its intersection with the set of states that are compatible/consistent with the observation/measurements (*update* step).

M. Khajenejad, Fatima Shoaib and S.Z. Yong are with the School for Engineering of Matter, Transport and Energy, Arizona State University, Tempe, AZ, USA (e-mail: {mkhajene, fshoaib, szyong}@asu.edu). This work is partially supported by NSF grants CNS-1932066 and CNS-1943545.

In the case of linear systems with polytopic initial sets, it is theoretically shown that tight (exact) enclosures can be obtained [8]. However, even for linear systems, the computational complexity of polytopic propagation is extensive and grows drastically with time [9]. Hence, simpler sets such as parallelotopes [6], [10], ellipsoids [11]–[13], intervals [14]–[17] or zonotopes [7], [18] have been used to characterize the enclosures. However, structural limitations of these sets sometimes lead to conservative enclosures. To address this, [19] introduced *constrained zonotopes* to ease some of the limitations imposed by zonotopes, while *zonotope bundles* were proposed in [20] to describe the intersection of zonotopes without explicit computations, both of which were shown to be equivalent representations of polytopes.

In contrast to linear systems, obtaining efficient set-valued estimates for nonlinear systems is still very challenging. A classical approach to tackle this problem has been to use interval arithmetic-based inclusion functions [21] to propagate the current enclosing sets through the nonlinear dynamics and then to apply interval-based set inversion techniques (e.g., SIVIA) to find upper approximations for the set of compatibles states with the current measurements [3], [22]. These approaches are computationally very efficient, but unfortunately, due to the nature of interval arithmetic, the resulting bounds are often conservative.

Further, for systems with linear observation functions, zonotopic propagation methods have been developed in [23]–[25], based on first-order Taylor expansion, mean value extension or DC programming. However, significant errors are caused in the update step due to the symmetry of zonotopes, even in the case with linear observation functions [19]. More recently, an interesting approach was proposed in [1] using constrained zonotopic propagation and update algorithms for discrete-time nonlinear systems with linear observation functions, based on mean value and first-order Taylor extensions.

Contributions. This paper proposes novel methods for recursive state estimation (consisting of propagation and update steps) using indirect representations of polytopes (specifically, constrained zonotopes or zonotope bundles) for nonlinear bounded-error discrete-time systems with nonlinear observation functions by leveraging remainder-form mixed-monotone decomposition functions [26] and the standard propagation and update approach. In particular, for the propagation step, we transform the prior ZBs/CZs into the space of CZ/ZB generators, which are interval-valued, and further transform the vector field into two components, one that is proven to attain tight image sets, as well as a linear remainder function, for which a family of remainder-form mixed-monotone decomposition functions [26] can be obtained. Each of the decomposition functions produces enclosures of the state trajectories; thus, we can intersect them to obtain the propagated ZB/CZ enclosures.

Moreover, we show that a similar idea, i.e., transformation from the state and uncertainty space to the space of generators of CZs/ZBs, can be used for the update step to find a family of enclosures of the *generalized nonlinear intersection* between the propagated set and the set of states that is compatible with observations, where the final enclosures are in the ZB/CZ representation of polytopes. Furthermore, we prove that the mean value extension approach used in [1] for the propagation step can also be leveraged for the update step when the observation function is nonlinear. Finally, we compare our proposed approaches with the mean value extension-based approach in [1] on two examples with a linear and a nonlinear observation function, respectively.

II. Preliminaries

In this section, we briefly introduce some of the main concepts that we use throughout the paper, as well as some important existing results that will be used for deriving our main results and for comparison.

Notation. $\mathbb{N}, \mathbb{N}_a, \mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ denote the set of positive integers, the first $a \in \mathbb{N}$ positive integers, the n-dimensional Euclidean space and the space of m by n real matrices, respectively. For $\mathcal{Z}, \mathcal{W} \subset \mathbb{R}^n, R \in \mathbb{R}^{m\times n}, \mathcal{Y} \subset \mathbb{R}^m$, and $\mu : \mathbb{R}^n \to \mathbb{R}^m$, $R\mathcal{Z} \triangleq \{Rz \mid z \in \mathcal{Z}\}, \mathcal{Z} \oplus \mathcal{W} \triangleq \{z + w \mid z \in \mathcal{Z}, w \in \mathcal{W}\}$, $\mathcal{Z} \ominus \mathcal{W} \triangleq \{z - w \mid z \in \mathcal{Z}, w \in \mathcal{W}\}$, $\mu(\mathcal{Z}) \triangleq \{\mu(z)|z \in \mathcal{Z}\}$ and $\mathcal{Z} \cup_\mu \mathcal{Y} \triangleq \{z \in \mathcal{Z} \mid \mu(z) \in \mathcal{Y}\}$ denote the linear mapping, Minkowski sum, Pontryagin difference, general (nonlinear) mapping and generalized (nonlinear) intersection, respectively. Further, the transpose, Moore-Penrose pseudoinverse and (i,j)-th element of R are given by $R^\top$, $R^\dagger$ and R_{ij}, while its row support is $r = \text{rowsupp}(R) \in \mathbb{R}^m$, where $r_i = 0$ if the i-th row of R is zero and $r_i = 1$ otherwise, $\forall i \in \mathbb{N}_m$. Moreover, $\mathbb{B}^n_\infty \triangleq \{z \in \mathbb{R}^n | \|z\|_\infty \le 1\}$ and $\mathbf{0}_n$ denote an ∞-norm hyperball and a zero vector in $\mathbb{R}^n$, respectively. For $x, z \in \mathbb{R}^n$, $\text{diag}(z)$ is a diagonal matrix in $\mathbb{R}^{n\times n}$ with z being its diagonal elements and $x \le z$ means that $z_i \le x_i$, $\forall i = 1,\dots,n$. Finally, $\langle\cdot,\cdot\rangle$ denotes the inner product operator.

Definition 1 (Intervals, H-Polytopes, Constrained Zonotopes (CZ) and Zonotope Bundles (ZB)). *A set $\mathcal{Z} \subset \mathbb{R}^n$ is (i) an interval, (ii) a polytope in hyperplane representation (H-polytope), (iii) a polytope in constrained zonotope representation (CZ), or (iv) a polytope in zonotope bundle representation (ZB), if*

(i) $\exists \underline{z}, \overline{z} \in \mathbb{R}^n$ such that $\mathcal{Z} = [\underline{z}, \overline{z}] \triangleq \{z \in \mathbb{R}^n \mid \underline{z} \le z \le \overline{z}\}$. An interval matrix can be defined similarly, in an element-wise manner;

(ii) $\exists A_p \in \mathbb{R}^{n_p\times n}, b_p \in \mathbb{R}^{n_p}$ such that $\mathcal{Z} = \{A_p, b_p\}_P \triangleq \{z \in \mathbb{R}^n \mid A_p z \le b_p\}$;

(iii) $\exists \tilde{G} \in \mathbb{R}^{n\times n_g}, \tilde{c} \in \mathbb{R}^n, \tilde{A} \in \mathbb{R}^{n_c\times n_g}, \tilde{b} \in \mathbb{R}^{n_c}$ such that $\mathcal{Z} = \{\tilde{G}, \tilde{c}, \tilde{A}, \tilde{b}\}_{CZ} \triangleq \{\tilde{G}\xi + \tilde{c} \mid \xi \in \mathbb{B}^{n_g}_\infty, \tilde{A}\xi = \tilde{b}\}$. n_g and n_c are called the number of generators and constraints of the CZ, respectively;

(iv) $\mathcal{Z}$ can be represented as an intersection of $S \in \mathbb{N}$ zonotopes, i.e., $\exists \{G_s \in \mathbb{R}^{n\times \hat{n}_s}, c_s \in \mathbb{R}^n\}_{s=1}^S$ such that $\mathcal{Z} = \bigcap_{s=1}^{S} \{G_s, c_s\}_Z \triangleq \bigcap_{s=1}^{S} \{G_s\zeta + c_s \mid \zeta \in \mathbb{B}^{\hat{n}_g}\}$, with $\hat{n}_s, s = 1,\dots,S$, being called the number of generators for each zonotope.

It is worth mentioning that a polytope $\mathcal{Z}$ can be equivalently given in the H-polytope, CZ or ZB representations and can be exactly transformed among these representations using off-the-shelf tools, e.g., CORA 2020 [27]. This is represented throughout this paper as:

$$\mathcal{Z} = \{A_p, b_p\}_P \equiv \{\tilde{G}, \tilde{c}, \tilde{A}, \tilde{b}\}_{CZ} \equiv \bigcap_{s=1}^{S} \{G_s, c_s\}_Z.$$

Proposition 1. *Consider an interval vector $\mathbb{IZ} \triangleq [\underline{z}, \overline{z}] \subset \mathbb{IR}^n$ and an interval matrix $\mathbb{J} \in \mathbb{IR}^{n\times m}$. Then, $\mathbb{IZ}$ and $\mathbb{J}$ can be equivalently represented as*

$$\mathbb{IZ} \triangleq [\underline{z}, \overline{z}] \equiv \text{mid}(\mathbb{IZ}) \oplus \tfrac{1}{2}\text{diag}(\text{diam}(\mathbb{IZ}))\mathbb{B}^n_\infty, \quad (1)$$

$$\mathbb{J} \triangleq [\underline{J}, \overline{J}] \equiv \text{mid}(\mathbb{J}) \oplus \mathbb{J}_\Delta, \quad (2)$$

where for $q \in \{\mathbb{IZ}, \mathbb{J}\}$, $\text{mid}(q) \triangleq \frac{1}{2}(\overline{q} + \underline{q})$, $\text{diam}(q) \triangleq (\overline{q} - \underline{q})$, and $\mathbb{J}_\Delta \in \mathbb{IR}^{n\times m}$ is an interval matrix that is defined as $[\mathbb{J}_\Delta]_{ij} \triangleq \frac{1}{2}[-\text{diam}(\mathbb{J})_{ij}\ \text{diam}(\mathbb{J})_{ij}], \forall i \in \mathbb{N}_n, \forall j \in \mathbb{N}_m$.

Proof. To prove (1), consider $z \in \mathbb{IZ} \Leftrightarrow \underline{z} \le z \le \overline{z} \Leftrightarrow \underline{z} - \text{mid}(\mathbb{IZ}) \le z - \text{mid}(\mathbb{IZ}) \le \overline{z} - \text{mid}(\mathbb{IZ}) \Leftrightarrow -\frac{1}{2}\text{diam}(\mathbb{IZ}) \le z - \text{mid}(\mathbb{IZ}) \le \frac{1}{2}\text{diam}(\mathbb{IZ}) \Leftrightarrow \text{mid}(\mathbb{IZ}) - \frac{1}{2}\text{diam}(\mathbb{IZ}) \le z \le \frac{1}{2}\text{diam}(\mathbb{IZ}) + \text{mid}(\mathbb{IZ}) \Leftrightarrow \exists \xi \in \mathbb{B}^n_\infty, s.t.\ z = \text{mid}(\mathbb{IZ}) + \frac{1}{2}\text{diag}(\text{diam}(\mathbb{IZ}))\xi \Leftrightarrow z \in \text{mid}(\mathbb{IZ}) \oplus \frac{1}{2}\text{diag}(\text{diam}(\mathbb{IZ}))\mathbb{B}^n_\infty$. The result in (2) is a straightforward extension of (1). ■

Proposition 2. *[1, Theorem 1] Let $\mathcal{X} = \{G, c, A, b\}_{CZ} \subset \mathbb{R}^m$ be a constrained zonotope with n_g generators and n_c constraints, and $\mathbb{J} \in \mathbb{IR}^{n\times m}$ be an interval matrix. Consider the set $S = \mathbb{J}\mathcal{X} \triangleq \{Jx | J \in \mathbb{J}, x \in \mathcal{X}\} \subset \mathbb{R}^n$. Let $\overline{\mathcal{X}} = \{\overline{G}, \overline{c}\}_Z$ be a zonotope satisfying $\mathcal{X} \subseteq \overline{\mathcal{X}}$ and $\overline{c} \in \mathbb{R}^{\overline{n}_g}$. Let $\mathbf{m} \in \mathbb{R}^n$ be an interval vector such that $\mathbf{m} \supset (\mathbb{J} - \text{mid}(\mathbb{J}))\overline{c}$ and $\text{mid}(\mathbf{m}) = \mathbf{0}_n$. Let $P \in \mathbb{R}^{n\times n}$ be a diagonal matrix defined as follows. $\forall i = 1,\dots,n$:*

$$P_{ii} = \frac{1}{2}\text{diam}(\mathbf{m}_i) + \tfrac{1}{2}\textstyle\sum_{j=1}^{\overline{n}_g}\sum_{k=1}^{m} \text{diam}(\mathbb{J}_{ik})|\overline{G}_{kj}|. \quad (3)$$

Then, $\mathcal{S} \subseteq \text{mid}(\mathbb{J})\mathcal{X} \oplus P\mathbb{B}^n_\infty$

$$= \{\left[\text{mid}(\mathbb{J})G\ \ P\right], \text{mid}(\mathbb{J})c, \left[A\ \ 0_{n_g\times n}\right], b\}_{CZ}. \quad (4)$$

Proposition 3 (RRSR Propagation Approach). *[1, Theorem 2] Let $f : \mathbb{R}^n \times \mathbb{R}^{n_w} \to \mathbb{R}^n$ be continuously differentiable and $\nabla_x f$ denote the gradient of f with respect to its first argument. Let $\mathcal{X} = \{G_x, c_x, A_x, b_x\}_{CZ} \subset \mathbb{R}^n$ and $\mathcal{W} \subset \mathbb{R}^{n_w}$ be constrained zonotopes (CZs). Choose any $h \in \mathcal{X}$. If $\mathcal{Z}$ is a CZ such that $f(h, \mathcal{W}) \subseteq \mathcal{Z}$ and $\mathbb{J} \in \mathbb{IR}^{n\times n}$ is an interval matrix satisfying $\nabla_x^\top f(\mathcal{X}, \mathcal{W}) \subseteq \mathbb{J}$, then*

$$f(\mathcal{X}, \mathcal{W}) \subseteq \mathcal{Z} \oplus \text{mid}(\mathbb{J})(\mathcal{X} \ominus \{h\}) \oplus \tilde{P}\mathbb{B}^n_\infty, \quad (5)$$

where $\tilde{P}$ can be computed using (3) with $\mathbb{J}$ and an enclosing zonotope $\overline{\mathcal{X}} = \{\overline{G}, \overline{c}\}_Z$ of $\mathcal{X} \ominus \{h\} \subseteq \overline{\mathcal{X}}$.

Definition 2 (Mixed-Monotone (One-Sided) Decomposition Functions For Discrete-Time Systems). *[26, Definitions 3–4] A mapping $f_d : \mathcal{Z} \times \mathcal{Z} \subset \mathbb{R}^{2n} \to \mathbb{R}^m$ is a discrete-time mixed-monotone decomposition function with respect to $f : \mathcal{Z} \subset \mathbb{R}^n \to \mathbb{R}^m$, over the set $\mathcal{Z}$, if it satisfies the following: (i) $f_d(x, x) = f(x)$, (ii) $x \geq x' \Rightarrow f_d(x, y) \geq f_d(x', y)$, and (iii) $y \geq y' \Rightarrow f_d(x, y) \leq f_d(x, y'), \forall x, y, x', y' \in \mathcal{Z}$. Further, if there exists two mixed-monotone mappings $\overline{f}_d, \underline{f}_d : \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}^m$, such that for any $\underline{z}, z, \overline{z} \in \mathcal{Z}$, the following holds: $\underline{z} \leq z \leq \overline{z} \Rightarrow \underline{f}_d(\underline{z}, \overline{z}) \leq f(z) \leq \overline{f}_d(\overline{z}, \underline{z})$, then $\overline{f}_d$ and $\underline{f}_d$ are called upper and lower decomposition functions for f over $\mathcal{Z}$, respectively.*

It is trivial to see that $\forall x \in [\underline{x}, \overline{x}], \underline{f}_d(\underline{x}, \overline{x}) \leq f(x) \leq \overline{f}_d(\overline{x}, \underline{x})$, where $\underline{f}_d, \overline{f}_d$ are lower and upper decomposition functions of f.

Proposition 4 (Tight and Tractable Remainder-Form Upper and Lower Decomposition Functions). *[26, Theorems 1–3] Consider a locally Lipschitz vector field $f_i : \mathbb{IZ} \triangleq [\underline{z}, \overline{z}] \subseteq \mathbb{IR}^{n_z} \to \mathbb{R}$. Let $\mathbb{N}_{n_z} \triangleq \{1, \ldots, n_z\}$ and $\overline{J}_i^{\tilde{f}}, \underline{J}_i^{\tilde{f}} \in \mathbb{R}^{n_z}$ denote the upper and lower bounds for the Jacobian matrix (vector) of f_i over $\mathbb{IZ}$. Suppose that Assumption 2 in Section III holds. Then, $f_i(\cdot)$ admits a family of mixed-monotone remainder-form decomposition functions denoted as $\{f_{d,i}(z, \hat{z}; m, h(\cdot))\}_{\mathbf{m} \in \mathbf{M}_i, h(\cdot) \in \mathcal{H}_{\mathbf{M}_i^c}}$, that is parametrized by a set of supporting vectors $\mathbf{m} \in \mathbf{M}_i^c$*

$$\begin{aligned}\mathbf{m} \in \mathbf{M}_i^c \triangleq \{\ & \mathbf{m} \in \mathbb{R}^{n_z} | \mathbf{m}_j = \min(\underline{J}_{ij}^f, 0) \vee \\ & \mathbf{m}_j = \max(\overline{J}_{i,j}^f, 0), \forall j \in \mathbb{N}_{n_z}\},\end{aligned} \tag{6}$$

and a locally Lipschitz remainder function $h(\cdot) \in \mathcal{H}_{M_i^c}$, where

$$\begin{aligned}f_{d,i}(z, \hat{z}; \mathbf{m}, h(\cdot)) = h_i(\zeta_{\mathbf{m}}(\hat{z}, z)) + f_i(\zeta_{\mathbf{m}}(z, \hat{z})) \\ - h_i(\zeta_{\mathbf{m}}(z, \hat{z})),\end{aligned} \tag{7}$$

and $\zeta_{\mathbf{m}}(z, \hat{z}) = [\zeta_{\mathbf{m},1}(z, \hat{z}), \ldots, \zeta_{\mathbf{m},n_z}(z, \hat{z})]^\top$, $\forall j \in \mathbb{N}_{n_z}$:

$$\zeta_{\mathbf{m},j}(z, \hat{z}) = \begin{cases} \hat{z}_j, \text{ if } \mathbf{m}_j = \max(\overline{J}_{i,j}^f, 0), \\ z_j, \text{ if } \mathbf{m}_j = \min(\underline{J}_{i,j}^f, 0), \end{cases} \tag{8}$$

and $\mathcal{H}_{\mathbb{M}_i} \triangleq \{h : \mathbb{IZ} \to \mathbb{R} | [\underline{J}^h(z), \overline{J}_C^h(z)] \subseteq \mathbb{M}_i, \forall z \in \mathbb{IZ}\}$. Moreover, the search for the tightest mixed-monotone upper and lower remainder-form decomposition functions in the form of (7) can be equivalently restricted to the set of "linear remainders," parametrized by $\mathbf{m} \in \mathbf{M}_i^c$, i.e., linear remainders $\{h(\cdot)\}_{\mathbf{m} \in \mathbf{M}_i^c} = \{\langle \mathbf{m}^i, \cdot \rangle\}_{\mathbf{m} \in \mathbf{M}_i^c}$.

Corollary 1. *Consider a locally Lipschitz mapping $\tilde{f}(\cdot) : \mathbb{I\Xi} \triangleq [\underline{\xi}, \overline{\xi}] \subseteq \mathbb{IR}^{n_\xi} \to \mathbb{R}^{n_x}$ that satisfies the assumptions in Proposition 4. Let us define: $\mathbb{N}_{n_x} \triangleq \{1, \ldots, n_x\}$ and*

$$\mathbf{H}_{\tilde{f}} \triangleq \{H \in \mathbb{R}^{n_x \times n_\xi} | H_{i,:}^\top \in \mathbf{M}_i^c, \forall i \in \mathbb{N}_{n_x}\}, \tag{9}$$

where $\mathbf{M}_i^c$ is defined in (6). Then, $\forall \xi \in \mathbb{I\Xi}, \forall H \in \mathbf{H}^{\tilde{f}}, \tilde{g}^H(\xi) \triangleq \tilde{f}(\xi) - H\xi$ is proven to be a Jacobian sign-stable (JSS) function, i.e., $\forall i \in \mathbb{N}_{n_x}, \forall j \in \mathbb{N}_{n_z}, J_{ij}^H(\xi) \triangleq \frac{\partial f_i}{\partial \xi_j}(\xi) \geq 0, \forall \xi \in \mathbb{I\Xi}$ or $J_{ij}^H(\xi) \triangleq \frac{\partial \tilde{g}_i^H}{\partial \xi_j}(\xi) \leq 0, \forall \xi \in \mathbb{I\Xi}$.. Consequently, $\tilde{g}^H(\cdot)$ can be tightly bounded in each dimension $i \in \mathbb{N}_{n_x}$ by remainder-form decomposition functions $\tilde{g}_{d,i}(\cdot, \cdot; H_{i,:}^\top, \langle H_{i,:}^\top, \cdot \rangle)$, constructed using (7)–(8), as follows:

$$\tilde{g}_{d,i}(\underline{\xi}, \overline{\xi}; H_{i,:}^\top, \langle H_{i,:}^\top, \cdot \rangle) \leq \tilde{g}_i(\xi) \leq \tilde{g}_{d,i}(\overline{\xi}, \underline{\xi}; H_{i,:}^\top, \langle H_{i,:}^\top, \cdot \rangle),$$

where, by [26, Lemma 3] and defining $m \triangleq H_{i,:}^\top$, we obtain $\tilde{g}_{d,i}(\overline{\xi}, \underline{\xi}; m, \langle m, \cdot \rangle) = \tilde{f}_i(\zeta_m^+) + m^\top(\zeta_m^- - \zeta_m^+)$, $\tilde{g}_{d,i}(\underline{\xi}, \overline{\xi}; m^\top, \langle m, \cdot \rangle) = \tilde{f}_i(\zeta_m^-) + m^\top(\zeta_m^+ - \zeta_m^-)$, $\zeta_m^+ \triangleq \zeta_m(\overline{\xi}, \underline{\xi})$, $\zeta_m^- \triangleq \zeta_m(\underline{\xi}, \overline{\xi})$, with $\zeta_m(\cdot, \cdot)$ given in (8).

Proof. The proof follows the lines of the proof of [26, Lemma 1, Proposition 10 and Corollary 2]. ■

III. PROBLEM FORMULATION

System Assumptions. Consider the following bounded-error nonlinear constrained discrete-time system:

$$\begin{aligned} x_{k+1} &= \hat{f}(x_k, w_k, u_k) = f(z_k), \\ \hat{\mu}(x_k, u_k) &= \mu(x_k) \in \mathcal{Y}_k,\ x_0 \in \hat{\mathcal{X}}_0, w_k \in \mathcal{W}_k, \end{aligned} \tag{10}$$

where $z_k \triangleq [x_k^\top w_k^\top]^\top$, $x_k \in \mathbb{R}^{n_x}$ is the state vector, $w_k \in \mathcal{W}_k \subset \mathbb{R}^{n_w}$ is the augmentation of all exogenous uncertain inputs, e.g., bounded process disturbance/noise and system uncertainties such as uncertain parameters and $u_k \in \mathcal{U}_k \subseteq \mathbb{R}^{n_u}$ is the *known* input signal. Furthermore, $f : \mathbb{R}^{n_z} \to \mathbb{R}^{n_x}$ (with $n_z \triangleq n_x + n_w$) and $\mu : \mathbb{R}^{n_x} \to \mathbb{R}^{n_\mu}$ are nonlinear state vector field and observation/constraint mappings, respectively, which are well-defined, given $\hat{f}(\cdot, \cdot)$ and $\hat{\mu}(\cdot, \cdot)$, as well as the fact that u_k is known. Note that the mapping $\mu(\cdot)$ along with the set $\mathcal{Y}_k$ characterize all existing and known or even manufactured/redundant constraints over the states, observations and measurement noise signals or uncertain parameters at time step k.

The unknown initial state x_0 is assumed to be in a given set $\hat{\mathcal{X}}_0$ and moreover, we assume the following:

Assumption 1. *The initial state set $\hat{\mathcal{X}}_0$, as well as $\mathcal{W}_k, \mathcal{Y}_k, \forall k \geq 0$ are known polytopes, or equivalently constrained zonotopes or zonotope bundles (cf. Definition 1).*

Assumption 2. *The nonlinear vector fields $f(\cdot)$ and $\mu(\cdot)$ are locally Lipschitz on their domains. Consequently, they are differentiable and have bounded Jacobian matrix elements, almost everywhere. We further assume that given any $\mathcal{Z} \subset \mathbb{R}^{n_z}$ and $\mathcal{X} \subset \mathbb{R}^{n_x}$, some upper and lower bounds for all elements of Jacobian matrices for $f(\cdot)$ and $\mu(\cdot)$ over $\mathcal{Z}$ and $\mathcal{X}$ are available or can be computed. In other words, $\exists \underline{J}^f, \overline{J}^f \in \mathbb{R}^{n_x \times n_z}, \underline{J}^\mu, \overline{J}^\mu \in \mathbb{R}^{n_\mu \times n_x}$, such that: $\underline{J}^f \leq J^f(z) \leq \overline{J}^f, \underline{J}^\mu \leq J^\mu(x) \leq \overline{J}^\mu, \forall z \in \mathcal{Z}, \forall x \in \mathcal{X}$, where $J^f(z)$ and $J^\mu(x)$ denote the Jacobian matrices of the mappings $f(\cdot)$ and $\mu(\cdot)$ at the points z and x, respectively.*

In this paper, we aim to propose novel set-membership approaches for obtaining polytopic-valued state estimates for bounded-error nonlinear systems (10) using indirect polytope representations, namely using zonotope bundles (ZBs) and constrained zonotopes (CZs). More formally, given the initial state set estimate $\hat{\mathcal{X}}_0$, where $x_0 \in \hat{\mathcal{X}}_0$, we consider a two-step approach for recursive state estimation by solving the following problems for the propagation and update steps, respectively, at each time step $k \in \mathbb{N}$:

Problem 1 (Propagation). *Given the 'updated set' $\mathcal{X}^u_{k-1}$ from the previous time step and $\mathcal{W}_{k-1}$ (with $\mathcal{Z}_{k-1} \triangleq \mathcal{X}^u_{k-1} \times \mathcal{W}_{k-1}$), find the 'propagated set' $\mathcal{X}^p_k$ that satisfies*

$$f(\mathcal{Z}_{k-1}) \triangleq \{\hat{f}(x, w, u_{k-1}) \mid x \in \mathcal{X}^u_{k-1}, w \in \mathcal{W}_k\} \subseteq \mathcal{X}^p_k. \quad (11)$$

Problem 2 (Update). *Given the 'propagated set' $\mathcal{X}^p_k$ and the uncertain observation/constraint set $\mathcal{Y}_k$ at time step k, find the 'updated set' $\mathcal{X}^u_k$ that satisfies*

$$\mathcal{X}^p_k \cap_\mu \mathcal{Y}_k \triangleq \{x \in \mathcal{X}^p_k \mid \mu(x) \in \mathcal{Y}_k\} \subseteq \mathcal{X}^u_k. \quad (12)$$

IV. Indirect Polytopic Set Computation

We consider a *recursive* two-step state estimation approach consisting of state propagation (prediction) and measurement update (refinement) steps, by solving Problems 1 and 2 in Sections IV-A and IV-B, respectively. Our recursive algorithm can be either initialized at time step 0 with the initial polytopic state estimate $\mathcal{X}_0$ as $\mathcal{X}^u_0 = \mathcal{X}_0$ or if $\mathcal{Y}_0$ is available/measured, with $\mathcal{X}^p_0 = \hat{\mathcal{X}}_0$ and the application of the update step by solving Problem 2 at time 0 to obtain $\mathcal{X}^u_0$.

A. Decomposition-Based ZB/CZ Propagation Step

In this section, we address Problem 1, assuming that the state estimate set from the previous time step is a zonotope bundle (Lemma 1) or a constrained zonotope (Lemma 2). The main idea is to "transform" the ZBs/CZs from the z-space, i.e., the space of augmented state x and process uncertainty w, to intervals in the ξ-space, i.e., the space of ZB/CZ generators. Then, based on our recent results in [26], we decompose the transformed vector fields in the ξ-space into two components, a Jacobian sign-stable (JSS) and a linear remainder mapping (cf. Corollary 1). Finally, we apply our recently developed approach to find a family of mixed-monotone remainder-form decomposition functions and to compute enclosures to the JSS components, which are proven to be tight by Corollary 1 for interval domains. Using these tight bounds and thanks to the linearity of the remainders, we show that by augmenting and intersecting all the obtained enclosures, the resulting set is a ZB/CZ. We formally summarize our proposed decomposition-based ZB/CZ approaches in the following Lemmas 1 and 2.

Lemma 1 (Decomposition-Based ZB Propagation). *Suppose $f : \mathcal{Z} \subset \mathbb{R}^{n_z} \to \mathbb{R}^{n_x}$ satisfies Assumption 2. Let $\mathcal{Z}$ be a ZB in $\mathbb{R}^{n_z}$, i.e., $\mathcal{Z} = \bigcap_{s=1}^{S} \{G_s, c_s\}_Z$, and $\forall s \in \mathbb{N}_S \triangleq \{1, \dots, S\}$, n_s be the number of generators of the corresponding zonotope. Then, the following set inclusion holds:*

$$f(\mathcal{Z}) \subseteq \mathcal{ZB}_f \triangleq \bigcap_{s=1}^{S} \bigcap_{H_s \in \mathbf{H}_{\tilde{f}_s}} \{G^{H_s}_s, c^{H_s}_s\}_Z, \quad (13)$$

where $G^{H_s}_s \triangleq [H_s \ \frac{1}{2}\mathrm{diag}(\overline{g}^{H_s}_s - \underline{g}^{H_s}_s)]$, $c^{H_s}_s \triangleq \frac{1}{2}(\overline{g}^{H_s}_s + \underline{g}^{H_s}_s)$,

$$\overline{g}^{H_s}_{s,i} \triangleq g^s_{i,d}(\mathbf{1}_{n_s}, -\mathbf{1}_{n_s}; {H_s^\top}_{(i,:)}, \langle {H_s^\top}_{(i,:)}, \cdot\rangle), \quad (14)$$

$$\underline{g}^{H_s}_{s,i} \triangleq g^s_{i,d}(-\mathbf{1}_{n_s}, \mathbf{1}_{n_s}; {H_s^\top}_{(i,:)}, \langle {H_s^\top}_{(i,:)}, \cdot\rangle), \quad (15)$$

while $g^s_{i,d}(\cdot, \cdot; {H_s^\top}_{(i,:)}, \langle {H_s^\top}_{(i,:)}, \cdot\rangle)$ is the tight mixed-monotone decomposition function (cf. Proposition 4) for the JSS mapping $g^{H_s}_{s,i}(\xi) \triangleq \tilde{f}_{s,i}(\xi) - \langle {H_s^\top}_{(i,:)}, \xi\rangle : \mathbb{B}^{n_s}_\infty \to \mathbb{R}^{n_x}$, $\mathbf{H}_{\tilde{f}_s}$ is defined in Corollary 1 (with the corresponding function being $\tilde{f}_s$) and $\tilde{f}_s(\xi) \triangleq f(c_s + G_s\xi)$.

Proof. To show (13), $\forall s \in \mathbb{N}_S$, consider the zonotope $\mathcal{Z}_s \triangleq \{G_s, c_s\}_Z \triangleq \{z = G_s\xi + c_s | \xi \in \mathbb{B}^{n_s}_\infty\}$ and let us define $\tilde{f}_s(\xi) : \mathbb{B}^{n_s}_\infty \to \mathbb{R}^{n_x} \triangleq f(G_s\xi + c_s)$, which implies that

$$f(\mathcal{Z}_s) \subseteq \tilde{f}_s(\mathbb{B}^{n_s}_\infty), \forall s \in \mathbb{N}_S. \quad (16)$$

On the other hand, note that by Corollary 1, $\forall H_s \in \mathbf{H}_{\tilde{f}_s}$, $\tilde{f}_s(\cdot)$ can be decomposed as

$$\tilde{f}_s(\xi) = g^{H_s}_s(\xi) + H_s\xi, \forall s \in \mathbb{N}_S, \forall \xi \in \mathbb{B}^{n_s}_\infty, \forall H_s \in \mathbf{H}_{\tilde{f}_s}, \quad (17)$$

where $g^{H_s}_s(\xi)$ is a JSS function in $\mathbb{B}^{n_s}_\infty$ and $\mathbf{H}_{\tilde{f}_s}$ can be computed from (9), with the corresponding function being $\tilde{f}_s$. Now (16) and (17) together imply:

$$f(\mathcal{Z}_s) \subseteq g^{H_s}_s(\mathbb{B}^{n_s}_\infty) \oplus H_s\mathbb{B}^{n_s}_\infty, \forall s \in \mathbb{N}_S, \forall H_s \in \mathbf{H}_{\tilde{f}_s}. \quad (18)$$

Again, it follows from Corollary 1 and the fact that $g^{H_s}_s(\xi)$ is a JSS function that in each dimension $i \in \mathbb{N}_{n_x}$, $g^{H_s}_{s,i}(\xi)$ can be tightly bounded as $\underline{g}^{H_s}_{s,i} \le g^{H_s}_{s,i}(\xi) \le \overline{g}^{H_s}_{s,i}, \forall \xi \in \mathbb{B}^{n_s}_\infty, \forall H_s \in \mathbf{H}_{\tilde{f}_s}$, with $\overline{g}^{H_s}_{s,i}, \underline{g}^{H_s}_{s,i}$ given in (14) and (15), respectively. Augmenting all these $\mathbb{N}_{nx}$ one-dimensional inequalities yields the following set inclusion for all $s \in \mathbb{N}_S$ and all $H_s \in \mathbf{H}_{\tilde{f}_s}$: $g^{H_s}_s(\mathbb{B}^{n_s}_\infty) \subseteq [\underline{g}^{H_s}_s, \overline{g}^{H_s}_s] = \frac{1}{2}((\underline{g}^{H_s}_s + \overline{g}^{H_s}_s) \oplus \mathrm{diag}(\overline{g}^{H_s}_s - \underline{g}^{H_s}_s)\mathbb{B}^{n_x}_\infty)$, where the last equality follows from Proposition 1. Combining this, (18) and the fact that the inclusion in (18) holds for all $s \in \mathbb{N}_S$ and all $H_s \in \mathbf{H}_{\tilde{f}_s}$ and hence for the intersection of all of them, we obtain (13). ∎

Lemma 2 (Decomposition-Based CZ Propagation). *Suppose $f : \mathcal{Z} \subset \mathbb{R}^{n_z} \to \mathbb{R}^{n_x}$ satisfies Assumption 2 and let $\mathcal{Z}$ be a CZ in $\mathbb{R}^{n_z}$, i.e., $\mathcal{Z} = \{\tilde{G}, \tilde{c}, \tilde{A}, \tilde{b}\}_{CZ}$, and n_g be the number of generators of $\mathcal{Z}$. Then, the following set inclusion holds:*

$$f(\mathcal{Z}) \subseteq \mathcal{CZ}_f \triangleq \bigcap_{H \in \mathbf{H}_{\tilde{f}}} \{\tilde{G}^H, \tilde{c}^H, \mathbb{A}, \tilde{b}\}_{CZ}, \quad (19)$$

where $\tilde{G}^H \triangleq [H \ \ \frac{1}{2}\mathrm{diag}(\overline{g}^H - \underline{g}^H)]$, $\mathbb{A} \triangleq [\tilde{A} \ \ 0_{n_g \times n_x}]$,

$$\overline{g}^H_i \triangleq \tilde{g}_{i,d}(\bar{\mathbf{l}}_{n_g}, \underline{\mathbf{l}}_{n_g}; H^\top_{i,:}, \langle H^\top_{i,:}, \cdot\rangle), \tilde{c}^H \triangleq \frac{1}{2}(\overline{g}^H + \underline{g}^H), \quad (20)$$

$$\underline{g}^H_i \triangleq \tilde{g}_{i,d}(\underline{\mathbf{l}}_{n_g}, \bar{\mathbf{l}}_{n_g}; H^\top_{i,:}, \langle H^\top_{i,:}, \cdot\rangle), \quad (21)$$

$\bar{\mathbf{l}}_{n_g} \triangleq \min(\mathbf{1}_{n_g}, \tilde{A}^\dagger\tilde{b} + \kappa\mathbf{r}_{n_g})$, $\underline{\mathbf{l}}_{n_g} \triangleq \max(-\mathbf{1}_{n_g}, \tilde{A}^\dagger\tilde{b} - \kappa\mathbf{r}_{n_g})$, $\tilde{g}_{i,d}(\cdot, \cdot; H^\top_{i,:}, \langle H^\top_{i,:}, \cdot\rangle)$ is the tight mixed-monotone decomposition function (cf. Proposition 4) for the JSS mapping $\tilde{g}_i(\xi) \triangleq \tilde{f}_i(\xi) - \langle H^\top_{i,:}, \xi\rangle : \mathbb{B}^{n_g}_\infty \to \mathbb{R}^{n_x}$, $\mathbf{H}_{\tilde{f}}$ is defined in Corollary 1 and $\tilde{f}(\xi) \triangleq f(\tilde{c} + \tilde{G}\xi)$, $\mathbf{r}_{n_g} \triangleq \mathrm{rowsupp}(I_{n_g} - \tilde{A}^\dagger\tilde{A})$ and κ is a very large positive real number (infinity).

Proof. To prove the inclusion in (19), consider the constrained zonotope representation of the set $\mathcal{Z}$, i.e., $\mathcal{Z} \triangleq \{\tilde{G}, \tilde{c}, \tilde{A}, \tilde{b}\}_{CZ} \triangleq \{z = \tilde{G}\xi + c \mid \xi \in \mathbb{B}^{n_g}_\infty, \tilde{A}\xi = \tilde{b}\}$. Using similar notation as in the proof of Lemma 1, let us define $\tilde{f}(\xi) : \mathbb{B}^{n_g}_\infty \to \mathbb{R}^{n_x} \triangleq f(\tilde{G}\xi + \tilde{c})$ that consequently returns

$$f(\mathcal{Z}) \subseteq \{\tilde{f}(\xi) \mid \xi \in \mathbb{B}^{n_g}_\infty, \tilde{A}\xi = \tilde{b}\}. \quad (22)$$

Note that by [28, Theorem 2], $\tilde{A}\xi = \tilde{b} \Rightarrow \xi \in \mathbb{I}\Xi \triangleq [\tilde{A}^\dagger\tilde{b} -$

$\kappa \mathbf{r}_{n_g}, \tilde{A}^\dagger \tilde{b} - \kappa \mathbf{r}_{n_g}]$, where $\mathbf{r}_{n_g} \triangleq \mathrm{rowsupp}(I_{n_g} - \tilde{A}^\dagger \tilde{A})$ and κ is a very large positive real number. Combining this with the fact that $\xi \in \mathbb{B}_\infty^{n_g}$ (cf. (22)), we find that $\xi \in \mathbb{I}\tilde{\Xi} \triangleq \mathbb{I}\Xi \cap \mathbb{B}_\infty^{n_g} = [\underline{\mathbf{l}}_{n_g}, \overline{\mathbf{l}}_{n_g}]$, where $\underline{\mathbf{l}}_{n_g}, \overline{\mathbf{l}}_{n_g}$ are defined below (21). On the other hand, similar to the proof of Lemma 1, we conclude by Corollary 1 that $\forall H \in \mathbf{H}_{\tilde{f}}$, $\tilde{f}(\cdot)$ can be decomposed as

$$\begin{aligned} \tilde{f}(\xi) &= \tilde{g}^H(\xi) + H\xi, \quad \forall H \in \mathbf{H}_{\tilde{f}}, \forall \xi \in \mathbb{I}\tilde{\Xi}, \\ \Rightarrow \tilde{f}(\mathbb{I}\tilde{\Xi}) &\subseteq \tilde{g}^H(\mathbb{I}\tilde{\Xi}) \oplus H\mathbb{I}\tilde{\Xi}, \forall H \in \mathbf{H}_{\tilde{f}}, \end{aligned} \tag{23}$$

where $\tilde{g}^H(\xi)$ is a JSS function in $\mathbb{I}\tilde{\Xi}$ and $\mathbf{H}_{\tilde{f}}$ is given in (9). By Corollary 1, in each dimension $i \in \mathbb{N}_{n_x}$, $\tilde{g}_i^H(\xi)$ can be tightly bounded as $\underline{g}_i^H \leq \tilde{g}_i^H(\xi) \leq \overline{g}_i^H, \forall \xi \in \mathbb{I}\tilde{\Xi}, \forall H \in \mathbf{H}_{\tilde{f}}$, with $\overline{g}_i^H, \underline{g}_i^H$ given in (20) and (21), respectively. Augmenting all these $\mathbb{N}_{nx}$ one-dimensional inequalities and applying Proposition 1 yield the following set inclusion: $\forall H \in \mathbf{H}_{\tilde{f}}$:

$$\tilde{g}^H(\mathbb{I}\tilde{\Xi}) \subseteq [\underline{g}^H, \overline{g}^H] = \frac{1}{2}((\underline{g}^H + \overline{g}^H) \oplus \mathrm{diag}(\overline{g}^H - \underline{g}^H)\mathbb{B}_\infty^{n_x}).$$

Combining this, (22), (23) and the fact that the inclusion in (23) holds for all $H \in \mathbf{H}_{\tilde{f}}$ and hence for the intersection of all of them, we obtain $f(\mathcal{Z}) \subseteq \{H\xi + \mathrm{diag}(\overline{g}^H - \underline{g}^H)\theta + \frac{1}{2}((\underline{g}^H + \overline{g}^H) \mid \xi \in \mathbb{B}_\infty^{n_g}, \theta \in \mathbb{B}_\infty^{n_x}, \tilde{A}\xi = \tilde{b}\}, \forall H \in \mathbf{H}_{\tilde{f}}$, where the set on the right hand side of the inclusion is equivalent to the intersection of the CZs on the right hand side of (19). ∎

Finally, for further improvement, we can take the intersection of the resulting propagated sets in Lemmas 1 and 2. This is formally summarized in the following Theorem 1.

Theorem 1 (Decomposition-Based ZB/CZ Propagation)**.** *Suppose all the assumptions in Lemmas 1 and 2 hold. Then, $f(\mathcal{Z}) \subseteq \mathcal{ZB}_f \cap \mathcal{CZ}_f$, where $\mathcal{ZB}_f, \mathcal{CZ}_f$ are computed in Lemmas 1 and 2, respectively.*

Proof. It follows from Lemmas 1 and 2 that $f(\mathcal{Z}) \subseteq \mathcal{ZB}_f$ and $f(\mathcal{Z}) \subseteq \mathcal{CZ}_f$, and so $f(\mathcal{Z}) \subseteq \mathcal{ZB}_f \cap \mathcal{CZ}_f$. ∎

B. Decomposition-Based CZ/ZB Update Step

In this section, we address Problem 2 for a given locally Lipschitz nonlinear vector field $\mu(\cdot)$ and assuming that the propagated and the observation/constraint sets at each time step k are zonotope bundles (Lemma 3) or constrained zonotopes (Lemma 4). Using a similar idea as in Section IV-A, i.e, considering the space of generators, decomposing the transformed observation function into a JSS and a linear component, applying the tight remainder-form decomposition functions [26] to bound the JSS component, augmenting and intersecting, as well as taking the advantage of linear remainder functions, we obtain ZB/CZ enclosures to the nonlinear generalized intersection in (12). The results of this section are summarized in Lemmas 3 and 4 and Theorem 2.

Lemma 3 (Decomposition-Based ZB Update)**.** *Suppose $\mu : \mathbb{R}^{n_x} \to \mathbb{R}^{n_\mu}$ satisfies Assumption 2. Let $\mathcal{Z}_f \subset \mathbb{R}^{n_x}$ and $Z_\mu \subset \mathbb{R}^{n_\mu}$ be two ZB sets, i.e., $\mathcal{Z}_f = \mathcal{ZB}_f = \bigcap_{r=1}^{R} \{G_f^r, c_f^r\}_Z$ and $\mathcal{Z}_\mu = \mathcal{ZB}_\mu = \bigcap_{t=1}^{T} \{G_\mu^t, c_\mu^t\}_Z$, and $\forall r \in \mathbb{N}_R \triangleq \{1, \ldots, R\}, \forall t \in \mathbb{N}_T \triangleq \{1, \ldots, T\}$, let n_r, n_t be the number of generators of the corresponding zonotopes, respectively. Then, the following set inclusion holds:*

$$\mathcal{ZB}_f \cap_\mu \mathcal{ZB}_\mu \subseteq \mathcal{ZB}_u \triangleq \bigcap_{r=1}^{R} \bigcap_{t=1}^{T} \bigcap_{Q_r \in \mathbf{Q}_{\tilde{\mu}_r}} \{\hat{G}_r^t, \hat{c}_r, \hat{A}_{r,t}^{Q_r}, \hat{b}_{r,t}^{Q_r}\}_{CZ}, \tag{24}$$

where $\hat{G}_r^t \triangleq [G_f^r \ \mathbf{0}^t], \hat{c}_r \triangleq c_f^r, \ \hat{b}_{r,t}^{Q_r} \triangleq c_\mu^t - \frac{1}{2}(\overline{p}_r^{Q_r} + \underline{p}_r^{Q_r})$,

$$\hat{A}_{r,t}^{Q_r} \triangleq \left[Q_r \ \ -G_\mu^t \ \ \tfrac{1}{2}\mathrm{diag}(\overline{p}_r^{Q_r} - \underline{p}_r^{Q_r})\right],$$

$$\overline{p}_{r,i}^{Q_r} \triangleq p_{i,d}^r(\mathbf{1}_{n_r}, -\mathbf{1}_{n_r}; Q_{r(i,:)}^\top, \langle Q_{r(i,:)}^\top, \cdot \rangle), \tag{25}$$

$$\underline{p}_{r,i}^{Q_r} \triangleq p_{i,d}^r(-\mathbf{1}_{n_r}, \mathbf{1}_{n_r}; Q_{r(i,:)}^\top, \langle Q_{r(i,:)}^\top, \cdot \rangle), \tag{26}$$

$p_{i,d}^r(\cdot, \cdot; Q_r, \langle Q_{r(i,:)}^\top, \cdot \rangle)$ is the tight mixed-monotone decomposition function (cf. Proposition 4) for the JSS mapping $p_{r,i}^{Q_r}(\alpha) \triangleq \tilde{\mu}_{r,i}(\alpha) - \langle Q_{r(i,:)}^\top, \alpha \rangle : \mathbb{B}_\infty^{n_r} \to \mathbb{R}^{n_\mu}$, $\mathbf{Q}_{\tilde{\mu}_r}$ is defined similar to $\mathbf{H}_f$ in Corollary 1 (with the corresponding function being $\tilde{\mu}_r(\alpha) \triangleq \mu(c_f^r + G_f^r \alpha)$) and $\mathbf{0}^t$ is a zero matrix in $\mathbb{R}^{n_x \times (n_t + n_\mu)}$.

Proof. Suppose $z \in \mathcal{ZB}_f \cap_\mu \mathcal{ZB}_\mu$. Then, by the definition of the operator $\cap_\mu$ (cf. (12)), $z \in \mathcal{ZB}_f$ and $\mu(z) \in \mathcal{ZB}_\mu$. The former implies that $\forall r \in \mathbb{N}_R, \exists \alpha \in \mathbb{B}_\infty^{n_r}$ such that $z = G_f^r \alpha + c_f^r$, while it follows from the latter that $\mu(z) = \mu(G_f^r \alpha + c_f^r) \triangleq \tilde{\mu}_r(\alpha) \in \mathcal{ZB}_\mu \Rightarrow \forall t \in \mathbb{N}_T, \exists \zeta \in \mathbb{B}_\infty^{n_t}$, such that $\tilde{\mu}_r(\alpha) = c_\mu^t + G_\mu^t \zeta$. Putting these two results in a set representation form, we obtain:

$$z \in \bigcap_{r=1}^{R} \bigcap_{t=1}^{T} \{G_f^r \alpha + c_f^r \,|\, \tilde{\mu}_r(\alpha) = c_\mu^t + G_\mu^t \zeta, \alpha \in \mathbb{B}_\infty^{n_r}, \zeta \in \mathbb{B}_\infty^{n_t}\}. \tag{27}$$

On the other hand, using Corollary 1, $\tilde{\mu}_r(\cdot)$ can be decomposed into a JSS and a linear mapping as follows: $\forall r \in \mathbb{N}_R, \forall Q_r \in \mathbf{Q}_{\tilde{\mu}_r}, \forall \alpha \in \mathbb{B}_\infty^{n_r}$:

$$\tilde{\mu}_r(\alpha) = p_r^{Q_r}(\alpha) + Q_r \alpha. \tag{28}$$

Moreover, by the same corollary, the JSS component $p_r^{Q_r}(\cdot)$ is tightly bounded as follows: $\forall i \in \mathbb{N}_{n_\mu}, \forall Q_r \in \mathbf{Q}_{\tilde{\mu}_r}, \underline{p}_{r,i}^{Q_r} \leq p_{r,i}^{Q_r}(\alpha) \leq \underline{p}_{r,i}^{Q_r}, \forall \alpha \in \mathbb{B}_\infty^{n_r}$, with $\underline{p}_{r,i}^{Q_r}, \overline{p}_{r,i}^{Q_r}$ given in (25) and (26), respectively. Combining this, as well as (28) and Proposition 1 results in: $\forall r \in \mathbb{N}_R$, $\forall Q_r \in \mathbf{Q}_{\tilde{\mu}_r}$, $\forall \alpha \in \mathbb{B}_\infty^{n_r}$, $\exists \theta \in \mathbb{B}_\infty^{n_\mu}$ such that $\tilde{\mu}_r(\alpha) = \frac{1}{2}(\underline{p}_{r,i}^{Q_r} + \overline{p}_{r,i}^{Q_r}) + \frac{1}{2}\mathrm{diag}(\overline{p}_{r,i}^{Q_r} - \underline{p}_{r,i}^{Q_r})\theta + Q_r \alpha$. Further, putting this together with (27) returns $z \in \bigcap_{r=1}^{R} \bigcap_{t=1}^{T} \bigcap_{Q_t \in \mathbf{Q}_{\tilde{\mu}_t}} \{G_f^r \alpha + c_f^r | \frac{1}{2}(\underline{p}_{r,i}^{Q_r} + \overline{p}_{r,i}^{Q_r}) + \frac{1}{2}\mathrm{diag}(\overline{p}_{r,i}^{Q_r} - \underline{p}_{r,i}^{Q_r})\theta + Q_r \alpha = c_\mu^t + G_\mu^t \zeta, \alpha \in \mathbb{B}_\infty^{n_r}, \zeta \in \mathbb{B}_\infty^{n_t}, \theta \in \mathbb{B}_\infty^{n_\mu}\}$, where the set on the right hand side is equivalent to the one on the right hand side of (24). ∎

Lemma 4 (Decomposition-Based CZ Update)**.** *Suppose $\mu : \mathbb{R}^{n_x} \to \mathbb{R}^{n_\mu}$ satisfies Assumption 2. Let $\mathcal{Z}_f \subset \mathbb{R}^{n_x}$ and $\mathcal{Z}_\mu \subset \mathbb{R}^{n_\mu}$ be two CZ sets, i.e., $\mathcal{Z}_f = \mathcal{CZ}_f = \{\tilde{G}_f, \tilde{c}_f, \tilde{A}_f, \tilde{b}_f\}_{CZ}$ and $\mathcal{Z}_\mu = \mathcal{CZ}_\mu = \{\tilde{G}_\mu, \tilde{c}_\mu, \tilde{A}_\mu, \tilde{b}_\mu\}_{CZ}$, and let n_c, n_τ be the number of generators of $\mathcal{Z}_f, \mathcal{Z}_\mu$, respectively. Then, the following set inclusion holds:*

$$\mathcal{CZ}_f \cap_\mu \mathcal{CZ}_\mu \subseteq \mathcal{CZ}_u \triangleq \bigcap_{\Omega \in \mathbf{\Omega}_\lambda} \{\mathbb{G}, \tilde{c}_f, \mathbb{A}_\Omega, \tilde{b}_\Omega\}_{CZ}, \tag{29}$$

where $\mathbb{G} \triangleq [\tilde{G}_f\ 0\ 0]$, $\tilde{b}_\Omega \triangleq [\tilde{b}_f^\top\ \tilde{b}_\mu^\top\ (\tilde{c}_f - \frac{1}{2}(\overline{\nu}^\Omega + \underline{\nu}^\Omega))^\top]^\top$,

$$\mathbb{A}_\Omega \triangleq \begin{bmatrix} \tilde{A}_f & 0 & 0 \\ 0 & \tilde{A}_\mu & \\ \Omega & -\tilde{G}_\mu & \frac{1}{2}\mathrm{diag}(\overline{\nu}^\Omega - \underline{\nu}^\Omega) \end{bmatrix},$$

$$\overline{\nu}_i^\Omega \triangleq \nu_{i,d}(\bar{\mathbf{l}}_{n_c}, \underline{\mathbf{l}}_{n_c}; \Omega_{(i,:)}^\top, \langle \Omega_{(i,:)}^\top, \cdot\rangle), \quad (30)$$

$$\underline{\nu}_i^\Omega \triangleq \nu_{i,d}(\underline{\mathbf{l}}_{n_c}, \bar{\mathbf{l}}_{n_c}; \Omega_{(i,:)}^\top, \langle \Omega_{(i,:)}^\top, \cdot\rangle), \quad (31)$$

$\bar{\mathbf{l}}_{n_c} \triangleq \min(\mathbf{1}_{n_c}, \tilde{A}_f^\dagger \tilde{b}_f + \kappa \mathbf{r}_{n_c})$, $\underline{\mathbf{l}}_{n_c} \triangleq \max(-\mathbf{1}_{n_c}, \tilde{A}_f^\dagger \tilde{b}_f - \kappa \mathbf{r}_{n_c})$, $\nu_{i,d}(\cdot, \cdot; \Omega_{(i,:)}^\top, \langle \Omega_{(i,:)}^\top, \cdot\rangle)$ *is the tight mixed-monotone decomposition function (cf. Proposition 4) for the JSS mapping* $\nu_i^\Omega(\beta) \triangleq \lambda_i(\beta) - \langle \Omega_{(i,:)}^\top, \beta\rangle : \mathbb{B}_\infty^{n_c} \to \mathbb{R}^{n_\mu}$, $\mathbf{\Omega}_\lambda$ *is defined similar to* $\mathbf{H}_f$ *in Corollary 1 (with the corresponding function being* $\lambda(\beta) \triangleq \mu(\tilde{c}_f + \tilde{G}_f \beta)$*),* $\mathbf{r}_{n_c} \triangleq \mathrm{rowsupp}(I_{n_c} - \tilde{A}_f^\dagger \tilde{A}_f)$ *and* κ *is a very large positive real number (infinity).*

Proof. Suppose $z \in \mathcal{CZ}_f \cap_\mu \mathcal{CZ}_\mu$. Then, by the definition of the operator $\cap_\mu$ (cf. (12)), $z \in \mathcal{CZ}_f$ and $\mu(z) \in \mathcal{CZ}_\mu$. The former implies that $\exists \beta \in \mathbb{B}_\infty^{n_c}$ such that $\tilde{A}_f \beta = \tilde{b}_f$ and $z = \tilde{G}_f \beta + \tilde{c}_f$, while it follows from the latter that $\mu(z) = \mu(\tilde{G}_f \beta + \tilde{c}_f) \triangleq \lambda(\beta) \in \mathcal{CZ}_\mu \Rightarrow \exists \gamma \in \mathbb{B}_\infty^{n_\tau}$ such that $\tilde{A}_\mu \gamma = \tilde{b}_\mu$ and $\lambda(\beta) = \tilde{c}_\mu + \tilde{G}_\mu \gamma$. Putting these two results into a set representation form, we obtain:

$$z \in \{\tilde{G}_f \beta + \tilde{c}_f \mid \lambda(\beta) = \tilde{c}_\mu + \tilde{G}_\mu \gamma, \tilde{A}_f \beta = \tilde{b}_f, \tilde{A}_\mu \gamma = \tilde{b}_\mu, \beta \in \mathbb{B}_\infty^{n_c}, \gamma \in \mathbb{B}_\infty^{n_\tau}\}. \quad (32)$$

On the other hand, using Corollary 1, $\lambda(\cdot)$ can be decomposed into a JSS and a linear mapping as follows:

$$\forall \Omega \in \mathbf{\Omega}_\lambda, \forall \beta \in \mathbb{B}_\infty^{n_c} : \lambda(\beta) = \nu^\Omega(\beta) + \Omega\beta. \quad (33)$$

Further, note that by [28, Theorem 2], $\tilde{A}_f \beta = \tilde{b}_f \Rightarrow \beta \in \mathbb{IB} \triangleq [\tilde{A}_f^\dagger \tilde{b}_f - \kappa \mathbf{r}_{n_c}, \tilde{A}_f^\dagger \tilde{b}_f - \kappa \mathbf{r}_{n_c}]$, where $\mathbf{r}_{n_c} \triangleq \mathrm{rowsupp}(I_{n_c} - \tilde{A}_f^\dagger \tilde{A}_f)$ and κ is a very large positive real number. Then, since $\beta \in \mathbb{B}_\infty^{n_c}$, we have $\beta \in \mathbb{IB} \cap \mathbb{B}_\infty^{n_c} = [\underline{\mathbf{l}}_{n_c}, \bar{\mathbf{l}}_{n_c}]$. Putting this and Corollary 1 together results in the JSS component $\nu^\Omega(\cdot)$ being tightly bounded, i.e., $\forall i \in \mathbb{N}_{n_\mu}, \forall \Omega \in \mathbf{\Omega}_\lambda, \underline{\nu}_i^\Omega \le \nu_i^\Omega(\beta) \le \underline{\nu}_i^\Omega, \forall \beta \in \mathbb{B}_\infty^{n_c}$, with $\underline{\nu}_i^\Omega, \overline{\nu}_i^\Omega$ given in (30), (31), respectively. Combining this, (33) and Proposition 1 leads to: $\forall \Omega \in \mathbf{\Omega}_\lambda, \forall \beta \in \mathbb{B}_\infty^{n_c}, \exists \rho \in \mathbb{B}_\infty^{n_\mu}$ such that $\lambda(\beta) = \frac{1}{2}(\underline{\nu}^\Omega + \overline{\nu}^\Omega) + \frac{1}{2}\mathrm{diag}(\overline{\nu}^\Omega - \underline{\nu}^\Omega)\rho + \Omega\alpha$, which along with (32) returns $z \in \cap_{\Omega \in \mathbf{\Omega}_\lambda} \{\tilde{G}_f \beta + \tilde{c}_f \mid \frac{1}{2}(\underline{\nu}^\Omega + \overline{\nu}^\Omega) + \frac{1}{2}\mathrm{diag}(\overline{\nu}^\Omega - \underline{\nu}^\Omega)\rho + \Omega\beta = \tilde{c}_\mu + \tilde{G}_\mu \gamma, \tilde{A}_f \beta = \tilde{b}_f, \tilde{A}_\mu \gamma = \tilde{b}_\mu, \beta \in \mathbb{B}_\infty^{n_c}, \gamma \in \mathbb{B}_\infty^{n_\tau}, \rho \in \mathbb{B}_\infty^{n_\mu}\}$, where the set on the right is equivalent to the right hand side of (24). ■

We conclude this subsection by combining the results in Lemmas 3 and 4 via the following theorem.

Theorem 2 (Decomposition-Based ZB/CZ Update)**.** *Suppose all the assumptions in Lemmas 3 and 4 hold. Then*

$$\mathcal{Z}_f \cap_\mu \mathcal{Z}_\mu \subseteq \mathcal{ZB}_u \cap \mathcal{CZ}_u,$$

where $\mathcal{ZB}_u, \mathcal{CZ}_u$ *are given in Lemmas 3 and 4, respectively.*

Proof. By Lemmas 3 and 4: $\mathcal{Z}_f \cap_\mu \mathcal{Z}_\mu \subseteq \mathcal{ZB}_u$ and $\mathcal{Z}_f \cap_\mu \mathcal{Z}_\mu \subseteq \mathcal{CZ}_u$, and hence, $\mathcal{Z}_f \cap_\mu \mathcal{Z}_\mu \subseteq \mathcal{ZB}_u \cap \mathcal{CZ}_u$. ■

C. Modifications to the Approach in [1]

The purpose of this subsection is twofold: i) We propose a potential refinement/improvement to the propagation approach in [1, Theorem 2] (recapped in Proposition 3) through the following Proposition 5, by applying our previously developed remainder-form decomposition functions to compute potentially tighter enclosing intervals to Jacobian matrix of $f(\cdot)$; ii) We propose an update method via Lemma 5, that is based on the "CZ-inclusion" introduced in [1, Theorem 1] (recapped in Proposition 2). The proposed update method is applicable to general nonlinear observation functions (similar to the proposed methods in Lemmas 3 and 4), as opposed to the update (i.e, linear intersection) approach in [1] that is only applicable when the observation function is linear.

Proposition 5 (Refinement to the Propagation Approach in [1])**.** *Suppose all the assumptions in Proposition 4 (i.e, [1, Theorem 2]) hold. Then, the set inclusion in* (5) *also holds when replacing* $\mathbb{J}$ *with* $\tilde{\mathbb{J}}$ *(or the best (tightest) of them), where* $\tilde{\mathbb{J}}$ *is an enclosing interval to* $g(x) \triangleq \nabla_x^\top f(X, W)$ *that can be computed by applying Proposition 4 to the function* $g(\cdot)$*.*

Proof. This directly follows from Proposition 4. ■

Lemma 5 (Update based on "CZ-Inclusion" in [1])**.** *Suppose all the assumptions in Lemma 4 hold. Let* $x_0 \in \mathcal{CZ}_f$ *and* $\mathbb{J}^\mu, \mathbb{J}_\Delta^\mu \in \mathbb{R}^{n_\mu \times n_x}$ *be interval matrices satisfying* $J^\mu(\mathcal{CZ}_f) \subseteq \mathbb{J}^\mu$ *and* $\forall i \in \mathbb{N}_{n_\mu}, \forall j \in \mathbb{N}_{n_x}, [\mathbb{J}_\Delta^\mu]_{ij} \triangleq \frac{1}{2}\left[-\mathrm{diam}(\mathbb{J}^\mu)_{ij}\ \ \mathrm{diam}(\mathbb{J}^\mu)_{ij}\right]$*, where* J^μ *denotes the Jacobian of* $\mu(\cdot)$*. Let* $\overline{\mathcal{Z}}_f = \{\overline{G}^f, \overline{c}^f\}_Z$ *be a zonotope satisfying* $\mathcal{CZ}_f \ominus x_0 \subseteq \overline{\mathcal{Z}}_f$ *with* $\overline{c}^f \in \mathbb{R}^{\overline{n}}$*, let* $\mathbf{m}^\mu \in \mathbb{R}^{n_\mu}$ *be an interval vector such that* $\mathbf{m}^\mu \supset \mathbb{J}_\Delta^\mu \overline{c}^f$ *and* $\mathrm{mid}(\mathbf{m}^\mu) = \mathbf{0}_{n_\mu}$ *and let* $P^\mu \in \mathbb{R}^{n_\mu \times n_\mu}$ *be a diagonal matrix defined as follows:* $\forall i = 1, \ldots, n_\mu$*:*

$$P_{ii}^\mu = \frac{1}{2}\mathrm{diam}(\mathbf{m}^\mu)_i + \tfrac{1}{2}\textstyle\sum_{j=1}^{\overline{n}} \sum_{k=1}^{n_x} \mathrm{diam}(\mathbb{J}_\Delta^\mu)_{ik} |\overline{G}_{kj}^f|. \quad (34)$$

Then, the following set inclusion holds:

$$\mathcal{CZ}_f \cap_\mu \mathcal{CZ}_\mu \subseteq \mathcal{CZ}_u^R \triangleq \{G_u, c_u, A_u, b_u\}_{CZ}, \quad (35)$$

where $G_u \triangleq [\tilde{G}_f\ 0\ 0]$,

$$A_u \triangleq \begin{bmatrix} \mathrm{mid}(\mathbb{J}^\mu)\tilde{G}_f & -\tilde{G}_\mu & G_R \\ \tilde{A}_f & 0 & 0 \\ 0 & \tilde{A}_\mu & 0 \\ 0 & 0 & A_R \end{bmatrix}, b_u \triangleq \begin{bmatrix} \tilde{c}_\mu - \mu(x_0) - c_R \\ + \mathrm{mid}(\mathbb{J}^\mu)(x_0 - \tilde{c}_f) \\ \tilde{b}_f \\ \tilde{b}_\mu \\ b_R \end{bmatrix},$$

$$G_R \triangleq [0\ P^\mu],\ c_R \triangleq 0,\ A_R \triangleq [\tilde{A}_f\ 0],\ b_R \triangleq \tilde{b}_f. \quad (36)$$

Proof. Let $z \in \mathcal{CZ}_f \cap_\mu \mathcal{CZ}_\mu$. Then, by the definition of the operator $\cap_\mu$ (cf. (12)), $z \in \mathcal{CZ}_f$ and $\mu(z) \in \mathcal{CZ}_\mu$. Further, by Proposition 1 and the mean value theorem, $z \in \mathcal{CZ}_f$ implies that $\mu(z) \in \mu(\mathcal{CZ}_f) \subseteq \mu(x_0) \oplus \mathbb{J}^\mu(\mathcal{CZ}_f \ominus x_0)$, where

$$\begin{aligned} &\mu(x_0) \oplus \mathbb{J}^\mu(\mathcal{CZ}_f \ominus x_0) \\ &= \mu(x_0) \oplus (\mathrm{mid}(\mathbb{J}^\mu) + \mathbb{J}_\Delta^\mu)(\mathcal{CZ}_f \ominus x_0) \quad (37) \\ &= (\mu(x_0) - \mathrm{mid}(\mathbb{J}^\mu)x_0) \oplus \mathrm{mid}(\mathbb{J}^\mu)\mathcal{CZ}_f \oplus \mathbb{J}_\Delta^\mu(\mathcal{CZ}_f \ominus x_0). \end{aligned}$$

On the other hand, by Proposition 2,

$$\mathbb{J}_\Delta^\mu(\mathcal{CZ}_f \ominus x_0) \subseteq \mathcal{CZ}_R \triangleq \{G_R, c_R, A_R, b_R\}_{CZ}, \quad (38)$$

with G_R, c_R, A_R, b_R given in (34) and (36) (note that $\mathrm{mid}(\mathbb{J}_\Delta^\mu) = 0$ by its definition) and where $\mathcal{CZ}_R$ has n_R generators. Then, the facts that $z \in \mathcal{CZ}_f \triangleq \{\tilde{G}_f \beta + \tilde{c}_f \mid \tilde{A}_f \beta = \tilde{b}_f, \beta \in \mathbb{B}_\infty^{n_c}\}$ and $\mu(z) \in \mathcal{CZ}_\mu \triangleq \{\tilde{G}_\mu \gamma + \tilde{c}_\mu \mid$

$\tilde{A}_\mu\gamma = \tilde{b}_\mu, \gamma \in \mathbb{B}_\infty^{n_\tau}\}$, along with (37) and (38), imply that $z \in \{\tilde{G}_f\beta + \tilde{c}_f | \tilde{c}_\mu + \tilde{G}_\mu\gamma = \mu(x_0) + \text{mid}(\mathbb{J}^\mu)(\tilde{c}_f - x_0) + \text{mid}(\mathbb{J}^\mu)\beta + C_R + G_R\xi_R, \tilde{A}_f\beta = \tilde{b}_f, \tilde{A}_\mu\gamma = \tilde{b}_\mu, A_R\xi_R = b_R, \beta \in \mathbb{B}_\infty^{n_c}, \gamma \in \mathbb{B}_\infty^{n_\tau}, \xi_R \in \mathbb{B}_\infty^{n_R}\}$, which is equivalent to the CZ on the right hand side of (35). ■

V. Simulations

In this section, we compare the performance of five approaches to guaranteed state estimation: i) RRSR, i.e., the mean value extension-based propagation introduced in [1] (recapped in Proposition 3) in addition to the update approach in [1] for the case when the observation function is linear (for Example I below) and its extension in Lemma 5 to nonlinear measurements (for Example II below), ii) D-RRSR, i.e, a modification to RRSR where the bounds for Jacobian matrices are computed using the reminder-form decomposition functions (cf. Proposition 5), iii) D-ZB, i.e., decomposition-based propagation and update with ZBs (cf. Lemmas 1 and 3), iv) D-CZ, i.e., decomposition-based propagation and update with CZs (cf. Lemmas 2 and 4) and v) COMB, i.e., a combination of i)–iv) via intersection (based on a similar idea as Theorems 1, 2). All simulations are performed on a 1.8 GHz (8 CPUS) i5-8250U, using MATLAB version 2020a and CORA 2020 [27].

A. Example I

Consider the following nonlinear discrete-time system from [1, Example 1]:

$$\begin{aligned} x_{1,k} &= 3x_{1,k-1} - \tfrac{x_{1,k-1}^2}{7} - \tfrac{4x_{1,k-1}x_{2,k-1}}{4+x_{1,k-1}} + w_{1,k-1}, \\ x_{2,k} &= -2x_{2,k-1} + \tfrac{3x_{1,k-1}x_{2,k-1}}{4+x_{1,k-1}} + w_{2,k-1}, \\ \begin{bmatrix} y_{1,k} \\ y_{2,k} \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} + \begin{bmatrix} v_{1,k} \\ v_{2,k} \end{bmatrix}, \end{aligned}$$

with $\|w_k\|_\infty \leq 0.1$, an unknown initial state $x_0 \in \mathcal{X}_0 = \left\{\begin{bmatrix} 0.1 & 0.2 & -0.1 \\ 0.1 & 0.1 & 0 \end{bmatrix}, \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}\right\}$ and $\mathcal{Y}_k \triangleq \{y_k - v_k \mid \|v_k\|_\infty \leq 0.4\}$.

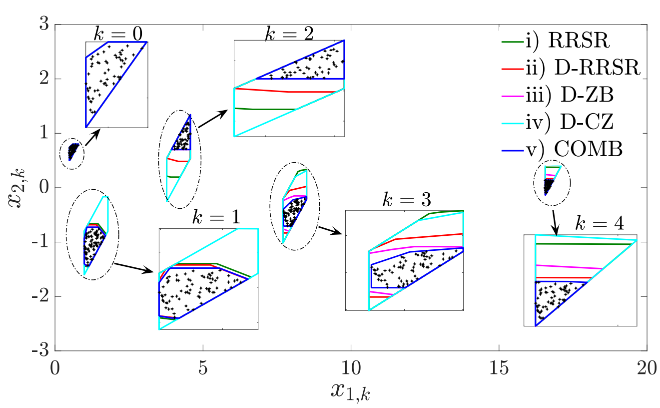

Fig. 1: Results for Example I from the first five time steps of set-valued state estimation, using five different approaches. Black dots are obtained from uniform sampling of the initial state and noise signals, and propagating through the system dynamics.

As can be seen from Figure 1, D-ZB provides less conservative enclosures compared to the other individual approaches, and further, the COMB approach results in a significant improvement by taking advantage of all approaches via intersection. Moreover, a more systematic comparison of the average computation times and enclosure set volumes of the five approaches is given in Table I. It can be observed that D-ZB is the fastest computationally, while the combination of all approaches, i.e., COMB, took the longest, as expected. Moreover, RRSR and D-RRSR took approximately the same time on average. In terms of average set volumes, D-ZB and D-RRSR generate the least conservative (smallest) enclosures when compared to the other approaches, while a further improvement is obtained using the intersection of all approaches (COMB).

TABLE I: Average total times (seconds) and average total volumes at each time step for five state estimators in Example I. Each average is taken over 50 simulations with uniformly sampled noise and initial state.

Methods:		$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
RRSR	Time:	0.0869	0.2496	0.1926	0.1960	0.2042
	Vol.:	0.2012	0.5002	0.6205	0.4811	0.3340
D-RRSR	Time:	0.0866	0.2251	0.1809	0.1977	0.2005
	Vol.:	0.2012	0.4758	0.6008	0.4385	0.1472
D-ZB	Time:	0.0882	0.0949	0.0906	0.0907	0.1226
	Vol.:	0.2012	0.4518	0.5729	0.32721	0.3175
D-CZ	Time:	0.0869	2.8245	2.9200	2.1183	3.3176
	Vol.:	0.2012	0.5673	0.6310	0.5061	0.4169
COMB	Time:	0.0872	6.1929	6.8815	6.2782	6.908
	Vol.:	0.2012	0.4485	0.5659	0.2841	0.1465

B. Example II (Unicycle System)

Now consider the following discretized unicycle-like mobile robot system [29]:

$$\begin{aligned} s_{x,k+1} &= s_{x,k} + T_0\phi_w \cos(\theta_k) + w_{1,k}, \\ s_{y,k+1} &= s_{y,k} + T_0\phi_w \sin(\theta_k) + w_{2,k}, \\ \theta_{k+1} &= \theta_k + T_0\phi_\theta + w_{3,k}, \\ y_k &= [d_{1,k}\ \phi_{1,k}\ d_{2,k}\ \phi_{2,k}]^\top + v_k, \end{aligned} \quad (39)$$

where $x_k \triangleq [s_{x,k}\ \ s_{y,k}\ \ \theta_k]^\top$, $w_k = [w_{x,k}\ \ w_{y,k}\ \ w_{\theta,k}]^\top, \phi_{\omega,k} = 0.3, \phi_{\theta,k} = 0.15, w_{x,k} = 0.2(0.5\rho_{x_{1,k}} - 0.3), w_{y,k} = 0.2(0.3\rho_{x_{2,k}} - 0.2)$ and $w_{\theta,k} = 0.2(0.6\rho_{x_{3,k}} - 0.4)$, with $\rho_{x_{l,k}} \in [0,1]$ $(l = 1,2,3)$ and initial state $x_0 = [0.1\ 0.2\ 1]^\top$. Moreover, $\forall i \in \{1,2\}$, $d_{i,k} = \sqrt{(s_{x_i} - s_{x,k})^2 + (s_{y_i} - s_{y,k})^2}$ and $\phi_{i,k} = \theta_k - \arctan(\frac{s_{y_i} - s_{y,k}}{s_{x_i} - s_{x,k}})$, with s_{x_i}, s_{y_i} being two known values. Furthermore, $\mathcal{Y}_k \triangleq \{y_k - v_k \mid v_{1,k} = 0.02\rho_{y_1,k} - 0.01, v_{2,k} = 0.03\rho_{y_2,k} - 0.01, v_{3,k} = 0.03\rho_{y_3,k} - 0.02, v_{4,k} = 0.05\rho_{y_4,k} - 0.03, \rho_{y_k,k} \in [0,1], \forall k = \{1,2,3,4\}\}$.

Applying all methods i) through v), one can observe from Figure 2 that the resulting set estimates appear comparable for all approaches. Upon closer examination, Table II shows that D-CZ takes the least average computation time followed by RRSR, D-RRSR, COMB and D-ZB, while in terms of average set volumes, the COMB approach results in the smallest volume followed by D-ZB, D-CZ, RRSR and D-RRSR. Note that the computation time for D-ZB is exceptionally large, presumably because of the specific implementation in CORA 2020 [27] for converting a polytope to its ZB representation that could result in a higher number of zonotopes than the minimal needed to exactly represent the same polytope. Thus, the reduction of the number of

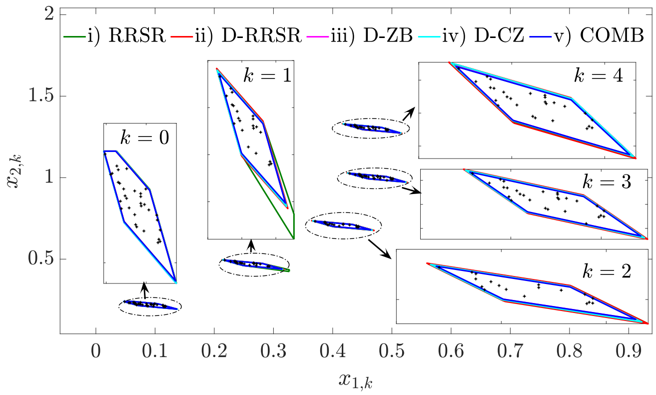

Fig. 2: Results for Example II from the first five time steps of set-valued state estimation, using five different approaches. Black dots are obtained from uniform sampling of the initial state and noise signals, and propagating through the system dynamics.

TABLE II: Average total times (seconds) and average total volumes (10^{-5}) at each time step for five state estimators in Example II. Each average is taken over 20 simulations with uniformly sampled noise and initial state.

Methods:		$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
RRSR	Time:	0.7719	4.2557	4.1883	2.9950	3.6747
	Vol.:	4.2924	3.7834	1.6171	4.5738	4.5558
D-RRSR	Time:	1.6690	42.905	45.571	28.642	50.539
	Vol.:	4.0527	3.2943	1.6600	5.1036	4.8001
D-ZB	Time:	1.3967	34.207	163.08	147.75	131.94
	Vol.:	4.2551	2.9697	1.3248	4.2917	4.2519
D-CZ	Time:	0.6020	2.1824	2.0195	2.2281	2.5908
	Vol.:	4.2551	3.0793	1.4337	4.6017	4.4564
COMB	Time:	0.2361	34.902	65.501	62.371	57.728
	Vol.:	4.0527	2.7726	1.2220	3.9126	3.9914

zonotopes in the bundle could be an interesting future topic, which could significantly decrease the computation time of the D-ZB approach.

VI. Conclusion

Novel methods were presented in this paper for guaranteed state estimation in bounded-error discrete-time nonlinear systems subject to nonlinear observations/constraints using indirect polytopic representations, i.e., using ZBs/CZs. By considering polytopes in the space of ZB/CZ's generators, our recent results on remainder-form mixed-monotone decomposition functions can be applied to compute enclosures that are guaranteed to enclose the set of all possible state trajectories. Further, the decomposition functions were leveraged to bound the nonlinear observation function to derive the updated set, i.e., to return enclosures to the intersection of the propagated set and the set of states that are consistent with noisy measurements. Finally, the mean value extension-based approach in [1] was also generalized to compute the updated set when the observation functions are nonlinear.

References

[1] B.S. Rego, G.V. Raffo, J.K. Scott, and D.M. Raimondo. Guaranteed methods based on constrained zonotopes for set-valued state estimation of nonlinear discrete-time systems. *Automatica*, 111:108614, 2020.

[2] C. Combastel, Q. Zhang, and A. Lalami. Fault diagnosis based on the enclosure of parameters estimated with an adaptive observer. *IFAC Proceedings Volumes*, 41(2):7314–7319, 2008.

[3] L. Jaulin. A nonlinear set membership approach for the localization and map building of underwater robots. *IEEE Transactions on Robotics*, 25(1):88–98, 2009.

[4] M.A. Dahleh and J.B. Pearson. ℓ_1-optimal feedback controllers for MIMO discrete-time systems. *IEEE Transactions on Automatic Control*, 32(4):314–322, 1987.

[5] M. Kieffer and E. Walter. Guaranteed nonlinear state estimator for cooperative systems. *Numerical algorithms*, 37(1-4):187–198, 2004.

[6] L. Chisci, G.A., and G. Zappa. Recursive state bounding by parallelotopes. *Automatica*, 32(7):1049–1055, 1996.

[7] V.T.H. Le, C. Stoica, T. Alamo, E.F. Camacho, and D. Dumur. Zonotopic guaranteed state estimation for uncertain systems. *Automatica*, 49(11):3418–3424, 2013.

[8] A. Girard and Le G.C. Efficient reachability analysis for linear systems using support functions. *IFAC Proceedings Volumes*, 41(2):8966–8971, 2008.

[9] J.S. Shamma and K. Tu. Set-valued observers and optimal disturbance rejection. *IEEE Trans. on Automatic Control*, 44(2):253–264, 1999.

[10] A. Vicino and G. Zappa. Sequential approximation of feasible parameter sets for identification with set membership uncertainty. *IEEE Transactions on Automatic Control*, 41(6):774–785, 1996.

[11] M. Khajenejad and S.Z. Yong. Simultaneous input and state set-valued $\mathcal{H}_\infty$-observers for linear parameter-varying systems. In *American Control Conference (ACC)*, pages 4521–4526. IEEE, 2019.

[12] B.T. Polyak, S.A. Nazin, Cé. Durieu, and E. Walter. Ellipsoidal parameter or state estimation under model uncertainty. *Automatica*, 40(7):1171–1179, 2004.

[13] M. Khajenejad and S.Z. Yong. Simultaneous mode, input and state set-valued observers with applications to resilient estimation against sparse attacks. In *Conference on Decision and Control (CDC)*, 2019.

[14] G. Zheng, D. Efimov, and W. Perruquetti. Design of interval observer for a class of uncertain unobservable nonlinear systems. *Automatica*, 63:167–174, 2016.

[15] M. Khajenejad and S.Z. Yong. Simultaneous input and state interval observers for nonlinear systems with full-rank direct feedthrough. In *2020 59th IEEE Conference on Decision and Control (CDC)*, pages 5443–5448. IEEE, 2020.

[16] Y. Wang, D.M. Bevly, and R. Rajamani. Interval observer design for LPV systems with parametric uncertainty. *Automatica*, 60:79–85, 2015.

[17] M. Khajenejad and S.Z. Yong. Simultaneous input and state interval observers for nonlinear systems with rank-deficient direct feedthrough. In *European Control Conference*, 2021, accepted.

[18] C. Combastel. Merging kalman filtering and zonotopic state bounding for robust fault detection under noisy environment. *IFAC-PapersOnLine*, 48(21):289–295, 2015.

[19] J.K. Scott, D.M. Raimondo, G.R. Marseglia, and R.D. Braatz. Constrained zonotopes: A new tool for set-based estimation and fault detection. *Automatica*, 69:126–136, 2016.

[20] M. Althoff and B.H Krogh. Zonotope bundles for the efficient computation of reachable sets. In *2011 50th IEEE conference on decision and control and European control conference*, pages 6814–6821. IEEE, 2011.

[21] R.E. Moore, R.B. Kearfott, and M.J. Cloud. *Introduction to interval analysis*. SIAM, 2009.

[22] L. Jaulin. Inner and outer set-membership state estimation. *Reliable Computing*, 22:47–55, 2016.

[23] C. Combastel. A state bounding observer for uncertain non-linear continuous-time systems based on zonotopes. In *Proceedings of the 44th IEEE Conference on Decision and Control*, pages 7228–7234. IEEE, 2005.

[24] T. Alamo, J.M. Bravo, and E.F. Camacho. Guaranteed state estimation by zonotopes. *Automatica*, 41(6):1035–1043, 2005.

[25] T. Alamo, J.M. Bravo, M.J. Redondo, and E.F. Camacho. A set-membership state estimation algorithm based on DC programming. *Automatica*, 44(1):216–224, 2008.

[26] M. Khajenejad and S.Z. Yong. Tight remainder-form decomposition functions with applications to constrained reachability and interval observer design. *IEEE Transactions on Automatic Control, submitted*, 2021, https://arxiv.org/pdf/2103.08638.pdf.

[27] M. Althoff. https://tumcps.github.io/cora/data/cora2020manual.pdf.

[28] M. James. The generalised inverse. *The Mathematical Gazette*, 62(420):109–114, 1978.

[29] B. Chen and G. Hu. Nonlinear state estimation under bounded noises. *Automatica*, 98:159–168, 2018.