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Sparse hop spanners for unit disk graphs [☆]

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ABSTRACT

A unit disk graph G on a given set P of points in the plane is a geometric graph where an edge exists between two points $p, q \in P$ if and only if $|pq| \le 1$. A spanning subgraph G' of G is a *k-hop spanner* if and only if for every edge $pq \in G$, there is a path between p, q in G' with at most k edges. We obtain the following results for unit disk graphs in the plane.

(i) Every n-vertex unit disk graph has a 5-hop spanner with at most $5.5n$ edges. We analyze the family of spanners constructed by Biniaz (2020) and improve the upper bound on the number of edges from $9n$ to $5.5n$.
(ii) Using a new construction, we show that every n-vertex unit disk graph has a 3-hop spanner with at most $11n$ edges.
(iii) Every n-vertex unit disk graph has a 2-hop spanner with $O(n \log n)$ edges. This is the first nontrivial construction of 2-hop spanners.
(iv) For every sufficiently large positive integer n, there exists a set P of n points on a circle, such that every plane hop spanner on P has hop stretch factor at least 4. Previously, no lower bound greater than 2 was known.
(v) For every finite point set on a circle, there exists a plane (i.e., crossing-free) 4-hop spanner. As such, this provides a tight bound for points on a circle.
(vi) The maximum degree of k-hop spanners cannot be bounded from above by a function of k for any positive integer k.

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1. Introduction

A *k-hop spanner* of a connected graph $G = (V, E)$ is a subgraph $G' = (V, E')$, where $E' \subseteq E$, with the additional property that the distance between any two vertices in G' is at most k times the distance in G [25,39], where the *distance* between two vertices is the number of edges on a shortest path between them. The graph G itself is a 1-hop spanner. The minimum k for which a subgraph G' is a k-hop spanner of G is referred to as the *hop stretch factor* (or *hop number*) of G'. An alternative characterization of k-hop spanners is given in the following lemma.

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Table 1
A summary of results on constructions of hop spanners for unit disk graphs in the plane.

Reference	k	$\lvert E'\rvert$	Guaranteed to be plane?
Catusse, Chepoi, and Vaxès (2010) [11]	5	$\leq 10n$	✗
Catusse, Chepoi, and Vaxès (2010) [11]	449	$\leq 3n$	✔
Biniaz (2020) [6]	5	$\leq 9n$	✗
Biniaz (2020) [6]	341	$\leq 3n$	✔
This paper	5	$\leq 5.5n$	✗
This paper	3	$\leq 11n$	✗
This paper	2	$O(n\log n)$	✗

Lemma 1 *(Peleg and Schäffer [39]). The subgraph $G' = (V, E')$ is a k-hop spanner of the graph $G = (V, E)$ if and only if the distance between u and v in G' is at most k for every edge $uv \in E$.*

If the subgraph G' has only $O(|V|)$ edges, then G' is called a *sparse* spanner. In this paper we are concerned with constructing sparse k-hop spanners (with small k) for unit disk graphs in the plane. Given a set P of n points $p_1, \ldots, p_n$ in the plane, the *unit disk graph* (UDG) is a geometric graph $G = G(P)$ on the vertex set P whose edges connect points that are at most unit distance apart. A *spanner of a point set P* is a spanner of its UDG.

Recognizing UDGs was shown to be NP-Hard by Breu and Kirkpatrick [9]. Unit disk graphs are commonly used to model network topology in ad hoc wireless networks and sensor networks. They are also used in multi-robot systems for practical purposes such as planning, routing, power assignment, search-and-rescue, information collection, and patrolling; refer to [2,19,24,29,35] for some applications of UDGs. For packet routing and other applications, a bounded-degree plane geometric spanner of the wireless network is often desired but not always feasible [7]. Since a UDG on n points can have a quadratic number of edges, a common desideratum is finding sparse subgraphs that approximate the respective UDG with respect to various criteria. *Plane* spanners, in which no two edges cross, are desirable for applications where edge crossings may cause interference.

Obviously, for every $k \geq 1$, every graph $G = (V, E)$ on n vertices has a k-hop spanner with $|E| = O(n^2)$ edges. If G is the complete graph, a star rooted at any vertex is a 2-hop spanner with $n - 1$ edges. However, the $O(n^2)$ bound on the size of a 2-hop spanner cannot be improved; a classic example [25] is that of a complete bipartite graph with $n/2$ vertices on each side. In general, if G has girth $k + 2$ or higher, then its only k-hop spanner is G itself. According to Erdős' girth conjecture [22], the maximum size of a graph with n vertices and girth $k+2$ is $\Theta(n^{1+1/\lceil k/2\rceil})$ for $k \geq 2$. (The *girth* of a graph is the length of a shortest cycle contained in the graph.) The conjecture has been confirmed for some small values of k, but remains open for $k > 9$. For any graph G with n vertices, a k-hop spanner with $O(n^{1+1/\lceil k/2\rceil})$ edges can be constructed in linear time [4,5]. We show that for unit disk graphs, we can do much better in terms of the number of edges for every $k \geq 2$.

Spanners in general and unit disk graph spanners in particular are used to reduce the size of a network and the amount of routing information. They are also used for maintaining network connectivity, improving throughput, and optimizing network lifetime [6,23,24,28,40].

Spanners for UDGs with hop stretch factors bounded by a constant were introduced by Catusse, Chepoi, and Vaxès in [11]. They constructed (i) 5-hop spanners with at most $10n$ edges for n-vertex UDGs; and (ii) plane 449-hop spanners with less than $3n$ edges. Recently, Biniaz [6] improved both these results and showed that for every n-vertex unit disk graph there exists (i) a 5-hop spanner with at most $9n$ edges, and (ii) a plane 341-hop spanner. The algorithms presented in [6,11] run in time that is polynomial in n. A summary of these results and our new results is included in Table 1.

Our results. The following are shown for unit disk graphs.

(i) Every n-vertex unit disk graph has a 5-hop spanner with at most $5.5n$ edges (Theorem 1 in Section 2). We carefully analyze the construction proposed by Biniaz [6] and improve the upper bound on the number of edges from $9n$ to $5.5n$.

(ii) Using a new construction, we show that every n-vertex unit disk graph has a 3-hop spanner with at most $11n$ edges (Theorem 2 in Section 2). Previously, no 3-hop spanner construction algorithm was known.

(iii) Every n-vertex unit disk graph has a 2-hop spanner with $O(n\log n)$ edges. This is the first construction with a sub-quadratic number of edges (Theorem 3 in Section 3) and our main result.

(iv) For every $n \geq 8$, there exists an n-element point set P such that every plane hop spanner on P has hop stretch factor at least 3. If n is sufficiently large, the lower bound can be raised to 4 (Theorems 4 and 5 in Section 4). A trivial lower bound of 2 can be easily obtained by placing four points at the four corners of a square of side-length 1/2.

(v) For every finite point set P on a circle C, there exists a plane 4-hop spanner (Theorem 6 in Section 5). The lower bound of 4 holds for some point-set on a circle.

(vi) For every pair of integers $k \geq 2$ and $\Delta \geq 2$, there exists a set P of $n = O(\Delta^k)$ points in the plane such that the unit disk graph $G = (P, E)$ on P has no k-hop spanner whose maximum degree is at most Δ (Theorem 7 in Section 6). An extension to dense graphs is given by Theorem 8 in Section 6. In contrast, Kanj and Perković [24] showed that UDGs admit bounded-degree *geometric* spanners (defined below).

Related work. Peleg and Schäffer [39] have shown that for a given graph G (not necessarily a UDG) and a positive integer m, it is NP-complete to decide whether there exists a 2-hop spanner of G with at most m edges. They also showed that for every graph on n vertices, a $(4k+1)$-hop spanner with $O(n^{1+1/k})$ edges can be constructed in polynomial time. In particular, every graph on n vertices has a $O(\log n)$-hop spanner with $O(n)$ edges. Their result was improved by Althöfer et al. [1], who showed that a $(2k-1)$-hop spanner with $O(n^{1+1/k})$ edges can be constructed in polynomial time; the run-time was later improved to linear [4,8]. Kortsarz and Peleg obtained approximation algorithms for the problem of finding, in a given graph, a 2-hop spanner of minimum size [25] or minimum maximum degree [26].

In the geometric setting, where the vertices are embedded in a metric space, spanners have been studied in [3,10,12,14,27,29] and many other papers. In particular, plane geometric spanners were studied in [7,8,17,18]. The reader is also referred to the surveys [8,21,31] and the monograph [34] dedicated to this subject.

Notation and terminology. For two points $p, q \in \mathbb{R}^2$, we denote the Euclidean distance by $d(p,q)$ or sometimes by $|pq|$. The distance between two sets, $A, B \subset \mathbb{R}^2$, is defined by $d(A,B) = \inf\{d(a,b) : a \in A, b \in B\}$. The diameter of a set A, denoted $\mathtt{diam}(A)$, is defined by $\mathtt{diam}(A) = \sup\{d(a,b) : a, b \in A\}$. For a set A, its boundary and interior are denoted by ∂A and $\mathrm{int}(A)$, respectively.

A geometric graph $G = (P, E)$ is a *geometric t-spanner*, for some $t \geq 1$, if for every pair of vertices $u, v \in P$, the Euclidean length of a shortest path $\pi_G(u,v)$ between u and v in G is at most t times $|uv|$, i.e., $\forall u, v \in V, |\pi_G(u,v)| \leq t|uv|$. When there is no need to specify t, we simply use the term *geometric spanner*.

Given a graph $G = (V, E)$ and a vertex $u \in V$, the *neighborhood* $N(u)$ is the set of vertices adjacent to u. For brevity, a *hop spanner for* a point set $P \subset \mathbb{R}^2$ is a hop spanner for the UDG on P. Assume we are given a subgraph $G' = (P, E')$ of the UDG for a point set P. For $p, q \in P$, let $\rho(p,q)$ denote a shortest path in G', i.e., a path containing the fewest edges; and $h(p,q)$ denote the corresponding hop distance (number of edges).

A geometric graph is *plane* if any two distinct edges are either disjoint or only share a common endpoint. Whenever we discuss plane graphs (plane spanners in particular), we assume that the points (vertices) are in *general position*, i.e., no three points are collinear.

A *unit disk* (resp., *circle*) is a disk (resp., circle) of unit radius. The complete bipartite graph with parts of size m and n is denoted by $K_{m,n}$; in particular, $K_{1,n}$ is a star on $n+1$ vertices. We use the shorthand notation $[n]$ for the set $\{1, 2, \ldots, n\}$.

2. Sparse (possibly nonplane) hop spanners

In this section we construct hop spanners with a linear number of edges that provide trade-offs between the two parameters of interest: hop stretch factor and total number of edges.

2.1. Construction of 5-hop spanners

We start with a brief summary of the 5-hop spanner construction by Biniaz [6, Theorem 3]. It is based on a regular hexagonal tiling of the plane with cells of unit diameter. Hence the UDG of a finite point set $P \subset \mathbb{R}^2$ contains every edge between points in the same cell. In every nonempty cell, a star rooted at an arbitrarily chosen point in the cell is created. Then, for every pair of cells, exactly one edge of the UDG is chosen, if such an edge exists. Biniaz showed that the resulting graph G' is a 5-hop spanner with at most $9n$ edges.

We next provide a more detailed description and an improved analysis of the construction. Consider a regular hexagonal tiling $\mathcal{T}$ in the plane with cells of unit diameter; refer to Fig. 1 (left). Let P be a finite set of points in the plane. We may assume that no point in P lies on a cell boundary. Every point in P lies in the interior of some cell of $\mathcal{T}$ (and so the distance between any two points in a cell is less than 1). Let $p \in P$ be a point in a cell σ. Denote by $H_1, \ldots, H_6$ the six cells adjacent to σ in counterclockwise order; these cells form the *first layer* around σ. Let $H_7, \ldots, H_{18}$ be the twelve cells at distance two from σ in counterclockwise order, forming the *second layer* around σ, such that H_7 is adjacent to only H_1 in the first layer.

For every two distinct cells $\sigma, \tau \in \mathcal{T}$, take an arbitrary edge $pq \in E$, $p \in \sigma$, $q \in \tau$, if such an edge exists; we call such an edge a *bridge*. Each cell σ can have bridges to at most 18 other cells, namely those in the two layers around σ. A bridge is *short* if it connects points in adjacent cells and *long* otherwise.

Lemma 2. *Let $p \in P$ be a point that lies in cell σ. The unit disk D centered at p intersects at most five cells from the second layer around σ.*

Proof. Let A be the center of σ (see Fig. 1 (right)). Subdivide σ into six regular triangles incident to A. By symmetry, we may assume that p lies in the regular triangle $\triangle ABC$, where $BC = \sigma \cap H_2$.

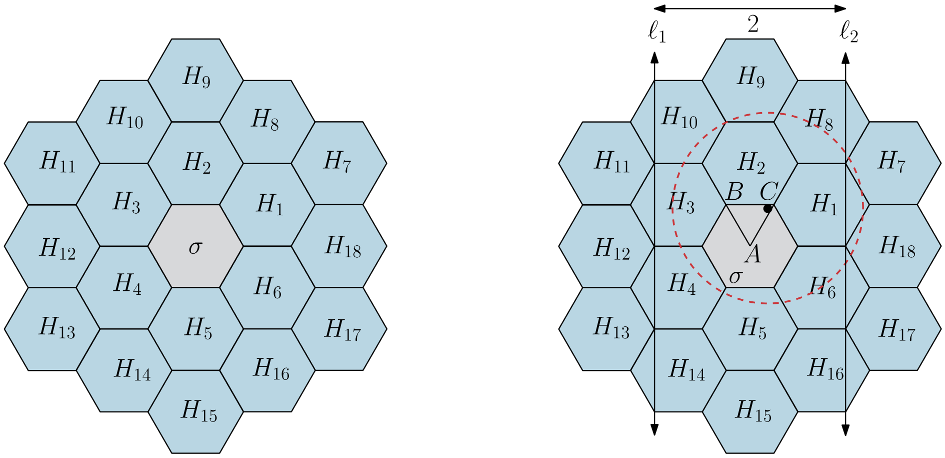

Fig. 1. Left: A regular hexagonal tiling with cells of unit diameter; the figure shows the two layers of cells around σ. Right: The unit disk centered at p intersects 11 cells $H_1, \ldots, H_{10}, H_{18}$.

Note that $d(\Delta ABC, H_i) > 1$ for $i \in \{13, 14, 15, 16, 17\}$, and D is disjoint from the five cells H_{13}, H_{14}, H_{15}, H_{16}, and H_{17}. Now, observe that $d(H_7 \cup H_{18}, H_{11} \cup H_{12}) = 2$. Hence, D intersects at most one of $H_7 \cup H_{18}$ and $H_{11} \cup H_{12}$. Consequently, D intersects at most $12 - 5 - 2 = 5$ cells from the second layer around σ. $\square$

Obviously, any two points in a cell σ are at most unit distance apart. Further, observe that the unit disk D centered at p intersects all six cells $H_1, \ldots, H_6$.

Let P be a set of n points and $G = (P, E)$ be the corresponding UDG. Lemma 2 immediately yields the following.

Corollary 1. *For every point $p \in P \cap \sigma$, every neighbor of p in G lies in σ or one of at most 11 cells around σ.*

The main result regarding 5-hop spanners is given below.

Theorem 1. *The (possibly nonplane) 5-hop spanner G' constructed by Biniaz [6, Theorem 3] has at most $5.5n$ edges.*

Proof. Let $\sigma \in \mathcal{T}$ be a nonempty hexagonal cell, and let $x = |P \cap \sigma|$ be the number of points in the cell. The graph G' contains a star induced by $P \cap \sigma$ with $x - 1$ inner edges, and at most 18 outer edges (i.e., bridges) connecting points in σ with points in other cells. We analyze the number of bridges depending on x.

If $x = 1$, there are no inner edges and at most 11 outer edges by Corollary 1. As such, the degree of the (unique) point in σ is at most 11 in G'.

If $x = 2$, there is one inner edge and at most 16 outer edges. Indeed, by Lemma 2, each point $p \in P \cap \sigma$ has neighbors in G in at most five cells from the second layer around σ (besides points in P in the six cells in the first layer). Two points in $P \cap \sigma$ can jointly have neighbors in G in at most $6 + 5 + 5 = 16$ other cells. As such, the average degree of points in σ is at most $(2 + 16)/2 = 9$ in G'.

If $x \geq 3$, there are $x - 1$ inner edges and at most 18 outer edges. As such, the average degree in G' of all points in σ is at most

$$\frac{2(x-1)+18}{x} = \frac{2x+16}{x} \leq \frac{22}{3}.$$

Summation over all cells implies that the average degree in the resulting 5-hop spanner G' is at most 11, thus G' has at most $5.5n$ edges. $\square$

2.2. Construction of 3-hop spanners

Here we show that every point set in the plane has a 3-hop spanner of linear size. This brings down the hop-stretch factor of Biniaz's construction from 5 to 3 at the expense of increasing the number of edges (from $5.5n$ to $11n$).

Theorem 2. *Every n-vertex unit disk graph has a (possibly nonplane) 3-hop spanner with at most $11n$ edges.*

Proof. Let P be a set of n points in the plane, and let $G = (P, E)$ be the UDG of P. Let G' be the 5-hop spanner described in Section 2.1, based on a hexagonal tiling $\mathcal{T}$ with cells of unit diameter. We construct a new graph G'' that consists of all bridges from G' and, for each nonempty cell $\sigma \in \mathcal{T}$, a spanning star of the points in σ defined as follows.

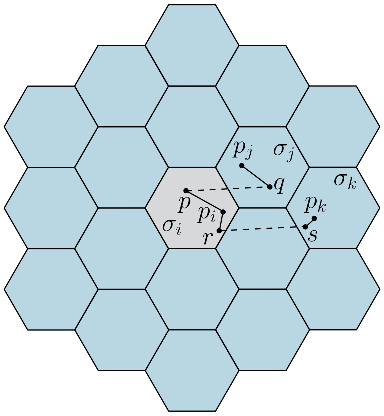

Fig. 2. Three points in P, $p_i \in \sigma_i$, $p_j \in \sigma_j$, and $p_k \in \sigma_k$ where $p_i p_j, p_i p_k \in E$. Edge pq is a short bridge connecting σ_i and σ_j. Edge rs is a long bridge connecting σ_i and σ_k.

For every nonempty cell $\sigma \in \mathcal{T}$ and every $p_i \in P \cap \sigma$, we add edges incident to p_i as follows. For every cell $\tau \in \mathcal{T}$ in the two layers around σ, if $d(p_i, \tau) \le 1$ and G' contains a bridge pq such that $p \in \sigma$, $p_i \ne p$, and $q \in \tau$, then we add the edge $p_i p$ to G''. Since $\mathtt{diam}(\sigma) = 1$, if pq is a short bridge, then p is the center of a spanning star on $P \cap \sigma$ in G'. In addition, if no short bridge is incident to any point in σ, then we add a spanning star of $P \cap \sigma$ to G'' (centered at an arbitrary point in σ if no point in $P \cap \sigma$ is incident to a long bridge, otherwise the star is centered at the endpoint of any long bridge).

It is easy to see that the hop distance between any two points within a cell is at most 2. Indeed, by construction, the points in each nonempty cell are connected by a spanning star. Consider now an edge $p_i p_j \in E$ of the UDG, where $p_i \in P \cap \sigma_i$, $p_j \in P \cap \sigma_j$ for $i \ne j$. Then, by Biniaz's construction, there is a unique bridge $pq \in G' \subset G''$ between some points $p \in P \cap \sigma_i$ and $q \in P \cap \sigma_j$. By our construction, we have either $p = p_i$ or the edge $p_i p$ is in G'', and similarly either $p_j = q$ or the edge $p_j q$ is in G''. As such, G'' contains a 3-hop path p_i, p, q, p_j between p_i and p_j. Refer to Fig. 2 for an illustration.

We derive an upper bound for the average degree of the points in σ as follows. Let $\sigma \in \mathcal{T}$ be a nonempty hexagonal cell, and let $x = |P \cap \sigma|$ be the number of points in the cell. By Corollary 1, the neighbors of each point $p_i \in \sigma$ lie in σ and at most 11 cells around σ. If p_i is not incident to any bridge, we add at most 11 edges between p_i and other points in σ; these edges increase the sum of degrees in σ by $2 \cdot 11 = 22$. Otherwise assume that p_i is incident to b_i bridges, for some $1 \le b_i \le 11$. Then we add edges from p_i to at most $11 - b_i$ other points in σ. The b_i bridges each have only one endpoint in σ. Overall, these edges contribute $2(11 - b_i) + b_i = 22 - b_i < 22$ to the sum of degrees in σ.

If no short bridge has an endpoint in σ, then by Lemma 2 we add at most 5 edges between each point $p_i \in \sigma$ and endpoints of long bridges; these edges increase the sum of degrees in σ by $2 \cdot 5 = 10$. However, we also add a spanning star that contributes $2(x-1)$ to the same sum. Overall, the sum of degrees in σ is bounded from above by

$$\begin{cases} 2 \cdot 11x = 22x, & \text{if some short bridge has an endpoint in } \sigma \\ 2(x-1) + 10x < 12x, & \text{otherwise.} \end{cases}$$

Thus, the average vertex degree is at most 22 in all $\sigma \in \mathcal{T}$. Consequently, the 3-hop spanner G'' has at most $11n$ edges. □

Remark. It is natural to ponder whether the UDG on any n points in the plane has a subgraph with $O(n)$ edges that is a k-hop spanner (for small k) and also a geometric spanner of G. Such subgraphs of UDGs can find practical uses in the real-world. Interestingly, the answer is yes. It is shown by Kanj and Perković [24] that the UDG of a point set P has a subgraph $G_1 = (P, E_1)$ with $O(n)$ edges that is a geometric t-spanner for some constant t. Let $G_2 = (P, E_2)$ be the 3-hop spanner generated by the construction in Theorem 2. Clearly, the graph $G' := (P, E_1 \cup E_2)$ is a subgraph of the UDG, it has $O(n)$ edges, and it is both a 3-hop spanner for the UDG of P and a geometric t-spanner for P with a constant t.

3. Construction of 2-hop spanners

In this section, we construct a 2-hop spanner with $O(n \log n)$ edges for a set P of n points in the plane. We begin with a construction in a bipartite setting (cf. Lemma 6), and then extend it to the general setup.

We briefly review the concept of ε-nets [33], which is crucial for our construction. Let $(P, \mathcal{R})$ be a set system (a.k.a. *range space*), where P is a finite set in an ambient space and $\mathcal{R}$ is a collection of subsets of that space (called *ranges*). For $\varepsilon > 0$, an *ε-net* for $(P, \mathcal{R})$ is a set $N \subset P$ such that for every $R \in \mathcal{R}$, $|P \cap R| \ge \varepsilon \cdot |P|$ implies $N \cap R \ne \emptyset$. When the ambient space is $\mathbb{R}^d$ for some $d \in \mathbb{N}$, and $\mathcal{R}$ is a collection of semi-algebraic sets, there exists an ε-net of size $O(\frac{d}{\varepsilon} \log \frac{d}{\varepsilon})$, and this bound is best possible in many cases [37]. However, for some geometric set systems, ε-nets of size $O(\frac{1}{\varepsilon})$ are possible. For

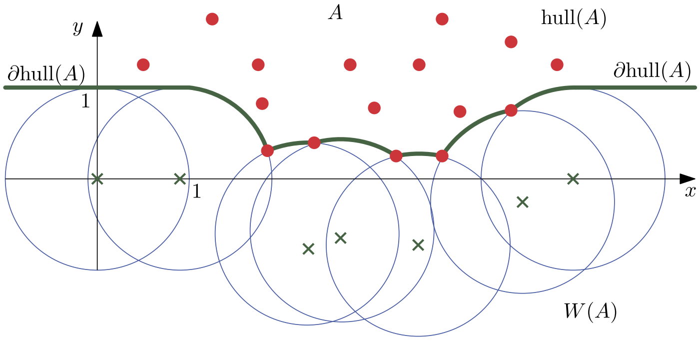

Fig. 3. A set A of 16 points above the x-axis, $W(A)$, and hull(A). The boundary ∂hull(A) is an x-monotone curve, which consists of horizontal segments and arcs of unit circles centered on or below the x-axis (the centers are marked with crosses).

example, if P is a finite set of points in the plane and $\mathcal{R}$ consists of halfplanes, then there exists an ε-net of size $O(\frac{1}{\varepsilon})$ [38]. We adapt this result to unit disks in a somewhat stronger form (cf. Lemma 5).

Alpha-shapes. As a generalization of convex hulls of a set of points, Edelsbrunner, Kirkpatrick, and Seidel [20] introduced α-shapes, using balls of radius $1/\alpha$ instead of halfplanes. We introduce a similar concept, in a bipartite setting, as follows; see Fig. 3 for an illustration. We consider the set system $(A, \mathcal{D})$, where A is a finite set of points in the plane above the x-axis and $\mathcal{D}$ is the set of all unit disks centered on or below the x-axis. Let $W(A)$ be the union of all unit disks $D \in \mathcal{D}$ such that $A \cap \mathrm{int}(D) = \emptyset$; and let $\mathrm{hull}(A) = \mathbb{R}^2 \setminus \mathrm{int}(W(A))$.

The following easy observation shows that disks in $\mathcal{D}$, restricted to the upper halfplane $\{(x, y) \in \mathbb{R}^2 : y > 0\}$, behave similarly to halfplanes in $\mathbb{R}^2$.

Lemma 3. *For any two points $p_1, p_2 \in \mathbb{R}^2$ above the x-axis, there is at most one unit circle centered at a point on or below the x-axis that is incident to both p_1 and p_2. Consequently, for any two unit disks $D_1, D_2 \in \mathcal{D}$, at most one point in $\partial D_1 \cap \partial D_2$ lies above the x-axis.*

Proof. Suppose that two unit circles, c_1 and c_2, are incident to both p_1 and p_2. Then the centers of c_1 and c_2 are on the orthogonal bisector of segment p_1p_2, on opposite sides of the line through p_1p_2. Hence one of the circle centers is above the x-axis, which is a contradiction. Therefore at most one of the circles is centered at a point on or below the x-axis. □

We continue with a few basic properties of the boundary of hull(A), which exhibits the same behavior as convex hulls with respect to lines in the plane.

Lemma 4. *The set system $(A, \mathcal{D})$ defined above has the following properties:*

1. *∂hull(A) lies above the x-axis;*
2. *every vertical line intersects ∂hull(A) in one point, thus ∂hull(A) is an x-monotone curve;*
3. *for every unit disk $D \in \mathcal{D}$, the intersection $D \cap (\partial\mathrm{hull}(A))$ is connected (possibly empty);*
4. *for every unit disk $D \in \mathcal{D}$, if $A \cap D \neq \emptyset$, then $A \cap D$ contains a point in ∂hull(A).*

Proof. Let h be the minimum of the y-coordinates of the points in A. If $h \geq 1$, then $W(A) = \{(x, y) : y \leq 1\}$ is a halfplane bounded by the line $y = 1$, so the lemma trivially holds. In the remainder of the proof, assume that $0 < h < 1$.
(1) Since $0 < h < 1$, the halfplane below the horizontal line $y = h$ lies in the interior of $W(A)$ (as every point below this line is in the interior of a unit disk whose center is below the x-axis and whose interior is disjoint from A). Property 1 follows.
(2) Let $p \in \partial$hull(A). Then p lies on the boundary of a unit disk D_p whose center is below the x-axis (and whose interior is disjoint from A). In particular $D_p \subset W(A)$. The vertical line segment from p to the x-axis lies in D_p, hence in $W(A)$. Consequently, $W(A)$ contains the vertical downward ray emanating from p. Property 2 follows.
(3) Let $D \in \mathcal{D}$. Suppose, to the contrary, that the intersection $D \cap (\partial\mathrm{hull}(A))$ has two or more components. By property 2, the x-coordinates of the components form disjoint intervals, and the components have a natural left-to-right ordering. Let p_1 be the rightmost point in the first component, and let p_2 be the leftmost point in the second component. Clearly $p_1, p_2 \in \partial D$. Let q be an arbitrary point in ∂hull(A) between p_1 and p_2. Then q lies on the boundary of a unit disk D_q whose center is below the x-axis (and whose interior is disjoint from A). Since $D_q \subset W(A)$, neither p_1 nor p_2 is in the interior of D_q. Since the center of D_q is below the x-axis, ∂D_q contains two interior-disjoint circular arcs between q and the x-axis; and both arcs must cross ∂D. We have found two intersection points in $\partial D \cap \partial D_q$ above the x-axis, contradicting Lemma 3. This completes the proof of Property 3.

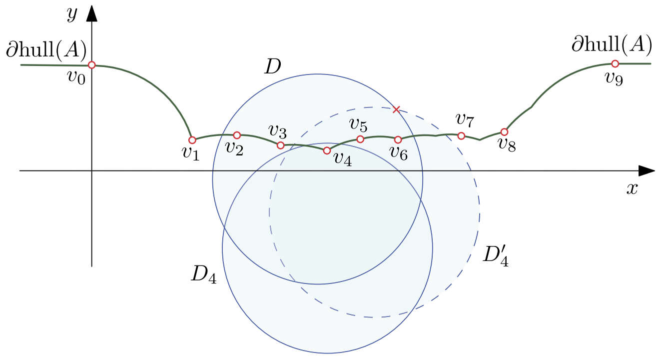

Fig. 4. Illustration for the proof of Lemma 5(iv) with $i = 2$ and $j = 4$. A unit disk D with $D \cap N = \{v_2, v_3, v_4, v_5, v_6\}$, and a unit disk D_4 with $v_4 \in D_4$ and $v_3, v_5 \notin D_4$. A hypothetical unit disk D'_4 (dashed) such that $v_4 \in D'_4$, and $\partial D'_4 \cap \text{hull}(A)$ crosses $\partial D \cap \text{hull}(A)$.

(4) Let $D \in \mathcal{D}$ such that $A \cap D \neq \emptyset$. By continuously translating D vertically down until its interior is disjoint from A, we obtain a unit disk D' such that $A \cap \text{int}(D') = \emptyset$ but $A \cap \partial D' \neq \emptyset$. Since the center of D' is vertically below the center of D, we have $A \cap \partial D' \subset A \cap D$ and $D' \subset W(A)$. This implies that $A \cap \partial D' \subset \partial\text{hull}(A)$, as required. $\square$

Lemma 5. *Consider the set system $(A, \mathcal{D})$ defined above. For every $\varepsilon \in (0, 1)$, we can construct an ε-net $N = \{v_1, \ldots, v_k\} \subset A$, labeled by increasing x-coordinates, such that*

(i) $|N| \leq \lfloor 2/\varepsilon \rfloor$;
(ii) $N \subset \partial\text{hull}(A)$;
(iii) *for every $D \in \mathcal{D}$, the points in $D \cap N$ are consecutive in N; and*
(iv) *for every $D \in \mathcal{D}$, $|N \cap D| \geq 5$ implies $|A \cap D| \geq 2\varepsilon|A|$.*

Proof. Let $M = A \cap \partial\text{hull}(A)$ be the set of points in A lying on the boundary of hull(A). By Lemma 4(4), if a unit disk $D \in \mathcal{D}$ contains any point in A, it contains a point from M. Consequently M is an ε-net for $(A, \mathcal{D})$ for every $\varepsilon > 0$. For a given $\varepsilon > 0$, let $N = N_\varepsilon$ be a minimal subset of M that is an ε-net for $(A, \mathcal{D})$ (obtained, for example, by successively deleting points from M while we maintain an ε-net).

Let $N = \{v_1, \ldots, v_k\}$, where we label the elements in N by increasing x-coordinates. For notational convenience, we introduce a point $v_0 \in \partial\text{hull}(A)$ on a vertical line one unit left of v_1, and $v_{k+1} \in \partial\text{hull}(A)$ on a vertical line one unit right of v_k. For $i = 1, \ldots k$, the minimality of N implies that $N \setminus \{v_i\}$ is not an ε-net, and so there exists a unit disk $D \in \mathcal{D}$ such that $|A \cap D| \geq \varepsilon|A|$ and $D \cap N = \{v_i\}$. Let $D_i \in \mathcal{D}$ be such a disk, with $|A \cap D_i| \geq \varepsilon|A|$ and $D_i \cap N = \{v_i\}$. By Lemma 4(3), D_i contains a connected arc of the x-monotone curve $\partial\text{hull}(A)$, but D_i contains neither v_{i-1} nor v_{i+1}. In particular, the x-coordinate of every point in $A \cap D_i$ lies between that of v_{i-1} and v_{i+1}. Consequently, every point in A lies in at most two disks D_i, $1 \leq i \leq k$. It follows that

$$k \cdot \varepsilon|A| = \sum_{i=1}^{k} \varepsilon|A| \leq \sum_{i=1}^{k} |A \cap D_i| \leq 2|A|,$$

hence $k \leq \lfloor 2/\varepsilon \rfloor$. This proves (i).

By construction, we have $N \subset M \subset \partial\text{hull}(A)$, which confirms (ii), and (iii) follows from Lemma 4(3). It remains to prove (iv); refer to Fig. 4. Assume that $D \in \mathcal{D}$ and $|N \cap D| \geq 5$. By (iii), we may assume that D contains five consecutive points in N, say, $v_i, \ldots, v_{i+4}$. For $j \in \{i+1, i+2, i+3\}$, consider the disk $D_j \in \mathcal{D}$ defined above, where $v_j \in D_j$ but $v_{j-1}, v_{j+1} \notin D_j$. In particular, $D_j \cap (\partial\text{hull}(A))$ lies between v_{j-1} and v_{j+1}. By Lemma 3, the circular arcs $\partial D \cap \text{hull}(A)$ and $\partial D_j \cap \text{hull}(A)$ cross at most once. However, if they cross once, then D_j contains one of the endpoints of $D \cap (\partial\text{hull}(A))$, and by Lemma 4(3) it contains $\{v_i, \ldots, v_j\}$ or $\{v_j, \ldots, v_{i+4}\}$, which is a contradiction. We conclude that $\partial D \cap \text{hull}(A)$ and $\partial D_j \cap \text{hull}(A)$ do not cross. Consequently, $D_j \cap \text{hull}(A) \subset D \cap \text{hull}(A)$, hence $A \cap D_j \subset A \cap D$. As noted above, $|A \cap D_j| \geq \varepsilon|A|$. Furthermore, $A \cap D_{i+1}$ and $A \cap D_{i+3}$ are disjoint as they are on opposite sides of the vertical line passing through v_{i+2}. Thus we obtain $|A \cap D| \geq |A \cap (D_{i+1} \cup D_{i+3})| \geq |A \cap D_{i+1}| + |A \cap D_{i+3}| \geq 2\varepsilon|A|$, as claimed. $\square$

Let A and B be two disjoint point sets above and below the x-axis, respectively. Denote by $U(A, B)$ the unit disk graph on $A \cup B$ and by $G(A, B)$ the bipartite subgraph of $U(A, B)$ consisting of all edges between A and B.

Lemma 6. *Let $P = A \cup B$ be a set of n points in the plane such that $\mathtt{diam}(A) \leq 1$, $\mathtt{diam}(B) \leq 1$, and A (resp., B) is above (resp., below) the x-axis. Then there is a subgraph H of $U(A, B)$ with $O(n \log n)$ edges such that for every edge ab of $G(A, B)$, H contains a path of length at most 2 between a and b.*

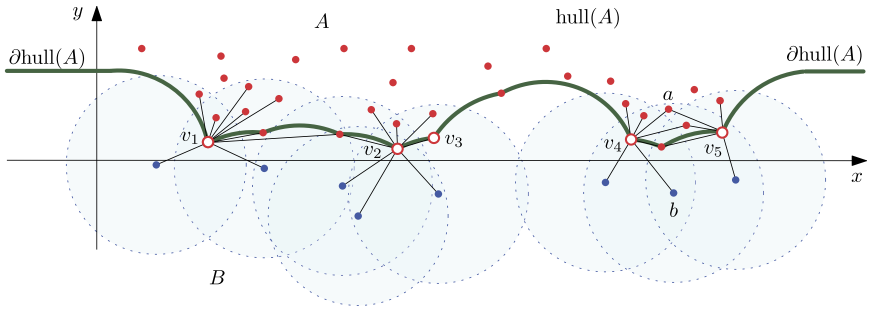

Fig. 5. Set A (resp., B) is above (resp., below) the x-axis. The points in an ε_i-net $N_i = \{v_1 \ldots, v_5\}$ are marked with hollow dots. The graph H_i is a union of stars centered at $v_1, \ldots, v_5$. (To avoid clutter, the depicted point set does not meet conditions $\mathtt{diam}(A) \le 1$ and $\mathtt{diam}(B) \le 1$ of Lemma 6.)

Proof. Our proof is constructive. For every point $b \in B$, let D_b be the unit disk centered at b. Consider the set system $(A, \mathcal{B})$, where $\mathcal{B} = \{D_b : b \in B\}$. We partition the set of disks $\mathcal{B}$ into $O(\log n)$ subsets based on the number of points of A contained in the disks. For every $i = 1, \ldots, \lceil \log n \rceil$, let

$$\mathcal{B}_i = \left\{ D \in \mathcal{B} : \frac{|A|}{2^i} \le |A \cap D| < \frac{|A|}{2^{i-1}} \right\}, \text{ and } \varepsilon_i = \frac{1}{2^i}.$$

Lemma 5 yields an ε_i-net $N_i \subset A$ of size at most $\lfloor 2/\varepsilon_i \rfloor = 2^{i+1}$ for the set system $(A, \mathcal{D})$. Since $\mathcal{B}_i \subset \mathcal{D}$, it follows that $N_i \subset A$ is also an ε_i-net for the set system $(A, \mathcal{B}_i)$.

We construct a graph H as a union of stars; see Fig. 5 for an illustration. For every $i = 1, \ldots, \lceil \log n \rceil$ and every $v \in N_i$, we create a star centered at v as follows. Let $B_i(v)$ be the set of points $b \in B$ such that $D_b \in \mathcal{B}_i$, $v \in D_b$, and v is the leftmost point in $N_i \cap D_b$. Let $A_i(v)$ be the set of points $a \in A$ such that $a \in D_b$ for some $b \in B_i(v)$. Note that $v \in A_i(v)$. Let $S_i(v)$ be the spanning star on the point set $A_i(v) \cup B_i(v)$ centered at v. By assumption, we have $\mathtt{diam}(A) \le 1$, and by the definition above, every point in $B_i(v)$ is at distance at most 1 from v. This implies that $S_i(v)$ is a subgraph of $U(A, B)$. Let H_i be the union of stars $S_i(v)$ for all $v \in N_i$; and let H be the union of the graphs H_i for $i = 1, \ldots, \lceil \log n \rceil$. Note that H is a union of stars in $U(A, B)$, and so it is a subgraph of $U(A, B)$.

To prove that H is a 2-hop spanner for $G(A, B)$, consider an edge ab of $G(A, B)$ with $a \in A$, $b \in B$. Since ab is an edge of $G(A, B)$, we have $|ab| \le 1$ hence $a \in D_b$. There exists an index $i \in \{1, \ldots, \lceil \log n \rceil\}$ for which $D_b \in \mathcal{B}_i$. As $|A \cap D_b| \ge |A|/2^i = \varepsilon_i |A|$, and N_i is an ε_i-net for $(A, \mathcal{B}_i)$, we have $D_b \cap N_i \ne \emptyset$. Let v be the leftmost point in $D_b \cap N_i$. Then by construction $a \in A_i(v)$ and $b \in B_i(v)$. If $a = v$, then the star $S_i(v)$ contains the edge ab, otherwise $S_i(v)$ contains the path a, v, b of length 2.

It remains to derive an upper bound on the number of edges in H. We claim that H_i has $O(n)$ edges for all $i = 1, \ldots, \lceil \log n \rceil$, which implies that H has $O(n \log n)$ edges overall.

Let $b \in B$. There is a unique index i such that $|A|/2^i \le |A \cap D_b| < |A|/2^{i-1}$; and there is a unique leftmost point v in $N_i \cap D_b$. Therefore, b is a leaf of a star $S_i(v)$ for at most one vertex $v \in A$, and so the degree of b is at most 1 in H_i, hence in H. Overall, H contains at most $|B|$ edges incident to B. We still need to bound the number of edges induced by A in H.

Let $i \in \{1, \ldots, \lceil \log n \rceil\}$. Assume that $N_i = \{v_1, \ldots, v_k\}$ is sorted by increasing x-coordinates. We also introduce points v_0 and v_{k+1} on $\partial \mathrm{hull}(A)$ as specified previously.

Let $a \in A$; refer to Fig. 5. Assume that a is in a star $S_i(v_j)$ for some $v_j \in N_i$. Assume further that the x-coordinate of a is between that of $v_{\ell-1}$ and v_ℓ for some $\ell \in \{1, \ldots, k+1\}$. Since a is in $S_i(v_j)$, then $a \in A_i(v_j)$ and there exists a point $b \in B$ such that $a \in D_b$, $D_b \in \mathcal{B}_i$, and v_j is the leftmost point in $D_b \cap N_i$. Since $D_b \in \mathcal{B}_i$, we have $|A \cap D_b| < 2\varepsilon_i |A|$.

By Lemma 5(iv), D_b contains at most 4 points in the net N_i. In particular, the unit circle ∂D_b intersects $\partial \mathrm{hull}(A)$ in two points: once between v_{j-1} and v_j, and once between v_j and v_{j+4}. Consequently, $0 \le \ell - j \le 4$, thus a is in at most 5 possible stars $S_i(v_j)$, $v_j \in N_i$. It follows that H_i has at most $5|A| + |B| \le 5n$ edges, as required. □

We now consider the general case.

Theorem 3. *Every n-vertex unit disk graph has a (possibly nonplane) 2-hop spanner with $O(n \log n)$ edges.*

Proof. Let P be a set of n points in the plane and let G denote the corresponding UDG. Consider a tiling of the plane with regular hexagons of unit diameter; and assume that no point in P lies on the boundary of any hexagon. Let $\mathcal{T}$ be the set of nonempty hexagons. Then P is partitioned into $O(n)$ sets $\{P \cap \sigma : \sigma \in \mathcal{T}\}$. As noted in Section 2.1, for every $\sigma \in \mathcal{T}$, there are 18 other cells within unit distance; see Fig. 1 (left).

For each cell $\sigma \in \mathcal{T}$, choose an arbitrary vertex $v_\sigma \in P \cap \sigma$, and create a star S_σ centered at v_σ on the vertex set $P \cap \sigma$. Since the stars are disjoint, they form a forest with $n - |\mathcal{T}|$ trees, thus the overall number of edges in all stars S_σ, $\sigma \in \mathcal{T}$, is $n - |\mathcal{T}| \le n$.

For every pair of cells $\sigma_i, \sigma_j \in \mathcal{T}$, where $d(\sigma_i, \sigma_j) \leq 1$, consider the bipartite subgraph of G: $G_{i,j} = G(P \cap \sigma_i, P \cap \sigma_j)$. By Lemma 6, there is a subgraph $H_{i,j}$ of $G_{i,j}$ of size

$$O\Big((|P \cap \sigma_i| + |P \cap \sigma_j|) \log(|P \cap \sigma_i| + |P \cap \sigma_j|)\Big) \subseteq O\Big((|P \cap \sigma_i| + |P \cap \sigma_j|) \log n\Big).$$

Since every vertex appears in at most 18 such bipartite graphs, the total number of edges in these graphs is at most $O\left(\sum_{\sigma \in \mathcal{T}} |P \cap \sigma| \log n\right) = O(n \log n)$.

We show that the union of the stars S_σ, $\sigma \in \mathcal{T}$, and the graphs $H_{i,j}$ is a 2-hop spanner. Let ab be an edge of the unit disk graph. If both a and b are in the same cell, say $\sigma \in \mathcal{T}$, then ab is an edge in the star or the star S_σ contains the path a, v_σ, b. Otherwise, a and b lie in two distinct cells, say $\sigma_i, \sigma_j \in \mathcal{T}$, such that $d(\sigma_i, \sigma_j) \leq |ab| \leq 1$. By Lemma 6 (where the role of the x-axis is taken by any separating line), $H_{i,j}$ contains a path of length at most 2 between a and b, as required. □

4. Lower bounds for plane hop spanners

A trivial lower bound of 2 for the hop stretch factor of plane subgraphs of UDGs can be easily obtained by taking the four corners of a square of side-length $\frac{1}{2}$. In this case, the UDG is the complete graph but a plane subgraph cannot contain both diagonals of the square. Our main result in this section is a lower bound of 4 for sufficiently large n (cf. Theorem 5). We begin with a lower bound of 3 that holds already for $n = 8$.

Theorem 4. *For every $n \geq 8$, there exists an n-element point set S on a circle such that every plane hop spanner on S has hop stretch factor at least* 3.

Proof. Let $P = \{p_1, \ldots, p_8\}$ be a set of 8 successive points on a circle of radius $r \geq 1$, so that $p_1 p_8$ is a horizontal chord, $|p_2 p_3| = |p_3 p_4| = |p_4 p_5| = |p_5 p_6| = |p_6 p_7|$, $|p_1 p_2| = |p_7 p_8| = 1.1 |p_2 p_3|$, $|p_1 p_4| < 1$, and $|p_2 p_6| = |p_3 p_7| = 1$. The UDG of P is shown in Fig. 6 (left). Note that $|p_1 p_5| = |p_4 p_8| > 1$; and that the orthogonal bisector of $p_1 p_8$ is a vertical axis of symmetry. Since P is in convex position we may assume that $p_i p_{i+1} \in E'$ for $i = 1, \ldots, 7$. Suppose that $G' = (P, E')$ is a plane hop spanner with hop stretch factor 2. Define the *span* of an edge $p_i p_j$ $(i < j)$, as $j - i$. We distinguish between two cases depending on whether E' contains at least one edge of span 2 whose endpoints are in $\{p_2, \ldots, p_7\}$.

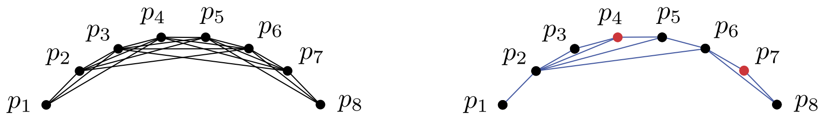

Fig. 6. Left: the 8-element point set P and its UDG. Right: a 3-hop plane spanner of P; for the hop distance between the two red points, p_4 and p_7, is 3. (For interpretation of the colors in the figures, the reader is referred to the web version of this article.)

Case 1: E' contains at least one edge of span 2 *whose endpoints are in* $\{p_2, \ldots, p_7\}$. Assume first that $p_3 p_5 \in E'$ or $p_4 p_6 \in E'$. Assume w.l.o.g. that $p_3 p_5 \in E'$. Since $h(p_1, p_4) \leq 2$, we have $p_1 p_3 \in E'$. Since $h(p_2, p_6) \leq 2$, we have $p_3 p_6 \in E'$. Since $h(p_4, p_7) \leq 2$, we have $p_3 p_7 \in E'$. Then $\rho(p_5, p_8)$ has at least 3 hops, a contradiction.

We can subsequently assume that $p_3 p_5, p_4 p_6 \notin E'$. Assume next that $p_2 p_4 \in E'$ or $p_5 p_7 \in E'$. Assume w.l.o.g. that $p_2 p_4 \in E'$. Since $h(p_3, p_6) \leq 2$, we have $p_2 p_6 \in E'$. Then $\rho(p_4, p_7)$ has at least 3 hops, a contradiction.

Case 2: E' contains no edge of span 2 *whose endpoints are in* $\{p_2, \ldots, p_7\}$. Since $h(p_3, p_6) \leq 2$, we have $p_3 p_6 \in E'$, $p_2 p_6 \in E'$, or $p_3 p_7 \in E'$. If $p_3 p_6 \in E'$, then $\rho(p_2, p_5)$ has at least 3 hops, a contradiction. Assume w.l.o.g. that $p_2 p_6 \in E'$. Then $\rho(p_1, p_4)$ has at least 3 hops, a contradiction.

Thus, we have shown that every plane hop spanner on P has hop stretch factor of at least 3. For every $n \geq 8$, we can add $n - 8$ points on the circle beyond p_8 such that every plane hop spanner on the resulting set S of n points has hop stretch factor of at least 3. □

We next derive a better bound assuming that n is sufficiently large.

Theorem 5. *For every sufficiently large n, there exists an n-element point set P on a circle such that every plane hop spanner on P has hop stretch factor at least* 4.

Proof. Consider a set P of n points that form the vertices of regular n-gon R inscribed in a circle C, where the circle is just a bit larger than the circumscribed circle of an equilateral triangle of unit edge length. Formally, for a given $\varepsilon \in (0, 1/50)$, set $n = \lceil 2\varepsilon^{-1} \rceil$ and choose the radius of C such that every sequence of $\left(\frac{1}{3} - \varepsilon\right) n$ consecutive points from P makes a subset of diameter at most 1; and any larger sequence makes a subset of diameter larger than 1. Note that $\varepsilon n \geq 2$. (We may set $\varepsilon = 0.02$, which yields $n = 100$.)

The short circular arc between two consecutive vertices of R is referred to as an *elementary arc*. (Its center angle is $2\pi/n$.) If A is a set of elementary arcs, $X(A)$ denotes its set of endpoints; obviously $|X(A)| \geq |A|$, with equality when A covers the entire circle C.

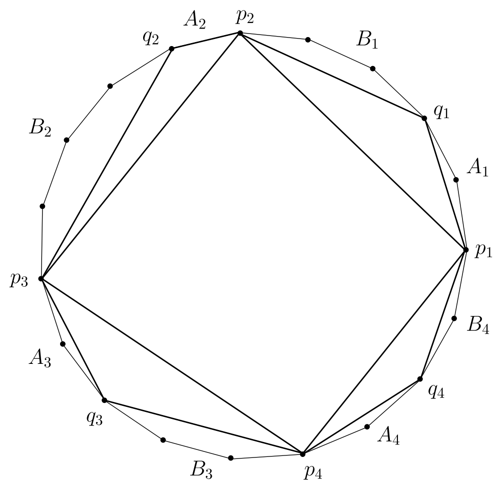

Fig. 7. The partition induced by the blocks for $n = 19$ and $k = 4$. The edges $p_i p_{i+1}$ are maximal edges of G' and $\Delta p_i p_{i+1} q_i$ is the unique triangle adjacent to $p_i p_{i+1}$ in the triangulation of the ith block. Since $n = 19$ is small, the figure only illustrates the notation used in the proof of Theorem 5; $|A_1| = 2$, $|B_1| = 3$, $|A_2| = 1$, $|B_2| = 4$, etc.

Suppose, for the sake of contradiction, that the unit disk graph G has a plane subgraph G' with hop number at most 3. First, augment G' to a maximal noncrossing subgraph of G, by successively adding edges from $G \setminus G'$ that do not introduce crossings. Adding edges does not increase the hop number of G', which remains at most 3.

We define *maximal* edges in G' as follows. Associate every edge of G' with the shorter circular arc between its endpoints. Observe that containment between arcs is a partial order (poset). An edge of G' is *maximal* if the associated arc is maximal in this poset. Due to planarity, if two arcs overlap, then one of the arcs contains the other. Hence the maximal edges correspond to nonoverlapping arcs. As such, the maximal edges form a convex cycle, i.e., a convex polygon $Q = p_1, p_2, \ldots, p_k$. Refer to Fig. 7. By the choice of C, we have $k \geq 4$. Each edge of the polygon Q determines a set of points, called *block*, that lie on the associated circular arc (both endpoints of the edge are included). Since the length of each edge of Q is at most 1, the restriction of G' to the vertices in a block is a triangulation.

Let $A_i \cup B_i$ be the sets of elementary arcs in counterclockwise order covering the ith block such that A_i and B_i are separated by a common vertex q_i, where the triangle $\Delta p_i p_{i+1} q_i$ is the (unique) triangle adjacent to the chord $p_i p_{i+1}$ in the triangulation of the ith block (where addition is modulo k, so that $k + 1 = 1$). In particular, q_i is the last endpoint of an elementary arc in A_i and the first endpoint of an elementary arc in B_i, in counterclockwise order. As such, we have

$$\sum_{i=1}^{k} (|A_i| + |B_i|) = n. \tag{1}$$

By definition, we have

$$|A_i| + |B_i| \leq \left(\frac{1}{3} - \varepsilon\right) n, \quad \text{for } i = 1, \ldots, k. \tag{2}$$

By the maximality of the blocks in G', we have

$$(|A_i| + |B_i|) + (|A_{i+1}| + |B_{i+1}|) \geq \left(\frac{1}{3} - \varepsilon\right) n, \quad \text{for } i = 1, \ldots, k. \tag{3}$$

By the maximality of G', we also have $k \leq 6$, since otherwise an averaging argument would yield two adjacent blocks, say, i and $i + 1$, that can be merged by adding one chord of length at most 1 and so that the merged sequence of points has size at most

$$|A_i| + |B_i| + |A_{i+1}| + |B_{i+1}| \leq \frac{2n}{7} < \left(\frac{1}{3} - \varepsilon\right) n,$$

which would be a contradiction. We next prove the following inequality:

$$|B_i| + |A_{i+1}| > \left(\frac{1}{3} - 3\varepsilon\right) n, \quad \text{for } i = 1, \ldots, k. \tag{4}$$

Suppose for contradiction that $|B_i| + |A_{i+1}| \leq \left(\frac{1}{3} - 3\varepsilon\right) n$ holds for some i. Consider the εn elementary arcs preceding the arcs in B_i and the εn elementary arcs following the arcs in A_{i+1}, in counterclockwise order. Denote these sets of arcs by U_i and V_i, respectively ($|U_i| = |V_i| = \varepsilon n$). Recall that $\varepsilon n \geq 2$ and thus $|X(U_i)|, |X(V_i)| \geq |U_i| = \varepsilon n \geq 2$.

We claim that there exist $u \in X(U_i)$ and $v \in X(V_i)$ such that $|uv| \le 1$ and $h(u,v) \ge 4$. Indeed, $\mathtt{diam}(X(U_i \cup B_i \cup A_{i+1} \cup V_i)) \le 1$ since $X(U_i \cup B_i \cup A_{i+1} \cup V_i)$ contains at most

$$\left(\frac{1}{3} - 3\varepsilon\right)n + 2\varepsilon n \le \left(\frac{1}{3} - \varepsilon\right)n$$

consecutive points. This proves the first part of the claim for any $u \in X(U_i)$ and $v \in X(V_i)$. For the second part, we can take u as one of the two vertices preceding q_i that is not p_i, and similarly we can take v as one of the two vertices following q_{i+1} that is not p_{i+2}. With this choice, we have $h(u, p_{i+1}) \ge 2$ and $h(p_{i+1}, v) \ge 2$, and $\rho(u,v)$ passes through p_{i+1}. Consequently,

$$h(u,v) \ge h(u, p_{i+1}) + h(p_{i+1}, v) \ge 2 + 2 = 4.$$

We have reached a contradiction, which proves (4). The summation of (4) over all $i = 1, \ldots, k$, in combination with (1) and the inequality $k \ge 4$ yields

$$n = \sum_{i=1}^{k}(|A_i| + |B_i|) = \sum_{i=1}^{k}(|B_i| + |A_{i+1}|) \ge k\left(\frac{1}{3} - 3\varepsilon\right)n \ge 0.27\,kn \ge 1.08\,n.$$

This last contradiction completes the proof of the theorem. □

5. An upper bound for points on a circle

For many problems dealing with finite point configurations in the plane, points in convex position or on a circle may allow for tighter bounds; see, e.g., [15,16,32,41]. We show that the lower bound of 4 for points on a circle is tight in this case.

Theorem 6. *For every finite point set S on a circle C, there exists a plane 4-hop spanner.*

Proof. Let C be a circle with center $o \in \mathbb{R}^2$ and radius $r > 0$. Let S be a set of n points on C, and let $G = G(S)$ be the corresponding UDG. We may assume w.l.o.g. that G is connected. If $r \le 1/2$, then $G = K_n$, we set $G' = K_{1,n-1}$, i.e., a star centered at an arbitrary point. This yields $h(s,s') \le 2$ for every $s, s' \in S$. We therefore subsequently assume that $r > 1/2$; this implies that no edge of G passes through o.

Let $\gamma \subset C$ be a shortest arc of C covering the points in S; and let $S = \{s_1, s_2, \ldots, s_n\}$ be a counterclockwise labeling of these points on γ. We claim that $|s_i s_{i+1}| \le 1$, for $i = 1, \ldots, n-1$. Indeed, let $1 \le i \le n-1$ be the smallest index such that $|s_i s_{i+1}| > 1$. Then $|s_1 s_n| \ge |s_i s_{i+1}| > 1$ and therefore $\{s_1, \ldots, s_i\}$ and $\{s_{i+1}, \ldots, s_n\}$ are disconnected in G, a contradiction. We construct a plane subgraph $G' = (S, E')$ of G in two phases, and then show the G' is a 4-hop spanner for S.

In the first phase, we incrementally construct a polygonal chain $Q = p_1, p_2, \ldots, p_k$, on a subset of k elements of S with the vertices chosen counterclockwise by a greedy algorithm starting with $p_1 = s_1$ (k is determined by the algorithm). The polygon Q will be part of the plane graph G'; the following properties will be satisfied.

- $p_i \in S$, for $i = 1, \ldots, k$,
- $|p_i p_{i+1}| \le 1$, $i = 1, \ldots, k-1$.

In the current step, assume that p_i has already been selected; here p_i precedes s_n. The algorithm checks subsequent points counterclockwise on C, say $s_j, s_{j+1}, \ldots$. As noted above, since G is connected, we have $|p_i s_j| \le 1$. The algorithm selects $p_{i+1} = s_{j+h}$, where $h \ge 0$ is the largest index such that $|p_i s_{j+h'}| \le 1$ for $h' = 0, 1, \ldots, h$, i.e., for *all* successive points until s_{j+h}; or $p_{i+1} = s_n$, if the last point is reached. If p_{i+1} precedes s_n, the algorithm updates $i \leftarrow i + 1$ and continues with the next iteration; if $p_{i+1} = s_n$, we set $k := i$. When this process terminates, k is set.

If $|p_k p_1| \le 1$, the edge $p_k p_1$ is added to close the chain, i.e., Q is a convex polygon whose k edges belong to E, in particular, $p_k p_1 \in E$; note that there may be points of S on the arc $\widehat{p_k p_1}$. It is possible that $|p_k p_1| > 1$, in which case $Q = p_1, \ldots, p_k$ is an open chain with $k - 1$ edges. In this case there are no other points of S on the arc $\widehat{p_k p_1}$. Each edge of the chain Q determines a set of points called *block* (endpoints of the edge are included). Depending on whether the chain Q is open or closed, there are either $k - 1$ blocks or k blocks.

In the second phase, for every edge $p_i p_{i+1} \in E$ (with wrap around), we connect p_i with all other points (if any) in that block (i.e., create a *star* whose apex is p_i); refer to Fig. 8 for an example. This completes the construction of the plane graph $G' = (S, E')$.

It remains to analyze its hop factor of G'. Let $uv \in E$ be any edge of G; we may assume w.l.o.g. that uv is horizontal and lies below the center o. Refer to Fig. 9(right). We show that uv can have at most one edge of Q strictly below it. Suppose that $e = p_i p_{i+1} \in E'$ is an edge of the polygon Q that lies strictly below uv. We claim that $i = k$, i.e., $e = p_k p_1$ and so this edge is unique if this occurs. Note that if $e = p_k p_1 \in E$, then the chain Q is closed.

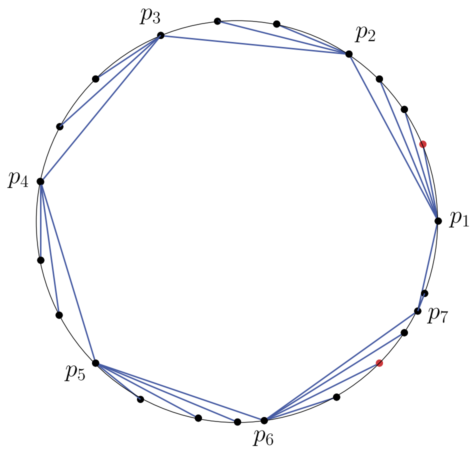

Fig. 8. An example of the 4-hop spanner constructed by the greedy algorithm; $P = p_1, \ldots, p_7$ is a closed chain. The two red points are at hop distance 4.

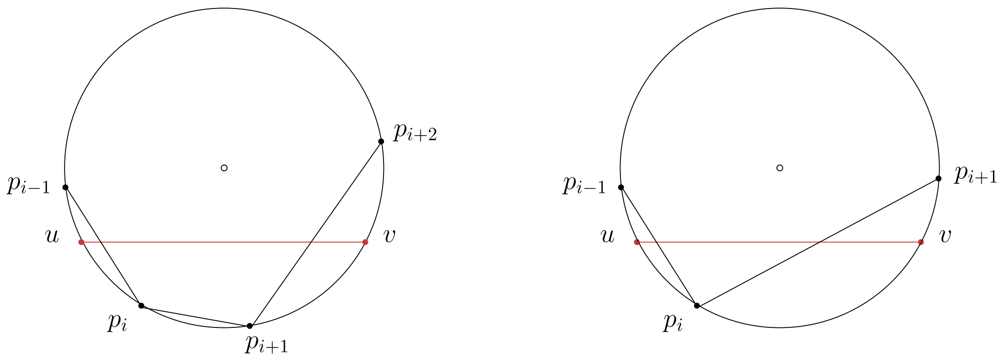

Fig. 9. Left: the path connecting u and v is $up_{i-1}p_ip_{i+1}v$. Right: the path connecting u and v is $up_{i-1}p_iv$.

Assume that $i \neq k$. Since uv is below the horizontal diameter of C, we have $|p_ip_{i+1}| < |p_iv| < |uv| \leq 1$, and thus the greedy algorithm would have chosen v or another vertex beyond v counterclockwise, instead of p_{i+1} as the other endpoint of the edge incident to p_i, a contradiction. This proves the claim.

By the claim, the endpoints of every edge $uv \in E$ lie either in the same block, in two adjacent blocks, or in two blocks that are separated by exactly one other block. Consequently, uv can be connected by a h-hop path, for some $h \leq 4$. Fig. 9 (left) shows the case when the endpoints u, v belong to two blocks that are separated by exactly one other block: the connecting path is $up_{i-1}p_ip_{i+1}v$. Fig. 9 (right) shows the case when the endpoints u, v belong to two adjacent blocks: the connecting path is $up_{i-1}p_iv$. When both u and v belong to the same block of the chain, they are connected either directly or by a path of length 2 via the center of the corresponding star. $\square$

6. The maximum degree of hop spanners cannot be bounded

It is not difficult to see that dense (abstract) graphs do not admit bounded degree hop spanners (irrespective of planarity). We start with an observation regarding the complete UDG K_n and then extend it and show that the maximum degree of hop spanners of sparse UDGs is also unbounded.

We use the fact that graphs of small diameter *and* maximum degree must be small. Indeed, a connected graph with diameter at most D and maximum degree is at most $\Delta \geq 3$ has fewer than $\frac{\Delta}{\Delta-2} \cdot (\Delta - 1)^D$ vertices [13, Proposition 1.3.3]; and a connected graph with diameter at most D and maximum degree is at most 2 has fewer than $2D + 2$ vertices. As such, a connected graph with diameter at most D and maximum degree at most $\Delta \geq 2$ has fewer than $2\Delta^D$ vertices.

Theorem 7. *For every pair of integers $k \geq 2$ and $\Delta \geq 2$, there exists a set S of $n \leq 2\Delta^k$ points such that the unit disk graph $G = (S, E)$ on S has no k-hop spanner whose maximum degree is at most Δ.*

Proof. Let S be a set of n points in a unit disk. Then the UDG G of S is the complete graph K_n. Suppose, to the contrary, that $G' = (S, E')$ is a k-hop spanner for G with maximum degree at most Δ. Then $h(p, q) \leq k$ for all $p, q \in S$, hence the diameter of G' is at most k. By the above observation we have $n < 2\Delta^k$, thus we obtain a contradiction if we set $n = 2\Delta^k$. $\square$

Theorem 8. *Let $t: \mathbb{N} \to \mathbb{N}$, $t(n) \le n$, be an integer function that tends to ∞ with n. For every pair of integers $k \ge 2$ and $\Delta \ge 2$, there exists $n_0 \in \mathbb{N}$ such that for every $n \ge n_0$, there is a set S of n points in the plane such that*

(i) *the unit disk graph $G = (S, E)$ on S has $\Theta(n \cdot t(n))$ edges, and*
(ii) *G has no k-hop spanner whose maximum degree is at most Δ.*

Proof. For a given t, partition n points into $\lfloor \frac{n}{t} \rfloor$ groups of size t and a remaining group (if any) of size $n - \lfloor \frac{n}{t} \rfloor t$. Place the groups in disjoint disks of unit diameter in the plane, so that the UDG of each group is a complete graph; and arrange the disks along a line such that the UDG G has exactly one edge between any two consecutive groups. Each group of size t induces $\binom{t}{2} = \Theta(t^2)$ edges, hence G has $\Theta(\frac{n}{t} \cdot t^2 + t) = \Theta(nt)$ edges.

Suppose that G has a k-hop spanner G' with maximum degree at most Δ. Then $h(p, q) \le k$ for all $p, q \in S$ within the same group, hence each group induces a subgraph of G' of diameter at most k. By the above observation we have $t < 2\Delta^k$, thus we obtain a contradiction if we choose n_0 such that $t(n) \ge 2\Delta^k$ for all $n \ge n_0$. $\square$

7. Conclusions

We have shown that the UDG of every set of n points in the plane admits a 5-hop spanner with at most $5.5n$ edges, a 3-hop spanner with at most $11n$ edges, and a 2-hop spanner with $O(n \log n)$ edges. The third bound leaves an interesting question: Are there n-element point sets for which every 2-hop spanner has $\omega(n)$ edges? Recent results show that unit disks may exhibit surprising behavior [30,36].

Finding nontrivial lower bounds for the size of k-hop spanners remains an open problem. We mention a few straightforward lower bounds. Observe that if the girth of an UDG G is $k \ge 4$, then the only $(k-2)$-hop spanner of G is G itself. In particular, for n points in a section of the square lattice $\mathbb{Z}^2$, the UDG has $(2 - o(1))n$ edges, its girth is 4, and so the only 2-hop spanner of G is G itself. For n points in a section of a hexagonal lattice, the UDG has $(\frac{3}{2} - o(1))n$ edges, its girth is 6, and so the only 3- or 4-hop spanner of G is G itself. Finally, for n points in $\mathbb{Z}^2 \setminus 2\mathbb{Z}^2$, the UDG has $(\frac{4}{3} - o(1))n$ edges, its girth is 8, and so the only 5- or 6-hop spanner of G is G itself.

Biniaz [6] showed that the UDG of every point set admits a plane hop spanner with hop stretch factor at most 341. For points on a circle, we have improved the upper bound to 4, and showed that this bound is the best possible. This is the first nontrivial lower bound on the hop stretch factor of any plane hop spanner (Theorem 6). Are there point sets for which every plane hop-spanner has hop stretch factor at least 5?

In this paper, we considered the UDG of a point set in terms of Euclidean distance (i.e., L_2-norm) in the plane. We can define UDG over any other norm over $\mathbb{R}^2$, where the unit disks are translates of a centrally symmetric convex body. Estimating the size of hop spanners over arbitrary normed spaces in $\mathbb{R}^2$ is another problem for consideration.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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