

On Shimura varieties for unitary groups

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To David Mumford on his 81st birthday

Abstract: This is a largely expository article based on our paper [31] on arithmetic diagonal cycles on unitary Shimura varieties. We define a class of Shimura varieties closely related to unitary groups which represent a moduli problem of abelian varieties with additional structure, and which admit interesting algebraic cycles. We generalize to arbitrary signature type the results of loc. cit. valid under special signature conditions. We compare our Shimura varieties with other unitary Shimura varieties.

Keywords: Shimura varieties, local models, Gan–Gross–Prasad cycles, Kudla–Rapoport divisors.

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1. Introduction

In [3], Deligne gives the definition of a Shimura variety $(S(G, \{h\})_K)$ (a tower of quasi-projective complex varieties indexed by sufficiently small compact open subgroups $K \subset G(\mathbb{A}_f)$) starting from a Shimura datum $(G, \{h\})$. He defines the associated Shimura field $E(G, \{h\})$ and proves that there is at most one *canonical model* of $(S(G, \{h\})_K)$ over $E(G, \{h\})$. He then goes on to construct the canonical model for some Shimura varieties associated to classical groups, by giving an interpretation of the varieties in the tower as moduli spaces of abelian varieties with additional structure. The basic example is given by the group G of symplectic similitudes with its natural conjugacy class $\{h\}$ (the *Siegel case*)—this case leads to the moduli space of principally polarized abelian varieties. Let p be a prime number. For open compact subgroups K of the form $K = K^p \times K_p$, where $K^p \subset G(\mathbb{A}_f^p)$ and where $K_p \subset G(\mathbb{Q}_p)$ is the stabilizer of a self-dual lattice (i.e., K_p is hyperspecial), this moduli description allows one to extend the model over \mathbb{Q} of $(S(G, \{h\})_K)$ to a model over the localization $\mathbb{Z}_{(p)}$ with good reduction modulo p .

Another class of examples is related to unitary groups. The case of the group of unitary similitudes is treated briefly by Deligne in [3] and in detail by Kottwitz in [14]. It is considerably more difficult than the Siegel case due to the failure of the Hasse principle for these groups. A special case of these Shimura varieties is used in all proofs of the local Langlands conjecture for p -adic fields (the *Harris-Taylor case*). It turns out that if n is even, the Shimura variety represents a moduli problem of abelian varieties with additional structure; if n is odd, the analogous moduli problem is represented by a finite disjoint sum of copies of the Shimura variety. If $K = K^p \times K_p$, where K_p is hyperspecial, Kottwitz defines a p -integral model of the corresponding model over $E(G, \{h\})$.

Another case is given by the unitary groups. This case is considered in [6], in the *fake Drinfeld signature* case (see Example 2.3 below), and is used in the formulation of the Arithmetic Gan–Gross–Prasad conjecture. This class of Shimura varieties is not of *PEL type*, i.e., is not represented by a moduli problem of abelian varieties with additional structure. It is, however, of *abelian type*. This entails that its canonical model is defined in Deligne [4]. It also implies that, by Kisin–Pappas [13], it has a p -integral model when

$K = K^p \times K_p$, where K_p is an arbitrary parahoric subgroup. However, both constructions are rather indirect and yield models which are difficult to analyze. This seems to be a major impediment to progress on these Shimura varieties.

In this paper, following [31], we formulate a new variant of the Deligne–Kottwitz Shimura varieties and compare it with the previous two classes. Our variant has the advantage of always representing a moduli problem of abelian varieties. By extending the moduli problem, we also define p -integral models when $K = K^p \times K_p$, with K_p a parahoric subgroup. In fact, under certain special circumstances, we even define global integral models (in the fake Drinfeld case). Another advantage of our variant is that it always accommodates the algebraic cycles that appear in the Gan–Gross–Prasad intersection problem and in the Kudla–Rapoport intersection problem. We refer to [31], where we give a variant of the Arithmetic Gan–Gross–Prasad conjecture and solve it in certain low-dimensional cases. In the case of an imaginary quadratic field, our variant Shimura variety appears also in [2], with similar aims. However, this special case does not bring out all the features of our definition; in particular, the *sign invariant* of [31] does not play any role in that case, comp. the table in Section 4.3 below.

Our paper is largely expository. One of our aims is to show that the definitions in [31] extend from the fake Drinfeld signature case to the case of general signature. We also go beyond [31] in that we also discuss the problem of flatness, resp. of smoothness, resp. of regularity, of the p -integral models in general. Our hope is that the global integral models constructed here will find applications in arithmetic intersection problems in analogy with those mentioned above.

The lay-out of the paper is as follows. In Section 2, we set the stage by recalling the Shimura varieties attached to groups of unitary similitudes and to unitary groups. In Section 3, we introduce the variant of these Shimura varieties introduced in [31]. In Section 4, we define p -integral models of these last Shimura varieties. In Section 5, we discuss flatness and smoothness of these p -integral models. In Section 6, we discuss global integral models. In Appendix A, we show how the formalism of the local model diagram of [32] holds for all primes p (including $p = 2$) in the case of unramified PEL data of type A.

The paper is a vastly extended version of the talk with the same title given by one of us (M.R.) at the *SuperAG* in Bonn 2017, at the occasion of his retirement from the University of Bonn. It is also related to his talk at the 2018 Simons conference *Periods and L-values of motives*, and to Appendix C in Y. Liu’s article [22].

We dedicate the paper to D. Mumford on the occasion of his 81st birthday. He is the founder of the theory of moduli spaces of abelian varieties and has been a main force in moving this subject to the forefront of mathematics. On a personal level, one of us (M.R.) owes more to him than can be said in a few lines; he is happy to express here his gratitude. We also thank the referee for his/her remarks on the paper.

Notation

In order to deal with *all* congruence subgroups K , not only ones that are small enough, we consider the tower $(S(G, \{h\})_K)$ as a tower of *orbifolds*. However, abusing language, we continue to refer to this tower as a Shimura variety.

We write \mathbb{A}_F , $\mathbb{A}_{F,f}$, and $\mathbb{A}_{F,f}^p$ for the respective rings of adèles, finite adèles, and finite adèles away from p of a number field F . When $F = \mathbb{Q}$, we abbreviate these to \mathbb{A} , \mathbb{A}_f , and \mathbb{A}_f^p , respectively, and we set $\overline{\mathbb{A}} := \mathbb{A} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$. Here $\overline{\mathbb{Q}}$ denotes the algebraic closure of \mathbb{Q} in \mathbb{C} . We fix once and for all an element $\sqrt{-1} \in \mathbb{C}$.

We take all hermitian forms to be linear in the first variable and conjugate-linear in the second, and we always assume that they are nondegenerate.

When working with vector spaces over F , we use a superscript $*$ to denote F -linear dual spaces and homomorphisms. We also use a superscript $*$ to denote dual lattices (in both the global and local contexts) with respect to a hermitian form. By contrast, we always use a superscript \vee to denote dual lattices with respect to a \mathbb{Q} - or \mathbb{Q}_p -valued bilinear form. The following situation (say, in the global context; the local context is completely analogous) will arise repeatedly throughout this paper. Let $(W, (\ , \))$ be a hermitian space for F with respect to an order 2 automorphism (in the paper, F will always be a CM field), let $\zeta \in F$ be a traceless element for this automorphism, and consider the alternating \mathbb{Q} -valued form $\text{tr}_{F/\mathbb{Q}} \zeta(\ , \)$ on W . Then for any O_F -lattice $\Lambda \subset W$, we have $\Lambda^\vee = \zeta^{-1} \mathfrak{D}^{-1} \Lambda^*$, where \mathfrak{D} denotes the different of F/\mathbb{Q} .

In the context of abelian schemes, we use a superscript \vee to denote the dual abelian scheme and dual morphisms. A *quasi-polarization* (sometimes called a \mathbb{Q} -*polarization* in the literature) on an abelian scheme A is a symmetric quasi-isogeny $\lambda: A \rightarrow A^\vee$ such that, Zariski-locally on the base, $n\lambda$ is an honest polarization for some positive integer n . We denote the Rosati involution of a quasi-polarization $\lambda: A \rightarrow A^\vee$ by Ros_λ . We denote by $\text{Hom}^0(A, B)$, resp. $\text{Hom}_{(p)}(A, B)$, the Zariski-sheafification of $U \mapsto \text{Hom}(A_U, B_U) \otimes_{\mathbb{Z}} \mathbb{Q}$, resp. $U \mapsto \text{Hom}(A_U, B_U) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$, for U an open subscheme of the base (the

homomorphism group in the isogeny category, resp. the prime-to- p isogeny category). We similarly define $\text{End}^0(A)$ and $\text{End}_{(p)}(A)$. When the base S locally noetherian, we denote by $\widehat{T}(A) = \prod_{\ell} T_{\ell}(A)$, resp. $\widehat{V}(A) = \widehat{T}(A) \otimes \mathbb{Q}$, resp. $\widehat{V}^p(A) = (\prod_{\ell \neq p} T_{\ell}(A)) \otimes \mathbb{Q}$, the Tate module, resp. the rational Tate module, resp. the rational Tate module prime to p , all regarded as smooth sheaves on S (assuming that S is a \mathbb{Q} -scheme in the case of $\widehat{T}(A)$ and $\widehat{V}(A)$, and a $\mathbb{Z}_{(p)}$ -scheme in the case of $\widehat{V}^p(A)$). Similarly, when a number field acts on A up to isogeny and v is a finite place of the number field whose residue characteristic ℓ is invertible on S , we denote by $V_v(A)$ the v -factor of $V_{\ell}(A)$. When furthermore the localization at ℓ of the ring of integers of the number field acts on A up to prime-to- ℓ isogeny, we denote by $T_v(A)$ the v -factor of $T_{\ell}(A)$.

We often use a subscript S to denote base change to S , and when $S = \text{Spec } A$, we often use the subscript A instead. Similarly, we sometimes write $X \otimes_B A$ to denote $X \times_{\text{Spec } B} \text{Spec } A$.

We write $(\text{LNSch})/R$ for the category of locally noetherian schemes over $\text{Spec } R$ for a ring R .

2. The Shimura varieties of Deligne and Kottwitz, and of Gan–Gross–Prasad

2.1. The group of symplectic similitudes

As motivation, we start with the Siegel case. Let $(W, \langle \ , \ \rangle)$ be a nonzero symplectic vector space of dimension $n = 2m$ over \mathbb{Q} . Let $\text{GSp} = \text{GSp}(W, \langle \ , \ \rangle)$ be the group of symplectic similitudes. Choose a symplectic basis of W , i.e., a basis with respect to which the matrix of $\langle \ , \ \rangle$ is given by

$$H_n = \begin{bmatrix} 0_m & -1_m \\ 1_m & 0_m \end{bmatrix},$$

where the displayed entries are $m \times m$ block matrices. The conjugacy class $\{h_{\text{GSp}}\}$ in the Shimura datum is the $\text{GSp}(\mathbb{R})$ -conjugacy class of the homomorphism

$$\begin{aligned} h_{\text{GSp}}: \quad \mathbb{C}^{\times} &\longrightarrow \text{GSp}(\mathbb{R}) \\ a + b\sqrt{-1} &\longmapsto a1_n + bH_n. \end{aligned}$$

Let $K \subset \text{GSp}(\mathbb{A}_f)$ be a compact open subgroup. Let \mathcal{F}_K be the category fibered in groupoids over $(\text{LNSch})/\mathbb{Q}$ which associates to each \mathbb{Q} -scheme S the groupoid of triples $(A, \lambda, \bar{\eta})$, where

- A is an abelian scheme over S ;
- λ is a quasi-polarization on A ; and
- $\bar{\eta}$ is a K -orbit of symplectic similitudes

$$(2.1) \quad \eta: \widehat{V}(A) \xrightarrow{\sim} W \otimes \mathbb{A}_f,$$

where K acts via the tautological representation of $\mathrm{GSp}(\mathbb{A}_f)$ on $W \otimes \mathbb{A}_f$, cf. [14, §5]. The morphisms $(A, \lambda, \bar{\eta}) \rightarrow (A', \lambda', \bar{\eta}')$ in this groupoid are the quasi-isogenies $\mu: A \rightarrow A'$ such that, Zariski-locally on S , the pullback of λ' is a \mathbb{Q}^\times -multiple of λ , and such that the pullback of $\bar{\eta}'$ is $\bar{\eta}$.

Note that the Weil form on the rational Tate module $\widehat{V}(A)$ defined by λ naturally takes values in $\mathbb{A}_f(1)$; to compare the symplectic forms on both sides of (2.1), it is necessary to choose a trivialization $\mathbb{A}_f(1) \xrightarrow{\sim} \mathbb{A}_f$, which is unique up to a factor in \mathbb{A}_f^\times .

The theorem in this context, which is the model of all other theorems in this paper, is the following.

Theorem 2.1. *The moduli problem \mathcal{F}_K is representable by a Deligne–Mumford stack M_K over $\mathrm{Spec} \mathbb{Q}$, and*

$$M_K(\mathbb{C}) = S(\mathrm{GSp}, \{h_{\mathrm{GSp}}\})_K,$$

compatible with changing K . □

In fact, the tower (M_K) is the canonical model of the Shimura variety $(S(\mathrm{GSp}, \{h_{\mathrm{GSp}}\})_K)$.

2.2. The group of unitary similitudes

Let F be a CM number field with maximal totally real subfield F_0 and nontrivial F/F_0 -automorphism $a \mapsto \bar{a}$. Let n be a positive integer. A *generalized CM type of rank n* is a function $r: \mathrm{Hom}_{\mathbb{Q}}(F, \overline{\mathbb{Q}}) \rightarrow \mathbb{Z}_{\geq 0}$, denoted $\varphi \mapsto r_\varphi$, such that

$$(2.2) \quad r_\varphi + r_{\bar{\varphi}} = n \quad \text{for all } \varphi,$$

comp. [19]. Here $\bar{\varphi}$ denotes the precomposition of φ by the nontrivial F/F_0 -automorphism. When n is understood, we also refer to r as a *signature type*. When $n = 1$, a generalized CM type is “the same” as a usual CM type (i.e., a half-system Φ of complex embeddings of F), via

$$\Phi = \{ \varphi \in \mathrm{Hom}_{\mathbb{Q}}(F, \overline{\mathbb{Q}}) \mid r_\varphi = 1 \}.$$

Fix a CM type Φ of F , and let $(W, (\ , \))$ be an F/F_0 -hermitian vector space of dimension n . The signatures of W at the archimedean places determine a generalized CM type r of rank n , by writing

$$\text{sig } W_\varphi = (r_\varphi, r_{\bar{\varphi}}), \quad \varphi \in \Phi, \quad W_\varphi := W \otimes_{F,\varphi} \mathbb{C}.$$

Let $G^\mathbb{Q}$ be the group of unitary similitudes of $(W, (\ , \))$, considered as a linear algebraic group over \mathbb{Q} (with similitude factor in \mathbb{G}_m). For each $\varphi \in \Phi$, choose a \mathbb{C} -basis of W_φ with respect to which the matrix of $(\ , \)$ is given by

$$(2.3) \quad \text{diag}(1_{r_\varphi}, -1_{r_{\bar{\varphi}}}).$$

The conjugacy class $\{h_{G^\mathbb{Q}}\}$ in the Shimura datum is the $G^\mathbb{Q}(\mathbb{R})$ -conjugacy class of the homomorphism $h_{G^\mathbb{Q}} = (h_{G^\mathbb{Q},\varphi})_{\varphi \in \Phi}$, where the components $h_{G^\mathbb{Q},\varphi}$ are defined with respect to the inclusion

$$(2.4) \quad G^\mathbb{Q}(\mathbb{R}) \subset \text{GL}_{F \otimes \mathbb{R}}(W \otimes \mathbb{R}) \xrightarrow[\sim]{\Phi} \prod_{\varphi \in \Phi} \text{GL}_{\mathbb{C}}(W_\varphi),$$

and where each component is defined on \mathbb{C}^\times by

$$h_{G^\mathbb{Q},\varphi}: z \longmapsto \text{diag}(z \cdot 1_{r_\varphi}, \bar{z} \cdot 1_{r_{\bar{\varphi}}}).$$

Then the reflex field $E(G^\mathbb{Q}, \{h_{G^\mathbb{Q}}\})$ is the reflex field E_r of r , which is the subfield of $\overline{\mathbb{Q}}$ defined by

$$(2.5) \quad \text{Gal}(\overline{\mathbb{Q}}/E_r) = \{ \sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \mid \sigma^*(r) = r \}.$$

Let $K \subset G^\mathbb{Q}(\mathbb{A}_f)$ be a compact open subgroup. Let \mathcal{F}_K be the category fibered in groupoids over $(\text{LNSch})/E_r$ which associates to each E_r -scheme S the groupoid of quadruples $(A, \iota, \lambda, \bar{\eta})$, where

- A is an abelian scheme over S ;
- $\iota: F \rightarrow \text{End}^0(A)$ is an action of F on A up to isogeny;
- λ is a quasi-polarization on A ; and
- $\bar{\eta}$ is a K -orbit of $\mathbb{A}_{F,f}$ -linear symplectic similitudes

$$\eta: \widehat{V}(A) \xrightarrow{\sim} W \otimes_{\mathbb{Q}} \mathbb{A}_f,$$

cf. [14, §5]. Here, as in the Introduction, we equip W with the \mathbb{Q} -symplectic form $\langle \cdot, \cdot \rangle = \text{tr}_{F/\mathbb{Q}} \zeta(\cdot, \cdot)$ for some fixed element $\zeta \in F^\times$ satisfying $\bar{\zeta} = -\zeta$. We impose the conditions that

$$(2.6) \quad \text{Ros}_\lambda(\iota(a)) = \iota(\bar{a}) \quad \text{for all } a \in F,$$

and that A satisfies the Kottwitz condition of signature type r ,

$$(2.7) \quad \text{char}(\iota(a) \mid \text{Lie } A) = \prod_{\varphi \in \text{Hom}(F, \overline{\mathbb{Q}})} (T - \varphi(a))^{r_\varphi} \quad \text{for all } a \in F.$$

Here the left-hand side in (2.7) denotes the characteristic polynomial of the action of $\iota(a)$ on the locally free \mathcal{O}_S -module $\text{Lie } A$; the right-hand side, which is a priori a polynomial with coefficients in E_r , is regarded as an element of $\mathcal{O}_S[T]$ via the structure morphism. The morphisms $(A, \iota, \lambda, \bar{\eta}) \rightarrow (A', \iota', \lambda', \bar{\eta}')$ in this groupoid are the F -linear quasi-isogenies $\mu: A \rightarrow A'$ such that, Zariski-locally on S , the pullback of λ' is a \mathbb{Q}^\times -multiple of λ , and such that the pullback of $\bar{\eta}'$ is $\bar{\eta}$.

The analog of Theorem 2.1 is as follows.

Theorem 2.2 (Kottwitz). *The moduli problem \mathcal{F}_K is representable by a Deligne–Mumford stack M_K over $\text{Spec } E_r$, and if n is even,*

$$M_K(\mathbb{C}) = S(G^\mathbb{Q}, \{h_{G^\mathbb{Q}}\})_K,$$

compatible with changing K . If n is odd, then $M_K(\mathbb{C})$ is a finite disjoint union of copies of $S(G^\mathbb{Q}, \{h_{G^\mathbb{Q}}\})_K$, again compatible with changing K ; these copies are enumerated by

$$\ker^1(\mathbb{Q}, G^\mathbb{Q}) := \ker[\text{H}^1(\mathbb{Q}, G^\mathbb{Q}(\overline{\mathbb{Q}})) \longrightarrow \text{H}^1(\mathbb{Q}, G^\mathbb{Q}(\overline{\mathbb{A}}))]. \quad \square$$

As in the Siegel case, the tower (M_K) is in fact the canonical model of the Shimura variety $(S(G^\mathbb{Q}, \{h_{G^\mathbb{Q}}\})_K)$.

Example 2.3. (i) (*Fake Drinfeld type*) We say that the generalized CM type r of rank n is of *fake Drinfeld type* relative to a distinguished element $\varphi_0 \in \text{Hom}_\mathbb{Q}(F, \overline{\mathbb{Q}})$ if

$$r_\varphi = \begin{cases} n - 1, & \varphi = \varphi_0; \\ 1, & \varphi = \bar{\varphi}_0; \\ 0 \text{ or } n, & \varphi \in \text{Hom}_\mathbb{Q}(F, \overline{\mathbb{Q}}) \setminus \{\varphi_0, \bar{\varphi}_0\}. \end{cases}$$

In this case, F embeds into E_r via φ_0 for $n \geq 3$. For $n = 2$, at least F_0 embeds into E_r via φ_0 . For $n = 1$, all we can say is that E_r is the reflex field of the CM type which is the support of r .

(ii) (*Strict fake Drinfeld type*) We say that r is of *strict fake Drinfeld type* for the CM type Φ and an element $\varphi_0 \in \Phi$, and we write $r = r^{(\Phi, \varphi_0)}$, if

$$(2.8) \quad r_{\varphi}^{(\Phi, \varphi_0)} = \begin{cases} n - 1, & \varphi = \varphi_0; \\ n, & \varphi \in \Phi \setminus \{\varphi_0\}. \end{cases}$$

(iii) (*Harris–Taylor type*) This is a special case of strict fake Drinfeld type. Suppose that $F = K_0 F_0$, where K_0 is an imaginary quadratic field embedded in $\overline{\mathbb{Q}}$. Let Φ be the induced CM type of F , i.e., the set of embeddings $F \rightarrow \overline{\mathbb{Q}}$ whose restriction to K_0 is the given embedding. We fix $\varphi_0 \in \Phi$. Then we define $r^{\text{HT}} := r^{(\Phi, \varphi_0)}$ and $h_{G^{\mathbb{Q}}}^{\text{HT}} := h_{G^{\mathbb{Q}}}$. In this case, we can be explicit about the reflex field: φ_0 identifies $F \xrightarrow{\sim} E_{r^{(\Phi, \varphi_0)}}$, unless $F_0 = \mathbb{Q}$ and $n = 2$ (then $E_{r^{(\Phi, \varphi_0)}} = \mathbb{Q}$) or F_0 is quadratic over \mathbb{Q} and $n = 1$ (then $E_{r^{(\Phi, \varphi_0)}}$ identifies via φ_0 with the unique quadratic subfield of F distinct from K_0 and F_0). Note that the book [9] is about the tower of moduli stacks (M_K) , and not about the Shimura variety $(S(G^{\mathbb{Q}}, \{h_{G^{\mathbb{Q}}}\})_K)$ (despite the title of the book!).

2.3. The unitary group

We continue with the notation of the last subsection, but this time we consider the group

$$(2.9) \quad G := \text{Res}_{F_0/\mathbb{Q}} \text{U}(W, (,)).$$

The conjugacy class in the Shimura datum for G is the conjugacy class of the homomorphism $h_G = (h_{G, \varphi})_{\varphi \in \Phi}$, where

$$h_{G, \varphi}: z \mapsto \text{diag}(1_{r_{\varphi}}, (\overline{z}/z)1_{r_{\overline{\varphi}}}),$$

and where the components are defined as in (2.4). The reflex field is the reflex field $E_{r^{\natural}}$ of the function r^{\natural} ,

$$(2.10) \quad \text{Gal}(\overline{\mathbb{Q}}/E_{r^{\natural}}) = \{ \sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \mid \sigma^*(r^{\natural}) = r^{\natural} \},$$

where we define

$$r^\natural: \text{Hom}_{\mathbb{Q}}(F, \overline{\mathbb{Q}}) \longrightarrow \mathbb{Z}_{\geq 0}$$

$$\varphi \longmapsto \begin{cases} 0, & \varphi \in \Phi; \\ r_\varphi, & \varphi \in \overline{\Phi}. \end{cases}$$

(Note that r^\natural need not be a generalized CM type.) The resulting Shimura variety $(S(G, \{h_G\})_K)$ is not of PEL type, i.e., it is not related to a moduli problem of abelian varieties (this can be seen already from the fact that the restriction of $\{h_G\}$ to $\mathbb{G}_m \subset \mathbb{S}$ is not mapped via the identity map to the center of G). However, this Shimura variety is of abelian type.

Example 2.4. In the fake Drinfeld case of Example 2.3(i), φ_0 embeds F into E_{r^\natural} for $n \geq 2$. In the strict fake Drinfeld case relative to (Φ, φ_0) of Example 2.3(ii), we have $\varphi_0: F \xrightarrow{\sim} E_{r^\natural}$ for all $n \geq 1$.

Remark 2.5. Suppose that $n \geq 2$, and let $u \in W$ be a totally positive vector, i.e., (u, u) is a totally positive element in F_0 . Consider the hermitian space $W^b := (u)^\perp$. Then the inclusion $W^b \subset W$ induces an inclusion $U(W^b) \subset U(W)$ of unitary groups (identifying $U(W^b)$ with the stabilizer of u). Let

$$H := \text{Res}_{F_0/\mathbb{Q}} U(W^b),$$

with its Shimura datum $\{h_H\}$. Using that u is totally positive, one verifies that the inclusion $H \subset G$ is compatible with the Shimura data $\{h_H\}$ and $\{h_G\}$. Hence there is an induced morphism of Shimura varieties,

$$(2.11) \quad \left(S(H, \{h_H\})_{K_H} \right) \hookrightarrow \left(S(G, \{h_G\})_{K_G} \right).$$

The resulting cycle leads to the *Kudla–Rapoport cycles* [18, 2] (in the context of locally symmetric spaces, these are the *Kudla–Millson cycles*, cf. [16]). These are cycles of codimension $\sum_{\varphi \in \Phi} r_\varphi$. The most interesting case is the strict fake Drinfeld case of Example 2.3(ii). In this case, we obtain divisors on the ambient variety.

Taking the graph morphism of (2.11), we obtain a closed embedding of towers,

$$(2.12) \quad \left(S(H, \{h_H\})_{K_H} \right) \hookrightarrow \left(S(H, \{h_H\})_{K_H} \right) \times \left(S(G, \{h_G\})_{K_G} \right).$$

The resulting cycle in the target of (2.12) is the *GGP cycle*, cf. [6]. This is a cycle of codimension $\sum_{\varphi \in \Phi} r_\varphi r_{\overline{\varphi}}$. The most interesting case is again the strict

fake Drinfeld case. In this case, the product variety has dimension $2n - 3$, and the cycle has codimension $n - 1$, i.e., the codimension is just more than half the dimension of the ambient variety.

Both of these constructions generalize to the case where the totally positive vector u is replaced by an m -tuple of totally positive vectors u_1, \dots, u_m which generate a totally definite subspace of W .

Remark 2.6. Let us discuss some of the advantages and disadvantages of the above Shimura varieties.

(i) First consider the Shimura varieties associated to $(G^{\mathbb{Q}}, \{h_{G^{\mathbb{Q}}}\})$. On the positive side we note the following.

- These Shimura varieties are close to moduli problems—but when n is odd, they are not quite represented by a moduli problem in general.
- The method of Kottwitz works for all Shimura varieties of PEL type.

On the negative side we note the following.

- For odd n , the Shimura varieties are not given by a moduli problem in general.
- It is difficult to construct integral or p -integral models of these Shimura varieties. More precisely, Kottwitz succeeds in constructing a p -integral model over $O_{E_r, (p)}$ only under the assumption that all data are unramified at p . This last condition means that p is unramified in F , that W is split at all places of F_0 over p , and that K is of the form $K = K^p \times K_p \subset G^{\mathbb{Q}}(\mathbb{A}_f) = G^{\mathbb{Q}}(\mathbb{A}_f^p) \times G^{\mathbb{Q}}(\mathbb{Q}_p)$, where K^p is arbitrary and K_p is the stabilizer of a self-dual lattice in $W \otimes_{\mathbb{Q}} \mathbb{Q}_p$. But allowing ramification in various forms leads to many new complications.
- In the context of Remark 2.5, the inclusion $U(W^b) \subset U(W)$ of unitary groups does not extend to an inclusion $GU(W^b) \subset GU(W)$ of groups of unitary similitudes, nor, after Weil restriction down to \mathbb{Q} , to an inclusion of the unitary \mathbb{G}_m -similitude group for W^b into $G^{\mathbb{Q}}$. Hence there is no Gan–Gross–Prasad set-up in the context of Kottwitz’s Shimura varieties, comp. [6, 31]. Similarly, there are no Kudla–Rapoport cycles on these Shimura varieties, cf. [18, 2]. See Section 3.5 for the analog of this discussion in the context of the RSZ Shimura varieties.

(ii) Now let us discuss the Shimura varieties associated to $(G, \{h_G\})$. On the positive side we note the following.

- The KR cycles and GGP cycles can be defined for them.
- In the strict fake Drinfeld case of Example 2.3(ii), the Shimura field is very simple: it identifies with F .

On the negative side we note the following.

- Since these Shimura varieties are not of PEL type, it is difficult to construct and control p -integral models of them. Since they are at least of abelian type, by Kisin–Pappas [13] they do have p -integral models when K is of the form $K = K^p K_p$, where K_p is a parahoric subgroup. However, these models are not very explicit. In particular, it seems difficult to address for these p -integral models the *Arithmetic Gan–Gross–Prasad conjecture* [6], the *Arithmetic intersection conjecture* of [31], and the *Kudla–Rapoport intersection conjecture* [18, Conj. 11.10].

3. The RSZ Shimura varieties

We continue with the notation F/F_0 , r , r^\natural , and $(W, (\cdot, \cdot))$ from Sections 2.2–2.3. Again we fix a CM type Φ of F .

3.1. The torus $Z^{\mathbb{Q}}$ and its Shimura variety

We are first going to consider the Shimura varieties of Section 2.2 in the special case that $n = 1$ and $(W, (\cdot, \cdot)) = (W_0, (\cdot, \cdot)_0)$ is totally positive definite, i.e., W_0 has signature $(1, 0)$ at each archimedean place. In this case, we write $Z^{\mathbb{Q}} := G^{\mathbb{Q}}$ (a torus over \mathbb{Q}) and $h_{Z^{\mathbb{Q}}} := h_{G^{\mathbb{Q}}}$. Explicitly,

$$Z^{\mathbb{Q}} = \{ z \in \text{Res}_{F/\mathbb{Q}} \mathbb{G}_m \mid \text{Nm}_{F/F_0}(z) \in \mathbb{G}_m \},$$

and the homomorphism $h_{Z^{\mathbb{Q}}}: \mathbb{C}^\times \rightarrow Z^{\mathbb{Q}}(\mathbb{R})$ identifies with the diagonal embedding into $(\mathbb{C}^\times)^\Phi$ with respect to the isomorphism

$$Z^{\mathbb{Q}}(\mathbb{R}) \xrightarrow{\sim} \{ (z_\varphi) \in (\mathbb{C}^\times)^\Phi \mid |z_\varphi| = |z_{\varphi'}| \text{ for all } \varphi, \varphi' \in \Phi \} \subset (\mathbb{C}^\times)^\Phi$$

induced by the isomorphism $F \otimes \mathbb{R} \xrightarrow{\Phi} \mathbb{C}^\Phi$.¹ The reflex field of $(Z^{\mathbb{Q}}, \{h_{Z^{\mathbb{Q}}}\})$ is E_Φ , the reflex field of Φ .

¹We note that [31] adopts the convention that $h_{Z^{\mathbb{Q}}}$ is the analogous embedding defined in the case that W_0 is totally *negative* definite, which means that it is our $h_{Z^{\mathbb{Q}}}$ precomposed by complex conjugation. This difference of convention results in a number of further differences with [31] throughout the rest of Section 3.

Let $K_{Z^{\mathbb{Q}}} \subset Z^{\mathbb{Q}}(\mathbb{A}_f)$ be a compact open subgroup. Then we obtain the Deligne–Mumford stack, which we denote by $M_{0,K_{Z^{\mathbb{Q}}}}$, representing the moduli problem of Section 2.2 for the hermitian space W_0 . It is a finite étale stack over $\text{Spec } E_{\Phi}$.

By Theorem 2.2, the complex fiber $M_{0,K_{Z^{\mathbb{Q}}}} \otimes_{E_{\Phi}} \mathbb{C}$ is isomorphic to a finite number of copies of the Shimura variety $S(Z^{\mathbb{Q}}, \{h_{Z^{\mathbb{Q}}}\})_{K_{Z^{\mathbb{Q}}}}$. To make this decomposition more explicit, let us first introduce the following definition.

Definition 3.1. An element $a \in F$ is Φ -adapted if $\varphi(a)$ is an $\mathbb{R}_{>0}$ -multiple of $\sqrt{-1} \in \mathbb{C}$ for all $\varphi \in \Phi$.

Thus any F/F_0 -traceless element² $\sqrt{\Delta} \in F^{\times}$ determines a unique CM type for F to which $\sqrt{\Delta}$ is adapted, and conversely, by weak approximation, any CM type admits elements adapted to it. In particular, let us fix a Φ -adapted element $\sqrt{\Delta}$ for our fixed CM type Φ . In the notation of the Introduction and Section 2.2, take $\zeta = \sqrt{\Delta}^{-1}$, so that we endow W_0 with the \mathbb{Q} -alternating form $\text{tr}_{F/\mathbb{Q}} \sqrt{\Delta}^{-1}(\cdot, \cdot)_0$ in the definition of the level structure for $M_{0,K_{Z^{\mathbb{Q}}}}$. Let $\mathcal{R}_{W_0, \sqrt{\Delta}}$ be the set of isometry classes of pairs $(U_0, \langle \cdot, \cdot \rangle_0)$ consisting of a one-dimensional F -vector space U_0 equipped with a nondegenerate \mathbb{Q} -alternating form $\langle \cdot, \cdot \rangle_0: U_0 \times U_0 \rightarrow \mathbb{Q}$ such that $\langle ax, y \rangle_0 = \langle x, \bar{a}y \rangle_0$ for all $x, y \in U_0$ and $a \in F$, such that $x \mapsto \langle \sqrt{\Delta}x, x \rangle_0$ is a positive definite quadratic form on U_0 , and such that for all finite primes p , the localization $U_0 \otimes \mathbb{Q}_p$ endowed with its \mathbb{Q}_p -alternating form is $F \otimes \mathbb{Q}_p$ -linearly similar to $(W_0, \text{tr}_{F/\mathbb{Q}} \sqrt{\Delta}^{-1}(\cdot, \cdot)_0) \otimes \mathbb{Q}_p$ up to a factor in \mathbb{Q}_p^{\times} . (Thus the pair $(W_0, \text{tr}_{F/\mathbb{Q}} \sqrt{\Delta}^{-1}(\cdot, \cdot)_0)$ tautologically defines a class in $\mathcal{R}_{W_0, \sqrt{\Delta}}$.) Then $\mathbb{Q}_{>0}$ acts on $\mathcal{R}_{W_0, \sqrt{\Delta}}$ by multiplying the form, and by [14, §8],

$$(3.1) \quad M_{0,K_{Z^{\mathbb{Q}}}} \otimes_{E_{\Phi}} \mathbb{C} \simeq \coprod_{\mathcal{R}_{W_0, \sqrt{\Delta}}/\mathbb{Q}_{>0}} S(Z^{\mathbb{Q}}, \{h_{Z^{\mathbb{Q}}}\})_{K_{Z^{\mathbb{Q}}}}.$$

(In terms of Theorem 2.2, the set $\mathcal{R}_{W_0, \sqrt{\Delta}}/\mathbb{Q}_{>0}$ is in bijection with $\ker^1(\mathbb{Q}, Z^{\mathbb{Q}})$ by taking the class of $(W_0, \text{tr}_{F/\mathbb{Q}} \sqrt{\Delta}^{-1}(\cdot, \cdot)_0)$ as basepoint.) Here the index associated to a \mathbb{C} -valued point $(A_0, \iota_0, \lambda_0, \bar{\eta}_0)$ of $M_{0,K_{Z^{\mathbb{Q}}}}$ is given by the \mathbb{Q}^{\times} -class of the F -vector space $H_1(A_0, \mathbb{Q})$ endowed with its natural \mathbb{Q} -valued Riemann form induced by λ_0 . The decomposition on the right-hand side of

²Here the notation $\sqrt{\Delta}$ reflects the fact that any Φ -adapted element must be a square root of some totally negative element $\Delta \in F_0^{\times}$, but we note that the element Δ itself will never play any explicit role for us.

(3.1) descends to E_Φ , and we accordingly write

$$(3.2) \quad M_{0,K_{Z^{\mathbb{Q}}}} = \coprod_{\tau \in \mathcal{R}_{W_0, \sqrt{\Delta}/\mathbb{Q}_{>0}}} M_{0,K_{Z^{\mathbb{Q}}}}^\tau.$$

3.2. The RSZ Shimura varieties

The Shimura varieties of [31] are attached to the group

$$(3.3) \quad \tilde{G} := Z^{\mathbb{Q}} \times_{\mathbb{G}_m} G^{\mathbb{Q}},$$

where the maps from the factors on the right-hand side to \mathbb{G}_m are respectively given by Nm_{F/F_0} and the similitude character. In terms of the Shimura data already defined, we obtain a Shimura datum for \tilde{G} by defining the Shimura homomorphism to be

$$h_{\tilde{G}}: \mathbb{C}^\times \xrightarrow{(h_{Z^{\mathbb{Q}}}, h_{G^{\mathbb{Q}}})} \tilde{G}(\mathbb{R}).$$

It is easy to see that $(\tilde{G}, \{h_{\tilde{G}}\})$ has reflex field $E \subset \overline{\mathbb{Q}}$ characterized by

$$(3.4) \quad \begin{aligned} \text{Gal}(\overline{\mathbb{Q}}/E) &= \{ \sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \mid \sigma \circ \Phi = \Phi \text{ and } \sigma^*(r) = r \} \\ &= \{ \sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \mid \sigma \circ \Phi = \Phi \text{ and } \sigma^*(r^\natural) = r^\natural \}. \end{aligned}$$

In other words, the reflex field is the common composite $E = E_\Phi E_r = E_\Phi E_{r^\natural}$.

Example 3.2. In the fake Drinfeld case of Example 2.3(i), φ_0 embeds F into E for $n \geq 2$ since $\varphi_0: F \rightarrow E_{r^\natural} \subset E$, cf. Example 2.4. When $n = 1$, the same statement holds in the strict fake Drinfeld case relative to Φ of Example 2.3(ii), but F may fail to embed in E for other signature types of fake Drinfeld type. In the Harris–Taylor case of Example 2.3(iii), we have $\varphi_0: F \xrightarrow{\sim} E$ for any $n \geq 1$.

The various relations between the groups we have introduced give rise to the following relations between Shimura varieties.

(i) By definition, the natural projection $\tilde{G} \rightarrow G^{\mathbb{Q}}$ induces a morphism of Shimura data $(\tilde{G}, \{h_{\tilde{G}}\}) \rightarrow (G^{\mathbb{Q}}, \{h_{G^{\mathbb{Q}}}\})$. Hence there is an induced morphism of Shimura varieties (i.e., a morphism of pro-varieties)

$$\left(S(\tilde{G}, \{h_{\tilde{G}}\})_{K_{\tilde{G}}} \right) \longrightarrow \left(S(G^{\mathbb{Q}}, \{h_{G^{\mathbb{Q}}}\})_{K_{G^{\mathbb{Q}}}} \right),$$

compatible with the inclusion $E_r \subset E$.

(ii) The torus $Z^{\mathbb{Q}}$ embeds naturally as a central subgroup of $G^{\mathbb{Q}}$, which gives rise to a product decomposition

$$(3.5) \quad \begin{aligned} \tilde{G} &\xrightarrow{\sim} Z^{\mathbb{Q}} \times G \\ (z, g) &\longmapsto (z, z^{-1}g), \end{aligned}$$

where $G \subset G^{\mathbb{Q}}$ is the unitary group (2.9). The isomorphism (3.5) extends to a product decomposition of Shimura data,

$$(3.6) \quad (\tilde{G}, \{h_{\tilde{G}}\}) \cong (Z^{\mathbb{Q}}, \{h_{Z^{\mathbb{Q}}}\}) \times (G, \{h_G\}),$$

and hence there is a product decomposition of Shimura varieties,

$$\left(S(\tilde{G}, \{h_{\tilde{G}}\})_{K_{\tilde{G}}} \right) \cong \left(S(Z^{\mathbb{Q}}, \{h_{Z^{\mathbb{Q}}}\})_{K_{Z^{\mathbb{Q}}}} \right) \times \left(S(G, \{h_G\})_{K_G} \right),$$

compatible with the inclusions $E_{\Phi} \subset E$ and $E_{\gamma^{\natural}} \subset E$.

3.3. The RSZ moduli problem in terms of isogeny classes

We are now going to give a moduli interpretation for the canonical model of the Shimura variety $S(\tilde{G}, \{h_{\tilde{G}}\})_{K_{\tilde{G}}}$ over $\text{Spec } E$. We will only consider subgroups $K_{\tilde{G}}$ which, with respect to the product decomposition (3.5), are of the form

$$(3.7) \quad K_{\tilde{G}} = K_{Z^{\mathbb{Q}}} \times K_G \subset \tilde{G}(\mathbb{A}_f) = Z^{\mathbb{Q}}(\mathbb{A}_f) \times G(\mathbb{A}_f),$$

for arbitrary open compact subgroups $K_{Z^{\mathbb{Q}}} \subset Z^{\mathbb{Q}}(\mathbb{A}_f)$ and $K_G \subset G(\mathbb{A}_f)$.

For the definition of level structures in the moduli problem, we fix a one-dimensional, totally positive definite F/F_0 -hermitian space $(W_0, (\ , \)_0)$ as in Section 3.1. We fix a Φ -adapted element $\sqrt{\Delta} \in F$ and a class $\tau \in \mathcal{R}_{W_0, \sqrt{\Delta}}/\mathbb{Q}_{>0}$, and we recall from (3.2) the stack $M_{0, K_{Z^{\mathbb{Q}}}}^{\tau}$ attached to W_0 . (Of course we may take τ to be the class of $(W_0, \text{tr}_{F/\mathbb{Q}} \sqrt{\Delta}^{-1}(\ , \)_0)$, but we do not require this.) We furthermore introduce the n -dimensional F -vector space

$$(3.8) \quad V := \text{Hom}_F(W_0, W).$$

The space V carries a natural F/F_0 -hermitian form, under which elements $x, y \in V$ pair to the composite

$$(3.9) \quad \left[W_0 \xrightarrow{x} W \xrightarrow{w \mapsto (-, w)} W^* \xrightarrow{y^*} W_0^* \xrightarrow{[w_0 \mapsto (-, w_0)_0]^{-1}} W_0 \right] \in \text{End}_F(W_0) \cong F.$$

The group \tilde{G} acts naturally by unitary transformations on V , given in terms of the defining presentation (3.3) by $(z, g) \cdot x = gxz^{-1}$. This action factors through the quotient $\tilde{G} \rightarrow G$ via (3.5) and induces $G \cong \text{Res}_{F_0/\mathbb{Q}} \text{U}(V)$.

We now define a category fibered in groupoids $\mathcal{F}_{K_{\tilde{G}}}(\tilde{G})$ over $(\text{LNSch})/E$. To lighten notation, we suppress the dependence of this category functor on the element τ .

Definition 3.3. The category functor $\mathcal{F}_{K_{\tilde{G}}}(\tilde{G})$ associates to each scheme S in $(\text{LNSch})/E$ the groupoid of tuples $(A_0, \iota_0, \lambda_0, \bar{\eta}_0, A, \iota, \lambda, \bar{\eta})$, where

- $(A_0, \iota_0, \lambda_0, \bar{\eta}_0)$ is an object of $M_{0, K_{\mathbb{Z}\mathbb{Q}}}^{\tau}(S)$;
- A is an abelian scheme over S ;
- $\iota: F \rightarrow \text{End}^0(A)$ is an action of F on A up to isogeny satisfying the Kottwitz condition (2.7);
- λ is a quasi-polarization on A whose Rosati involution satisfies condition (2.6); and
- $\bar{\eta}$ is a K_G -orbit (equivalently, a $K_{\tilde{G}}$ -orbit, where $K_{\tilde{G}}$ acts through its projection $K_{\tilde{G}} \rightarrow K_G$) of isometries of $\mathbb{A}_{F,f}/\mathbb{A}_{F_0,f}$ -hermitian modules

$$(3.10) \quad \eta: \widehat{V}(A_0, A) \xrightarrow{\sim} V \otimes_F \mathbb{A}_{F,f}.$$

Here

$$(3.11) \quad \widehat{V}(A_0, A) := \text{Hom}_{\mathbb{A}_{F,f}}(\widehat{V}(A_0), \widehat{V}(A)),$$

endowed with its natural $\mathbb{A}_{F,f}$ -valued hermitian form h ,

$$(3.12) \quad h(x, y) := \lambda_0^{-1} \circ y^{\vee} \circ \lambda \circ x \in \text{End}_{\mathbb{A}_{F,f}}(\widehat{V}(A_0)) = \mathbb{A}_{F,f}, \quad x, y \in \widehat{V}(A_0, A),$$

cf. [18, §2.3].³ Furthermore, for any geometric point $\bar{s} \rightarrow S$, the orbit $\bar{\eta}$ is required to be $\pi_1(S, \bar{s})$ -stable with respect to the $\pi_1(S, \bar{s})$ -action on the fiber $\widehat{V}(A_0, A)(\bar{s}) = \text{Hom}_{\mathbb{A}_{F,f}}(\widehat{V}(A_{0,\bar{s}}), \widehat{V}(A_{\bar{s}}))$ (a condition which holds for all \bar{s} on a given connected component S_0 of S as soon as it holds for a single \bar{s} on S_0); comp. [14, §5] or [18, Rem. 4.2]. A morphism $(A_0, \iota_0, \lambda_0, \bar{\eta}_0, A, \iota, \lambda, \bar{\eta}) \rightarrow (A'_0, \iota'_0, \lambda'_0, \bar{\eta}'_0, A', \iota', \lambda', \bar{\eta}')$ in this groupoid is given by a pair of F -linear quasi-isogenies $\mu_0: A_0 \rightarrow A'_0$ and $\mu: A \rightarrow A'$ such that μ_0 is an isomorphism

³To be clear, $y^{\vee}: \widehat{V}(A^{\vee}) \rightarrow \widehat{V}(A_0^{\vee})$ denotes the adjoint of y with respect to the Weil pairings on $\widehat{V}(A) \times \widehat{V}(A^{\vee})$ and $\widehat{V}(A_0) \times \widehat{V}(A_0^{\vee})$.

$(A_0, \iota_0, \lambda_0, \bar{\eta}_0) \xrightarrow{\sim} (A'_0, \iota'_0, \lambda'_0, \bar{\eta}'_0)$ in $M_{0, K_{\mathbb{Z}^Q}}^r(S)$, such that $\mu^*(\lambda')$ is the same \mathbb{Q}^\times -multiple of λ as $\mu_0^*(\lambda'_0)$ is of λ_0 at each point of S (the *multiplier condition*), and such that under the natural isomorphism $\widehat{V}(A_0, A) \xrightarrow{\sim} \widehat{V}(A'_0, A')$ sending $x \mapsto \mu \circ x \circ \mu_0^{-1}$ (which is an isometry by the multiplier condition), $\bar{\eta}'$ pulls back to $\bar{\eta}$.

Remark 3.4. Let us comment further on the space V introduced in (3.8). The hermitian forms $(\ , \)_0$ and $(\ , \)$ determine a conjugate-linear isomorphism $V \xrightarrow{\sim} \text{Hom}_F(W, W_0)$, $x \mapsto x^{\text{ad}}$, characterized by the formula

$$(xw_0, w) = (w_0, x^{\text{ad}}w)_0, \quad x \in V, w_0 \in W_0, w \in W.$$

Then the pairing (3.9) on V can be expressed succinctly as sending x, y to $y^{\text{ad}}x$. Note that the adjoint x^{ad} defined in this way is the same as the adjoint with respect to the \mathbb{Q} -valued forms $\text{tr}_{F/\mathbb{Q}} \sqrt{\Delta}^{-1}(\ , \)$ and $\text{tr}_{F/\mathbb{Q}} \sqrt{\Delta}^{-1}(\ , \)_0$.

Concretely, upon choosing a basis vector in W_0 , the hermitian form on W_0 is represented by a totally positive element $a \in F_0$, and we obtain an F -linear isomorphism $V \simeq W$. With respect to this isomorphism, the form on V is then given by $a^{-1}(\ , \)$. In particular, V has the same signature at each archimedean place as W , and in the special case that W_0 equals F endowed with its norm form, there is a canonical isometry $V \cong W$. This last case recovers the case taken in the definition of level structures in [31, §3.2] (modulo the sign conventions alluded to previously in footnote 1). But even in this special case, it is often helpful to distinguish between V and W , and more generally, it can be desirable to allow other possibilities for W_0 .

The following theorem is the analog for $(\tilde{G}, \{h_{\tilde{G}}\})$ of Theorems 2.1 and 2.2.

Theorem 3.5. *The moduli problem $\mathcal{F}_{K_{\tilde{G}}}(\tilde{G})$ is representable by a Deligne–Mumford stack $M_{K_{\tilde{G}}}(\tilde{G})$ over $\text{Spec } E$, and*

$$M_{K_{\tilde{G}}}(\tilde{G})(\mathbb{C}) = S(\tilde{G}, \{h_{\tilde{G}}\})_{K_{\tilde{G}}},$$

compatible with changing $K_{\tilde{G}}$ of the form (3.18).

Proof. This is the extension to the case of arbitrary signature types of [31, Prop. 3.7].⁴ The key point is the following. Define for $(A_0, \iota_0, \lambda_0, \bar{\eta}_0, A, \iota, \lambda, \bar{\eta})$

⁴Strictly speaking, the statement and proof in loc. cit. is for the moduli problem given in Definition 3.8 below, in the case of a particular signature type. But the argument transposes to the present situation almost unchanged.

in $M_{K_{\tilde{G}}}(\tilde{G})(\mathbb{C})$ a hermitian space $V(A_0, A)$ over F in analogy with $\widehat{V}(A_0, A)$, but by using Betti homology groups instead of rational Tate modules. Then $\widehat{V}(A_0, A) = V(A_0, A) \otimes_F \mathbb{A}_{F,f}$. By the level structure $\bar{\eta}$, the two hermitian spaces V and $V(A_0, A)$ are isomorphic at all finite places. At an archimedean place corresponding to $\varphi \in \Phi$, by the Kottwitz condition (2.7) and the analogous condition of signature $((1, 0)_{\varphi \in \Phi})$ for $M_{0, K_{\mathbb{Z}^{\mathbb{Q}}}}$, the signature of $V(A_0, A)$ is $(r_{\varphi}, r_{\bar{\varphi}})$. Hence, by the Hasse principle for hermitian spaces, $V(A_0, A)$ and V are isomorphic. The choice of an isomorphism j between them allows one to define a map $M_{K_{\tilde{G}}}(\tilde{G})(\mathbb{C}) \rightarrow S(\tilde{G}, \{h_{\tilde{G}}\})_{K_{\tilde{G}}}$ which one shows to be an isomorphism independent of the choice of j . \square

3.4. Variant moduli problems in terms of isomorphism classes

In this section we give some “isomorphism class” variants of the moduli problems introduced above.

We begin with the moduli problem for $Z^{\mathbb{Q}}$. For simplicity, we restrict to the case that the level subgroup $K_{Z^{\mathbb{Q}}} = K_{Z^{\mathbb{Q}}}^{\circ} \subset Z^{\mathbb{Q}}(\mathbb{A}_f)$ is the (unique) maximal compact open subgroup,

$$(3.13) \quad K_{Z^{\mathbb{Q}}}^{\circ} := \{ z \in (O_F \otimes \widehat{\mathbb{Z}})^{\times} \mid \text{Nm}_{F/F_0}(z) \in \widehat{\mathbb{Z}}^{\times} \}.$$

We define the following category fibered in groupoids \mathcal{F}_0 over $(\text{LNSch})_{/E_{\Phi}}$.

Definition 3.6. The category functor \mathcal{F}_0 associates to each scheme S in $(\text{LNSch})_{/E_{\Phi}}$ the groupoid of triples $(A_0, \iota_0, \lambda_0)$, where

- A_0 is an abelian scheme over S ;
- $\iota_0: O_F \rightarrow \text{End}(A_0)$ is an O_F -action satisfying the Kottwitz condition (2.7) in the case of signature $((1, 0)_{\varphi \in \Phi})$ for elements in O_F ,

$$(3.14) \quad \text{char}(\iota(a) \mid \text{Lie } A_0) = \prod_{\varphi \in \Phi} (T - \varphi(a)) \quad \text{for all } a \in O_F;$$

and

- λ_0 is a principal polarization on A_0 whose Rosati involution satisfies condition (2.6) on O_F with respect to ι_0 .

A morphism $(A_0, \iota_0, \lambda_0) \rightarrow (A'_0, \iota'_0, \lambda'_0)$ in this groupoid is an O_F -linear isomorphism of abelian schemes $\mu_0: A_0 \xrightarrow{\sim} A'_0$ such that the pullback of λ'_0 is λ_0 .

By the proof of [11, Prop. 3.1.2], \mathcal{F}_0 is representable by a DM stack M_0 which is finite and étale over $\text{Spec } E_\Phi$.

Unfortunately, it may happen that M_0 is empty. In order to circumvent this issue, we introduce the following variant of M_0 , cf. [11, Def. 3.1.1]. Fix a non-zero ideal \mathfrak{a} of O_{F_0} . Then we define the Deligne–Mumford stack $M_0^\mathfrak{a}$ of triples $(A_0, \iota_0, \lambda_0)$ as in Definition 3.6, except that we replace the condition that λ_0 is principal by the condition that λ_0 is a polarization satisfying $\ker \lambda_0 = A_0[\mathfrak{a}]$. Then, again, $M_0^\mathfrak{a}$ is finite and étale over $\text{Spec } E_\Phi$, cf. [11, Prop. 3.1.2].

If $M_0^\mathfrak{a}$ is non-empty, then, as in the case of the moduli stack $M_{0, K_{Z^{\mathbb{Q}}}}^\circ$ in Section 3.1, its complex fiber is a finite union of copies of $S(Z^{\mathbb{Q}}, \{h_{Z^{\mathbb{Q}}}\})_{K_{Z^{\mathbb{Q}}}}^\circ$. More precisely, let $\mathcal{L}_\Phi^\mathfrak{a}$ be the set of isomorphism classes of pairs $(\Lambda_0, \langle \cdot, \cdot \rangle_0)$ consisting of a locally free O_F -module Λ_0 of rank one equipped with a nondegenerate alternating form $\langle \cdot, \cdot \rangle_0: \Lambda_0 \times \Lambda_0 \rightarrow \mathbb{Z}$ such that $\langle ax, y \rangle_0 = \langle x, \bar{a}y \rangle_0$ for all $x, y \in \Lambda_0$ and $a \in O_F$, such that the dual lattice Λ_0^\vee of Λ_0 inside $\Lambda_0 \otimes_{\mathbb{Z}} \mathbb{Q}$ equals $\mathfrak{a}^{-1}\Lambda_0$, and such that $x \mapsto \langle \sqrt{\Delta}x, x \rangle_0$ is a positive definite quadratic form on Λ_0 for some (equivalently, any) Φ -adapted element $\sqrt{\Delta} \in F$. Then $\mathcal{L}_\Phi^\mathfrak{a}$ is a finite set, in natural bijection with the isomorphism classes of objects in $M_0^\mathfrak{a}(\mathbb{C})$, cf. [31, §3.2]. Given $\Lambda_0, \Lambda'_0 \in \mathcal{L}_\Phi^\mathfrak{a}$ (as is customary, we often suppress the pairings when denoting elements in $\mathcal{L}_\Phi^\mathfrak{a}$), define $\Lambda_0 \sim \Lambda'_0$ if $\Lambda_0 \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ and $\Lambda'_0 \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ are \widehat{O}_F -linearly similar up to a factor in $\widehat{\mathbb{Z}}^\times$ and $\Lambda_0 \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\Lambda'_0 \otimes_{\mathbb{Z}} \mathbb{Q}$ are F -linearly similar up to a (necessarily positive) factor in \mathbb{Q}^\times . Then

$$(3.15) \quad M_0^\mathfrak{a} \otimes_{E_\Phi} \mathbb{C} \simeq \coprod_{\mathcal{L}_\Phi^\mathfrak{a}/\sim} S(Z^{\mathbb{Q}}, \{h_{Z^{\mathbb{Q}}}\})_{K_{Z^{\mathbb{Q}}}}^\circ,$$

cf. Lem. 3.4 and the paragraph following it in [31]. Here the index associated to a \mathbb{C} -valued point $(A_0, \iota_0, \lambda_0)$ of $M_0^\mathfrak{a}$ is given by the class in $\mathcal{L}_\Phi^\mathfrak{a}/\sim$ of the O_F -module $H_1(A_0, \mathbb{Z})$ endowed with its natural \mathbb{Z} -valued Riemann form induced by the polarization. The decomposition on the right-hand side of (3.15) descends to E_Φ , and we accordingly write

$$(3.16) \quad M_0^\mathfrak{a} = \coprod_{\xi \in \mathcal{L}_\Phi^\mathfrak{a}/\sim} M_0^{\mathfrak{a}, \xi}.$$

Remark 3.7. (i) If F/F_0 is ramified at some finite place, then $M_0^\mathfrak{a}$ is non-empty for any \mathfrak{a} , cf. [11, pf. of Prop. 3.1.6]. A special case of this is when $F = K_0F_0$, where K_0 is an imaginary quadratic field and the discriminants of K_0/\mathbb{Q} and F_0/\mathbb{Q} are relatively prime.

- (ii) If F/F_0 is unramified at every finite place and $M_0^{O_{F_0}} = \emptyset$, then it is easy to deduce from loc. cit. and class field theory that $M_0^{\mathfrak{p}}$ is non-empty for any prime ideal $\mathfrak{p} \subset O_{F_0}$ which is inert in F . For example, this case arises when $F_0 = \mathbb{Q}(\sqrt{3})$ and $F = F_0(\sqrt{-1})$.
- (iii) In particular, given finitely many prime numbers p_1, \dots, p_r , there always exists \mathfrak{a} relatively prime to p_1, \dots, p_r such that $M_0^{\mathfrak{a}}$ is non-empty.
- (iv) When $F_0 = \mathbb{Q}$, the set $\mathcal{L}_{\Phi}^{\mathfrak{a}}/\sim$ has only one element, so that the decomposition (3.16) is trivial.

To directly compare $M_0^{\mathfrak{a}}$ and $M_{0, K_{\mathbb{Z}\mathbb{Q}}^{\circ}}$ (or more precisely, the summands occurring on the respective right-hand sides of (3.2) and (3.16)), let \mathfrak{a} be such that $M_0^{\mathfrak{a}} \neq \emptyset$, and let $\sqrt{\Delta}$ be any Φ -adapted element in F . Fix a class $\xi \in \mathcal{L}_{\Phi}^{\mathfrak{a}}/\sim$. Let $(\Lambda_0, \langle \cdot, \cdot \rangle_0)$ be a representative of ξ in $\mathcal{L}_{\Phi}^{\mathfrak{a}}$, and set $W_0 := \Lambda_0 \otimes_{\mathbb{Z}} \mathbb{Q}$. Let τ denote the class of $(W_0, \langle \cdot, \cdot \rangle_0 \otimes \mathbb{Q})$ in $\mathcal{R}_{W_0, \sqrt{\Delta}}/\mathbb{Q}_{>0}$. (Here we are implicitly endowing W_0 with the unique F/F_0 -hermitian form $(\cdot, \cdot)_0$ such that $\langle \cdot, \cdot \rangle_0 \otimes \mathbb{Q} = \text{tr}_{F/\mathbb{Q}} \sqrt{\Delta}^{-1}(\cdot, \cdot)_0$.) Then the set $\mathcal{R}_{W_0, \sqrt{\Delta}}$ and the class τ are independent of the choice of representative of ξ . We define an isomorphism

$$(3.17) \quad M_0^{\mathfrak{a}, \xi} \xrightarrow{\sim} M_{0, K_{\mathbb{Z}\mathbb{Q}}^{\circ}}^{\tau}$$

as follows. Let S be a locally noetherian E_{Φ} -scheme, and let $(A_0, \iota_0, \lambda_0)$ be an S -point on $M_0^{\mathfrak{a}, \xi}$. By the definition of the summands in the decomposition (3.16) (see [31, pf. of Lem. 3.4]), at each geometric point \bar{s} of S there exists an \widehat{O}_F -linear symplectic similitude (up to a factor in $\widehat{\mathbb{Z}}$)

$$\widehat{T}(A_0, \bar{s}) \xrightarrow{\sim} \Lambda_0 \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}.$$

The set of all such similitudes is a $K_{\mathbb{Z}\mathbb{Q}}^{\circ}$ -orbit, and upon extending scalars to \mathbb{A}_f they define a level structure $\overline{\eta}_0$ of similitudes

$$\widehat{V}(A_0) \xrightarrow{\sim} W_0 \otimes_{\mathbb{Q}} \mathbb{A}_f.$$

Then the morphism (3.17) sends $(A_0, \iota_0, \lambda_0) \mapsto (A_0, \iota_0, \lambda_0, \overline{\eta}_0)$. This morphism is an isomorphism by an obvious modification of the argument in [18, Prop. 4.4], or see [21, Prop. 1.4.3.4].

Keeping W_0 fixed, it is not hard to show that every class $\tau' \in \mathcal{R}_{W_0, \sqrt{\Delta}}/\mathbb{Q}_{>0}$ is represented by a space of the form $\Lambda'_0 \otimes_{\mathbb{Z}} \mathbb{Q}$ for some $\Lambda'_0 \in \mathcal{L}_{\Phi}^{\mathfrak{a}}$. (Since we will make no essential use of this fact later in the paper, we leave the details to the reader.) Choosing such a Λ'_0 for each τ' , and taking the \sim -class of Λ'_0 ,

we obtain an injection $\mathcal{R}_{W_0, \sqrt{\Delta}/\mathbb{Q}_{>0}} \hookrightarrow \mathcal{L}_{\mathbb{F}}^{\mathfrak{a}}/\sim$. In this way, combined with the previous paragraph, we may identify $M_{0, K_{Z^{\mathbb{Q}}}}^{\circ}$ with an open and closed substack of $M_0^{\mathfrak{a}}$. (Note however that the choice of each Λ'_0 , and hence the embedding $M_{0, K_{Z^{\mathbb{Q}}}}^{\circ} \hookrightarrow M_0^{\mathfrak{a}}$, is not canonical.)

We now turn to a couple of variants of the moduli problem attached to \tilde{G} in Definition 3.3. We consider a subgroup $K_{\tilde{G}}$ of the form (3.7) with $K_{Z^{\mathbb{Q}}} = K_{Z^{\mathbb{Q}}}^{\circ}$, so that

$$(3.18) \quad K_{\tilde{G}} = K_{Z^{\mathbb{Q}}}^{\circ} \times K_G,$$

still with $K_G \subset G(\mathbb{A}_f)$ an arbitrary open compact subgroup. Fix \mathfrak{a} , $\sqrt{\Delta}$, ξ , and W_0 all as before (3.17). Set $V := \text{Hom}_F(W_0, W)$, endowed with its natural hermitian form. We define the following category fibered in groupoids $\mathcal{F}'_{K_{\tilde{G}}}(\tilde{G})$ over $(\text{LNSch})/E$. To lighten notation, we suppress the dependence on the ideal \mathfrak{a} and the element ξ .

Definition 3.8. The category functor $\mathcal{F}'_{K_{\tilde{G}}}(\tilde{G})$ associates to each scheme S in $(\text{LNSch})/E$ the groupoid of tuples $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \bar{\eta})$, where

- $(A_0, \iota_0, \lambda_0)$ is an object of $M_0^{\mathfrak{a}, \xi}(S)$; and
- the tuple $(A, \iota, \lambda, \bar{\eta})$ is as in Definition 3.3.

A morphism $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \bar{\eta}) \rightarrow (A'_0, \iota'_0, \lambda'_0, A', \iota', \lambda', \bar{\eta}')$ in this groupoid is given by an isomorphism $\mu_0: (A_0, \iota_0, \lambda_0) \xrightarrow{\sim} (A'_0, \iota'_0, \lambda'_0)$ in $M_0^{\mathfrak{a}, \xi}(S)$ and an F -linear quasi-isogeny $\mu: A \rightarrow A'$ pulling λ' back to λ and $\bar{\eta}'$ back to $\bar{\eta}$.

The morphism (3.17) induces a natural comparison morphism of category functors,

$$(3.19) \quad \begin{array}{ccc} \mathcal{F}'_{K_{\tilde{G}}}(\tilde{G}) & \longrightarrow & \mathcal{F}_{K_{\tilde{G}}}(\tilde{G}) \\ (A_0, \iota_0, \lambda_0, A, \iota, \lambda, \bar{\eta}) & \longmapsto & (A_0, \iota_0, \lambda_0, \bar{\eta}_0, A, \iota, \lambda, \bar{\eta}). \end{array}$$

The fact that (3.17) is an isomorphism easily implies that (3.19) is an isomorphism as well. In this way, $\mathcal{F}'_{K_{\tilde{G}}}(\tilde{G})$ gives another moduli interpretation of the stack $M_{K_{\tilde{G}}}(\tilde{G})$ over $\text{Spec } E$, cf. Theorem 3.5.

We give a third moduli interpretation of $M_{K_{\tilde{G}}}(\tilde{G})$, in which *all* of the data is taken up to isomorphism, as follows. We continue with \mathfrak{a} , ξ , $\sqrt{\Delta}$, W_0 , and

V as above. Let $\Lambda_0 \in \mathcal{L}_{\mathfrak{F}}^{\mathfrak{a}}$ denote the representative of ξ used to define W_0 as before (3.17). Fix any O_F -lattice $\Lambda \subset W$, and define the O_F -lattice

$$L := \text{Hom}_{O_F}(\Lambda_0, \Lambda) \subset V.$$

We again take the subgroup $K_{\tilde{G}} = K_{Z^{\circ}} \times K_G$ of the form (3.18), and we assume that $L \otimes_{O_F} \widehat{O}_F$ is K_G -stable inside $V \otimes_F \mathbb{A}_{F,f}$ (which is equivalent to $\Lambda \otimes_{O_F} \widehat{O}_F$ being K_G -stable inside $W \otimes_F \mathbb{A}_{F,f}$). Let N be a positive integer such that the principal congruence subgroup mod N for L ,

$$(3.20) \quad K^{L,N} := \{g \in G(\mathbb{A}_f) \mid (g - 1) \cdot L \otimes_{O_F} \widehat{O}_F \subset NL \otimes_{O_F} \widehat{O}_F\},$$

is contained in K_G . We define the following moduli problem. As before we suppress the dependence on \mathfrak{a} and ξ in the notation.

Definition 3.9. The category functor $\mathcal{F}_{K_{\tilde{G}}}^{L,N}(\tilde{G})$ associates to each scheme S in $(\text{LNSch})/E$ the groupoid of tuples $(A_0, \iota_0, \lambda_0, B, \iota, \lambda, \overline{\eta}_N)$, where

- $(A_0, \iota_0, \lambda_0)$ is an object of $M_0^{\mathfrak{a},\xi}(S)$;
- B is an abelian scheme over S ;
- $\iota: O_F \rightarrow \text{End}(B)$ is an action of O_F on B satisfying the Kottwitz condition (2.7) for all $a \in O_F$;
- λ is a quasi-polarization on B whose Rosati involution satisfies condition (2.6) for all $a \in O_F$; and
- $\overline{\eta}_N$ is an étale closed subscheme

$$\overline{\eta}_N \subset \underline{\text{Isom}}_{O_F}(\underline{\text{Hom}}_{O_F}(A_0[N], B[N]), (L/NL)_S)$$

over S such that for every geometric point $\overline{s} \rightarrow S$, the fiber $\overline{\eta}_N(\overline{s})$ identifies with a $K_G/K^{L,N}$ -orbit of isomorphisms

$$\eta_N(\overline{s}): \text{Hom}_{O_F}(A_0[N](\overline{s}), B[N](\overline{s})) \xrightarrow{\sim} L/NL$$

which lift to \widehat{O}_F -linear isometries of hermitian modules⁵

$$(3.21) \quad \widehat{\text{T}}(A_0, B)(\overline{s}) \xrightarrow{\sim} L \otimes_{O_F} \widehat{O}_F.$$

⁵Note that we have made no assumption on the restriction of the hermitian form on V to L ; all we can say is that this restriction takes values in some fractional ideal \mathfrak{d} of F . Similarly, since λ_0 need not be principal and λ is only required to be a quasi-polarization, the hermitian form on $\widehat{\text{T}}(A_0, B)$ need not be \widehat{O}_F -valued.

Here

$$(3.22) \quad \widehat{\mathbb{T}}(A_0, B) := \text{Hom}_{\widehat{O}_F}(\widehat{\mathbb{T}}(A_0), \widehat{\mathbb{T}}(B)),$$

regarded as a smooth \widehat{O}_F -sheaf on S , and endowed with its natural hermitian form as in (3.12). Furthermore, the notion of “lift” is with respect to the evident reduction-mod- N maps $\widehat{\mathbb{T}}(A_0, B)(\bar{s}) \rightarrow \text{Hom}_{O_F}(A_0[N](\bar{s}), B[N](\bar{s}))$ and $L \otimes_{O_F} \widehat{O}_F \rightarrow L/NL$. A morphism

$$(A_0, \iota_0, \lambda_0, B, \iota, \lambda, \bar{\eta}_N) \longrightarrow (A'_0, \iota'_0, \lambda'_0, B', \iota', \lambda', \bar{\eta}'_N)$$

in this groupoid is given by an isomorphism $\mu_0: (A_0, \iota_0, \lambda_0) \xrightarrow{\sim} (A'_0, \iota'_0, \lambda'_0)$ in $M_0^{\mathfrak{a}, \xi}(S)$ and an O_F -linear isomorphism of abelian schemes $\mu: B \xrightarrow{\sim} B'$ pulling λ' back to λ and $\bar{\eta}'_N$ back to $\bar{\eta}_N$.

Remark 3.10. As usual, the condition on the level structure $\bar{\eta}_N$ in Definition 3.9 holds for all geometric points $\bar{s} \rightarrow S$ as soon as it holds for a single geometric point on each connected component of S . The proof is similar to [21, Lems. 1.3.6.5, 1.3.6.6, Cor. 1.3.6.7].

Remark 3.11. Note that the (quasi-)polarization type of λ in Definition 3.9 is determined by Λ , in the sense that the existence of the isometries (3.21) implies that $\widehat{\mathbb{T}}(A)$ and $\widehat{\mathbb{T}}(A)^\vee$ (the dual lattice of $\widehat{\mathbb{T}}(A)$ inside $\widehat{V}(A)$ with respect to λ and the Weil pairing) have the same relative position as Λ and Λ^\vee (the dual lattice of Λ inside W with respect to $\text{tr}_{F/\mathbb{Q}} \sqrt{\Delta}^{-1}(\cdot, \cdot)$) have in W . In particular, λ is an honest polarization if and only if $\Lambda \subset \Lambda^\vee$.

Remark 3.12. In the special case that $W_0 = F$ with $(x, y)_0 = \text{Nm}_{F/F_0}(x\bar{y})$, take $\mathfrak{a} = \sqrt{\Delta}^{-1} \mathfrak{D}$, where \mathfrak{D} denotes the different of F/\mathbb{Q} , and where $\sqrt{\Delta}$ is any Φ -adapted element such that \mathfrak{a} is an integral ideal. Then \mathfrak{a} is the image in O_F of an ideal in O_{F_0} , and $O_F^\vee = \mathfrak{a}^{-1} O_F$. Hence we may take ξ to be the class defined by $\Lambda_0 = O_F$ in the above discussion, and we obtain canonical isometries $V \cong W$ and $L \cong \Lambda$. Thus in this case, one may formulate Definition 3.9 purely in terms of Λ , without needing to introduce L .

The moduli problem $\mathcal{F}_{K_G}^{L, N}(\tilde{G})$ is related to $\mathcal{F}'_{K_G}(\tilde{G})$ via a natural comparison equivalence

$$(3.23) \quad \begin{array}{ccc} \varphi: & \mathcal{F}_{K_G}^{L, N}(\tilde{G}) & \xrightarrow{\sim} \mathcal{F}'_{K_G}(\tilde{G}) \\ & (A_0, \iota_0, \lambda_0, B, \iota, \lambda, \bar{\eta}_N) & \longmapsto (A_0, \iota_0, \lambda_0, B, \iota, \lambda, \bar{\eta}), \end{array}$$

where $\bar{\eta}$ is the K_G -orbit of isometries $\widehat{V}(A_0, B) \xrightarrow{\sim} V \otimes_F \mathbb{A}_{F,f}$ induced by extension of scalars from the lifts (3.21) of the sections of $\bar{\eta}_N$. (Note that, given any geometric point $\bar{s} \rightarrow S$, stability of the orbit $\bar{\eta}$ under the action of $\pi_1(S, \bar{s})$ follows from finite étaleness of $\bar{\eta}_N$.) The inverse of φ can be explicitly described in a way similar to the proof of [18, Prop. 4.4]; see also [3, §4.12] or [21, Prop. 1.4.3.4]. Let S be a locally noetherian scheme over $\text{Spec } E$, and let $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \bar{\eta})$ be an S -valued point on $\mathcal{F}'_{K_G}(\tilde{G})$. Since we assume that $L \otimes_{O_F} \widehat{O}_F$ is K_G -stable, there exists a unique \widehat{O}_F -submodule $T \subset \widehat{V}(A)$ such that the submodule $\text{Hom}_{\widehat{O}_F}(\widehat{\text{T}}(A_0), T) \subset \widehat{V}(A_0, A)$ identifies with $L \otimes_{O_F} \widehat{O}_F \subset V \otimes_F \mathbb{A}_{F,f}$ under each $\eta \in \bar{\eta}$. The submodule T gives rise to an abelian scheme B over S with O_F -action ι_B and an F -linear quasi-isogeny $\mu: B \rightarrow A$ such that $\mu_*(\widehat{\text{T}}(B)) = T$ inside $\widehat{V}(A)$. The pullback of λ along μ defines the quasi-polarization λ_B , and the reduction mod N of the isometries

$$\widehat{\text{T}}(A_0, B) \xrightarrow{\mu_*} \text{Hom}_{O_F}(\widehat{\text{T}}(A_0), T) \xrightarrow{\eta} L \otimes_{O_F} \widehat{O}_F,$$

for $\eta \in \bar{\eta}$, defines the finite étale scheme $\bar{\eta}_N$ (using that $\bar{\eta}$ is $\pi_1(S, \bar{s})$ -stable with respect to any geometric point $\bar{s} \rightarrow S$). Then $(A_0, \iota_0, \lambda_0, B, \iota_B, \lambda_B, \bar{\eta}_N)$ is the image of $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \bar{\eta})$ under the inverse of φ . The equivalence (3.23) shows that $\mathcal{F}_{K_G}^{L,N}(\tilde{G})$ gives a third moduli interpretation of $M_{K_G}(\tilde{G})$, and that, up to canonical equivalence, $\mathcal{F}_{K_G}^{L,N}(\tilde{G})$ is independent of the choice of L and N such that $K^{L,N} \subset K_G$.

3.5. GGP and KR cycles

To conclude Section 3, we give the definition of GGP and KR cycles in the context of the RSZ Shimura varieties. In the case of the GGP cycles, let $n \geq 2$. We return to the situation of Remark 2.5, with $u \in W$ a fixed totally positive vector and $W^b = (u)^\perp$. Replacing W by W^b in the discussion in Section 3.2, we get a Shimura datum $(\tilde{H}, \{h_{\tilde{H}}\})$, where \tilde{H} denotes the analog of the group \tilde{G} for W^b . The inclusion $W^b \subset W$ then induces a natural inclusion $\tilde{H} \subset \tilde{G}$.

Since u is totally positive, the inclusion $\tilde{H} \subset \tilde{G}$ induces a morphism of the Shimura data we have defined. Hence there is a morphism of towers

$$(3.24) \quad \left(S(\tilde{H}, \{h_{\tilde{H}}\})_{K_{\tilde{H}}} \right) \longrightarrow \left(S(\tilde{G}, \{h_{\tilde{G}}\})_{K_{\tilde{G}}} \right),$$

which lies over the tower $(S(Z^{\mathbb{Q}}, \{h_{Z^{\mathbb{Q}}}\})_{K_{Z^{\mathbb{Q}}})$. Taking the graph morphism,

we obtain a closed embedding of towers,

$$(3.25) \quad \left(S(\tilde{H}, \{h_{\tilde{H}}\})_{K_{\tilde{H}}} \right) \hookrightarrow \left(S(\tilde{H}, \{h_{\tilde{H}}\})_{K_{\tilde{H}}} \right) \times_{(S(\mathbb{Z}^{\mathbb{Q}}, \{h_{\mathbb{Z}^{\mathbb{Q}}}\})_{K_{\mathbb{Z}^{\mathbb{Q}}})} \left(S(\tilde{G}, \{h_{\tilde{G}}\})_{K_{\tilde{G}}} \right).$$

The resulting cycle in the target of (3.25) is the *GGP cycle* in the present context.

In terms of the moduli description of Theorem 3.5, the above morphisms admit the following simple descriptions. Using that u is totally positive, it is easy to see that E is also the reflex field of $(\tilde{H}, \{h_{\tilde{H}}\})$. Take $W_0 := (u)$ in moduli problem of Definition 3.3 for both groups, so that we consider the spaces $V = \text{Hom}_F(W_0, W)$ and $V^b = \text{Hom}_F(W_0, W^b)$. Similarly fix the same class τ in both moduli problems. Then for level subgroups $K_{\tilde{G}} = K_{\mathbb{Z}^{\mathbb{Q}}} \times K_G$ and $K_{\tilde{H}} = K_{\mathbb{Z}^{\mathbb{Q}}} \times K_H$ of the form (3.7), and such that $K_H \subset K_G$, the morphism (3.24) is given by the finite and unramified morphism over E ,

$$(3.26) \quad \begin{aligned} M_{K_{\tilde{H}}}(\tilde{H}) &\longrightarrow M_{K_{\tilde{G}}}(\tilde{G}) \\ (A_0, \iota_0, \lambda_0, \bar{\eta}_0, A^b, \iota^b, \lambda^b, \bar{\eta}^b) &\longmapsto (A_0, \iota_0, \lambda_0, \bar{\eta}_0, A^b \times A_0, \iota^b \times \iota_0, \lambda^b \times \lambda_0, \bar{\eta}). \end{aligned}$$

Here the level structure $\bar{\eta}$ is the K_G -orbit of the isometries

$$(3.27) \quad \begin{aligned} \widehat{V}(A_0, A^b \times A_0) &= \widehat{V}(A_0, A^b) \oplus \text{End}_{\mathbb{A}_{F,f}}(\widehat{V}(A_0)) \xrightarrow[\sim]{\eta^b \oplus [\text{id} \rightarrow \text{id}]} \\ &= (V^b \otimes_F \mathbb{A}_{F,f}) \oplus (\text{End}_F(W_0) \otimes_F \mathbb{A}_{F,f}) = V \otimes_F \mathbb{A}_{F,f} \end{aligned}$$

for $\eta^b \in \bar{\eta}^b$. The GGP cycle for the given levels is the graph of (3.26),

$$(3.28) \quad M_{K_{\tilde{H}}}(\tilde{H}) \hookrightarrow M_{K_{\tilde{H}}}(\tilde{H}) \times_{M_{0, K_{\mathbb{Z}^{\mathbb{Q}}}}^{\tau}} M_{K_{\tilde{G}}}(\tilde{G}).$$

The morphisms (3.26) and (3.28) admit analogous descriptions in terms of the alternative moduli interpretations of Section 3.4. In the case of the moduli problems $\mathcal{F}'_{K_{\tilde{H}}}(\tilde{H})$ and $\mathcal{F}'_{K_{\tilde{G}}}(\tilde{G})$ of Definition 3.8, let the level subgroups $K_{\tilde{G}} = K_{\mathbb{Z}^{\mathbb{Q}}}^{\circ} \times K_G$ and $K_{\tilde{H}} = K_{\mathbb{Z}^{\mathbb{Q}}}^{\circ} \times K_H$ be of the form (3.18), and again assume that $K_H \subset K_G$. Fix an ideal \mathfrak{a} such that $M_0^{\mathfrak{a}} \neq \emptyset$, and again take $W_0 = (u)$ in both moduli problems. The proof of [11, Prop. 3.1.6] shows that for an appropriate Φ -adapted $\sqrt{\Delta} \in F$, there exists an O_F -lattice $\Lambda_0 \subset W_0$ such that $\Lambda_0^{\vee} = \mathfrak{a}^{-1} \Lambda_0$ with respect to the form $\text{tr}_{F/\mathbb{Q}} \sqrt{\Delta}^{-1}(\ , \)|_{W_0}$; fix such a $\sqrt{\Delta}$

and Λ_0 , and let ξ denote the class of Λ_0 in $\mathcal{L}_{\mathbb{F}}^{\mathfrak{a}}/\sim$. Then the morphism (3.26) is given by the morphism of moduli problems $\mathcal{F}'_{K_{\tilde{H}}}(\tilde{H}) \rightarrow \mathcal{F}'_{K_{\tilde{G}}}(\tilde{G})$ sending

$$(3.29) \quad (A_0, \iota_0, \lambda_0, A^b, \iota^b, \lambda^b, \tilde{\eta}^b) \longmapsto (A_0, \iota_0, \lambda_0, A^b \times A_0, \iota^b \times \iota_0, \lambda^b \times \lambda_0, \tilde{\eta}),$$

where the level structure $\tilde{\eta}$ is defined exactly as in (3.27). The GGP cycle is again the graph of this morphism, as in (3.28) (with the fibered product over $M_0^{\mathfrak{a}, \xi}$). The descriptions of the morphisms (3.26) and (3.28) in terms of the moduli problems $\mathcal{F}_{K_{\tilde{H}}}^{L, N}(\tilde{H})$ and $\mathcal{F}_{K_{\tilde{G}}}^{L, N}(\tilde{G})$ are completely analogous.

To define KR cycles, we again give differing versions according to our differing moduli interpretations of the RSZ Shimura varieties. First consider the moduli problem $\mathcal{F}_{K_{\tilde{G}}}(\tilde{G})$, for any choice of defining data in Definition 3.3. Let $(A_0, \iota_0, \lambda_0, \tilde{\eta}_0, A, \iota, \lambda, \tilde{\eta})$ be an S -point on $\mathcal{F}_{K_{\tilde{G}}}(\tilde{G})$ for a connected, locally noetherian E -scheme S . Then $\text{Hom}_F^0(A_0, A)$ is well-defined, and by the multiplier condition in Definition 3.3, it carries a natural well-defined F/F_0 -hermitian form h' , in analogy with (3.12),

$$(3.30) \quad h'(x, y) := \lambda_0^{-1} \circ y^\vee \circ \lambda \circ x \in \text{End}_{O_F}^0(A_0) \cong F, \quad x, y \in \text{Hom}_F^0(A_0, A).$$

Note that passing to Tate modules defines an isometric embedding

$$\text{Hom}_F^0(A_0, A) \hookrightarrow \widehat{V}(A_0, A).$$

Now let m be a positive integer, let $T \in \text{Herm}_m(F)$ be an $m \times m$ hermitian matrix which is positive semidefinite at all archimedean places, and let L be an O_F -lattice in the vector space V of Definition 3.3. The *KR cycle* $Z(T, L)$ is the stack of tuples

$$(3.31) \quad (A_0, \iota_0, \lambda_0, \tilde{\eta}_0, A, \iota, \lambda, \tilde{\eta}; \mathbf{x}),$$

where $(A_0, \iota_0, \lambda_0, \tilde{\eta}_0, A, \iota, \lambda, \tilde{\eta})$ is an object in $\mathcal{F}_{K_{\tilde{G}}}(\tilde{G})$ and $\mathbf{x} = (x_1, \dots, x_m) \in \text{Hom}_F^0(A_0, A)^m$ is an m -tuple of quasi-homomorphisms such that $(h'(x_i, x_j)) = T$, and such that for $i = 1, \dots, m$ and each $\eta \in \tilde{\eta}$, the quasi-homomorphism x_i identifies with an element of $L \otimes_{O_F} \widehat{O}_F$ under the composite

$$\text{Hom}_F^0(A_0, A) \hookrightarrow \widehat{V}(A_0, A) \xrightarrow[\sim]{\eta} V \otimes_F \mathbb{A}_{F, f}.$$

(Note that if $L \otimes_{O_F} \widehat{O}_F$ is stable inside $V \otimes_F \mathbb{A}_{F, f}$ under the action of the subgroup K_G of (3.7), then this last condition is independent of $\eta \in \tilde{\eta}$.)

A morphism $(A_0, \iota_0, \lambda_0, \bar{\eta}_0, A, \iota, \lambda, \bar{\eta}; \mathbf{x}) \rightarrow (A'_0, \iota'_0, \lambda'_0, \bar{\eta}'_0, A', \iota', \lambda', \bar{\eta}'; \mathbf{x}')$ consists of quasi-isogenies $\mu_0: A_0 \rightarrow A'_0$ and $\mu: A \rightarrow A'$ as in Definition 3.3 which pull \mathbf{x}' back to \mathbf{x} . The proof of [18, Prop. 2.9] transposes to the present setting to show that $Z(T, L)$ is representable by a DM stack which is finite and unramified over $\mathcal{F}_{K_{\tilde{G}}}^{L,N}(\tilde{G}) \cong M_{K_{\tilde{G}}}(\tilde{G})$ (in the present setting, one uses the lattice L in the moduli problem to deduce finiteness). If T is totally positive definite and $1 \leq m \leq \min\{r_\varphi \mid \varphi \in \Phi\}$, then $Z(T, L)$ has codimension $m \sum_{\varphi \in \Phi} r_\varphi$ over $M_{K_{\tilde{G}}}(\tilde{G})$. In particular, in the strict fake Drinfeld case relative to Φ of Example 2.3(ii), the codimension is m .

In the case of the moduli problem $\mathcal{F}'_{K_{\tilde{G}}}(\tilde{G})$ (for any choice of defining data in Definition 3.8), the *KR cycle* $Z'(T, L)$, for T and L as above, is the stack of tuples $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \bar{\eta}; \mathbf{x})$, where $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \bar{\eta})$ is an object in $\mathcal{F}'_{K_{\tilde{G}}}(\tilde{G})$ and \mathbf{x} is exactly as in (3.31). Then the equivalence $\mathcal{F}'_{K_{\tilde{G}}}(\tilde{G}) \xrightarrow{\sim} \mathcal{F}_{K_{\tilde{G}}}(\tilde{G})$ of (3.19) induces a natural equivalence $Z'(T, L) \xrightarrow{\sim} Z(T, L)$.

In the case of the moduli problem $\mathcal{F}_{K_{\tilde{G}}}^{L,N}(\tilde{G})$ (for any choice of defining data in Definition 3.9), let $(A_0, \iota_0, \lambda_0, B, \iota, \lambda, \bar{\eta}_N)$ be an object in $\mathcal{F}_{K_{\tilde{G}}}^{L,N}(\tilde{G})$. Then the group $\text{Hom}_{O_F}(A_0, B)$ is well-defined (since both A_0 and B are taken up to isomorphism as abelian schemes), and this group carries a natural O_F/O_{F_0} -hermitian form h' , defined as in (3.30), which takes values in some fractional ideal \mathfrak{d} of F . In this case, passing to Tate modules defines an isometric embedding $\text{Hom}_{O_F}(A_0, B) \hookrightarrow \widehat{\text{T}}(A_0, B)$. (Therefore, by the existence of a level structure in the moduli problem, if the hermitian form on V is O_F -valued on L , then h' will be O_F -valued too.) The *KR cycle* $Z^{L,N}(T)$, for T as above, is the stack of tuples $(A_0, \iota_0, \lambda_0, B, \iota, \lambda, \bar{\eta}_N; \mathbf{x})$, where $(A_0, \iota_0, \lambda_0, B, \iota, \lambda, \bar{\eta}_N)$ is an object in $\mathcal{F}_{K_{\tilde{G}}}^{L,N}(\tilde{G})$ and $\mathbf{x} = (x_1, \dots, x_m) \in \text{Hom}_{O_F}(A_0, B)^m$ is an m -tuple of homomorphisms such that $(h'(x_i, x_j)) = T$. In this case, the equivalence $\mathcal{F}_{K_{\tilde{G}}}^{L,N}(\tilde{G}) \xrightarrow{\sim} \mathcal{F}'_{K_{\tilde{G}}}(\tilde{G})$ of (3.23) induces a natural equivalence $Z^{L,N}(T) \xrightarrow{\sim} Z'(T, L)$.

4. p -integral models of RSZ Shimura varieties

4.1. Semi-global models

In this subsection we define a “semi-global” integral model of the stack $M_{K_{\tilde{G}}}(\tilde{G})$ over $\text{Spec } O_{E,(p)}$ for certain level subgroups $K_{\tilde{G}}$.

We first need some preparatory notions. Let k be any algebraically closed field which is an O_E -algebra. Let $(A_0, \iota_0, \lambda_0)$ and (A, ι, λ) be two triples,

each consisting of an abelian variety over k , an F -action up to isogeny, and a quasi-polarization whose Rosati involution induces the nontrivial F/F_0 -automorphism on F . Suppose that (A_0, ι_0) satisfies the Kottwitz condition (3.14), and that (A, ι) satisfies the Kottwitz condition (2.7) relative to a fixed choice of a generalized CM type r of rank n ; in particular, this implies that A_0 and A have respective dimensions $[F_0 : \mathbb{Q}]$ and $n \cdot [F_0 : \mathbb{Q}]$. Let v be a finite place of F_0 which does not split in F . Then [31, App. A] defines a sign invariant $\text{inv}_v^r(A_0, \iota_0, \lambda_0, A, \iota, \lambda) \in \{\pm 1\}$.⁶ If the residue characteristic of v does not equal $\text{char } k$, then inv_v^r is simply the Hasse invariant of the $F_v/F_{0,v}$ -hermitian space

$$(4.1) \quad V_v(A_0, A) := \text{Hom}_{F_v}(V_v(A_0), V_v(A)),$$

where the hermitian form is the obvious v -adic analog of (3.12) (and hence $V_v(A_0, A)$ is the v -factor of (3.11) when $\text{char } k = 0$). If the residue characteristic of v equals $\text{char } k$, then inv_v^r is defined similarly in terms of the highest exterior power of the Hom space of the rational Dieudonné modules of A_0 and A , with a further correction factor in terms of the function r . The sign invariant depends only on the tuple $(A_0, \iota_0, \lambda_0; A, \iota, \lambda)$ up to isogeny, and it is locally constant in families over O_E -schemes [31, Prop. A.1].

We next note that the definition of the moduli space $M_0^{\mathfrak{a}}$ over $\text{Spec } E$ of Section 3.4 extends word-for-word to a moduli space $\mathcal{M}_0^{\mathfrak{a}}$ over $\text{Spec } O_E$. Then $\mathcal{M}_0^{\mathfrak{a}}$ is a Deligne–Mumford stack, finite and étale over $\text{Spec } O_E$, cf. [11, Prop. 3.1.2]. It follows that the decomposition (3.16) extends to a disjoint union decomposition of $\mathcal{M}_0^{\mathfrak{a}}$,

$$(4.2) \quad \mathcal{M}_0^{\mathfrak{a}} = \coprod_{\xi \in \mathcal{L}_{\Phi}^{\mathfrak{a}}/\sim} \mathcal{M}_0^{\mathfrak{a}, \xi}.$$

For the rest of this section we fix a prime number p . We denote by \mathcal{V}_p the set of places of F_0 over p . If $p = 2$, then we assume that every $v \in \mathcal{V}_p$ is unramified in F . We fix \mathfrak{a} , $\sqrt{\Delta}$, ξ , Λ_0 , and W_0 as before (3.17). We continue with the n -dimensional hermitian space W , and as usual we set $V = \text{Hom}_F(W_0, W)$. For each $v \in \mathcal{V}_p$, we endow the $F_v/F_{0,v}$ -hermitian space $W_v := W \otimes_F F_v$ with the \mathbb{Q}_p -valued alternating form $\text{tr}_{F_v/\mathbb{Q}_p} \sqrt{\Delta}^{-1}(\ , \)$, and we fix a *vertex lattice* $\Lambda_v \subset W_v$ with respect to this form, i.e., Λ_v is an

⁶Since we take the Kottwitz condition (3.14) for A_0 to be the opposite of the one used in loc. cit., we need to use the version of inv_v^r modified as in [31, Rem. A.2] when the residue characteristic of v equals $\text{char } k$.

$O_{F,v}$ -lattice such that

$$\Lambda_v \subset \Lambda_v^\vee \subset \pi_v^{-1}\Lambda_v.$$

Here π_v denotes a uniformizer in F_v (if v splits in F , this means the image in F_v of a uniformizer for $F_{0,v}$), and $\Lambda_v^\vee \subset W_v$ denotes the dual lattice with respect to $\text{tr}_{F_v/\mathbb{Q}_p} \sqrt{\Delta}^{-1}(\ , \)$.⁷

We consider a subgroup $K_{\tilde{G}} = K_{\mathbb{Z}^{\mathbb{Q}}}^\circ \times K_G$ as in (3.18). We assume that $K_G \subset G(\mathbb{A}_f)$ is of the form $K_G = K_G^p \times K_{G,p}$, where $K_G^p \subset G(\mathbb{A}_f^p)$ is arbitrary and where

$$K_{G,p} = \prod_{v \in \mathcal{V}_p} K_{G,v} \subset G(\mathbb{Q}_p) = \prod_{v \in \mathcal{V}_p} \text{U}(W)(F_{0,v}),$$

with

$$(4.3) \quad K_{G,v} := \text{Stab}_{\text{U}(W)(F_{0,v})}(\Lambda_v).$$

We note that if v is unramified in F , then $K_{G,v}$ is a maximal parahoric subgroup of $\text{U}(W)(F_{0,v})$. If v ramifies in F (recall that in this case we assume that $v \nmid 2$), then $K_{G,v}$ is a maximal compact subgroup of $\text{U}(W)(F_{0,v})$ which contains a (maximal) parahoric subgroup with index 2, unless n is even and Λ_v is π_v -modular, in which case $K_{G,v}$ is itself maximal parahoric; see [27, §4.a]. Here π_v -modular means that $\Lambda_v^\vee = \pi_v^{-1}\Lambda_v$.

We now define the following category fibered in groupoids $\mathcal{F}_{K_{\tilde{G}}}^{\text{naive}}(\tilde{G})$ over $(\text{LNSch})_{/O_{E,(p)}}$. As before, to lighten notation, we suppress the ideal \mathfrak{a} and the element ξ .

Definition 4.1. The category functor $\mathcal{F}_{K_{\tilde{G}}}^{\text{naive}}(\tilde{G})$ associates to each scheme S in $(\text{LNSch})_{/O_{E,(p)}}$ the groupoid of tuples $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \bar{\eta}^p)$, where

- $(A_0, \iota_0, \lambda_0)$ is an object of $\mathcal{M}_0^{\mathfrak{a},\xi}(S)$;
- A is an abelian scheme over S ;
- $\iota: O_{F,(p)} \rightarrow \text{End}_{(p)}(A)$ is an action up to prime-to- p isogeny satisfying the Kottwitz condition (2.7) on $O_{F,(p)}$;

⁷We remind the reader that Λ_v^\vee and the dual lattice Λ_v^* with respect to the hermitian form on W_v need not be equal, but they are at least scalar multiples of each other. The lattice Λ_v^\vee is more natural to use in connection with polarizations. We also point out that vertex lattices in [31] are always taken with respect to *hermitian* forms.

- $\lambda \in \text{Hom}_{(p)}(A, A^\vee)$ is a quasi-polarization on A whose Rosati involution satisfies condition (2.6) on $O_{F,(p)}$; and
- $\bar{\eta}^p$ is a K_G^p -orbit of isometries of $\mathbb{A}_{F,f}^p/\mathbb{A}_{F_0,f}^p$ -hermitian modules

$$(4.4) \quad \eta^p: \widehat{V}^p(A_0, A) \xrightarrow{\sim} V \otimes_F \mathbb{A}_{F,f}^p,$$

where

$$(4.5) \quad \widehat{V}^p(A_0, A) := \text{Hom}_{\mathbb{A}_{F,f}^p}(\widehat{V}^p(A_0), \widehat{V}^p(A)),$$

and where the hermitian form on $\widehat{V}^p(A_0, A)$ is the obvious prime-to- p analog of (3.12).

We impose the following further conditions on the above tuples.

- (i) Consider the decomposition of p -divisible groups

$$(4.6) \quad A[p^\infty] = \prod_{v \in \mathcal{V}_p} A[v^\infty]$$

induced by the action of $O_{F_0} \otimes \mathbb{Z}_p \cong \prod_{v \in \mathcal{V}_p} O_{F_0,v}$. Since Ros_λ is trivial on O_{F_0} , λ induces a polarization $\lambda_v: A[v^\infty] \rightarrow A^\vee[v^\infty] \cong A[v^\infty]^\vee$ of p -divisible groups for each v . The condition we impose is that $\ker \lambda_v$ is contained in $A[\iota(\pi_v)]$ of rank $\#(\Lambda_v^\vee/\Lambda_v)$ for each $v \in \mathcal{V}_p$.

- (ii) We require that at every geometric point \bar{s} of S the following *sign condition* holds for every non-split place $v \in \mathcal{V}_p$,

$$(4.7) \quad \text{inv}_v^r(A_{0,\bar{s}}, \iota_{0,\bar{s}}, \lambda_{0,\bar{s}}, A_{\bar{s}}, \iota_{\bar{s}}, \lambda_{\bar{s}}) = \text{inv}_v(V),$$

where the right-hand denotes the Hasse invariant of the hermitian space V at v .

A morphism $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \bar{\eta}^p) \rightarrow (A'_0, \iota'_0, \lambda'_0, A', \iota', \lambda', \bar{\eta}'^p)$ in this groupoid is given by an isomorphism $\mu_0: (A_0, \iota_0, \lambda_0) \xrightarrow{\sim} (A'_0, \iota'_0, \lambda'_0)$ in $\mathcal{M}_0^{\alpha,\xi}(S)$ and an $O_{F,(p)}$ -linear quasi-isogeny $\mu: A \rightarrow A'$ inducing an isomorphism $A[p^\infty] \xrightarrow{\sim} A'[p^\infty]$, pulling λ' back to λ , and pulling $\bar{\eta}'^p$ back to $\bar{\eta}^p$.

Remark 4.2. Let $\mathbf{A} = (A_{0,\bar{s}}, \iota_{0,\bar{s}}, \lambda_{0,\bar{s}}, A_{\bar{s}}, \iota_{\bar{s}}, \lambda_{\bar{s}}, \bar{\eta}^p)$ be a tuple as in Definition 4.1, except where we don't impose the sign condition in (ii), and suppose that \mathbf{A} lifts to characteristic zero. Then, by the product formula and the Hasse principle for hermitian vector spaces, by the Kottwitz condition and

the existence of the prime-to- p level structure $\bar{\eta}^p$, and by local constancy of inv_v^r , the sign condition for all non-split $v \in \mathcal{V}_p$ except one implies the sign condition for \mathbf{A} at the remaining place. In particular, when $F_0 = \mathbb{Q}$, the sign condition is empty, provided that $\mathcal{F}_{K_{\tilde{G}}}^{\text{naive}}(\tilde{G})$ is topologically flat over $O_{E,(p)}$. We refer to Remark 6.5(i) for other instances when the sign condition is redundant. In these instances the *naive* moduli problem considered here is replaced by more sophisticated integral models which are flat over $O_{E,(p)}$.

Remark 4.3. As in Remarks 3.4 and 3.12, when $(W_0, (\cdot, \cdot)_0) = (F, \text{Nm}_{F/F_0})$, we may replace V by W everywhere in Definition 4.1.

The following theorem shows that the moduli functor $\mathcal{F}_{K_{\tilde{G}}}^{\text{naive}}(\tilde{G})$ defines an extension of $M_{K_{\tilde{G}}}(\tilde{G})$ with reasonable properties.

Theorem 4.4. *The moduli problem $\mathcal{F}_{K_{\tilde{G}}}^{\text{naive}}(\tilde{G})$ is representable by a Deligne–Mumford stack $\mathcal{M}_{K_{\tilde{G}}}^{\text{naive}}(\tilde{G})$ over $\text{Spec } O_{E,(p)}$, and*

$$\mathcal{M}_{K_{\tilde{G}}}^{\text{naive}}(\tilde{G}) \times_{\text{Spec } O_{E,(p)}} \text{Spec } E \cong M_{K_{\tilde{G}}}(\tilde{G}).$$

Furthermore,

- (i) *If $M_{K_{\tilde{G}}}(\tilde{G})$ is proper over $\text{Spec } E$, then $\mathcal{M}_{K_{\tilde{G}}}^{\text{naive}}(\tilde{G})$ is proper over $\text{Spec } O_{E,(p)}$.*
- (ii) *If p is unramified in F and the vertex lattice Λ_v is self-dual for all $v \in \mathcal{V}_p$, then $\mathcal{M}_{K_{\tilde{G}}}^{\text{naive}}(\tilde{G})$ is smooth over $\text{Spec } O_{E,(p)}$.*

Proof. This is the extension of [31, Th. 4.1] to the case of arbitrary signature type. The key point is the statement which compares $M_{K_{\tilde{G}}}(\tilde{G})$ with the generic fiber of $\mathcal{M}_{K_{\tilde{G}}}^{\text{naive}}(\tilde{G})$. Using the moduli interpretation $\mathcal{F}'_{K_{\tilde{G}}}(\tilde{G})$ of $M_{K_{\tilde{G}}}(\tilde{G})$ of Definition 3.8, this comes down to completing the prime-to- p level structure $\bar{\eta}^p$ as in (4.4) to a level structure $\bar{\eta}$ as in (3.10), for a point $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \bar{\eta}^p)$ of $\mathcal{M}_{K_{\tilde{G}}}^{\text{naive}}(\tilde{G})$ over an E -scheme. Indeed, by the sign condition (4.7), there exists an isomorphism between $V_v(A_0, A) = \widehat{V}(A_0, A)_v$ and V_v for all $v \in \mathcal{V}_p$. Now, recall that prior to defining $\mathcal{F}_{K_{\tilde{G}}}^{\text{naive}}(\tilde{G})$, we fixed a lattice $\Lambda_0 \subset W_0$ such that $\Lambda_0^\vee = \mathfrak{a}^{-1}\Lambda_0$. The $K_{G,v}$ -equivalence class $\bar{\eta}_v$ of the isomorphism $V_v(A_0, A) \simeq V_v$ is then singled out by stipulating that it takes the lattice $\text{Hom}_{O_{F,v}}(T_v(A_0), T_v(A))$ in $V_v(A_0, A)$ to the lattice $L_v := \text{Hom}_{O_{F,v}}(\Lambda_{0,v}, \Lambda_v)$ in V_v , where $\Lambda_{0,v} := \Lambda \otimes_{O_F} O_{F,v}$; note that $K_{G,v}$ is the

stabilizer of L_v with respect to the isomorphism $U(W)(F_{0,v}) \cong U(V)(F_{0,v})$. We remark that in loc. cit., this part of the argument is carried out when $p = 2$ under the assumption that all $v \in \mathcal{V}_2$ are split in F . The argument extends to the case that all $v \in \mathcal{V}_2$ are unramified in F by [12, Th. 7.1], which says that all vertex lattices of the same type are conjugate under the unitary group in any hermitian space attached to an unramified extension of local fields of characteristic not 2. We also remark that the smoothness assertion in (ii) follows as in the proof of [31, Th. 4.1], using Theorem A.1 in the appendix below to extend the formalism of local models to the case $p = 2$ under the assumption that all $v \in \mathcal{V}_2$ are unramified in F .

The properness assertion in (i) follows as in [14, end of §5]. □

Remark 4.5. The lattice stabilizer groups $K_{G,p}$ appearing as p -factors of the level subgroups in Theorem 4.4 are all maximal. It is possible to extend the definitions above to general *lattice multichain* stabilizer groups at p (the v -factors of which, for each $v \in \mathcal{V}_p$, continue to contain a parahoric subgroup with index 1 or 2, as after (4.3)), by replacing the entry A in $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \overline{\eta}^p)$ by a *multichain* of abelian varieties, cf. [32, Def. 6.9] (in the context of general PEL moduli problems) or [28, §1.4] (in the context of unitary moduli problems with $F_0 = \mathbb{Q}$).

4.2. Semi-global GGP and KR cycles

In this section we give semi-global versions of the GGP and KR cycles of Section 3.5.

Let us again start with the GGP cycles. As before, we take $n \geq 2$, we fix a vector $u \in W$ of totally positive norm, we set $W^b = (u)^\perp$, and we consider the resulting Shimura datum $(\tilde{H}, \{h_{\tilde{H}}\})$ for W^b . We may then define a semi-global integral model $\mathcal{M}_{K_{\tilde{H}}}^{\text{naive}}(\tilde{H})$ over $\text{Spec } O_{E,(p)}$ as in Section 4.1. The definition of this stack depends on the choice of a vertex lattice $\Lambda_v^b \subset W_v^b$ for each $v \in \mathcal{V}_p$; to define a semi-global version $\mathcal{M}_{K_{\tilde{H}}}^{\text{naive}}(\tilde{H}) \rightarrow \mathcal{M}_{K_{\tilde{G}}}^{\text{naive}}(\tilde{G})$ of the morphism (3.29), these lattices and the lattices $\Lambda_v \subset W_v$ in the definition of $\mathcal{M}_{K_{\tilde{G}}}^{\text{naive}}(\tilde{G})$ need to be suitably related. As in Section 3.5, we set $W_0 = (u)$, $V = \text{Hom}_F(W_0, W)$, and $V^b = \text{Hom}_F(W_0, W^b)$. By Remark 3.7(i)(ii), the stack $\mathcal{M}_0^{\mathfrak{a}}$ is non-empty for \mathfrak{a} equal to O_{F_0} or to an inert prime ideal; we fix such an \mathfrak{a} , and we further fix $\sqrt{\Delta}$, Λ_0 , and ξ as before (3.29). Then the localization $\Lambda_{0,v} = \Lambda_0 \otimes_{O_F} O_{F,v}$ is a vertex lattice in $W_{0,v}$ for every place

$v \in \mathcal{V}_p$. For simplicity, let us now assume that Λ_v and Λ_v^b satisfy the relation

$$(4.8) \quad \Lambda_v = \Lambda_v^b \oplus \Lambda_{0,v} \subset W_v = W_v^b \oplus W_{0,v}$$

for each $v \in \mathcal{V}_p$. We further assume that the prime-to- p level subgroups satisfy $K_H^p \subset K_G^p$. Then the morphism (3.29) extends to a morphism of p -integral models,

$$(4.9) \quad \begin{aligned} \mathcal{M}_{K_{\tilde{H}}}^{\text{naive}}(\tilde{H}) &\longrightarrow \mathcal{M}_{K_{\tilde{G}}}^{\text{naive}}(\tilde{G}) \\ (A_0, \iota_0, \lambda_0, A^b, \iota^b, \lambda^b, \bar{\eta}^{p,b}) &\longmapsto (A_0, \iota_0, \lambda_0, A^b \times A_0, \iota^b \times \iota_0, \lambda^b \times \lambda_0, \bar{\eta}^p). \end{aligned}$$

Here the K_G^p -orbit $\bar{\eta}^p$ is defined in terms of the obvious prime-to- p analog of (3.27), and it is easy to see that the sign condition on $\mathcal{M}_{K_{\tilde{H}}}^{\text{naive}}(\tilde{H})$ implies that the formula in (4.9) indeed produces points satisfying the sign condition on $\mathcal{M}_{K_{\tilde{G}}}^{\text{naive}}(\tilde{G})$. The p -integral GGP cycle for the given levels is the graph of (4.9),

$$(4.10) \quad \mathcal{M}_{K_{\tilde{H}}}^{\text{naive}}(\tilde{H}) \hookrightarrow \mathcal{M}_{K_{\tilde{H}}}^{\text{naive}}(\tilde{H}) \times_{\mathcal{M}_0^{\mathfrak{a},\mathfrak{e}}} \mathcal{M}_{K_{\tilde{G}}}^{\text{naive}}(\tilde{G}).$$

Remark 4.6. Our assumption that the lattices in question satisfy the relation (4.8) for all $v \in \mathcal{V}_p$ is, in certain cases, a serious one. For example, if n is even and there is a place $v \in \mathcal{V}_p$ which ramifies in F , then it is impossible to choose Λ_v and Λ_v^b in this way such that $\mathcal{M}_{K_{\tilde{G}}}^{\text{naive}}(\tilde{G})$ has good reduction (at least outside of zero dimensional cases). We refer to [31, §4.4] for more general definitions of GGP cycles in this and further such contexts.

Now let us define the semi-global KR cycles. In fact, we will give two versions of the definition. The first is based directly on the moduli problem $\mathcal{F}_{K_{\tilde{G}}}^{\text{naive}}(\tilde{G})$, for any choice of defining data in Definition 4.1. Fix a global O_F -lattice $\Lambda \subset W$ whose localization $\Lambda \otimes_{O_F} O_{F,v}$, for each $v \in \mathcal{V}_p$, equals the vertex lattice $\Lambda_v \subset W_v$ we fixed before Definition 4.1. Set $L := \text{Hom}_{O_F}(\Lambda_0, \Lambda) \subset V$. Let m be a positive integer, and let $T \in \text{Herm}_m(F)$ be a hermitian matrix which is positive semidefinite at all archimedean places. Then the p -integral KR cycle $\mathcal{Z}^{\text{naive}}(T, L)$ is the stack of tuples $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \bar{\eta}^p; \mathbf{x})$, where $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \bar{\eta}^p)$ is an object in $\mathcal{F}_{K_{\tilde{G}}}^{\text{naive}}(\tilde{G})$ and $\mathbf{x} = (x_1, \dots, x_m) \in \text{Hom}_{(p), O_{F,(p)}}(A_0, A)^m$ is an m -tuple of $O_{F,(p)}$ -linear quasi-homomorphisms such that $(h'(x_i, x_j)) = T$ and such that each x_i identifies with an element of

$L \otimes_{O_F} \widehat{O}_F^p$ under the composite

$$\mathrm{Hom}_{(p), O_{F,(p)}}(A_0, A) \hookrightarrow \widehat{V}^p(A_0, A) \xrightarrow{\eta^p} V \otimes_F \mathbb{A}_{F,f}^p,$$

for each $\eta^p \in \overline{\eta}^p$. Here the hermitian form h' on $\mathrm{Hom}_{(p), O_{F,(p)}}(A_0, A)$ is defined as in (3.30), and $\widehat{O}_F^p := (O_F[\frac{1}{p}])^\wedge \subset \mathbb{A}_{F,f}^p$. The proof of Theorem 4.4 extends to show that the generic fiber of $\mathcal{Z}^{\mathrm{naive}}(T, L)$ is canonically isomorphic to the KR cycle $Z'(T, L)$ defined in Section 3.5.

To give the second version of the p -integral KR cycle, we first need to introduce a p -integral version of Definition 3.9. Keep all the notation of the previous paragraph, and assume that $L \otimes_{O_F} \widehat{O}_F^p$ is K_G^p -stable inside $V \otimes_F \mathbb{A}_{F,f}^p$ (which is equivalent to $\Lambda \otimes_{O_F} \widehat{O}_F^p$ being K_G^p -stable inside $W \otimes_F \mathbb{A}_{F,f}^p$). For N a positive integer prime to p , define $K^{p,L,N} \subset G(\mathbb{A}_{F,f}^p)$ as the obvious prime-to- p analog of $K^{L,N}$ in (3.20). Then $K^{L,N} = K^{p,L,N} \times K_{G,p}$. Choose N such that $K^{p,L,N} \subset K_G^p$. We define the following moduli problem.

Definition 4.7. The category functor $\mathcal{F}_{K_{\widetilde{G}}}^{\mathrm{naive}, L, N}(\widetilde{G})$ associates to each scheme S in $(\mathrm{LNSch})_{/O_{E,(p)}}$ the groupoid of tuples $(A_0, \iota_0, \lambda_0, B, \iota, \lambda, \overline{\eta}_N^p)$, where

- $(A_0, \iota_0, \lambda_0)$ is an object of $\mathcal{M}_0^{a,\xi}(S)$;
- B is an abelian scheme over S ;
- $\iota: O_F \rightarrow \mathrm{End}(B)$ is an action of O_F on B satisfying the Kottwitz condition (2.7) on O_F ;
- $\lambda \in \mathrm{Hom}_{(p)}(B, B^\vee)$ is a quasi-polarization on B whose Rosati involution satisfies condition (2.6) on O_F ; and
- $\overline{\eta}_N^p$ is a closed étale subscheme

$$\overline{\eta}_N^p \subset \underline{\mathrm{Isom}}_{O_F}(\underline{\mathrm{Hom}}_{O_F}(A_0[N], B[N]), (L/NL)_S)$$

over S such that for every geometric point $\overline{s} \rightarrow S$ (or equivalently, for a single geometric point on each connected component of S), the fiber $\overline{\eta}_N^p(\overline{s})$ identifies with a $K_G^p/K^{p,L,N}$ -orbit of isomorphisms

$$\eta_N^p(\overline{s}): \mathrm{Hom}_{O_F}(A_0[N](\overline{s}), B[N](\overline{s})) \xrightarrow{\sim} L/NL$$

which lift to \widehat{O}_F^p -linear isometries of hermitian modules

$$\widehat{T}^p(A_0, B)(\overline{s}) \xrightarrow{\sim} L \otimes_{O_F} \widehat{O}_F^p.$$

Here

$$(4.11) \quad \widehat{T}^p(A_0, B) := \text{Hom}_{\widehat{O}_F}(\widehat{T}^p(A_0), \widehat{T}^p(B))$$

is the obvious prime-to- p analog of (3.22).

We require that the tuples $(A_0, \iota_0, \lambda_0, B, \iota, \lambda, \overline{\eta}_N^p)$ satisfy conditions (i) and (ii) from Definition 4.1 (with B in place of A). A morphism

$$(A_0, \iota_0, \lambda_0, B, \iota, \lambda, \overline{\eta}_N^p) \longrightarrow (A'_0, \iota'_0, \lambda'_0, B', \iota', \lambda', \overline{\eta}'_N)$$

in this groupoid is given by an isomorphism $\mu_0: (A_0, \iota_0, \lambda_0) \xrightarrow{\sim} (A'_0, \iota'_0, \lambda'_0)$ in $\mathcal{M}_0^{\alpha, \xi}(S)$ and an O_F -linear isomorphism of abelian schemes $\mu: B \xrightarrow{\sim} B'$ pulling λ' back to λ and $\overline{\eta}'_N$ back to $\overline{\eta}_N^p$.

The obvious prime-to- p analog of the morphism (3.23) defines a natural equivalence of moduli problems

$$\mathcal{F}_{K_{\widetilde{G}}}^{\text{naive}, L, N}(\widetilde{G}) \xrightarrow{\sim} \mathcal{F}_{K_{\widetilde{G}}}^{\text{naive}}(\widetilde{G}).$$

Hence we obtain another moduli interpretation of the stack $\mathcal{M}_{K_{\widetilde{G}}}^{\text{naive}}(\widetilde{G})$.

In terms of the moduli functor $\mathcal{F}_{K_{\widetilde{G}}}^{\text{naive}, L, N}(\widetilde{G})$, we now define the p -integral KR cycle $\mathcal{Z}^{L, N}(T)$ word-for-word as in the case of $Z^{L, N}(T)$ in Section 3.5, simply replacing $\mathcal{F}_{K_{\widetilde{G}}}^{L, N}(\widetilde{G})$ everywhere by $\mathcal{F}_{K_{\widetilde{G}}}^{\text{naive}, L, N}(\widetilde{G})$. Then $\mathcal{Z}^{L, N}(T)$ is canonically equivalent to $\mathcal{Z}(T, L)$, and its generic fiber canonically identifies with $Z^{L, N}(T)$.

4.3. Summary table

The following table summarizes some properties of the various unitary Shimura varieties we have introduced above. In the last column, by “cycle property” we mean whether there exists a KR cycle in the Shimura variety and a GGP cycle in an appropriate product of the Shimura varieties, in analogy with the discussion in Section 3.5. In the last row, by “BHKRY” we mean the special case of RSZ Shimura varieties where $F_0 = \mathbb{Q}$, and of (strict fake) Drinfeld type, cf. [2]. (In loc. cit. only the case of principal polarization is considered.) In this case, $F = K_0$ is an imaginary quadratic field. The term “no sign necessary” refers to the case when $\mathcal{F}_{K_{\widetilde{G}}}^{\text{naive}}(\widetilde{G})$ is topologically flat over $O_{K_0, (p)}$, cf. Remark 4.2.

Name	Shimura datum	Reflex field	Moduli problem	p -integral moduli problem	Cycle property
D/K (§2.2)	$(G^{\mathbb{Q}}, \{h_{G^{\mathbb{Q}}}\})$	E_r	<i>yes</i> for n even, <i>almost</i> for n odd	<i>yes</i> if p totally unramified	<i>no</i>
GGP (§2.3)	$(G, \{h_G\})$	$E_{r, \natural}$	<i>no</i>	<i>no</i>	<i>yes</i>
RSZ (§3.2)	$(\tilde{G}, \{h_{\tilde{G}}\})$	$E = E_{\Phi} E_r = E_{\Phi} E_{r, \natural}$	<i>yes</i>	<i>yes</i> for $K_{G,p}$ a lattice multichain stabilizer	<i>yes</i>
HT (Ex. 2.3(iii))	$(G^{\mathbb{Q}}, \{h_{G^{\mathbb{Q}}}^{\text{HT}}\})$	Usually $F = K_0 F_0$; see Ex. 2.3(iii)	as in D/K	as in D/K, all levels if p split in K_0	as in D/K
BHKRY	as in RSZ	K_0	as in RSZ	as in RSZ, but no sign necessary	as in RSZ

5. Flat and smooth p -integral models of RSZ Shimura varieties

The p -integral model $\mathcal{M}_{K_{\tilde{G}}}^{\text{naive}}(\tilde{G})$ of $M_{K_{\tilde{G}}}(\tilde{G})$ defined in Section 4 is not always flat over $\text{Spec } O_{E,(p)}$. In this section, we first give some cases where it is known to be flat. We then give some cases where, upon imposing further conditions on the Lie algebra of the abelian variety (A, ι, λ) in the moduli problem, we obtain a closed substack $\mathcal{M}_{K_{\tilde{G}}}(\tilde{G})$ of $\mathcal{M}_{K_{\tilde{G}}}^{\text{naive}}(\tilde{G})$ which is flat, with the same generic fiber. Finally, we give some cases, beyond the totally unramified case appearing in Theorem 4.4(ii), where $\mathcal{M}_{K_{\tilde{G}}}(\tilde{G})$ is even regular or smooth. We continue to assume that every $v \in \mathcal{V}_p$ is unramified in F if $p = 2$, and we again take $K_{\tilde{G}}$ as in Section 4.1, i.e., of the form $K_{\mathbb{Z}^{\mathbb{Q}}}^{\circ} \times K_G^p \times K_{G,p}$, with K_G^p arbitrary and $K_{G,p}$ a product of stabilizers of vertex lattices Λ_v for $v \in \mathcal{V}_p$.

Let us note at the outset that when using terminology relating a lattice to its dual in this section, we will always mean the dual with respect to the \mathbb{Q}_p -valued form, e.g., Λ_v being self-dual means that $\Lambda_v = \Lambda_v^{\vee}$. Strictly speaking, this usage differs from all the papers on unitary local models we will refer to (where the dual is with respect to the hermitian form). But since Λ_v^{\vee} and Λ_v^* are scalar multiples of each other, the periodic lattice chains generated by $\{\Lambda_v, \Lambda_v^{\vee}\}$ and $\{\Lambda_v, \Lambda_v^*\}$ are the same, and there is ultimately no essential difference.

5.1. Flatness of $\mathcal{M}_{K_{\tilde{G}}}^{\text{naive}}(\tilde{G})$

The following result gives some cases where $\mathcal{M}_{K_{\tilde{G}}}^{\text{naive}}(\tilde{G})$ is known to be flat.

Theorem 5.1. *Suppose that p is unramified in F (without any condition on the vertex lattices Λ_v), or that the following three conditions hold:*

- (1) *each place $v \in \mathcal{V}_p$ which is unramified in F has ramification index $e \leq 2$ over p ,*⁸
- (2) *each place $v \in \mathcal{V}_p$ which ramifies in F is unramified over p , and the lattice Λ_v for such v is self-dual;*
- (3) *the integers r_{φ} for varying $\varphi \in \text{Hom}_{\mathbb{Q}}(F, \overline{\mathbb{Q}})$ differ by at most one.*

Then $\mathcal{M}_{K_{\tilde{G}}}^{\text{naive}}(\tilde{G})$ is flat over $\text{Spec } O_{E,(p)}$.

⁸This hypothesis can now be removed; see Remark 5.3(i) and footnote 10 below.

In fact, Theorem 5.1 is a consequence of the following more precise statement, which however requires some more notation to set up. Let ν be a place of E over p , and choose an embedding $\alpha: \mathbb{Q} \rightarrow \overline{\mathbb{Q}}_p$ inducing ν . Then α induces an identification

$$(5.1) \quad \begin{array}{ccc} \alpha_*: \operatorname{Hom}(F, \overline{\mathbb{Q}}) & \xrightarrow{\sim} & \operatorname{Hom}(F, \overline{\mathbb{Q}}_p) \\ \varphi & \longmapsto & \alpha \circ \varphi. \end{array}$$

For each p -adic place w of F , let

$$(5.2) \quad \operatorname{Hom}_w(F, \overline{\mathbb{Q}}) := \{ \varphi \in \operatorname{Hom}(F, \overline{\mathbb{Q}}) \mid \alpha \circ \varphi \text{ induces } w \}.$$

Then, under the identification α_* , the sets $\operatorname{Hom}_w(F, \overline{\mathbb{Q}})$ are the $\operatorname{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ -orbits in $\operatorname{Hom}(F, \overline{\mathbb{Q}})$, and hence are independent of the choice of α inducing ν . For each w , let F_w^t denote the maximal unramified extension of \mathbb{Q}_p in F_w . For each $\psi \in \operatorname{Hom}_{\mathbb{Q}_p}(F_w^t, \overline{\mathbb{Q}}_p)$, let $\operatorname{Hom}_{w,\psi}(F, \overline{\mathbb{Q}}) \subset \operatorname{Hom}_w(F, \overline{\mathbb{Q}})$ denote the fiber over ψ of the composite

$$(5.3) \quad \operatorname{Hom}_w(F, \overline{\mathbb{Q}}) \xrightarrow[\sim]{\alpha_*} \operatorname{Hom}_{\mathbb{Q}_p}(F_w, \overline{\mathbb{Q}}_p) \xrightarrow{\text{restrict}} \operatorname{Hom}_{\mathbb{Q}_p}(F_w^t, \overline{\mathbb{Q}}_p).$$

Then, under the identification α_* , the sets $\operatorname{Hom}_{w,\psi}(F, \overline{\mathbb{Q}})$ are the I_p -orbits in $\operatorname{Hom}(F, \overline{\mathbb{Q}})$, where $I_p \subset \operatorname{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ denotes the inertia subgroup. The label ψ in $\operatorname{Hom}_{w,\psi}(F, \overline{\mathbb{Q}})$ therefore generally depends on the choice of α inducing ν , but the partition

$$(5.4) \quad \operatorname{Hom}(F, \overline{\mathbb{Q}}) = \coprod_{\substack{w|p \\ \psi: F_w^t \rightarrow \overline{\mathbb{Q}}_p}} \operatorname{Hom}_{w,\psi}(F, \overline{\mathbb{Q}})$$

depends only on ν up to labeling. We now have the following result on flatness of the base change $\mathcal{M}_{K_G}^{\text{naive}}(\tilde{G})_{O_{E,(\nu)}}$.

Theorem 5.2. *Suppose that the following three conditions hold.*

(1) *For each $v \in \mathcal{V}_p$ which is unramified in F , the ramification index e of v over p satisfies $e \leq 2$ or, for each of the one or two places w of F over v and each $\psi \in \operatorname{Hom}_{\mathbb{Q}_p}(F_w^t, \overline{\mathbb{Q}}_p)$,*⁹

$$e \geq \min \left\{ \sum_{\varphi \in \operatorname{Hom}_{w,\psi}(F, \overline{\mathbb{Q}})} r_\varphi, \sum_{\varphi \in \operatorname{Hom}_{w,\psi}(F, \overline{\mathbb{Q}})} r_{\overline{\varphi}} \right\}.$$

⁹This hypothesis can now be removed; see Remark 5.3(i) and footnote 10 below.

(2) For each place $v \in \mathcal{V}_p$ which ramifies in F , v is unramified over p and the lattice Λ_v is self-dual.

(3) For each place w of F over p and each $\psi \in \text{Hom}_{\mathbb{Q}_p}(F_w^t, \overline{\mathbb{Q}}_p)$, the integers r_φ for varying $\varphi \in \text{Hom}_{w,\psi}(F, \overline{\mathbb{Q}})$ differ by at most one.

Then $\mathcal{M}_{K\tilde{G}}^{\text{naive}}(\tilde{G})_{O_{E,(\nu)}}$ is flat over $\text{Spec } O_{E,(\nu)}$.

Proof. After extending scalars to the ν -adic completion $O_{E,(\nu)} \rightarrow O_{E,\nu}$, this follows from the local model diagram over $O_{E,\nu}$,

$$(5.5) \quad \begin{array}{ccc} & \mathcal{M}_{K\tilde{G}}^{\text{naive}}(\tilde{G})_{O_{E,\nu}} & \\ \swarrow \pi & & \searrow \tilde{\varphi} \\ \mathcal{M}_{K\tilde{G}}^{\text{naive}}(\tilde{G})_{O_{E,\nu}} & & (\mathcal{M}_0^{\mathfrak{a}})_{O_{E,\nu}} \times_{O_{E,\nu}} \mathbb{M}^{\text{naive}}. \end{array}$$

Let us briefly remark on the notation; see [32, Ch. 6] or [26, §15] for more details. Let

$$\Lambda_p := \bigoplus_{v \in \mathcal{V}_p} \Lambda_v \subset W \otimes_{\mathbb{Q}} \mathbb{Q}_p = \bigoplus_{v \in \mathcal{V}_p} W_v.$$

Let \mathcal{L} be the self-dual periodic multichain of $O_{F,p}$ -lattices in $W \otimes_{\mathbb{Q}} \mathbb{Q}_p$ generated by Λ_p and its dual. A triple (A, ι, λ) as in the moduli problem for $\mathcal{M}_{K\tilde{G}}^{\text{naive}}(\tilde{G})$ then gives rise in a natural way to a polarized \mathcal{L} -set of abelian varieties $\{A_\Lambda\}_{\Lambda \in \mathcal{L}}$. For S in $(\text{LNSch})_{/O_{E,\nu}}$, $\tilde{\mathcal{M}}_{K\tilde{G}}^{\text{naive}}(\tilde{G})_{O_{E,\nu}}(S)$ is then the groupoid of objects $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \bar{\eta})$ in $\mathcal{M}_{K\tilde{G}}^{\text{naive}}(\tilde{G})_{O_{E,\nu}}(S)$ equipped with an isomorphism of polarized multichains $\{H_1^{\text{dR}}(A_\Lambda)\}_{\Lambda \in \mathcal{L}} \xrightarrow{\sim} \mathcal{L} \otimes_{\mathbb{Z}_p} \mathcal{O}_S$. The morphism π in (5.5) is the natural forgetful morphism; it is a torsor under $\mathcal{P}_{O_{E,\nu}}$, where \mathcal{P} is the automorphism scheme of \mathcal{L} (as a polarized multichain) over \mathbb{Z}_p , which is a smooth affine group scheme (in fact, a \mathbb{Z}_p -model of the unitary similitude group $G^{\mathbb{Q}}$ for W). The naive local model $\mathbb{M}^{\text{naive}}$ is a projective $O_{E,\nu}$ -scheme attached to the multichain \mathcal{L} and the Shimura datum $(G^{\mathbb{Q}}, \{h_{G^{\mathbb{Q}}}\})$. The group scheme $\mathcal{P}_{O_{E,\nu}}$ acts naturally on $\mathbb{M}^{\text{naive}}$, and the morphism $\tilde{\varphi}$ is $\mathcal{P}_{O_{E,\nu}}$ -equivariant and formally smooth of the same relative dimension as π .

The flatness of $\mathcal{M}_{K\tilde{G}}^{\text{naive}}(\tilde{G})_{O_{E,\nu}}$ now follows from the flatness of $\mathbb{M}^{\text{naive}}$ (and étaleness of $\mathcal{M}_0^{\mathfrak{a}}$). More precisely, by its definition, $\mathbb{M}^{\text{naive}}$ is a moduli space for certain $O_F \otimes_{\mathbb{Z}} \mathcal{O}_S$ -linear quotient bundles of $\Lambda_p \otimes_{\mathbb{Z}_p} \mathcal{O}_S$. The decomposition

$O_{F_0} \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \prod_{v \in \mathcal{V}_p} O_{F_0, v}$ then induces a natural decomposition

$$(5.6) \quad \mathbb{M}^{\text{naive}} \cong \prod_{v \in \mathcal{V}_p} \mathbb{M}(v)_{O_{E, \nu}}^{\text{naive}},$$

where each $\mathbb{M}(v)_{O_{E, \nu}}^{\text{naive}}$ is the base change to $O_{E, \nu}$ of a naive local model attached to the local $F_v/F_{0, v}$ -hermitian space W_v , the lattice Λ_v , and the function $r|_{\cup_w|_v \text{Hom}_w(F, \overline{\mathbb{Q}})}$. Thus flatness of $\mathbb{M}^{\text{naive}}$ follows from flatness of each $\mathbb{M}(v)_{O_{E, \nu}}^{\text{naive}}$. Now let $F_{0, v}^t$ denote the maximal unramified extension of \mathbb{Q}_p in $F_{0, v}$, and let $E(v)^{\text{un}} \subset \overline{\mathbb{Q}_p}$ denote the maximal unramified extension of the reflex field $E(v)$ of $\mathbb{M}(v)_{O_{E, \nu}}^{\text{naive}}$. Consider the decomposition

$$O_{F_{0, v}^t} \otimes_{\mathbb{Z}_p} O_{E(v)^{\text{un}}} \cong \prod_{\psi_0 \in \text{Hom}_{\mathbb{Q}_p}(F_{0, v}^t, \overline{\mathbb{Q}_p})} O_{E(v)^{\text{un}}}.$$

After extending scalars to $O_{E(v)^{\text{un}}}$, the action of $O_{F_{0, v}^t} \otimes_{\mathbb{Z}_p} O_{E(v)^{\text{un}}}$ on the lattice $\Lambda_v \otimes_{\mathbb{Z}_p} O_{E(v)^{\text{un}}}$ induces a natural decomposition

$$(5.7) \quad \mathbb{M}(v)_{O_{E(v)^{\text{un}}}}^{\text{naive}} \cong \prod_{\psi_0} \mathbb{M}(v, \psi_0)_{O_{E(v)^{\text{un}}}}^{\text{naive}},$$

where each $\mathbb{M}(v, \psi_0)_{O_{E(v)^{\text{un}}}}^{\text{naive}}$ is the base change to $O_{E(v)^{\text{un}}}$ of a naive local model attached to the tower $F_v/F_{0, v}/F_{0, v}^t$. Thus the problem of flatness of $\mathbb{M}^{\text{naive}}$ reduces to flatness of each $\mathbb{M}(v, \psi_0)_{O_{E(v)^{\text{un}}}}^{\text{naive}}$.

When v is unramified in F , for each ψ_0 , there is a further decomposition

$$O_{F_v^t} \otimes_{O_{F_{0, v}^t}, \psi_0} O_{E(v)^{\text{un}}} \cong \prod_{\psi} O_{E(v)^{\text{un}}},$$

where the product is over the two homomorphisms $\psi: F_v^t \rightarrow \overline{\mathbb{Q}_p}$ extending ψ_0 . Picking one of these ψ , this decomposition induces an identification of $\mathbb{M}(v, \psi_0)_{O_{E(v)^{\text{un}}}}^{\text{naive}}$ with the base change to $O_{E(v)^{\text{un}}}$ of a naive local model \mathbb{M}' attached to the totally ramified extension F_w/F_w^t , the group $\text{Res}_{F_w/F_w^t} \text{GL}_n$, and the function $r|_{\text{Hom}_{w, \psi}(F, \overline{\mathbb{Q}})}$; here w is the place of F over v determined by ψ (of course there is only ambiguity in w when v splits in F), and $\text{Hom}_{w, \psi}(F, \overline{\mathbb{Q}})$ identifies with the embeddings of F_w into $\overline{\mathbb{Q}_p}$ extending ψ as in (5.3) above. (Replacing the choice of ψ by $\overline{\psi}$ results in an isomorphic naive local model for $\text{Res}_{F_w/F_w^t} \text{GL}_n$.) When Λ_v is self-dual, \mathbb{M}' is the local model defined in [25] in the case of a single lattice; by the hypotheses in (1) and (3), \mathbb{M}' is flat by Th. B and the following paragraph in loc. cit. (which we note relies, in turn,

on a result of Weyman [36]). For general Λ_v (still with v unramified in F), \mathbb{M}' is a naive local model for $\text{Res}_{F_w/F_w^t} \text{GL}_n$ and $r|_{\text{Hom}_{w,\psi}(F,\overline{\mathbb{Q}})}$ in the case of a periodic lattice chain \mathcal{L}_ψ generated by one or two lattices. By Görtz [8, §1 Th.], flatness of \mathbb{M}' in this case (or more generally, in the case of an arbitrary periodic lattice chain \mathcal{L}_ψ) follows from flatness of the naive local model in the single lattice case. This concludes the proof when v is unramified in F .

When v ramifies in F (subject to the hypotheses in (2) and (3)), flatness of each $\mathbb{M}(v, \psi_0)^{\text{naive}}$ is proved in [35]. □

Remark 5.3. (i) It is conjectured just after Th. B in [25] that when v is unramified in F and Λ_v is self-dual, the naive local model $\mathbb{M}(v, \psi_0)^{\text{naive}}$ appearing in the proof of Theorem 5.2 is flat with reduced special fiber, without any assumption on the ramification of v over p .¹⁰ This would imply (again using Görtz [8] to pass to the case of general Λ_v) that hypothesis (1) in both of Theorems 5.1 and 5.2 can be removed.

(ii) We do not know if the conclusion of Theorem 5.2 remains valid if one allows the places v which ramify in F to have any ramification over p . However, the assumption for such v that Λ_v is self-dual is necessary if $n > 1$.

(iii) Condition (3) in Theorem 5.2 is necessary for the conclusion to hold, cf. [25, Cor. 3.3].

5.2. Flat subscheme of $\mathcal{M}_{K_{\tilde{G}}}^{\text{naive}}(\tilde{G})$

Even if $\mathcal{M}_{K_{\tilde{G}}}^{\text{naive}}(\tilde{G})$ is not flat over $\text{Spec } O_{E,(p)}$, we sometimes can strengthen the conditions on the abelian varieties (A, ι, λ) occurring in the moduli problem to define a closed substack $\mathcal{M}_{K_{\tilde{G}}}(\tilde{G})$ of $\mathcal{M}_{K_{\tilde{G}}}^{\text{naive}}(\tilde{G})$ which is flat with the same generic fiber. For simplicity, let us consider this question after base change to the completed local ring $O_{E,\nu}$ for ν a p -adic place of E . Then, similarly to the proof of Theorem 5.2, the O_F -action on the abelian variety A induces an action of $O_F \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \prod_w O_{F,w}$ on $\text{Lie } A$, and hence a canonical decomposition

$$(5.8) \quad \text{Lie } A = \bigoplus_w \text{Lie}_w A,$$

¹⁰ Since the preprint version of this paper appeared, Muthiah–Weekes–Yacobi have posted a proof [23] of this conjecture in full generality, by proving [25, Conj. 5.8].

where w runs through the p -adic places of F . For each w , using the notation of Section 5.1, the $O_{F_w^t}$ -action on $\text{Lie}_w A$ similarly induces a decomposition

$$(5.9) \quad \text{Lie}_w A = \bigoplus_{\psi \in \text{Hom}_{\mathbb{Q}_p}(F_w^t, \overline{\mathbb{Q}}_p)} \text{Lie}_{w,\psi} A;$$

here in fact we choose an embedding $\alpha: \overline{\mathbb{Q}} \rightarrow \overline{\mathbb{Q}}_p$ inducing ν as just before (5.1), and the decomposition (5.9) is defined after base changing the moduli problem to $\text{Spec } O_{E_\nu^{\text{un}}}$, where E_ν^{un} denotes the maximal unramified extension of E_ν (embedded via α) in $\overline{\mathbb{Q}}_p$.

We now consider conditions on $\text{Lie}_{w,\psi} A$ under various assumptions on w and ψ , as follows. In some cases the conditions are quite technical to formulate and we only give references. We again write $\text{Hom}_{w,\psi}(F, \overline{\mathbb{Q}}) \subset \text{Hom}(F, \overline{\mathbb{Q}})$ for the fiber over ψ in the diagram (5.3). Furthermore, we note that cases (3)–(5) below are disallowed when $p = 2$ by our standing assumption that the places $v \in \mathcal{V}_2$ are unramified in F .

(1) Suppose that the restricted function $r|_{\text{Hom}_{w,\psi}(F, \overline{\mathbb{Q}})}$ takes values in $\{0, n\}$ (a *banal* signature type at ψ). Then there is the *Eisenstein condition* on $\text{Lie}_{w,\psi} A$ of [31, (4.10)].¹¹ (Note that, in contrast to cases (2)–(5) below, here we make no ramification assumptions on w ; however, the Eisenstein condition at ψ is already implied by the Kottwitz condition (2.7) if w is unramified over p .)

(2) Suppose that w is unramified over F_0 and suppose that the restricted function $r|_{\text{Hom}_{w,\psi}(F, \overline{\mathbb{Q}}) \cup \text{Hom}_{\overline{w}, \overline{\psi}}(F, \overline{\mathbb{Q}})}$ is of the form

$$r_\varphi = \begin{cases} n - 1, & \text{for some } \varphi = \varphi_0 \in \text{Hom}_{w,\psi}(F, \overline{\mathbb{Q}}) \cup \text{Hom}_{\overline{w}, \overline{\psi}}(F, \overline{\mathbb{Q}}); \\ 1, & \varphi = \overline{\varphi}_0; \\ 0 \text{ or } n, & \varphi \neq \varphi_0, \overline{\varphi}_0. \end{cases}$$

Here $\overline{\psi}: F_w^t \rightarrow \overline{\mathbb{Q}}_p$ denotes the conjugate embedding.

If $\overline{\varphi}_0 \in \text{Hom}_{w,\psi}(F, \overline{\mathbb{Q}})$, then there is the *Eisenstein condition* on $\text{Lie}_{w,\psi} A$ of [33, (8.2)]. If $\overline{\varphi}_0 \in \text{Hom}_{\overline{w}, \overline{\psi}}(F, \overline{\mathbb{Q}})$, then there is the *Eisenstein condition* on $\text{Lie}_{\overline{w}, \overline{\psi}} A$ of [33, (8.2)]. (This condition is again already implied by the Kottwitz condition if w is unramified over p .)

¹¹Strictly speaking, here we mean that the expression $Q_{A_\psi}(\iota(\pi))$ defined in loc. cit. is the zero endomorphism on $\text{Lie}_{w,\psi} A$.

(3) Suppose that w is ramified over F_0 and the place v of F_0 under w is unramified over p ; or, equivalently, that $\text{Hom}_{w,\psi}(F, \overline{\mathbb{Q}})$ is of the form $\{\varphi_\psi, \overline{\varphi}_\psi\}$ for some $\varphi_\psi \in \Phi$. Then there is the *wedge condition* of Pappas [24] at ψ : if $r_{\varphi_\psi} \neq r_{\overline{\varphi}_\psi}$, then

$$(5.10) \quad \left. \begin{aligned} \bigwedge^{r_{\overline{\varphi}_\psi}+1} (\iota(a) - \varphi_\psi(a) \mid \text{Lie}_{w,\psi} A) = 0 \\ \bigwedge^{r_{\varphi_\psi}+1} (\iota(a) - \overline{\varphi}_\psi(a) \mid \text{Lie}_{w,\psi} A) = 0 \end{aligned} \right\} \text{ for all } a \in O_{F,w}.$$

Here, using that $r_{\varphi_\psi} \neq r_{\overline{\varphi}_\psi}$, it is easy to see that φ_ψ and $\overline{\varphi}_\psi$ map F_w into E_v^{un} , and the expressions $\varphi_\psi(a)$ and $\overline{\varphi}_\psi(a)$ are then viewed as sections of the structure sheaf of the base scheme, as in (2.7). (There is no condition at ψ when $r_{\varphi_\psi} = r_{\overline{\varphi}_\psi}$. There is an analogous condition when w is unramified over F_0 , but it is already implied by the Kottwitz condition.)

(4) In the same situation as in (3), suppose in addition that n is even. Then there is the *spin condition* of [28, §8.2] on $\text{Lie}_{w,\psi} A$.¹² We note that in the special case that Λ_v is π_v -modular (recall this means that $\Lambda_v^\vee = \pi_v^{-1}\Lambda_v$) and $\{r_{\varphi_\psi}, r_{\overline{\varphi}_\psi}\} = \{1, n-1\}$, and in the presence of the wedge condition at ψ , the spin condition at ψ admits the simple formulation that the endomorphism $\iota(\pi_v) \mid \text{Lie}_{w,\psi} A$ is nonvanishing at each point of the base, cf. [30, §6] or [31, (4.28)].

(5) In the same situation as in (3), suppose in addition that n is odd. Then there is the *refined spin condition* of [34, §2.5], translated to a condition on $\text{Lie}_{w,\psi} A$ as in [30, §7] and just before Rem. 4.6 in [31]. We note that these latter two references treat on the nose the special case that Λ_v is *almost* π_v -modular (i.e., $\Lambda_v \subset \Lambda_v^\vee \subset \pi_v^{-1}\Lambda_v$ with $\dim_{O_{F,v}/\pi_v O_{F,v}} \pi_v^{-1}\Lambda_v/\Lambda_v^\vee = 1$) and $\{r_{\varphi_\psi}, r_{\overline{\varphi}_\psi}\} = \{1, n-1\}$.

Theorem 5.4. *Suppose that for every p -adic place w of F and every embedding $\psi: F_w^t \rightarrow \overline{\mathbb{Q}}_p$, one of the following hypotheses is satisfied and, in each case, impose the given condition on the object (A, ι, λ) in the moduli problem for $\mathcal{M}_{K_G}^{\text{naive}}(\tilde{G})_{O_{E_v^{\text{un}}}}$. Throughout, let v denote the place of F_0 under w .*

(a) *w is unramified over F_0 and v satisfies the ramification hypothesis in Theorem 5.2(1); or w is ramified over F_0 , v is unramified over p , and the*

¹²Strictly speaking, loc. cit. only formulates the spin condition on the local model. We will not spell out the translation of the spin condition to $\text{Lie}_{w,\psi} A$ more explicitly; it is entirely analogous to the translation of the refined spin condition of [34, §2.5] to the Lie algebra of a p -divisible group given in [30, §7] and just before Rem. 4.6 in [31].

lattice Λ_v is self-dual. Furthermore, the r_φ 's for varying $\varphi \in \text{Hom}_{w,\psi}(F, \overline{\mathbb{Q}})$ differ by at most one. In this case, impose no further condition.

(b) w and ψ satisfy the assumption in (1) above. Then impose the Eisenstein condition of (1) on $\text{Lie}_{w,\psi} A$.

(c) w and ψ satisfy the assumptions in (2) above and the lattice Λ_v is self-dual or π_v -modular.¹³ In the notation of (2), if $\overline{\varphi}_0 \in \text{Hom}_{w,\psi}(F, \overline{\mathbb{Q}})$, then impose the Eisenstein condition of (2) on $\text{Lie}_{w,\psi} A$; if $\overline{\varphi}_0 \in \text{Hom}_{\overline{w},\overline{\psi}}(F, \overline{\mathbb{Q}})$, then impose the Eisenstein condition of (2) on $\text{Lie}_{\overline{w},\overline{\psi}} A$.

(d) w and ψ satisfy the assumptions in (3) above and the lattice Λ_v is self-dual. Then impose the wedge condition on $\text{Lie}_{w,\psi} A$.

(e) w and ψ satisfy the assumptions in (4) above, Λ_v is π_v -modular, and $\{r_{\varphi_\psi}, r_{\overline{\varphi}_\psi}\} = \{1, n - 1\}$. Then impose the wedge condition and the spin condition on $\text{Lie}_{w,\psi} A$.

(f) w and ψ satisfy the assumptions in (5) above, Λ_v is almost π_v -modular, and $\{r_{\varphi_\psi}, r_{\overline{\varphi}_\psi}\} = \{1, n - 1\}$. Then impose the refined spin condition on $\text{Lie}_{w,\psi} A$.

Then these conditions descend to define a closed substack $\mathcal{M}_{K_{\tilde{G}}}(G)_{O_{E,\nu}}$ of $\mathcal{M}_{K_{\tilde{G}}}^{\text{naive}}(\tilde{G})_{O_{E,\nu}}$ which is flat over $\text{Spec } O_{E,\nu}$ with the same generic fiber as $\mathcal{M}_{K_{\tilde{G}}}^{\text{naive}}(\tilde{G})_{O_{E,\nu}}$.

Proof. As in the proof of Theorem 5.2, the statements on flatness and the generic fiber reduce to statements on the local model $\mathbb{M}_{O_{E_v^{\text{un}}}} \subset \mathbb{M}_{O_{E_v^{\text{un}}}}^{\text{naive}}$ defined by the analogous conditions on $\mathbb{M}_{O_{E_v^{\text{un}}}}^{\text{naive}}$. As in (5.6) and (5.7), there is a product decomposition

$$(5.11) \quad \mathbb{M}_{O_{E_v^{\text{un}}}} \cong \prod_{\substack{v \in \mathcal{V}_p \\ \psi_0: F_{0,v}^t \rightarrow \overline{\mathbb{Q}}_p}} \mathbb{M}(v, \psi_0)_{O_{E_v^{\text{un}}}},$$

and we reduce to proving flatness factor-by-factor on the right-hand side. For each p -adic place w of F and embedding $\psi: F_w^t \rightarrow \overline{\mathbb{Q}}_p$, the place v of F_0 under w and the restriction $\psi|_{F_0^t}$ indexes one of the factors in (5.11), and all indices in the product arise in this way as w and ψ vary. Thus we consider cases based on the type of w and ψ . If w and ψ are as in (a), then the factor

¹³The definition of π_v -modular when v is unramified in F is word-for-word the same as when v ramifies in F , namely that $\Lambda_v^\vee = \pi_v^{-1}\Lambda_v$.

$\mathbb{M}(v, \psi|_{F_{0,v}^t})_{O_{E_{\nu}^{\text{un}}}} = \mathbb{M}(v, \psi|_{F_{0,v}^t})_{O_{E_{\nu}^{\text{un}}}}^{\text{naive}}$ is flat by the proof of Theorem 5.2. If w and ψ are as in (b), then the factor $\mathbb{M}(v, \psi|_{F_{0,v}^t})_{O_{E_{\nu}^{\text{un}}}}$ is flat (in fact, trivial) with the same generic fiber as $\mathbb{M}(v, \psi|_{F_{0,v}^t})_{O_{E_{\nu}^{\text{un}}}}^{\text{naive}}$ by [31, App. B]. If w and ψ are as in (c), then $\mathbb{M}(v, \psi|_{F_{0,v}^t})_{O_{E_{\nu}^{\text{un}}}}$ is flat (in fact, smooth) with the same generic fiber by [33, Lem. 8.6]. If w and ψ are as in (d), then $\mathbb{M}(v, \psi|_{F_{0,v}^t})_{O_{E_{\nu}^{\text{un}}}}$ is flat with the same generic fiber by [35] (when $\{r_{\varphi_{\psi}}, r_{\overline{\varphi}_{\psi}}\} = \{1, n-1\}$, this was proved by Pappas in [24]). If w and ψ are as in (e), then $\mathbb{M}(v, \psi|_{F_{0,v}^t})_{O_{E_{\nu}^{\text{un}}}}$ is flat (in fact, smooth) with the same generic fiber by [28, §5.3]. If w and ψ are as in (f), then $\mathbb{M}(v, \psi|_{F_{0,v}^t})_{O_{E_{\nu}^{\text{un}}}}$ is flat (in fact, smooth) with the same generic fiber by [34, Th. 1.4]. By assumption, every w and ψ is of one of these types, and this completes the proof of the statements on flatness and the generic fiber. The statement on descent is easy to verify. \square

Remark 5.5. The refined spin condition in (5) above is defined in [34] for n even as well as odd, and it is shown there to imply the wedge condition and the spin condition (without changing the generic fiber). Therefore one may treat (3)–(5) above in a uniform way by imposing the refined spin condition in each case; the advantage of the wedge condition and (to a lesser extent) the spin condition is only that they are simpler to state. It is conjectured in loc. cit. that the refined spin condition produces flatness for any signature type $\{r_{\varphi_{\psi}}, r_{\overline{\varphi}_{\psi}}\}$ and any lattice type (still with the place v unramified over p). When v ramifies in F and is *ramified* over p , nothing is known about characterizing the (flat) local model in terms of an explicit moduli problem.

Remark 5.6. Suppose that $v \in \mathcal{V}_p$ splits in F , say $v = w\overline{w}$, and suppose that the restricted function $r|_{\text{Hom}_w(F, \overline{\mathbb{Q}})}$ is of the form

$$(5.12) \quad r_{\varphi} = \begin{cases} n-1, & \varphi = \varphi_0 \text{ for some } \varphi_0 \in \text{Hom}_w(F, \overline{\mathbb{Q}}); \\ n, & \varphi \in \text{Hom}_w(F, \overline{\mathbb{Q}}) \setminus \{\varphi_0\}. \end{cases}$$

Then it is possible to impose a *Drinfeld level structure* at v in the moduli problem appearing in Theorem 5.4. More precisely, let m be a nonnegative integer, and define $K_{G,v}^m$ to be the principal congruence subgroup mod \mathfrak{p}_v^m inside $K_{G,v}$, where \mathfrak{p}_v denotes the prime ideal in O_{F_0} determined by v . Let

$$K_G^m := K_{Z^{\mathbb{Q}}}^{\circ} \times K_G^p \times K_{G,v}^m \times \prod_{v' \in \mathcal{V}_p \setminus \{v\}} K_{G,v'} \subset K_{\tilde{G}}.$$

Then one can extend the definition of $\mathcal{M}_{K_G^{\sim}(\tilde{G})_{O_{E,\nu}}}$ to the case of the level subgroup K_G^m by adding a Drinfeld level- m structure at v . Briefly, (5.12)

implies that in the decomposition (5.8) of $\text{Lie } A$, the summand $\text{Lie}_w A$ has rank $n[F_w : \mathbb{Q}_p] - 1$, and the summand $\text{Lie}_{\bar{w}} A$ has rank 1. The datum we add to the moduli problem is an $O_{F, \bar{w}}$ -linear homomorphism of finite flat group schemes,

$$\varphi: \pi_{\bar{w}}^{-m} \Lambda_{\bar{w}} / \Lambda_{\bar{w}} \longrightarrow \underline{\text{Hom}}_{O_{F, \bar{w}}} (A_0[\bar{w}^m], A[\bar{w}^m]),$$

which is a Drinfeld \bar{w}^m -structure on the target. Here $\Lambda_{\bar{w}}$ is the summand attached to \bar{w} in the natural decomposition $\Lambda_v = \Lambda_w \oplus \Lambda_{\bar{w}}$, with Λ_v the vertex lattice at v chosen prior to Definition 4.1. See [31, §4.3] (which we note interchanges the roles of w and \bar{w}) for more details.

5.3. Exotic smoothness and regularity

In some special cases the conditions introduced in Section 5.2 define integral models with good or semi-stable reduction, beyond the totally unramified situation in Theorem 4.4(ii). For simplicity, we will again consider these questions after base change to the completed local ring $O_{E, \nu}$ for ν a p -adic place of E . We again choose an embedding $\alpha: \bar{\mathbb{Q}} \rightarrow \bar{\mathbb{Q}}_p$ inducing ν and use the notation introduced before Theorem 5.2.

The following gives conditions when $\mathcal{M}_{K_G^\sim(\tilde{G})_{O_{E, \nu}}}$ is known to be smooth.

Theorem 5.7. *In the setting of Theorem 5.4, suppose that every pair (w, ψ) is of type (b), (c), (e), (f), or the following special case of type (a):*

(a') *w is unramified over p and the lattice Λ_v is self-dual or π_v -modular.*

Then the integral model $\mathcal{M}_{K_G^\sim(\tilde{G})_{O_{E, \nu}}}$ defined in Theorem 5.4 is smooth over $\text{Spec } O_{E, \nu}$.

Proof. Similarly to the proof of Theorem 5.4, this reduces to smoothness of the local model \mathbb{M} . For (w, ψ) of type (b), (c), (e), or (f), the factor $\mathbb{M}(v, \psi|_{F_{0, v}^t})_{O_{E, \nu}^{\text{un}}}$ in the decomposition (5.11) (again denoting by v the place of F_0 under w) is smooth by the references given in the proof of Theorem 5.4 (additionally using that smoothness of this factor in type (f) is due to Richarz [1, Prop. 4.16]). Smoothness of this factor in type (a') is standard (it is isomorphic to a Grassmannian for GL_n). \square

The fact that smoothness can occur in types (c), (e), and (f), when ramification is present, is a surprising phenomenon termed *exotic smoothness* in [29, 30].

The following theorem gives conditions when $\mathcal{M}_{K_G^\sim(\tilde{G})_{O_{E, \nu}}}$ is known to have semi-stable reduction, and hence to be regular.

Theorem 5.8. *Let $n \geq 2$. In the setting of Theorem 5.4, suppose that there is a p -adic place w_0 of F , with place v_0 of F_0 under it, of the following special case of type (a):*

(a'') w_0 is unramified over p , and the function $r|_{\text{Hom}_{w_0}(F, \overline{\mathbb{Q}}) \cup \text{Hom}_{\overline{w}_0}(F, \overline{\mathbb{Q}})}$ is of the form

(5.13)

$$r_\varphi = \begin{cases} \text{arbitrary,} & \text{for some } \varphi = \varphi_0, \overline{\varphi}_0 \in \text{Hom}_{w_0}(F, \overline{\mathbb{Q}}) \cup \text{Hom}_{\overline{w}_0}(F, \overline{\mathbb{Q}}); \\ 0 \text{ or } n, & \varphi \neq \varphi_0, \overline{\varphi}_0. \end{cases}$$

Furthermore, Λ_{v_0} is almost self-dual (i.e., $\Lambda_{v_0} \subset \Lambda_{v_0}^\vee \subset \pi_{v_0}^{-1} \Lambda_{v_0}$ with $\text{rank}_{\mathcal{O}_{F, v_0} / \pi_v \mathcal{O}_{F, v_0}} \Lambda_{v_0}^\vee / \Lambda_{v_0} = 1$) or almost π_{v_0} -modular (i.e., $\Lambda_{v_0} \subset \Lambda_{v_0}^\vee \subset \pi_{v_0}^{-1} \Lambda_{v_0}$ with $\text{rank}_{\mathcal{O}_{F, v_0} / \pi_v \mathcal{O}_{F, v_0}} \Lambda_{v_0}^\vee / \Lambda_{v_0} = n - 1$), or $\{r_{\varphi_0}, r_{\overline{\varphi}_0}\} = \{1, n - 1\}$.

In addition, suppose that every pair (w, ψ) as in Theorem 5.4 for which $w \neq w_0, \overline{w}_0$ is of type (b), (c), or type (a') in Theorem 5.7. Furthermore, suppose that E_ν is unramified over \mathbb{Q}_p . Then $\mathcal{M}_{K_G}(\tilde{G})_{O_{E, \nu}}$ has semi-stable reduction over $\text{Spec } O_{E, \nu}$.

Proof. As before, the proof is via the local model, using in particular the product decomposition (5.11). We first consider the factors in (5.11) corresponding to v_0 . Note that $F_{0, v_0} = F_{0, v_0}^t$ by hypothesis (a''). Regarding φ_0 as an embedding $F_{w_0} \rightarrow \overline{\mathbb{Q}}_p$ via α_* as in (5.3), consider the restriction $\varphi_0|_{F_{0, v_0}}$, and let $E_0 \subset \overline{\mathbb{Q}}_p$ denote the reflex field of the factor $\mathbb{M}(v_0, \varphi_0|_{F_{0, v_0}})$. By Görtz [7, §4.4.5], $\mathbb{M}(v_0, \varphi_0|_{F_{0, v_0}})$ has semi-stable reduction over O_{E_0} under the assumption that Λ_{v_0} is almost self-dual or or almost π_{v_0} -modular, or under the assumption that $\{r_{\varphi_0}, r_{\overline{\varphi}_0}\} = \{1, n - 1\}$. Since E_ν is unramified over \mathbb{Q}_p , $\mathbb{M}(\varphi_0|_{F_{0, v_0}})_{O_{E_\nu^{\text{un}}}}$ has semi-stable reduction over $O_{E_\nu^{\text{un}}}$. It follows from the assumption on r in (5.13) that the factors $\mathbb{M}(v_0, \psi_0)_{O_{E_\nu^{\text{un}}}}$ for $\psi_0 \neq \varphi_0|_{F_{0, v_0}}$ are isomorphic to $\text{Spec } O_{E_\nu^{\text{un}}}$ [31, App. B]. By Theorem 5.7, the factors $\mathbb{M}(v, \psi_0)_{O_{E_\nu^{\text{un}}}}$ for $v \neq v_0$ are smooth, and the theorem follows. \square

Remark 5.9. The proof of Theorem 5.7 shows that if (w, ψ) is of type (e) or (f), then the factor of the local model in (5.11) obtained from (w, ψ) is also smooth. However, we cannot allow the presence of such types in Theorem 5.8, since they would result in E_ν being ramified over \mathbb{Q}_p , which would destroy semi-stable reduction of the factor $\mathbb{M}(v_0, \varphi_0|_{F_{0, v_0}})$ after extending scalars from O_{E_0} to O_{E_ν} .

Remark 5.10. As is transparent from the above, smoothness and semi-stability of the p -integral models of the Shimura variety follow from the

corresponding property of the local models. We refer to [10] for a classification, under certain hypotheses, of general local models which are smooth, resp. have semi-stable reduction. In particular, let us single out one case of a deeper level subgroup (cf. Remark 4.5) in which semi-stable reduction arises. Let $n, w_0,$ and v_0 be as in Theorem 5.8, and modify the definition of type (a'') to require that $\{r_{\varphi_0}, r_{\overline{\varphi_0}}\} = \{1, n-1\}$ and to allow the level subgroup K_{G, v_0} at v_0 to be the stabilizer in $U(W)(F_{0, v_0})$ of *any* self-dual periodic lattice chain. Then, by work of Drinfeld [5] (see also [7, §4.4.5]), the factor $\mathbb{M}(v_0, \varphi_0|_{F_{0, v_0}})$ of the local model has semi-stable reduction. Hence, provided that the other factors of the local model are smooth and that E_ν is unramified over $\mathbb{Q}_p,$ the corresponding moduli stack will have semi-stable reduction over $\text{Spec } O_{E, \nu}.$

6. Global integral models of RSZ Shimura varieties

It is sometimes of interest to construct models of $M_{K_G}(\widetilde{G})$ over $\text{Spec } O_E.$ Rather than striving for maximal generality, in this section we single out two situations where this can be done. In both cases, we take the signature function r for the n -dimensional space W to be of fake Drinfeld type relative to a fixed element $\varphi_0 \in \Phi,$ cf. Example 2.3(i). Recall that, in this case, φ_0 embeds $F \rightarrow E$ for $n \geq 2,$ cf. Example 3.2. Hence each finite place ν of E induces a place w_ν of F and a place v_ν of F_0 via φ_0 for such $n.$ We set

$$\mathcal{V}_{\text{ram}} := \{ \text{finite places } v \text{ of } F_0 \mid v \text{ ramifies in } F \},$$

and we assume that all $v \in \mathcal{V}_{\text{ram}}$ are unramified over \mathbb{Q} and do not divide 2.

6.1. Integral models with exotic good reduction

In this subsection we define global integral models which, when $n \geq 2,$ have so-called exotic good reduction at all places ν of E such that the induced place v_ν of F_0 ramifies in F (and which, when $n = 1,$ are étale over $\text{Spec } O_E).$ We fix $\mathfrak{a}, \sqrt{\Delta}, \xi, \Lambda_0,$ and W_0 as before (3.17). As usual, we set $V = \text{Hom}_F(W_0, W),$ and we endow W with the \mathbb{Q} -valued alternating form $\text{tr}_{F/\mathbb{Q}} \sqrt{\Delta}^{-1}(\ , \).$ We fix an O_F -lattice $\Lambda \subset W$ whose localization is a vertex lattice with respect to this form at every finite place $v:$

$$\Lambda_v \subset \Lambda_v^\vee \subset \pi_v^{-1} \Lambda_v.$$

Of course, as for any O_F -lattice in $W,$ the localization Λ_v is necessarily self-dual for all but finitely many $v.$ We define the finite set

$$(6.1) \quad \mathcal{V}_{\text{un}}^\Lambda := \{ \text{finite places } v \text{ of } F_0 \mid v \text{ is unramified in } F \text{ and } \Lambda_v \subsetneq \Lambda_v^\vee \subsetneq \pi_v^{-1} \Lambda_v \}.$$

In addition to our assumptions on \mathcal{V}_{ram} at the beginning of this section, we impose on the tuple $(F/F_0, W, \Lambda)$ the following conditions.

- All $v \in \mathcal{V}_{\text{un}}^\Lambda$ are unramified over \mathbb{Q} .
- If $v \in \mathcal{V}_{\text{ram}}$, then the localization Λ_v of Λ is π_v -modular if n is even, and almost π_v -modular if n is odd (see Theorem 5.4(e)(f) for the definitions of these terms).

Starting from an arbitrary CM extension F/F_0 and n -dimensional hermitian space W , we note that if n is odd, then such a Λ always exists in W ; whereas if n is even, then such a Λ exists if and only if W_v is a split hermitian space for all $v \in \mathcal{V}_{\text{ram}}$ and for all finite v which are inert in F and ramified over \mathbb{Q} . We set

$$K_G^\circ := \{ g \in G(\mathbb{A}_f) \mid g(\Lambda \otimes \widehat{\mathbb{Z}}) = \Lambda \otimes \widehat{\mathbb{Z}} \},$$

and, as usual, we define $K_G^\circ := K_{\mathbb{Z}^\mathbb{Q}}^\circ \times K_G^\circ$.

We formulate a moduli problem $\mathcal{F}_{K_G^\circ}(\widetilde{G})$ over $\text{Spec } O_E$ as follows. As earlier in the paper, to lighten notation, we suppress the dependence on the ideal \mathfrak{a} and the element ξ .

Definition 6.1. The category functor $\mathcal{F}_{K_G^\circ}(\widetilde{G})$ associates to each O_E -scheme S the groupoid of tuples $(A_0, \iota_0, \lambda_0, A, \iota, \lambda)$, where

- $(A_0, \iota_0, \lambda_0)$ is an object of $\mathcal{M}_0^{\mathfrak{a}, \xi}(S)$;
- A is an abelian scheme over S ;
- $\iota: O_F \rightarrow \text{End}(A)$ is an action satisfying the Kottwitz condition (2.7) of signature type r on O_F ; and
- λ is a polarization on A whose Rosati involution satisfies condition (2.6) on O_F .

We also impose that the kernel of the polarization λ is of the type prescribed in Definition 4.1(i) for every p , relative to the lattice Λ fixed above. Furthermore, we impose for every finite place ν of E that after base-changing (A, ι, λ) to $S \otimes_{O_E} O_{E, \nu}$, the resulting triple satisfies the conditions on $\text{Lie } A$ imposed in the definition of $\mathcal{M}_{K_G^\circ}(\widetilde{G})_{O_{E, \nu}}$ in Theorem 5.4. In particular, we note that when $n \geq 2$ and ν is such that the induced place w_ν of F is ramified over F_0 , this entails imposing the wedge condition and spin condition in Theorem 5.4(e) when n is even, and the refined spin condition in Theorem 5.4(f) when n is odd, on the appropriate summand of $\text{Lie } A$.

Finally, we impose the sign condition that at every geometric point \bar{s} of S ,

$$(6.2) \quad \text{inv}_v^r(A_{0,\bar{s}}, \iota_{0,\bar{s}}, \lambda_{0,\bar{s}}, A_{\bar{s}}, \iota_{\bar{s}}, \lambda_{\bar{s}}) = \text{inv}_v(V),$$

for every finite place v of F_0 which is non-split in F .

A morphism $(A_0, \iota_0, \lambda_0, A, \iota, \lambda) \rightarrow (A'_0, \iota'_0, \lambda'_0, A', \iota', \lambda')$ in this groupoid is given by an isomorphism $\mu_0: (A_0, \iota_0, \lambda_0) \xrightarrow{\sim} (A'_0, \iota'_0, \lambda'_0)$ in $\mathcal{M}_0^{\alpha,\xi}(S)$ and an O_F -linear isomorphism $\mu: A \xrightarrow{\sim} A'$ of abelian schemes pulling λ' back to λ .

The following theorem (the extension of [31, Th. 5.2] to the present setting) shows that the moduli functor $\mathcal{F}_{K_G^\circ}(\tilde{G})$ defines an extension of $M_{K_G^\circ}(\tilde{G})$ over $\text{Spec } O_E$ with good properties. It follows immediately from Theorem 5.4 and Theorems 5.7 and 5.8.

Theorem 6.2. *The moduli problem $\mathcal{F}_{K_G^\circ}(\tilde{G})$ is representable by a Deligne–Mumford stack $\mathcal{M}_{K_G^\circ}(\tilde{G})$ flat over $\text{Spec } O_E$. For every finite place ν of E , the base change of $\mathcal{M}_{K_G^\circ}(\tilde{G})$ to $\text{Spec } O_{E,\nu}$ is isomorphic to the ν -adic integral moduli space of Theorem 5.4 in the case of the level subgroup K_G° . Hence:*

(i) *If $n \geq 2$, then $\mathcal{M}_{K_G^\circ}(\tilde{G})$ is smooth of relative dimension $n - 1$ over the open subscheme of $\text{Spec } O_E$ obtained by removing all finite places ν for which the induced place v_ν of F_0 lies in $\mathcal{V}_{\text{un}}^\Lambda$. If $n = 1$, then $\mathcal{M}_{K_G^\circ}(\tilde{G})$ is finite étale over all of $\text{Spec } O_E$.*

(ii) *If $n \geq 2$, then $\mathcal{M}_{K_G^\circ}(\tilde{G})$ has semi-stable reduction over the open subscheme of $\text{Spec } O_E$ obtained by removing all finite places ν ramified over \mathbb{Q} for which the induced place v_ν lies in $\mathcal{V}_{\text{un}}^\Lambda$. \square*

Remark 6.3. The isomorphism in Theorem 6.2 between the base change $\mathcal{M}_{K_G^\circ}(\tilde{G}) \otimes_{O_E} O_{E,\nu}$ and the moduli space of Theorem 5.4 can be made *canonical* in terms of the lattices $\Lambda_0 \subset W_0$ and $\Lambda \subset W$ fixed prior to Definition 6.1. Indeed, given an object $(A_0, \iota_0, \lambda_0, A, \iota, \lambda)$ in the moduli problem of Definition 6.1 over an $O_{E,\nu}$ -scheme, one defines the prime-to- p level $\tilde{\mathcal{T}}^p$ to be the set of all isometries $\tilde{V}^p(A_0, A) \xrightarrow{\sim} V \otimes_F \mathbb{A}_{F,f}^p$ carrying $\widehat{T}^p(A_0, A)$ to $\text{Hom}_{O_F}(\Lambda_0, \Lambda) \otimes_{O_F} \widehat{O}_F^p$.

Remark 6.4. Consider the moduli problem that associates to each O_E -scheme S the groupoid of triples (A, ι, λ) as in the last three bullet points of Definition 6.1, where the kernel of λ is of the type prescribed in Definition

4.1(i) for every p with respect to our fixed Λ , and such that for every finite place ν of E , the base change of (A, ι, λ) to $S \otimes_{O_E} O_{E,\nu}$ satisfies the conditions on Lie A imposed in Theorem 5.4. Via essentially the same proof as for Theorem 5.4, this moduli problem is represented by a Deligne–Mumford stack \mathcal{M}_r which is flat over $\text{Spec } O_E$. Then the stack $\mathcal{M}_{K_G^\circ}(\tilde{G})$ of Theorem 6.2 admits the simple description as the open and closed substack of

$$\mathcal{M}_0^{a,\xi} \times_{\text{Spec } O_E} \mathcal{M}_r$$

where the sign condition (6.2) holds pointwise.

We note that \mathcal{M}_r is an integral model for a finite disjoint union of copies of the Shimura variety $S(G^\mathbb{Q}, \{h_{G^\mathbb{Q}}\})$ of Kottwitz type for *maximal level structure*; therefore the previous description explains the relation between the integral model \mathcal{M}_r and the integral model $\mathcal{M}_{K_G^\circ}(\tilde{G})$ of the RSZ Shimura variety $S(\tilde{G}, \{h_{\tilde{G}}\})$ for level structure $K_{\tilde{G}}^\circ$.

Remark 6.5. (i) Let v be a finite place of F_0 which is non-split in F . Let ℓ denote the residue characteristic of v . Let k be an O_E -algebra which is an algebraically closed field of characteristic not ℓ . Generalizing somewhat from the setting of Definition 6.1, let $(A_0, \iota_0, \lambda_0, A, \iota, \lambda)$ be a tuple consisting of an abelian variety A_0 over $\text{Spec } k$ of dimension $[F_0 : \mathbb{Q}]$, an action $\iota_0 : O_{F,(\ell)} \rightarrow \text{End}_{(\ell)}(A_0)$ up to prime-to- ℓ isogeny, a quasi-polarization λ_0 on A_0 such that $\text{Ros}_\lambda(\iota_0(a)) = \iota_0(\bar{a})$ for all $a \in O_{F,(\ell)}$, and a triple (A, ι, λ) of the same form, except where A has dimension $n \cdot [F_0 : \mathbb{Q}]$. Consider the $O_{F,v}$ -lattice $T_v(A_0)$ inside the one-dimensional F_v -vector space $V_v(A_0)$, and let $T_v(A_0)^\vee$ denote the dual lattice with respect to λ_0 and the Weil pairing. Similarly define $T_v(A)^\vee$ inside the n -dimensional vector space $V_v(A)$. Say that

$$(6.3) \quad T_v(A_0)^\vee = \pi_v^m T_v(A_0)$$

inside $V_v(A_0)$. Recall $V_v(A_0, A) = \text{Hom}_{F_v}(V_v(A_0), V_v(A))$, which by (4.1) is a $F_v/F_{0,v}$ -hermitian space. Then the $O_{F,v}$ -lattice

$$T_v(A_0, A) := \text{Hom}_{O_{F,v}}(T_v(A_0), T_v(A)) \subset V_v(A_0, A)$$

has hermitian dual

$$(6.4) \quad T_v(A_0, A)^* = \text{Hom}_{O_{F,v}}(T_v(A_0)^\vee, T_v(A)^\vee) = \pi_v^m \text{Hom}_{O_{F,v}}(T_v(A_0), T_v(A)^\vee).$$

Similarly, for the moment let $\Lambda_{0,v} \subset W_{0,v}$ and $\Lambda_v \subset W_v$ be any $O_{F,v}$ -lattices, and suppose that $\Lambda_{0,v}^\vee = \pi_v^m \Lambda_{0,v}$. Then the $O_{F,v}$ -lattice

$$L_v := \text{Hom}_{O_{F,v}}(\Lambda_{0,v}, \Lambda_v) \subset V_v$$

has hermitian dual

$$(6.5) \quad L_v^* = \text{Hom}_{O_{F,v}}(\Lambda_{0,v}^\vee, \Lambda_v^\vee) = \pi_v^m \text{Hom}_{O_{F,v}}(\Lambda_{0,v}, \Lambda_v^\vee).$$

If the quasi-polarization λ is such that the relative position of $T_v(A)$ and $T_v(A)^\vee$ in $V_v(A)$ is the same as that of Λ_v and Λ_v^\vee in W_v , then it follows from (6.4) and (6.5) that the relative position of $T_v(A_0, A)$ and $T_v(A_0, A)^*$ in $V_v(A_0, A)$ is the same as that of L_v and L_v^* in V_v . In particular, this will be the case if $(A_0, \iota_0, \lambda_0)$ arises from a k -point on $\mathcal{M}_0^{a,\xi}$, the lattice $\Lambda_{0,v}$ is the localization at v of the global lattice $\Lambda_0 \subset W_0$ fixed prior to Definition 6.1, and λ induces an honest polarization of v -divisible groups $A[v^\infty] \rightarrow A^\vee[v^\infty]$ whose kernel satisfies the condition in Definition 4.1(i) relative to Λ_v .

Now suppose that v is *inert* in F . Then the split and non-split n -dimensional $F_v/F_{0,v}$ -hermitian spaces are distinguished by whether $\text{ord}_v(\det B)$ is respectively even or odd, for B the change-of-basis matrix going from any F_v -basis of the vector space to its dual basis with respect to the hermitian form (independent of the choice of basis). Hence $T_v(A_0, A)$ and $T_v(A_0, A)^*$ having the same relative position as L_v and L_v^* implies that $V_v(A_0, A)$ and V_v are isometric hermitian spaces. Hence the sign condition (6.2) for v is automatically satisfied at all points \bar{v} of residue characteristic not ℓ . It then follows from flatness of the moduli problem¹⁴ over $\text{Spec } O_E$ in Theorem 6.2 and local constancy of inv_v^r as a function on general base schemes [31, Prop. A.1] that the sign condition is automatically satisfied everywhere, for all inert v .

Now suppose that n is *even* and v *ramifies* in F . Then the split and non-split n -dimensional $F_v/F_{0,v}$ -hermitian spaces are distinguished by whether they respectively do or do not contain a π_v^i -modular lattice M with respect to the hermitian form (meaning that $M^* = \pi_v^{-i}M$) for some, or equivalently any, odd integer i . Similarly to the previous paragraph, this implies that the sign condition (6.2) is automatically satisfied at v (the main new point being that at points of the moduli space of residue characteristic not ℓ , the integer m in (6.3) must be even when v is ramified, and the condition on λ in Definition 4.1(i) at v relative to a π_v -modular Λ_v then forces $V_v(A_0, A)$ to be split). We conclude that *the full sign condition (6.2) is automatically satisfied when n is even*. In particular, the open and closed embedding

$$\mathcal{M}_{K_G^2}(\tilde{G}) \subset \mathcal{M}_0^{a,\xi} \times_{\text{Spec } O_E} \mathcal{M}_r$$

of Remark 6.4 is an equality when n is even.

¹⁴More precisely, the analogous moduli problem defined without the sign condition is also flat over $\text{Spec } O_E$, since flatness is purely a question of the local model at each finite place of E .

(ii) Suppose that $F_0 = \mathbb{Q}$. Then the sign condition can be replaced by the condition that for every geometric point \bar{s} of S , there exists an isomorphism of hermitian $O_{F,\ell}$ -lattices

$$\text{Hom}_{O_{F,\ell}}(\text{T}_\ell(A_{0,\bar{s}}), \text{T}_\ell(A_{\bar{s}})) \simeq \text{Hom}_{O_{F,\ell}}(\Lambda_{0,\ell}, \Lambda_\ell)$$

for every prime number $\ell \neq \text{char } \kappa(\bar{s})$, cf. [2, §2.3]. Indeed, this follows from the product formula and the Hasse principle for hermitian forms, cf. Remark 4.2.

We finally note that these remarks using flatness are also applicable to the semi-global moduli problems in Theorem 5.4.

6.2. Integral models for principal polarization

In this subsection we generalize the integral models of [2] from the case $F_0 = \mathbb{Q}$ to the case of arbitrary totally real F_0 . Throughout, we take $\mathfrak{a} = O_{F_0}$ and assume that $M_0^{O_{F_0}} \neq \emptyset$; recall from Remark 3.7(i) that this assumption is satisfied whenever F/F_0 is ramified at some finite place. We let $\sqrt{\Delta}$, ξ , Λ_0 , W_0 , and V all be as in Section 6.1. We assume that W contains an O_F -lattice Λ which is *self-dual* for $\text{tr}_{F/\mathbb{Q}} \sqrt{\Delta}^{-1}(\ , \)$, and we fix such a Λ once and for all. (Note that this implies that the set $\mathcal{V}_{\text{un}}^\Lambda$ defined as in (6.1) is empty in the present case.) We set, as in the previous subsection,

$$K_G^\circ := \{ g \in G(\mathbb{A}_f) \mid g(\Lambda \otimes \widehat{\mathbb{Z}}) = \Lambda \otimes \widehat{\mathbb{Z}} \},$$

and, as usual, we define $K_G^\circ = K_{Z\mathbb{Q}}^\circ \times K_G^\circ$.

In the present situation, we formulate the following variant of the moduli problem in Definition 6.1. As usual, we suppress the ideal $\mathfrak{a} = O_{F_0}$ and the element ξ in the notation.

Definition 6.6. The category functor $\mathcal{F}_{K_G^\circ}(\widetilde{G})$ associates to each O_E -scheme S the groupoid of tuples $(A_0, \iota_0, \lambda_0, A, \iota, \lambda)$, where $(A_0, \iota_0, \lambda_0, A, \iota)$ is as in Definition 6.1, and where

- λ is a *principal* polarization whose Rosati involution satisfies condition (2.6) on O_F .

Again, we impose the sign condition (6.2) for every finite non-split place of F_0 . Likewise, we impose for every finite place ν of E that after base-changing (A, ι, λ) to $S \otimes_{O_E} O_{E,\nu}$, the resulting triple satisfies the conditions on Lie A imposed in the definition of $\mathcal{M}_{K_G^\circ}(\widetilde{G})_{O_{E,\nu}}$ in Theorem 5.4. In particular, we note

that for the places w of F which are ramified over F_0 and of the same residue characteristic as ν , this entails imposing the wedge condition in Theorem 5.4(d) on each summand $\text{Lie}_{w,\psi} A$ (when the signature function $r|_{\text{Hom}_{w,\psi}(F,\overline{\mathbb{Q}})}$ is of banal type, it is equivalent to impose the Eisenstein condition in Theorem 5.4(b)).

The morphisms in this groupoid are as in Definition 6.1.

Theorem 6.7. *The moduli problem $\mathcal{F}_{K_G^\circ}(\tilde{G})$ is representable by a Deligne–Mumford stack $\mathcal{M}_{K_G^\circ}(\tilde{G})$ flat over $\text{Spec } O_E$. For every finite place ν of E , the base change of $\mathcal{M}_{K_G^\circ}(\tilde{G})$ to $\text{Spec } O_{E,\nu}$ is canonically isomorphic to the ν -adic integral moduli space of Theorem 5.4 in the case of the level subgroup K_G° . Furthermore:*

(i) *If $n \geq 2$, then $\mathcal{M}_{K_G^\circ}(\tilde{G})$ is smooth of relative dimension $n - 1$ over the open subscheme of $\text{Spec } O_E$ obtained by removing the set $\mathcal{V}_{\text{ram}}(E)$ of finite places ν for which the induced place v_ν of F_0 lies in \mathcal{V}_{ram} . If $n = 1$, then $\mathcal{M}_{K_G^\circ}(\tilde{G})$ is finite étale over all of $\text{Spec } O_E$.*

(ii) *If $n \geq 2$, then the fiber of $\mathcal{M}_{K_G^\circ}(\tilde{G})$ over a place $\nu \in \mathcal{V}_{\text{ram}}(E)$ has only isolated singularities. If $n \geq 3$, then blowing up these isolated points for all such ν yields a model $\mathcal{M}_{K_G^\circ}^\natural(\tilde{G})$ which has semi-stable reduction, and hence is regular, over the open subscheme of $\text{Spec } O_E$ obtained by removing the places $\nu \in \mathcal{V}_{\text{ram}}(E)$ which are ramified over F . This model represents the moduli problem $\mathcal{F}_{K_G^\circ}^\natural(\tilde{G})$ formulated below.*

Proof. Representability of $\mathcal{F}_{K_G^\circ}(\tilde{G})$ is standard, and the statement on the base change to $\text{Spec } O_{E,\nu}$ is obvious, with level structures defined as in Remark 6.3 (thus the isomorphism over $\text{Spec } O_{E,\nu}$ is made canonical via the choice of Λ_0 and Λ). Flatness follows from Theorem 5.4. Assertion (i) follows from Theorem 5.7. Assertion (ii) reduces to a statement on the local model. More precisely, let $\nu \in \mathcal{V}_{\text{ram}}(E)$, let p denote the residue characteristic of ν , and consider the decomposition of the local model $\mathbb{M}_{O_{E^{\text{un}}}}$ in (5.11) relative to the choice of an embedding $\alpha: \overline{\mathbb{Q}} \rightarrow \overline{\mathbb{Q}}_p$ inducing ν . Since the signature type is of fake Drinfeld type, the factors $\mathbb{M}(v, \psi_0)_{O_{E^{\text{un}}}}$ for all $v \neq v_\nu$ are trivial by the Eisenstein condition. Since we assume that every $v \in \mathcal{V}_{\text{ram}}$ is unramified over \mathbb{Q} , we have $F_{0,v_\nu} = F_{0,v_\nu}^t$, and similarly $\mathbb{M}(v_\nu, \psi_0)_{O_{E^{\text{un}}}}$ is trivial for all embeddings $\psi_0: F_{0,v_\nu} \rightarrow \overline{\mathbb{Q}}_p$ except the single embedding $\psi_{0,\alpha}$ induced by $\alpha \circ \varphi_0$. The factor $\mathbb{M}(v_\nu, \psi_{0,\alpha})_{O_{E^{\text{un}}}}$ is then the base change to $\text{Spec } O_{E^{\text{un}}}$ of the

local model for $\mathrm{GU}_n(F_{v_\nu}/F_{0,v_\nu})$ in the case of a self-dual lattice and signature type $(n - 1, 1)$. By Pappas [24, Th. 4.5 & its proof], when $n \geq 2$, this local model is singular at a single point, with blowup at this point of semi-stable reduction. When $n \geq 3$, the reflex field of this local model identifies with F_{v_ν} , and therefore semi-stable reduction is preserved after the base change $O_{F,v_\nu} \rightarrow O_{E_\nu^{\mathrm{un}}}$ provided ν is unramified over F . \square

Remark 6.8. The reflex field of the local model for $\mathrm{GU}_2(F_{v_\nu}/F_{0,v_\nu})$ appearing in the proof of Theorem 6.7 is F_{0,v_ν} , not F_{v_ν} . In this case, the local model itself has semi-stable reduction over $\mathrm{Spec} O_{F_0,v_\nu}$, without needing to blow up. However, since F_{v_ν} maps into E_ν , the extension $O_{F_0,v_\nu} \rightarrow O_{E_\nu^{\mathrm{un}}}$ is necessarily ramified, and hence semi-stable reduction is lost upon base change.

Remark 6.9. There is an obvious analog of Remark 6.4 in the present situation, where the moduli problem for \mathcal{M}_r is replaced by the analogous one with respect to our self-dual lattice Λ (in particular, the polarization λ in the resulting moduli problem is principal; when $F_0 = \mathbb{Q}$, the resulting stack is denoted $\mathcal{M}_{(n-1,1)}^{\mathrm{Pap}}$ in [2, §2.3]). Furthermore, by the same argument as in Remark 6.5(i), the sign condition (6.2) imposed in Definition 6.6 is redundant at all inert places v . However, it is no longer the case that the sign condition is redundant at ramified places v when n is even (since in the ramified case, for any $n \in \mathbb{Z}_{>0}$, both isometry types of n -dimensional $F_v/F_{0,v}$ -hermitian spaces contain a self-dual lattice). Finally, Remark 6.5(ii) transposes word-for-word to the present situation, cf. [2, §2.3].

Here is the moduli problem mentioned in Theorem 6.7(ii) (the *Kramer model*). Let $n \geq 2$.

Definition 6.10. The category functor $\mathcal{F}_{K_G^{\circlearrowleft}}^{\natural}(\tilde{G})$ associates to each O_E -scheme S the groupoid of tuples $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \mathcal{P})$, where $(A_0, \iota_0, \lambda_0, A, \iota, \lambda) \in \mathcal{F}_{K_G^{\circlearrowleft}}(\tilde{G})(S)$, and where $\mathcal{P} \subset \mathrm{Lie} A$ is an O_F -stable \mathcal{O}_S -submodule which, Zariski-locally on S , is an \mathcal{O}_S -free direct summand satisfying the *rank condition* (6.7) and the *Kramer–Eisenstein condition* (6.10) below.

The rank condition and the Kramer–Eisenstein condition mentioned in Definition 6.10 are conditions for every finite place ν of E . Fixing ν , let p denote the residue characteristic of ν , and choose an embedding $\alpha: \overline{\mathbb{Q}} \rightarrow \overline{\mathbb{Q}}_p$ inducing ν as before (5.1). For any p -adic place w of F and any \mathbb{Q}_p -embedding $\psi: F_w^t \rightarrow \overline{\mathbb{Q}}_p$, define $\mathrm{Hom}_{w,\psi}(F, \overline{\mathbb{Q}}) \subset \mathrm{Hom}(F, \overline{\mathbb{Q}})$ as in (5.3). We recall that the resulting partition $\mathrm{Hom}(F, \overline{\mathbb{Q}}) = \coprod_{w,\psi} \mathrm{Hom}_{w,\psi}(F, \overline{\mathbb{Q}})$ as in (5.4) depends

only on ν up to labeling of the sets on the right-hand side. Let

$$r_{w,\psi}^\Phi := \sum_{\varphi \in \text{Hom}_{w,\psi}(F, \overline{\mathbb{Q}}) \cap \Phi} r_\varphi.$$

Then the rank condition for ν on the $O_F \otimes_{\mathbb{Z}} \mathcal{O}_S$ -module \mathcal{P} states that in the decomposition analogous to (5.8) and (5.9),

$$(6.6) \quad \mathcal{P} = \bigoplus_{w,\psi} \mathcal{P}_{w,\psi},$$

we have

$$(6.7) \quad \text{rank}_{\mathcal{O}_S} \mathcal{P}_{w,\psi} = r_{w,\psi}^\Phi$$

for all w, ψ . Here, as in (5.9), the decomposition (6.6) is defined when S is an $O_{E_\nu^n}$ -scheme, and the rank condition for all ν then descends to a condition on O_E -schemes.

The Kramer–Eisenstein condition for ν is similarly a condition on each summand $\mathcal{P}_{w,\psi}$ in (6.6). The statement of the condition depends on the restricted function $r|_{w,\psi} := r|_{\text{Hom}_{w,\psi}(F, \overline{\mathbb{Q}})}$, and involves polynomials closely related to those in the formulation of the Eisenstein condition in [33, §8]. For each w, ψ , let

$$(6.8) \quad \begin{aligned} C_{w,\psi}^\Phi &:= \{ \varphi \in \text{Hom}_{w,\psi}(F, \overline{\mathbb{Q}}) \cap \Phi \mid r_\varphi \neq 0 \}, \\ C_{w,\psi}^{\overline{\Phi}} &:= \{ \varphi \in \text{Hom}_{w,\psi}(F, \overline{\mathbb{Q}}) \cap \overline{\Phi} \mid r_\varphi \neq 0 \}. \end{aligned}$$

Let π be a uniformizer in F_w , and define the polynomials in $\overline{\mathbb{Q}}_p[T]$,

$$(6.9) \quad Q_{C_{w,\psi}^\Phi}(T) := \prod_{\varphi \in C_{w,\psi}^\Phi} (T - \varphi(\pi)) \in \overline{\mathbb{Q}}_p[T] \quad \text{and} \quad Q_{C_{w,\psi}^{\overline{\Phi}}}(T) := \prod_{\varphi \in C_{w,\psi}^{\overline{\Phi}}} (T - \varphi(\pi)).$$

Here and below we implicitly use α to identify $C_{w,\psi}^\Phi$ and $C_{w,\psi}^{\overline{\Phi}}$ with subsets of $\text{Hom}_{\mathbb{Q}_p}(F_w, \overline{\mathbb{Q}}_p)$. Then the Kramer–Eisenstein condition on $\mathcal{P}_{w,\psi}$ is that

$$(6.10) \quad Q_{C_{w,\psi}^\Phi}(\pi \otimes 1)|_{\mathcal{P}_{w,\psi}} = 0 \quad \text{and} \quad Q_{C_{w,\psi}^{\overline{\Phi}}}(\pi \otimes 1)|_{\text{Lie}_{w,\psi} A/\mathcal{P}_{w,\psi}} = 0;$$

here the condition is defined when S is a scheme over $\text{Spec } O_L$ for any subfield $L \subset \overline{\mathbb{Q}}_p$ large enough to contain the image of E_ν under α , the image of F_w^t under ψ , and the coefficients of the polynomials $Q_{C_{w,\psi}^\Phi}(T)$ and $Q_{C_{w,\psi}^{\overline{\Phi}}}(T)$.

The Kramer–Eisenstein condition (6.10) is independent of the choice of uniformizer π . Indeed, this follows immediately from the next lemma, upon taking the field L to be large enough to contain $\varphi(F_w)$ for all $\varphi \in C_{w,\psi}^\Phi \cup C_{w,\psi}^{\overline{\Phi}}$.

Lemma 6.11. *Let π and π' be two uniformizers of F_w . Then the elements $\pi \otimes 1 - 1 \otimes \pi$ and $\pi' \otimes 1 - 1 \otimes \pi'$ are unit multiples of each other in the ring $O_{F_w} \otimes_{O_{F_w^t}} O_{F_w}$.*

Proof. Say $\pi' = u\pi$, with $u \in O_{F_w}^\times$. Let $e := [F_w : F_w^t]$. Then there exist unique $u_0, \dots, u_{e-1} \in O_{F_w^t}$, with $u_0 \in O_{F_w^t}^\times$, such that $u = u_0 + u_1\pi + \dots + u_{e-1}\pi^{e-1}$. Hence

$$\pi' \otimes 1 - 1 \otimes \pi' = u_0(\pi \otimes 1 - 1 \otimes \pi) + \dots + u_{e-1}(\pi^e \otimes 1 - 1 \otimes \pi^e).$$

Every term on the right-hand side is divisible by $\pi \otimes 1 - 1 \otimes \pi$. Factoring out, we obtain an equation of the form

$$\pi' \otimes 1 - 1 \otimes \pi' = (\pi \otimes 1 - 1 \otimes \pi)(u_0 + a),$$

where $a \in O_{F_w} \otimes_{O_{F_w^t}} O_{F_w}$ lies in the maximal ideal generated by $\pi \otimes 1$ and $1 \otimes \pi$. This completes the proof. \square

The Kramer–Eisenstein condition for ν is that (6.10) holds for all w, ψ with w of the same residue characteristic as ν . The (full) Kramer–Eisenstein condition is that the Kramer–Eisenstein condition for ν holds for all ν ; this condition again descends to O_E -schemes.

As a first step towards understanding the rank and Kramer–Eisenstein conditions, let \tilde{E} denote the composite of E and the normal closure of F in $\overline{\mathbb{Q}}$, and let $O'_{\tilde{E}}$ be the ring obtained from $O_{\tilde{E}}$ by inverting the finitely many rational primes p which ramify in F . Then

$$O_F \otimes_{\mathbb{Z}} O'_{\tilde{E}} \cong \prod_{\varphi \in \text{Hom}(F, \overline{\mathbb{Q}})} O'_{\tilde{E}}.$$

If S is an $O'_{\tilde{E}}$ -scheme and $(A_0, \iota_0, \lambda_0, A, \iota, \lambda)$ is an S -point on $\mathcal{F}_{K_G^{\circ}}(\tilde{G})$, the O_F -action ι thus induces a canonical decomposition

$$\text{Lie } A = \bigoplus_{\varphi \in \text{Hom}(F, \overline{\mathbb{Q}})} \text{Lie}_{\varphi} A.$$

By the Kottwitz condition, $\text{rank}_{O_S} \text{Lie}_{\varphi} A = r_{\varphi}$. By the definition of $O'_{\tilde{E}}$, the set $\text{Hom}_{w,\psi}(F, \overline{\mathbb{Q}})$ is a singleton set $\{\varphi\}$ for all places w such that S

has a nonempty fiber of the same residue characteristic as w . Therefore the rank condition imposes that \mathcal{P}_φ equals $\text{Lie}_\varphi A$ or 0 according as $\varphi \in \Phi$ or $\varphi \notin \Phi$. Furthermore, the resulting \mathcal{O}_S -module $\mathcal{P} = \bigoplus_\varphi \mathcal{P}_\varphi$ obviously satisfies the Kramer–Eisenstein condition. We conclude that the datum of \mathcal{P} in the moduli problem $\mathcal{F}_{K_G^{\natural}}(\tilde{G})$ is redundant over $\text{Spec } \mathcal{O}'_E$ —and hence, by descent, over $\text{Spec } \mathcal{O}'_E$, where \mathcal{O}'_E is the ring obtained from \mathcal{O}_E by inverting the primes p that ramify in F . In fact, a stronger statement is true.

Lemma 6.12. *Keep the notation above, and assume that S is a scheme over $\text{Spec } \mathcal{O}_L$ for $L \subset \overline{\mathbb{Q}}_p$ sufficiently large. If $r|_{w,\psi}$ is banal or w is unramified over F_0 , then there exists a unique summand $\mathcal{P}_{w,\psi} \subset \text{Lie}_{w,\psi} A$ satisfying the rank condition (6.7) and the Kramer–Eisenstein condition (6.10).*

Proof. To show existence and uniqueness of $\mathcal{P}_{w,\psi}$, it suffices to solve the analogous problem on the local model, as in the proofs of Theorems 5.2 and 5.4. The local model is a moduli functor of $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathcal{O}_S$ -linear quotients $\Lambda \otimes_{\mathbb{Z}} \mathcal{O}_S \twoheadrightarrow \mathcal{Q}$ satisfying certain conditions. Since L is sufficiently large, we have the usual direct sum decompositions

$$\bigoplus_{w',\psi'} \Lambda_{w',\psi',S} \twoheadrightarrow \bigoplus_{w',\psi'} \mathcal{Q}_{w',\psi'},$$

where $\Lambda_{w',\psi',S} := \Lambda \otimes_{\mathcal{O}_F} \mathcal{O}_{F,w'} \otimes_{\mathcal{O}_{F^t,w'},\psi'} \mathcal{O}_S$. Our problem is to show that there exists a unique subbundle $\mathcal{P}'_{w,\psi} \subset \mathcal{Q}_{w,\psi}$ satisfying the (analogs on the local model of) the rank condition and the Kramer–Eisenstein condition. Throughout the rest of the proof, we will make use of the set

$$A_{w,\psi} := \{ \varphi \in \text{Hom}_{w,\psi}(F, \overline{\mathbb{Q}}) \mid r_\varphi = n \}$$

and the polynomial

$$(6.11) \quad Q_{A_{w,\psi}}(T) := \prod_{\varphi \in A_{w,\psi}} (T - \varphi(\pi)),$$

cf. [33, (2.3), (2.8)] (in the case of the extension F_w/\mathbb{Q}_p). As before, in (6.11) we have implicitly used α to identify $A_{w,\psi}$ with a subset of $\text{Hom}_{\mathbb{Q}_p}(F_w, \overline{\mathbb{Q}}_p)$.

First suppose that $r|_{w,\psi}$ is banal. Then $C_{w,\psi}^\Phi \cup C_{w,\psi}^{\overline{\Phi}} = A_{w,\psi}$, and, by the definition of the moduli problem, $\mathcal{Q}_{w,\psi}$ is required to satisfy the Eisenstein condition $Q_{A_{w,\psi}}(\pi \otimes 1)|_{\mathcal{Q}_{w,\psi}} = 0$, cf. [31, (B.5)]. Furthermore, the Kottwitz condition in the banal case implies that $\mathcal{Q}_{w,\psi}$ has \mathcal{O}_S -rank $n \cdot \#A_{w,\psi}$. This forces

$$\ker[\Lambda_{w,\psi,S} \twoheadrightarrow \mathcal{Q}_{w,\psi}] = Q_{A_{w,\psi}}(\pi \otimes 1) \cdot \Lambda_{w,\psi,S} \subset Q_{C_{w,\psi}^\Phi}(\pi \otimes 1) \cdot \Lambda_{w,\psi,S},$$

cf. [31, (B.6)]. Hence

$$(6.12) \quad \text{cok}(Q_{C_{w,\psi}^\Phi}(\pi \otimes 1)|_{\mathcal{Q}_{w,\psi}}) \cong \text{cok}(Q_{C_{w,\psi}^\Phi}(\pi \otimes 1)|_{\Lambda_{w,\psi,S}}).$$

Since the right-hand side of (6.12) is a free \mathcal{O}_S -module of rank $n \cdot \#C_{w,\psi}^\Phi = r_{w,\psi}^\Phi$, we conclude that $\ker(Q_{C_{w,\psi}^\Phi}(\pi \otimes 1)|_{\mathcal{Q}_{w,\psi}})$ is a direct summand of $\mathcal{Q}_{w,\psi}$ of the same rank $r_{w,\psi}^\Phi$. The rank condition (6.7) and the first relation in the Krämer–Eisenstein condition (6.10) then force the equality

$$(6.13) \quad \mathcal{P}'_{w,\psi} = \ker(Q_{C_{w,\psi}^\Phi}(\pi \otimes 1)|_{\mathcal{Q}_{w,\psi}}).$$

(One sees easily that (6.13) also equals $Q_{C_{w,\psi}^\Phi}(\pi \otimes 1) \cdot \mathcal{Q}_{w,\psi}$.) This completes the proof in the banal case.

Now suppose that w is unramified over F_0 and $r|_{w,\psi}$ is non-banal. Then (w, ψ) is one of the pairs (w_ν, ψ_α) and $(\bar{w}_\nu, \bar{\psi}_\alpha)$, where ψ_α denotes the embedding $F_{w_\nu}^t \rightarrow \bar{\mathbb{Q}}_p$ induced by $\alpha \circ \varphi_0$. (These pairs are distinct, by unramifiedness). In the case of $(\bar{w}_\nu, \bar{\psi}_\alpha)$, since the signature type is of fake Drinfeld type, we have $\bar{\varphi}_0 \in \text{Hom}_{\bar{w}_\nu, \bar{\psi}_\alpha}(F, \bar{\mathbb{Q}})$ with $r_{\bar{\varphi}_0} = 1$, and $\mathcal{Q}_{\bar{w}_\nu, \bar{\psi}_\alpha}$ satisfies the Eisenstein condition given in [33, (8.2)]. Let

$$\mathcal{K}_{\bar{w}_\nu, \bar{\psi}_\alpha} := \ker[\Lambda_{\bar{w}_\nu, \bar{\psi}_\alpha, S} \xrightarrow{f} \mathcal{Q}_{\bar{w}_\nu, \bar{\psi}_\alpha}].$$

Taking $V = Q_{A_{\bar{w}_\nu, \bar{\psi}_\alpha}}(\pi \otimes 1) \cdot \Lambda_{\bar{w}_\nu, \bar{\psi}_\alpha, S}$ and $W = \Lambda_{\bar{w}_\nu, \bar{\psi}_\alpha, S}$ in [33, Lem. 4.10], and using the Eisenstein condition, we conclude that $\mathcal{K}_{\bar{w}_\nu, \bar{\psi}_\alpha} \subset Q_{A_{\bar{w}_\nu, \bar{\psi}_\alpha}}(\pi \otimes 1) \cdot \Lambda_{\bar{w}_\nu, \bar{\psi}_\alpha, S}$; comp. the proof of [33, Lem. 8.6]. From here, since $C_{\bar{w}_\nu, \bar{\psi}_\alpha}^\Phi \subset A_{\bar{w}_\nu, \bar{\psi}_\alpha}$ in the present case, the same argument as in the previous paragraph shows that we again must have $\mathcal{P}'_{\bar{w}_\nu, \bar{\psi}_\alpha} = \ker(Q_{C_{\bar{w}_\nu, \bar{\psi}_\alpha}^\Phi}(\pi \otimes 1)|_{\mathcal{Q}_{\bar{w}_\nu, \bar{\psi}_\alpha}})$.

In the case of the pair (w_ν, ψ_α) , we have $C_{w_\nu, \psi_\alpha}^\Phi \subset A_{w_\nu, \psi_\alpha}$. Therefore the rank condition (6.7) and the second relation in the Krämer–Eisenstein condition (6.10) imply that the inverse image of $\mathcal{P}'_{w_\nu, \psi_\alpha}$ in $\Lambda_{w_\nu, \psi_\alpha, S}$ is equal to $Q_{C_{w_\nu, \psi_\alpha}^\Phi}(\pi \otimes 1) \cdot \Lambda_{w_\nu, \psi_\alpha, S}$. This uniquely determines $\mathcal{P}'_{w_\nu, \psi_\alpha}$ if it exists. In turn, existence holds if and only if

$$(6.14) \quad \mathcal{K}_{w_\nu, \psi_\alpha} \subset Q_{C_{w_\nu, \psi_\alpha}^\Phi}(\pi \otimes 1) \cdot \Lambda_{w_\nu, \psi_\alpha, S},$$

where $\mathcal{K}_{w_\nu, \psi_\alpha} := \ker[\Lambda_{w_\nu, \psi_\alpha, S} \rightarrow \mathcal{Q}_{w_\nu, \psi_\alpha}]$. To show the containment (6.14), it suffices to show that

$$(6.15) \quad \mathcal{K}_{w_\nu, \psi_\alpha} \subset Q_{A_{w_\nu, \psi_\alpha}}(\pi \otimes 1) \cdot \Lambda_{w_\nu, \psi_\alpha, S}.$$

Now, by the formalism of local models, self-duality of the lattice Λ gives rise to a perfect pairing

$$\Lambda_{w_\nu, \psi_\alpha, S} \times \Lambda_{\bar{w}_\nu, \bar{\psi}_\alpha, S} \longrightarrow \mathcal{O}_S$$

under which the O_F -actions on the two factors are conjugate-adjoint, and such that $\mathcal{K}_{w_\nu, \psi_\alpha} \subset \Lambda_{w_\nu, \psi_\alpha, S}$ and $\mathcal{K}_{\bar{w}_\nu, \bar{\psi}_\alpha} \subset \Lambda_{\bar{w}_\nu, \bar{\psi}_\alpha, S}$ are the perp-modules of each other. Thus (6.15) is equivalent to the containment

$$(6.16) \quad (Q_{A_{w_\nu, \psi_\alpha}}(\pi \otimes 1) \cdot \Lambda_{w_\nu, \psi_\alpha, S})^\perp \subset \mathcal{K}_{\bar{w}_\nu, \bar{\psi}_\alpha}.$$

It is a pleasant exercise to show that the left-hand side in (6.16) equals

$$Q_0(\bar{\pi} \otimes 1) Q_{A_{\bar{w}_\nu, \bar{\psi}_\alpha}}(\bar{\pi} \otimes 1) \cdot \Lambda_{\bar{w}_\nu, \bar{\psi}_\alpha, S},$$

where the polynomials $Q_{A_{\bar{w}_\nu, \bar{\psi}_\alpha}}$ and $Q_0(T) := T - \bar{\varphi}_0(\bar{\pi})$ are defined for the field $F_{\bar{w}_\nu}$ with respect to the uniformizer $\bar{\pi}$. Thus the containment (6.16) holds by the Eisenstein condition [33, (8.2)], which completes the proof. \square

It follows from Lemma 6.12 that the natural forgetful morphism

$$(6.17) \quad \mathcal{F}_{K_G^\circ}^{\natural}(\tilde{G}) \longrightarrow \mathcal{F}_{K_G^\circ}(\tilde{G})$$

is an isomorphism over the open locus $\text{Spec}(O_E[\mathcal{V}_{\text{ram}}^{-1}]) \subset \text{Spec}(O_E)$. On the other hand, let ν now be a place lying over some $v_\nu \in \mathcal{V}_{\text{ram}}$. Then the functor $\mathcal{F}_{K_G^\circ}^{\natural}(\tilde{G})_{O_{E,(\nu)}}$ can be understood via the corresponding local model for $\mathcal{M}_{K_G^\circ}(\tilde{G})_{O_{E,(\nu)}}$. As in the proof of Theorem 6.7, the base change to $\text{Spec } O_{E_\nu^{\text{un}}}$ of the local model \mathbb{M} is a product

$$\mathbb{M}_{O_{E_\nu^{\text{un}}}} = \prod_{v, \psi_0} \mathbb{M}(v, \psi_0)_{O_{E_\nu^{\text{un}}}}.$$

By Lemma 6.12, for all factors except the one indexed by $(v_\nu, \psi_{0,\alpha})$, the datum of \mathcal{P} is redundant; here $\psi_{0,\alpha}$ is as in the proof of Theorem 6.7. The factor $\mathbb{M}(v_\nu, \psi_{0,\alpha})_{O_{E_\nu^{\text{un}}}}$ is the base change to $O_{E_\nu^{\text{un}}}$ of the Pappas local model for $\text{GU}_n(F_{w_\nu}/F_{0,v_\nu})$, for a self-dual lattice and signature type $(n - 1, 1)$, cf. [24]. The datum of \mathcal{P} corresponds to a point of the *Kramer local model* [15] mapping to the Pappas local model.

Appendix A. Local model diagram for unramified PEL data of type A

In this appendix, we change notation from the main body of the paper. Let $(F, B, V, (\cdot, \cdot))$ be rational data of PEL type over \mathbb{Q}_p in the sense of [32, §1.38]. Let $*$ be the induced involution on B . Let O_B be a maximal order of B invariant under $*$. Let F_0 denote the invariants of $*$ in F , and let O_{F_0} and O_F denote the respective rings of integers. Our main purpose is to prove the following theorem, which is a special case of [32, Th. 3.16] when $p \neq 2$. We adopt the terminology of loc. cit.

Theorem A.1. *Assume that O_F is an étale O_{F_0} -algebra free of rank 2.*

Let \mathcal{L} be a self-dual multichain of O_B -lattices in V . Let T be a \mathbb{Z}_p -scheme on which p is locally nilpotent. Let $\{M_\Lambda \mid \Lambda \in \mathcal{L}\}$ be a polarized multichain of $O_B \otimes_{\mathbb{Z}_p} \mathcal{O}_T$ -modules of type (\mathcal{L}) . Then locally for the étale topology on T , the polarized multichain $\{M_\Lambda\}$ is isomorphic to $\mathcal{L} \otimes_{\mathbb{Z}_p} \mathcal{O}_T$.

Furthermore, if $\{M'_\Lambda\}$ is a second polarized multichain of $O_B \otimes_{\mathbb{Z}_p} \mathcal{O}_T$ -modules of type (\mathcal{L}) , then the functor of isomorphisms of polarized multichains on the category of T -schemes,

$$T' \longmapsto \text{Isom}(\{M_\Lambda \otimes_{\mathcal{O}_T} \mathcal{O}_{T'}\}, \{M'_\Lambda \otimes_{\mathcal{O}_T} \mathcal{O}_{T'}\}),$$

is representable by a smooth affine T -scheme.

Proof. We may assume that F_0 is a field. If F/F_0 is split, i.e., we are in the case (I) of [32, p. 135], then the first claim follows from [32, Th. 3.11] by [32, Lem. A.8], which reduces this case to the unpolarized case. The proofs of these facts are valid for $p = 2$, and the trivialization even exists locally for the Zariski topology on T . That the Isom-functor is representable by an affine scheme of finite type is trivial. Smoothness follows from [32, Th. 3.11].

Now let F/F_0 be an unramified field extension. Let F^t denote the maximal unramified subextension of \mathbb{Q}_p in F , and let O_{F^t} denote its ring of integers. Set

$$\begin{aligned} \tilde{F}_0 &:= F_0 \otimes_{\mathbb{Q}_p} F^t, & \tilde{F} &:= F \otimes_{\mathbb{Q}_p} F^t, & \tilde{B} &:= B \otimes_{\mathbb{Q}_p} F^t, \\ \tilde{V} &:= V \otimes_{\mathbb{Q}_p} F^t, & \widetilde{(\cdot, \cdot)} &:= (\cdot, \cdot) \otimes \text{tr}_{F^t/\mathbb{Q}_p}, \\ O_{\tilde{F}_0} &:= O_{F_0} \otimes_{\mathbb{Z}_p} O_{F^t}, & O_{\tilde{F}} &:= O_F \otimes_{\mathbb{Z}_p} O_{F^t}, & O_{\tilde{B}} &:= O_B \otimes_{\mathbb{Z}_p} O_{F^t}, \\ & & \tilde{\mathcal{L}} &:= \mathcal{L} \otimes_{\mathbb{Z}_p} O_{F^t}. \end{aligned}$$

Then $(\tilde{F}, \tilde{B}, \tilde{V}, (\tilde{}, \tilde{}), O_{\tilde{B}}, \tilde{\mathcal{L}})$ are integral data of PEL type¹⁵ such that $\tilde{F} = \tilde{F}_0 \times \tilde{F}_0$, i.e., \tilde{F}/\tilde{F}_0 is a product of split quadratic extensions. Set $\tilde{M}_\Lambda := M_\Lambda \otimes_{\mathbb{Z}_p} O_{F^t}$. Then $\{\tilde{M}_\Lambda\}$ is a polarized multichain of $O_{\tilde{B}} \otimes_{\mathbb{Z}_p} \mathcal{O}_T$ -modules of type $(\tilde{\mathcal{L}})$. From the previous case, we obtain that locally for the Zariski topology on T , there exists an isomorphism between $\{\tilde{M}_\Lambda\}$ and $\tilde{\mathcal{L}} \otimes_{\mathbb{Z}_p} \mathcal{O}_T$. Set $\tilde{T} := T \times_{\text{Spec } \mathbb{Z}_p} \text{Spec } O_{F^t}$. Then \tilde{T} is an étale covering of T , and there are natural isomorphisms of polarized multichains of $O_B \otimes_{\mathbb{Z}_p} \mathcal{O}_{\tilde{T}}$ -modules of type (\mathcal{L}) ,

$$\{M_\Lambda \otimes_{\mathcal{O}_T} \mathcal{O}_{\tilde{T}}\} \cong \{\tilde{M}_\Lambda\}, \quad \{\mathcal{L} \otimes_{\mathbb{Z}_p} \mathcal{O}_{\tilde{T}}\} \cong \{\tilde{\mathcal{L}} \otimes_{\mathbb{Z}_p} \mathcal{O}_T\}.$$

Therefore there is a trivialization of the multichain $\{M_\Lambda \otimes_{\mathcal{O}_T} \mathcal{O}_{\tilde{T}}\}$, locally on \tilde{T} . In the same way, the smoothness of the Isom-scheme follows from the previous case. \square

As a consequence, the entire formalism of local models and the local model diagram in [32] carries over to the $p = 2$ case for PEL data as in Theorem A.1. In particular, in the context of the main body of the paper, this applies when all 2-adic places of the totally real field are unramified in the CM field.

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¹⁵In particular, $O_{\tilde{B}}$ is indeed a maximal order since the extension F^t/\mathbb{Q}_p is unramified. Also, strictly speaking, $\tilde{\mathcal{L}} = \{\Lambda \otimes_{\mathbb{Z}_p} O_{F^t} \mid \Lambda \in \mathcal{L}\}$ is not a multichain of $O_{\tilde{B}}$ -lattices in the precise sense of [32, Def. 3.4], since \tilde{B} has more simple factors than B . But $\tilde{\mathcal{L}}$ gives rise to a notion of multichain of $O_{\tilde{B}} \otimes_{\mathbb{Z}_p} \mathcal{O}_T$ -modules of type $(\tilde{\mathcal{L}})$ in exactly the same way as [32, Def. 3.10], and this notion is the same as the one for the honest multichain of lattices generated by $\tilde{\mathcal{L}}$ in \tilde{V} .

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