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of finite-order diffeomorphisms

JIANFENG LIN
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Let K be a knot in an integral homology 3–sphere Y and Σ the corresponding n–fold cyclic branched cover. Assuming that Σ is a rational homology sphere (which is always the case when n is a prime power), we give a formula for the Lefschetz number of the action that the covering translation induces on the reduced monopole homology of Σ. The proof relies on a careful analysis of the Seiberg–Witten equations on 3–orbifolds and of various η–invariants. We give several applications of our formula: (1) we calculate the Seiberg–Witten and Furuta–Ohta invariants for the mapping tori of all semifree actions of $\mathbb{Z}/n$ on integral homology 3–spheres; (2) we give a novel obstruction (in terms of the Jones polynomial) for the branched cover of a knot in S^3 being an L–space; and (3) we give a new set of knot concordance invariants in terms of the monopole Lefschetz numbers of covering translations on the branched covers.

1 Introduction

Monopole Floer homology, defined by Kronheimer and Mrowka [23] and Frøyshov [18], is a powerful invariant of 3–manifolds. Orientation-preserving self-diffeomorphisms of the 3–manifold act on the monopole Floer homology. We initiated a study of this action in our paper [33], where we calculated the monopole Lefschetz number of an involution on a rational homology 3–sphere that makes it into a double branched cover over a link in the 3–sphere. Here we continue this study and extend our calculations to all finite-order diffeomorphisms making a rational homology 3–sphere into a branched cover of a knot in an integral homology 3–sphere. Note that the case of free diffeomorphisms of finite order was dealt with by the second and third authors in [46].

Given a knot K in an integral homology 3–sphere Y and an integer $n \geq 2$, consider the n–fold cyclic branched cover $\Sigma \to Y$ with branch set K and denote by $\tau\colon \Sigma \to \Sigma$ the

covering translation. Assume that Σ is a rational homology 3–sphere (which is always true when n is a prime power). Observe that Σ admits a unique spin^c structure $\mathfrak{s}$ such that $\tau_*(\mathfrak{s}) = \mathfrak{s}$, and that this spin^c structure is actually a spin structure; see Remark 6.3. Our main theorem can now be stated as follows:

Theorem A   *Let* $\tau_* \colon \mathrm{HM}^{\mathrm{red}}(\Sigma, \mathfrak{s}) \to \mathrm{HM}^{\mathrm{red}}(\Sigma, \mathfrak{s})$ *be the map in the reduced monopole Floer homology induced by the covering translation* τ. *Then*

$$(1) \qquad \mathrm{Lef}(\tau_*) = n \cdot \lambda(Y) + \frac{1}{8} \sum_{m=1}^{n-1} \mathrm{sign}_{m/n}(K) - h(\Sigma, \mathfrak{s}),$$

where $\lambda(Y)$ *is the Casson invariant of* Y, $\mathrm{sign}_{m/n}(K)$ *are the Tristram–Levine signatures of* K, *and* $h(\Sigma, \mathfrak{s})$ *is the Frøyshov invariant.*

The formula (1) was first conjectured in [33] but its proof here will use methods completely different from those in [33]: we will first calculate the Seiberg–Witten invariant $\lambda_{\mathrm{SW}}(X)$ for the mapping torus X of τ using the original definition of $\lambda_{\mathrm{SW}}(X)$ from Mrowka, Ruberman and Saveliev [40], and then apply the splitting theorem of our paper [34] to derive (1).

Theorem B   *Let* Σ *be a rational homology sphere as above and* X *the mapping torus of the covering translation* $\tau \colon \Sigma \to \Sigma$ *with the standard orientation and homology orientation. Then*

$$(2) \qquad -\lambda_{\mathrm{SW}}(X) = n \cdot \lambda(Y) + \frac{1}{8} \sum_{m=1}^{n-1} \mathrm{sign}_{m/n}(K).$$

We should mention that formula (2) was independently proved by Langte Ma [36] using a different set of techniques. His result actually holds without the assumption that Σ is a rational homology sphere.

1.1   Motivation

As in our previous work, the research here is motivated by a conjectural relationship between two gauge-theoretic invariants of 4–manifolds with the homology of $S^1 \times S^3$, one from the Donaldson theory and the other from the Seiberg–Witten theory. In this section, we discuss our motivation and provide some further implications.

1.1.1   The Witten-style conjecture for finite-order mapping tori   Recall that classical gauge-theoretic invariants of a smooth closed oriented 4–manifold X (the Seiberg–Witten invariants and the Donaldson polynomial invariants) are only defined when X

satisfies the condition $b_2^+(X) \geq 1$. However, with suitable modifications, some of these invariants can be defined for other 4–manifolds. In particular, let X be a smooth closed oriented 4–manifold such that

(3) $H_*(X; \mathbb{Z}) = H_*(S^1 \times S^3; \mathbb{Z})$ and $H_*(\widetilde{X}; \mathbb{Q}) = H_*(S^3; \mathbb{Q})$,

where $\widetilde{X}$ is the universal abelian cover of X. Then there are two gauge-theoretic invariants for X. The first one is the invariant $\lambda_{\mathrm{SW}}(X)$ we mentioned earlier. It was defined in [40] by counting Seiberg–Witten monopoles on X and modifying this count by an index-theoretic correction term. The second invariant, called $\lambda_{\mathrm{FO}}(X)$, was defined by Furuta and Ohta [19] as one quarter times the degree zero Donaldson polynomial of X. Furuta and Ohta used somewhat more restrictive hypotheses on X; we extended their definition to all manifolds X satisfying (3) in our paper [33].

Conjecture C   [40; 33]   *For any X satisfying* (3),

$$\lambda_{\mathrm{SW}}(X) = -\lambda_{\mathrm{FO}}(X).$$

Note that Conjecture C relates a Seiberg–Witten-type invariant to a Donaldson-type invariant. Therefore, it can be thought of as a Witten-style conjecture [50] for 4–manifolds with the homology of $S^1 \times S^3$.

The main results of this paper are inspired by Conjecture C for finite-order mapping tori, and we confirm this conjecture for the mapping tori of Theorem B. More precisely, we have the following theorem:

Theorem D   *Let Σ be a rational homology sphere which is a cyclic branched cover of an integral homology sphere, with branch set a knot. Let X be the mapping torus of the covering transformation $\tau: \Sigma \to \Sigma$. Then X satisfies* (3) *and we have*

$$\lambda_{\mathrm{SW}}(X) = -\lambda_{\mathrm{FO}}(X).$$

Note that, in their previous work [46], the second and third authors verified Conjecture C for the mapping tori of all finite-order diffeomorphisms $\tau: \Sigma \to \Sigma$ of integral homology spheres that generate a free group action on Σ.

Corollary E   *Let Σ be an integral homology sphere and $\tau: \Sigma \to \Sigma$ an orientation-preserving diffeomorphism of order n generating a semifree action of $\mathbb{Z}/n$ on Σ. If X is the mapping torus of τ, then $\lambda_{\mathrm{SW}}(X) = -\lambda_{\mathrm{FO}}(X)$.*

1.1.2 Normalized Lefschetz number in monopole and instanton Floer homology

Theorem D has an intriguing Floer-theoretic interpretation in terms of an expected comparison of the reduced monopole homology groups $\mathrm{HM}^{\mathrm{red}}_*(Y)$ of Kronheimer and Mrowka [23] with the reduced instanton homology groups $\widehat{\mathrm{HF}}_*(Y)$ of Frøyshov [17]. As an extension of Witten's conjecture [50] on closed 4–manifolds, such a comparison should also include the maps between the homology groups in question induced by cobordisms. For the product cobordism between *integral* homology spheres, this manifests itself in the known relationship between the Casson invariant and the Euler characteristic of the reduced homologies normalized by the respective h–invariants:

$$(4) \qquad \chi(\mathrm{HM}^{\mathrm{red}}(\Sigma)) + h(\Sigma) = \lambda(\Sigma), \quad \tfrac{1}{2}\chi(\widehat{\mathrm{HF}}(\Sigma)) - h_D(\Sigma) = \lambda(\Sigma),$$

where h and h_D are respectively the monopole and instanton Frøyshov invariants. Taking (4) as a model, let us define the normalized Lefschetz numbers of a homology cobordism W from an integral homology sphere Σ to itself to be

$$\mathrm{Lef}(W_*\colon \mathrm{HM}^{\mathrm{red}}(\Sigma) \to \mathrm{HM}^{\mathrm{red}}(\Sigma)) + h(\Sigma),$$
$$\tfrac{1}{2}\mathrm{Lef}(W_*\colon \widehat{\mathrm{HF}}(\Sigma) \to \widehat{\mathrm{HF}}(\Sigma)) - h_D(\Sigma)$$

in the monopole and instanton cases, respectively. In the special case of an orientation-preserving self-diffeomorphism $\tau\colon \Sigma \to \Sigma$, the normalized Lefschetz numbers of its mapping cylinder are referred to as the normalized Lefschetz numbers of τ.

Corollary F *If Σ is an integral homology sphere and $\tau\colon \Sigma \to \Sigma$ an orientation-preserving diffeomorphism of order n generating a semifree action of $\mathbb{Z}/n$ on Σ, then the normalized monopole and instanton Lefschetz numbers of τ agree.*

This can be seen as follows. Let X be the homology $S^1 \times S^3$ obtained by gluing up the two boundary components of W via the identity map. Then the normalized monopole Lefschetz number of W equals $-\lambda_{\mathrm{SW}}(X)$ by the splitting formula [34], and the normalized instanton Lefschetz number of W equals $\lambda_{\mathrm{FO}}(X)$ by the splitting formula of Anvari [1]. Comparing this with Corollary E completes the proof.

Note that for any 3–manifold, Kronheimer and Mrowka [24] conjecture that the sutured versions of monopole Floer homology and instanton Floer homology coincide. However, relations between all other versions remain mysterious. It is possible that Corollary F is a shadow of some deeper relation between these two theories.

1.2 Applications

Theorem A can be used to study knots in S^3 and smooth concordance between them. Given a knot $K \subset S^3$ and an integer $n > 1$, we let $\Sigma_n(K)$ be the n–fold cyclic branched cover of K. Denote by $L_n(K)$ the Lefschetz number of the map $\tau_* \colon \mathrm{HM}^{\mathrm{red}}(\Sigma_n(K)) \to \mathrm{HM}^{\mathrm{red}}(\Sigma_n(K))$ induced by the covering transformation τ. Then we have the following corollary of Theorem A, which generalizes [33, Corollary F]:

Corollary G *Let* $n = p^m$ *for* p *a prime number. Then the integer* $L_n(K)$ *is an additive smooth concordance invariant.*

Recall that a closed oriented 3–manifold Σ is called an L–space if $H_1(\Sigma; \mathbb{Q}) = 0$ and $\mathrm{HM}^{\mathrm{red}}(\Sigma; \mathbb{Z}) = 0$. Recent work of Boileau, Boyer and Gordon [5; 6] has focused attention on the question of which branched covers of knots are L–spaces. This is of interest in its own right but also as a test case for the L–space conjecture of Boyer, Gordon and Watson [7] equating the property of a rational homology sphere *not* being an L–space with the left-orderability of its fundamental group. Using $L_n(K)$, we can show that the property of a knot $K \subset S^3$ not having an L–space branched cover can sometimes hold for an entire concordance class of K.

Theorem H *Let* $n = p^m$ *for* p *a prime number. There is a knot* K_n *such that, for any knot* K *that is smoothly concordant to* K_n, *its* n–*fold cyclic branched cover* $\Sigma_n(K)$ *is not an* L–*space.*

In a further application in this vein, we give a systematic obstruction of branched covers being an L–space in terms of the Jones polynomial.

Theorem I *Let* $K \subset S^3$ *be a knot with* $\det(K) = 1$ *and* $J'_K(-1) \neq 0$, *where* $J_K(t)$ *is the Jones polynomial of* K. *Then, for any* $m \geq 1$, *the* $2m$–*fold cyclic branched cover* $\Sigma_{2m}(K)$ *is not an* L–*space.*

There are, of course, plenty of examples of knots K whose cyclic branched covers $\Sigma_n(K)$ are not L–spaces for all n. The novelty of our result is that it produces such examples in a systematic way. Note that $\Sigma_{2m}(K)$ in the above theorem may fail to be an L–space simply because it is not a rational homology sphere. However, since $\det(K) = 1$, the manifold $\Sigma_{2m}(K)$ is an m–fold cyclic branched cover of the integral homology sphere $\Sigma_2(K)$ and so it is automatically a rational homology sphere whenever m is a prime power.

Finally, recall the conjectural behavior [46, Remark 4.2] of the invariant λ_{SW} under orientation reversal.

Conjecture J *Let X be a smooth spin rational homology $S^1 \times S^3$ which is oriented and homology oriented, and denote by $-X$ the manifold X with reversed orientation but the same homology orientation. Then $\lambda_{SW}(-X) = -\lambda_{SW}(X)$.*

Note that proving this conjecture would provide an alternative route to the resolution of the triangulation conjecture, proved by Manolescu [37]. We can verify Conjecture J in a special case.

Theorem K *Let X be the mapping torus of an orientation-preserving diffeomorphism (not necessarily of finite order) of a rational homology sphere. Then, for any choice of spin structure on X, we have*

$$-\lambda_{SW}(X) = \lambda_{SW}(-X).$$

Theorem K will actually follow from our splitting formula [34]. We decided to include Theorem K here because of its relevance to the main result of this paper, Theorem B.

1.3 An outline of the proof

We will first prove Theorem B by computing the invariant $\lambda_{SW}(X)$ directly from its definition [40],

$$\lambda_{SW}(X) = \#\mathcal{M}(X, g, \beta) + w(X, g, \beta),$$

where g and β are generic metric and perturbation, $\mathcal{M}(X, g, \beta)$ the Seiberg–Witten moduli space, and $w(X, g, \beta)$ an index-theoretic correction term. In the special case of the mapping torus X at hand, we have an orbifold circle bundle $\pi \colon X \to Y^o$, where Y^o is the orbifold with the underlying space Y, the singular set K and the cone angle $2\pi/n$ along K. According to Baldridge [4], for the right choice of metrics and perturbations, the moduli space $\mathcal{M}(X, g, \beta)$ splits into a disjoint union of the orbifold Seiberg–Witten moduli spaces corresponding to all possible orbifold spin^c structures on Y^o. We relate these orbifold moduli spaces to the Seiberg–Witten moduli space on Y in Section 2 using an argument reminiscent of the pillowcase argument of Herald [21] in Donaldson's theory. The correction term $w(X, g, \beta)$ for the mapping torus X is just a combination of the η–invariants of Y, which we calculate in Sections 3 and 4 using surgery techniques and the splitting formula of Mazzeo and Melrose [38]. All of this gives us a formula for $\lambda_{SW}(X)$ in terms of certain invariants of Y. It is

converted into (2) in Section 5 using the formula of Lim [31] for the Casson invariant in the Seiberg–Witten theory.

Theorem A follows easily from Theorem B using the splitting theorem of [34]. Theorem D is proved in Section 6. It can be viewed as a generalization of the work of the second and third authors [45] to rational homology spheres, or as a generalization of [33, Theorem 7.1] to $n \geq 2$, and it is proved by essentially the same methods. Proofs for all of the applications are contained in Section 7.

2 Counting Seiberg–Witten monopoles

Let $K \subset Y$ be a knot in an oriented integral homology 3–sphere. Denote by Y^o the orbifold with the underlying space Y, the singular set K and the cone angle $2\pi/n$ along K. In this section, we study monopoles on Y^o using the Seiberg–Witten theory on manifolds with product ends. For the latter, we follow closely the exposition in Lim [32; 31].

2.1 Monopoles on product-end manifolds

Let $D(K)$ be a tubular neighborhood of $K \subset Y$ and $N = Y - \text{Int}(D(K))$ the knot exterior, which is a compact 3–manifold with boundary a 2–torus T. Associate with N the product-end manifold

$$N^* = N \cup_T \big([0, \infty) \times T\big).$$

Fix a metric $g(N)$ on N that restricts to a flat metric on the boundary and is a product metric in its collar neighborhood. This metric extends in an obvious fashion to a product-end metric on the manifold N^*.

The manifold Y has a unique spin structure. It restricts to a spin structure on N, which in turn extends to a spin structure on N^* with spinor bundle E. Let A be a unitary

connection in the determinant bundle of E, and φ a spinor on N^*. Let us consider the (ω, α)–perturbed Seiberg–Witten equations

$$(5) \qquad F_A + \omega = \tau(\varphi), \quad \mathcal{D}_A(\varphi) + \alpha \cdot \varphi = 0,$$

where ω is a closed 2–form on N^* and α is a 1–form on N^*, both with coefficients in $i\mathbb{R}$ and with compact support in $\operatorname{Int}(N)$. The solutions (A, φ) of these equations will be called *monopoles*. Denote by $\mathcal{M}_{\alpha,\omega}(N^*)$ the L^2–moduli space of monopoles on N^* with respect to the gauge group action. The monopoles in $\mathcal{M}_{\alpha,\omega}(N^*)$ are known to have asymptotic values at infinity, with a flat connection and the zero spinor (a proof can be derived by crossing N^* with S^1 as in Taubes [48]; compare with Nicolaescu [43, Chapter 4.2]). This gives rise to a map

$$R: \mathcal{M}_{\alpha,\omega}(N^*) \to \chi(T),$$

where $\chi(T)$ is the moduli space of flat $U(1)$ connections on $\det(E|_T)$, modulo gauge transformation on $E|_T$. Equivalently, we can view $\chi(T)$ as the $U(1)$–character variety of $\pi_1(T)$. One can easily see that $\chi(T)$ is a 2–torus; we will introduce a set of coordinates on $\chi(T)$ as follows.

Choose simple closed curves m and ℓ on the 2–torus T so that m bounds a disk in $D(K)$ and ℓ bounds a Seifert surface in N. Let $\mathcal{D}(T)$ be the Dirac operator on the 2–torus T associated to the spin connection. An easy calculation shows that there exists a unique point $[A_0] \in \chi(T)$ for which the coupled Dirac operator $\mathcal{D}_{A_0}(T)$ has nonzero kernel. Then, for any $[A] \in \chi(T)$, we can write $A - A_0 = \alpha \in \Omega^1(T; i\mathbb{R})$ and define

$$m(A) = -2i \int_m \alpha \quad \text{and} \quad \ell(A) = -2i \int_\ell \alpha.$$

The assignment of $(m(A), \ell(A))$ to $[A]$ gives a homeomorphism $\chi(T) \to \mathbb{R}^2/(2\mathbb{Z})^2$. For $a, b \in \mathbb{R}/2\mathbb{Z}$, denote by $[A_{(a,b)}]$ the point in $\chi(T)$ with coordinates $m(A_{(a,b)}) = a$ and $\ell(A_{(a,b)}) = b$. Note that the restriction of the unique flat $U(1)$ connection on Y to the torus T is the flat connection $A_{(1,1)}$.

We will treat $[A_{(0,0)}] = [A_0]$ as a singular point of $\chi(T)$. For sufficiently small $\epsilon > 0$, denote by U_ϵ the complement in $\chi(T)$ of the closed ϵ–disk centered at $[A_{(0,0)}]$. Also, for future use, denote by $\mathcal{S}_a$ the circle in the torus $\chi(T)$ which consists of the points $[A_{(a,b)}]$ with the fixed a and arbitrary b.

The monopoles in $\mathcal{M}_{\alpha,\omega}(N^*)$ having $\varphi = 0$ will be called *reducible*, and all other monopoles will be called *irreducible*. Denote by $\mathcal{M}_{\alpha,\omega}^{\mathrm{red}}(N^*)$ and $\mathcal{M}_{\alpha,\omega}^{\mathrm{irr}}(N^*)$ the

reducible and irreducible loci of $\mathcal{M}_{\alpha,\omega}(N^*)$, respectively. The restrictions of the map R to $\mathcal{M}^{\mathrm{irr}}_{\alpha,\omega}(N^*)$ and to $\mathcal{M}^{\mathrm{red}}_{\alpha,\omega}(N^*)$ will be called, respectively,

$$R^{\mathrm{irr}} \colon \mathcal{M}^{\mathrm{irr}}_{\alpha,\omega}(N^*) \to \chi(T) \quad \text{and} \quad R^{\mathrm{red}} \colon \mathcal{M}^{\mathrm{red}}_{\alpha,\omega}(N^*) \to \chi(T).$$

The following structure theorem for the moduli space $\mathcal{M}_{\alpha,\omega}(N^*)$ is proved in Lim [32, Theorem 1.3; 31, Theorem 3]:

Theorem 2.1 *For any sufficiently small $\epsilon > 0$ there are arbitrarily small perturbations α and ω such that the following statements hold:*

(1) *The map R^{red} is a diffeomorphism onto its image, which is a circle contained in U_ϵ.*

(2) *The closure of $(R^{\mathrm{irr}})^{-1}(U_\epsilon)$ in $\mathcal{M}_{\alpha,\omega}(N^*)$ is a smooth compact 1–manifold with boundary; its boundary points lie in $\mathcal{M}^{\mathrm{red}}_{\alpha,\omega}(N^*) \cup R^{-1}(\partial\overline{U}_\epsilon)$.*

(3) *Any boundary point of the closure of $(R^{\mathrm{irr}})^{-1}(U_\epsilon)$ that lies in $\mathcal{M}^{\mathrm{red}}_{\alpha,\omega}(N^*)$ has a neighborhood in $\mathcal{M}_{\alpha,\omega}(N^*)$ which is modeled on the zeroes of the map $\mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ sending (t,z) to tz, with $\mathbb{R} \times \{0\}$ corresponding to the reducibles.*

(4) *The map R is smooth on $(R^{\mathrm{irr}})^{-1}(U_\epsilon)$.*

(5) *Both $(R^{\mathrm{irr}})^{-1}(U_\epsilon)$ and $\mathcal{M}^{\mathrm{red}}_{\alpha,\omega}(N^*)$ are canonically oriented by the choice of orientation on the real line $H^1(N^*;\mathbb{R})$.*

From now on, we will always work with ϵ, ω and α that satisfy the conditions of Theorem 2.1. There will be further conditions that will require ϵ, ω and α to be sufficiently small. These conditions will be summarized in Remark 3.5.

One can say more about the map R^{red}; see discussion after Theorem 3 in [32] and Section 7 of [31]. The reducible monopoles are given by the equation $F_A + \omega = 0$. If $\omega = 0$, these are just flat connections on N^*, which are mapped to the circle $b = -1$ in the torus $\chi(T)$. The same is true for any ω with the vanishing cohomology class in $H^2_c(N^*;\mathbb{R}) = H^2(N,\partial N;\mathbb{R}) = \mathbb{R}$. In general, it follows by a direct calculation that

$$\ell(A) = -1 + i \int_F \omega,$$

where F is a Seifert surface for $\ell \subset N$. In particular, the image of $R^{\mathrm{red}} \colon \mathcal{M}^{\mathrm{red}}_{\alpha,\omega}(N^*) \to \chi(T)$ is a circle given by the equation $b = -1 + c(\omega)$, where $c(\omega) \in \mathbb{R}$ can be made arbitrarily small by choosing small ω. Such a circle is indicated near the bottom of Figure 1.
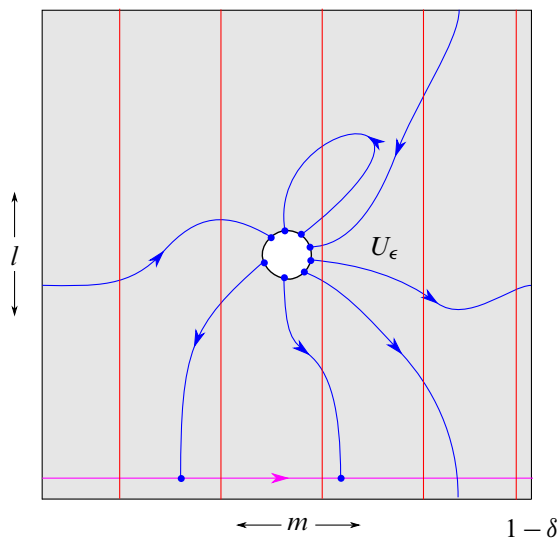
Figure 1: The image of $\mathcal{M}_{\alpha,\omega}(N^*)$ in U_ϵ.

2.2 Monopoles on orbifolds

Recall that we denoted by Y^o the orbifold with the underlying space Y, the singular set K and the cone angle $2\pi/n$ along K, and that we equipped the knot exterior $N = Y^o - D^o(K)$ with a metric $g(N)$ which restricts to a flat metric on the boundary T. We will further assume that, with respect to this flat metric, the meridian m and the longitude ℓ are orthogonal geodesics of length u and v, respectively. We will equip Y^o with an orbifold metric obtained by gluing $g(N)$ to an orbifold metric $g^o_{u,v}$ on $D^o(K)$. The latter metric is defined as follows:

Definition 2.2 Let $D^2 \times S^1$ be a solid torus with the polar coordinates (r, θ, ξ), where $0 \le r \le \frac{1}{2}\pi$ and $\theta, \xi \in \mathbb{R}/2\pi$. For any $u, v > 0$, define the metric $g_{u,v}$ on $D^2 \times S^1$ by the formula

$$\frac{v^2}{4\pi^2} d\xi \otimes d\xi + \frac{u^2}{4\pi^2}(dr \otimes dr + h(r)^2 d\theta \otimes d\theta),$$

where $h: \left[0, \frac{1}{2}\pi\right] \to \mathbb{R}$ is a smooth function such that

- $h(r) = \sin r$ when $0 \le r \le \frac{1}{6}\pi$,
- $h(r) = 1$ when $\frac{5}{12}\pi \le r \le \frac{1}{2}\pi$, and
- $h''(r) < 0$ when $\frac{1}{6}\pi \le r \le \frac{5}{12}\pi$.

We call $g_{u,v}$ the *smooth bullet-type metric*. With respect to this metric, the boundary of D^2 has length u and the circle factor has length v. We define the *orbifold bullet-type metric* $g_{u,v}^o$ as the quotient metric of $g_{nu,v}$ under the finite-order isometry ι of $D^2 \times S^1$ taking (r, θ, ξ) to $(r, \theta + 2\pi/n, \xi)$. Note that both $g_{u,v}$ and $g_{u,v}^o$ are flat near the boundary and have positive scalar curvature $\text{scal}(g_{u,v})$ and $\text{scal}(g_{u,v}^o)$ elsewhere.

We will need to perturb the Seiberg–Witten equations near K. To do this, we let ν be the 2–form on $D^2 \times S^1$ obtained by pulling back an $i\mathbb{R}$–valued 2–form on the unit disk D^2 which is supported in the region $r \leq \frac{1}{6}\pi$ and integrates to $\frac{1}{2}i$. We define the orbifold 2–form ν^o to be the quotient of the form $n \cdot \nu$ by the action of the isometry ι. The following lemma is straightforward:

Lemma 2.3 *There exists a constant $\delta_0 > 0$ such that the inequalities*

$$(6) \qquad |\delta_0 \nu(x)| \leq \text{scal}(g_{u,v})(x) \quad \text{and} \quad |\delta_0 \nu^o(y)| \leq \text{scal}(g_{u,v}^o)(y)$$

hold for all $x \in D(K)$ and $y \in D^o(K)$.

In what follows, we will choose $\epsilon > 0$ (the radius of the disk around the singularity in $\chi(T^2)$) smaller than the constant $\delta_0 > 0$ of Lemma 2.3.

Denote by $g(Y^o)$ the orbifold metric on $Y^o = N \cup_T D^o(K)$ obtained by gluing together the metrics $g(N)$ and $g_{u,v}^o$. Recall that the underlying space of Y^o is an integral homology 3–sphere. Therefore, the orbifold spin^c structures $\mathfrak{s}_k$ on Y^o can be canonically parametrized by integers $0 \leq k \leq n-1$; see Baldridge [4, Theorems 7 and 4]. Given an orbifold spin^c structure $\mathfrak{s}_k$, consider the orbifold Seiberg–Witten equations as in Baldridge [4, Section 2.4],

$$(7) \qquad F_A + \omega^o = \tau(\varphi), \quad \mathcal{D}_A(\varphi) = 0,$$

where ω^o is a closed orbifold 2–form on Y^o with coefficients in $i\mathbb{R}$. Equations (7) give rise to the orbifold Seiberg–Witten moduli space, which will be called

$$\mathcal{M}(Y^o, \mathfrak{s}_k, g(Y^o), \omega^o).$$

This moduli space can be recovered from $\mathcal{M}_{\alpha,\omega}(N^*)$ by using the right choice of metric and perturbation on Y^o. The metric will be the metric $g(Y^o)_L$ on the orbifold

$$Y^o = N \cup_T ([-L, L] \times T) \cup_T D^o(K)$$

obtained from $g(Y^o)$ by "neck stretching", with the perturbation 2–form $\omega_\delta^o = \omega + \delta \cdot \nu^o$ being the sum of the 2–form ω supported in $\text{Int}(N)$, and the orbifold 2–form $\delta \cdot \nu^o$ supported in $\text{Int}(D^o(K))$, which integrates to $\frac{1}{2}i\delta$ on each meridional disk.

Proposition 2.4 *For a generic $\delta \in (\epsilon, \delta_0)$, the map R^{irr} is transversal to $\mathcal{S}_{2k/n+1-\delta}$ for all k. Moreover, for all sufficiently large $L > 0$, the zero-dimensional manifolds*

$$\mathcal{M}^{\mathrm{irr}}(Y^o, \mathfrak{s}_k, g(Y^o)_L, \omega_\delta^o - d\alpha) \quad \text{and} \quad (R^{\mathrm{irr}})^{-1}(\mathcal{S}_{2k/n+1-\delta})$$

are orientation-preserving diffeomorphic.

Proof The transversality assertion follows from Theorem 2.1 and Sard's theorem. The identification between $\mathcal{M}^{\mathrm{irr}}(Y^o, \mathfrak{s}_k, g(Y^o)_L, \omega_\delta^o - d\alpha)$ and $(R^{\mathrm{irr}})^{-1}(\mathcal{S}_{2k/n+1-\delta})$ is an orbifold version of [32, Theorem 1.4; 31, Theorem 4], which is essentially a gluing argument along a torus, hence the presence of orbifold points makes no difference. The key ingredient is the inequality (6), which implies that the $(\delta \cdot v^o)$–perturbed Seiberg–Witten equations on the cylindrical-end orbifold $\big((-\infty, 0] \times T\big) \cup_T D^o(K)$ have no irreducible solutions with finite energy. □

An argument similar to that in [31, page 637] then shows that the oriented count of points in $(R^{\mathrm{irr}})^{-1}(\mathcal{S}_{2k/n+1-\delta})$ is given by the intersection number of $R(\mathcal{M}_{\alpha,\omega}^{\mathrm{irr}}(N^*))$ with the circle $\mathcal{S}_{2k/n+1-\delta}$ in the torus $\chi(T)$, where the torus $\chi(T)$ is oriented by $\partial/\partial a \wedge \partial/\partial b$ and the circle $\mathcal{S}_{2k/n+1-\delta}$ is oriented by $\partial/\partial b$ (again, the presence of orbifold points makes no difference). By combining this observation with Proposition 2.4, we obtain the following result:

Corollary 2.5 *For a generic $\delta \in (\epsilon, \delta_0)$ and for all $L > 0$ sufficiently large, the oriented count of points in the moduli space $\mathcal{M}^{\mathrm{irr}}(Y^o, \mathfrak{s}_k, g(Y^o)_L, \omega_\delta^o - d\alpha)$ equals the intersection number of $R(\mathcal{M}_{\alpha,\omega}^{\mathrm{irr}}(N^*))$ with the circle $\mathcal{S}_{2k/n+1-\delta}$ in the torus $\chi(T)$.*

2.3 Monopoles on Y

Now we study the perturbed Seiberg–Witten equations on the homology sphere Y. Let us consider the splitting $Y = N \cup_T D(K)$ with a metric $g(Y)$ which restricts to the metric $g(N)$ on N and a smooth bullet-type metric $g_{u,v}$ on $D(K) = S^1 \times D^2$ (see Definition 2.2). For any constant $L > 0$, equip the manifold

$$Y = N \cup_T ([-L, L] \times T) \cup_T D(K)$$

with the metric $g(Y)_L$ obtained from $g(Y)$ by "neck stretching". In addition, let $\omega_\delta = \omega + \delta \cdot v$ be the sum of the form ω supported on N and the 2–form $\delta \cdot v$ supported on $D(K)$ (as defined in Section 2.2), which integrates to $\frac{1}{2}i\delta$ on each

meridional disk. Then the (ω_δ, α)–perturbed Seiberg–Witten equations, as studied by Lim [31], are of the form

$$(8) \qquad\qquad F_A + \omega_\delta = \tau(\varphi), \quad \mathcal{D}_A(\varphi) + \alpha \cdot \varphi = 0.$$

The corresponding moduli space $\mathcal{M}_{\alpha,\omega_\delta}(Y, g(Y))$ contains exactly one reducible solution. We wish to compare $\mathcal{M}_{\alpha,\omega_\delta}(Y, g(Y))$ with the moduli space $\mathcal{M}(Y, g(Y), \beta)$ (which was used in [40] to define λ_{SW}) given by the equations

$$(9) \qquad\qquad F_A = \tau(\varphi) + d\beta, \quad \mathcal{D}_A(\varphi) = 0.$$

To this end, observe that any closed 2–form ω_δ is exact on the homology sphere Y, hence can be written in the form $\omega_\delta = d\gamma_\delta$, where γ_δ is a 1–form with coefficients in $i\mathbb{R}$.

Proposition 2.6 *The choice of $\beta_\delta = \alpha - \gamma_\delta$ establishes an orientation-preserving diffeomorphism between the moduli spaces $\mathcal{M}_{\alpha,\omega_\delta}(Y, g(Y))$ and $\mathcal{M}(Y, g(Y), \beta_\delta)$.*

Proof The change of variables $B = A + \alpha$ in the Seiberg–Witten equations (8) results in the equations $F_B - d\alpha + d\gamma_\delta = \tau(\varphi)$ and $\mathcal{D}_B(\varphi) = 0$, which match the equations (9) defining $\mathcal{M}(Y, g(Y), \beta_\delta)$ once we set $\beta_\delta = \alpha - \gamma_\delta$. The result now follows by comparing the orientation conventions for the two moduli spaces. □

The following result is a special case of Corollary 2.5 when $n = 1$:

Corollary 2.7 *For a generic $\delta \in (\epsilon, \delta_0)$, the map R^{irr} is transversal to $\mathcal{S}_{1-\delta}$. Moreover, for all $L > 0$ sufficiently large, the oriented count of points in the moduli space $\mathcal{M}^{\mathrm{irr}}(Y, g(Y)_L, \beta_\delta)$ equals the intersection number of $R(\mathcal{M}^{\mathrm{irr}}_{\alpha,\omega}(N^*))$ with the circle $\mathcal{S}_{1-\delta}$ in the torus $\chi(T)$.*

2.4 The spectral flow formula

According to Theorem 2.1(1), the moduli space $\mathcal{M}^{\mathrm{red}}_{\alpha,\omega}(N^*)$ is a circle for any sufficiently small generic perturbations α and ω. This circle admits a parametrization by $\theta(N^*, a) = (R^{\mathrm{red}})^{-1}(\mathcal{S}_a)$, thereby giving rise to a family of twisted Dirac operators $\mathcal{D}_{\theta(N^*,a)}$ on the product-end manifold N^*. The L^2 completions of these operators are self-adjoint, which leads to a well-defined notion of spectral flow; see for instance Cappell, Lee and Miller [10]. The points where the operators $\mathcal{D}_{\theta(N^*,a)}$ have nonzero kernels are precisely the points in $\mathcal{M}^{\mathrm{red}}_{\alpha,\omega}(N^*)$ which serve as the boundary points

of the closure of $(R^{\mathrm{irr}})^{-1}(U_\epsilon)$. It then follows from Theorem 2.1(3) that, for any choice of perturbations α and ω as in that theorem, the spectral flow is transverse; that is, the spectral curves of the family $\mathcal{D}_{\theta(N^*,a)}$ intersect the a–axis transversely with multiplicity 1.

Lemma 2.8 *For sufficiently small α and ω, the spectral flow of the family of twisted Dirac operator $\mathcal{D}_{\theta(N^*,a)}$ around the circle is equal to zero.*

Proof Let $f: N^* \to S^1$ be an arbitrary smooth function which induces an isomorphism $f_*: H_1(X) \to \mathbb{Z}$ and consider the family $\mathcal{D}_a = \mathcal{D} + ia\,df$ of twisted Dirac operators on N^*. The unperturbed spin Dirac operator $\mathcal{D}$ is quaternionic linear, hence the j–conjugate of $\mathcal{D}_a$ is $\mathcal{D}_{-a}$, which makes the picture of the spectral curves of $\mathcal{D}_a$ symmetric with respect to the involution sending a to $-a$. This ensures that the spectral flow of the family $\mathcal{D}_a$, which was defined in [3, Section 7] as the intersection number of the spectral curves with a small vertical shift of the a–axis, vanishes. The family $\mathcal{D}_{\theta(N^*,a)}$ is a deformation of the family $\mathcal{D}_a$, hence its spectral flow vanishes as well, as long as the perturbations α and ω are sufficiently small. $\square$

In what follows, we will choose a generic $\delta > 0$ such that the following condition is satisfied:

(10)   The operators $\mathcal{D}_{\theta(N^*,a)}$ are invertible at $a = 2k/n - 1 - \delta$ for all k.

Proposition 2.9 *For all sufficiently large $L > 0$, sufficiently small generic perturbations α and ω, and generic $\delta > 0$, one has the relation*

$$\#\mathcal{M}^{\mathrm{irr}}(Y^o, \mathfrak{s}_k, g(Y^o)_L, \omega_\delta^o - d\alpha)$$
$$= \#\mathcal{M}^{\mathrm{irr}}(Y, g(Y)_L, \beta_\delta) - \mathrm{SF}(\mathcal{D}_{\theta(N^*,1-\delta)}, \mathcal{D}_{\theta(N^*,2k/n+1-\delta)}).$$

The spectral flow in this formula is calculated along any path in the circle $\theta(N^*,a)$ that leads from $\theta(N^*, 1-\delta)$ to $\theta(N^*, 2k/n+1-\delta)$; according to Lemma 2.8, this is a well-defined quantity.

Proof According to Corollaries 2.5 and 2.7, we have the identities

$$\#\mathcal{M}^{\mathrm{irr}}(Y^o, \mathfrak{s}_k, g(Y^o)_L, \omega_\delta^o - d\alpha) = \#\big(R(\mathcal{M}^{\mathrm{irr}}_{\alpha,\omega}(N^*)) \cap \mathcal{S}_{2k/n+1-\delta}\big),$$
$$\#\mathcal{M}^{\mathrm{irr}}(Y, g(Y)_L, \beta_\delta) = \#\big(R(\mathcal{M}^{\mathrm{irr}}_{\alpha,\omega}(N^*)) \cap \mathcal{S}_{1-\delta}\big).$$

According to Theorem 2.1, the one-dimensional moduli space $\mathcal{M}^{\mathrm{irr}}_{\alpha,\omega}(N^*)$ provides an oriented cobordism between the points in $\mathcal{M}^{\mathrm{irr}}_{\alpha,\omega}(N^*) \cap R^{-1}(\mathcal{S}_{1-\delta})$, the points

in $\mathcal{M}^{\mathrm{irr}}_{\alpha,\omega}(N^*) \cap R^{-1}(\mathcal{S}_{2k/n-1-\delta})$, and the reducible boundary points in $\mathcal{M}^{\mathrm{red}}_{\alpha,\omega}(N^*)$ that fit into the interval between $\theta(N^*, 1-\delta)$ and $\theta(N^*, 2k/n + 1 - \delta)$; of the two intervals in the circle $\mathcal{M}^{\mathrm{red}}_{\alpha,\omega}(N^*)$ having the same endpoints, we choose the one that does not contain $\theta(N^*, 0)$ so as to stay within U_ϵ. The reducible boundary points in $\mathcal{M}^{\mathrm{red}}_{\alpha,\omega}(N^*)$ correspond to the points where the Dirac operators $\mathcal{D}_{\theta(N^*,a)}$ have nonzero kernels, hence these are precisely the points that contribute to the spectral flow in the statement of the proposition. That they contribute with the right sign follows from the description of the signs in [32, Section 11.3; 31, page 635]: when moving in the direction of the orientation of $\mathcal{M}^{\mathrm{red}}_{\alpha,\omega}(N^*)$, the contribution to the spectral flow at $\theta(N^*, a)$ is positive if and only if the orientation of $\mathcal{M}^{\mathrm{irr}}_{\alpha,\omega}(N^*)$ is into $\theta(N^*, a)$. $\square$

3 Eta-invariant of the Dirac operator

Let X be the mapping torus of the covering translation $\tau \colon \Sigma \to \Sigma$ with the standard orientation and homology orientation. A choice of spin structure on X induces a spin structure on Σ. This induced spin structure is invariant under τ, hence it is the same spin structure that lifts the unique spin structure on the integral homology sphere Y. We will have these particular spin structures in mind when talking about the spin manifolds Σ and Y. In this section, we will compare the η–invariants of the (twisted) spin Dirac operators on Σ and Y. The idea for this comparison comes from Lim [31].

For the knot $K \subset Y$, consider its preimage $\widetilde{K} \subset \Sigma$ under the branched cover projection $\Sigma \to Y$. The knot $\widetilde{K}$ can also be viewed as the fixed-point set of the covering translation $\tau \colon \Sigma \to \Sigma$. Write

$$(11) \qquad\qquad Y = N \cup D(K) \quad \text{and} \quad \Sigma = N_n \cup \widetilde{D}(K),$$

where $N_n \to N$ is a regular n–fold cover. Let $Y(0)$ be the manifold obtained from Y by 0–surgery on the knot K, and similarly $\Sigma(0)$ the manifold obtained from Σ by 0–surgery on $\widetilde{K}$. Write

$$(12) \qquad\qquad Y(0) = N \cup D(0) \quad \text{and} \quad \Sigma(0) = N_n \cup \widetilde{D}(0),$$

where $D(0)$ is a solid torus $D^2 \times S^1$ glued to N along its torus boundary by sending its longitude and meridian to the curve m and l in $T = \partial N$, respectively (see Section 2.1), and similarly for $\widetilde{D}(0)$. The regular n–fold cover $N_n \to N$ extends to a regular n–fold cover $\Sigma(0) \to Y(0)$. The aforementioned spin structures on Y and Σ give rise to spin structures on $Y(0)$ and $\Sigma(0)$. Denote by $g(Y(0))$ the metric on the manifold Y_0 that

equals g_N on N and the bullet-type metric $g_{v,u}$ on $D(0)$. We use the symbols $g(\Sigma)$ and $g(\Sigma(0))$ to denote the pullback of the metrics $g(Y^o)$ and $g(Y(0))$. For any positive constant L, denote by $g(*)_L$ the metrics on these manifolds obtained from $g(*)$ by neck stretching as in Section 2; these metrics will sometimes be suppressed in our notation.

Let $\theta(Y)$ be the unique reducible monopole in $\mathcal{M}_{\alpha,\omega_\delta}(Y, g(Y)_L)$ for the choices of α, ω_δ and L as Corollary 2.7; this monopole is the unique extension to Y of the reducible monopole $\theta(N^*, 1-\delta)$ on N^* whose limiting value is the flat connection $A_{(1-\delta, -1+c(\omega))}$. Denote by $\theta(\Sigma)$ the lift of $\theta(Y)$ to Σ.

Note that each reducible monopole $\theta(N^*, a)$ on N^* admits a unique extension to a reducible monopole on $Y(0)$, which will be called $\theta(Y(0), a)$. This follows by applying Theorem 2.1 to the manifold $D(0)$ in place of N, and matching the limiting values in $\chi(T)$ by choosing an appropriate perturbation on $D(0)$. This perturbation can be made arbitrary small by choosing small α and ω. A similar construction applied to $\Sigma(0)$ in place of $Y(0)$ gives rise to the monopole $\theta(\Sigma(0), a)$ on $\Sigma(0)$. The monopoles $\theta(Y(0), 1-\delta)$ and $\theta(\Sigma(0), 1-\delta)$ will be referred to as simply $\theta(Y(0))$ and $\theta(\Sigma(0))$.

Lemma 3.1 *For any generic $\delta > 0$ that satisfies (10), there exist constants C_1 and C_2 which are independent of Y, Σ, K and L (but may depend on δ) such that, for all sufficiently large $L > 0$,*

$$\tag{13} \eta(\mathcal{D}_{\theta(Y)}) = \eta(\mathcal{D}_{\theta(Y(0))}) + C_1 + o(1),$$

$$\tag{14} \eta(\mathcal{D}_{\theta(\Sigma)}) = \eta(\mathcal{D}_{\theta(\Sigma(0))}) + C_2 + o(1),$$

where $o(1)$ as usual denotes a quantity that limits to zero as L goes to infinity.

Proof The restrictions of the monopoles $\theta(Y)$ and $\theta(Y(0))$ to the tubular neighborhoods $D(K)$ and $D(0)$ give rise to the monopoles on the product-end manifolds

$$D^*(K) = \big((-\infty, 0] \times T\big) \cup_T D(K) \quad \text{and} \quad D^*(0) = \big((-\infty, 0] \times T\big) \cup_T D(0),$$

which will be called, respectively, $\theta(D^*(K))$ and $\theta(D^*(0))$. Since both manifolds $D^*(K)$ and $D^*(0)$ have nonnegative scalar curvature, the Dirac operators $\mathcal{D}_{\theta(D^*(K))}$ and $\mathcal{D}_{\theta(D^*(0))}$ are invertible. Combined with the invertibility of $\mathcal{D}_{\theta(N^*,1-\delta)}$, this implies that, for all sufficiently large $L > 0$, the operators $\mathcal{D}_{\theta(Y)}$ and $\mathcal{D}_{\theta(Y(0))}$ are invertible. Moreover, it follows from Mazzeo and Melrose [38] that

$$\eta(\mathcal{D}_{\theta(Y)}) = \eta(\mathcal{D}_{\theta(N^*,1-\delta)}) + \eta(\mathcal{D}_{\theta(D^*(K))}) + o(1),$$

$$\eta(\mathcal{D}_{\theta(Y(0))}) = \eta(\mathcal{D}_{\theta(N^*,1-\delta)}) + \eta(\mathcal{D}_{\theta(D^*(0))}) + o(1).$$

Subtracting these two formulas, we obtain formula (13) with the constant $C_1 = \eta(\mathcal{D}_{\theta(D^*(K))}) - \eta(\mathcal{D}_{\theta(D^*(0))})$, which is a linear combination of the η–invariants of standard solid tori and hence is independent of Y, K and L. Formula (14) is proved similarly. $\qquad\square$

Lemma 3.2 *One has the identity*

$$\eta(\mathcal{D}_{\theta(\Sigma(0))}) = \sum_{k=0}^{n-1} \eta(\mathcal{D}_{\theta(Y(0),2k/n+1-\delta)}).$$

Proof Since $\theta(\Sigma(0))$ is the lift of $\theta(Y(0))$ under the regular n–fold covering map $\Sigma(0) \to Y(0)$, the covering translation $\tau\colon \Sigma(0) \to \Sigma(0)$ induces an action on the spinors, splitting each operator $\mathcal{D}_{\theta(\Sigma(0))}$ into a direct sum of the operators $\mathcal{D}^{\alpha}_{\theta(\Sigma(0))}$ on its eigenspaces. This in turn leads to the identity

$$\eta(\mathcal{D}_{\theta(\Sigma(0))}) = \sum_{\alpha} \eta(\mathcal{D}^{\alpha}_{\theta(\Sigma(0))}).$$

One can easily check that the operators $\mathcal{D}^{\alpha}_{\theta(\Sigma(0))}$ in this formula are precisely the operators $\mathcal{D}_{\theta(Y(0),2k/n+1-\delta)}$, which completes the proof. $\qquad\square$

Lemma 3.3 *For all sufficiently large $L > 0$ and all k, one has*

$$\eta(\mathcal{D}_{\theta(Y(0),2k/n+1-\delta)}) = \eta(\mathcal{D}_{\theta(Y(0))}) + 2\,\mathrm{SF}(\mathcal{D}_{\theta(N^*,1-\delta)}, \mathcal{D}_{\theta(N^*,2k/n+1-\delta)}).$$

Proof According to [3], the difference $\eta(\mathcal{D}_{\theta(Y(0),2k/n+1-\delta)}) - \eta(\mathcal{D}_{\theta(Y(0))})$ equals twice the spectral flow $\mathrm{SF}(\mathcal{D}_{\theta(Y(0))}, \mathcal{D}_{\theta(Y(0),2k/n+1-\delta)})$. According to [10], for all sufficiently large $L > 0$, this spectral flow is the sum of the spectral flow over $D^*(0)$ and the spectral flow over N^*. The former vanishes because the metric on $D^*(0)$ has nonnegative scalar curvature and the perturbation is small, and the latter equals the spectral flow $\mathrm{SF}(\mathcal{D}_{\theta(N^*,1-\delta)}, \mathcal{D}_{\theta(N^*,2k/n+1-\delta)})$. $\qquad\square$

Corollary 3.4 *For any δ that satisfies (10), there exists a constant C' which, for all sufficiently large $L > 0$, is independent of Y, Σ, K and L, and has the property that*

$$\eta(\mathcal{D}_{\theta(\Sigma)}) = n \cdot \eta(\mathcal{D}_{\theta(Y)}) + 2\sum_{k=0}^{n-1} \mathrm{SF}(\mathcal{D}_{\theta(N^*,1-\delta)}, \mathcal{D}_{\theta(N^*,2k/n+1-\delta)}) + C' + o(1).$$

Remark 3.5 So far, we have made choices of the following parameters:

- $\epsilon > 0$, the radius of the disk around the singular point $[A_{(0,0)}] \in \chi(T)$;

- α and ω, differential forms on N^* used to perturb the Seiberg–Witten equations; and

- $\delta > 0$, a positive number whose products with the fixed 2–forms ν and ν^o serve as the perturbations on $D(K)$ and $D^o(K)$.

(Other parameters, such as the bullet-type metric $g_{u,v}$ and the 2–form ν on $D(K)$, have been fixed since the moment they were defined.) We have imposed several constraints on these parameters. To clarify that these constrains do not contradict each other, we summarize them here: First, we choose $\epsilon > 0$ to be small enough that Theorem 2.1 applies. The parameter $\epsilon > 0$ also needs to be smaller than the constant $\delta_0 > 0$ of Lemma 2.3. Then we choose generic α and ω that meet the requirements of Theorem 2.1. The perturbations α and ω also need to be small enough that Lemma 2.8, Proposition 2.9 and all results of Section 3 hold. Finally, we choose a generic $\delta \in (\epsilon, \delta_0)$ to satisfy the condition (10) and the condition of Corollary 2.5. We will fix all these choices from now on and will not discuss them further.

4   Eta-invariant of the signature operator

In this section, we will analyze the eta-invariants of the odd signature operators on the manifolds $M = Y$, Σ, $Y(0)$ and $\Sigma(0)$. As before, we fix the long neck metrics $g(M)_L$ on these manifolds and use $\eta_{\text{sign}}(M)$ to denote the corresponding eta-invariants. Since $\eta_{\text{sign}}(M)$ is closely related to a topological invariant (the signature), its analysis is much easier than that of the eta-invariant of the Dirac operator as in Meyerhoff and Ruberman [39].

Lemma 4.1   *One has the identities*

$$\eta_{\text{sign}}(\Sigma) = \eta_{\text{sign}}(\Sigma(0)) \quad and \quad \eta_{\text{sign}}(Y) = \eta_{\text{sign}}(Y(0)).$$

Proof   This will follow from an excision argument for $\eta_{\text{sign}}(M)$. We will focus on the second equality since the first one is similar. To keep better track of notation, we will denote by

$$f_1 \colon \partial D(K) \to T = \partial N \quad and \quad f_0 \colon \partial D(0) \to T = \partial N$$

the gluing maps for Y and $Y(0)$, and denote by $g(T)$ the restriction of the metric $g(N)$ to T. We form a Riemannian 4–manifold with corners,

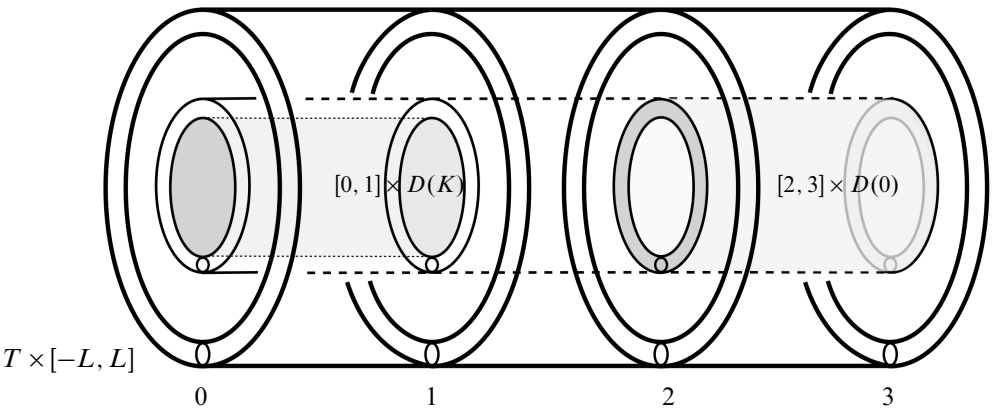$$W = ([0, 3] \times [-L, L] \times T) \cup ([0, 1] \times D(K)) \cup ([2, 3] \times D(0)),$$

Figure 2: $g(W)$.

where $[0, 1] \times D(K)$ is glued to $[0, 3] \times [-L, L] \times T$ via the map

$$\mathrm{id}_{[0,1]} \times f_1 : [0, 1] \times \partial D(K) \to [0, 1] \times \{L\} \times T,$$

and $[2, 3] \times D(0)$ is glued to $[0, 3] \times [-L, L] \times T$ via the map

$$\mathrm{id}_{[2,3]} \times f_0 : [2, 3] \times \partial D(K) \to [2, 3] \times \{L\} \times T.$$

A schematic picture of this region is depicted in Figure 2.

The metric $g(W)$ on W is obtained by gluing together the product metrics

$$[0, 3] \times [-L, L] \times g(T), \quad [0, 1] \times g_{u,v} \quad \text{and} \quad [2, 3] \times g_{v,u}.$$

The connected component

$$(\{1\} \times D(K)) \cup ([1, 2] \times \{L\} \times T) \cup (\{2\} \times D(0))$$

of the boundary of W is the sphere S^3 with a nonsmooth metric. We modify the metric $g(W)$ in a collar neighborhood of this S^3 to obtain a new metric $g'(W)$ that equals the product metric with the standard smooth round metric $g(S^3)$. Next, we form the manifold $W' = W \cup ([0, 3] \times N)$ by gluing the two pieces together via the map

$$\mathrm{id}_{[0,3] \times T} : [0, 3] \times \{-L\} \times T \to [0, 3] \times \partial N.$$

The metric $g(W')$ is obtained by gluing together $g'(W)$ and the product metric $[0, 3] \times g(N)$. Note that $(W', g(W'))$ is a smooth Riemannian manifold (without corners). It has three boundary components: $(Y, g(Y)_L)$, $(Y(0), g(Y(0))_L)$ and $(S^3, g(S^3))$. The signature of W' is zero because the map $H^2(W', \partial W'; \mathbb{R}) \to H^2(W'; \mathbb{R})$ is trivial. The eta-invariant of S^3 vanished because $(S^3, g(S^3))$ admits

an orientation-reversing isometry. The Pontryagin form on $[0, 3] \times N$ also vanishes with respect to the product metric. Now, using the Atiyah–Patodi–Singer index theorem [2], we obtain the equality

$$\eta_{\mathrm{sign}}(Y) - \eta_{\mathrm{sign}}(Y(0)) = -\frac{1}{3} \int_{W'} p_1(g'(W)),$$

whose right-hand side is a constant independent of N. To compute this constant, we can set $N = D^2 \times S^1$ and $g(N) = g_{u,v}$. Then both $(Y, g(Y)_L)$ and $(Y(0), g(Y(0))_L)$ admit orientation-reversing isometries, hence their eta-invariants vanish. Thus we conclude that, for any N, the equality $\eta_{\mathrm{sign}}(Y) - \eta_{\mathrm{sign}}(Y(0)) = 0$ holds.   □

Lemma 4.2 *One has the identity*

$$\eta_{\mathrm{sign}}(\Sigma(0)) = n \cdot \eta_{\mathrm{sign}}(Y(0)) - \sum_{m=1}^{n-1} \mathrm{sign}_{m/n}(K).$$

Proof The manifold $\Sigma(0)$ is a regular (unbranched) n–fold cyclic cover of $Y(0)$. By splitting the forms on $\Sigma(0)$ into eigenspaces for the induced action of τ, we obtain the relation

$$\eta_{\mathrm{sign}}(\Sigma(0)) = \eta_{\mathrm{sign}}(Y(0)) + \sum_{\alpha \neq 1} \eta_{\mathrm{sign},\alpha}(Y(0)).$$

Since $\rho_\alpha(Y(0)) = \eta_{\mathrm{sign}}(Y(0)) - \eta_{\mathrm{sign},\alpha}(Y(0))$, this immediately implies that

$$\eta_{\mathrm{sign}}(\Sigma(0)) = n \cdot \eta_{\mathrm{sign}}(Y(0)) - \sum_{\alpha \neq 1} \rho_\alpha(Y(0)).$$

Now, if α sends the meridian of K to $m \in \mathbb{Z}/n$, it follows from Theorem 3.6 of Gilmer [20] that $\rho_\alpha(Y(0)) = \mathrm{sign}_{m/n}(K)$. Gilmer only states his result for $Y = S^3$ but his method will work more generally for any integral homology sphere.   □

Corollary 4.3 *The eta-invariants of Σ and Y are related by the formula*

$$\eta_{\mathrm{sign}}(\Sigma) = n \cdot \eta_{\mathrm{sign}}(Y) - \sum_{m=1}^{n-1} \mathrm{sign}_{m/n}(K).$$

5 Proof of Theorem B

Let X be the mapping torus of $\tau \colon \Sigma \to \Sigma$. Since τ has finite order, X admits an obvious circle action and the orbifold circle bundle $\pi \colon X \to Y^o$. Let $i\eta$ be the connection

form of this bundle. Any orbifold metric $g(Y^o)_L$ on Y^o as in Proposition 2.4 defines a metric $g = \eta^2 + \pi^* g(Y^o)_L$ on X. Similarly, a perturbation form $\omega^o_\delta - d\alpha$ as in Proposition 2.4 lifts to a τ–invariant form ω on Σ. Since Σ is a rational homology sphere, we can write $\omega = d\gamma$ for some 1–form γ on Σ and let $\beta = \pi^*\gamma$. With respect to the metric g and perturbation β, we conclude as in Baldridge [4] that

$$\#\mathcal{M}^{\text{irr}}(X, g, \beta) = \sum_{k=0}^{n-1} \#\mathcal{M}^{\text{irr}}(Y^o, \mathfrak{s}_k, g(Y^o)_L, \omega^o_\delta - d\alpha).$$

Remark 5.1 A fine point in this kind of argument is the discrepancy between the spinorial connection and the Levi-Civita connection associated with the metric. This discrepancy does not arise in our case because the two Dirac operators in question differ by the Clifford multiplication by the 3–form $\eta \wedge d\eta$ (see Baldridge [4, Lemma 17]), which vanishes because η is a flat connection.

Remark 5.2 It follows from Baldridge [4] that the irreducible part of the Seiberg–Witten moduli space on X is nondegenerate. The definition of $\lambda_{\text{SW}}(X)$ in [40] further requires that the (perturbed) blown-up Seiberg–Witten moduli space on X be free of reducibles. That this is the case for our mapping torus X follows by a simple Fourier analysis argument from a similar property of the blown-up Seiberg–Witten moduli space on Y^o.

Proposition 2.9 now implies that

$$(15) \quad \#\mathcal{M}^{\text{irr}}(X, g, \beta) = n \cdot \#\mathcal{M}^{\text{irr}}(Y, g_L, \beta_\delta) - \sum_{k=0}^{n-1} \text{SF}(\mathcal{D}_{\theta(N^*, 1-\delta)}, \mathcal{D}_{\theta(N^*, 2k/n+1-\delta)}).$$

The correction terms of Lim [32] and Mrowka, Ruberman and Saveliev [40] in the product-end case are, respectively,

$$c(Y, g_L, \beta_\delta) = \tfrac{1}{2}\eta(\mathcal{D}_{\theta(Y)}) + \tfrac{1}{8}\eta_{\text{sign}}(Y) \quad \text{and} \quad w(X, g, \beta) = \tfrac{1}{2}\eta(\mathcal{D}_{\theta(\Sigma)}) + \tfrac{1}{8}\eta_{\text{sign}}(\Sigma).$$

It follows from Corollaries 3.4 and 4.3 that there is a constant C, which is independent of Y, Σ, N, K and L but may depend on δ, such that

$$(16) \quad w(X, g, \beta) = n \cdot c(Y, g_L, \beta_\delta) + \sum_{k=0}^{n-1} \text{SF}(\mathcal{D}_{\theta(N^*, 1-\delta)}, \mathcal{D}_{\theta(N^*, 2k/n+1-\delta)})$$
$$- \frac{1}{8} \sum_{k=0}^{n-1} \text{sign}_{k/n}(K) + C + o(1).$$

Combining formulas (15) and (16) with the Lim formula [32] for the Casson invariant, $-\lambda(Y) = \#\mathcal{M}^{\mathrm{irr}}(Y, g_L, \beta_\delta) + c(Y, g_L, \beta_\delta)$, we obtain[1]

$$\lambda_{\mathrm{SW}}(X) = \#\mathcal{M}^{\mathrm{irr}}(X, g, \beta) + w(X, g, \beta) = -n \cdot \lambda(Y) - \frac{1}{8} \sum_m \mathrm{sign}_{m/n}(K) + C + o(1).$$

Since $\lambda_{\mathrm{SW}}(X)$, $\lambda(Y)$ and $\mathrm{sign}_{k/n}(K)$ are metric-independent and C is independent of L, by passing to the limit as $L \to \infty$, we obtain the formula

$$\lambda_{\mathrm{SW}}(X) = -n \cdot \lambda(Y) - \frac{1}{8} \sum_m \mathrm{sign}_{m/n}(K) + C,$$

which further implies that C is a truly universal constant (independent, in particular, of Y, K and δ). By applying this formula to $Y = S^3$ and an unknot K, we conclude that $C = 0$. This finishes the proof.

6   Computing the Furuta–Ohta invariant

Let as before K be a knot in an integral homology sphere Y and Σ the n–fold cyclic cover of Y with branch set K for $n \geq 2$. We continue to assume that Σ is a rational homology sphere. It comes equipped with the covering translation $\tau \colon \Sigma \to \Sigma$ which is an orientation-preserving diffeomorphism of order n. The mapping torus of τ is the smooth 4–manifold $X = ([0, 1] \times \Sigma)/(0, x) \sim (1, \tau(x))$ with the product orientation.

Proposition 6.1   *The manifold X has the integral homology of $S^1 \times S^3$.*

Proof   Let N be the knot K exterior, $N_n \to N$ the n–fold cyclic cover of N and $N_\infty \to N$ its infinite cyclic cover. Denote by $\tau \colon N_\infty \to N_\infty$ the covering translation; it descends to the covering translations $\tau \colon N_n \to N_n$ and $\tau \colon \Sigma \to \Sigma$. It follows from the Wang exact sequence

$$\cdots \to H_1(\Sigma) \xrightarrow{\tau_* - 1} H_1(\Sigma) \to H_1(X) \to H_0(\Sigma) = \mathbb{Z} \xrightarrow{\tau_* - 1} H_0(\Sigma) = \mathbb{Z}$$

that all we need to prove is to show that $\tau_* - 1 \colon H_1(\Sigma) \to H_1(\Sigma)$ is surjective. To this end, choose a Seifert surface for K and a basis for its first homology yielding the $2h \times 2h$ Seifert matrix V. Then we have the following presentations [8, Theorem 8.8 and Proposition 8.20] for the first homology of N_∞ and Σ as modules over, respectively,

---

[1] Our orientation conventions differ from those of Lim [31], hence the Casson invariant $\lambda(Y)$ shows up with the negative sign. This is the same issue we dealt in [46, Section 7].

$\mathbb{Z}[t, t^{-1}]$ and $\mathbb{Z}[\mathbb{Z}/n]$:

$$H_1(N_\infty) = (\mathbb{Z}[t, t^{-1}])^{2h}/(V^\top - tV), \quad H_1(\Sigma) = (\mathbb{Z}[\mathbb{Z}/n])^{2h}/(V^\top - tV).$$

It is a standard fact [30] that $\tau_* - 1 \colon H_1(N_\infty) \to H_1(N_\infty)$ is an isomorphism. Writing π for the projection homomorphism

$$\pi \colon \mathbb{Z}[t, t^{-1}] \to \mathbb{Z}[t, t^{-1}]/(t^n - 1) = \mathbb{Z}[\mathbb{Z}/n],$$

we obtain a commutative diagram

$$
\begin{array}{ccc}
(\mathbb{Z}[t, t^{-1}])^{2h} & \xrightarrow{V^\top - tV} & (\mathbb{Z}[t, t^{-1}])^{2h} \\
\downarrow{\pi} & & \downarrow{\pi} \\
(\mathbb{Z}[\mathbb{Z}/n])^{2h} & \xrightarrow{V^\top - tV} & (\mathbb{Z}[\mathbb{Z}/n])^{2h}
\end{array}
$$

which readily implies that the map $\tau_* - 1 \colon H_1(\Sigma) \to H_1(\Sigma)$ is surjective. $\qquad\square$

Corollary 6.2 *The automorphism $\tau_* \colon H_1(\Sigma) \to H_1(\Sigma)$ has zero fixed-point set.*

Remark 6.3 The manifold X has a unique spin^c structure. Its restriction to Σ is the unique spin^c structure $\mathfrak{s}$ with the property $\tau_*(\mathfrak{s}) = \mathfrak{s}$ which appears in the statement of Theorem A. This spin^c structure is in fact a spin structure, obtained by restricting to Σ either of the two distinct spin structures on X; see the discussion at the beginning of Section 3.

The infinite cyclic cover $\widetilde{X} \to X$, which is just the product $\mathbb{R} \times \Sigma$, has the rational homology of S^3, hence the conditions (3) are satisfied. According to [33, Section 7.1], the manifold X has a well-defined Furuta–Ohta invariant,

$$(17) \qquad\qquad \lambda_{\mathrm{FO}}(X) = \tfrac{1}{4} \cdot \#\mathcal{M}^*(X) \in \mathbb{Q},$$

where $\#\mathcal{M}^*(X)$ stands for the signed count of points in the (possibly perturbed) moduli space $\mathcal{M}^*(X)$ of irreducible ASD connections in a trivial SU(2)–bundle $E \to X$, with the signs determined by a choice of orientation and homology orientation on X. In this section, we will prove the following formula for this invariant:

Theorem 6.4 *Let $\lambda(Y)$ be the Casson invariant of Y and denote by $\mathrm{sign}_{m/n}(K)$ the Tristram–Levine equivariant signatures of the knot K. Then*

$$\lambda_{\mathrm{FO}}(X) = n \cdot \lambda(Y) + \frac{1}{8} \sum_{m=1}^{n-1} \mathrm{sign}_{m/n}(K).$$

This formula was proved in Collin and Saveliev [15] and [45] under the assumption that Σ is an integral homology sphere, and in [33] under the assumption that $n = 2$. Our proof here will rely on the extension of those techniques to the general case at hand.

6.1 Equivariant theory

We will first describe $\mathcal{M}^*(X)$ in terms of $\mathcal{R}(\Sigma)$, the SU(2)–character variety of $\pi_1(\Sigma)$. To this end, consider the splitting

$$\mathcal{R}(\Sigma) = \{\theta\} \sqcup \mathcal{R}_{\mathrm{ab}}(\Sigma) \sqcup \mathcal{R}_{\mathrm{irr}}(\Sigma),$$

whose three components consist of the trivial representation θ and the conjugacy classes of abelian (that is, nontrivial reducible) and irreducible representations, respectively. This decomposition is preserved by the map $\tau^* : \mathcal{R}(\Sigma) \to \mathcal{R}(\Sigma)$ induced by the covering translation. Denote by $\mathcal{R}^\tau(\Sigma)$ the fixed-point set of the map τ^* acting on $\mathcal{R}(\Sigma) \setminus \{\theta\} = \mathcal{R}_{\mathrm{ab}}(\Sigma) \sqcup \mathcal{R}_{\mathrm{irr}}(\Sigma)$, so that

$$\mathcal{R}^\tau(\Sigma) = \mathcal{R}^\tau_{\mathrm{ab}}(\Sigma) \sqcup \mathcal{R}^\tau_{\mathrm{irr}}(\Sigma).$$

The following algebraic lemma will allow us to obtain useful information about the action of τ^* on $\mathcal{R}_{\mathrm{ab}}(\Sigma)$:

Lemma 6.5 *Let G be a finite abelian group and $s : G \to G$ an automorphism with zero fixed-point set. Then the induced automorphism $s^* : \mathrm{Hom}(G, U(1)) \to \mathrm{Hom}(G, U(1))$ of the character group of G has the trivial character as its only fixed point.*

Proof Let us consider the homomorphism $u : G \to G$ given by the formula $u(g) = s(g) - g$. Since s has zero fixed-point set, u is injective, and, since G is finite, u is an isomorphism. This implies that u^* is also an isomorphism, which completes the proof because $u^*(\chi) = s^*(\chi) \cdot \chi^{-1}$ on characters $\chi : G \to U(1)$.   □

The significance of the character group $\mathrm{Hom}(H_1(\Sigma), U(1))$ to us is that its quotient by the equivalence relation identifying α with α^{-1} is precisely $\mathcal{R}_{\mathrm{ab}}(\Sigma)$. We call a character $\alpha \in \mathrm{Hom}(H_1(\Sigma), U(1))$ *central* if the representation obtained by composing α with the inclusion of $U(1)$ in SU(2) as a maximal torus is central; the latter simply means that the image of α is contained in $\{\pm 1\} \subset U(1)$.

Proposition 6.6 *The fixed-point set $\mathcal{R}^\tau_{\mathrm{ab}}(\Sigma)$ consists of the equivalence classes of noncentral characters $\alpha \in \mathrm{Hom}(H_1(\Sigma), U(1))$ such that $\tau^*\alpha = \alpha^{-1}$. In particular, $\mathcal{R}^\tau_{\mathrm{ab}}(\Sigma)$ is empty for odd n, and $\mathcal{R}^\tau_{\mathrm{ab}}(\Sigma) = \mathcal{R}_{\mathrm{ab}}(\Sigma)$ for $n = 2$.*

Proof A character α gives rise to a point in $\mathcal{R}_{ab}^{\tau}(\Sigma)$ if and only if $\tau^*\alpha = \alpha$ or $\tau^*\alpha = \alpha^{-1}$. The former only occurs for the trivial character by Lemma 6.5. For a central character α, the condition $\tau^*\alpha = \alpha^{-1}$ is equivalent to $\tau^*\alpha = \alpha$, hence α is again trivial. If n is odd, it follows that $\alpha = \alpha^{-1}$, so α is central and hence trivial. If $n = 2$, it follows from Lemma 6.5 that τ^* acts as the negative identity on the character group and therefore as the identity on $\mathcal{R}_{ab}(\Sigma)$. $\qquad\square$

Proposition 6.7 *Let $i : \Sigma \to X$ be the inclusion map given by the formula $i(x) = [0, x]$. Then the induced map*

(18) $$i^* : \mathcal{M}^*(X) \to \mathcal{R}^{\tau}(\Sigma)$$

is well defined, and is a one-to-one correspondence over $\mathcal{R}_{ab}^{\tau}(\Sigma)$ and a two-to-one correspondence over $\mathcal{R}_{irr}^{\tau}(\Sigma)$.

Proof The natural projection $X \to S^1$ is a locally trivial bundle whose homotopy exact sequence

$$0 \to \pi_1(\Sigma) \to \pi_1(X) \to \mathbb{Z} \to 0$$

splits, making $\pi_1(X)$ into a semidirect product of $\pi_1(\Sigma)$ and $\mathbb{Z}$. Let t be a generator of $\mathbb{Z}$; then every representation $A : \pi_1(X) \to SU(2)$ determines and is uniquely determined by the pair (α, u), where $u = A(t)$ and $\alpha = i^*A : \pi_1(\Sigma) \to SU(2)$ is a representation such that $\tau^*\alpha = u\alpha u^{-1}$. In particular, the conjugacy class of α is fixed by τ^*.

If α is trivial, A must be reducible. If α is nontrivial abelian, it cannot be central by Proposition 6.6. Given a noncentral abelian α, conjugate it to a representation whose image is in the group $U(1)$ of unit complex numbers in $SU(2)$. Then $\tau^*\alpha = \alpha^{-1}$ by Proposition 6.6, hence $\alpha = i^*A$ with $u = A(t)$ in the circle $j \cdot U(1)$. In particular, A is irreducible and $u^2 = -1$. Since any two elements of $j \cdot U(1)$ are conjugate to each other by a unit complex number, the map i^* is a one-to-one correspondence over $\mathcal{R}_{ab}^{\tau}(\Sigma)$. Finally, let α be an irreducible representation with conjugacy class in $\mathcal{R}_{irr}^{\tau}(\Sigma)$. Then there is a unit quaternion u such that $\tau^*\alpha = u\alpha u^{-1}$, and therefore α is in the image of i^*. Moreover, there are exactly two different choices of u such that $\tau^*\alpha = u\alpha u^{-1}$, because, if $u_1\alpha u_1^{-1} = u_2\alpha u_2^{-1}$, then $u_1 = \pm u_2$ since α is irreducible. Therefore, the map i^* is a two-to-one correspondence in this case. Also note that the irreducibility of α implies that $u^n = \pm 1$. $\qquad\square$

Remark 6.8 It follows from the above proof that the characters in $\mathcal{M}^*(X)$ that are mapped by i^* to $\mathcal{R}_{ab}^{\tau}(\Sigma)$ are binary dihedral, while those mapped to $\mathcal{R}_{irr}^{\tau}(\Sigma)$ are not.

The Zariski tangent space to $\mathcal{R}^\tau(\Sigma)$ at a point $[\alpha] \in \mathcal{R}^\tau(\Sigma)$ is the fixed-point set of the map $\tau^* \colon T_{[\alpha]}\mathcal{R}(\Sigma) \to T_{[\alpha]}\mathcal{R}(\Sigma)$. Using an identification $T_{[\alpha]}\mathcal{R}(\Sigma) = H^1(\Sigma, \operatorname{ad}\alpha)$ and the fact that $\tau^*\alpha = u\alpha u^{-1}$, this set can be described in cohomological terms as the fixed-point set of the map

$$\operatorname{Ad} u \circ \tau^* \colon H^1(\Sigma, \operatorname{ad}\alpha) \to H^1(\Sigma, \operatorname{ad}\alpha).$$

We call $\mathcal{R}^\tau(\Sigma)$ *nondegenerate* if the equivariant cohomology groups $H^1_\tau(\Sigma, \operatorname{ad}\alpha) = \operatorname{Fix}(\operatorname{Ad} u \circ \tau^* \colon H^1(\Sigma, \operatorname{ad}\alpha) \to H^1(\Sigma, \operatorname{ad}\alpha))$ vanish for all $[\alpha] \in \mathcal{R}^\tau(\Sigma)$. The moduli space $\mathcal{M}^*(X)$ is called *nondegenerate* if $\operatorname{coker}(d_A^* \oplus d_A^+) = 0$ for all $[A] \in \mathcal{M}^*(X)$. Since $\operatorname{ind}(d^* \oplus d_A^+) = \dim \mathcal{M}^*(X) = 0$, this is equivalent to $\ker(d_A^* \oplus d_A^+) = 0$ and, since A is flat and irreducible, to simply $H^1(X, \operatorname{ad} A) = 0$.

Proposition 6.9 *The moduli space $\mathcal{M}^*(X)$ is nondegenerate if and only if $\mathcal{R}^\tau(\Sigma)$ is nondegenerate.*

Proof The group $H^1(X, \operatorname{ad} A)$ can be computed with the help of the Leray–Serre spectral sequence of the fibration $X \to S^1$ with fiber Σ. The E_2–page of this spectral sequence is

$$E_2^{pq} = H^p(S^1, \mathcal{H}^q(\Sigma, \operatorname{ad}\alpha)),$$

where $\alpha = i^*A$ and $\mathcal{H}^q(\Sigma, \operatorname{ad}\alpha)$ is the local coefficient system associated with the fibration. The groups E_2^{pq} vanish for all $p \geq 2$, hence the spectral sequence collapses at the E_2–page, and

(19) $H^1(X, \operatorname{ad} A) = H^1(S^1, \mathcal{H}^0(\Sigma, \operatorname{ad}\alpha)) \oplus H^0(S^1, \mathcal{H}^1(\Sigma, \operatorname{ad}\alpha)).$

The generator of $\pi_1(S^1)$ acts on the cohomology groups $H^*(\Sigma, \operatorname{ad}\alpha)$ as

$$\operatorname{Ad} u \circ \tau^* \colon H^*(\Sigma, \operatorname{ad}\alpha) \to H^*(\Sigma, \operatorname{ad}\alpha),$$

where u is such that $\tau^*\alpha = u\alpha u^{-1}$. If α is irreducible, $H^0(\Sigma, \operatorname{ad}\alpha) = 0$ and the first summand in (19) vanishes. If α is nontrivial abelian, we may assume without loss of generality that it takes values in the group $U(1)$ of unit complex numbers. Then $\tau^*\alpha = u\alpha u^{-1}$ for some $u \in j \cdot U(1)$ and $H^0(\Sigma, \operatorname{ad}\alpha) = i \cdot \mathbb{R}$ as a subspace of $\mathfrak{su}(2)$, with $\tau^* = \operatorname{id}$. One can easily check that $\operatorname{Ad} u$ acts as minus identity on $i \cdot \mathbb{R}$, hence the first summand in (19) again vanishes. The second summand in (19) is the fixed-point set of τ^* acting on $H^1(\Sigma, \operatorname{ad}\alpha)$, which is the equivariant cohomology $H^1_\tau(\Sigma, \operatorname{ad}\alpha)$. Thus we conclude that $H^1(X, \operatorname{ad} A) = H^1_\tau(\Sigma, \operatorname{ad}\alpha)$, which completes the proof. $\square$

Let us assume that $\mathcal{R}^\tau(\Sigma)$ is nondegenerate. For any $[\alpha] \in \mathcal{R}^\tau(\Sigma)$, its orientation will be given by

$$(-1)^{\operatorname{SF}^\tau(\theta, \alpha)},$$

where $\mathrm{SF}^\tau(\theta, \alpha)$ is the mod 2 equivariant spectral flow defined in [45, Section 3.4] for irreducible α. That definition extends word for word to abelian α after one resolves the technical issue of the existence of a constant lift, which we will do next.

Let P be an SU(2)–bundle over Σ with a fixed trivialization and α an abelian flat connection in P with holonomy in $\mathcal{R}^\tau_{\mathrm{ab}}(\Sigma)$; we are abusing notation by using the same symbol for the connection and its holonomy. Since $\mathcal{R}^\tau_{\mathrm{ab}}(\Sigma)$ is empty for odd n (see Proposition 6.6), we will assume without loss of generality that n is even. Then τ admits a lift $\widetilde{\tau} \colon P \to P$ such that $\widetilde{\tau}^*\alpha = \alpha$. Since α is abelian and noncentral, this lift is defined uniquely up to the stabilizer of α, which is a copy of $U(1)$ in SU(2). The lift $\widetilde{\tau}$ can be written in the base–fiber coordinates as $\widetilde{\tau}(x, y) = (\tau(x), \rho(x) \cdot y)$ for some function $\rho \colon \Sigma \to \mathrm{SU}(2)$. We call it *constant* if there exists $u \in \mathrm{SU}(2)$ such that $\rho(x) = u$ for all $x \in \mathrm{SU}(2)$.

Lemma 6.10 *By changing α within its gauge equivalence class, one may assume that $\widetilde{\tau}$ is a constant lift with $u^2 = -1$.*

Proof The equation $\widetilde{\tau}^*\alpha = \alpha$ implies that $(\widetilde{\tau}^n)^*\alpha = \alpha$, so the gauge transformation $\widetilde{\tau}^n$ belongs to the stabilizer of the connection α. If $x \in \mathrm{Fix}(\tau)$ then $\widetilde{\tau}^n(x, y) = (x, \rho(x)^n \cdot y)$, hence $\rho(x)^n$ is a unit complex number independent of x. This implies that $\rho(x)$ itself is a unit complex number unless $\rho(x)^n = \pm 1$. The latter equation actually implies that $\rho(x)^2 = -1$ because, at the level of holonomy representations, $\tau^*\alpha = \alpha^{-1}$ is conjugate to α by an element $u \in \mathrm{SU}(2)$ with $u^2 = -1$; see the proof of Proposition 6.7. Since $\rho(x)^2 = -1$ describes a single conjugacy class $\mathrm{tr}\,\rho(x) = 0$ in SU(2), we may assume that $\rho(x) = u$ for all $x \in \mathrm{Fix}(\tau)$.

To finish the proof, we will follow the argument of [45, Section 2.2]. Let $u \colon P \to P$ be the constant lift $u(x, y) = (\tau(x), u \cdot y)$ and consider the SO(3)–orbifold bundles $P/\widetilde{\tau}$ and P/u over the integral homology sphere Y. All such bundles are classified by the holonomy around the singular set in Y. Since this holonomy equals $\mathrm{Ad}(u)$ in both cases, the bundles $P/\widetilde{\tau}$ and P/u must be isomorphic, with any isomorphism pulling back to a gauge transformation $g \colon P \to P$ relating the lifts $\widetilde{\tau}$ and u. $\square$

Proposition 6.11 *Assuming that the moduli space $\mathcal{R}^\tau(\Sigma)$ is nondegenerate, the map (18) is orientation-preserving.*

Proof The proof from [45, Section 3] extends to the current situation with no change. $\square$

6.2 Orbifold theory

Under the continued nondegeneracy assumption, we will now describe $\mathcal{R}^\tau(\Sigma)$ in terms of orbifold representations. Let us consider the orbifold fundamental group $\pi_1^V(Y, K) = \pi_1(N)/\langle \mu^n \rangle$, where $N = Y - \text{Int}(D(K))$ is the knot exterior and μ is a meridian of K. This group can be included into the split orbifold exact sequence

$$1 \to \pi_1 \Sigma \xrightarrow{\pi_*} \pi_1^V(Y, K) \xrightarrow{j} \mathbb{Z}/n \to 1,$$

where j is the abelianization homomorphism. Denote by $\mathcal{R}^V(Y, K; \text{SO}(3))$ the character variety of irreducible SO(3) representations of the group $\pi_1^V(Y, K)$, and also introduce the character variety $\mathcal{R}^\tau(\Sigma; \text{SO}(3))$ of nontrivial representations $\pi_1 \Sigma \to \text{SO}(3)$ fixed by τ^*.

Proposition 6.12 *The pullback of representations via the map π_* in the orbifold exact sequence gives rise to a one-to-one correspondence*

$$\pi^*: \mathcal{R}^V(Y, K; \text{SO}(3)) \to \mathcal{R}^\tau(\Sigma; \text{SO}(3)).$$

Proof One can easily see that a representation $\alpha': \pi_1^V(Y, K) \to \text{SO}(3)$ pulls back to a trivial representation $\theta: \pi_1 \Sigma \to \text{SO}(3)$ if and only if α' is reducible. The same argument as in [15, Proposition 3.3] shows that all pullback representations belong to $\mathcal{R}^\tau(\Sigma, \text{SO}(3))$. The inverse map for π^* is constructed as follows: given $[\alpha] \in \mathcal{R}^\tau(\Sigma, \text{SO}(3))$, choose $v \in \text{SO}(3)$ such that $\tau^*\alpha = v\alpha v^{-1}$, and define a representation α' of $\pi_1^V(Y, K) = \pi_1 \Sigma \rtimes \mathbb{Z}/n$ by the formula

(20) $$\alpha'(g \cdot \mu^k) = \alpha(g) \cdot v^k.$$

If α is irreducible, the element v is unique, hence (20) gives an inverse map. If α is nontrivial abelian, an argument similar to that in the proof of Proposition 6.6 shows that $v = \text{Ad}\, u$ for some $u \in j \cdot U(1)$. Since any two elements of $j \cdot U(1)$ are conjugate to each other by a unit complex number, formula (20) again gives an inverse map. $\square$

Representations $\pi_1^V(Y, K) \to \text{SO}(3)$ need not lift to SU(2) representations; however, they lift to projective representations $\pi_1^V(Y, K) \to \text{SU}(2)$ sending μ^n to ± 1; see [44, Section 3.1]. The character variety of such projective representations will be denoted by $\mathcal{R}^V(Y, K)$, and it will be oriented using the orbifold spectral flow.

Proposition 6.13 *The correspondence of Proposition 6.12 gives rise to an orientation-preserving correspondence $\mathcal{R}^V(Y, K) \to \mathcal{R}^\tau(\Sigma)$ which is one-to-one over $\mathcal{R}_{\text{ab}}^\tau(\Sigma)$ and two-to-one over $\mathcal{R}_{\text{irr}}^\tau(\Sigma)$.*

Proof We first need to check that the map $\mathcal{R}^V(Y, K) \to \mathcal{R}^\tau(\Sigma)$ is well defined because the pullback of a projective representation $\pi_1^V(Y, K) \to \mathrm{SU}(2)$ is a priori a projective representation $\alpha \colon \Sigma \to \mathrm{SU}(2)$. The only obstruction to it being an actual representation is the second Stiefel–Whitney class $w_2(\mathrm{Ad}\,\alpha) \in H^2(\Sigma; \mathbb{Z}/2)$. That this class vanishes can be seen as follows. The conjugacy class of α is fixed by τ; therefore, $w_2(\mathrm{Ad}\,\alpha)$ must belong to the fixed-point set of $\tau^* \colon H^2(\Sigma; \mathbb{Z}/2) \to H^2(\Sigma; \mathbb{Z}/2)$. Using Poincaré duality, this fixed-point set can be identified with the kernel of the map $\tau_* - 1 \colon H_1(\Sigma; \mathbb{Z}/2) \to H_1(\Sigma; \mathbb{Z}/2)$. This kernel vanishes because the map $\tau_* - 1 \colon H_1(\Sigma) \to H_1(\Sigma)$ is injective by Corollary 6.2 and is therefore an automorphism of the finite abelian group $H_1(\Sigma)$; compare the proof of Lemma 6.5.

Let us now consider the adjoint representation $\mathrm{Ad}\colon \mathrm{SU}(2) \to \mathrm{SO}(3)$ and the induced maps on character varieties,

$$(21) \qquad \mathcal{R}^\tau(\Sigma) \to \mathcal{R}^\tau(\Sigma; \mathrm{SO}(3)) \quad \text{and} \quad \mathcal{R}^V(Y, K) \to \mathcal{R}^V(Y, K; \mathrm{SO}(3)).$$

The first map is a one-to-one correspondence, which can be seen as follows. We showed in the previous paragraph that every equivariant representation $\pi_1 \Sigma \to \mathrm{SO}(3)$ admits a lift to a representation $\pi_1 \Sigma \to \mathrm{SU}(2)$. The number of all such lifts is known to equal the cardinality of $H^1(\Sigma; \mathbb{Z}/2)$. However, we are only interested in equivariant lifts, and their number equals the cardinality of the fixed-point set of $\tau^* \colon H^1(\Sigma; \mathbb{Z}/2) \to H^1(\Sigma; \mathbb{Z}/2)$. The latter cardinality is easily seen to be 1 using Corollary 6.2.

The second map in (21) is the quotient map by the action of $\mathbb{Z}/2$ sending the image of the meridian μ to its negative. The fixed points of this action are precisely the binary dihedral projective representations $\alpha' \colon \pi_1^V(Y, K) \to \mathrm{SU}(2)$. Now, the proof will be finished as soon as we show that an irreducible projective representation $\alpha' \colon \pi_1^V(Y, K) \to \mathrm{SU}(2)$ is binary dihedral if and only if its pullback representation $\pi^*\alpha' \colon \pi_1 \Sigma \to \mathrm{SU}(2)$ is abelian.

If $\pi^*\alpha'$ is abelian, its image belongs to $U(1) \subset \mathrm{SU}(2)$ and the image of α' to its $\mathbb{Z}/n$ extension. Since the only finite extension of $U(1)$ inside $\mathrm{SU}(2)$ is the binary dihedral group $U(1) \cup j \cdot U(1)$, we conclude that α' must be binary dihedral. In particular, if n is odd, the representation $\pi^*\alpha'$ cannot be abelian, which matches the fact that $\mathcal{R}^\tau_{\mathrm{ab}}(\Sigma)$ is empty by Proposition 6.6. Conversely, it follows from the orbifold exact sequence that $\pi_1 \Sigma$ is the commutator subgroup of $\pi_1^V(Y, K)$. Therefore, if α' is binary dihedral, the image of $\pi^*\alpha'$ must belong to the commutator subgroup of $U(1) \cup j \cdot U(1)$, which is of course the group $U(1)$.

Since the orbifold spectral flow matches the equivariant spectral flow used to orient $\mathcal{R}^\tau(\Sigma)$, the above correspondence is orientation-preserving. $\qquad\square$

6.3 Perturbations

In this section, we will remove the assumption that $\mathcal{R}^\tau(\Sigma)$ is nondegenerate which we used until now. To accomplish that, we will switch from the language of representations to the language of connections. Let P a trivialized SU(2)–bundle over Σ. Any endomorphism $\widetilde{\tau}\colon P \to P$ which lifts the involution τ induces an action on the space of connections $\mathcal{A}(\Sigma)$ by pullback. Since any two such lifts are related by a gauge transformation, this action gives a well defined action on the configuration space $\mathcal{B}(\Sigma) = \mathcal{A}(\Sigma)/\mathcal{G}(\Sigma)$. The fixed-point set of this action will be denoted by $\mathcal{B}^\tau(\Sigma)$.

The irreducible part of $\mathcal{B}^\tau(\Sigma)$ was studied in [45], hence we will only deal with reducible connections. In fact, we will further restrict ourselves to constant lifts u because any flat abelian connection α admits such a lift; see Lemma 6.10.

Let $\mathcal{A}^u(\Sigma) \subset \mathcal{A}(\Sigma)$ consist of all nontrivial connections A such that $u^*A = A$, and $\mathcal{G}^u(\Sigma) \subset \mathcal{G}(\Sigma)$ of all gauge transformations g such that $gu = ug$. The quotient space $\mathcal{A}^u(\Sigma)/\mathcal{G}^u(\Sigma)$ will be denoted by $\mathcal{B}^u(\Sigma)$. The following lemma is a key to making the arguments of [45] work in the case of abelian connections:

Lemma 6.14 *The group $\mathcal{G}^u(\Sigma)$ acts on $\mathcal{A}^u(\Sigma)$ with the stabilizer $\{\pm 1\}$. Moreover, the natural map $\mathcal{B}^u(\Sigma) \to \mathcal{B}^\tau(\Sigma)$ is a two-to-one correspondence to its image on the irreducible part of $\mathcal{B}^u(\Sigma)$, and a one-to-one correspondence on the reducible part.*

Proof For the sake of simplicity, we will assume that reducible connections have their holonomy in the subgroup $U(1)$ of unit complex numbers in SU(2), and that $u \in j \cdot U(1)$. Let us suppose that $g^*A = A$ for a connection $A \in \mathcal{A}^u(\Sigma)$ and a gauge transformation $g \in \mathcal{G}^u(\Sigma)$. If A is irreducible, we automatically have $g = \pm 1$. If A is nontrivial abelian, then g is a complex number, and the condition $ug = gu$ implies that $g = \pm 1$.

To prove the second statement, consider a connection A such that $u^*A = A$ and consider its gauge equivalence class in $\mathcal{B}^\tau(\Sigma)$. It consists of all connections g^*A such that $u^*g^*A = g^*A$. Since $A = u^*A$, we immediately conclude that $u^*g^*A = g^*u^*A$, so that ug and gu differ by an element in the stabilizer of A. If A is irreducible, its stabilizer consists of ± 1, hence $ug = \pm gu$. The group of gauge transformations

satisfying this condition contains $\mathcal{G}^u(\Sigma)$ as a subgroup of index 2, which leads to the desired two-to-one correspondence. If A is nontrivial abelian, its stabilizer consists of unit complex numbers. Therefore, we can write $ug = c^2 gu$ with $c \in U(1)$ or, equivalently, $ucg = cgu$. This provides us with a gauge transformation $cg \in \mathcal{G}^u(\Sigma)$ such that $(cg)^*A = g^*A$, yielding the one-to-one correspondence on the reducible part. □

With this lemma in place, the proof of Proposition 6.7 can be restated in gauge-theoretic terms as in [45, Proposition 3.1]. The treatment of perturbations in our case is then essentially identical to that in [15; 45], one important observation being that the orbifold representations α' that pull back to abelian representations of $\pi_1(\Sigma)$ are in fact irreducible. This fact is used in the proof of [15, Lemma 3.8], which supplies us with sufficiently many admissible perturbations.

6.4 Proof of Theorem 6.4

The outcome of Sections 6.1 and 6.2 is that, perhaps after perturbing as in Section 6.3, we have two orientation-preserving correspondences,

$$\mathcal{M}^*(X) \to \mathcal{R}^\tau(\Sigma) \leftarrow \mathcal{R}^V(Y, K),$$

both of which are one-to-one over $\mathcal{R}_{ab}(\Sigma)$ and two-to-one over $\mathcal{R}^\tau_{irr}(\Sigma)$ (we omit perturbations in our notation). These correspondences imply the existence of an orientation-preserving one-to-one correspondence between $\mathcal{M}^*(X)$ and $\mathcal{R}^V(Y, K)$. The proof of Theorem 6.4 will be complete after we express the signed count of points in $\mathcal{R}^V(Y, K)$ in terms of the Casson invariant of Y and the equivariant knot signatures of K.

The character variety $\mathcal{R}^V(Y, K)$ of projective representations α' splits into two components corresponding, to whether $(\alpha'(\mu))^n$ equals $+1$ or -1. Let N be the exterior of the knot K; then this splitting corresponds to the splitting

$$(22) \qquad \mathcal{R}^V(Y, K) = \bigcup_{k=0}^{n} \mathcal{S}_{k/n}(N, \mathrm{SU}(2)),$$

where $\mathcal{S}_a(N, \mathrm{SU}(2))$ comprises the conjugacy classes of representations $\gamma \colon \pi_1 N \to \mathrm{SU}(2)$ such that $\mathrm{tr}\, \gamma(\mu) = 2\cos(\pi a)$. According to Herald [21] — see also Collin and Saveliev [15] — the combined signed count of points in (22) equals

$$4n \cdot \lambda(Y) + \frac{1}{2} \sum_{m=1}^{n-1} \mathrm{sign}_{m/n}(K).$$

Dividing this formula by 4, we obtain the formula for the Furuta–Ohta invariant $\lambda_{\mathrm{FO}}(X)$ claimed in Theorem 6.4.

7 Applications

In this section we supply proofs for the applications of our main theorem discussed in Section 1.2 of the introduction; for the convenience of the reader we will restate each result before giving the proof.

Theorem H *Let $n = p^m$ for p a prime number. There is a knot K_n such that, for any knot K that is smoothly concordant to K_n, its n–fold cyclic branched cover $\Sigma_n(K)$ is not an L–space.*

Proof Since n is a prime power, it is standard that all of the Tristram–Levine signatures are knot concordance invariants. Similarly, the h–invariant of the n–fold cyclic branched cover, with the specified spin^c structure, is a knot concordance invariant [22]. It now follows from Theorem A that $\mathrm{Lef}(\tau_*)$ is a knot concordance invariant. Therefore, if K_n is a knot for which $\mathrm{Lef}(\tau_*) \neq 0$, then the same is true for any knot K in the concordance class of K_n. In particular, the n–fold cyclic branched cover of K is not an L–space.

All that remains to prove the theorem is to find a knot K_n with $\mathrm{Lef}(\tau_*) \neq 0$. Pick relatively prime integers q and r both of which are greater than or equal to 2 and are relatively prime with p; we will exclude the triple $(2, 3, 5)$ to avoid dealing with the exceptional case of the Poincaré homology sphere. The n–fold cyclic branched cover of the right-handed torus knot $T(q, r)$ is the Brieskorn homology sphere

$$\Sigma(n, q, r) = \{x^n + y^q + z^r = 0\} \cap S^5$$

with its canonical link orientation and with the covering translation $\tau(x, y, z) = (e^{2\pi i/n} x, y, z)$. The homology sphere $\Sigma(n, q, r)$ admits a fixed-point-free circle action $t(x, y, z) = (t^{qr} x, t^{nr} y, t^{nq} z)$ making it into a Seifert fibered manifold; see Neumann and Raymond [42]. The covering translation τ is actually contained in this circle action: it corresponds to the choice of $t = e^{2\pi i p/n}$ for any integer p such that $pqr \equiv 1 \pmod{n}$. This implies that τ is isotopic to the identity and that $\mathrm{Lef}(\tau_*) = \chi(\mathrm{HM}^{\mathrm{red}}(\Sigma(n, q, r)))$.

To show that $\chi(\mathrm{HM}^{\mathrm{red}}(\Sigma(n, q, r))) \neq 0$, we will use the identification of the monopole and Heegaard Floer homology due to Kutluhan, Lee and Taubes [25; 26; 27; 28; 29], or,

alternatively, Colin, Ghiggini and Honda [11; 12; 13; 14] and Taubes [49]. The nonvanishing of the Euler characteristic of $\mathrm{HM}^{\mathrm{red}}(\Sigma(n,q,r))$ follows from this identification and the corresponding result in Heegaard Floer homology [9] as well as [16; 47]. $\qquad\square$

For any prime power n, we can define a subgroup $\mathcal{BL}_n$ in the smooth concordance group $\mathcal{C}$ generated by knots that are concordant to a knot whose n–fold branched cover is an L–space. One would expect that, for a given n, the group $\mathcal{BL}_n$ is rather small; our Theorem H provides some evidence for that.

Corollary 7.1 *Let n be a prime power. Then $\mathcal{C}/\mathcal{BL}_n$ has a $\mathbb{Z}$ summand.*

Proof It follows from the proof of Theorem H that $\mathrm{Lef}(\tau_*)\colon\mathcal{C}\to\mathbb{Z}$ is a well-defined nonzero map. Since both $h(\Sigma,\mathfrak{s})$ and the Tristram–Levine signatures are homomorphisms, Theorem A implies that $\mathrm{Lef}(\tau_*)$ is a homomorphism as well. Thus there is a surjection from $\mathcal{C}/\mathcal{BL}_n$ to the image of this homomorphism, which is isomorphic to $\mathbb{Z}$. $\qquad\square$

Remark 7.2 One could alternatively deduce that the Lefschetz number is a homomorphism from the splitting formula of [34] and the additivity of λ_{SW} proved in [35].

Theorem I *Let $K\subset S^3$ be a knot with $\det(K)=1$ and $J'(-1)\neq 0$, where $J_K(t)$ is the Jones polynomial of K. Then, for any $m\geq 1$, the $2m$–fold cyclic branched cover $\Sigma_{2m}(K)$ is not an L–space.*

Proof We may assume without loss of generality that $\Sigma_{2m}(K)$ is a rational homology sphere. Since $\det(K)=1$, $\Sigma_2(K)$ is a homology sphere, so this is automatic when m is a prime power. Suppose $\Sigma_{2m}(K)$ is an L–space and apply the formula of Theorem A to the covering translation τ and its square τ^2. We obtain the formulas

$$h(\Sigma_{2m}(K),\mathfrak{s}) = \frac{1}{8}\sum_{j=1}^{2m-1}\mathrm{sign}_{j/2m}(K,S^3),$$

$$h(\Sigma_{2m}(K),\mathfrak{s}) = m\cdot\lambda(\Sigma_2(K)) + \frac{1}{8}\sum_{j=1}^{m-1}\mathrm{sign}_{j/m}(K,\Sigma_2(K)).$$

Comparing them, we obtain

$$m\cdot\lambda(\Sigma_2(K)) = \frac{1}{8}\sum_{j=1}^{2m-1}\mathrm{sign}_{j/2m}(K,S^3) - \frac{1}{8}\sum_{j=1}^{m-1}\mathrm{sign}_{j/m}(K,\Sigma_2(K)).$$

On the other hand, we have the equality

$$\frac{1}{8} \sum_{j=1}^{2m-1} \text{sign}_{j/2m}(K, S^3) - \frac{1}{8} \sum_{j=1}^{m-1} \text{sign}_{j/m}(K, \Sigma_2(K)) = \frac{m}{8} \text{sign}_{1/2}(K, S^3),$$

which can be proved as follows. By pushing a Seifert surface of K into the interior of the 4–ball and taking the 2–fold and $2m$–fold branched covers, we obtain 4–manifolds W_2 and W_{2m} with

$$\sum_{j=1}^{2m-1} \text{sign}_{j/2m}(K, S^3) = \text{sign}(W_{2m}),$$

$$\sum_{j=1}^{m-1} \text{sign}_{j/m}(K, \Sigma_2(K)) = \text{sign}(W_{2m}) - m \cdot \text{sign}(W_2),$$

$$\text{sign}_{1/2}(K, S^3) = \text{sign}(W_2).$$

This gives the desired formula. We therefore conclude that

$$\lambda(\Sigma_2(K)) = \tfrac{1}{8} \text{sign}_{1/2}(K, S^3).$$

Comparing this with Mullins' theorem [41]

$$\lambda(\Sigma_2(K)) = -\tfrac{1}{12} J'(-1) + \tfrac{1}{8} \text{sign}_{1/2}(K, S^3),$$

we obtain $J'(-1) = 0$, a contradiction. □

Theorem K *Let X be the mapping torus of an orientation-preserving diffeomorphism (not necessary of finite order) of a rational homology sphere. Then, for any choice of spin structure on X, we have*

$$-\lambda_{\text{SW}}(X) = \lambda_{\text{SW}}(-X).$$

Proof Let X be the mapping torus of a finite-order diffeomorphism $\tau: \Sigma \to \Sigma$ of a rational homology sphere Σ. Given a spin structure on X denote by $\mathfrak{s}$ its restriction to Σ and observe that $\tau^*(\mathfrak{s}) = \mathfrak{s}$. Let $-\Sigma$ be the manifold Σ with reversed orientation and denote by $-\mathfrak{s}$ and $-\tau$ the corresponding spin structure and diffeomorphism, respectively. By the splitting theorem of λ_{SW} [34, Theorem A], we have

$$\lambda_{\text{SW}}(X) = -\text{Lef}(\tau_*: \text{HM}^{\text{red}}(\Sigma, \mathfrak{s}) \to \text{HM}^{\text{red}}(\Sigma, \mathfrak{s})) - h(\Sigma, \mathfrak{s}),$$

$$\lambda_{\text{SW}}(-X) = -\text{Lef}((-\tau)_*: \text{HM}^{\text{red}}(-\Sigma, -\mathfrak{s}) \to \text{HM}^{\text{red}}(-\Sigma, -\mathfrak{s})) - h(-\Sigma, -\mathfrak{s}).$$

Since h is a homology cobordism invariant, it vanishes on the manifold $\Sigma \# (-\Sigma)$. It now follows from the additivity of h (see [18, Theorem 3]) that $h(-\Sigma, -\mathfrak{s}) = -h(\Sigma, \mathfrak{s})$.

Therefore, all we need is to check is that

$$\mathrm{Lef}((-\tau)_*) = -\mathrm{Lef}(\tau_*).$$

Lemma 7.3 *There is a duality isomorphism* $\widetilde{\mathrm{HM}}_a(-\Sigma, -\mathfrak{s}) \cong \widehat{\mathrm{HM}}_{1+a}(\Sigma, \mathfrak{s})^*$ *with respect to the canonical mod 2 grading in monopole homology.*

Proof The monopole homology has two canonical gradings: the rational grading $\mathrm{gr}^{\mathbb{Q}}$ and the mod 2 grading $\mathrm{gr}^{(2)}$. Kronheimer and Mrowka [23, Proposition 28.3.4] construct a duality isomorphism $\widetilde{\mathrm{HM}}(-\Sigma, -\mathfrak{s}) \to \widehat{\mathrm{HM}}(\Sigma, \mathfrak{s})^*$ which maps elements of rational grading j to elements of rational grading $-1 - j$. Since the relative rational grading matches modulo 2 the relative mod 2 grading, there is a universal constant $c(\Sigma, \mathfrak{s}) \in \mathbb{Q}/2\mathbb{Z}$ such that $\mathrm{gr}^{\mathbb{Q}} = \mathrm{gr}^{(2)} + c(\Sigma, \mathfrak{s}) \pmod 2$. Therefore, the above duality isomorphism maps elements of mod 2 grading a to elements of mod 2 grading $1 + a + c(\Sigma, \mathfrak{s}) + c(-\Sigma, -\mathfrak{s}) \pmod 2$. The calculation of [34, Lemma 2.6] implies that $c(\Sigma, \mathfrak{s}) + c(-\Sigma, -\mathfrak{s}) = 0 \pmod 2$, thereby completing the argument. □

Recall from [23, (3.4)] that $\mathrm{HM}_a^{\mathrm{red}}(\Sigma, \mathfrak{s}) = \mathrm{im}\, j_{\Sigma,a}$ for the connecting homomorphism $j_{\Sigma,a} \colon \widetilde{\mathrm{HM}}_a(\Sigma, \mathfrak{s}) \to \widehat{\mathrm{HM}}_a(\Sigma, \mathfrak{s})$. Moreover, it follows from the definition of $j_{\Sigma,a}$ that, under the duality isomorphisms

$$\widetilde{\mathrm{HM}}_a(-\Sigma, -\mathfrak{s}) \cong \widehat{\mathrm{HM}}_{1+a}(\Sigma, \mathfrak{s})^* \quad \text{and} \quad \widehat{\mathrm{HM}}_a(-\Sigma, -\mathfrak{s}) \cong \widetilde{\mathrm{HM}}_{1+a}(\Sigma, \mathfrak{s})^*$$

of Lemma 7.3, the map $j_{-\Sigma,a}$ is the dual of the map $j_{\Sigma,1+a}$. As a consequence, we obtain an isomorphism

$$\mathrm{HM}_a^{\mathrm{red}}(-\Sigma, -\mathfrak{s}) \cong \mathrm{HM}_{1+a}^{\mathrm{red}}(\Sigma, \mathfrak{s})^*.$$

With respect to this isomorphism, $(-\tau)_*$ is the dual of τ_*, which implies that

$$\mathrm{Lef}((-\tau)_*) = -\mathrm{Lef}(\tau_*). \qquad \square$$

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Yau Mathematical Sciences Center, Tsinghua University
Beijing, China

Department of Mathematics, Brandeis University
Waltham, MA, United States

Department of Mathematics, University of Miami
Coral Gables, FL, United States

linjian5477@gmail.com, ruberman@brandeis.edu, saveliev@math.miami.edu

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