



WAVE TURBULENCE AND COLLECTIVE BEHAVIOR MODELS FOR WAVE EQUATIONS WITH SHORT- AND LONG-RANGE INTERACTIONS

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Dedicated to the memory of Professor Roland Glowinski

Abstract. In this paper, we discuss a situation, which could lead to both wave turbulence and collective behavior kinetic equations. The wave turbulence kinetic models appear in the kinetic limit when the wave equations have local differential operators. Viewing wave equations on the lattice as chains of anharmonic oscillators and replacing the local differential operators (short-range interactions) by non-local ones (long-range interactions), we arrive at a new Vlasov-type kinetic model in the mean field limit under the molecular chaos assumption reminiscent of models for collective behavior in which anharmonic oscillators replace individual particles.

Keywords. Collective behavior; Kinetic equation, Wave turbulence.

1. INTRODUCTION

Having the origin in the works of Peierls [71, 72], Hasselmann [50, 51], Benney-Saffman-Newell [11, 12], and Zakharov [88], wave turbulence (WT) theory describes the dynamics of weakly nonlinear and dispersive waves (classical or non-classical) out of thermal equilibrium. Even though wave fields describing the processes of random wave interactions in nature are enormously diverse, a common mathematical framework can be used to model the dynamics of spectral energy transfer in both quantum or classical wave systems. In this mathematical framework, the probability density functions associated with weakly nonlinear wave interactions are solutions of wave kinetic (WK) equations. Over the years, WK equations have been shown to play important roles in a vast range of physical applications, as discussed in the books [67, 88]. We also mention closely related kinetic models developed when the interest focuses in the interaction of particles and oscillators; see, e.g., [32].

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In addition, since the realization of Bose-Einstein condensation (BEC) in trapped atomic vapors of [6, 17, 30], a period of intense theoretical and experimental research has been initiated. A theoretical Quantum Kinetic (QK) theory, which takes into account the coupled non-equilibrium dynamics of both the thermal cloud of the Bose gas and the BEC under investigation, is needed to support the experimental results. Although being used to describe different physical phenomena, QK kinetic equations are quite similar with WT ones [73, 84, 85]. During the last few years, there has been a growing interest in rigorously understanding those kinetic equations. Starting with the pioneering work of Lukkarinen and Spohn [63], there have been a lot of recent works in rigorously deriving WK equations (see, e.g., [5, 18, 19, 25, 26, 34, 35, 38, 39, 40, 41, 79] and the references therein). The analysis of WK and QK equations is also a topic of current interest. We list here an incomplete list and refer to the references therein for more detailed descriptions of the literature [2, 3, 7, 8, 9, 20, 31, 42, 43, 44, 45, 46, 47, 54, 60, 61, 62, 68, 69, 76, 77, 78, 82, 83].

Collective behavior of self-propelled particles such as swarming of bacteria, schooling of fishes, flocking of birds and mobile agents, appears in many contexts [14, 27, 28, 29, 37, 53, 65, 70, 86, 89]. How and when clusters emerge, and what type of rules of engagement in influences clusters are among the questions that have been attracted the attention of scientists for decades. Over the last years, there have been growing interests in the mathematical community in studying those models rigorously. We list here only a few of those works and refer the readers to the references therein as the list is quite incomplete [1, 10, 15, 21, 22, 33, 36, 48, 49, 52, 55, 58, 64, 66, 75, 80, 81].

In this paper, we discuss a connection between wave turbulence and collective behavior kinetic models. Starting from the weakly nonlinear wave equation on the lattice, it has been showed [63, 79] that the wave kinetic equations can be derived rigorously, under suitable assumptions on the randomization of the initial condition and the wave equations. This procedure is summarized in Section 2 for a wave equation with a quadratic nonlinearity and the kinetic equation under consideration is the 3-wave kinetic equation (2.7). However, by using other type of nonlinearities, we could arrive at different wave kinetic equations, with the same procedure. It is well-known in the physical community that chains of anharmonic oscillators, such as the Fermi-Pasta-Ulam-Tsingou (FPTU), also exhibit collective behaviors, if the interactions are long-range (see for instance [23, 24]). As wave equations on the lattice could also be viewed as chains of anharmonic oscillators, by replacing the local differential operators (for example, the Laplace or Biharmonic operators) in the wave equations by operators that describe long-range interactions (for example, the fractional Laplace operators), we could expect to obtain models that exhibit collective behaviors. One of the key difference between our chains of anharmonic oscillators and models for consensus, flocking and swarming [29, 28, 49] is that, in our case, the system under consideration will need to “label” the location of the oscillator in the lattice. This is done by adding a new kinetic variable in the density distribution of anharmonic oscillators representing such location.

In Section 3, we derive formally three Vlasov-type kinetic equations (3.4), starting from a wave equation whose differential operator is a fractional Laplacian. Here, the density g is not only a function of the position r (of the oscillation), the velocity v (of the oscillation), the time t variables but also of an additional continuous variable x , which “labels” the location of the anharmonic oscillator in the lattice. This framework is reminiscent of polyatomic and

multicomponent models that add a kinetic variable to differentiate species, see for example [16] and references therein.

Let us mention, however, that the concept of “label” has been previously introduced in consensus models in the work of Biccari, Ko and Zuazua [13], which considers “networked consensus models” and has an inspiration from the previous work of Kawamura [56] on the nonlocal Kuramoto-Sakaguchi equation, where the “label” is indeed as a location vector. A different form of the Kuramoto-Sakaguchi equation is used in [4], where the distribution function f depends on the state θ and the natural frequency Ω , which represents the “label”.

These type of Vlasov-type kinetic equations are recent and increasingly important in the literature and their mathematical properties are open for investigation. Due to the scope of our paper, in Section 3, we only focus on an example of a linear wave equation.

In considering collective behavior, we are reminded of the surprising recurrence result in FPUT, that countered the expected thermalization. It is perhaps the nearest neighbor coupling (i.e. local interaction as in classical random walk) combined with nonlinearity that triggers this behavior. Alternatively, in the canonical model for global coupling, the Kuramoto model [59], at sufficiently high coupling strength, collective synchronous behavior emerges, which overcomes the expected deviations from the natural frequency of the ideally identical oscillators. The question is, if these observations are representative in our newly obtained Vlasov-type kinetic equations. A perhaps subtle but important difference between the Kuramoto model and wave-like and corresponding Schrödinger-like models, is that in the first one the state variable referred to as an angle is real, whereas in particular in applications on electromagnetism and quantum mechanics, in the second case the state variable is complex. In this second case recent research in quantum mechanics and photonics suggests the importance of long range (global) coupling of for example photonic resonators of fiber amplifiers, to enhance coherence. In fact, it is coherence that one views as the order parameter than in the Kuramoto models measures the degree of synchronization.

2. WAVE TURBULENCE KINETIC MODELS FOR DISCRETE NONLINEAR WAVE EQUATIONS WITH SHORT-RANGE INTERACTIONS

Let us first start with a nonlinear wave equation, with a quadratic nonlinearity. However, our discussion could be extended to cubic and higher order nonlinearities

$$\begin{aligned} \frac{\partial^2 \psi}{\partial t^2}(x, t) + \mathcal{L}\psi(x, t) + \lambda \psi^2(x, t) &= 0, \\ \psi(x, 0) = \psi_0(x), \quad \frac{\partial \psi}{\partial t}(x, 0) &= \psi_1(x), \end{aligned}$$

for x being on the torus $[0, 1]^d$, $t \in \mathbb{R}_+$, and λ is a small constant describing the smallness of the nonlinearity. We suppose that the interactions are short-range, which is, the operator \mathcal{L} is a standard local differential operator, for instance, when $\mathcal{L} = \Delta$, we obtain the Klein-Gordon equation and when $\mathcal{L} = -\Delta^2$, we obtain the beam wave equation. Similar with [79], we introduce the finite volume mesh, namely,

$$\Lambda = \Lambda(D) = \left\{ 0, \frac{1}{2D+1}, \dots, \frac{2D}{2D+1} \right\}^d,$$

for some constant $D \in \mathbb{N}$. As we will work on the Fourier transform, we define the mesh size of the frequency space to be $h = \frac{1}{2D+1}$. We follow [79] and introduce discretized equation

$$\begin{aligned} \partial_{tt} \psi(x, t) &= - \sum_{y \in \Lambda} O_1(x-y) \psi(y, t) - \lambda (\psi(x, t))^2, \\ \psi(x, 0) &= \psi_0(x), \quad \partial_t \psi(x, 0) = \psi_1(x), \quad \forall (x, t) \in \Lambda \times \mathbb{R}_+, \end{aligned} \quad (2.1)$$

in which $O_1(x-y)$ is the finite difference operator obtained from the continuum operator \mathcal{L} . We now introduce the discrete Fourier transform

$$\hat{\psi}(k) = h^d \sum_{x \in \Lambda} \psi(x) e^{-2\pi i k \cdot x}, \quad k \in \Lambda^* = \Lambda^*(D) = \{-D, \dots, 0, \dots, D\}^d.$$

At the end of this standard procedure, (2.1) can be rewritten in the Fourier space as a system of ODEs

$$\begin{aligned} \partial_t \hat{\psi}(k, t) &= - (\bar{\omega}(k))^2 \hat{\psi}(k, t) - \lambda \sum_{k=k_1+k_2; k_1, k_2 \in \Lambda^*} \hat{\psi}(k_1, t) \hat{\psi}(k_2, t), \\ \hat{\psi}(k, 0) &= \hat{\psi}_0(k), \quad \partial_t \hat{\psi}(k, 0) = \hat{\psi}_1(k). \end{aligned} \quad (2.2)$$

In the beam wave case, $\mathcal{L} = -\Delta^2$, and the dispersion relation takes the discretized form (see [74])

$$\bar{\omega}(k) = \sin^2(2\pi h k^1) + \dots + \sin^2(2\pi h k^d),$$

with $k = (k^1, \dots, k^d)$. Later, we will also need the rescaled dispersion relation

$$\omega(k) = \sin^2(2\pi k^1) + \dots + \sin^2(2\pi k^d).$$

We define the inverse Fourier transform $f(x) = \sum_{k \in \Lambda^*} \hat{f}(k) e^{2\pi i k \cdot x}$, as well as the shorthand notations

$$\int_{\Lambda} dx = h^d \sum_{x \in \Lambda}, \quad \langle f, g \rangle = h^d \sum_{x \in \Lambda} f(x)^* g(x), \quad \langle x \rangle = \sqrt{1 + |x|^2}, \quad \forall x \in \mathbb{R}^d,$$

where if $z \in \mathbb{C}$, then z^* is the complex conjugate. We also denote $\sum_{k \in \Lambda^*} = \int_{\Lambda^*} dk$. In addition, for any $N \in \mathbb{N} \setminus \{0\}$, similar with [79], we define the delta function δ_N on $(\mathbb{Z}/N)^d$ as

$$\delta_N(k) = |N|^d \mathbf{1}(k \bmod 1 = 0), \quad \forall k \in (\mathbb{Z}/N)^d,$$

in which the sub-index N is commonly omitted and written as

$$\delta(k) = |N|^d \mathbf{1}(k \bmod 1 = 0), \quad \forall k \in (\mathbb{Z}/N)^d.$$

Equation (2.2) can now be expressed as a coupling system

$$\begin{aligned} \partial_t q(k, t) &= p(k, t), \\ \partial_t p(k, t) &= - (\bar{\omega}(k))^2 q(k, t) \\ &\quad - \lambda \int_{(\Lambda^*)^2} dk_1 dk_2 \delta(k - k_1 - k_2) q(k_1, t) q(k_2, t), \\ q(k, 0) &= \hat{\psi}_0(k), \quad p(k, 0) = \hat{\psi}_1(k), \quad \forall (k, t) \in \Lambda^* \times \mathbb{R}_+, \end{aligned} \quad (2.3)$$

which, under the transformation (cf. [87])

$$a(k, t) = \bar{\omega}(k) q(k, t) + \frac{\mathbf{i}}{\bar{\omega}(k)} p(k, t),$$

with the inverse

$$\begin{aligned} q(k, T) &= \frac{1}{2\bar{\omega}(k)} \left[a(k) + a^*(-k) \right], \\ p(k, T) &= \mathbf{i} \frac{\bar{\omega}(k)}{2} \left[-a(k) + a^*(-k) \right], \end{aligned}$$

leads to the following system of ordinary differential equations

$$\begin{aligned} \partial_t a(k, t) &= -\mathbf{i}\bar{\omega}(k)a(k, t) - \mathbf{i}\lambda \int_{(\Lambda^*)^2} dk_1 dk_2 \delta(k - k_1 - k_2) \times \\ &\quad \times [8\bar{\omega}(k)\bar{\omega}(k_1)\bar{\omega}(k_2)]^{-1} \left[a(k_1, t) + a^*(-k_1, t) \right] \left[a(k_2, t) + a^*(-k_2, t) \right], \quad (2.4) \\ a(k, 0) &= a_0(k) = \frac{1}{2} \left[\bar{\omega}(k)q(k, 0) + \frac{\mathbf{i}}{\bar{\omega}(k)}p(k, 0) \right], \forall (k, t) \in \Lambda^* \times \mathbb{R}_+. \end{aligned}$$

Let a, a^* denote the vectors $(a_k)_{k \in \Lambda^*}$, $(a_k^*)_{k \in \Lambda^*}$, and let us set

$$H(a, a^*) = H_1(a, a^*) + \lambda H_2(a, a^*), \quad (2.5)$$

with

$$\begin{aligned} H_1(a, a^*) &= \sum_{k \in \Lambda_B^*} \frac{1}{2} \bar{\omega}(k) |a_k|^2, \\ H_2(a, a^*) &= \sum_{k, k_1, k_2 \in \Lambda^*} \mathscr{W}(k, k_1, k_2) \delta(k - k_1 - k_2) \left[a(k_1, t) + a^*(-k_1, t) \right] \\ &\quad \times \left[a(k_2, t) + a^*(-k_2, t) \right] a_k^*, \end{aligned}$$

$$\mathscr{W}(k, k_1, k_2) = [\bar{\omega}(k)\bar{\omega}(k_1)\bar{\omega}(k_2)]^{-1}, \quad \mathscr{M}(k, k_1, k_2) = [8\bar{\omega}(k)\bar{\omega}(k_1)\bar{\omega}(k_2)]^{-1}.$$

We then obtain the system

$$\partial_t a_k = \mathbf{i} \frac{\partial H(a, a^*)}{\partial a_k^*}. \quad (2.6)$$

By defining $\hat{a}(k, 1, t) = a_k(t)$ and $\hat{a}(k, -1, t) = a_k^*(t)$, we rewrite system (2.4) as

$$\begin{aligned} \partial_t \hat{a}(k, \sigma, t) &= -\mathbf{i}\sigma\bar{\omega}(k)\hat{a}(k, \sigma, t) - \mathbf{i}\sigma\lambda \sum_{\sigma_1, \sigma_2 \in \{\pm 1\}} \sum_{k_1, k_2 \in \Lambda^*} \delta(\sigma k - \sigma_1 k_1 - \sigma_2 k_2) \\ &\quad \times \mathscr{M}(k, k_1, k_2) \hat{a}(k_1, \sigma_1, t) \hat{a}(k_2, \sigma_2, t), \end{aligned}$$

$$\hat{a}(k, 1, 0) = a_0(k), \quad \forall (k, t) \in \Lambda^* \times \mathbb{R}_+.$$

For sake of simplicity, we also denote $\hat{a}(k, \sigma, t)$ as $\hat{a}_t(k, \sigma)$

$$\begin{aligned} \partial_t \hat{a}_t(k, \sigma) &= -\mathbf{i}\sigma\bar{\omega}(k)\hat{a}_t(k, \sigma) - \mathbf{i}\sigma\lambda \sum_{\sigma_1, \sigma_2 \in \{\pm 1\}} \sum_{k_1, k_2 \in \Lambda^*} \delta(\sigma k - \sigma_1 k_1 - \sigma_2 k_2) \\ &\quad \times \mathscr{M}(k, k_1, k_2) \hat{a}_t(k_1, \sigma_1) \hat{a}_t(k_2, \sigma_2), \end{aligned}$$

$$\hat{a}_0(k, 1) = a_0(k), \quad \forall (k, t) \in \Lambda^* \times \mathbb{R}_+.$$

By setting $f_{\lambda,D}(k,t) = \langle \hat{a}_t(k,-1), \hat{a}_t(h^{-1}k,1) \rangle$, and scaling $k \rightarrow hk$, in the kinetic limit of $D \rightarrow \infty$, $\lambda \rightarrow 0$ and $t = \lambda^{-2}\tau = \mathcal{O}(\lambda^{-2})$, under suitable randomization of the system, we obtain ([79]) $\lim_{\lambda \rightarrow 0, D \rightarrow \infty} f_{\lambda,D}(k, \lambda^{-2}\tau) = f(k, \tau)$, which solves the wave turbulence model

$$\begin{aligned} \partial_\tau f(k,t) &= \mathcal{C}[f](k), \quad f(k,0) = f_0(k), \quad \forall k \in \mathbb{T}^d, \\ \mathcal{C}[f](k) &= \int_{\mathbb{T}^6} K(\omega, \omega_1, \omega_2) \delta(k - k_1 - k_2) \delta(\omega - \omega_1 - \omega_2) [f_1 f_2 - f f_1 - f f_2] dk_1 dk_2 \\ &\quad - 2 \int_{\mathbb{T}^6} K(\omega, \omega_1, \omega_2) \delta(k_1 - k - k_2) \delta(\omega_1 - \omega - \omega_2) [f_2 f - f f_1 - f_1 f_2] dk_1 dk_2, \end{aligned} \quad (2.7)$$

where $f = f(k)$, $f_1 = f(k_1)$, $f_2 = f(k_2)$, $\omega = \omega(k)$, $\omega_1 = \omega(k_1)$, $\omega_2 = \omega(k_2)$ and $K(k, k_1, k_2) = [8\omega(k)\omega(k_1)\omega(k_2)]^{-1}$.

3. COLLECTIVE BEHAVIOR KINETIC MODELS OF DISCRETE NON-LOCAL WAVE EQUATIONS WITH LONG-RANGE INTERACTIONS

Model (2.6) is indeed a chain of anharmonic oscillators, in which the Hamiltonian H given by (2.5). The collective behavior of chains of anharmonic oscillators is a subject of growing interests in the physical community. For instance, in the case of the Fermi-Pasta-Ulam chains, the collective behavior can be obtained via a long-range interaction generalisation, in which the interactions are chosen to be non-local (see, e.g., [23, 24]). Inspired by this idea, we replace the local operator \mathcal{L} by a non-local one. As an illustration, we consider $\mathcal{L} = (-\Delta)^\alpha$ with $0 < \alpha < 1$ and obtain the following discrete wave equation with long-range lattice interactions (see [57] for the same setting for the nonlinear Schrödinger equation)

$$\begin{aligned} \partial_{tt} \psi(x,t) &= h^d \sum_{y \in \Lambda, y \neq x} \frac{\psi(y,t) - \psi(x,t)}{|y-x|^{d+2\alpha}} - \lambda (\psi(x,t))^2, \\ \psi(x,0) &= \psi_0(x), \quad \partial_t \psi(x,0) = \psi_1(x), \quad \forall (x,t) \in \Lambda \times \mathbb{R}_+. \end{aligned}$$

Note that $h^d \sum_{y \in \Lambda, y \neq x} \frac{\psi(y,t) - \psi(x,t)}{|y-x|^{d+2\alpha}}$ is the discretized version of the fractional Laplacian

$$(-\Delta \psi)^\alpha = - C_{d,\alpha} \int_{\mathbb{T}^d} dy \frac{\psi(y,t) - \psi(x,t)}{|y-x|^{d+2\alpha}},$$

with $C_{d,\alpha} = \frac{4^\alpha \Gamma(d/2 + \alpha)}{\pi^{\frac{d}{2}} |\Gamma(-\alpha)|}$.

In the scope of our paper, we restrict our considerations to the linear case $\lambda = 0$

$$\begin{aligned} \partial_{tt} \psi(x,t) &= h^d \sum_{y \in \Lambda, y \neq x} \frac{\psi(y,t) - \psi(x,t)}{|y-x|^{d+2\alpha}}, \\ \psi(x,0) &= \psi_0(x), \quad \partial_t \psi(x,0) = \psi_1(x), \quad \forall (x,t) \in \Lambda \times \mathbb{R}_+, \end{aligned} \quad (3.1)$$

or equivalently, if we set $r_x(t) = \psi(x,t)$ and $v_x(t) = \partial_t \psi(x,t)$, the following system can be obtained ((2.3))

$$\begin{aligned} \partial_t r_x &= v_x, \quad \partial_t v_x = h^d \sum_{y \in \Lambda, y \neq x} \frac{r_y - r_x}{|y-x|^{d+2\alpha}}, \\ r_x(0) &= \psi_0(x), \quad v_x(0) = \psi_1(x), \quad \forall (x,t) \in \Lambda \times \mathbb{R}_+. \end{aligned}$$

Under suitable randomization of the initial conditions $\psi_0(x)$ and $\psi_1(x)$, this system of equations describes the long-range interactions of the lattice points Λ and $(\psi(x,t), \partial_t \psi(x,t))$ represents

the phase space position of the x -particle at time t . Due to the long-range interactions between the $N = (2D + 1)^d$ particles, a collective behavior dynamics is expected for (3.1), similar to what happens for the Fermi-Pasta-Ulam chains with long-range interactions [23, 24].

Next, we will discuss “mean-field limit” of the above system by taking the limit $h \rightarrow 0$ or, equivalently $h^{-d} = N = (2D + 1)^d \rightarrow \infty$. Formally, it is not a difficult task to derive the mean-field limit equation for Hamiltonian dynamics; see, e.g., [49] for the case of the Cucker-Smale model. Rigorously, such derivations are challenging, especially when the interaction potentials are singular which is the current case. We assume that the initial data are chosen in a way that the empirical measure $N^{-1} \sum_{x \in \Lambda} \delta_{Q_x} \delta_{P_x}$ weakly converges in the limit $N \rightarrow \infty$ to the absolutely continuous measure $g_0(r, v) dr dv$ with some smooth density $g_0(r, v)$. Here, r and v are numbers in \mathbb{R} . We ask whether at some positive time $t > 0$ the empirical measure $N^{-1} \sum_{x \in \Lambda} \delta_{Q_x(t)} \delta_{P_x(t)}$ weakly converges to $g(r, v, t) dr dv$ with a density $g(r, v, t)$ satisfying some limiting evolution equation. Physically, the equation follows from the Liouville theorem, assuming that the number of particles is large enough such that it becomes meaningful to observe the distribution function $g^N = g^N(t, (x, r_x, v_x)_{x \in \Lambda})$. Defining the one-particle marginal distribution

$$\rho^N(t, x, r_x, v_x) = \int_{\mathbb{R}^{(d+2)(|\Lambda|-1)}} \prod_{z \in \Lambda \setminus \{x\}} dz dr_z dv_z g^N(t, (z, r_z, v_z)_{z \in \Lambda}),$$

where $|\Lambda|$ denotes the number of grid points. We now follow the BBGKY hierarchy to derive formally the kinetic description. To this end denote Δ^h the discrete Laplacian in d -dimensions with $d \geq 2$. Set $\Phi_\beta^h(x) = (\Delta^h)^{-1} |x|^\beta$ for $\beta \neq -2$ and $x \in \Lambda$. Since $\Delta |\cdot|^{\beta+2} = (\beta+2)(d+\beta) |\cdot|^\beta$ we have that

$$\Phi_\beta^h(x) \rightarrow c_{\beta,d} |x|^{\beta+2} \quad \text{as} \quad h \rightarrow 0, \quad c_{\beta,d} = \frac{1}{(\beta+2)(d+\beta)},$$

in which the boundary condition of the problem $\Delta \Phi_\beta = |x|^\beta$ is chosen appropriately such that the discretised sequence $\{\Phi_\beta^h(x)\}_{h \in \Lambda}$ has the desired limit. The Liouville equation reads, setting in the sequel $\beta = -d - 2\alpha$,

$$\partial_t g^N + \sum_{x \in \Lambda} v_x \partial_{r_x} g^N + h^d \sum_{x \in \Lambda} \partial_{v_x} \left(\sum_{y \in \Lambda, y \neq x} (r_y - r_x) (\Delta^h \Phi_\beta^h)(y - x) g^N \right) = 0. \quad (3.2)$$

We now integrate both sides of (3.2) with respect to $dr_y dv_y$, with $y \in \Lambda \setminus \{x\}$, to study the marginal distribution $\rho^N(x, r_x, v_x)$. Under the assumption that g^N is rapidly decaying at infinity, the transport term in (3.2) amounts to

$$\int_{\mathbb{R}^{(d+2)(|\Lambda|-1)}} \prod_{y \in \Lambda \setminus \{x\}} dy dr_y dv_y \left(\sum_{z \in \Lambda} v_z \partial_{r_z} g^N(t, (z, r_z, v_z)_{z \in \Lambda}) \right) = v_x \partial_{r_x} \rho^N(t, x, r_x, v_x).$$

We next study the forcing term, which by integration by parts reads

$$\begin{aligned} & h^d \int_{\mathbb{R}^{(d+2)(|\Lambda|-1)}} \prod_{y \in \Lambda \setminus \{x\}} dy dr_y dv_y \left[\sum_{z \in \Lambda} \partial_{v_z} \left(\sum_{s \in \Lambda, s \neq z} (r_s - r_z) (\Delta^h \Phi_\beta^h)(s - z) g^N \right) \right] \\ &= h^d \int_{\mathbb{R}^{(d+2)(|\Lambda|-1)}} \prod_{y \in \Lambda \setminus \{x\}} dy dr_y dv_y \left[\partial_{v_x} \left(\sum_{s \in \Lambda, s \neq x} (r_s - r_x) (\Delta^h \Phi_\beta^h)(s - x) g^N \right) \right]. \end{aligned}$$

We move ∂_{v_x} and the quantity $\sum_{s \in \Lambda, s \neq x} \Phi_\beta^h(s-x)$ outside of the integral and obtain

$$\begin{aligned} & h^d \int_{\mathbb{R}^{(d+2)(|\Lambda|-1)}} \prod_{y \in \Lambda \setminus \{x\}} dy dr_y dv_y \left[\sum_{z \in \Lambda} \partial_{v_z} \left(\sum_{y \in \Lambda, y \neq z} (r_y - r_z) (\Delta^h \Phi_\beta^h)(y-z) g^N \right) \right] \\ &= \partial_{v_x} \left(h^d \sum_{s \in \Lambda, s \neq x} (\Delta^h \Phi_\beta^h)(s-x) \int_{\mathbb{R}^{(d+2)(|\Lambda|-1)}} \prod_{y \in \Lambda \setminus \{x\}} dy dr_y dv_y (r_s - r_x) g^N \right). \end{aligned}$$

We define the two-particle marginal function

$$\rho^N(t, x, r_x, v_x, y, r_y, v_y) = \int_{\mathbb{R}^{(d+2)(|\Lambda|-1)}} \prod_{z \in \Lambda \setminus \{x, y\}} dz dr_z dv_z g^N(t, (z, r_z, v_z)_{z \in \Lambda}),$$

and find

$$\begin{aligned} & h^d \int_{\mathbb{R}^{(d+2)(|\Lambda|-1)}} \prod_{y \in \Lambda \setminus \{x\}} dy dr_y dv_y \sum_{z \in \Lambda} \partial_{v_z} \left(\sum_{y \in \Lambda, y \neq z} (r_y - r_z) (\Delta^h \Phi_\beta^h)(y-z) g^N \right) \\ &= \partial_{v_x} \left(h^d \sum_{s \in \Lambda, s \neq x} (\Delta^h \Phi_\beta^h)(s-x) \int_{\mathbb{R}^{2+d}} ds dr_s dv_s (r_s - r_x) \rho^N \right). \end{aligned}$$

This leads to the following equation for the one-particle marginal function ρ^N

$$\begin{aligned} & \partial_t \rho^N + v_x \partial_{r_x} \rho^N \\ &+ \partial_{v_x} \left(h^d \sum_{s \in \Lambda, s \neq x} (\Delta^h \Phi_\beta^h)(s-x) \int_{\mathbb{R}^{2+d}} ds dr_s dv_s (r_s - r_x) \rho^N \right) = 0. \end{aligned}$$

Recall that Δ^h is a self-adjoint operator acting on the lattice location variable $s \in \Lambda$, consequently

$$\begin{aligned} & h^d \sum_{s \in \Lambda, s \neq x} (\Delta^h \Phi_\beta^h)(s-x) \int_{\mathbb{R}^{2+d}} ds dr_s dv_s (r_s - r_x) \rho^N \\ &= h^d \sum_{s \in \Lambda, s \neq x} (\Delta^h \Phi_\beta^h)(s-x) \int_{\mathbb{R}^{2+d}} ds d\tilde{r} d\tilde{v} (\tilde{r} - r_x) \rho^N \\ &= \int_{\mathbb{R}^{2+d}} ds d\tilde{r} d\tilde{v} (\tilde{r} - r_x) \langle \Phi_\beta^h(\cdot - x), (\Delta^h \rho^N)(t, x, r, v, \cdot, \tilde{r}, \tilde{v}) \rangle \end{aligned}$$

Passing to the mean-field limit $N \rightarrow \infty$, we obtain the one- and two-particle density functions (dropping the sub- x notation)

$$\begin{aligned} \lim_{N \rightarrow \infty} \rho^N(t, x, r_x, v_x) &= g(t, x, r, v), \\ \lim_{N \rightarrow \infty} \rho^N(t, x, r_x, v_x, s, \tilde{r}, \tilde{v}) &= \tilde{g}(t, x, r, v, s, \tilde{r}, \tilde{v}), \end{aligned}$$

and the formal limit (note that $\beta + 2 > -d$)

$$\langle \Phi_\beta^h(x - \cdot), (\Delta^h \rho^N)(t, x, \cdot, r, v, \tilde{r}, \tilde{v}) \rangle \rightarrow c_{\beta, d} \int_{\mathbb{R}^d} |s-x|^{\beta+2} (\Delta_s \tilde{g})(t, x, r, v, s, \tilde{r}, \tilde{v}) ds,$$

which leads to the mean-field equation

$$\begin{aligned} & \partial_t g(t, x, r, v) + v \partial_r g(t, x, r, v) \\ &+ c_{\beta, d} \partial_v \left(\int_{\mathbb{R}^d} d\tilde{r} d\tilde{v} (\tilde{r} - r) \int_{\mathbb{T}^d} \frac{ds}{|s-x|^{d+2\alpha-2}} (\Delta_s \tilde{g})(t, x, s, r, v, \tilde{r}, \tilde{v}) \right) = 0. \end{aligned} \tag{3.3}$$

If, in addition, we make the molecular chaos assumption $\tilde{g}(t, x, s, r, v, \tilde{r}, \tilde{v}) = g(t, x, r, v)g(t, s, \tilde{r}, \tilde{v})$, then,

$$\begin{aligned} & c_{\beta,d} \int_{\mathbb{R}^d} d\tilde{r} d\tilde{v} (\tilde{r} - r) \int_{\mathbb{T}^d} \frac{ds}{|s-x|^{d+2\alpha-2}} (\Delta_s \tilde{g})(t, x, s, r, v, \tilde{r}, \tilde{v}) \\ &= c_{\beta,d} g(t, x, r, v) \int_{\mathbb{R}^d} d\tilde{r} d\tilde{v} (\tilde{r} - r) \int_{\mathbb{T}^d} \frac{ds}{|s-x|^{d+2\alpha-2}} (\Delta_s g)(t, s, \tilde{r}, \tilde{v}) \\ &= C_{d,\alpha}^{-1} g(t, x, r, v) \int_{\mathbb{R}^d} d\tilde{r} d\tilde{v} (r - \tilde{r}) (-\Delta_x)^\alpha g(t, x, \tilde{r}, \tilde{v}) \\ &=: C_{d,\alpha}^{-1} \Sigma_g(t, x, r) g(t, x, r, v), \end{aligned}$$

and deduce from (3.3) the Vlasov-type equation

$$\partial_t g(t, x, r, v) + v \partial_r g(t, x, r, v) + \mathbb{C}_1[g, g](t, x, r, v) = 0, \quad (3.4)$$

with

$$\mathbb{C}_1[g, g] := C_{d,\alpha}^{-1} \Sigma_g(t, x, r) \partial_v g(t, x, r, v)$$

and

$$\Sigma_g(t, x, r) = (-\Delta_x)^\alpha \int_{\mathbb{R}^d} d\tilde{r} d\tilde{v} (r - \tilde{r}) g(t, x, \tilde{r}, \tilde{v}).$$

where $(-\Delta_x)^\alpha$ is the fractional Laplacian in the torus. When the wave equation is set in the whole space, we obtain, by an identical formal argument, the fractional Laplacian in \mathbb{R}^d .

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REFERENCES

- [1] G. Albi, et al., Vehicular traffic, crowds, and swarms: From kinetic theory and multiscale methods to applications and research perspectives, *Math. Models Methods Appl. Sci.* 29 (2019) 1901-2005.
- [2] R. Alonso, I. M. Gamba, M.-B. Tran, The Cauchy problem and BEC stability for the quantum Boltzmann-Gross-Pitaevskii system for bosons at very low temperature, arXiv preprint arXiv: 1609.07467, 2018.
- [3] R. Alonso, T. Goudon, A. Vasseur, Damping of particles interacting with a vibrating medium, *Annales de l'Institut Henri Poincaré C, Analyse non linéaire*, 34 (2017) 1727-1758.
- [4] D. Amadori, S.-Y. Ha, J. Park, On the global well-posedness of bv weak solutions to the kuramoto-sakaguchi equation, *J. Differential Equations* 262 (2017) 978-1022.
- [5] I. Ampatzoglou, C. Collot, P. Germain, Derivation of the kinetic wave equation for quadratic dispersive problems in the inhomogeneous setting, arXiv preprint arXiv: 2107.11819, 2021.
- [6] M.H. Anderson, et al., Observation of Bose-Einstein Condensation in a dilute atomic vapor, *Science* 269 (1995) 198-201.
- [7] L. Arkeryd, A. Nouri, Bose condensates in interaction with excitations: a kinetic model, *Comm. Math. Phys.* 310 (2012) 765-788.

- [8] L. Arkeryd, A. Nouri, A Milne problem from a Bose condensate with excitations, *Kinet. Relat. Models* 6 (2013) 671-686.
- [9] L. Arkeryd, A. Nouri, Bose condensates in interaction with excitations: a two-component space-dependent model close to equilibrium, *J. Stat. Phys.* 160 (2015) 209-238.
- [10] N. Bellomo, J. Soler, On the mathematical theory of the dynamics of swarms viewed as complex systems, *Math. Models Methods Appl. Sci.* 22 (2012) 1140006.
- [11] D.J. Benney, A.C. Newell, Random wave closures, *Stud. Appl. Math.* 48 (1969) 29-53.
- [12] D.J. Benney, P.G. Saffman, Nonlinear interactions of random waves in a dispersive medium, *Proc. R. Soc. London A* 289 (1996) 301-320.
- [13] U. Biccari, D. Ko, E. Zuazua, Dynamics and control for multi-agent networked systems: A finite-difference approach, *Math. Models Methods Appl. Sci.* 29 (2019) 755-790.
- [14] V.D. Blondel, J.M. Hendrickx, J.N. Tsitsiklis, On krause's multi-agent consensus model with state-dependent connectivity, *IEEE Trans. Automatic Control* 54 (2009) 2586-2597.
- [15] F. Bolley, J. A. Cañizo, J. A. Carrillo, Stochastic mean-field limit: Non-Lipschitz forces and swarming, *Math. Models Methods Appl. Sci.* 21 (2011) 2179-2210.
- [16] T. Borsoni, M. Bisi, M. Groppi, A general framework for the kinetic modeling of polyatomic gases, arXiv: 2107.11432, 2021.
- [17] C.C. Bradley, et al., Evidence of Bose-Einstein condensation in an atomic gas with attractive interactions, *Phys. Rev. Lett.* 75 (1995) 1687.
- [18] T. Buckmaster, et al., On the kinetic wave turbulence description for nls, *Quart. Appl. Math.* to appear.
- [19] T. Buckmaster, et al., Onset of the wave turbulence description of the longtime behavior of the nonlinear schrodinger equation Submitted, 2019.
- [20] S. Cai, X. Lu, The spatially homogeneous boltzmann equation for bose-einstein particles: Rate of strong convergence to equilibrium, arXiv preprint arXiv: 1808.04038, 2018.
- [21] M. Caponigro, et al., Sparse stabilization and control of alignment models, *Math. Models Methods Appl. Sci.* 25 (2015) 521-564.
- [22] J.A. Carrillo, Y.-P. Choi, M. Hauray, The derivation of swarming models: mean-field limit and wasserstein distance, In: *Collective Dynamics from Bacteria to Crowds*, pp. 1-46, Springer, 2014.
- [23] H. Christodoulidi, et al., Dynamics and statistics of the fermi-pasta-ulam β -model with different ranges of particle interactions, *J. Stat. Mech. Theory Exp.* 12 (2016) 123206.
- [24] H. Christodoulidi, C. Tsallis, T. Bountis, Fermi-pasta-ulam model with long-range interactions: Dynamics and thermostatics, *EPL Europhysics Lett.* 108 (2014) 40006.
- [25] C. Collot, P. Germain, On the derivation of the homogeneous kinetic wave equation, arXiv preprint arXiv: 1912.10368, 2019.
- [26] C. Collot, P. Germain, Derivation of the homogeneous kinetic wave equation: longer time scales, arXiv preprint arXiv: 2007.03508, 2020.
- [27] I.D. Couzin, et al., Effective leadership and decision-making in animal groups on the move, *Nature* 433 (2005) 513-516.
- [28] F. Cucker, S. Smale, Emergent behavior in flocks, *IEEE Trans. Auto. Control* 52 (2007) 852-862.
- [29] F. Cucker, S. Smale, On the mathematics of emergence, *Japanese J. Math.* 2 (2007) 197-227.
- [30] K.B. Davis, et al., Bose-Einstein condensation in a gas of sodium atoms, *Phys. Rev. Lett.* 75 (1995), 3969.
- [31] S. De Bièvre, T. Goudon, A. Vasseur, Particles interacting with a vibrating medium: existence of solutions and convergence to the Vlasov-Poisson system, *SIAM J. Math. Anal.* 48 (2016) 3984-4020.
- [32] S. De Bièvre, P.E. Parris, A. Silvius, Chaotic dynamics of a free particle interacting linearly with a harmonic oscillator, *Phys. D* 208 (2005) 96-114.
- [33] P. Degond, A. Frouvelle, J.-G. Liu, Phase transitions, hysteresis, and hyperbolicity for self-organized alignment dynamics, *Arch. Ration. Mech. Anal.* 216 (2015) 63-115.
- [34] Y. Deng, Z. Hani, On the derivation of the wave kinetic equation for nls, arXiv preprint arXiv: 1912.09518, 2019.
- [35] Y. Deng, Z. Hani, Full derivation of the wave kinetic equation, arXiv preprint arXiv: 2104.11204, 2021.
- [36] R. Duan, M. Fornasier, G. Toscani, A kinetic flocking model with diffusion, *Commun. Math. Phys.* 300 (2010) 95-145.

- [37] B. Düring, P. Markowich, J.-F. Pietschmann, M.-T. Wolfram, Boltzmann and fokker–planck equations modelling opinion formation in the presence of strong leaders, *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 465 (2009) 3687-3708.
- [38] A. Dymov, S. Kuksin, Formal expansions in stochastic model for wave turbulence 1: kinetic limit, arXiv preprint arXiv: 1907.04531, 2019.
- [39] A. Dymov, S. Kuksin, Formal expansions in stochastic model for wave turbulence 2: method of diagram decomposition, arXiv preprint arXiv: 1907.02279, 2019.
- [40] A. Dymov, S. Kuksin, On the zakharov-l’vov stochastic model for wave turbulence, In: *Doklady Mathematics*, vol. 101, pp. 102-109, Springer, 2020.
- [41] A. Dymov, S. Kuksin, A. Maiocchi, S. Vladuts, The large-period limit for equations of discrete turbulence, arXiv preprint arXiv: 2104.11967, 2021.
- [42] M. Escobedo, S. Mischler, M.A. Valle, Homogeneous Boltzmann equation in quantum relativistic kinetic theory, volume 4 of *Electronic Journal of Differential Equations Monograph*, Southwest Texas State University, San Marcos, TX, 2003.
- [43] M. Escobedo, M.-B. Tran, Convergence to equilibrium of a linearized quantum Boltzmann equation for bosons at very low temperature, *Kinetic and Related Models* 8 (2015) 493–531.
- [44] M. Escobedo, J.J.L. Velázquez, Finite time blow-up and condensation for the bosonic Nordheim equation, *Invent. Math.* 200 (2015) 761-847.
- [45] M. Escobedo, J.J.L. Velázquez, On the theory of weak turbulence for the nonlinear Schrödinger equation, *Mem. Amer. Math. Soc.* 238 (2015) doi: 10.1090/memo/1124.
- [46] I.M. Gamba, L.M. Smith, M.-B. Tran, On the wave turbulence theory for stratified flows in the ocean, *Math. Models Methods Appl. Sci.* 30 (2020) 105-137.
- [47] P. Germain, A.D. Ionescu, M.-B. Tran, Optimal local well-posedness theory for the kinetic wave equation, *J. Funct. Anal.* 279 (2020) 108570.
- [48] S.-Y. Ha, K. Lee, D. Levy, Emergence of time-asymptotic flocking in a stochastic cucker-smale system, *Commun. Math. Sci.* 7 (2009) 453-469.
- [49] S.-Y. Ha, E. Tadmor, From particle to kinetic and hydrodynamic descriptions of flocking, *Kinetic & Related Models* 1 (2008) 415.
- [50] K. Hasselmann, On the non-linear energy transfer in a gravity-wave spectrum part 1. general theory, *J. Fluid Mech.* 12 (1962) 481-500.
- [51] K. Hasselmann, On the spectral dissipation of ocean waves due to white capping, *Boundary-Layer Meteorology* 6 (1974) 107-127.
- [52] S. He, E. Tadmor, A game of alignment: collective behavior of multi-species, In: *Annales de l’Institut Henri Poincaré C, Analyse non linéaire*, vol. 38, pp. 1031-1053, Elsevier, 2021.
- [53] R. Hegselmann, Opinion dynamics and bounded confidence models, analysis, and simulation, *J. Artificial Soc. Social Simul.* 5 (2002) 1-33.
- [54] S. Jin, M.-B. Tran, Quantum hydrodynamic approximations to the finite temperature trapped bose gases, *Physica D: Nonlinear Phenomena* 380-381 (2018) 45-57.
- [55] T.K. Karper, A. Mellet, K. Trivisa, Hydrodynamic limit of the kinetic cucker–smale flocking model, *Math. Models Methods Appl. Sci.* 25 (2015) 131-163.
- [56] Y. Kawamura, From the kuramoto-sakaguchi model to the kuramoto-sivashinsky equation, *Phys. Rev. E* 89 (2014) 010901.
- [57] K. Kirkpatrick, E. Lenzmann, G. Staffilani, On the continuum limit for discrete nls with long-range lattice interactions, *Commun. Math. Phys.* 317 (2013) 563-591.
- [58] D. Ko, E. Zuazua, Asymptotic behavior and control of a “guidance by repulsion” model, *Math. Models Methods Appl. Sci.* 30 (2020) 765-804.
- [59] Y. Kuramoto, International symposium on mathematical problems in theoretical physics, *Lecture Notes in Physics* 30 (1975) 420.
- [60] X. Lu, On isotropic distributional solutions to the Boltzmann equation for Bose-Einstein particles, *J. Statist. Phys.* 116 (2004) 1597-1649.
- [61] X. Lu, The Boltzmann equation for Bose-Einstein particles: velocity concentration and convergence to equilibrium, *J. Stat. Phys.* 119 (2015) 1027-1067.

- [62] X. Lu, The Boltzmann equation for Bose-Einstein particles: condensation in finite time, *J. Stat. Phys.* 150 (2013) 1138-1176.
- [63] J. Lukkarinen, H. Spohn, Weakly nonlinear Schrödinger equation with random initial data, *Invent. Math.* 183 (2011) 79-188.
- [64] J. Morales, J. Peszek, E. Tadmor, Flocking with short-range interactions, *J. Stat. Phys.* 176 (2019) 382-397.
- [65] S. Motsch, E. Tadmor, A new model for self-organized dynamics and its flocking behavior, *J. Stat. Phys.* 144 (2011) 923-947.
- [66] S. Motsch, E. Tadmor, Heterophilous dynamics enhances consensus, *SIAM Rev.* 56 (2014) 577-621.
- [67] S. Nazarenko, *Wave Turbulence*, volume 825 of *Lecture Notes in Physics*, Springer, Heidelberg, 2011.
- [68] T.T. Nguyen, M.-B. Tran, On the kinetic equation in Zakharov's wave turbulence theory for capillary waves, *SIAM J. Math. Anal.* 50 (2018) 2020-2047.
- [69] T.T. Nguyen, M.-B. Tran, Uniform in time lower bound for solutions to a quantum boltzmann equation of bosons, *Arch. Arch. Ration. Mech. Anal.* 231 (2019) 63-89.
- [70] M.R. D'Orsogna, Y.L. Chuang, A.L. Bertozzi, L. Chayes, Self-propelled particles with soft-core interactions: patterns, stability, and collapse, *Phys. Rev. Lett.* 96 (2006) 104302.
- [71] R. Peierls, Zur kinetischen theorie der wärmeleitung in kristallen, *Annalen der Physik* 395 (1929) 1055-1101.
- [72] R.E. Peierls, *Quantum theory of solids*, *Theoretical physics in the twentieth century (Pauli memorial volume)*, pp. 140–160. Interscience, New York, 1960.
- [73] Y. Pomeau, M.-B. Tran, *Statistical physics of non equilibrium quantum phenomena*, *Lecture Notes in Physics*, Springer, 2019.
- [74] B. Rumpf, A. Soffer, M.-B. Tran, On the wave turbulence theory: ergodicity for the elastic beam wave equation, arXiv preprint arXiv: 2108.13223, 2021.
- [75] R. Shvydkoy, E. Tadmor, Eulerian dynamics with a commutator forcing ii: Flocking, *Dyn. Sys.* 37 (2017) 5503-5520.
- [76] A. Soffer, M.-B. Tran, On coupling kinetic and schrodinger equations, *J. Differential Equations* 265 (2018) 2243-2279.
- [77] A. Soffer, M.-B. Tran, On the dynamics of finite temperature trapped bose gases, *Adv. Math.* 325 (2018) 533-607.
- [78] A. Soffer, M.-B. Tran, On the energy cascade of 3-wave kinetic equations: beyond kolmogorov–zakharov solutions, *Commun. Math. Phys.* 376 (2020) 2229–2276.
- [79] G. Staffilani, M.-B. Tran, On the wave turbulence theory for stochastic and random multidimensional kdv type equations, arXiv preprint arXiv: 2106.09819, 2021.
- [80] E. Tadmor, On the mathematics of swarming: Emergent behavior in alignment dynamics, arXiv preprint arXiv: 2102.09134, 2021.
- [81] E. Tadmor, C. Tan, Critical thresholds in flocking hydrodynamics with non-local alignment, *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences* 372 (2014) 20130401.
- [82] M.-B. Tran, Convergence to equilibrium of some kinetic models, *J. Differential Equations* 255 (2013) 405-440.
- [83] M.-B. Tran, G. Craciun, L.M. Smith, S. Boldyrev, A reaction network approach to the theory of acoustic wave turbulence, *J. Differential Equations* 269 (2020) 4332-4352.
- [84] M.-B. Tran, Y. Pomeau, Boltzmann-type collision operators for bogoliubov excitations of bose-einstein condensates: A unified framework, *Physical Rev. E* 101 (2020), 032119.
- [85] M.-B. Tran, Y. Pomeau, On a thermal cloud-bose–einstein condensate coupling system, *The Eur. Phys. J. Plus* 136 (2021) 1-11.
- [86] T. Vicsek, A. Czirók, E. Ben-Jacob, I. Cohen, O. Shochet, Novel type of phase transition in a system of self-driven particles, *Phys. Rev. Lett.* 75 (1995) 1226.
- [87] V.E. Zakharov, N.N. Filonenko, Weak turbulence of capillary waves, *J. Appl. Mech. Tech. Phys.* 8 (1967) 37-40.
- [88] V.E. Zakharov, V.S. L'vov, G. Falkovich, *Kolmogorov spectra of turbulence I: Wave turbulence*, Springer Science & Business Media, 2012.

- [89] M.M. Zavlanos, M.B. Egerstedt, G.J. Pappas, Graph-theoretic connectivity control of mobile robot networks, *Proceedings of the IEEE*, 99 (2011) 1525-1540.