D. BCD (2021) "On the Thom Conjecture in $\mathbb{CP}^3$,"
International Mathematics Research Notices, Vol. 00, No. 0, pp. 1–14
https://doi.org/10.1093/imrn/rnab343

On the Thom Conjecture in $\mathbb{CP}^3$

Daniel Ruberman[1,*], Marko Slapar[2,3,4] and Sašo Strle[3,4]

[1]Department of Mathematics, MS 050, Brandeis University, Waltham, MA 02454, USA, [2]Faculty of Education, University of Ljubljana, Kardeljeva ploščad 16, 1000 Ljubljana, Slovenia, [3]IMFM, Jadranska 19, 1000 Ljubljana, Slovenia, and [4]Faculty of Mathematics and Physics, University of Ljubljana, Jadranska 19, 1000 Ljubljana, Slovenia

Correspondence to be sent to: e-mail: ruberman@brandeis.edu

What is the simplest smooth simply connected 4-manifold embedded in $\mathbb{CP}^3$ homologous to a degree d hypersurface V_d? A version of this question associated with Thom asks if V_d has the smallest b_2 among all such manifolds. While this is true for degree at most 4, we show that for all $d \geq 5$, there is a manifold M_d in this homology class with $b_2(M_d) < b_2(V_d)$. This contrasts with the Kronheimer–Mrowka solution of the Thom conjecture about surfaces in $\mathbb{CP}^2$ and is similar to results of Freedman for $2n$-manifolds in $\mathbb{CP}^{n+1}$ with n odd and greater than 1.

1 Introduction

A conjecture attributed to René Thom states that a nonsingular algebraic hypersurface V_d of degree d in $\mathbb{CP}^{n+1}$ is the "simplest" representative of its homology class. The notion of complexity of a (real) submanifold M of dimension $2n$ in $\mathbb{CP}^{n+1}$ is motivated by the Lefschetz hyperplane theorem, which implies that the homology and homotopy groups of V_d are determined by the ambient manifold $\mathbb{CP}^{n+1}$ below the middle dimension n. We are looking for manifolds that closely resemble the behavior of algebraic hypersurfaces, so the appropriate class of submanifolds within which to look for least complexity

Received September 25, 2021; Revised November 8, 2021; Accepted November 11, 2021
Communicated by Prof. Simon Donaldson

representatives of a given codimension-2 homology class in $\mathbb{CP}^{n+1}$ is the class of manifolds M for which the relative (homology and) homotopy groups of the pair $(\mathbb{CP}^{n+1}, M)$ are trivial up to the middle dimension. The free parameter is therefore the middle dimensional Betti number, $b_n(M)$.

The conjecture is true for $n = 1$, so in $\mathbb{CP}^2$, which was proved by Kronheimer and Mrowka using Seiberg–Witten theory.

Theorem 1.1 (Kronheimer–Mrowka [9]). A nonsingular algebraic curve of degree $d \in \mathbb{N}$ is a minimal genus smooth surface representing $d[\mathbb{CP}^1] \in H_2(\mathbb{CP}^2; \mathbb{Z})$; the 1st Betti number of such a surface is $d^2 - 3d + 2$.

For larger odd n the conjecture is false as was shown by Freedman [4]. The idea of the proof is to perform ambient surgery on V_d to reduce the middle Betti number. For technical reasons it is necessary to replace the condition that the relative homotopy groups vanish up to the middle dimension with a stronger condition $\pi_k(C, \partial C) = 0$ for $k \leq n$, where C is the closed complement of a regular neighborhood of a submanifold $M^{2n} \subset N^{2n+2}$. When this condition holds for the pair (N, M), we say that M is a *taut* submanifold of N. Such embeddings were studied by Thomas and Wood [14] who, in particular, showed that V_d are taut and also established lower bounds for $b_n(M)$ for a taut representative M of the class of V_d. For $V_d \subset \mathbb{CP}^{2m}$ these lower bounds come from the G-signature theorem, are smaller than $b_n(V_d)$, and are almost attained by Freedman's taut manifolds (moreover, he shows they can be realized by rationally taut submanifolds, which satisfy the same bounds).

Theorem 1.2 (Freedman [4]). For any $m \geq 2$, V_d is not minimal taut in $\mathbb{CP}^{2m}$ for prime d, where $d \neq 2, 3$ for $m = 2$ and $d \neq 2$ for $m = 3$.

This leaves open the case of V_d in $\mathbb{CP}^{2m+1}$ with $m \geq 1$. It seems likely that an adaptation of Freedman's method would establish a result analogous to Theorem 1.2 when $m > 1$, so we concentrate on $m = 1$. Thus, for any positive integer d we study smooth simply connected 4-dimensional submanifolds M of $\mathbb{CP}^3$ representing the homology class $d[\mathbb{CP}^2]$, which also carry the class of $[\mathbb{CP}^1]$ in $H_2(\mathbb{CP}^3)$. We show that, analogous to the higher-dimensional cases, one can find such manifolds M with $b_2(M) < b_2(V_d)$, hence the conjecture does not hold in $\mathbb{CP}^3$. Although we also use ambient surgery we do not need tautness of the embedding due to the special geometric situation in which we perform the construction.

Recall that for a nonsingular V_d the following hold ($\sim$ indicates asymptotic behavior for large d):

- V_d is simply connected;
- $b_2(V_d) = d^3 - 4d^2 + 6d - 2 \sim d^3$;
- $\sigma(V_d) = -d(d^2-4)/3 \sim -d^3/3$;
- V_d is even (spin) for d even, odd for d odd.

For small values of d this yields the following:

d	$b_2(V_d)$	$\sigma(V_d)$	V_d
1	1	1	$\mathbb{CP}^2$
2	2	0	$S^2 \times S^2$
3	7	-5	$\mathbb{CP}^2\#6\overline{\mathbb{CP}^2}$
4	22	-16	K3
5	53	-35	quintic

We show in Proposition 2.1 that the signature and the parity of its intersection form for a 4-dimensional submanifold of interest in $\mathbb{CP}^3$ are determined by the class it represents and that its b_2^+ is at least 1. Inspecting the list above it is then clear that one cannot reduce b_2 in any class with $d \leq 3$. For $d = 4$ the same conclusion follows from the 10/8 Theorem of Furuta [7] and in fact from Donaldson's Theorems B and C [2].

Theorem A. V_d is not minimal in its homology class for $d \geq 5$. There exist simply connected smooth submanifolds M_d of $\mathbb{CP}^3$ homologous to V_d with $b_2(M_d) < b_2(V_d)$. Moreover, for large d we can choose M_d so that $b_2(M_d)$ grows as $3d^3/4$.

Using our construction we can reduce $b_2(V_5)$ by 8, so we obtain a M_5 with $b_2(M_5) = 45$. The smallest $b_2(M_d)$ our method could possibly produce is $\sim d^3/2$, which yields $b_2/|\sigma| \sim 3/2$. In contrast to Freedman's work, which is restricted to prime degrees, our results work for arbitrary $d \geq 5$.

As in the work of Freedman and Matsumoto, the proof of the theorem relies on ambient surgery to reduce the 2nd Betti number of the manifold. However, we do not know how to directly implement the approach in [4, 10], so we take a somewhat different route. Using the results of Baader et al. [1] we identify a large subgroup of $H_2(V_d)$ on which the intersection pairing is hyperbolic. If these classes were represented by smoothly embedded spheres, then they would be candidates for performing surgery on V_d to reduce b_2. However, it follows from Donaldson's work (see [6, Corollary 6.4.2]) that no (non-trivial) homology class of self-intersection 0 in V_d is represented by a smoothly embedded sphere. On the other hand, Wall [15, 16] showed that these classes can be represented by spheres after stabilization. Using this, we can perform ambient surgery to remove a part of the 2nd homology while preserving the characteristic properties of the submanifold. If in place of smooth embeddings one was content with a topologically locally flat embedding of M_d in $\mathbb{CP}^3$, the proof is somewhat simpler. See the remarks on the proof at the conclusion of the paper.

2 Basic Properties

Proposition 2.1. Let d be a positive integer and $M \subset \mathbb{CP}^3$ be a smooth simply connected 4-dimensional submanifold representing the homology class $d[\mathbb{CP}^2]$ and such that $H_2(M) \to H_2(\mathbb{CP}^3)$ is onto. Then $\sigma(M) = -d(d^2-4)/3$, M is spin iff d is even, and $b_2^+(M) \geq 1$.

Proof. By the signature theorem, the signature of M is determined by its 1st Pontryagin class. This, in turn, is determined by the ambient manifold $\mathbb{CP}^3$ and the homology class of M, so it agrees with the signature of V_d.

The 2nd Stiefel–Whitney class of the normal bundle of M in $\mathbb{CP}^3$ factors through $H^2(\mathbb{CP}^3;\mathbb{Z}/2)$ and hence is equal to dx, where x is the image of the generator of $H^2(\mathbb{CP}^3;\mathbb{Z}/2)$ in $H^2(M;\mathbb{Z}/2)$. Since $\mathbb{CP}^3$ is spin, it follows that $w_2(M) = dx$.

Because the class of M is a positive multiple of $[\mathbb{CP}^2]$ and the inclusion of M into $\mathbb{CP}^3$ induces a surjection on H_2, $H_2(M)$ contains a class of positive square. ■

3 A Model Manifold

We first choose a smooth algebraic hypersurface V_d of degree d in $\mathbb{CP}^3$ that intersects a 6-ball in a submanifold carrying a large part of the 2nd homology. Recall we are only interested in $d > 4$.

Proposition 3.1. V_d can be chosen so that its intersection F_d with a 6-ball B can be isotoped (rel boundary) to the boundary of B. Moreover, F_d is the d-fold branched cover of the 4-ball branched along a pushed-in Seifert surface Σ_d for the $(d-1,d)$ torus knot and $b_2(V_d) = b_2(F_d) + d$.

Proof. Let W_d be the singular variety representing the codimension-2 class of multiplicity d in $\mathbb{CP}^3$ given by the equation

$$z_0 z_1^{d-1} + z_2^d = z_3^d.$$

Hence, $[1 : 0 : 0 : 0] \in W_d$ is the unique singular point (for $d > 2$); let B be a small ball about the singularity so that $W_d \cap B$ is the cone on $W_d \cap \partial B$. Clearly, W_d is the d-fold branched cover of $\mathbb{CP}^2$ with branch set a singular sphere with a unique singular point whose link of singularity is the $(d-1, d)$ torus knot $T_{d-1,d}$. To obtain a smooth representative V_d of the same homology class, we choose a nearby nonsingular surface, for example, the one given by

$$z_0 z_1^{d-1} + z_2^d = \varepsilon z_0^d + z_3^d$$

for a small enough $\varepsilon \neq 0$. In V_d the neighborhood of the singularity $W_d \cap B$ is replaced by the Milnor fiber F_d, which can be thought of as the branched cover of B^4 with branch set a pushed-in Seifert surface Σ_d for $T_{d-1,d}$. A Euler characteristic computation shows that $b_2(V_d) = b_2(F_d) + d$. Moreover, the Milnor fiber F_d can be isotoped into the boundary sphere of B while fixing its boundary. ■

Next we show there exists a large subgroup of $H_2(F_d)$ (all homology groups from now on have integer coefficients) on which the intersection pairing is hyperbolic. The intersection form of F_d is determined by the Seifert form θ_d of the Seifert surface Σ_d. Moreover, θ_d also determines the linking form Θ_d on $H_2(F_d) \cong H_1(\Sigma_g) \otimes \mathbb{Z}^{d-1}$ for the embedding of F_d into $\partial B = S^5$; indeed, $\Theta_d = \theta_d \otimes \Lambda_{d-1}$ [3], where Λ_k is the $k \times k$ matrix of the form

$$\Lambda_k = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & -1 \\ 0 & \cdots & 0 & 0 & 1 \end{bmatrix}.$$

Even though the smooth slice genus of a torus knot is equal to its genus, the same is not true of its topological (locally flat) slice genus as was first demonstrated by Rudolph [12]. The main tool in the construction is Freedman's result that an Alexander polynomial 1 knot is topologically slice. A systematic study of the topological slice genus of torus knots was conducted by Baader et al. [1]. They construct subsurfaces of Seifert surfaces whose boundaries are Alexander polynomial 1 knots. We only need the following property of the Seifert form.

Theorem 3.2 ([1]). $H_1(\Sigma_d)$ contains a subgroup G_d of rank $2r_d \sim d^2/4$ such that the restriction of the Seifert form θ_d to G_d is of the following form, consisting of four $r_d \times r_d$

blocks:

$$\begin{bmatrix} 0 & I + U_d \\ L_d & * \end{bmatrix},$$

where U_d and L_d are strictly upper- and lower-triangular matrices, respectively.

Corollary 3.3. The restriction of the Seifert form Θ_d for F_d to the subgroup $\widehat{G}_d = G_d \otimes \mathbb{Z}^{d-1}$ of $H_2(F_d)$ has the same form as θ_d in the previous theorem with the blocks of size $\widehat{r}_d = r_d(d-1)$. Hence, the restriction of the intersection form of F_d to $\widehat{G}_d$ is equivalent to $\oplus \widehat{r}_d H$, where $H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ denotes the hyperbolic form. The rank of $\widehat{G}_d$ for large d behaves as $d^3/4$.

Proof. Let $(x_i)_i$ be the generators of $\widehat{G}_d$ corresponding to the first half of the generators for G_d relative to which the Seifert form is given by the matrix in the above theorem and let $(x_i')_i$ be the generators corresponding to the second half. It follows from the structure of the matrix Λ_{d-1} that Θ_d has the same form as θ_d so it, in particular, vanishes on the subgroup generated by the $(x_i)_i$. Since the intersection form of F_d is given by $\Theta_d + \Theta_d^\top$, it follows that $x_i \cdot x_j = 0$ and $x_i \cdot x_i' = 1$ for all i and j. To make all the other pairings vanish we inductively change the basis elements (x_i') by adding to them linear combinations of x_j for $j \leq i$ and x_j' for $j < i$. ■

4 Spherical Classes

In order to reduce the rank of $H_2(F_d)$, we would like to show that some set of generators for the subgroup $\widehat{G}_d$ of $H_2(F_d)$ can be represented by embedded spheres in V_d and that in fact a regular neighborhood of representatives for a pair of generators giving an H summand as above is diffeomorphic to a punctured $S^2 \times S^2$ and the spheres corresponding to different H summands are disjoint. In general the classes in $\widehat{G}_d$ may not be represented by embedded spheres (though they are of course spherical) but by Wall's stable diffeomorphism results they are after stabilizing.

Theorem 4.1 (Wall [15, 16]). Let M and N be smooth simply connected closed 4-manifolds with isomorphic intersection forms. Then the following hold:

(1) for all large enough $\ell > 0$, the stabilized manifolds $M\#\ell(S^2 \times S^2)$ and $N\#\ell(S^2 \times S^2)$ are diffeomorphic;

(2) if the intersection form of M is indefinite, any automorphism of the intersection form of $M\#S^2 \times S^2$ is induced by a diffeomorphism.

Choose a standard model manifold realizing the intersection form of V_d:

$$M_d = \frac{b_2+\sigma}{2} S^2 \times S^2 \# |\sigma| \, \overline{\mathbb{CP}^2} \quad \text{for } d > 1 \text{ odd},$$
$$M_d = \frac{8b_2+11\sigma}{16} S^2 \times S^2 \# \frac{|\sigma|}{16} K3 \quad \text{for } d \text{ even},$$

where $b_2 = b_2(V_d)$ and $\sigma = \sigma(V_d)$. Fix ℓ so that V_d and M_d become diffeomorphic after ℓ stabilizations. We can realize this stabilization of V_d in $\mathbb{CP}^3$ by internal connected sum of F_d with ℓ trivial copies of $S^2 \times S^2 \subset S^5 = \partial B$ (each contained in its own 5-disk); denote the stabilized F_d and V_d by F_d^s and V_d^s, respectively. We add to $\widehat{G}_d$ the stabilization classes thus obtaining $\widehat{G}_d^s \leq H_2(V_d^s)$ with $H_2(V_d^s)/\widehat{G}_d^s \cong H_2(V_d)/\widehat{G}_d$.

Denote by h_d the number of $S^2 \times S^2$ summands in M_d. Note that h_d for d odd grows as $d^3/3$ whereas for d even as $13d^3/48$, so in any case faster than $\widehat{r}_d \sim d^3/8$. The comparison for small values of d is given in the table below where the data for r_d comes from [1, Table 1].

d	5	6	7	8	9
r_d	1	2	4	$[5, 6]$	$[6, 9]$
$\widehat{r}_d$	4	10	24	$[35, 42]$	$[48, 72]$
h_d	9	9	41	41	113

We will assume in what follows that $\widehat{r}_d \leq h_d$, which is clearly true for large d. For those small values of d for which this is not the case we replace $\widehat{G}_d$ by one of its subgroups satisfying the condition.

Proposition 4.2. The restriction of the intersection pairing of V_d^s to $\widehat{G}_d^s$ is equivalent to the sum of hyperbolic forms H. The classes in $\widehat{G}_d^s$ can be represented by smoothly embedded spheres in V_d^s so that for each summand H the corresponding representatives intersect geometrically once and the spheres corresponding to different H summands are disjoint.

Proof. The first claim follows from the construction. By the choice of ℓ, $V_d^s = V_d\#\ell(S^2 \times S^2)$ is diffeomorphic to the stabilization $M_d^s = M_d\#\ell(S^2 \times S^2)$. We can choose an isomorphism between the intersection pairings of V_d^s and M_d^s that maps the generators of the subgroup $\widehat{G}^s$ into the generators of the subgroup supported by the sum of $S^2 \times S^2$'s. Since this isomorphism is by Theorem 4.1 induced by a diffeomorphism, the second claim follows. ■

5 Ambient Surgery

In order to reduce the 2nd Betti number of V_d^s we wish to perform ambient surgery along the spheres guaranteed by Proposition 4.2. Let Σ_i be the spheres representing the first half of the generators of $\widehat{G}^s$ on which the linking pairing Θ_d^s vanishes identically (so this collection of spheres contains a representative for one of the generators for each H summand of the restriction of the intersection form to $\widehat{G}^s$). If Σ_i is contained in $F_d^s \subset \partial B$, then it bounds an embedded disk D_i in the 6-ball B and the normal disk bundle of D_i contains an embedded 5-dimensional 3-handle with core D_i. The vanishing of the linking pairing guarantees that these handles may be chosen to be disjoint. Since we do not have the control over the action of the diffeomorphism in Wall's stabilization theorem, the spheres might not be contained in F_d. Our main lemma shows that we can arrive at the same conclusion.

Lemma 5.1. Let $\Sigma_i \subset V_d^s$ be the 2-spheres described above. Then there exist pairwise disjoint smoothly embedded 3-disks $D_i \subset \mathbb{CP}^3$ with $D_i \cap V_d^s = \Sigma_i$. Moreover, the disks D_i are not tangent to V_d^s.

Proof. Denote by $x_i \in H_2(V_d^s)$ the homology class of Σ_i. Since this class comes from F_d^s it may be represented by an immersed sphere $\Sigma_i^1 \subset F_d^s$ with transverse double points. Then Σ_i^1 and $\Sigma_i = \Sigma_i^0$ are homotopic in V_d^s (since it is simply connected). According to [8, Theorem 8.3], this homotopy may be replaced by a smooth regular homotopy $\varphi_i\colon S^2 \times I \to V_d^s$ (i.e., a homotopy of immersions) if the normal bundle of the immersed sphere Σ_i^1 is trivial. Since the class x_i has square 0, the latter condition is equivalent to Σ_i^1 having the same number of positive and negative (transverse) double points. This condition can be satisfied since double points of either sign may be added locally to Σ_i^1 by replacing a disk with the trace of a homotopy of arcs in $\mathbb{R}^3$ obtained by the sequence of a 1st Reidemeister move, followed by a crossing change and another 1st Reidemeister move. (See [13, Figure 2] for a picture of this process.) We may further assume that the regular homotopy is in general position, so it is a sequence of isotopies, finger moves, and Whitney moves [5, §1.6]. The spheres $\Sigma_i^t = \varphi_i(S^2 \times \{t\})$ for $t \in I$ then have transverse double points with the exception of finitely many points; each of these is either the first

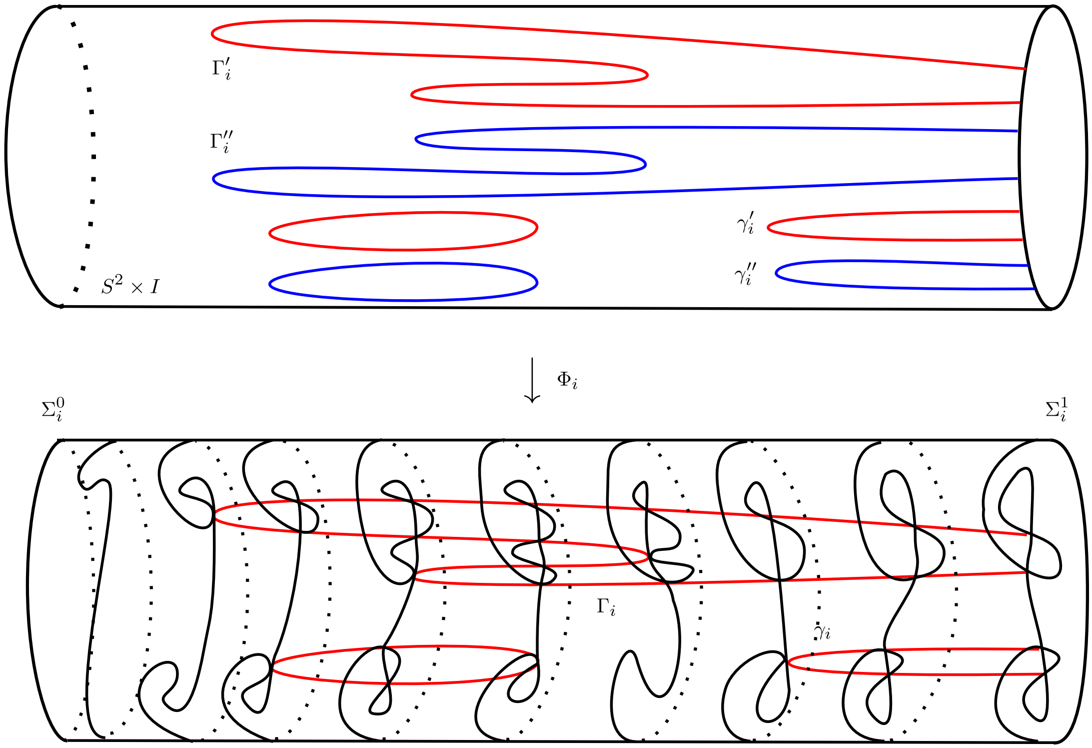

Fig. 1. Regular homotopy with separated stages and arcs of double points.

point of self-intersection for a finger move or the last point of self-intersection for a Whitney move, where the sphere is tangent to itself. Let Γ_i be the set of double points of immersed spheres for the map

$$\Phi_i\colon S^2 \times I \to V_d^s \times I, \quad \Phi_i(x,t) = (\varphi_i(x,t), t).$$

Note that Γ_i is the union of properly embedded arcs (with endpoints in Σ_i^1) and circles; the preimage of Γ_i in $S^2 \times I$ consists of two copies of Γ_i, written $\Gamma_i' \sqcup \Gamma_i''$.

Further, we may assume that the regular homotopies corresponding to different spheres are in general position. This implies that for any time t at most two of the spheres Σ_i^t intersect in the same point and this point is not a double point of one of the spheres. The intersections of different spheres are transverse except at tangencies corresponding to finger and Whitney moves. Then Δ_{ij}, the intersection of the images of Φ_i and Φ_j, is the union of properly embedded arcs (with endpoints in $\Sigma_i^1 \cap \Sigma_j^1$) and circles. The preimages $\Delta_{ij}^i = \Phi_i^{-1}(\Delta_{ij})$ and $\Delta_{ij}^j = \Phi_j^{-1}(\Delta_{ij}) \subset S^2 \times I$ are two copies of Δ_{ij}.

We will push the traces of the homotopies φ_i into $\mathbb{CP}^3$ (more precisely, into a tubular neighborhood of V_d^s) separating their stages and resolving the (self-)intersections of the immersed spheres thus obtaining disjoint embedded annuli W_i connecting Σ_i^0 to a push-off of Σ_i^1. Equip the normal bundle ν of $V_d^s \subset \mathbb{CP}^3$ with a Riemannian metric; we may assume that the metric over F_d^s is induced from the metric on the 6-ball B, which we identify with the ball of radius 2 in $\mathbb{R}^6$. By rescaling the metric (by a constant factor) we may assume that the unit disk bundle of ν is identified with a tubular neighborhood of V_d^s; we will use this identification implicitly in what follows.

We fix a trivialization of the pull-back of ν via φ_i. This bundle is the pull-back of the trivial bundle over $S^2 \times \{1\}$, and there is a particular choice of trivialization over this sphere given by the vector fields E_1^i, the pull-back of the inner normal to S^5 in B^6, and E_2^i, the pull-back of the normal vector field to F_d^s in S^5. Choosing a trivialization of the bundle over $S^2 \times I$, we extend (E_1^i, E_2^i) to orthonormal trivializing sections (E_1^i, E_2^i) of the whole bundle. Let $\lambda\colon [0,1] \to [0,1]$ be a smooth increasing surjective function that is constant in some neighborhoods of the endpoints. Then $\psi_i\colon S^2 \times I \to \mathbb{CP}^3$, given by

$$(x,t) \mapsto tE_1^i(\varphi_i(x,\lambda(t))),$$

is an embedding of the image of Φ_i in $\mathbb{CP}^3$ (with collars added at each end). Denote the image of ψ_i by Z_i. Note that φ_i factors through Z_i, where Z_i maps to V_d^s by the projection. In particular, the pull-back of ν via φ_i factors through its pull-back to Z_i. So for any component γ_i of $\Gamma_i \subset Z_i$ we may identify the pull-back of ν to γ_i with its pull-back to either component $\gamma_i' \subset \Gamma_i'$ or $\gamma_i'' \subset \Gamma_i''$ of its preimage.

In order to get embedded annuli W_i we first need to remove the double points of immersed spheres. Note that over any corresponding pair of components $\gamma_i' \sqcup \gamma_i''$ in $\Gamma_i' \sqcup \Gamma_i'' \subset S^2 \times I$ that map to $\gamma_i \subset \Gamma_i \subset Z_i$, the two trivializations of the pull-back of the normal bundle ν to γ_i determined by (E_1^i, E_2^i) restricted to either γ_i' or γ_i'' are homotopic as any such component is null-homotopic in $S^2 \times I$. We now change E_1^i over γ_i'' to agree with the restriction of E_1^i to γ_i' rotated by a small angle $\delta > 0$ in the direction of $E_2^i|\gamma_i'$. Choose small pairwise disjoint compact regular neighborhoods $K_i = K_i' \sqcup K_i''$ for $\Gamma_i' \sqcup \Gamma_i''$ and $L_{ij}^i \sqcup L_{ij}^j$ for $\Delta_{ij}^i \sqcup \Delta_{ij}^j$. Using $(E_1^i, E_2^i)|K_i'$ to trivialize the normal bundle over K_i, we choose the fiberwise universal cover of the corresponding circle bundle in which $E_1^i|K_i'$ corresponds to the zero and $E_2^i|K_i'$ to a positive angle. Then the lift of $E_1^i|K_i''$ may be smoothly spliced with the constant section δ and then pushed down into the circle bundle to give the new section $E_1^i|K_i''$; then rotating $E_2^i|K_i''$ appropriately we obtain an orthonormal frame. In fact, when γ_i is an arc, we complete γ_i'' to a circle by

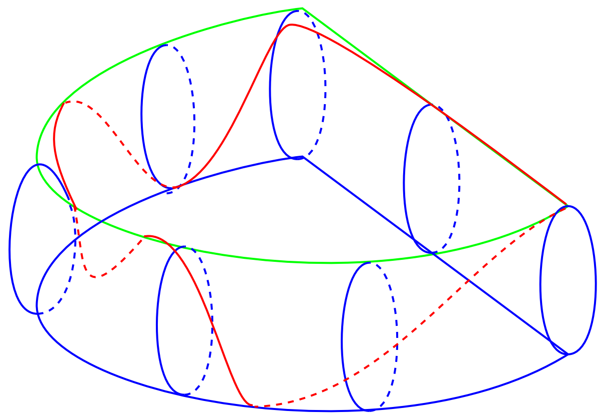

Fig. 2. Comparison of the two trivializations of the pull-back of ν to γ_i, given by the sections $E_1^i|\gamma_i'$ (green) and $E_1^i|\gamma_i''$ (red). In the picture, γ_i is assumed to be an arc and is completed to a circle by an arc over which the two sections agree.

adding to it an arc in $S^2 \times \{1\}$. Since over $S^2 \times \{1\}$ the section E_1^i is determined as the inner normal to the boundary of B, we choose the lift of $E_1^i|K_i''$ over this arc to be zero. This shows that the double points of Σ_i^1 are removed by a small homotopy inside the ball B.

To remove the intersections between different spheres we just repeat the same procedure with any Δ_{ij}, where we assume $i < j$, by changing the section E_1^j over L_{ij}^j. Denote the resulting embeddings obtained in this way from ψ_i by $\Psi_i \colon S^2 \times I \to \mathbb{CP}^3$; then one boundary component of $W_i = \Psi_i(S^2 \times I)$ is Σ_i and the other is the push-in of Σ_i^1, which we denote $\widetilde{\Sigma}_i = \Psi_i(S^2 \times \{1\})$. Note that the spheres $\widetilde{\Sigma}_i$ are essentially contained in a 5-sphere S concentric with the boundary of the ball B; they only deviate from S in small neighborhoods of double points and intersection points between different spheres (more precisely, over the images of $K_i'' \cap \widetilde{\Sigma}_i$, and over $L_{ij}^j \cap \widetilde{\Sigma}_j$ for $i < j$). But as noted above, the removal of intersection points in Σ_i^1 is realized by a small homotopy. Hence, the projection of $\widetilde{\Sigma}_i$ into S along the normal vector field is a diffeomorphism and we may and will assume that $\widetilde{\Sigma}_i$ is contained in S. Then $\widetilde{\Sigma}_i$ bounds a properly immersed 3-disk D_i in the ball B' bounded by S. Assuming D_i is in general position, it may have transverse self-intersections, but pairs of double points in D_i of opposite sign can be canceled using the Whitney trick. Note that the number of double points of either sign may be increased by adding kinks (analogous to 1st Reidemeister move) into $\widetilde{\Sigma}_i$. Thus, we may assume that D_i is embedded.

Since $\Theta^s_d(\Sigma^1_i, \Sigma^1_j) = 0$ and $\widetilde{\Sigma}_i$ is just a push-in of Σ^1_i into a concentric sphere, the linking number of $\widetilde{\Sigma}_i$ and $\widetilde{\Sigma}_j$ (where the latter can be considered as a perturbation of the normal push-off of Σ^1_j) is trivial. Hence, the intersection number of D_i and D_j is zero for all $i \neq j$, so we may assume that they are geometrically disjoint (by using the Whitney trick). ■

Theorem 5.2. The homology class of V_d in $\mathbb{CP}^3$ is represented by a simply connected manifold N_d with $H_2(N_d) \cong H_2(V_d)/\widehat{G}$.

Proof. We first show that the disk D_i may be thickened to a 5-dimensional 3-handle h_i in $\mathbb{CP}^3$ with the attaching region contained in V^s_d and whose attaching sphere is equal to Σ_i. The normal bundle of Σ_i in V^s_d is trivial, so its normal disk bundle in $\mathbb{CP}^3$ admits a splitting $\Sigma_i \times B^2 \times B^2$, where the first B^2 corresponds to the normal directions in V^s_d, and the second corresponds to the restriction of the normal bundle ν of $V^s_d \subset \mathbb{CP}^3$ to Σ_i. The latter is trivialized by (E^1_i, E^2_i) and the normal disk bundle of D_i over Σ_i is given by $\Sigma_i \times B^2 \times B^1E^2_i$. This trivialization extends over the normal bundle of D_i in $\mathbb{CP}^3$ since $\pi_2(\mathrm{GL}_3\mathbb{R})$ is trivial. The required handle h_i is $D_i \times B^2 \times 0$.

Using the handles h_i we perform ambient surgery on V^s_d along the Σ_i to obtain a manifold N_d, homologous to V^s_d and hence to V_d. Clearly, $H_2(N_d)$ is isomorphic to $H_2(V^s_d)/\widehat{G}^s \cong H_2(V_d)/\widehat{G}$ since surgery on Σ_i kills also its dual class.

That N_d is simply connected follows since the fundamental group of the complement of Σ_i is normally generated by its meridian which is trivial in N_d, because the dual class to Σ_i is also represented by a sphere. ■

The final question to address is whether the manifolds N_d are taut in $\mathbb{CP}^3$. We show below that $\pi_k(\mathbb{CP}^3, N_d)$ is trivial for $k \leq 2$. In fact, it also follows by general position arguments that $\pi_1(C, \partial C)$ is trivial, where C is the closed complement of a tubular neighborhood of N_d.

Proposition 5.3. The pair $(\mathbb{CP}^3, N_d)$ is 2-connected.

Proof. Since $\mathbb{CP}^3$ and N_d are simply connected, we only need to verify that the inclusion induced homomorphism is surjective on π_2 or equivalently on H_2. Since V_d is taut, so is the stabilized manifold V^s_d (by the argument as in the previous sentence). Hence, the generator $x \in H_2(\mathbb{CP}^3)$ is the image of an element $\widetilde{x} \in H_2(V^s_d)$. Suppose now we do the surgery on a sphere Σ_i representing the class $x_i \in H_2(V^s_d)$. Let $y_i \in H_2(V^s_d)$ be the homology class of the dual sphere to Σ_i. Then x_i has trivial algebraic intersection with the class

$$\widetilde{x}' := \widetilde{x} - (\widetilde{x} \cdot x_i) y_i.$$

Since y_i is supported by the Milnor fiber $F_d \subset B$, it maps to the trivial class in $H_2(\mathbb{CP}^3)$, so the surgery preserves surjectivity. ■

In general, the ambient surgery construction may destroy tautness, but based on a theorem of Quinn [11] as quoted by Freedman [4, Theorem 2.5], it seems that one can re-embed N_d as a taut submanifold.

Remarks on the proof. The overall strategy used to prove Theorem A is similar to that in the work of Freedman and Matsumoto, but with an important difference. In our work and also in [4, 10] an algebraic form on a subspace of the middle homology is an obstruction to doing ambient surgery. (For us it is essentially the Seifert form, whereas [4, 10] use a Wall-type form denoted (λ, μ).) However, the technique in [4, 10], applied in our setting, would be to immerse a 3-disk with boundary on V_d, and use the vanishing of (λ, μ) to push the singularities out to the boundary 2-spheres; these would, in principle, be removed by an application of the Whitney trick. Since the Whitney trick does not apply in dimension 4, we modified the procedure to get embedded 2-spheres (after stabilization) and then remove the singularities of the 3-handles by the Whitney trick in dimension 6.

In another direction, the topological version of Theorem A, in which one demands only that M_d have a locally flat topological embedding, follows directly from the smooth case. On the other hand, the proof in the topological case is easier, as the classes we want to surger would be represented by embedded spheres (with embedded duals) in F_d. In this setting, the 5-dimensional 3-handles needed for the ambient surgeries will lie in the 6-ball.

Funding

This work was partially supported by NSF [DMS-1811111 and DMS-1952755 to D.R.]; Slovenian Research Agency (ARRS) [P1-0291 and J1-9104 to M.S.]; and Slovenian Research Agency (ARRS) [P1-0288 to S.S.].

References

[1] Baader, S., P. Feller, L. Lewark, and L. Liechti. “On the topological 4-genus of torus knots.” *Trans. Amer. Math. Soc.* 370 (2018): 2639–56.

[2] Donaldson, S. K. “Connections, cohomology and the intersection forms of 4-manifolds.” *J. Differential Geom.* 24 (1986): 275–341.

[3] Durfee, A. and L. Kauffman. “Periodicity of branched cyclic covers.” *Math. Ann.* 218 (1975): 15–174.

[4] Freedman, M. H. “Surgery on codimension 2 submanifolds.” *Memoirs of the AMS* 12, no. 191 (1977) iv+93 pp.

[5] Freedman, M. H. and F. Quinn. *Topology of 4-Manifolds*. Princeton Mathematical Series 39. Princeton: Princeton University Press, 1990.

[6] Friedman, R. and J. W. Morgan. *Smooth Four-Manifolds and Complex Surfaces*. Springer: Berlin, Heidelberg, 1994.

[7] Furuta, M. “Monopole equation and the $\frac{11}{8}$-conjecture.” *Math. Res. Lett.* 8 (2001): 279–91.

[8] Hirsch, M. W. “Immersions of manifolds.” *Trans. Amer. Math. Soc.* 93 (1959): 242–76.

[9] Kronheimer, P. B. and T. S. Mrowka. “The genus of embedded surfaces in the projective plane.” *Math. Res. Lett.* 1 (1994): 797–808.

[10] Matsumoto, Y. “Knot cobordism groups and surgery in codimension two.” *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* 20 (1973): 253–317.

[11] Quinn, F. “Almost canonical inverse images.” *Comment. Math. Helv.* 49 (1974): 168–74.

[12] Rudolph, L. “Some topologically locally-flat surfaces in the complex projective plane.” *Comment. Math. Helv.* 59 (1984): 592–9.

[13] Schwartz, H. R. “A note on the complexity of h–cobordisms.” *Algebraic Geom. Topol.* 20 (2020): 3313–27.

[14] Thomas, E. and J. Wood. “On manifolds representing homology classes in codimension 2.” *Invent. Math.* 25 (1974): 63–89.

[15] Wall, C. T. C. “Diffeomorphisms of 4-manifolds.” *J. Lond. Math. Soc.* 39 (1964): 131–40.

[16] Wall, C. T. C. “On simply-connected 4-manifolds.” *J. Lond. Math. Soc.* 39 (1964): 141–9.