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On a thermal cloud—Bose–Einstein condensate coupling system

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Abstract Starting with an N body system describing a Bose gas at finite temperatures, we derive a new model that contains a Gross–Pitaevskii equation of motion for the condensate wave function and a quantum Boltzmann equation for the excitations. The model is valid for a wide range of finite temperatures and approaches the standard ZNG model when the temperature of the system T is close to the Bose–Einstein condensate (BEC) transition temperature T_c.

1 Introduction

The realization of Bose–Einstein condensation (BEC) in trapped atomic vapors of ^{23}Na [5], ^{87}Rb [2] and ^{7}Li [3] has initiated a period of intense theoretical and experimental research. The experimental results need a theoretical support which takes into account the coupled non-equilibrium dynamics of both the BEC and the thermal cloud of the Bose gas under investigation. In the pioneering work [10–12], Kirkpatrick and Dorfman (KD) started to develop such a theory, based on the Bogoliubov mean field approach. Their theory includes a mean field kinetic equation for the thermal cloud that describes the relaxation in terms of "collisions" between excitations. This theory was then extended by Zaremba et al. [23], in which the full coupling system of a quantum Boltzmann equation for the density function of the normal fluid/thermal cloud and a Gross–Pitaevskii equation for the wavefunction of the BEC has been introduced. The model is named ZNG, after the authors. Independently, the same mean field model was also derived by Pomeau et al. [22] (PBMR) by a different method. Since our work is mainly focused on the mean field approach, we refer the readers to the book [15] for further discussions on the other theories.

The ZNG model has been remarkably successful in describing a wide range of BEC phenomena [6]. In the ZNG theory, there are two types of collisional processes: the $1 \leftrightarrow 2$ interactions between the condensate and the excited atoms and the $2 \leftrightarrow 2$ interactions between the excited atoms themselves. A third, previously missing, collisional process, which takes into account $1 \leftrightarrow 3$ type collisions between the excitations, has been suggested by Gust and Reichl [7,19].

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In the previous work [21], we have provided a mathematical justification of the new collisional process as well as a unified framework to explain the origins of all the three collision operators. The mathematical justification of [21] is based on a precise calculation of all of commutators for the Bogoliubov excitations in the system, without dropping any terms. In [21], it has also been shown that the $1\leftrightarrow 3$ collision operator indeed becomes important at lower temperature ranges, while being negligible when T is closed to T_c. Experimental evidence for this new collision operator has also been provided in [8,14,18].

The goal of our work is to mathematically derive a new mean field coupling system, with all the three collision operators, at all finite temperature ranges, based on the framework provided in [21]. This model becomes the standard ZNG model when the temperature of the system is high enough. In our derivation, we have tried to rely on exact mathematical computations, in which, most of the terms are kept and only a few approximations are employed. Before our work, another lower temperature system, in which only the $1 \leftrightarrow 2$ collision is included, was also derived by Imamovic-Tomasovic and Griffin (IG) [9], motivated by the reason that the ZNG model is based on particle-like Hatree-Fock excitations and ultimately breaks down at low temperatures as its thermal excitations do not include the phonon part of the Bogoliubov spectrum. Since in [9], the two collisional processes $2 \leftrightarrow 2$ and $1 \leftrightarrow 3$ are both missing, the IG model does not approach the ZNG model when $T \approx Tc$.

The coupling system

We will now write our *final* system, that couples the generalized Gross–Pitaevskii and the quantum Boltzmann equations, in the local rest frame. The derivation of this system will be given in the next sections.

Denote by $n_c(x,t)$ and $\phi(x,t)$, the local density of particles in the condensate and the condensate phase, and set

$$\Upsilon = n_c^{1/2} \exp(i\phi). \tag{1}$$

The generalized Gross–Pitaevskii equation

The generalized Gross–Pitaevskii equation reads

$$\begin{aligned} i\hbar\partial_t \Upsilon(x,t) = \Big[&- \frac{\hbar^2\nabla^2}{2m} + g[n_c(x,t) + 2\tilde{n}(x,t)] \\ &+ U(x) - \frac{i\hbar}{2n_c} \int' \mathrm{d}p C_{12}[f](x,p,t) \Big] \Upsilon(x,t), \end{aligned} \tag{2}$$

where

$$\tilde{n}(x,t) = \int' \mathrm{d}p\, f(x,p,t). \tag{3}$$

In the above equations, $f(x,p,t)$ is the solution of the quantum Boltzmann equation (5), the operator C_{12} can be found in (5), $\hbar$ is the reduced Planck constant, g is the interaction coupling constant, $U(x)$ is the confinement potential, $\int'$ stands for $\int_{\mathbb{R}^3\setminus\{O\}}$, m is the mass of the particles. The velocity of the condensate is

$$v(x,t) = \frac{\hbar}{m}\nabla\phi(x,t). \tag{4}$$

The quantum Boltzmann equation

The quantum Boltzmann equation for the density $f(x, p, t)$ of the non-condensate atoms reads

$$\left(\frac{\partial}{\partial t} + \frac{1}{\hbar}\nabla_p(\omega_p + \hbar p \cdot v)\cdot\nabla_x - \frac{1}{\hbar}\nabla_x(\omega_p + \hbar p \cdot v)\cdot\nabla_p\right) f = C_{12}[f] + C_{22}[f] + C_{31}[f], \tag{5}$$

where ω_p is the Bogoliubov dispersion relation defined in (18) and the forms of C_{12}, C_{22}, C_{31} are given explicitly below

$$\begin{aligned} C_{12}[f](p) = {} & \frac{4\pi g^2 n_c}{\hbar(2\pi)^3}\int'\int'\int' \mathrm{d}p_1\mathrm{d}p_2\mathrm{d}p_3\delta(p_1 - p_2 - p_3) \\ & \times(\delta(p-p_1) - \delta(p-p_2) - \delta(p-p_3))\delta(\omega_1 - \omega_2 - \omega_3) \\ & \times (K^{1,2}_{1,2,3})^2\Big[f_2 f_3(f_1+1) - f_1(f_2+1)(f_3+1)\Big], \end{aligned} \tag{6}$$

$$\begin{aligned} C_{22}[f](p) = {} & \frac{\pi g^2}{\hbar(2\pi)^6}\int'\int'\int'\int' \mathrm{d}p_1\mathrm{d}p_2\mathrm{d}p_3\mathrm{d}p_4 \\ & \times(\delta(p-p_1) + \delta(p-p_2) - \delta(p-p_3) - \delta(p-p_4)) \\ & \times\delta(\omega_1 + \omega_2 - \omega_3 - \omega_4)\delta(p_1 + p_2 - p_3 - p_4)(K^{2,2}_{1,2,3,4})^2 \\ & \times\Big[f_3 f_4(f_2+1)(f_1+1) - f_1 f_2(f_3+1)(f_4+1)\Big], \end{aligned} \tag{7}$$

and

$$\begin{aligned} C_{31}[f](t,p) = {} & \frac{3\pi g^2}{\hbar(2\pi)^6}\int'\int'\int'\int' \mathrm{d}p_1\mathrm{d}p_2\mathrm{d}p_3\mathrm{d}p_4 \\ & \times(\delta(p-p_1) - \delta(p-p_2) - \delta(p-p_3) - \delta(p-p_4)) \\ & \times\delta(p_1 - p_2 - p_3 - p_4)\delta(\omega_1 - \omega_2 - \omega_3 - \omega_4)(K^{3,1}_{1,2,3,4})^2 \\ & \times\Big[f_3 f_4 f_2(f_1+1) - f_1(f_2+1)(f_3+1)(f_4+1)\Big], \end{aligned} \tag{8}$$

in which ω_i, f_i stand for $\omega(p_i)$, $f(p_i)$, $p \in \mathbb{R}^3\backslash\{O\}$ is the 3-dimensional non-zero momentum variable. In the above collision operators, the kernels are defined as follows

$$\begin{aligned} K^{1,2}_{1,2,3} = {} & u_{p_1}u_{p_2}u_{p_3} - v_{p_1}v_{p_2}v_{p_3} - u_{p_1}u_{p_2}v_{p_3} \\ & + v_{p_1}v_{p_2}u_{p_3} - u_{p_1}v_{p_2}u_{p_3} + v_{p_1}u_{p_2}v_{p_3}, \end{aligned} \tag{9}$$

$$\begin{aligned} K^{2,2}_{1,2,3,4} = {} & u_{p_1}u_{p_2}u_{p_3}u_{p_4} + u_{p_1}v_{p_2}u_{p_3}v_{p_4} + u_{p_1}v_{p_2}v_{p_3}u_{p_4} \\ & + v_{p_1}u_{p_2}v_{p_3}u_{p_4} + v_{p_1}u_{p_2}u_{p_3}v_{p_4} + v_{p_1}v_{p_2}v_{p_3}v_{p_4}, \end{aligned} \tag{10}$$

and

$$K^{3,1}_{1,2,3,4} = 2\Big[u_{p_1}u_{p_2}v_{p_3}u_{p_4} + v_{p_1}v_{p_2}u_{p_3}v_{p_4}\Big], \tag{11}$$

with u_p and v_p being defined in (19).

When $T \approx T_c$, the Bogoliubov dispersion relation can be approximated by the Hatree–Fock energy. In this regime, $u_p \backsim 1$ and $v_p \backsim 0$. Therefore, $K^{1,2}_{1,2,3} \backsim 1$, $K^{2,2}_{1,2,3,4} \backsim 1$, while $K^{3,1}_{1,2,3,4} \backsim 0$. As a result, when $T \approx T_c$, C_{31} is negligible while the two collision operators C_{12} and C_{22} dominate the collisional processes. In lower temperature regimes, both u_p and v_p are large, making all quantities $K^{1,2}_{1,2,3}$, $K^{2,2}_{1,2,3,4}$, $K^{3,1}_{1,2,3,4}$ large. Thus, the contribution of

all collision operators C_{12}, C_{22} and C_{31} needs to be taken into account. It is discussed in [14][Section 8.2.3] that using all the three collision operators C_{12}, C_{22} and C_{31}, the speed of both the fast mode and the slow mode can be computed and they turn out to approach finite values in the limit $T \to 0$ K. using all the three collision operators C_{12}, C_{22} and C_{31}, the speed of both the fast mode and the slow mode can be computed and they turn out to approach finite values in the limit $T \to 0$ K. This computation is consistent with the findings of Lee and Yang [13] using a very different approach. Moreover, it is also discussed in [14] that the value of the sound mode lifetime, computed by using all the three collision operators C_{12}, C_{22} and C_{31}, is consistent with that reported in the Steinhauer experiment [20].

In the integral on the momenta $\int_{\mathbb{R}^3\setminus\{O\}} \mathrm{d}p$, the origin is removed due to the fact that the condensate has been factored out in the Bogoliubov diagonalization. In a mathematical point of view, if the origin is not removed from the domain of integration, the solution can develop a singular part supported at $\{O\}$.

2 The quantum system and the three unitary transformations

To derive the coupling system (1)–(11), we consider a system of weakly interacting, spinless bosons at finite temperatures. We introduce the boson field operator $\hat{\Psi}(x)$, and its conjugate $\hat{\Psi^\dagger}(x)$. These operators satisfy the the commutation relation $[\hat{\Psi}(x), \hat{\Psi}(x')] = [\hat{\Psi^\dagger}(x), \hat{\Psi^\dagger}(x')] = 0$; $[\hat{\Psi}(x), \hat{\Psi^\dagger}(x')] = \delta(x - x')$. The Hamiltonian of the system is now written

$$\hat{H} = \int_{\mathbb{T}_L^d} \mathrm{d}x \hat{\Psi^\dagger}(x)\Big[-\frac{\hbar^2}{2m}\nabla^2 + U(x) + \frac{1}{2}\hat{\Psi^\dagger}(x)\mathcal{V}(x,x')\hat{\Psi}(x')\Big]\hat{\Psi}(x), \tag{12}$$

where $\mathbb{T}_L^3$ is the 3-dimensional periodic torus $\left[-\frac{L}{2}, \frac{L}{2}\right]^3$; $\mathcal{V}(x, x')$ is the interaction potential between two particles at locations x, x'. We also take $\mathcal{V}(x, x') = g\delta(x - x')$. Inserting these two forms for the external and interaction potentials into (12), we find

$$\hat{H} = \hat{H}' + \hat{V}, \tag{13}$$

where

$$\hat{H}' = \frac{\hbar^2}{2m}\int \mathrm{d}x \hat{\nabla}\Psi^\dagger(x) \cdot \nabla\hat{\Psi}(x), \tag{14}$$

and

$$\hat{V} = \frac{g}{2}\int \mathrm{d}x \hat{\Psi^\dagger}(x)\hat{\Psi^\dagger}(x)\hat{\Psi}(x)\hat{\Psi}(x), \tag{15}$$

with the shorthand notations $\int_{\mathbb{T}_L^d} = \int$.

We introduce the non-equilibrium statistical density operator $\hat{\rho}(t)$ of the spatially inhomogeneous Bose gas at time t. This density operator then satisfies the quantum Liouville equation

$$\frac{\partial\hat{\rho}}{\partial t} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}].$$

We follow the approach of Kirkpatrick and Dorfman [10,12]. First, we introduce the following local unitary transformations to the quantum Liouville equation. The first unitary

operator changes the reference frame of the system to one in which the superfluid velocity is zero

$$\hat{O}_1[\phi] \;=\; \exp\left[-i\int \mathrm{d}x'\phi(x',t)\hat{\Psi}^{\dagger}(x',t)\hat{\Psi}(x',t)\right].$$

The second unitary transformation replaces the boson field operators in the Hamiltonians by new ones without the contribution of the condensate

$$\hat{O}_2[n_c] = \exp\left[\int \mathrm{d}x'[\hat{\Psi}(x',t) - \hat{\Psi}^{\dagger}(x',t)]n_c^{\frac{1}{2}}(x',t)\right].$$

Now, we write the time-dependent field operator $\hat{\Psi}$ as the sum of a condensate part Φ and a non-condensate part $\hat{\psi}$

$$\hat{\Psi}(x,t) \;=\; \hat{\psi}(x,t) \;+\; \Phi(x,t). \tag{16}$$

Under the assumption that the difference between the condensate field operator and the average value are approximately the same in the thermodynamics limit, we replace $\Phi(x,t)$ by the c-number $n_c^{1/2}(x,t)\exp(i\phi(x,t))$. Thus, $\hat{\psi}(x,t)$ can be expressed as follows

$$\hat{O}_1^{\dagger}\hat{O}_2^{\dagger}\hat{\Psi}\hat{O}_2\hat{O}_1 = \Psi\,\hat{\exp}[-i\phi] - n_c^{1/2} = \psi\,\hat{\exp}[-i\phi].$$

Using the above two unitary transformations, we obtain the Liouville equation for the density operator $\hat{\rho}'$ of the non-condensate part in the superfluid rest-frame

$$\frac{\partial\hat{\rho}'}{\partial t} \;=\; -\frac{i}{\hbar}[\hat{H}'', \hat{\rho}'].$$

We introduce the last unitary transformation - the local Bogoliubov transformation, which changes the field operators from those for particles to those for Bogoliubov excitations. To this end, we define the Wigner operator

$$\hat{f}(x,\,p) \;=\; \int \mathrm{d}x'[\exp(ip\cdot x')]\hat{\psi^{\dagger}}(x+x'/2)\hat{\psi}(x-x'/2). \tag{17}$$

Since $\hat{\rho}'$ is defined in the superfluid rest-frame, we can define the Fourier series of the fields operators, under the assumption that the gas is slightly inhomogeneous in space

$$\hat{\psi}(x) = \frac{1}{\sqrt{\Omega}}\tilde{\sum_p}\exp(ikp\cdot x)\hat{a}_p,$$

where we employ the shorthand notation $\tilde{\sum}_p \;=\; \sum_{p\in\mathbb{Z}_L^d, p\neq 0}$, with $\mathbb{Z}_L^3 = (\mathbb{Z}/L)^3$ and Ω is the volume of the box under consideration.

We now define the Bogoliubov unitary transformation operator

$$\hat{O}_3(x,t) = \exp\left[\frac{1}{2}\tilde{\sum_p}\vartheta_p(x,t)\left(\hat{a}_p\hat{a}_{-p} - \hat{a}_p^{\dagger}\hat{a}_{-p}^{\dagger}\right)\right].$$

In the above formulation, $\vartheta_p(x,t)$ depends on $\tilde{\omega}_p$, which, by neglecting completely the quantum pressure, can be approximated by the Bogoliubov dispersion relation

$$\tilde{\omega}_p \approx \omega_p = \left[\frac{gn_c\hbar^2}{m}p^2 + \left(\frac{\hbar^2p^2}{2m}\right)^2\right]^{\frac{1}{2}}. \tag{18}$$

This is often referred to as Thomas–Fermi approximation [4,6]. We now write down the Bogoliubov excitation annihilation and creation operators

$$\hat{a}_p \;=\; u_p\hat{b}_p \;-\; v_p\hat{b}^{\dagger}_{-p}, \quad \hat{a}^{\dagger}_p \;=\; u_p\hat{b}^{\dagger}_p \;-\; v_p\hat{b}_{-p}, \tag{19}$$

with

$$u_p, v_p = \left(\frac{\epsilon_p + gn_c}{2\omega_p} \pm \frac{1}{2}\right)^{\frac{1}{2}},$$

$$u_p = \cosh\vartheta_p, v_p = -\sinh\vartheta_p.$$

We also define the new statistical density operator $\hat{\rho}_b(x,t)$ for the Bogoliubov excitations

$$\hat{\rho}_b(x,t) \;=\; \hat{O}_3(x,t)\hat{\rho}'(t)\hat{O}^{\dagger}_3(x,t). \tag{20}$$

We arrive at a new quantum Liouville equation for the Bogoliubov excitations

$$\frac{\partial\hat{\rho}_b}{\partial t} \;=\; -\frac{i}{\hbar}[\hat{H}_b, \hat{\rho}_b]. \tag{21}$$

The operator $\hat{H}_b$ takes the form

$$\hat{H}_b \;=\; \hat{H}_{T,1} \;+\; \hat{H}_{T,2} \;+\; \hat{H}_2 \;+\; \hat{H}_3 \;+\; \hat{H}_4, \tag{22}$$

in which $\hat{H}_{T,1}$, $\hat{H}_{T,2}$ will contribute to the transport part of the kinetic equation

$$\begin{aligned}
\hat{H}_{T,1} &= \tilde{\sum_p}\omega_p\hat{b}^{\dagger}\hat{b}_p \;+\; \hbar\sum_p(v\cdot p)\hat{b}^{\dagger}_p\hat{b}_p,\\
\hat{H}_{T,2} &= \sum_{j,l=1}^{3}\tilde{\sum_{p_1,p_2}}\frac{\partial v_j}{\partial x_l}\frac{\hbar}{2V}\int \mathrm{d}x'(x'_l - x_l)\exp[ix'\cdot(p_2-p_1)]\\
&\quad\times(p_{1j}+p_{2j})[(u_{p_1}u_{p_2} - v_{p_1}v_{p_2})\hat{b}^{\dagger}_{p_1}\hat{b}_{p_2} - u_{p_2}v_{p_1}\hat{b}_{-p_1}\hat{b}_{p_2}\\
&\quad - u_{p_1}v_{p_2}\hat{b}^{\dagger}_{p_1}\hat{b}_{-p_2}] + \sum_{j,l=1}^{3}\tilde{\sum_{p_1,p_2}}\frac{\partial n_c}{\partial x_j}\frac{g}{2V}\int \mathrm{d}x'(x'_j - x_j)\\
&\quad\times\exp[ix'\cdot(p_2-p_1)][2(u_{p_1}u_{p_2} - u_{p_2}v_{p_1} - u_{p_1}v_{p_2}\\
&\quad + v_{p_1}v_{p_2})\hat{b}^{\dagger}_{p_1}\hat{b}_{p_2} + (u_{p_1}u_{p_2} + v_{p_1}v_{p_2} - 2u_{p_1}v_{p_2})\hat{b}^{\dagger}_{p_1}\hat{b}^{\dagger}_{-p_2}\\
&\quad + (u_{p_1}u_{p_2} + v_{p_1}v_{p_2} - 2u_{p_2}v_{p_1})\hat{b}_{-p_1}\hat{b}_{p_2}]\\
&\quad - \frac{i\hbar}{2}\frac{\partial(n_cv_j)}{\partial x_j}\tilde{\sum_p}\frac{\partial\vartheta_p}{\partial n_c}(\hat{b}_p\hat{b}_{-p} - \hat{b}^{\dagger}_p\hat{b}^{\dagger}_{-p}).
\end{aligned} \tag{23}$$

Define the distribution function for the Bogoliubov excitations by

$$f(x,p,t) = \mathrm{Tr}[\hat{\rho}_b(x,t)\hat{f}(x,t)] = \langle\hat{f}(x,t)\rangle,$$

it follows from the quantum Liouville equation (21) that

$$\frac{\partial f(x,p,t)}{\partial t} = -\frac{i}{\hbar}\sum_{q\neq\pm 2p}\exp(iq\cdot x)\langle[\hat{b}^{\dagger}_{p-q/2}\hat{b}_{p+q/2}, \hat{H}_b]\rangle. \tag{24}$$

The three Hamiltonians $\hat{H}_2$, $\hat{H}_3$, $\hat{H}_4$ will only contribute to the collisional processes, similar to the spatial homogeneous case [21]. As a result, the techniques introduced in [21] can be

reused to derive the collision operators. The explicit forms of $\hat{H}_2$, $\hat{H}_3$, $\hat{H}_4$ will be given in the Appendix.

Notice that in the superfluid rest frame, the condensate wave function $\Phi(x,t)$ becomes $n_c(x,t)^{1/2}$, and the phase $\phi(x,t)$ is removed. As a result, instead of writing the equation for the dynamics of $\Phi(x,t)$, we write the equation for $\Upsilon = \Phi\exp(i\phi)$

$$i\hbar\frac{\partial\Upsilon(x,t)}{\partial t} = \Big(-\frac{\hbar\nabla^2}{2m} + gn_c(x,t) + 2g\tilde{n}(x,t) + U(x)\Big) \times \Upsilon(x,t) + g\tilde{m}(x,t)\Upsilon^*(x,t) + g\langle\hat{\psi}^\dagger(x,t)\hat{\psi}(x,t)\hat{\psi}(x,t)\rangle, \quad (25)$$

where $\tilde{n}(x,t) = \langle\hat{\psi}^\dagger(x,t)\hat{\psi}(x,t)\rangle$ is the non-equilibrium non-condensate density, $\tilde{m}(x,t) = \langle\hat{\psi}(x,t)\hat{\psi}(x,t)\rangle$ is the off-diagonal non-condensate density. All of the terms in (25) are non-zero due to the Bose broken symmetry.

3 The quantum Boltzmann equation

If we replace the right hand side of (24) by $\hat{H}_{T,1}$ and label this term by $\mathcal{A}$, we find

$$\mathcal{A} = \frac{i}{\hbar}\sum_{q\neq\pm 2p}\exp(iq\cdot x)[\omega_{p+q/2} - \omega_{p-q/2} + \hbar q\cdot v]\langle\hat{b}^\dagger_{p-q/2}\hat{b}_{p+q/2}\rangle.$$

We now introduce the key approximation of our derivation. We suppose that the hydrodynamic variables are slowly varying in space and time. As a result, we will expand the microscopic quantities n_c and v about their values in x and the gradients $|\nabla_x n_c(x,t)|^k$, $|\nabla_x v(x,t)|^k$ become the parameters describing the smallness in our asymptotic expansion. We denote these smallness parameters by $O(|\nabla|^k)$, which are the only smallness parameters being used. Now, in (26), expanding $\omega_{p+q/2} - \omega_{p-q/2}$ and v in powers of q, we obtain the approximation

$$\mathcal{A} = -\sum_{j=1}^{3}\frac{1}{\hbar}\frac{\partial}{\partial p_j}(\omega_p + \hbar p\cdot v)\frac{\partial f(x,p,t)}{\partial x_j} + O(|\nabla|). \quad (26)$$

Similarly, if we replace the right hand side of (24) by $\hat{H}_{T,2}$ and label this term by $\mathcal{B}$, the same asymptotic expansion also gives

$$\mathcal{B} = \sum_{j=1}^{3}\frac{1}{\hbar}\frac{\partial}{\partial x_j}(\omega_p + \hbar p\cdot v)\frac{\partial f(x,p,t)}{\partial p_j} + O(|\nabla|). \quad (27)$$

The above two approximations give the transport part of the kinetic equation

$$\mathcal{T}[f] = \left(\frac{\partial}{\partial t} + \frac{1}{\hbar}\nabla_p(\omega_p + \hbar p\cdot v)\cdot\nabla_x - \frac{1}{\hbar}\nabla_x(\omega_p + \hbar p\cdot v)\cdot\nabla_p\right) f. \quad (28)$$

The derivation of the collision operators from $\hat{H}_2$, $\hat{H}_3$ and $\hat{H}_4$ follows verbatim the argument of the homogeneous case [21], based on the method by Akhiezer and Peletminskii [1]. We note that Kirkpartrick–Dorfman's method [12] is also based on the same principles. Indeed, the effects of $\hat{H}_2$ and $\hat{H}_3$ have already been studied in [21, Sections IV, V]. On the other hand, the role of $\hat{H}_4$ is exactly the same as that of $\hat{H}_{1,1}$, $\hat{H}'_{3,1}$ and, therefore, is negligible, following [21, Section IV]. As a consequence, we will not repeat the derivation of

the collision operators here, but rather recall the main ideas of the computations. The key assumption of the Akhiezer–Peletminskii method is that in the long time $t >> t_0 \equiv r_0/v_0$, where r_0 is the radius of the initial correlations and v_0 is the average quasi-particle velocity, the state of the system of weakly interacting Bogoliubov quasiparticles is played by the single-particle density matrix. By applying twice the quantum Liouville equation (24), we arrive at a new form of the type $\langle[\hat{H}_2+\hat{H}_3+\hat{H}_4,[\hat{b}^\dagger\hat{b},\hat{H}_2+\hat{H}_3+\hat{H}_4]]\rangle$. We then compute all of the commutators, without doing any approximations by dropping terms. Among all of the possible combinations, there are only three that really contribute into the collisional processes $\langle[\hat{H}_{1,2},[\hat{b}^\dagger\hat{b},\hat{H}_{1,2}]\rangle$, $\langle[\hat{H}_{2,2},[\hat{b}^\dagger\hat{b},\hat{H}_{2,2}]\rangle$ and $\langle[\hat{H}_{3,1},[\hat{b}^\dagger\hat{b},\hat{H}_{3,1}]\rangle$, while the other terms can be proved to vanish due to the violation of the conservation of energy. As a result, the C_{12} collision operator arises from commutators of the type $[\hat{b}^\dagger\hat{b}^\dagger\hat{b},[\hat{b}^\dagger\hat{b},\hat{b}^\dagger\hat{b}\hat{b}]]$ and $[\hat{b}^\dagger\hat{b}\hat{b},[\hat{b}^\dagger\hat{b},\hat{b}^\dagger\hat{b}^\dagger\hat{b}]]$, coming from $\langle[\hat{H}_{1,2},[\hat{b}^\dagger\hat{b},\hat{H}_{1,2}]\rangle$. The C_{22} collision operator arises from commutators of the type $[\hat{b}^\dagger\hat{b}^\dagger\hat{b}\hat{b},[\hat{b}^\dagger\hat{b},\hat{b}^\dagger\hat{b}^\dagger\hat{b}\hat{b}]]$, coming from $\langle[\hat{H}_{2,2},[\hat{b}^\dagger\hat{b},\hat{H}_{2,2}]\rangle$. The C_{31} collision operator arises from commutators of the types $[\hat{b}^\dagger\hat{b}^\dagger\hat{b}^\dagger\hat{b},[\hat{b}^\dagger\hat{b},\hat{b}^\dagger\hat{b}\hat{b}\hat{b}]]$ and $[\hat{b}^\dagger\hat{b}\hat{b}\hat{b},[\hat{b}^\dagger\hat{b},\hat{b}^\dagger\hat{b}^\dagger\hat{b}^\dagger\hat{b}]]$, coming from $\langle[\hat{H}_{3,1},[\hat{b}^\dagger\hat{b},\hat{H}_{3,1}]\rangle$. Collecting (28) and the three collision operators C_{12}, C_{22}, C_{31}, we finally obtain the quantum Boltzmann equation

$$
\begin{aligned}
\mathcal{T}[f] = {} & \frac{4\pi g^2 n_c}{\hbar(2\pi)^3}\int'\int'\int' \mathrm{d}p_1\mathrm{d}p_2\mathrm{d}p_3\delta(p_1-p_2-p_3)\\
& \times(\delta(p-p_1)-\delta(p-p_2)-\delta(p-p_3))\delta(\omega_1-\omega_2-\omega_3)\\
& \times(K^{1,2}_{1,2,3})^2\Big[f_2f_3(f_1+1)-f_1(f_2+1)(f_3+1)\Big]\\
& +\frac{\pi g^2}{\hbar(2\pi)^6}\int'\int'\int'\int' \mathrm{d}p_1\mathrm{d}p_2\mathrm{d}p_3\mathrm{d}p_4\\
& \times(\delta(p-p_1)+\delta(p-p_2)-\delta(p-p_3)-\delta(p-p_4))\\
& \times\delta(\omega_1+\omega_2-\omega_3-\omega_4)\delta(p_1+p_2-p_3-p_4)(K^{2,2}_{1,2,3,4})^2\\
& \times\Big[f_3f_4(f_2+1)(f_1+1)-f_1f_2(f_3+1)(f_4+1)\Big]\\
& +\frac{3\pi g^2}{\hbar(2\pi)^6}\int'\int'\int'\int' \mathrm{d}p_1\mathrm{d}p_2\mathrm{d}p_3\mathrm{d}p_4\\
& \times(\delta(p-p_1)-\delta(p-p_2)-\delta(p-p_3)-\delta(p-p_4))\\
& \times\delta(p_1-p_2-p_3-p_4)\delta(\omega_1-\omega_2-\omega_3-\omega_4)(K^{3,1}_{1,2,3,4})^2\\
& \times\Big[f_3f_4f_2(f_1+1)-f_1(f_2+1)(f_3+1)(f_4+1)\Big],
\end{aligned}
\tag{29}
$$

in which, the quantities $K^{1,2}_{1,2,3}$, $K^{2,2}_{1,2,3,4}$ and $K^{3,1}_{1,2,3,4}$ are defined in (9)–(11). The quantity n_c can be deduced from the solution of the Gross–Pitaevskii equation by (1).

4 The Gross–Pitaevskii equation

Similar with Griffin et al. [6] and Pomeau and Tran [14], we also limit our analysis to the Popov approximation by setting $\tilde{m}=0$. From (25), we obtain

$$
\begin{aligned}
i\hbar\frac{\partial\Upsilon(x,t)}{\partial t} = {} & \Big(-\frac{\hbar\nabla^2}{2m}+gn_c(x,t)+2g\tilde{n}(x,t)+U(x)\Big)\\
& \times\Upsilon(x,t)+g\langle\hat{\psi}^\dagger(x,t)\hat{\psi}(x,t)\hat{\psi}(x,t)\rangle.
\end{aligned}
\tag{30}
$$

To derive the generalized Gross–Pitaevskii equation, we only need to compute $g\langle\hat{\psi}^\dagger(x,t)\hat{\psi}(x,t)\hat{\psi}(x,t)\rangle$ in (30). This quantity can be computed by exactly the same strategy used to compute C_{12} (see also [16,17]). As a consequence, we skip the details of this calculation and display only the final result, which also involves C_{12} in its expression

$$g\langle\hat{\psi}^\dagger(x,t)\hat{\psi}(x,t)\hat{\psi}(x,t)\rangle = -\frac{i\hbar\Upsilon}{2n_c}\int' \mathrm{d}pC_{12}[f]. \tag{31}$$

Plugging (31) into (30) yields

$$\begin{aligned} i\hbar\frac{\partial\Upsilon(x,t)}{\partial t} &= \Big(-\frac{\hbar\nabla^2}{2m} + gn_c(x,t) + 2g\tilde{n}(x,t) + U(x)\Big) \\ &\quad\times\Upsilon(x,t) - \frac{i\hbar\Upsilon}{2n_c}\int' \mathrm{d}pC_{12}[f], \end{aligned} \tag{32}$$

which is the same with (2). This is our Gross–Pitaevskii equation, in which, $\tilde{n}$ is computed using the solution of the quantum Boltzmann equation via (3) and n_c is determined by (1). The form of C_{12} is given in (6).

5 Conclusion

In this work, we have derived a coupling system that includes a Gross–Pitaevskii equation of motion for the condensate wave function and a quantum Boltzmann equation for the excitations. The model approaches the standard ZNG model when the temperature of the system T is close to the Bose–Einstein Condensate (BEC) transition temperature T_c.

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6 Appendix 1

We present below the explicit forms of $\hat{H}_2$, $\hat{H}_3$ and $\hat{H}_4$, which can be computed following the same argument with Tran and Pomeau [21]

$$\begin{aligned} \hat{H}_2 &= \hat{H}_{1,2} + \hat{H}_{3,0}, \\ \hat{H}_{1,2} &= g\sqrt{\frac{n_c}{V}}\sum_{p_1,p_2,p_3\neq 0}\delta(p_1-p_2-p_3)K^{1,2}_{1,2,3} \\ &\quad\times(\hat{b}^\dagger_{p_1}\hat{b}_{p_2}\hat{b}_{p_3} + \hat{b}^\dagger_{p_3}\hat{b}^\dagger_{p_2}\hat{b}_{p_1}), \\ \hat{H}_{3,0} &= g\sqrt{\frac{n_c}{V}}\sum_{p_1,p_2,p_3\neq 0}\delta(p_1+p_2+p_3) \\ &\quad\times\Big[K^{3,0}_{1,2,3}(\hat{b}^\dagger_{p_3}\hat{b}^\dagger_{p_2}\hat{b}^\dagger_{p_1} + \hat{b}_{p_1}\hat{b}_{p_2}\hat{b}_{p_3})\Big], \\ K^{3,0}_{1,2,3} &= u_{p_1}v_{p_2}v_{p_3} - v_{p_1}u_{p_2}u_{p_3}; \end{aligned} \tag{33}$$

and

$$\begin{aligned}
\hat{H}_3 &= \hat{H}_{2,2} + \hat{H}_{1,1} + \hat{H}_{3,1} + \hat{H}'_{3,1} + \hat{H}_{4,0},\\
\hat{H}_{2,2} &= \frac{g}{2V}\sum_{p_1,p_2,p_3,p_4\neq 0}\delta(p_1+p_2-p_3-p_4)K^{2,2}_{1,2,3,4}\\
&\quad\times \hat{b}^{\dagger}_{p_1}\hat{b}^{\dagger}_{p_2}\hat{b}_{p_3}\hat{b}_{p_4},\\
\hat{H}_{1,1} &= \frac{g}{2V}\sum_{p_1,p_2\neq 0}K^{1,1}_{1,2}\hat{b}^{\dagger}_{p_1}\hat{b}_{p_1},\\
K^{1,1}_{1,2} &= 4v^2_{p_1}v^2_{p_2}+4u^2_{p_1}v^2_{p_2}+4u_{p_1}v_{p_1}u_{p_2}v_{p_2},\\
\hat{H}_{3,1} &= \frac{g}{2V}\sum_{p_1,p_2,p_3,p_4\neq 0}\delta(p_1-p_2-p_3-p_4)\\
&\quad\times K^{3,1}_{1,2,3,4}\Big[\hat{b}^{\dagger}_{p_1}\hat{b}_{p_2}\hat{b}_{p_3}\hat{b}_{p_4}+\hat{b}^{\dagger}_{p_4}\hat{b}^{\dagger}_{p_3}\hat{b}^{\dagger}_{p_2}\hat{b}_{p_1}\Big],\\
\hat{H}'_{3,1} &= \frac{g}{2V}\sum_{p_1,p_2\neq 0}\Big[\hat{b}_{p_1}\hat{b}_{-p_1}K^{2,0}_{1,2}+\hat{b}^{\dagger}_{p_1}\hat{b}^{\dagger}_{-p_1}K^{2,0}_{1,2}\Big],\\
K^{2,0}_{1,2} &= u^2_{p_1}u_{p_2}v_{p_2}+v^2_{p_1}u_{p_2}v_{p_2}+4u_{p_1}v_{p_1}v^2_{p_2},\\
\hat{H}_{4,0} &= \frac{g}{2V}\sum_{p_1,p_2,p_3,p_4\neq 0}\delta(p_1+p_2+p_3+p_4)\\
&\quad\times K^{4,0}_{1,2,3,4}\Big[\hat{b}^{\dagger}_{p_1}\hat{b}^{\dagger}_{p_2}\hat{b}^{\dagger}_{p_3}\hat{b}^{\dagger}_{p_4}+\hat{b}_{p_1}\hat{b}_{p_2}\hat{b}_{p_3}\hat{b}_{p_4}\Big],\\
K^{4,0}_{1,2,3,4} &= u_{p_1}u_{p_2}v_{p_3}v_{p_4};
\end{aligned}\tag{34}$$

Finally

$$\begin{aligned}
\hat{H}_4 &= i\hbar n_c^{1/2}L_{n_c}\sum_{p\neq 0}\frac{\partial\vartheta_p}{\partial n_c}(\hat{b}_p\hat{b}_{-p}-\hat{b}^{\dagger}_p\hat{b}^{\dagger}_{-p})\\
&\quad-\sum_{p\neq 0}m\varrho[u^2_p+v^2_p]\hat{b}^{\dagger}_p\hat{b}_p+\sum_{p\neq 0}m\varrho u_pv_p[\hat{b}_p\hat{b}_{-p}+\hat{b}^{\dagger}_p\hat{b}^{\dagger}_{-p}]\\
\varrho &= -\frac{v^2}{2}-\frac{\hbar}{m}\frac{\partial\phi}{\partial t}-gn_c,\quad L_{n_c}=\hbar^{-1}\mathrm{Im}S,\\
S &= g\mathrm{Tr}\Big[\hat{\rho}_b\hat{\psi}^{\dagger}\hat{\psi}\hat{\psi}\Big]+2n_c^{1/2}g\mathrm{Tr}\Big[\hat{\rho}_b\hat{\psi}^{\dagger}\hat{\psi}\Big]+n_c^{1/2}g\mathrm{Tr}\Big[\hat{\rho}_b\hat{\psi}\hat{\psi}\Big].
\end{aligned}\tag{35}$$

References

1. A.I. Akhiezer, S.V. Peletminskii, *Methods of Statistical Physics, Volume 104 of International Series in Natural Philosophy* (Elsevier, New York, 1981)
2. M.H. Anderson, J.R. Ensher, M.R. Matthews, C.E. Wieman, E.A. Cornell, Observation of Bose–Einstein condensation in a dilute atomic vapor. Science **269**(5221), 198–201 (1995)
3. C.C. Bradley, C.A. Sackett, J.J. Tollett, R.G. Hulet, Evidence of Bose–Einstein condensation in an atomic gas with attractive interactions. Phys. Rev. Lett. **75**(9), 1687 (1995)
4. F. Dalfovo, S. Giorgini, L.P. Pitaevskii, S. Stringari, Theory of Bose–Einstein condensation in trapped gases. Rev. Mod. Phys. **71**(3), 463 (1999)
5. K.B. Davis, M.-O. Mewes, M.R. Andrews, N.J. van Druten, D.S. Durfee, D.M. Kurn, W. Ketterle, Bose–Einstein condensation in a gas of sodium atoms. Phys. Rev. Lett. **75**(22), 3969 (1995)

6. A. Griffin, T. Nikuni, E. Zaremba, *Bose-Condensed Gases at Finite Temperatures* (Cambridge University Press, Cambridge, 2009)
7. E. D. Gust and L. E. Reichl. Collision integrals in the kinetic equations of dilute Bose–Einstein condensates. arXiv:1202.3418 (2012)
8. E.D. Gust, L.E. Reichl, Decay of hydrodynamic modes in dilute Bose–Einstein condensates. Phys. Rev. A **90**(4), 043615 (2014)
9. M. Imamovic-Tomasovic, A. Griffin, Quasiparticle kinetic equation in a trapped Bose gas at low temperatures. J. Low Temp. Phys. **122**, 617–655 (2001)
10. T.R. Kirkpatrick, J.R. Dorfman, Transport theory for a weakly interacting condensed Bose gas. Phys. Rev. A (3) **28**(4), 2576–2579 (1983)
11. T.R. Kirkpatrick, J.R. Dorfman, Transport coefficients in a dilute but condensed Bose gas. J. Low Temp. Phys. **58**, 399–415 (1985)
12. T.R. Kirkpatrick, J.R. Dorfman, Transport in a dilute but condensed nonideal Bose gas: kinetic equations. J. Low Temp. Phys. **58**, 301–331 (1985)
13. T.-D. Lee, C.-N. Yang, Question of parity conservation in weak interactions. Phys. Rev. **104**(1), 254 (1956)
14. Y. Pomeau, M.-B. Tran, *Statistical Physics of Non Equilibrium Quantum Phenomena, Volume of 967 Lecture Notes in Physics* (Springer, Berlin, 2019)
15. N. Proukakis, S. Gardiner, M. Davis, M. Szymanska, *Cold Atoms: Volume 1 Quantum Gases Finite Temperature and Non-Equilibrium Dynamics* (Imperial College Press, 2013)
16. N.P. Proukakis, K. Burnett, Generalized mean fields for trapped atomic Bose–Einstein condensates. J. Res. Natl. Inst. Stand. Technol. **101**(4), 457 (1996)
17. N.P. Proukakis, K. Burnett, H.T.C. Stoof, Microscopic treatment of binary interactions in the nonequilibrium dynamics of partially Bose-condensed trapped gases. Phys. Rev. A **57**(2), 1230 (1998)
18. L.E. Reichl, E.D. Gust, Transport theory for a dilute Bose-Einstein condensate. Phys. Rev. A **88**(5), 053603 (2013)
19. L.E. Reichl, M.-B. Tran, A kinetic equation for ultra-low temperature Bose-Einstein condensates. J. Phys. A: Math. Theor. **52**(6), 063001 (2019)
20. I. Shammass, S. Rinott, A. Berkovitz, R. Schley, J. Steinhauer, Phonon dispersion relation of an atomic bose-einstein condensate. Phys. Rev. Lett. **109**(19), (2012)
21. M.-B. Tran, Y. Pomeau, Boltzmann-type collision operators for Bogoliubov excitations of Bose-Einstein condensates: a unified framework. Phys. Rev. E **101**(3), (2020)
22. Pomeau, Y., Brachet, M.É., Métens, S. and Rica, S. Théorie cinétique d'un gaz de bose dilué avec condensat. C. R. Acad Sci. Paris S'er. IIb M'ec. Phys. Astr. **327**, 791–798 (1999)
23. E. Zaremba, T. Nikuni, A. Griffin, Dynamics of trapped bose gases at finite temperatures. J. Low Temp. Phys. **116**, 277–345 (1999)