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Solomon-Tukachinsky’s Versus Welschinger’s Open Gromov-Witten Invariants of Symplectic Six-Folds

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Our previous paper describes a geometric translation of the construction of open Gromov–Witten invariants by Solomon and Tukachinsky from a perspective of A_∞-algebras of differential forms. We now use this geometric perspective to show that these invariants reduce to Welschinger’s open Gromov–Witten invariants in dimension 6, inline with their and Tian’s expectations. As an immediate corollary, we obtain a translation of Solomon–Tukachinsky’s open WDVV equations into relations for Welschinger’s invariants.

1 Introduction

Fukaya–Oh–Ohta–Ono [5] associated A_∞-algebras with Lagrangian submanifolds of symplectic manifolds and extensively studied the theory of bounding chains in A_∞-algebras, from both the point of view of differential forms and geometrically. Numerical counts of J-holomorphic disks have since been extracted from the A_∞-algebras of [5] in some settings. In the present paper, we compare certain disk invariants of symplectic six-folds arising from [5] with the disk invariants constructed differently by Welschinger [17].

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Let (X,ω) be a compact symplectic six-fold and $Y\subset X$ be a compact Lagrangian submanifold. We denote by

$$\mu_\omega\colon H_2(X,Y;\mathbb{Z})\longrightarrow\mathbb{Z} \tag{1.1}$$

the Maslov index of the pair (X,Y). This pair is called positive if

$$\omega(\beta)>0,\ \ \mu_Y^\omega(\beta)\geqslant 0 \qquad\Longrightarrow\qquad \mu_Y^\omega(\beta)>0 \tag{1.2}$$

for every $\beta\in H_2(X,Y;\mathbb{Z})$ representable by a continuous map from $(\mathbb{D}^2,S^1)$. If this condition holds, then every non-constant J-holomorphic map from $(\mathbb{D}^2,S^1)$ to (X,Y) has a positive Maslov index for a generic ω-compatible almost complex structure J. The same happens along a generic path of almost complex structures if in addition Y is orientable (which implies that μ_Y^ω takes only even values). Suppose $\mathfrak{os}\equiv(\mathfrak{o},\mathfrak{s})$ is a relative OSpin-structure on Y, that is, a pair consisting of an orientation $\mathfrak{o}$ on Y and a relative Spin-structure $\mathfrak{s}$ on the oriented manifold $(Y,\mathfrak{o})$.

Let R be a commutative ring with unity 1. If the homomorphism

$$\iota_{Y*}\colon H_1(Y;R)\longrightarrow H_1(X;R) \tag{1.3}$$

induced by the inclusion $\iota_Y\colon Y\longrightarrow X$ is injective, the boundaries of maps from $(\mathbb{D}^2,S^1)$ to (X,Y) are homologically trivial in Y and thus have well-defined linking numbers with values in R. Under this injectivity assumption and the positivity condition (1.2), Welschinger [17] defines open Gromov–Witten (GW) invariants

$$\langle\cdot,\ldots,\cdot\rangle_{\beta,k}^{\omega,\mathfrak{os}}\colon\bigoplus_{l=0}^{\infty}H^{2*}(X,Y;R)^{\oplus l}\longrightarrow R,\quad \beta\in H_2(X,Y;\mathbb{Z})-\{0\},\ k\in\mathbb{Z}^+, \tag{1.4}$$

enumerating J-holomorphic multi-disks of total degree β weighted by R-valued linking numbers of their boundaries; see [17, Section 4.1] and (3.4). In “real settings” (when Y is a topological component of the fixed locus of anti-symplectic involution on X), these invariants encode the real Gromov–Witten invariants defined in [11, 15, 16]; see Remark 3.3. A special case of the setting of [17] is when Y is an R-homology S^3.

Based on A_∞-algebra considerations, Fukaya [4] defines a count $\langle\rangle_{\beta,0}^{\omega,\mathfrak{os}}\in\mathbb{Q}$ of J-holomorphic degree β disks in X with boundary in Y under the assumption that (X,ω) is a Calabi–Yau three-fold and the Maslov index μ_ω^Y of (X,Y) vanishes; this count may in general depend on the ω-compatible almost complex structure J on X. Motivated by [4], Solomon and Tukachinsky [12] further extend the theory of bounding chains of differential forms to define counts

$$\langle\cdot,\ldots,\cdot\rangle^{\omega,\mathfrak{os}}_{\beta,k}:\bigoplus_{l=0}^{\infty}H^{2*}(X,Y;\mathbb{R})^{\oplus l}\longrightarrow\mathbb{R},\quad \beta\in H_2(X,Y;\mathbb{Z}),\ k\in\mathbb{Z}^{\geqslant 0},\tag{1.5}$$

of J-holomorphic disks in symplectic manifolds X of arbitrary dimension $2n$ and show that bounding chains that are equivalent in a suitable sense define the same counts. They also prove that bounding chains exist and any two relevant bounding chains are equivalent if n is odd and Y is an $\mathbb{R}$-homology sphere. The resulting open Gromov–Witten invariants (1.5) of (X,Y) depend on the choice of a (relative) OSpin-structure $\mathfrak{os}$ on Y but are independent of all other auxiliary choices (such as J).

Well before [12], Tian expressed a belief that the construction of [17] is a geometric realization of the algebraic considerations behind the construction of [4]. The same sentiment is expressed in [12, Section 1.2.7]. The present paper uses the geometric interpretation of the construction of [12] described in [2], which is applicable over any commutative ring R with unity under the positivity condition (1.2) in the case of symplectic six-folds, to confirm Tian's and Solomon–Tukachinsky's expectations; see Theorem 1.2 below. Proposition 3.2, a kind of open divisor relation that trades real codimension 1 bordered insertions at boundary marked points for linking numbers of their boundaries with the boundaries of the disks, provides a transition from bounding chains to linking numbers. As an immediate consequence of Theorem 1.2, the basic structural properties and the Witten-Dijkgraaf-Verlinde-Verlinde (WDVV)-type relations for open Gromov–Witten invariants obtained in [13] yield similar properties and relations for Welschinger's invariants (1.4); see Corollaries 1.4 and 1.6.

The positivity condition (1.2) ensures that virtual techniques are not necessary for the purposes of this paper. However, the reasoning extends to general symplectic six-folds satisfying the injectivity condition (1.2) via the setup of Appendix A in [2] if $R\supset\mathbb{Q}$. This setup is compatible with standard virtual class approaches, such as in [6, 8–10]. As we only need evaluation maps from the (J,ν)-spaces to be pseudocycles, a full virtual cycle construction and gluing across all strata of the (J,ν)-spaces are not necessary.

Remark 1.1. In [17, Corollary 2], the insertions for the open invariants (1.4) are taken in $H^*(X;R)$ instead of $H^{2*}(X,Y;R)$. However, there is a minor omission in [17, Section 4.2.3]. For this reason, the insertions do need to be taken in $H^{2*}(X,Y;R)$ for the invariants to be well defined.

1.1 Comparison theorem

Let (X,ω) be a compact symplectic manifold of dimension $2n$, $Y \subset X$ be a compact Lagrangian submanifold, and $\mathfrak{os} \equiv (\mathfrak{o},\mathfrak{s})$ be a relative OSpin-structure on Y. Let $\alpha \equiv (\beta,K,L)$ be a tuple consisting of $\beta \in H_2(X,Y;\mathbb{Z})$, a generic finite subset K of Y, and a generic set L of pseudocycles $\Gamma_1,\dots,\Gamma_l$ in $X-Y$ representing Poincaré duals of some cohomology classes $\gamma_1,\dots,\gamma_l$ on (X,Y). For a generic ω-compatible almost complex structure J on X, a bounding chain $(\mathfrak{b}_{\alpha'})_{\alpha'\in\mathcal{C}_{\omega;\alpha}(Y)}$ on (α,J) in the geometric perspective of [2] is a tuple of bordered pseudocycles with certain properties specified in the $\dim X=6$ case by Definition 2.6 in the present paper. Such a tuple determines a pseudocycle $\mathfrak{bb}_\alpha$ into Y; see (2.11). It has a well-defined degree, and we set

$$\langle\gamma_1,\dots,\gamma_l\rangle^{\omega,\mathfrak{os}}_{\beta,|K|+1} \equiv \langle L\rangle^{\omega,\mathfrak{os}}_{\beta;K} \equiv \deg \mathfrak{bb}_\alpha; \tag{1.6}$$

if the dimension of $\mathfrak{bb}_\alpha$ is not n, the above numbers are defined to be 0. This degree may depend on the choices of K, L, J, and $(\mathfrak{b}_{\alpha'})_{\alpha'\in\mathcal{C}_{\omega;\alpha}(Y)}$. Bounding chains differing by a pseudo-isotopy of [2, Definition 2.2] determine the same degrees (1.6); see [2, Section 2.2]. This guarantees that the numbers in (1.6) depend only on ω, $\mathfrak{os}$, β, $|K|$, and $\gamma_1,\dots,\gamma_l$ if n is odd and Y is an R-homology sphere. However, the injectivity of (1.3) does *not* guarantee the existence of a pseudo-isotopy between a pair of bounding chains associated even with the same K, L, and J. Nevertheless, the following statement shows that the open invariants (1.6) are equivalent to Welschinger's invariants $\langle\dots\rangle^{\omega,\mathfrak{os}}_{\beta,|K|+1}$ and thus are independent of the choice of bounding chain.

Theorem 1.2. Suppose R is a commutative ring with unity, (X,ω) is a compact symplectic six-fold, $Y\subset X$ is a compact Lagrangian submanifold so that the positivity condition (1.2) holds and the homomorphism (1.3) is injective, and $\mathfrak{os}$ is a relative OSpin-structure on Y. Let $\beta\in H_2(X,Y;\mathbb{Z})$, K be a generic finite subset of Y, $L\equiv\{\Gamma_1,\dots,\Gamma_l\}$ be a generic set of even-dimensional pseudocycles to $X-Y$, $\alpha\equiv(\beta,K,L)$, and J be a generic ω-compatible almost complex structure on X.

(W1) There exists a bounding chain $(\mathfrak{b}_{\alpha'})_{\alpha'\in\mathcal{C}_{\omega;\alpha}(Y)}$ on (α,J).

(W2) If $\gamma_1,\dots,\gamma_l\in H^{2*}(X,Y;R)$ are the Poincaré duals of $\Gamma_1,\dots,\Gamma_l$, then

$$\langle\gamma_1,\dots,\gamma_l\rangle^{\omega,\mathfrak{os}}_{\beta,|K|+1} = (-1)^{|K|+1}\langle L\rangle^{\omega,\mathfrak{os}}_{\beta;K}$$

for any bounding chain $(\mathfrak{b}_{\alpha'})_{\alpha'\in\mathcal{C}_{\omega;\alpha}(Y)}$ on (α,J).

For each $\alpha' \in \mathcal{C}_{\omega;\alpha}(Y)$ satisfying the dimension condition in Definition 2.6, the right-hand side of (2.9) consists of the boundaries of some maps from $(\mathbb{D}^2, S^1)$ to (X,Y); see Proposition 3.1. By the injectivity of (1.3), we can thus choose a bordered pseudocycle $\mathfrak{b}_{\alpha'}$ to Y satisfying (2.9). This implies that a bounding chain $(\mathfrak{b}_{\alpha'})_{\alpha' \in \mathcal{C}_{\omega;\alpha}(Y)}$ on (α, J) can be constructed by induction on the partially ordered set $\mathcal{C}_{\omega;\alpha}(Y)$ and establishes Theorem 1.2(W1).

A key ingredient in the proof of Proposition 3.1 is the open divisor relation of Proposition 3.2, which replaces real codimension 1 bordered insertions at boundary marked points with linking numbers. This relation is also combined with Proposition 3.1 to obtain the identification of the disk counts (1.6) with Welschinger's invariants (1.4) stated in Theorem 1.2. This identification in turn implies that the numbers (1.6) depend only on ω, $\mathfrak{os}$, β, $|K|$, and $\gamma_1, \ldots, \gamma_l$.

We denote by

$$q_Y : H_2(X;\mathbb{Z}) \longrightarrow H_2(X,Y;\mathbb{Z}) \tag{1.7}$$

the natural homomorphism. A bounding chain $(\mathfrak{b}_\alpha)_{\alpha \in \mathcal{C}_{\omega;\alpha}(Y)}$ as in Definition 2.6 can also be used to define a count of J-holomorphic degree β disks through $|K|$ (rather than $|K|+1$) points in Y if

$$k \equiv |K| \neq 0 \qquad \text{or} \qquad \beta \notin \operatorname{Im}\big(q_Y : H_2(X;\mathbb{Z}) \longrightarrow H_2(X,Y;\mathbb{Z})\big)\,; \tag{1.8}$$

see (2.12). The definition of the invariants (1.4) in [17] immediately extends to counts of multi-disks with $k=0$ points in Y if β satisfies the 2nd condition in (1.8). The proof of Theorem 1.2 can be slightly modified to cover this case. It can also be readily extended to the open invariants with insertions from $H_2(Y;R)$, which are defined in [17].

Remark 1.3. The definition (1.6) of open invariants in [2] is a geometric translation of the construction of the invariants (1.5) in [12], in the sense that we use submanifolds instead of differential forms. The two invariants are not explicitly shown to be the same, but we expect this to be straightforward.

It is immediate from (3.4) that Welschinger's open invariants are symmetric linear functionals that satisfy an open divisor relation:

$$\langle \gamma, \gamma_1, \ldots, \gamma_l \rangle^{\omega,\mathfrak{os}}_{\beta,k} = \langle \gamma, \beta \rangle \, \langle \gamma_1, \ldots, \gamma_l \rangle^{\omega,\mathfrak{os}}_{\beta,k} \qquad \forall\, \gamma \in H^2(X,Y;R).$$

Combining Theorem 1.2 above with the last three statements of [2, Theorem 2.9], which in turn are analogues of [3, Proposition 2.1] and [13, Corollary 1.5], we obtain below additional properties of Welschinger's open invariants.

Suppose Y is connected. The kernel of the homomorphism

$$H_2(X-Y;R) \longrightarrow H_2(X;R)$$

is then generated by the homology class $[S(\mathcal{N}_y Y)]_{X-Y}$ of a unit sphere $S(\mathcal{N}_y Y)$ in the fiber of $\mathcal{N}Y$ over any $y \in Y$. We orient $S(\mathcal{N}_y Y)$ as in [3, Section 2.5] and denote the image of $[S(\mathcal{N}_y Y)]_{X-Y}$ under the Lefschetz duality isomorphism

$$\mathrm{PD}_{X,Y}\colon H_2\left(X-Y;R\right) \xrightarrow{\approx} H^4\left(X,Y;R\right)$$

by $\eta^{\circ}_{X,Y}$. For $B \in H_2(X;\mathbb{Z})$, let

$$\langle\cdot,\ldots,\cdot\rangle^{\omega}_{B}\colon \bigoplus_{l=0}^{\infty} H^*(X;R)^{\oplus l} \longrightarrow R$$

be the standard GW-invariants of (X,ω). We denote by $[Y]_X$ the homology class on X determined by Y and by $w_2(\mathfrak{os})$ the 2nd Stiefel–Whitney class of $\mathfrak{os}$ ($w_2(\mathfrak{os}) = w_2(V)$ in the notation of [5, Definition 8.1.2]).

Corollary 1.4. Let (X,ω,Y), $\mathfrak{os}$, β, k, and $\gamma_1,\ldots,\gamma_l$ be as in Theorem 1.2. If the pair (k,β) satisfies (1.8) and Y is connected, Welschinger's open invariants (1.4) satisfy the following properties.

(WGW1) $\langle \gamma^{\circ}_{X,Y},\gamma_1,\ldots,\gamma_l\rangle^{\omega,\mathfrak{os}}_{\beta,k} = -\langle \gamma_1,\ldots,\gamma_l\rangle^{\omega,\mathfrak{os}}_{\beta,k+1}$.

(WGW2) If $k=1$ and $\gamma_0 \in H^3(X;R)$,

$$\langle\gamma_0,[Y]_X\rangle\,\langle \gamma_1,\ldots,\gamma_l\rangle^{\omega,\mathfrak{os}}_{\beta,k} = -\sum_{B\in q_Y^{-1}(\beta)}(-1)^{\langle w_2(\mathfrak{os}),B\rangle}\left\langle \mathrm{PD}_X\left([Y]_X\right),\gamma_0,\gamma_1|_X,\ldots,\gamma_l|_X\right\rangle^{\omega}_{B}.$$

(WGW3) If $[Y]_X \neq 0$ and $k \geqslant 2$, then $\langle \gamma_1,\ldots,\gamma_l\rangle^{\omega,\mathfrak{os}}_{\beta,k} = 0$.

1.2 WDVV-type relations

Let R, (X,ω,Y) and $\mathfrak{os}$ be as in Theorem 1.2. We now use this theorem to translate the WDVV-type relations for the open GW-invariants (1.5) obtained in [13] to relations for

Welschinger's open invariants (1.4) under the assumptions that R is a field and the homomorphism

$$\iota_{Y*}: H_2(Y;R) \longrightarrow H_2(X;R) \tag{1.9}$$

induced by the inclusion $\iota_Y: Y \longrightarrow X$ is trivial. For $k \in \mathbb{Z}^{\geqslant 0}$, define

$$[k] = \{1, \ldots, k\}.$$

Under the assumption (1.8), we extend the invariants (1.4) to the degree $\beta = 0$ and inputs from $H^0(X;R)$ by

$$\langle \gamma_1, \ldots, \gamma_l \rangle_{0,k}^{\omega,\mathfrak{os}} = \begin{cases} -\langle \gamma_1, \mathrm{pt} \rangle, & \text{if } (k,l) = (1,1); \\ 0, & \text{otherwise;} \end{cases}$$

$$\langle \gamma_1, \gamma_2, \ldots, \gamma_l, 1 \rangle_{\beta,k}^{\omega,\mathfrak{os}} = \begin{cases} -1, & \text{if } (\beta,k,l) = (0,1,0); \\ 0, & \text{otherwise.} \end{cases}$$

In light of Theorem 1.2 and the symmetry of the open invariants (1.5), these extensions are consistent with (OGW2) and (OGW3) in [2, Theorem 2.9]. If $[Y]_X = 0$ and γ is a two-dimensional pseudocycle to $X - Y$ bounding a pseudocycle $\mathfrak{b}$ transverse to Y, we define

$$\mathrm{lk}_{\mathfrak{os}}(\gamma) \equiv \left|\mathfrak{b} \times_{\mathrm{fb}} \iota_Y\right|^{\pm};$$

see Section 2.1 for the sign conventions for fiber products. This linking number of γ and Y with the orientation determined by the relative OSpin-structure $\mathfrak{os}$ does not depend on the choice of $\mathfrak{b}$. We set $\mathrm{lk}_{\mathfrak{os}}(\gamma) = 0$ if γ is not a two-dimensional pseudocycle.

For $l \in \mathbb{Z}^{\geqslant 0}$, $B \in H_2(X;\mathbb{Z})$, and an ω-tame almost complex structure J, we denote by $\mathfrak{M}^{\mathbb{C}}_{\{0\}\sqcup[l]}(B;J)$ the moduli space of stable J-holomorphic degree B maps from $\mathbb{P}^1$ into X with marked points indexed by the set $\{0\}\sqcup[l]$. It carries a canonical orientation. For each $i \in \{0\}\sqcup[l]$, let

$$\mathrm{ev}_i: \mathfrak{M}^{\mathbb{C}}_{\{0\}\sqcup[l]}(B;J) \longrightarrow X$$

be the evaluation morphism at the i-th marked point. If in addition $\Gamma_1, \ldots, \Gamma_l$ are maps to X, let

$$\mathfrak{M}^{\mathbb{C}}_{\{0\}\sqcup[l]}(B;J)\times_{\mathrm{fb}}\big((i,\Gamma_i)_{i\in[l]}\big) \equiv \mathfrak{M}^{\mathbb{C}}_{\{0\}\sqcup[l]}(B;J)_{(\mathrm{ev}_1,\ldots,\mathrm{ev}_l)}\times_{\Gamma_1\times\ldots\times\Gamma_l}\big((\mathrm{dom}\,\Gamma_1)\times\ldots\times(\mathrm{dom}\,\Gamma_l)\big)\,.$$

If J is generic and $\Gamma_1, \ldots, \Gamma_l$ are pseudocycles in general position, then

$$f^{\mathbb{C}}_{B,(\Gamma_i)_{i\in[l]}} \equiv \Big(\mathrm{ev}_0\colon \mathfrak{M}^{\mathbb{C}}_{\{0\}\sqcup[l]}(B;J)\times_{\mathrm{fb}}\big((i,\Gamma_i)_{i\in[l]}\big) \longrightarrow X\Big)$$

is a pseudocycle of dimension

$$\dim f^{\mathbb{C}}_{B,(\Gamma_i)_{i\in[l]}} = \mu_\omega\left(q_Y(B)\right) - \sum_{i=1}^{l}\big(\mathrm{codim}\,\Gamma_i - 2\big) + 2$$

transverse to Y.

Since $\dim Y = 3$, the cohomology long exact sequence for the pair (X, Y) implies that the restriction homomorphism

$$H^p(X,Y;R) \longrightarrow H^p(X;R) \tag{1.10}$$

is surjective for $p=4,6$. Since R is a field and the homomorphism (1.9) is trivial, (1.10) is also surjective for $p=2$. Let

$$\gamma_1^\bigstar \equiv 1 \in H^0(X;R) \qquad \text{and} \qquad \gamma_2^\bigstar, \ldots, \gamma_N^\bigstar \in H^{2*}(X,Y;R)$$

be such that $\gamma_1^\bigstar, \gamma_2^\bigstar|_X, \ldots, \gamma_N^\bigstar|_X$ is a basis for $H^{2*}(X;R)$, $(g_{ij})_{i,j}$ be the $N\times N$-matrix given by

$$g_{ij} = \Big\langle \gamma_i^\bigstar \gamma_j^\bigstar, [X]\Big\rangle$$

and $(g^{ij})_{i,j}$ be its inverse. Let $\Gamma_1^\bigstar = \mathrm{id}_X$ and $\Gamma_2^\bigstar, \ldots, \Gamma_N^\bigstar$ be pseudocycles to $X - Y$ representing the Poincaré duals of $\gamma_2^\bigstar, \ldots, \gamma_N^\bigstar$.

Remark 1.5. $\sum\limits_{i,j\in[N]} g^{ij}\gamma_i^\bigstar \times \gamma_j^\bigstar \in H^{2n}(X\times X;R)$ is the Poincaré dual to the diagonal modulo $H^{\mathrm{odd}}(X;R)\otimes H^{\mathrm{odd}}(X;R)$. The corresponding terms in Corollary 1.6 with odd-degree insertions $\gamma_i^\bigstar$ would vanish for dimensional reasons and thus do not need to be considered.

For the purpose of WDVV-type equations for the invariants (1.4), we extend the signed counts (3.4) to the pairs (k,β) not satisfying (1.8), that is, $k=0$ and $\beta\in H_2(X,Y;\mathbb{Z})$ is in the image of the homomorphism q_Y in (1.7), as follows. Let $\gamma_1,\ldots,\gamma_l$ be elements of $\{1\}\sqcup H^{2*}(X,Y;R)$. If $[Y]_X\neq 0$, we define

$$\langle \gamma_1,\ldots,\gamma_l\rangle^{\omega,\mathfrak{os}}_{\beta,0}=0.$$

Suppose next that $[Y]_X=0$. Let $\Gamma_1,\ldots,\Gamma_l$ be generic pseudocycles to X so that $\Gamma_i=\mathrm{id}_X$ if $\gamma_i=1$ and Γ_i is a pseudocycle into $X-Y$ representing the Poincaré dual of γ_i otherwise. For $B\in H_2(X;\mathbb{Z})$, let $(\lambda^j_{B,(\gamma_i)_{i\in[l]}})_{j\in[N]}\in R^N$ be such that

$$\Big[f^{\mathbb{C}}_{B,(\Gamma_i)_{i\in[l]}}\Big]=\sum_{j=1}^{N}\lambda^j_{B,(\gamma_i)_{i\in[l]}}\mathrm{PD}_X\left(\gamma_j^{\star}|_X\right)\in H_*(X;R);$$

the tuple $(\lambda^j_{B,(\gamma_i)_{i\in[l]}})_{j\in[N]}$ depends only on B, $\gamma_1,\ldots,\gamma_l$, and $\gamma_2^{\star},\ldots,\gamma_N^{\star}$. Define

$$\langle \gamma_1,\ldots,\gamma_l\rangle^{\omega,\mathfrak{os}}_{\beta,0}=\text{RHS of (3.4)}+\sum_{B\in q_Y^{-1}(\beta)}(-1)^{\langle w_2(\mathfrak{os}),B\rangle}\mathrm{lk}_{\mathfrak{os}}\left(f^{\mathbb{C}}_{B,(\Gamma_i)_{i\in[l]}}-\sum_{j=1}^{N}\lambda^j_{B,(\gamma_i)_{i\in[l]}}\Gamma_j^{\star}\right)$$

in this case. This number depends on the span of the elements $\gamma_2^{\star},\ldots,\gamma_N^{\star}$ in $H^{2*}(X,Y;R)$, but not on the choice of pseudocycles $\Gamma_1,\ldots,\Gamma_l$ and $\Gamma_2^{\star},\ldots,\Gamma_N^{\star}$ representing the Poincaré duals of $\gamma_1,\ldots,\gamma_l$ and $\gamma_2^{\star},\ldots,\gamma_N^{\star}$, respectively. For example,

$$\langle \gamma_1,\gamma_2\rangle^{\omega,\mathfrak{os}}_{0,0}=\mathrm{lk}_{\mathfrak{os}}\left(\Gamma_1\cap\Gamma_2-\sum_{j=1}^{N}\lambda^j_{\gamma_1\gamma_2}\Gamma_j^{\star}\right),\quad\text{where }\ \gamma_1\gamma_2\equiv\sum_{j=1}^{N}\lambda^j_{\gamma_1\gamma_2}\gamma_j^{\star}|_X\in H^*(X;\mathbb{Q}).$$

Let $\gamma\equiv(\gamma_1,\ldots,\gamma_l)$ be a tuple of elements of $\{1\}\sqcup H^{2*}(X,Y;R)$. For $I\subset\{1,2,\ldots,l\}$, we denote by γ_I the $|I|$-tuple consisting of the entries of γ indexed by I. If in addition $\beta\in H_2(X,Y;\mathbb{Z})$, define

$$k_\beta(\gamma_I)\equiv\frac{1}{2}\left(\mu_\omega(\beta)-\sum_{i\in I}(\deg\gamma_i-2)\right),\quad \langle\gamma_I\rangle^{\omega,\mathfrak{os}}_{\beta}=\begin{cases}\langle\gamma_I\rangle^{\omega,\mathfrak{os}}_{\beta,k_\beta(\gamma_I)}, & \text{if } k_\beta(\gamma_I)\geqslant 0;\\ 0, & \text{otherwise.}\end{cases}$$

For $i,j=1,2,\ldots,l$, we define

$$\mathcal{P}(l)=\{(I,J):\{1,2,\ldots,l\}=I\sqcup J,\ 1\in I\},\quad \mathcal{P}_{i;}(l)=\{(I,J)\in\mathcal{P}(l): i\in I\},$$
$$\mathcal{P}_{;j}(l)=\{(I,J)\in\mathcal{P}(l): j\in J\},\qquad \mathcal{P}_{i;j}(l)=\mathcal{P}_{i;}(l)\cap\mathcal{P}_{;j}(l).$$

For $\beta\in H_2(X,Y;\mathbb{Z})$, let

$$\mathcal{P}_{\mathbb{C}}(\beta)=\left\{(\beta',B)\in H_2(X,Y;\mathbb{Z})\oplus H_2(X;\mathbb{Z}):\ \beta'+q_Y(B)=\beta\right\},$$
$$\mathcal{P}_{\mathbb{R}}(\beta)=\left\{(\beta_1,\beta_2)\in H_2(X,Y;\mathbb{Z})\oplus H_2(X,Y;\mathbb{Z}):\ \beta_1+\beta_2=\beta\right\}.$$

For a relative OSpin-structure $\mathfrak{os}$ on (X,Y) and $B\in H_2(X;\mathbb{Z})$, we define a twisted version of the standard Gromov–Witten invariants

$$\langle\cdot,\ldots,\cdot\rangle_B^{\omega,\mathfrak{os}}\equiv(-1)^{w_2(\mathfrak{os})}\langle\cdot,\ldots,\cdot\rangle_B^{\omega}.$$

Combining Theorem 1.2 above with [2, Theorem 2.10], we obtain relations between Welschinger's open invariants (1.4) as stated below.

Corollary 1.6. Let R be a field and (X,ω,Y), $\mathfrak{os}$, β, and $\gamma\equiv(\gamma_1,\ldots,\gamma_l)$ be as in Theorem 1.2 with

$$k\equiv\frac{1}{2}\left(\mu_\omega(\beta)-\sum_{i=1}^{l}\left(\deg\mu_i-2\right)\right)-1\geqslant 0.$$

Suppose in addition that the homomorphism (1.9) is trivial.

(OWDVV1) If $l\geqslant 2$ and $k\geqslant 1$, then

$$\sum_{\substack{(\beta',B)\in\mathcal{P}_{\mathbb{C}}(\beta)\\(I,J)\in\mathcal{P}_{2;}(l)}}\sum_{i,j\in[N]}\Big\langle\gamma_I|_X,\gamma_i^{\star}|_X\Big\rangle_B^{\omega,\mathfrak{os}}g^{ij}\left\langle\gamma_j^{\star},\gamma_J\right\rangle_{\beta'}^{\omega,\mathfrak{os}}-\sum_{\substack{(\beta_1,\beta_2)\in\mathcal{P}_{\mathbb{R}}(\beta)\\(I,J)\in\mathcal{P}_{2;}(l)}}\binom{k-1}{k_{\beta_1}(\gamma_I)}\left\langle\gamma_I\right\rangle_{\beta_1}^{\omega,\mathfrak{os}}\left\langle\gamma_J\right\rangle_{\beta_2}^{\omega,\mathfrak{os}}$$
$$=-\sum_{\substack{(\beta_1,\beta_2)\in\mathcal{P}_{\mathbb{R}}(\beta)\\(I,J)\in\mathcal{P}_{;2}(l)}}\binom{k-1}{k_{\beta_1}(\gamma_I)-1}\left\langle\gamma_I\right\rangle_{\beta_1}^{\omega,\mathfrak{os}}\left\langle\gamma_J\right\rangle_{\beta_2}^{\omega,\mathfrak{os}}.$$

(OWDVV2) If $l \geqslant 3$, then

$$\sum_{\substack{(\beta',B)\in\mathcal{P}_{\mathbb{C}}(\beta)\\(I,J)\in\mathcal{P}_{2;3}(l)}} \sum_{i,j\in[N]} \big\langle \gamma_I|_X, \gamma_i^{\star}|_X \big\rangle_B^{\omega,os} g^{ij} \left\langle \gamma_j^{\star}, \gamma_J \right\rangle_{\beta'}^{\omega,os} - \sum_{\substack{(\beta_1,\beta_2)\in\mathcal{P}_{\mathbb{R}}(\beta)\\(I,J)\in\mathcal{P}_{2;3}(l)}} \binom{k}{k_{\beta_1}(\gamma_I)} \left\langle \gamma_I \right\rangle_{\beta_1}^{\omega,os} \left\langle \gamma_J \right\rangle_{\beta_2}^{\omega,os}$$

$$= \sum_{\substack{(\beta',B)\in\mathcal{P}_{\mathbb{C}}(\beta)\\(I,J)\in\mathcal{P}_{3;2}(l)}} \sum_{i,j\in[N]} \big\langle \gamma_I|_X, \gamma_i^{\star}|_X \big\rangle_B^{\omega,os} g^{ij} \left\langle \gamma_j^{\star}, \gamma_J \right\rangle_{\beta'}^{\omega,os} - \sum_{\substack{(\beta_1,\beta_2)\in\mathcal{P}_{\mathbb{R}}(\beta)\\(I,J)\in\mathcal{P}_{3;2}(l)}} \binom{k}{k_{\beta_1}(\gamma_I)} \left\langle \gamma_I \right\rangle_{\beta_1}^{\omega,os} \left\langle \gamma_J \right\rangle_{\beta_2}^{\omega,os}.$$

Remark 1.7. The open WDVV equations in [13] are stated for $R = \mathbb{R}$ only. However, a translation of its proof into the geometric language of [2] applies with any field R without any changes.

2 Preliminaries

Section 2.1 recalls the orientation conventions for fiber products and some of their properties from [2, Section 5.1]; their proofs are included at the referee's suggestion. The combinatorial objects needed for the geometric presentation of the open invariants of [12] in symplectic six-folds are gathered in Section 2.2. We describe the relevant moduli spaces of stable disk maps and specify their orientations in Section 2.3. Section 2.4 specializes the geometric definition of bounding chain from [2] to symplectic six-folds and uses it to define counts J-holomorphic disks.

2.1 Fiber products

We say a short exact sequence of oriented vector spaces

$$0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0$$

is orientation compatible if for an oriented basis $(v_1', \ldots, v_m')$ of V', an oriented basis $(v_1'', \ldots, v_n'')$ of V'', and a splitting $j\colon V'' \longrightarrow V$, $(v_1', \ldots, v_m', j(v_1''), \ldots, j(v_n''))$ is an oriented basis of V. We say it has sign $(-1)^{\epsilon}$ if it becomes orientation compatible after twisting the orientation of V by $(-1)^{\epsilon}$. We use the analogous terminology for short exact sequences of Fredholm operators with respect to orientations of their determinants; see [19, Section 2].

Let M be an oriented manifold with boundary ∂M. We orient the normal bundle $\mathcal{N}$ to ∂M by the outer normal direction and orient ∂M so that the short exact sequence

$$0 \longrightarrow T_p\partial M \longrightarrow T_pM \longrightarrow \mathcal{N} \longrightarrow 0 \tag{2.1}$$

is orientation compatible at each point $p \in \partial M$. We refer to this orientation of ∂M as the boundary orientation.

We orient $M \times M$ by the usual product orientation and the diagonal $\Delta_M \subset M \times M$ by the diffeomorphism

$$M \longrightarrow \Delta_M, \qquad p \longrightarrow (p,p).$$

We orient the normal bundle $\mathcal{N}\Delta_M$ of Δ_M so that the short exact sequence

$$0 \longrightarrow T_{(p,p)}\Delta_M \longrightarrow T_{(p,p)}(M \times M) \longrightarrow \mathcal{N}\Delta_M|_{(p,p)} \longrightarrow 0$$

is orientation compatible for each point $p \in M$. Thus, the isomorphism

$$\mathcal{N}\Delta_M|_{(p,p)} \longrightarrow T_pM, \qquad [v,w] \longrightarrow w-v,$$

respects the orientations.

For maps $f\colon M \longrightarrow X$ and $g\colon \Gamma \longrightarrow X$, we denote by

$$f \times_{\mathrm{fb}} g \equiv M_f \times_g \Gamma \equiv \{(p,q) \in M \times \Gamma\colon f(p) = g(q)\}$$

their fiber product. If M, Γ, and X are oriented manifolds (M, Γ possibly with boundary) and $f, f|_{\partial M}$ are transverse to $g, g|_{\partial\Gamma}$, we orient $M_f \times_g \Gamma$ so that the short exact sequence

$$0 \longrightarrow T_{(p,q)}(M_f \times_g \Gamma) \longrightarrow T_{(p,q)}(M \times \Gamma) \xrightarrow{[d_pf, d_qg]} \mathcal{N}\Delta_X|_{(f(p),g(q))} \longrightarrow 0$$

is orientation compatible for every $(p,q) \in M_f \times_g \Gamma$. The exact sequence

$$0 \longrightarrow T_{(p,q)}(M_f \times_g \Gamma) \longrightarrow T_{(p,q)}(M \times \Gamma) \xrightarrow{d_qg - d_pf} T_{f(p)}X \longrightarrow 0 \tag{2.2}$$

is then orientation compatible as well. We refer to this orientation of $M_f \times_g \Gamma$ as the fiber product orientation. The next lemma is straightforward.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_{11} \arrow[r] \arrow[d] & A_{12} \arrow[r] \arrow[d] & A_{13} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_{21} \arrow[r] \arrow[d] & A_{22} \arrow[r] \arrow[d] & A_{23} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_{31} \arrow[r] \arrow[d] & A_{32} \arrow[r] \arrow[d] & A_{33} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Fig. 1. Commutative square of exact sequences of vector spaces for the statement of Lemma 2.1.

Lemma 2.1. Suppose A_{ij} with $i,j \in [3]$ are oriented finite-dimensional vector spaces, the rows and columns in the diagram in Figure 1 are exact sequences of vector-space homomorphisms, and this diagram commutes. The total number of rows and columns in this diagram that (do not) respect the orientations is congruent to $\dim(A_{13})\dim(A_{31})$ mod 2.

Lemma 2.2. With the assumptions as above,

$$\partial(M_f \times_g \Gamma) = (-1)^{\dim X}\left((-1)^{\dim\Gamma}(\partial M)_f \times_g \Gamma \sqcup M_f \times_g \partial\Gamma\right).$$

Proof. We denote the normal bundles of ∂M in M and of $\partial\Gamma$ in Γ by $\mathcal{N}\partial M$ and $\mathcal{N}\partial\Gamma$, respectively.

Suppose $(p,q) \in \partial M_f \times_g \Gamma$. The exact sequences (2.1) and (2.2) then induce the 1st commutative square of exact sequences in Figure 2. The top (resp. middle) row in this diagram respects the fiber product orientation on $\partial M_f \times_g \Gamma$ (resp. $M_f \times_g \Gamma$), the boundary orientation on ∂M (resp. given orientation on M), and the given orientations on Γ and X. The left column respects the boundary orientation of $T_{(p,q)}(\partial M_f \times_g \Gamma)$, the fiber product orientation of $T_{(p,q)}(M_f \times_g \Gamma)$, and the orientation of $\mathcal{N}_p\partial M$. The middle column respects the boundary orientation of $T_{(p,q)}(M_f \times_g \Gamma)$, the product orientation of $T_{(p,q)}(M \times \Gamma)$, and the orientation of $\mathcal{N}_p\partial M$ if and only if $\dim\Gamma \in 2\mathbb{Z}$. Along with Lemma 2.1, this implies

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & T_{(p,q)}(\partial M_f \times_g \Gamma) \arrow[r] \arrow[d] & T_{(p,q)}(\partial M \times \Gamma) \arrow[r, "d_q g - d_p f"] \arrow[d] & T_{f(p)}X \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & T_{(p,q)}(M_f \times_g \Gamma) \arrow[r] \arrow[d] & T_{(p,q)}(M \times \Gamma) \arrow[r, "d_q g - d_p f"] \arrow[d] & T_{f(p)}X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{N}_p \partial M \arrow[r] \arrow[d, "\mathrm{id}"] & \mathcal{N}_p \partial M \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0 & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & T_{(p,q)}(M_f \times_g \partial\Gamma) \arrow[r] \arrow[d] & T_{(p,q)}(M \times \partial\Gamma) \arrow[r, "d_q g - d_p f"] \arrow[d] & T_{f(p)}X \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & T_{(p,q)}(M_f \times_g \Gamma) \arrow[r] \arrow[d] & T_{(p,q)}(M \times \Gamma) \arrow[r, "d_q g - d_p f"] \arrow[d] & T_{f(p)}X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{N}_q \partial\Gamma \arrow[r, "\mathrm{id}"] \arrow[d] & \mathcal{N}_q \partial\Gamma \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0 & &
\end{tikzcd}$$

Fig. 2. Commutative squares of exact sequences for the proofs of Lemma 2.2.

that the boundary and fiber product orientations of

$$T_{(p,q)}\left(\partial(M_f \times_g \Gamma)\right) = T_{(p,q)}\left(\partial M_f \times_g \Gamma\right)$$

are the same if and only if $\dim X + \dim \Gamma \in 2\mathbb{Z}$.

Suppose $(p,q) \in M_f \times_g \partial\Gamma$. The exact sequences (2.1) and (2.2) then induce the 2nd commutative square of exact sequences in Figure 2. The same statements as in the previous paragraph apply, except now the middle column always respects the relevant orientations. Along with Lemma 2.1, this implies that the boundary and fiber product orientations of

$$T_{(p,q)}\left(\partial(M_f\times_g\Gamma)\right)=T_{(p,q)}\left(M_f\times_g\partial\Gamma\right)$$

are the same if and only if $\dim X\in 2\mathbb{Z}$. ■

For a diffeomorphism $\sigma : M \longrightarrow M'$ between oriented manifolds, we define $\operatorname{sgn}\sigma=1$ if σ is everywhere orientation preserving and $\operatorname{sgn}\sigma=-1$ if σ is everywhere orientation reversing.

Lemma 2.3. Suppose $\sigma_M,\sigma_\Gamma,\sigma_X$ are self-diffeomorphisms of M,Γ,X, respectively, with well-defined signs. If the diagram

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\sigma_M"'] & X \arrow[d, "\sigma_X"'] & \Gamma \arrow[l, "g"'] \arrow[d, "\sigma_\Gamma"] \\ M \arrow[r, "f"] & X & \Gamma \arrow[l, "g"'] \end{tikzcd}$$

commutes, then the sign of the diffeomorphism

$$M_f\times_g\Gamma\longrightarrow M_f\times_g\Gamma,\qquad (p,q)\longrightarrow\left(\sigma_M(p),\sigma_\Gamma(q)\right),\tag{2.3}$$

is $(\operatorname{sgn}\sigma_M)(\operatorname{sgn}\sigma_\Gamma)(\operatorname{sgn}\sigma_X)$.

Proof. Let $(p,q)\in M_f\times_g\Gamma$. The diagram

$$\begin{tikzcd} 0 \arrow[r] & T_{(p,q)}(M_f\times_g\Gamma) \arrow[r] \arrow[d, "(2.3)"'] & T_{(p,q)}(M\times\Gamma) \arrow[r, "d_qg-d_pf"] \arrow[d, "d_p\sigma_M\times d_q\sigma_\Gamma"] & T_{f(p)}X \arrow[r] \arrow[d, "d_{f(p)}\sigma_X"] & 0 \\ 0 \arrow[r] & T_{(p,q)}(M_f\times_g\Gamma) \arrow[r] & T_{(p,q)}(M\times\Gamma) \arrow[r, "d_qg-d_pf"] & T_{f(p)}X \arrow[r] & 0 \end{tikzcd}$$

of vector space homomorphisms then commutes. Since the signs of the isomorphisms given by the middle and right vertical arrows in the above diagram are $(\operatorname{sgn}\sigma_M)(\operatorname{sgn}\sigma_\Gamma)$ and $\operatorname{sgn}\sigma_X$, respectively, the sign of the isomorphism given by the left vertical arrow is $(\operatorname{sgn}\sigma_M)(\operatorname{sgn}\sigma_\Gamma)(\operatorname{sgn}\sigma_X)$. ■

Let M,Γ,X and f,g be as above Lemma 2.2. Suppose in addition that $e\colon M\longrightarrow Y$ and $h\colon C\longrightarrow Y$. Let $e'\colon M_f\times_g\Gamma\longrightarrow Y$ be the composition of the projection $M_f\times_g\Gamma\longrightarrow M$

$$
\begin{tikzcd}
 & \Gamma \arrow[d, "g"] & \Gamma\times C \arrow[l] \arrow[r] \arrow[d] & C \arrow[d, "h"'] \\
M_f\times_g\Gamma \arrow[ur] \arrow[dr] \arrow[rrr, bend right=60, "e'"] & X & X\times Y \arrow[l] \arrow[r] & Y \\
 & M \arrow[u, "f"'] \arrow[ur] \arrow[urr, "e"] & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & T_{((p,q),r)}((M_f\times_g\Gamma)_{e'}\times_h C) \arrow[r] \arrow[d, "(2.4)"'] & T_{((p,q),r)}((M_f\times_g\Gamma)\times C) \arrow[r, "d_rh-d_pe"] \arrow[d] & T_{e(p)}Y \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T_{(p,(q,r))}(M_{(f,e)}\times_{g\times h}(\Gamma\times C)) \arrow[r] \arrow[d] & T_{(p,(q,r))}(M\times(\Gamma\times C)) \arrow[r] \arrow[d, "d_qg-d_pf"] & T_{(f(p),e(p))}(X\times Y) \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & T_{f(p)}X \arrow[r, "\mathrm{id}"] \arrow[d] & T_{f(p)}X \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}
$$

Fig. 3. The maps of Lemma 2.4 and a commutative square of exact sequences for its proof.

with e; see Figure 3. There is then a natural bijection

$$\Big(M_f\times_g\Gamma\Big)_{e'}\times_h C\approx M_{(f,e)}\times_{g\times h}(\Gamma\times C)\,. \tag{2.4}$$

If C, Y are oriented manifolds and all relevant maps are transverse, then both sides of this bijection inherit fiber product orientations. For any map $h: M\longrightarrow Z$ between manifolds, let

$$\operatorname{codim} h=\dim Z-\dim M\,.$$

Lemma 2.4. The diffeomorphism (2.4) has sign $(-1)^{(\dim X)(\operatorname{codim} h)}$ with respect to the fiber product orientations on the two sides.

Proof. Let $((p,q),r)\in(M_f\times_g\Gamma)_{e'}\times_h C$. The exact sequence (2.2) then induces the commutative square of exact sequences in Figure 3. The top and middle rows in this diagram respect the fiber product and product orientations of the relevant spaces. The middle and right columns have signs $(-1)^{(\dim X)(\dim C)}$ and $(-1)^{(\dim X)(\dim Y)}$, respectively. Along with Lemma 2.1, this implies the claim. ■

2.2 Combinatorial notation

Let (X,ω) be a compact symplectic six-fold, $Y\subset X$ be a compact Lagrangian submanifold,

$$H_2^\omega(X,Y)=\big\{\beta\in H_2(X,Y;\mathbb{Z}):\omega(\beta)>0 \text{ or } \beta=0\big\},$$

and $\mathcal{J}_\omega$ be the space of ω-compatible almost complex structures on X. Let

$$\begin{aligned}\mathcal{C}_\omega(Y)=\big\{(\beta,K,L):\beta\in H_2^\omega(X,Y),\ &K \text{ is a finite collection of points in} Y,\\ &L \text{ is a finite collection of pseudocycles in} X-Y \text{with} R\text{-coefficients},\\ &(\beta,K,L)\neq(0,\emptyset,\emptyset)\big\}.\end{aligned}$$

This collection has a natural partial order:

$$(\beta',K',L')\preceq(\beta,K,L)\qquad\text{if}\qquad \beta-\beta'\in H_2^\omega(X,Y),\quad K'\subset K,\quad\text{and}\quad L'\subset L.$$

The elements $(0,K,L)$ of $\mathcal{C}_\omega(Y)$ with $|K|+|L|=1$ are minimal with respect to this partial order. For each element $\alpha\equiv(\beta,K,L)$ of $\mathcal{C}_\omega(Y)$, we define

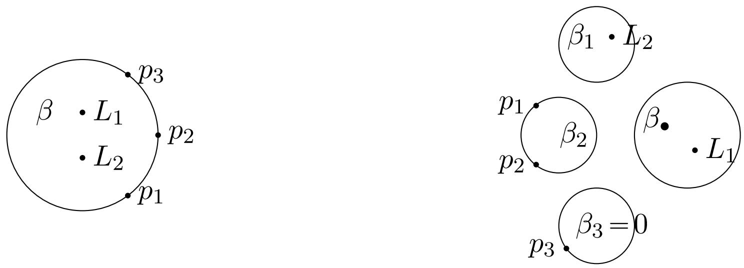
$$\beta(\alpha)\equiv\beta,\qquad K(\alpha)\equiv K,\qquad L(\alpha)\equiv L,$$
$$\dim(\alpha)=\mu_Y^\omega(\beta)-2|K|-\sum_{\Gamma\in L}(\operatorname{codim}\Gamma-2),\quad \mathcal{C}_{\omega;\alpha}(Y)=\big\{\alpha'\in\mathcal{C}_\omega(Y):\ \alpha'\prec\alpha\big\}.$$

For $\alpha\in\mathcal{C}_\omega(Y)$, let

$$\begin{aligned}\mathcal{D}_\omega(\alpha)=\Big\{&\big(\beta_\bullet,k_\bullet,L_\bullet,(\alpha_i)_{i\in[k_\bullet]}\big):\ \beta_\bullet\in H_2^\omega(X,Y),\ k_\bullet\in\mathbb{Z}^{\geqslant0},\ L_\bullet\subset L(\alpha),\ (\beta_\bullet,k_\bullet,L_\bullet)\neq(0,1,\emptyset),\\ &\alpha_i\in\mathcal{C}_\omega(Y)\ \forall\, i\in[k_\bullet],\ \beta_\bullet+\sum_{i=1}^{k_\bullet}\beta(\alpha_i)=\beta(\alpha),\ \bigsqcup_{i=1}^{k_\bullet}K(\alpha_i)=K(\alpha),\ L_\bullet\sqcup\bigsqcup_{i=1}^{k_\bullet}L_i(\alpha)=L(\alpha)\Big\}.\end{aligned}$$

An element in $\mathcal{D}_\omega(\alpha)$ is a "degeneration" of α into a central piece and $k_\bullet$ many branches; see Figure 4. Since $\alpha_i\prec\alpha$ for every

$$\eta\equiv\big(\beta_\bullet,k_\bullet,L_\bullet,(\alpha_i)_{i\in[k_\bullet]}\big)\equiv\big(\beta_\bullet,k_\bullet,L_\bullet,(\beta_i,K_i,L_i)_{i\in[k_\bullet]}\big)\in\mathcal{D}_\omega(\alpha)\tag{2.5}$$



$\alpha\in\mathcal{C}_\omega(Y)$ $\eta\in\mathcal{D}_\omega(\alpha)$

$$\alpha=\big(\beta,\{p_1,p_2,p_3\},\{L_1,L_2\}\big)$$

$$\eta=\Big(\beta_\bullet,3,\{L_1\},\big(\beta_1,\varnothing,\{L_1\}\big),\big(\beta_2,\{p_1,p_2\},\varnothing\big),\big(0,\{p_3\},\varnothing\big)\Big),\quad \beta_\bullet+\beta_1+\beta_2+0=\beta$$

Fig. 4. An element $\alpha\in\mathcal{C}_\omega(Y)$ and a "degeneration" $\eta\in\mathcal{D}_\omega(\alpha)$ of it.

and every $i\in[k_\bullet]$, $k_\bullet=0$ if α is a minimal element of $\mathcal{C}_\omega(Y)$. Thus,

$$\mathcal{D}_\omega\left(0,\{\mathrm{pt}\},\emptyset\right)=\emptyset \quad \text{and} \quad \mathcal{D}_\omega\left(0,\emptyset,\{\Gamma\}\right)=\{(0,0,\{\Gamma\},())\}$$

for any point $\mathrm{pt}\in Y$ and any R-pseudocycle Γ in $X-Y$. For $\eta\in\mathcal{D}_\omega(\alpha)$ as in (2.5) and $i\in[k_\bullet]$, we define

$$\beta_\bullet(\eta)=\beta_\bullet,\quad k_\bullet(\eta)=k_\bullet,\quad L_\bullet(\eta)=L_\bullet,$$
$$\alpha_i(\eta)=\alpha_i=(\beta_i,K_i,L_i),\quad \beta_i(\eta)=\beta_i,\quad K_i(\eta)=K_i,\quad L_i(\eta)=L_i.$$

2.3 Moduli spaces

We denote by $\mathbb{D}^2\subset\mathbb{C}$ the unit disk with the induced complex structure, by $\mathbb{D}^2\vee\mathbb{D}^2$ the union of two disks joined at a pair of boundary points, and by $S^1\subset\mathbb{D}^2$ and $S^1\vee S^1\subset\mathbb{D}^2\vee\mathbb{D}^2$ the respective boundaries. We orient the boundaries counterclockwise; thus, starting from a smooth point x_0 of $S^1\vee S^1$, we proceed counterclockwise to the node nd, then circle the 2nd copy of S^1 counterclockwise back to nd, and return to x_0 counterclockwise from nd. We call smooth points $x_0,x_1,\ldots,x_k$ on S^1 or $S^1\vee S^1$ ordered by position if they are traversed counterclockwise.

Let $k,l\in\mathbb{Z}^{\geqslant 0}$ with $k+2l\geqslant 3$. We denote by $\mathcal{M}^{\mathrm{uo}}_{k,l}$ the moduli space of k distinct boundary marked points $x_1,\ldots,x_k$ and l distinct interior marked points $z_1,\ldots,z_l$ on the unit disk $\mathbb{D}$ (the superscript *uo* ("unordered") means that $x_1,\ldots,x_k$ do not necessarily lie in cyclic order on $S^1\subset\mathbb{D}^2$). We orient $\mathcal{M}^{\mathrm{uo}}_{1,1}$ as a plus point. The space $\mathcal{M}^{\mathrm{uo}}_{3,0}$ consists of two points, $\mathcal{C}_{3,0}^{+}$ with the three boundary points ordered by position and $\mathcal{C}_{3,0}^{-}$ with the three boundary points not ordered by position. We orient $\mathcal{C}_{3,0}^{+}$ as a plus point and $\mathcal{C}_{3,0}^{-}$ as a minus point. We identify $\mathcal{M}_{0,2}^{\mathrm{uo}}$ with the interval $(0,1)$ by taking $z_1=0$ and $z_2\in(0,1)$ and orient it by the negative orientation of $(0,1)$.

We orient other $\mathcal{M}_{k,l}^{\mathrm{uo}}$ inductively. If $k\geqslant 1$, we orient $\mathcal{M}_{k,l}^{\mathrm{uo}}$ so that the short exact sequence

$$0\longrightarrow T_{x_k}S^1\longrightarrow T\mathcal{M}_{k,l}^{\mathrm{uo}}\xrightarrow{\mathrm{d}\mathfrak{f}_k^{\mathbb{R}}}T\mathcal{M}_{k-1,l}^{\mathrm{uo}}\longrightarrow 0 \tag{2.6}$$

induced by the forgetful morphism $\mathfrak{f}_k^{\mathbb{R}}$ dropping x_k has sign $(-1)^k$ with respect to the counterclockwise orientation of S^1. Thus,

$$T\mathcal{M}_{k,l}^{\mathrm{uo}}\approx T\mathcal{M}_{k-1,l}^{\mathrm{uo}}\oplus T_{x_k}S^1.$$

If $l\geqslant 1$, we orient $\mathcal{M}_{k,l}^{\mathrm{uo}}$ so that the short exact sequence

$$0\longrightarrow T_{z_l}\mathbb{D}\longrightarrow T\mathcal{M}_{k,l}^{\mathrm{uo}}\xrightarrow{\mathrm{d}\mathfrak{f}_l^{\mathbb{C}}}T\mathcal{M}_{k,l-1}^{\mathrm{uo}}\longrightarrow 0 \tag{2.7}$$

induced by the forgetful morphism $\mathfrak{f}_l^{\mathbb{C}}$ dropping z_l is orientation compatible with respect to the complex orientation of $\mathbb{D}$. By a direct check, the orientations of $\mathcal{M}_{1,2}^{\mathrm{uo}}$ induced from $\mathcal{M}_{0,2}^{\mathrm{uo}}$ via (2.6) and from $\mathcal{M}_{1,1}^{\mathrm{uo}}$ via (2.7) are the same, and the orientations of $\mathcal{M}_{3,1}^{\mathrm{uo}}$ induced from $\mathcal{M}_{1,1}^{\mathrm{uo}}$ via (2.6) and from $\mathcal{M}_{3,0}^{\mathrm{uo}}$ via (2.7) are also the same. Since the fibers of $\mathfrak{f}_l^{\mathbb{C}}$ are even dimensional, it follows that the orientation on $\mathcal{M}_{k,l}^{\mathrm{uo}}$ above is well defined.

Let (X,ω) be a symplectic manifold, $Y\subset X$ be a Lagrangian submanifold, $\beta\in H_2^{\omega}(X,Y)$, and $J\in\mathcal{J}_{\omega}$. For a finite ordered set K and a finite set L, we denote by $\mathfrak{M}_{K,L}^{\mathrm{uo},\bigstar}(\beta;J)$ the moduli space of stable J-holomorphic degree β maps from $(\mathbb{D}^2,S^1)$ and $(\mathbb{D}^2\vee\mathbb{D}^2,S^1\vee S^1)$ to (X,Y) with boundary and interior marked points indexed by K and L, respectively. Let

$$\mathfrak{M}_{K,L}^{\mathrm{uo}}(\beta;J)\subset\mathfrak{M}_{K,L}^{\mathrm{uo},\bigstar}(\beta;J)$$

be the subspace of maps from $(\mathbb{D}^2,S^1)$. If $K=[k]$ for $k\in\mathbb{Z}^{\geqslant 0}$ (resp. $L=[l]$ for $l\in\mathbb{Z}^{\geqslant 0}$), we write k for K (resp. l for L) in the subscripts of these moduli spaces. For

$$[\mathbf{u}]\equiv\Big[u\colon(\mathbb{D},S^1)\longrightarrow(X,Y),(x_i)_{i\in[k]},(z_i)_{i\in[l]}\Big]\in\mathfrak{M}_{k,l}^{\mathrm{uo}}(\beta;J), \tag{2.8}$$

let

$$D_{J;\mathbf{u}}\colon \Gamma\left(u^*TX, u|_{S^1}^*TY\right) \longrightarrow \Gamma\left(T^*\mathbb{D}^{0,1}\otimes_{\mathbb{C}} u^*(TX,J)\right)$$

be the linearization of the $\{\bar\partial_J\}$-operator on the space of maps from $(\mathbb{D},S^1)$ to (X,Y). By [5, Proposition 8.1.1], a relative OSpin-structure $\mathfrak{os}$ on Y determines an orientation on $\det(D_{J;\mathbf{u}})$.

We orient $\mathfrak{M}^{\mathrm{uo}}_{k,l}(\beta;J)$ by requiring the short exact sequence

$$0\longrightarrow \ker D_{J;\mathbf{u}} \longrightarrow T_{\mathbf{u}}\mathfrak{M}^{\mathrm{uo}}_{k,l}(\beta;J) \xrightarrow{\mathrm{d}\mathfrak{f}} T_{\mathfrak{f}(\mathbf{u})}\mathcal{M}^{\mathrm{uo}}_{k,l} \longrightarrow 0$$

to be orientation compatible, where $\mathfrak{f}$ is the forgetful morphism dropping the map part of $\mathbf{u}$. This orientation extends over $\mathfrak{M}^{\mathrm{uo},\bigstar}_{k,l}(\beta;J)$. If K is a finite ordered set and L is a finite set, we orient $\mathfrak{M}^{\mathrm{uo},\bigstar}_{K,L}(\beta;J)$ from $\mathfrak{M}^{\mathrm{uo},\bigstar}_{|K|,|L|}(\beta;J)$ by identifying K with $[|K|]$ as ordered sets and L with $[|L|]$ as sets.

Remark 2.5. The above paragraph endows $\mathfrak{M}^{\mathrm{uo},\bigstar}_{K,L}(\beta;J)$ with an orientation under the assumption that $|K|+2|L|\geqslant 3$. If $|K|+2|L|<3$, one first stabilizes the domain of $\mathbf{u}$ by adding one or two interior marked points, then orients the tangent space of the resulting map as above, and finally drops the added marked points using the canonical complex orientation of $\mathbb{D}$; see the proof of [7, Corollary 1.8].

For $i\in K$ and $j\in L$, let

$$\mathrm{evb}_i\colon \mathfrak{M}^{\mathrm{uo},\bigstar}_{K,L}(\beta;J)\longrightarrow Y \qquad\text{and}\qquad \mathrm{evi}_j\colon \mathfrak{M}^{\mathrm{uo},\bigstar}_{K,L}(\beta;J)\longrightarrow X$$

be the evaluation morphisms at the i-th boundary marked point and the i-th interior marked point, respectively. If $M\subset \mathfrak{M}^{\mathrm{uo},\bigstar}_{K,L}(\beta;J)$, we denote the restrictions of evb_i and evi_j to M also by evb_i and evi_j. If in addition $m,m'\in\mathbb{Z}^{\geqslant 0}$,

$$\left(\mathfrak{b}_s\colon Z_{\mathfrak{b}_s}\longrightarrow Y\right)_{s\in[m]} \qquad\text{and}\qquad \left(\Gamma_s\colon Z_{\Gamma_s}\longrightarrow X\right)_{s\in[m']}$$

are tuples of maps and $i_1,\ldots,i_m\in[k]$ and $j_1,\ldots,j_{m'}\in L$ are distinct elements, let

$$\begin{aligned}
&M\times_{\mathrm{fb}}\left((i_s,\mathfrak{b}_s)_{s\in[m]};(j_s,\Gamma_s)_{s\in[m']}\right)\\
&\equiv M_{(\mathrm{evb}_{i_1},\ldots,\mathrm{evb}_{i_m},\mathrm{evi}_{j_1},\ldots,\mathrm{evi}_{j_{m'}})}\times_{\mathfrak{b}_1\times\ldots\times\mathfrak{b}_m\times\Gamma_1\times\ldots\times\Gamma_{m'}}\left(Z_{\mathfrak{b}_1}\times\ldots\times Z_{\mathfrak{b}_m}\times Z_{\Gamma_1}\times\ldots\times Z_{\Gamma_{m'}}\right)
\end{aligned}$$

be their fiber product with M. If M is an oriented manifold and $\mathfrak{b}_s$ and Γ_s are smooth maps from oriented manifolds satisfying the appropriate transversality conditions, then we orient this space as in Section 2.1. For $i \in [k]$ with $i \neq i_s$ for any $s \in [m]$ (resp. $j \in L$ with $j \neq j_s$ for any $s \in [m']$), we define

$$\mathrm{evb}_i \text{ (resp. } \mathrm{evi}_j): M \times_{\mathrm{fb}} \big((i_s, \mathfrak{b}_s)_{s\in[m]}; (j_s, \Gamma_s)_{s\in[m']}\big) \longrightarrow Y \text{ (resp.} X)$$

to be the composition of the evaluation map evb_i (resp. evi_j) defined above with the projection to the 1st component.

2.4 Open Gromov–Witten invariants

In the remainder of this paper, we use the term (bordered) pseudocycle to mean (bordered) pseudocycle in the usual sense taken with R-coefficients; see the last part of [3, Section 3] for precise definitions. We recall that every R-homology class in a manifold can be represented by a pseudocycle in this sense, which is unique up to equivalence; see [18, Theorem 1.1].

Let (X,ω) be a symplectic six-fold and $Y \subset X$ be a Lagrangian submanifold. For a point $\mathrm{pt} \in Y$, we denote its inclusion into Y also by pt. For $\beta \in H_2^{\omega}(X,Y)$, $k \in \mathbb{Z}^{\geqslant 0}$, a finite set L, and $J \in \mathcal{J}_{\omega}$, let

$$\mathfrak{M}^{\bigstar}_{k,L}(\beta;J) \subset \mathfrak{M}^{\mathrm{uo},\bigstar}_{k,L}(\beta;J)$$

be the subspace of maps with the boundary marked points ordered by position. If in addition $\eta \in \mathcal{D}_{\omega}(\alpha)$ for some $\alpha \in \mathcal{C}_{\omega}(Y)$, define

$$\mathfrak{M}_{\eta;J} \equiv \mathfrak{M}^{\bigstar}_{k_\bullet(\eta),L_\bullet(\eta)}(\beta_\bullet(\eta);J), \quad \mathfrak{M}^{+}_{\eta;J} \equiv \mathfrak{M}^{\bigstar}_{k_\bullet(\eta)+1,L_\bullet(\eta)}(\beta_\bullet(\eta);J).$$

Definition 2.6. Let R, (X,ω), Y, $\mathfrak{os}$, and $\alpha \equiv (\beta,K,L)$ be as in Theorem 1.2. A bounding chain on (α,J) is a collection $(\mathfrak{b}_{\alpha'})_{\alpha'\in\mathcal{C}_{\omega;\alpha}(Y)}$ of bordered pseudocycles into Y such that

(BC1) $\dim \mathfrak{b}_{\alpha'} = \dim(\alpha')+2$ for all $\alpha' \in \mathcal{C}_{\omega;\alpha}(Y)$;

(BC2) $\mathfrak{b}_{\alpha'} = \emptyset$ unless $\alpha' = (0,\{\mathrm{pt}\},\emptyset)$ for some $\mathrm{pt} \in K$ or $\dim(\alpha')=0$;

(BC3) $\mathfrak{b}_{(0,\{\mathrm{pt}\},\emptyset)} = \mathrm{pt}$ for all $\mathrm{pt} \in K$;

(BC4) for all $\alpha' \in \mathcal{C}_{\omega;\alpha}(Y)$ such that $\dim(\alpha')=0$,

$$\partial\mathfrak{b}_{\alpha'} = \left(\mathrm{evb}_1 : \bigcup_{\eta\in\mathcal{D}_{\omega}(\alpha')} (-1)^{k_\bullet(\eta)} \mathfrak{M}^{+}_{\eta;J} \times_{\mathrm{fb}} \big((i+1, \mathfrak{b}_{\alpha_i(\eta)})_{i\in[k_\bullet(\eta)]}; (i,\Gamma_i)_{\Gamma_i\in L_\bullet(\eta)}\big) \longrightarrow Y\right). \tag{2.9}$$

Since the dimension of every pseudocycle $\Gamma_i \in L$ is even, the oriented morphism

$$\mathfrak{bb}_\eta \equiv \left(\mathrm{evb}_1 : (-1)^{k_\bullet(\eta)}\mathfrak{M}^+_{\eta;J} \times_{\mathrm{fb}} \left((i+1, \mathfrak{b}_{\alpha_i(\eta)})_{i\in[k_\bullet(\eta)]}; (i, \Gamma_i)_{\Gamma_i \in L_\bullet(\eta)}\right) \longrightarrow Y\right) \tag{2.10}$$

in (2.9) does not depend on the choice of identification of $L_\bullet(\eta)$ with $[|L_\bullet(\eta)|]$. By [2, Lemma 3.1], the map

$$\mathfrak{bb}_{\alpha'} \equiv \bigcup_{\eta \in \mathcal{D}_\omega(\alpha')} \mathfrak{bb}_\eta \tag{2.11}$$

with orientation induced by the relative OSpin-structure $\mathfrak{os}$ on Y is a pseudocycle for every $\alpha' \in \mathcal{C}_{\omega;\alpha}(Y) \cup \{\alpha\}$. If in addition $\dim(\alpha) = 2$, then $\mathfrak{bb}_\alpha$ is a pseudocycle of codimension 0. Its degree determines a count of J-holomorphic disks in (X, Y) through $|K|+1$ points in Y as in (1.6).

A bounding chain $(\mathfrak{b}_{\alpha'})_{\alpha' \in \mathcal{C}_{\omega;\alpha}(Y)}$ as in Definition 2.6 can also be used to define the counts (1.6) of J-holomorphic disks in the case of no boundary constraints in the following way. We denote the signed cardinality of a finite set S of signed points by $|S|^\pm$. If S is not a finite set of signed points, we set $|S|^\pm \equiv 0$. If $\eta \in \mathcal{D}_\omega(\alpha)$, let

$$s^*(\eta) \equiv \begin{cases} \frac{1}{k_\bullet(\eta)} - \frac{1}{2}, & \text{if } k_\bullet(\eta) \neq 0, \\ 1, & \text{if } k_\bullet(\eta) = 0. \end{cases}$$

Define

$$\langle L \rangle^{\omega,\mathfrak{os}}_{\beta;K} \equiv \sum_{\eta \in \mathcal{D}_\omega(\alpha)} (-1)^{k_\bullet(\eta)} s^*(\eta) \left| \mathfrak{M}_{\eta;J} \times_{\mathrm{fb}} \left((i, \mathfrak{b}_{\alpha_i(\eta)})_{i \in [k_\bullet(\eta)]}; (i, \Gamma_i)_{\Gamma_i \in L_\bullet(\eta)}\right) \right|^\pm + \frac{1}{2} \sum_{p \in K} \langle L \rangle^{\omega,\mathfrak{os}}_{\beta;K-\{p\}}. \tag{2.12}$$

This number vanishes unless $\dim(\alpha) = 0$. Unlike (1.6), (2.12) provides a definition of the counts (1.5) with $k=0$. By [2, Theorem 2.7(2)],

$$\langle L \rangle^{\omega,\mathfrak{os}}_{\beta;K} \text{ in (2.12)} = \langle L \rangle^{\omega,\mathfrak{os}}_{\beta;K-\{\mathrm{pt}\}} \text{ in (1.6)}$$

for any $\mathrm{pt} \in K$ if $\langle L \rangle_{\beta;K-\{\mathrm{pt}\}}$ does not depend on $\mathrm{pt} \in K$.

3 Proof of Theorem 1.2

For the remainder of the paper, we take (X,ω,Y) and $\mathfrak{os}$ as in Theorem 1.2. Let γ_1,γ_2 be smooth maps from oriented closed one-manifolds into the oriented closed three-manifold Y with disjoint images. If $\gamma_1=\partial\mathfrak{b}_1$ and $\gamma_2=\partial\mathfrak{b}_2$ for some bordered pseudocycles $\mathfrak{b}_1$ and $\mathfrak{b}_2$ into Y so that $\mathfrak{b}_1$ is transverse to γ_2 and $\mathfrak{b}_2$ is transverse to γ_1, we define

$$\mathrm{lk}(\gamma_1,\gamma_2)\equiv\big|\mathfrak{b}_1\times_{\mathrm{fb}}\gamma_2\big|^{\pm}=-\big|\gamma_1\times_{\mathrm{fb}}\mathfrak{b}_2\big|^{\pm}=\big|\mathfrak{b}_2\times_{\mathrm{fb}}\gamma_1\big|^{\pm}=-\big|\gamma_2\times_{\mathrm{fb}}\mathfrak{b}_1\big|^{\pm}; \tag{3.1}$$

the first and last equalities above hold by Lemma 2.2, while the middle one follows from Lemma 2.3. The sign of a point (p,q) of $\mathfrak{b}_1\times_{\mathrm{fb}}\gamma_2$ is the sign of the isomorphism

$$T_p\mathrm{dom}(\mathfrak{b}_1)\oplus T_q\mathrm{dom}(\gamma_2)\longrightarrow T_{\mathfrak{b}_1(p)}Y=T_{\gamma_2(q)}Y,\qquad (v,w)\longrightarrow \mathrm{d}_p\mathfrak{b}_1(v)+\mathrm{d}_q\gamma_2(w).$$

The linking number (3.1) of the one-cycles γ_1 and γ_2 that bound in Y does not depend on the choice of $\mathfrak{b}_1,\mathfrak{b}_2$. In this section, we take linking numbers of the boundaries ∂u of J-holomorphic maps u from $(\mathbb{D}^2,S^1)$ to (X,Y). By the injectivity of (1.3), these boundaries also bound in Y and thus have well-defined linking numbers.

3.1 Bounding chains and Welschinger's invariants

For $\beta_1,\ldots,\beta_m\in H_2^{\omega}(X,Y)$, we denote by $\mathfrak{M}^{\mathrm{uo}}_{K,L}(\beta_1,\ldots,\beta_m;J)$ the moduli space of unions of m J-holomorphic disks in classes $\beta_1,\ldots,\beta_m$ with L-labeled interior marked points and K-labeled boundary marked points between the m disks. In contrast to [17, Section 2.4], we do *not* order the disks or orient this moduli space. Let

$$\mathfrak{M}^{\mathrm{uo},\circ}_{K,L}(\beta_1,\ldots,\beta_m;J)\subset\mathfrak{M}^{\mathrm{uo}}_{K,L}(\beta_1,\ldots,\beta_m;J) \tag{3.2}$$

be the dense open subset of the multi-disks whose m components have pairwise disjoint boundaries in Y. We extend the definitions of the evaluations maps evb_i and evi_i and of the associated fiber product $\times_{\mathrm{fb}}$ of Section 2.3 to the moduli spaces in (3.2).

Let $\alpha\equiv(\beta,K,L)$ and J be as in Theorem 1.2 and $p_1,\ldots,p_k$ be an ordering of the elements of K. For any element $\alpha'\equiv(\beta',K',L')$ of $\mathcal{C}_{\omega;\alpha}(Y)\cup\{\alpha\}$, we endow $K'\subset K$ with the order induced from K. We define the spaces of (constrained) single α'-disks and

α'-multi-disks by

$$\mathrm{SD}(\alpha') \equiv \mathfrak{M}^{\mathrm{uo}}_{K',L'}(\beta';J){\times_{\mathrm{fb}}}\Big((i,p_i)_{i\in K'};(i,\Gamma_i)_{\Gamma_i\in L'}\Big) \qquad \text{and}$$

$$\mathrm{MD}(\alpha') \equiv \bigsqcup_{m=1}^{\infty}\ \bigsqcup_{\substack{\beta_1,\ldots,\beta_m\in H_2^{\omega}(X,Y)\\ \text{unordered}\\ \beta_1+\ldots+\beta_m=\beta'}} \mathfrak{M}^{\mathrm{uo},\circ}_{K',L'}(\beta_1,\ldots,\beta_m;J){\times_{\mathrm{fb}}}\Big((i,p_i)_{i\in K'};(i,\Gamma_i)_{\Gamma_i\in L'}\Big),$$

respectively. Note that we do not orient $\mathrm{MD}(\alpha')$ but define the sign of a single element of $\mathrm{MD}(\alpha')$ below via the spaces $\mathrm{SD}(\alpha'')$.

We write an element $\mathbf{u}$ of $\mathrm{MD}(\alpha')$ as

$$\mathbf{u}\equiv\big[\mathbf{u}_1,\ldots,\mathbf{u}_m\big] \quad \text{with} \quad \mathbf{u}_r\in\mathfrak{M}^{\mathrm{uo}}_{K_r,L_r}(\beta_r;J){\times_{\mathrm{fb}}}\Big((i,p_i)_{i\in K_r};(i,\Gamma_i)_{\Gamma_i\in L_r}\Big) \tag{3.3}$$

$$\text{for some}\quad m,\beta_r,K_r,L_r\ \text{ with }\ \beta_1+\ldots+\beta_m=\beta',\ \bigsqcup_{r=1}^{m}K_r=K',\ \bigsqcup_{r=1}^{m}L_r=L'.$$

For such an element $\mathbf{u}$ of $\mathrm{MD}(\alpha')$, we write $\mathbf{u}_r\in\mathbf{u}$ to indicate that $\mathbf{u}_r$ is a component of the multi-disk $\mathbf{u}$. Let

$$\partial\mathbf{u}:\underbrace{S^1\sqcup\ldots\sqcup S^1}_{m}\longrightarrow Y$$

be the boundary of the components of $\mathbf{u}$ with the orientation induced by the complex orientation on the unit disk. If $\dim(\alpha')=0$ and $\mathbf{u}_r\in\mathbf{u}$, we denote by $\mathrm{sgn}(\mathbf{u}_r)$ the sign of $\mathbf{u}_r$ as an element of the fiber product in (3.3) and set

$$\mathrm{sgn}(\mathbf{u})\equiv\prod_{\mathbf{u}_r\in\mathbf{u}}\mathrm{sgn}(\mathbf{u}_r);$$

this sign does not depend on the order on K. If $\dim(\alpha')\neq 0$, we define $\mathrm{sgn}(\mathbf{u})\equiv 0$.

For $\mathbf{u}\in\mathrm{MD}(\alpha')$ as in (3.3), we denote by $K_{\mathbf{u}}$ the complete graph with vertices $\mathbf{u}_1,\ldots,\mathbf{u}_m$. We call a tree $T\subset K_{\mathbf{u}}$, that is, a connected subgraph without loops, spanning if T contains all vertices of $K_{\mathbf{u}}$ and denote by $\mathrm{ST}(\mathbf{u})$ the set of all spanning trees $T\subset K_{\mathbf{u}}$. Let

$$\mathrm{lk}(\mathbf{u};T)\equiv\prod_{\substack{\text{edge } e\in T\\ \text{connecting } \mathbf{u}_r,\mathbf{u}_s}}\mathrm{lk}\big(\partial\mathbf{u}_r,\partial\mathbf{u}_s\big)\quad\forall\,T\in\mathrm{ST}(\mathbf{u}) \quad\text{and}\quad \mathrm{lk}(\mathbf{u})\equiv\sum_{T\in\mathrm{ST}(\mathbf{u})}\mathrm{lk}(\mathbf{u};T).$$

Welschinger's definition of the open GW-invariant (1.4) in [17, Section 4.1] is equivalent to

$$\langle \Gamma_1, \ldots, \Gamma_l \rangle^{\omega,\mathfrak{os}}_{\beta,|K|} = \sum_{\mathbf{u}\in \mathrm{MD}(\alpha)} \mathrm{sgn}(\mathbf{u})\mathrm{lk}(\mathbf{u})\,. \tag{3.4}$$

The 1st statement of the next proposition establishes (W1). We deduce (W2) from the 2nd statement of this proposition and Proposition 3.2. The two propositions are proved in Sections 3.2 and 3.3.

Proposition 3.1. Let $\alpha \in \mathcal{C}_\omega(Y)$ and J be as in Theorem 1.2.

(1) There exists a bounding chain $(\mathfrak{b}_{\alpha'})_{\alpha'\in\mathcal{C}_{\omega;\alpha}(Y)}$ on (α, J).

(2) For every such bounding chain $(\mathfrak{b}_{\alpha'})_{\alpha'\in\mathcal{C}_{\omega;\alpha}(Y)}$ and $\alpha' \in \mathcal{C}_{\omega;\alpha}(Y)$ with $\dim(\alpha')=0$, the associated closed pseudocycle (2.11) satisfies

$$\partial\mathfrak{b}_{\alpha'} = \mathfrak{b}\mathfrak{b}_{\alpha'} = (-1)^{|K(\alpha')|} \bigsqcup_{\mathbf{u}\in\mathrm{MD}(\alpha')} \mathrm{sgn}(\mathbf{u})\mathrm{lk}(\mathbf{u})\partial\mathbf{u}\,. \tag{3.5}$$

If $k\in\mathbb{Z}^+$ and $K\subset[k]$, let

$$\mathfrak{f}^b_k\colon \mathfrak{M}^{\mathrm{uo},\bigstar}_{k,l}(\beta;J) \longrightarrow \mathfrak{M}^{\mathrm{uo},\bigstar}_{k-1,l}(\beta;J) \quad \text{and} \quad \mathfrak{f}^b_{k;K}\colon \mathfrak{M}^{\mathrm{uo},\bigstar}_{k,l}(\beta;J) \longrightarrow \mathfrak{M}^{\mathrm{uo},\bigstar}_{[k]-K,l}(\beta;J)$$

be the forgetful morphisms dropping the boundary marked point with index k and the boundary marked points indexed by the set K, respectively.

Proposition 3.2 (Open divisor relation). Suppose $K' \subset K \subset [k]$, $L \subset [l]$, and $(\mathfrak{b}_i)_{i\in K}$ and $(\Gamma_i)_{i\in L}$ are tuples of bordered pseudocycles into Y and X, respectively. If the codimension of $\mathfrak{b}_i$ is 1 for every $i\in K'$ and

$$K' \subset \big\{k', \ldots, k\big\} \subset K$$

for $k' \in [k]$, then there exists a dense open subset $\mathfrak{M}^*_{[k]-K',l}$ of the target of the induced forgetful morphism

$$\mathfrak{f}^b_k\colon \mathfrak{M}^{\mathrm{uo},\bigstar}_{k,l}(\beta;J)\times_{\mathrm{fb}}\big((i,\mathfrak{b}_i)_{i\in K}; (i,\Gamma_i)_{i\in L}\big) \longrightarrow \mathfrak{M}^{\mathrm{uo},\bigstar}_{k-1,l}(\beta;J)\times_{\mathrm{fb}}\Big((i,\mathfrak{b}_i)_{i\in K-\{k\}}; (i,\Gamma_i)_{i\in L}\Big) \tag{3.6}$$

so that (3.6) restricts to a covering map over each connected component $\mathfrak{M}$ of $\mathfrak{M}^*_{[k]-K',l}$. If in addition the codimensions of all $\mathfrak{b}_i$ are odd and the codimensions of Γ_i are even, then

$$\deg\left(\mathfrak{f}^{b}_{k;K'}\big|_{\mathfrak{M}}\right) = (-1)^{|K'|}\prod_{i\in K'}\mathrm{lk}(\partial\mathbf{u},\partial\mathfrak{b}_i)$$

for any $\mathbf{u}\in\mathfrak{M}$.

Remark 3.3. Suppose that ϕ is an anti-symplectic involution on (X,ω), that is, $\phi^2=\mathrm{id}_X$ and $\phi^*\omega=-\omega$, $Y\subset X$ is a topological component of the fixed locus X^ϕ of ϕ, $\phi^*J=-J$, and $\mathfrak{os}$ is an OSpin-structure on Y, that is, a relative OSpin-structure $\mathfrak{os}$ with $w_2(\mathfrak{os})=0$. Suppose also that for every $i\in[l]$ there exists a diffeomorphism ϕ_{Γ_i} of $\mathrm{dom}\,\Gamma_i$ such that

$$\phi\circ\Gamma_i=\Gamma_i\circ\phi_{\Gamma_i} \qquad \text{and} \qquad \mathrm{sgn}\,\phi_{\Gamma_i}=(-1)^{(\dim\Gamma_i)/2}. \tag{3.7}$$

Let $H_{2;\phi}(X,Y)$ be the quotient of $H_2(X,Y;\mathbb{Z})$ by the image of the endomorphism $\{\mathrm{Id}+\phi_*\}$. For $B\in H_{2;\phi}(X,Y)$, we denote by $\{\Gamma_1,\ldots,\Gamma_l\}^{\omega,\mathfrak{os}}_{B,|K|}$ the real genus 0 GW-invariant of (X,ω,ϕ) enumerating degree B real J-holomorphic spheres through the constraints $\Gamma_1,\ldots,\Gamma_l$ and $|K|$ generic points in Y as defined in [3]. This invariant is a re-interpretation of the invariants defined in [11, 15, 16]; see also [2, Remark B.4]. We show below that

$$\{\Gamma_1,\ldots,\Gamma_l\}^{\omega,\mathfrak{os}}_{B,|K|}=2^{l-1}\sum_{\beta\in B}\langle\Gamma_1,\ldots,\Gamma_l\rangle^{\omega,\mathfrak{os}}_{\beta,|K|}, \tag{3.8}$$

that is, Welschinger's invariants (3.4) sum up to the real genus 0 GW-invariants as expected.

Proof. Define

$$\mathrm{SD}(B,K,L)=\bigsqcup_{\substack{\alpha'\in\mathcal{C}_\omega(Y),\beta(\alpha')\in B\\ K(\alpha')=K,L(\alpha')=L}}\mathrm{SD}(\alpha'), \qquad \mathrm{MD}(B,K,L)=\bigsqcup_{\substack{\alpha'\in\mathcal{C}_\omega(Y),\beta(\alpha')\in B\\ K(\alpha')=K,L(\alpha')=L}}\mathrm{MD}(\alpha').$$

We show below that the collection

$$\{(\mathbf{u},T)\colon \mathbf{u}\in\mathrm{MD}(B,K,L)-\mathrm{SD}(B,K,L),\ T\in\mathrm{ST}(\mathbf{u})\} \tag{3.9}$$

splits into subcollections $\mathcal{A}_i$ so that

$$\mathrm{sgn}(\mathbf{u})=\mathrm{sgn}(\mathbf{u}') \quad \forall\,(\mathbf{u},T),(\mathbf{u}',T')\in\mathcal{A}_i \qquad \text{and} \qquad \sum_{(\mathbf{u},T)\in\mathcal{A}_i}\mathrm{lk}(\mathbf{u};T)=0. \tag{3.10}$$

In light of (3.4), this implies that

$$\sum_{\beta\in B}\langle\Gamma_1,\ldots,\Gamma_l\rangle^{\omega,\mathfrak{os}}_{\beta,|K|}\equiv\sum_{\mathbf{u}\in\mathrm{MD}(B,K,L)}\;\sum_{T\in\mathrm{ST}(\mathbf{u})}\mathrm{sgn}(\mathbf{u})\mathrm{lk}(\mathbf{u};T)=\sum_{\mathbf{u}\in\mathrm{SD}(B,K,L)}\mathrm{sgn}(\mathbf{u})\,. \tag{3.11}$$

By the next paragraph, the elements of $\mathrm{SD}(B,K,L)$ come in pairs of the same sign. The union of the elements in such a pair is a degree B real J-holomorphic sphere through the constraints $\Gamma_1,\ldots,\Gamma_l$. Conversely, a degree B real J-holomorphic sphere through $\Gamma_1,\ldots,\Gamma_l$ determines a pair of elements of $\mathrm{SD}(B,K,L)$. However, $\{\Gamma_1,\ldots,\Gamma_l\}^{\omega,\mathfrak{os}}_{B,|K|}$ counts each degree B real J-holomorphic sphere along with its contact points with $\Gamma_1,\ldots,\Gamma_l$ and thus 2^l times (because the contact points come in conjugate pairs). By the 2nd condition in (3.7), all such decorated J-holomorphic spheres contribute to $\{\Gamma_1,\ldots,\Gamma_l\}^{\omega,\mathfrak{os}}_{B,|K|}$ with the same sign. Thus, $\{\Gamma_1,\ldots,\Gamma_l\}^{\omega,\mathfrak{os}}_{B,|K|}$ is $2^l/2$ times the right-hand side of (3.11).

Let $\mathfrak{c}$ be the complex conjugation on $\mathbb{D}^2$. The replacement of $\mathbf{u}_r\in\mathbf{u}$ as in (3.3) with

$$\mathbf{u}'_r\equiv\Big(\big[\phi\circ u_r\circ\mathfrak{c},(x_i)_{i\in K_r},(\mathfrak{c}(z_i))_{\Gamma_i\in L_r}\big],(i,p_i)_{i\in K_r};(i,\phi_{\Gamma_i}(q_i))_{\Gamma_i\in L_r}\Big)$$
$$\in\mathfrak{M}^{\mathrm{uo}}_{K_r,L_r}(-\phi_*(\beta_r);J)\times_{\mathrm{fb}}\Big((i,p_i)_{i\in K_r};(i,\Gamma_i)_{\Gamma_i\in L_r}\Big)$$

preserves $\mathrm{MD}(B,K,L)$. Let $\mathbf{u}'\in\mathrm{MD}(B,K,L)$ be the resulting element. By [11, Proposition 5.1], $\mathrm{sgn}(\mathbf{u}'_r)=\mathrm{sgn}(\mathbf{u}_r)$; see also [2, Lemma B.7]. Thus, $\mathrm{sgn}(\mathbf{u}')=\mathrm{sgn}(\mathbf{u})$. However, $\partial\mathbf{u}_r$ and $\partial\mathbf{u}'_r$ are the same circles with the opposite orientations. If precisely one edge of a tree $T\in\mathrm{ST}(\mathbf{u})$ contains $\mathbf{u}_r\in\mathbf{u}$ as a vertex, that is, $\mathbf{u}_r$ is a "leaf" of T, this implies that $\mathrm{lk}(\mathbf{u};T)=-\mathrm{lk}(\mathbf{u}';T')$, where T' is the tree obtained from T by replacing the vertex $\mathbf{u}_r$ with $\mathbf{u}'_r$.

Given an element $(\mathbf{u},T)$ of the collection (3.9), let $\mathcal{A}_i$ be the subcollection of (3.9) consisting of all pairs $(\mathbf{u}',T')$ obtained from $(\mathbf{u},T)$ by replacing some of the leaves $\mathbf{u}_r\in T$ by $\mathbf{u}'_r$ as in the previous paragraph. If T contains N leaves, then $|\mathcal{A}_i|=2^N$. By the previous paragraph, $\mathcal{A}_i$ satisfies the 1st condition in (3.10) and

$$\sum_{(\mathbf{u}',T')\in\mathcal{A}_i}\mathrm{lk}(\mathbf{u}';T')=\sum_{k=0}^{N}(-1)^k\mathrm{lk}(\mathbf{u};T)\binom{N}{k}=\mathrm{lk}(\mathbf{u};T)\,(1+(-1))^N=0.$$

Thus, $\mathcal{A}_i$ satisfies the 2nd condition in (3.10), as required. ■

3.2 Main argument

We continue with the setting of Theorem 1.2 and Proposition 3.1. Let $\alpha' \in \mathcal{C}_{\omega;\alpha}(Y)\cup\{\alpha\}$. Recall that an element $\eta \in \mathcal{D}_\omega(\alpha')$ is a "degeneration" of α' into a center and branches. For $\eta\in\mathcal{D}_\omega(\alpha')$, define

$$K_\bullet^*(\eta) \equiv \big\{i\in[k_\bullet(\eta)] : \alpha_i(\eta)\neq(0,\{\mathrm{pt}\},\emptyset)\ \forall\,\mathrm{pt}\in K\big\}\,,$$
$$K_\bullet^{\mathrm{pt}}(\eta) \equiv \big\{\mathrm{pt}\in K : (0,\{\mathrm{pt}\},\emptyset)=\alpha_i(\eta)\ \text{for some}\ i\in[k_\bullet(\eta)]\big\}\,;$$

these are the set of indices of the non-point branches and the set of point branches. Let

$$\alpha_\bullet^{\mathrm{pt}}(\eta) \equiv \Big(\beta_\bullet(\eta), K_\bullet^{\mathrm{pt}}(\eta), L_\bullet(\eta)\Big) \in \mathcal{C}_{\omega;\alpha}(Y)\cup\{\alpha\}, \quad \mathfrak{M}^{\mathrm{uo},+}_{\eta;J} = \mathfrak{M}^{\mathrm{uo},\bigstar}_{[k_\bullet(\eta)+1],L_\bullet(\eta)}\big(\beta_\bullet(\eta);J\big)\,.$$

For $\eta,\eta'\in\mathcal{D}_\omega(\alpha')$, define $\eta\sim\eta'$ if

$$\big(\beta_\bullet(\eta),k_\bullet(\eta),L_\bullet(\eta)\big) = \big(\beta_\bullet(\eta'),k_\bullet(\eta'),L_\bullet(\eta')\big) \qquad \text{and}$$
$$\big(\alpha_i(\eta)\big)_{i\in[k_\bullet(\eta)]}\ \text{is a permutation of}\ \big(\alpha_i(\eta')\big)_{i\in[k_\bullet(\eta)]}\,,$$

that is, they have the same center and their branches differ by a permutation. Denote by $[\eta]$ the equivalence class of η. With $\mathfrak{bb}_\eta$ as in (2.10), let

$$\mathfrak{bb}_{[\eta]} = \bigsqcup_{\eta'\in[\eta]}\mathfrak{bb}_{\eta'}\,.$$

We define

$$\begin{aligned}\mathrm{DMD}(\alpha') &\equiv \big\{(\mathbf{u},\mathbf{u}_\bullet,T) : \mathbf{u}\in\mathrm{MD}(\alpha'),\ \mathbf{u}_\bullet\in\mathbf{u},\ T\in\mathrm{ST}(\mathbf{u})\big\}\,,\\ \widetilde{\mathrm{DMD}}(\alpha') &\equiv \Big\{\Big(\eta,\mathbf{u}_\bullet,(\tilde{\mathbf{u}}_i)_{i\in K_\bullet^*(\eta)}\Big) : \eta\in\mathcal{D}_\omega(\alpha'),\ \mathbf{u}_\bullet\in\mathrm{SD}(\alpha_\bullet^{\mathrm{pt}}(\eta)),\\ &\qquad\qquad \tilde{\mathbf{u}}_i\in\mathrm{DMD}(\alpha_i(\eta))\ \forall\,i\in K_\bullet^*(\eta)\Big\}.\end{aligned} \tag{3.12}$$

The notation "DMD" stands for "decorated multi-disk": an element in $\mathrm{DMD}(\alpha')$ is an α'-multi-disk $\mathbf{u}$ with a choice of "base component" $\mathbf{u}_\bullet$ and a spanning tree T. An element in $\widetilde{\mathrm{DMD}}(\alpha')$ is an element in $\mathrm{DMD}(\alpha')$ with the branches ordered; this is the content of the bijection (3.13). See Figure 5. We define elements $(\eta,\mathbf{u}_\bullet,(\tilde{\mathbf{u}}_i)_{i\in K_\bullet^*(\eta)})$ and $(\eta',\mathbf{u}'_\bullet,(\tilde{\mathbf{u}}'_i)_{i\in K_\bullet^*(\eta')})$

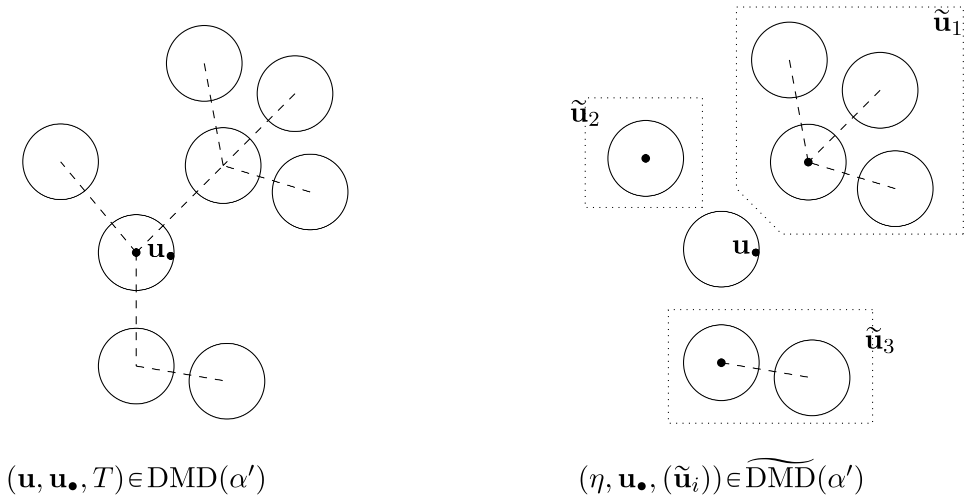


Fig. 5. An element in $\mathrm{DMD}(\alpha')$ and its corresponding element in $\widetilde{\mathrm{DMD}}(\alpha')$.

of $\widetilde{\mathrm{DMD}}(\alpha')$ to be equivalent if $\mathbf{u}_\bullet = \mathbf{u}'_\bullet$, $k_\bullet(\eta) = k_\bullet(\eta')$, and there exists a permutation σ of $[k_\bullet(\eta)]$ such that

$$\alpha_i(\eta) = \alpha_{\sigma(i)}(\eta') \ \ \forall\, i \in \big[k_\bullet(\eta)\big], \quad \sigma\left(K^*_\bullet(\eta)\right) \subset K^*_\bullet(\eta'), \quad \text{and} \quad \widetilde{\mathbf{u}}_i = \widetilde{\mathbf{u}}'_{\sigma(i)} \ \ \forall\, i \in K^*_\bullet(\eta),$$

that is, the branches are permuted. We denote by $\overline{\mathrm{DMD}}(\alpha')$ the quotient of the space in (3.12) by this equivalence relation.

Let $(\mathbf{u}, \mathbf{u}_\bullet, T) \in \mathrm{DMD}(\alpha')$. For each $\mathbf{u}_r \in \mathbf{u}$, let

$$\beta(\mathbf{u}_r) \in H_2^\omega(X, Y), \qquad L(\mathbf{u}_r) \subset L(\alpha'), \quad \text{and} \quad K(\mathbf{u}_r) \subset K(\alpha')$$

be the degree of $\mathbf{u}_r$, the interior marked points carried by $\mathbf{u}_r$, and the boundary marked points carried by $\mathbf{u}_r$, respectively. We denote by $\mathrm{Br}(\mathbf{u}_\bullet; T)$ the set of branches of T at $\mathbf{u}_\bullet$, that is, the trees T_i obtained by removing the vertex $\mathbf{u}_\bullet$ from the graph T. For each $i \in \mathrm{Br}(u_\bullet; T)$, we denote by $\mathbf{u}'_i$ the set of all vertices of T_i and by $\mathbf{u}'_{i\bullet} \in \mathbf{u}'_i$ the vertex connected by an edge of T to $\mathbf{u}_\bullet$. Define

$$\beta_i = \sum_{\mathbf{u}_r \in \mathbf{u}'_i} \beta(\mathbf{u}_r), \quad K_i = \bigsqcup_{\mathbf{u}_r \in \mathbf{u}'_i} K(\mathbf{u}_r), \quad L_i = \bigsqcup_{\mathbf{u}_r \in \mathbf{u}'_i} L(\mathbf{u}_r), \quad \alpha_i = (\beta_i, K_i, L_i) \in \mathcal{C}_{\omega;\alpha}(Y).$$

Let $\alpha_{\mathrm{pt}}=(0,\{\mathrm{pt}\},\emptyset)$ for each $\mathrm{pt}\in K(\mathbf{u}_\bullet)$ and

$$k_\bullet=\big|K(\mathbf{u}_\bullet)\big|+\big|\mathrm{Br}(\mathbf{u}_\bullet;T)\big|.$$

Identifying $K(\mathbf{u}_\bullet)\sqcup\mathrm{Br}(\mathbf{u}_\bullet;T)$ with $[k_\bullet]$, we obtain an element

$$\big(\eta\equiv\big(\beta(\mathbf{u}_\bullet),k_\bullet,L(\mathbf{u}_\bullet),(\alpha_i)_{i\in[k_\bullet]}\big),\mathbf{u}_\bullet,(\mathbf{u}'_i,\mathbf{u}'_{i\bullet},T_i)_{i\in\mathrm{Br}(\mathbf{u}_\bullet;T)}\big)\in\widetilde{\mathrm{DMD}}(\alpha').$$

The induced element of $\overline{\mathrm{DMD}}(\alpha')$ does not depend on the choice of this identification. In this way, we obtain a natural bijection

$$\mathrm{DMD}(\alpha')\longrightarrow\overline{\mathrm{DMD}}(\alpha'). \tag{3.13}$$

For $k\in\mathbb{Z}^{\geqslant 0}$, we denote by $\mathbb{S}_k$ the k-th symmetric group. For $\sigma\in\mathbb{S}_{k_\bullet(\eta)}$, define

$$\iota^+_{\eta;\sigma}:\mathfrak{M}^+_{\eta;J}\longrightarrow\mathfrak{M}^{\mathrm{uo},+}_{\eta;J},$$
$$\iota^+_{\eta;\sigma}\Big(u;x_1,x_2,\ldots,x_{k_\bullet(\eta)+1},(z_i)_{\Gamma_i\in L_\bullet(\eta)}\Big)=\Big(u;x_1,x_{1+\sigma(1)},\ldots,x_{1+\sigma(k_\bullet(\eta))},(z_i)_{\Gamma_i\in L_\bullet(\eta)}\Big).$$

This map is an open embedding and

$$\mathfrak{M}^{\mathrm{uo},+}_{\eta;J}=\bigsqcup_{\sigma\in\mathbb{S}_{k_\bullet(\eta)}}(\mathrm{sgn}\,\sigma)\Big(\mathrm{Im}\,\iota^+_{\eta;\sigma}\Big).$$

If $\eta\sim\eta'$ are such that $\alpha_i(\eta)=\alpha_{\sigma(i)}(\eta')$ for all $i\in[k_\bullet(\eta)]$, then

$$(-1)^{k_\bullet(\eta)}\mathfrak{bb}_{\eta'}\approx\Big(\mathrm{evb}_1:(\mathrm{sgn}\,\sigma)\Big(\mathrm{Im}\,\iota^+_{\eta;\sigma}\Big)\times_{\mathrm{fb}}\Big((i+1,\mathfrak{b}_{\alpha_i(\eta)})_{i\in[k_\bullet(\eta)]};(i,\Gamma_i)_{\Gamma_i\in L_\bullet(\eta)}\Big)\longrightarrow Y\Big)$$

by Lemma 2.3. Therefore,

$$\mathfrak{bb}_{[\eta]}\approx(-1)^{k_\bullet(\eta)}\Big(\mathrm{evb}_1:\mathfrak{M}^{\mathrm{uo},+}_{\eta;J}\times_{\mathrm{fb}}\Big((i+1,\mathfrak{b}_{\alpha_i(\eta)})_{i\in[k_\bullet(\eta)]};(i,\Gamma_i)_{\Gamma_i\in L_\bullet(\eta)}\Big)\longrightarrow Y\Big). \tag{3.14}$$

Proof of (W2). We establish this statement with K replaced by $K-\{p_1\}$ under the assumption that $\dim(\alpha)=0$.

Let α' and η be as above with $1\notin K(\alpha')$. With $\mathfrak{p}_1\equiv(0,\{p_1\},\emptyset)$,

$$\mathfrak{bb}_{[\eta]}\times_{\mathrm{fb}}\mathfrak{b}_{\mathfrak{p}_1}=\underbrace{(-1)^{k_\bullet(\eta)}}_{(3.14)}\underbrace{(-1)^{k_\bullet(\eta)}}_{\text{Lemma 2.3}}\underbrace{(-1)^{k_\bullet(\eta)}}_{\text{Lemma 2.4}}\mathfrak{M}^{\mathrm{uo},+}_{\eta;J}\times_{\mathrm{fb}}\Big((1,\mathfrak{b}_{\mathfrak{p}_1}),(i+1,\mathfrak{b}_{\alpha_i(\eta)})_{i\in[k_\bullet(\eta)]};(i,\Gamma_i)_{\Gamma_i\in L_\bullet(\eta)}\Big).$$

Suppose in addition $\dim(\alpha')=2$. By the above identity, Proposition 3.2 with $K'=K^*_\bullet(\eta)$, and (3.5) with α' replaced by $\alpha_i(\eta)$,

$$\begin{aligned}\left|\mathfrak{b}\mathfrak{b}_{[\eta]}\times_{\mathrm{fb}}\mathfrak{b}_{\mathfrak{p}_1}\right|^{\pm} &= (-1)^{|K^{\mathrm{pt}}_\bullet(\eta)|}\sum_{\mathbf{u}_\bullet\in\mathrm{SD}(\alpha^{\mathrm{pt}}_\bullet(\eta)+\mathfrak{p}_1)}\left(\mathrm{sgn}(\mathbf{u}_\bullet)\prod_{i\in K^*_\bullet(\eta)}\mathrm{lk}(\partial\mathbf{u}_\bullet,\partial\mathfrak{b}_{\alpha_i(\eta)})\right)\\ &= (-1)^{|K(\alpha')|}\sum_{\mathbf{u}_\bullet\in\mathrm{SD}(\alpha^{\mathrm{pt}}_\bullet(\eta)+\mathfrak{p}_1)}\left(\mathrm{sgn}(\mathbf{u}_\bullet)\prod_{i\in K^*_\bullet(\eta)}\left(\sum_{\substack{\mathbf{u}_i\in\mathrm{MD}(\alpha_i(\eta))\\ T_i\in\mathrm{ST}(\mathbf{u}_i)}}\mathrm{sgn}(\mathbf{u}_i)\mathrm{lk}(\mathbf{u}_i;T_i)\mathrm{lk}(\partial\mathbf{u}_\bullet,\partial\mathbf{u}_i)\right)\right).\end{aligned} \tag{3.15}$$

Since $\mathrm{lk}(\partial\mathbf{u}_\bullet,\partial\mathbf{u}_i)=\sum_{\mathbf{u}_{i\bullet}\in\mathbf{u}_i}\mathrm{lk}(\partial\mathbf{u}_\bullet,\partial\mathbf{u}_{i\bullet})$, the last expression equals to

$$(-1)^{|K(\alpha')|}\sum_{\substack{[\eta]\in\mathcal{D}_\omega(\alpha')/\sim\\ \mathbf{u}_\bullet\in\mathrm{SD}(\alpha^{\mathrm{pt}}_\bullet(\eta)+\mathfrak{p}_1)}}\left(\sum_{\substack{(\mathbf{u}_i,\mathbf{u}_{i\bullet},T_i)\in\mathrm{DMD}(\alpha_i(\eta))\\ \text{for each } i\in K^*_\bullet(\eta)}}\mathrm{sgn}(\mathbf{u}_\bullet)\prod_{i\in K^*_\bullet(\eta)}\mathrm{sgn}(\mathbf{u}_i)\mathrm{lk}(\mathbf{u}_i;T_i)\mathrm{lk}(\partial\mathbf{u}_\bullet,\partial\mathbf{u}_{i\bullet})\right).$$

Using the bijectivity of the map (3.13), we thus obtain

$$-\deg\mathfrak{b}\mathfrak{b}_{\alpha'}=(-1)^{|K(\alpha')|}\sum_{\substack{(\mathbf{u},\mathbf{u}_\bullet,T)\in\mathrm{DMD}(\alpha'+\mathfrak{p}_1)\\ \mathbf{u}_\bullet\text{ passes thr. } p_1}}\mathrm{sgn}(\mathbf{u})\mathrm{lk}(\mathbf{u};T)=(-1)^{|K(\alpha')|}\sum_{\mathbf{u}\in\mathrm{MD}(\alpha'+\mathfrak{p}_1)}\mathrm{sgn}(\mathbf{u})\mathrm{lk}(\mathbf{u}).$$

Taking $\alpha'=(\beta,K-\{p_1\},L)$ above and using (1.6) and (3.4), we obtain

$$\langle L\rangle^{\omega,\mathfrak{os}}_{\beta,K-\{p_1\}}=(-1)^{|K|}\langle\Gamma_1,\ldots,\Gamma_l\rangle^{\omega,\mathfrak{os}}_{\beta,|K|}$$

and establish the claim. ■

Proof of Proposition 3.1. We prove both statements by induction on the set $\mathcal{C}_\omega(Y)$ with respect to the partial order $\prec$ defined in Section 2.2. It is sufficient to consider the elements $\alpha'\in\mathcal{C}_\omega(Y)$ with $\dim(\alpha')=0$ only.

Suppose $\alpha\in\mathcal{C}_\omega(Y)$ with $\dim(\alpha)=0$ and $(\mathfrak{b}_{\alpha'})_{\alpha'\in\mathcal{C}_{\omega;\alpha}(Y)}$ is a collection of bordered pseudocycles into Y satisfying the conditions of Definition 2.6 as well as the 2nd equality in (3.5) if $\dim(\alpha')=0$. By (3.14), Proposition 3.2 with $K'=K^*_\bullet(\eta)$, and (3.5) with

α' replaced by $\alpha_i(\eta)$,

$$\begin{aligned}\mathfrak{bb}_{[\eta]} &= (-1)^{|K^{\mathrm{pt}}_{\bullet}(\eta)|}\bigsqcup_{\mathbf{u}_\bullet\in\mathrm{SD}(\alpha^{\mathrm{pt}}_{\bullet}(\eta))}\Bigg(\mathrm{sgn}(\mathbf{u}_\bullet)\prod_{i\in K^*_\bullet(\eta)}\mathrm{lk}(\partial\mathbf{u}_\bullet,\partial\mathfrak{b}_{\alpha_i(\eta)})\Bigg)\,\partial\mathbf{u}_\bullet \\ &= (-1)^{|K(\alpha)|}\bigsqcup_{\mathbf{u}_\bullet\in\mathrm{SD}(\alpha^{\mathrm{pt}}_{\bullet}(\eta))}\Bigg(\mathrm{sgn}(\mathbf{u}_\bullet)\prod_{i\in K^*_\bullet(\eta)}\Bigg(\sum_{\substack{\mathbf{u}_i\in\mathrm{MD}(\alpha_i(\eta))\\ T_i\in\mathrm{ST}(\mathbf{u}_i)}}\mathrm{sgn}(\mathbf{u}_i)\mathrm{lk}(\mathbf{u}_i;T_i)\mathrm{lk}(\partial\mathbf{u}_\bullet,\partial\mathbf{u}_i)\Bigg)\Bigg)\,\partial\mathbf{u}_\bullet\,.\end{aligned}\tag{3.16}$$

Summing up (3.16) over the equivalence classes $[\eta]$ of η in $\mathcal{D}_\omega(\alpha)$ and using the bijectivity of the map (3.13), we obtain

$$\mathfrak{bb}_\alpha \equiv \bigsqcup_{[\eta]\in\mathcal{D}_\omega(\alpha)/\sim}\mathfrak{bb}_{[\eta]} = (-1)^{|K(\alpha)|}\bigsqcup_{\substack{\mathbf{u}\in\mathrm{MD}(\alpha)\\ \mathbf{u}_\bullet\in\mathbf{u},T\in\mathrm{ST}(\mathbf{u})}}\mathrm{sgn}(\mathbf{u})\mathrm{lk}(\mathbf{u};T)\partial\mathbf{u}_\bullet = (-1)^{|K(\alpha)|}\bigsqcup_{\mathbf{u}\in\mathrm{MD}(\alpha)}\mathrm{sgn}(\mathbf{u})\mathrm{lk}(\mathbf{u})\partial\mathbf{u}\,.$$

This establishes the 2nd equality in (3.5) with α' replaced by α. Along with the injectivity of (1.3), it implies that $\mathfrak{bb}_\alpha$ bounds in Y. Thus, we can choose a bordered pseudocycle $\mathfrak{b}_\alpha$ into Y satisfying the 1st equality in (3.5) with α' replaced by α. ■

3.3 Open divisor relation

We deduce Proposition 3.2 from the following lemma, which confirms the $K'=\{k\}$ case of this proposition.

Lemma 3.4. Let β, k, l, K, L, $(\mathfrak{b}_i)_{i\in K}$, and $(\Gamma_i)_{i\in L}$ be as in Proposition 3.2. If $k\in K$ and the codimension of $\mathfrak{b}_k$ is 1, then there exists a dense open subset $\mathfrak{M}^*_{k-1,l}$ of the target of the induced forgetful morphism

$$\mathfrak{f}^b_k\colon \mathfrak{M}^{\mathrm{uo},\bigstar}_{k,l}(\beta;J)\times_{\mathrm{fb}}\big((i,\mathfrak{b}_i)_{i\in K};(i,\Gamma_i)_{i\in L}\big)\longrightarrow \mathfrak{M}^{\mathrm{uo},\bigstar}_{k-1,l}(\beta;J)\times_{\mathrm{fb}}\Big((i,\mathfrak{b}_i)_{i\in K-\{k\}};(i,\Gamma_i)_{i\in L}\Big)\tag{3.17}$$

so that (3.17) restricts to a covering map over each connected component $\mathfrak{M}$ of $\mathfrak{M}^*_{k-1,l}$. If in addition the codimensions of all $\mathfrak{b}_i$ are odd and the codimensions of Γ_i are even, then the degree of this restriction is $-\mathrm{lk}(\partial\mathbf{u},\partial\mathfrak{b}_k)$ for any $\mathbf{u}\in\mathfrak{M}$.

Proof. We denote the right-hand side of (3.17) by M and define

$$K''=K-\{k\},\qquad \widetilde{M}\equiv\mathfrak{M}^{\mathrm{uo},\bigstar}_{k,l}(\beta;J)\times_{\mathrm{fb}}\big((i,\mathfrak{b}_i)_{i\in K''};(i,\Gamma_i)_{i\in L}\big)\,.$$

By Lemma 2.4,

$$\text{LHS of (3.17)} = (-1)^{|K|-1}\left(\widetilde{M}_{\mathrm{evb}_k}\times_{\mathfrak{b}_k}(\mathrm{dom}\,\mathfrak{b}_k)\right). \tag{3.18}$$

If the pseudocycles $\mathfrak{b}_i$ with $i\in K$ have odd codimensions and the pseudocycles Γ_i have even codimensions, then

$$\dim \widetilde{M}_{\mathrm{evb}_k}\times_{\mathfrak{b}_k}(\mathrm{dom}\,\mathfrak{b}_k) \cong k-|K| \mod 2. \tag{3.19}$$

Let $M'\subset M$ be the image of the elements of the left-hand side of (3.17), which meet the boundary of any of the pseudocycles $\mathfrak{b}_i$ and Γ_i or the pairwise intersection of any pair of these pseudocycles. The dense open subset $\mathfrak{M}^*_{k-1,l}$ of M is the open subset of $M-M'$ consisting of the maps $\mathbf{u}$ from $\mathbb{D}^2$ with $\partial\mathbf{u}$ transverse to $\mathfrak{b}_k$.

We compute the sign of $\mathfrak{f}^b_k$ at a preimage $(\widetilde{\mathbf{u}},q_k)$ of $\mathbf{u}$ in the fiber product space in (3.18) under (3.17). Denote the k-th boundary marked point of $\widetilde{\mathbf{u}}$ by x_k and the image of $\mathbf{u}$ in $Y^{K''}\times X^I$ by y. All rows and the right column in the 1st diagram of Figure 6 are orientation compatible. The short exact sequence

$$0\longrightarrow T_{x_k}S^1\longrightarrow T_{\widetilde{\mathbf{u}}'}\mathfrak{M}^{\mathrm{uo}}_{k,l}(\beta;J)\longrightarrow T_{\mathbf{u}'}\mathfrak{M}^{\mathrm{uo}}_{k-1,l}(\beta;J)\longrightarrow 0,$$

where $\widetilde{\mathbf{u}}'$ and $\mathbf{u}'$ are the projections of $\widetilde{\mathbf{u}}$ and $\mathbf{u}$, respectively, to the corresponding disk moduli spaces, has sign $(-1)^{k-1}$. Along with Lemma 2.1, this implies that the middle and left columns in the 1st diagram also have signs $(-1)^{k-1}$. Thus, the middle column in the 2nd diagram has sign $(-1)^{k-1}$ as well. The middle row and the side columns in this diagram are orientation compatible. The sign of the top row is the sign of (x_k,q_k) in the fiber product $(\partial\mathbf{u})\times_{\mathrm{fb}}\mathfrak{b}_k$. Along with Lemma 2.1 and (3.19), this implies that the sign of the bottom row is $(-1)^{|K|-1}$ times the sign of (x_k,q_k) in the fiber product $(\partial\mathbf{u})\times_{\mathrm{fb}}\mathfrak{b}_k$.

Combining the last conclusion with (3.18), we obtain

$$\sum_{(\widetilde{\mathbf{u}},q_k)\in\{\mathfrak{f}^b_k\}^{-1}(\mathbf{u})}\mathrm{sgn}\left(\mathrm{d}_{(\widetilde{\mathbf{u}},q_k)}\mathfrak{f}^b_k\right) = \left|(\partial\mathbf{u})\times_{\mathrm{fb}}\mathfrak{b}_k\right|^{\pm}.$$

Along with (3.1), this establishes the degree claim. ■

Proof of Proposition 3.2. The 1st claim follows immediately from the 1st claim of Lemma 3.4. By Lemma 2.3, a reordering of the pseudocycles $\mathfrak{b}_i$s with $i=k',\ldots,k$ does not change the oriented space on the left-hand side of (3.6). We can thus assume that

$$K' = \{k-|K'|+1,\ldots,k\}\subset[k].$$

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & T_{x_k}S^1 \arrow[r, "\mathrm{Id}"] \arrow[d] & T_{x_k}S^1 \arrow[r] \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & T_{\tilde{\mathbf{u}}}\widetilde{M} \arrow[r] \arrow[d] & T_{\tilde{\mathbf{u}}}\Big(\mathfrak{M}^{\mathrm{uo}}_{k,l}(\beta;J)\times\prod\limits_{i\in K''}(\mathrm{dom}\,\mathfrak{b}_i)\times\prod\limits_{i\in L}(\mathrm{dom}\,\Gamma_i)\Big) \arrow[r] \arrow[d] & \mathcal{N}\Delta_{Y^{K''}\times X^L}\big|_y \arrow[r] \arrow[d, "\mathrm{Id}"'] & 0 \\
0 \arrow[r] & T_{\mathbf{u}}M \arrow[r] \arrow[d] & T_{\mathbf{u}}\Big(\mathfrak{M}^{\mathrm{uo}}_{k-1,l}(\beta;J)\times\prod\limits_{i\in K''}(\mathrm{dom}\,\mathfrak{b}_i)\times\prod\limits_{i\in L}(\mathrm{dom}\,\Gamma_i)\Big) \arrow[r] \arrow[d] & \mathcal{N}\Delta_{Y^{K''}\times X^L}\big|_y \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] \arrow[d] & T_{x_k}S^1\oplus T_{q_k}(\mathrm{dom}\,\mathfrak{b}_k) \arrow[r, "\mathrm{d}_{q_k}\mathfrak{b}_k-\mathrm{d}_{x_k}\partial\mathbf{u}"] \arrow[d] & T_{\mathfrak{b}_k(q_k)}Y \arrow[r] \arrow[d, "\mathrm{Id}"'] & 0 \\
0 \arrow[r] & T_{(\tilde{\mathbf{u}},q_k)}(\widetilde{M}_{\mathrm{evb}_k}\times_{\mathfrak{b}_k}(\mathrm{dom}\,\mathfrak{b}_k)) \arrow[r] \arrow[d, "\mathrm{Id}"] & T_{\tilde{\mathbf{u}}}\widetilde{M}\oplus T_{q_k}\mathrm{dom}(\mathfrak{b}_k) \arrow[r, "\mathrm{d}_{q_k}\mathfrak{b}_k-\mathrm{d}_{\tilde{\mathbf{u}}}\mathrm{evb}_k"] \arrow[d] & T_{\mathfrak{b}_k(q_k)}Y \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T_{(\tilde{\mathbf{u}},q_k)}(\widetilde{M}_{\mathrm{evb}_k}\times_{\mathfrak{b}_k}(\mathrm{dom}\,\mathfrak{b}_k)) \arrow[r, "\mathrm{d}_{(\tilde{\mathbf{u}},q_k)}\mathfrak{f}^b_k"] \arrow[d] & T_{\mathbf{u}}M \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0 & &
\end{tikzcd}
$$

Fig. 6. Commutative squares of exact sequences for the proof of Lemma 3.4.

The 2nd claim then follows from the 2nd claim of Lemma 3.4 by induction. ■

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References

[1] Chen, X. "Steenrod pseudocycles, lifted cobordisms, and Solomon's relations for Welschinger's invariants." (forthcoming) math/1809.08919v2.

[2] Chen, X. "A geometric depiction of Solomon–Tukachinsky's construction of open GW-invariants." (forthcoming) math/1912.04119.

[3] Chen, X. and A. Zinger. "WDVV-type relations for disk Gromov–Witten invariants in dimension 6." (forthcoming) math/1904.04254.

[4] Fukaya, K. "Counting pseudo-holomorphic discs in Calabi–Yau 3-folds." *Tohoku Math. J.* 63, no. 4 (2011): 697–727.

[5] Fukaya, K., Y.-G. Oh, H. Ohta, and K. Ono. *Lagrangian Intersection Floer Theory: Anomaly and Obstruction*. AMS/IP Studies in Advanced Mathematics 46. Somerville, MA: American Mathematical Society, Providence, RI; International Press, 2009.

[6] Fukaya, K. and K. Ono. "Arnold conjecture and Gromov–Witten invariant." *Topology* 38, no. 5 (1999): 933–1048.

[7] Georgieva, P. "The orientability problem in open Gromov–Witten theory." *Geom. Topol.* 17, no. 4 (2013): 2485–512.

[8] Hofer, H., K. Wysocki, and E. Zehnder. "Applications of polyfold theory I: the polyfolds of Gromov–Witten theory." *Mem. Amer. Math. Soc.* 248 (2017): 1179.

[9] Li, J. and G. Tian. "Virtual Moduli Cycles and Gromov–Witten Invariants of General Symplectic Manifolds." In *Topics in symplectic 4-manifolds (Irvine, CA, 1996)*, 47–83. First Int. Press Lect. Ser. I. Internat. Press, Cambridge, MA, 1998.

[10] Pardon, J. "An algebraic approach to virtual fundamental cycles on moduli spaces of pseudo-holomorphic curves." *Geom. Topol.* 20, no. 2 (2016): 779–1034.

[11] Solomon, J. "Intersection theory on the moduli space of holomorphic curves with Lagrangian boundary conditions." (forthcoming) math/0606429v3.

[12] Solomon, J. and S. Tukachinsky. "Point-like bounding chains in open Gromov–Witten theory." (forthcoming) math/1608.02495v2.

[13] Solomon, J. and S. Tukachinsky. "Relative quantum cohomology." (forthcoming) math/1906.04795v2.

[14] Welschinger, J.-Y. "Invariants of real symplectic 4-manifolds and lower bounds in real enumerative geometry." *Invent. Math.* 162, no. 1 (2005): 195–234.

[15] Welschinger, J.-Y. "Spinor states of real rational curves in real algebraic convex 3-manifolds and enumerative invariants." *Duke Math. J.* 127, no. 1 (2005): 89–121.

[16] Welschinger, J.-Y. "Enumerative invariants of strongly semipositive real symplectic six-manifolds." (forthcoming) math/0509121v2.

[17] Welschinger, J.-Y. "Open Gromov–Witten invariants in dimension six." *Math. Ann.* 356, no. 3 (2013): 1163–82.

[18] Zinger, A. "Pseudocycles and integral homology." *Trans. Amer. Math. Soc.* 360, no. 5 (2008): 2741–65.

[19] Zinger, A. "The determinant line bundle for Fredholm operators: construction, properties, and classification." *Math. Scand.* 118, no. 2 (2016): 203–68.