



On integral models of Shimura varieties

Georgios Pappas¹

Received: 13 June 2021 / Revised: 8 February 2022 / Accepted: 4 March 2022

© The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2022

Abstract

We show how to characterize integral models of Shimura varieties over places of the reflex field where the level subgroup is parahoric by formulating a definition of a “canonical” integral model. We then prove that in Hodge type cases and under a tameness hypothesis, the integral models constructed by the author and Kisin in previous work are canonical and, in particular, independent of choices. A main tool is a theory of displays with parahoric structure that we develop in this paper.

Contents

1	Introduction	1
1.10	Notations	1
2	Algebraic preliminaries	2
2.1	Witt vectors and variants	2
2.4	Some perfectoid rings	2
2.5	Period rings	2
3	Shimura pairs and $\mathcal{G}$ -torsors	3
2.1	Shimura pairs	3
3.2	Torsors	3
4	Displays with $\mathcal{G}$ -structure	4
4.1	The construction of the modification	4
4.2	$(\mathcal{G}, \mathcal{M})$ -displays	4
4.3	Dieudonné $(\mathcal{G}, \mathcal{M})$ -displays	4
4.4	Relation with Zink’s displays	4
4.5	Rigidity and locally universal displays	4
5	Crystalline $\mathcal{G}$ -representations	5
5.2	The Breuil–Kisin $\mathcal{G}$ -module	5

Communicated by Roseline Periyanyagam.

Partially supported by NSF grants DMS-1701619, DMS-2100743, and the Bell Companies Fellowship Fund through the Institute for Advanced Study.

✉ Georgios Pappas
pappasg@msu.edu

¹ Department of Mathematics, Michigan State University, East Lansing, MI 48824, USA

5.3	The Dieudonné $\mathcal{G}$ -display	
5.4	The Breuil–Kisin–Fargues $\mathcal{G}$ -module	
6	Associated systems	
6.1	Local systems and associated systems	
6.2	Local systems and associated displays	
6.3	Rigidity and uniqueness	
6.4	Existence of associated systems	
6.5	Independence	
7	Canonical integral models	
8	Shimura varieties at tame parahoric primes	
	References	

1 Introduction

1.1. In this paper, we show how to uniquely characterize integral models of Shimura varieties over some primes where non-smooth reduction is expected. More specifically, we consider integral models over primes p at which the level subgroup is parahoric. Then, under some further assumptions, we provide a notion of a “canonical” integral model.

At such primes, the Shimura varieties have integral models with complicated singularities [31, 32]. This happens even for the most commonly used Shimura varieties with level structure, such as Siegel varieties, and it foils attempts to characterize the models by simple conditions. The main observation of this paper is that we can characterize these integral models by requiring that they support suitable “ $\mathcal{G}$ -displays”, i.e. filtered Frobenius modules with $\mathcal{G}$ -structure, where $\mathcal{G}$ is the smooth integral p -adic group scheme which corresponds to the level subgroup. We then prove that these modules exist in most Hodge type cases treated by the author and Kisin in [20]. As a corollary, we show that these integral models of Shimura varieties with parahoric level structure, are independent of the choices made in their construction.

Let us first recall the story over “good” places, i.e. over primes at which the level subgroup is hyperspecial. One expects that there is an integral model with smooth reduction at such primes. This expectation was first spelled out by Langlands in the 80’s. Later, it was pointed out by Milne [26] that one can uniquely characterize smooth integral models over the localization of the reflex field at such places by requiring that they satisfy a Neron-type extension property. Milne calls smooth integral models with this property “canonical”. The natural integral models of Siegel Shimura varieties, at good primes, are smooth and satisfy the extension property. Therefore, they are canonical. In this case, the extension property follows by the Neron-Ogg-Shafarevich criterion and a purity result of Vasiu and Zink [38] about extending abelian schemes over codimension ≥ 2 subschemes of smooth schemes. This argument extends to the very general class of Shimura varieties of abelian type at good primes, provided we can show there is a smooth integral model which is, roughly speaking, constructed using moduli of abelian varieties. This existence of such a canonical smooth integral model for Shimura varieties of abelian type at places over good primes was shown by Kisin ([19], see also earlier work of Vasiu [37]).

The problem becomes considerably harder over other primes. Here, we are considering primes p at which the level subgroup is parahoric. For the most part, we also

require that the reductive group splits over a tamely ramified extension, although our formulation is more general. Under these assumptions, models for Shimura varieties of abelian type, integral at places over such p , were constructed by Kisin and the author [20]. This follows work of Rapoport and Zink [32], of Rapoport and the author, and of many others, see [29]. The construction in [20] uses certain simpler schemes, the “local models” that depend only on the local Shimura data. Then, integral models for Shimura varieties of Hodge type are given by taking the normalization of the Zariski closure of a well-chosen embedding of the Shimura variety in a Siegel moduli scheme over the integers. More generally, models of Shimura varieties of abelian type are obtained from those of Hodge type by a quotient construction that uses Deligne’s theory of connected Shimura varieties. All these integral models of Shimura varieties have the same étale local structure as the corresponding local models. However, the problem of characterizing them globally or showing that they are independent of choices was not addressed in loc. cit.¹ Here, we give a broader notion of “canonical” integral model and solve these problems when the varieties are of Hodge type. Such a characterization was not known before, not even for general PEL type Shimura varieties.

1.2. Let us now explain these results more carefully.

Let (G, X) be a Shimura datum [10] with corresponding conjugacy class of minuscule cocharacters $\{\mu\}$ and reflex field E . To fix ideas, we will always assume that the center $Z(G)$ of G has the same $\mathbb{Q}$ -split rank as $\mathbb{R}$ -split rank. (This condition holds for Shimura data of Hodge type.) For an open compact subgroup $K \subset G(\mathbb{A}_f)$ of the finite adelic points of G , the Shimura variety

$$\mathrm{Sh}_K(G, X) = G(\mathbb{Q}) \backslash (X \times G(\mathbb{A}_f) / K)$$

has a canonical model over E .

Fix an odd prime p . Suppose $K = K_p K^p$, with $K_p \subset G(\mathbb{Q}_p)$ and $K^p \subset G(\mathbb{A}_f^p)$, both compact open, with K^p sufficiently small. Denote by $\mathcal{L}_K$ the pro-étale $\mathcal{G}(\mathbb{Z}_p)$ -cover

$$\mathcal{L}_K := \varprojlim_{K'_p \subset K_p} \mathrm{Sh}_{K'_p K^p}(G, X) \rightarrow \mathrm{Sh}_K(G, X),$$

with K'_p running over all compact open subgroups of K_p .

Assume that:

(a) The level K_p is a *parahoric* subgroup in the sense of Bruhat-Tits [36], i.e. K_p is the neutral component of the stabilizer of a point in the (enlarged) building of $G(\mathbb{Q}_p)$. Then $K_p = \mathcal{G}(\mathbb{Z}_p)$, where $\mathcal{G}$ is the corresponding parahoric smooth connected affine group scheme over $\mathbb{Z}_p$ with $\mathcal{G} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = G_{\mathbb{Q}_p}$ ([36]).

We will occasionally assume the slightly stronger:

(a+) The level K_p is a *connected parahoric* subgroup. By definition, this means that K_p is parahoric and is the stabilizer of a point in the enlarged building, i.e. we can choose the point such that the stabilizer is actually connected. (Such a subgroup is sometimes also called a *parahoric stabilizer*.)

¹ with the exception of the very restricted result [20, Prop. 4.6.28].

Now choose a place v of E over p . Let $\mathcal{O}_{E,(v)}$ be the localization of the ring of integers $\mathcal{O}_E$ at v . Denote by $\widehat{E}$ the completion of E at v , by $\widehat{\mathcal{O}_E}$ the integers of $\widehat{E}$ and fix an algebraic closure k of the residue field k_E of E . We can also consider $\{\mu\}$ as a conjugacy class of cocharacters which is defined over E . Under some mild assumptions (see Sect. 2.1), we have the local model

$$M^{\text{loc}} = M^{\text{loc}}(\mathcal{G}, \{\mu\}),$$

as characterized by [35, Conj. 21.4.1]. This is a flat and projective $\widehat{\mathcal{O}_E}$ -scheme with $\mathcal{G}$ -action. Its generic fiber is G_E -equivariantly isomorphic to the variety X_μ of parabolic subgroups of G_E of type μ , and its special fiber is reduced.

We will assume:

(b) There is a closed group scheme immersion $\iota : \mathcal{G} \hookrightarrow \text{GL}_n$ over $\widehat{\mathbb{Z}_p}$ such that $\iota(\mu)$ is conjugate to one of the standard minuscule cocharacters of GL_n , $\iota(\mathcal{G})$ contains the scalars, and the map ι gives an equivariant closed immersion

$$\iota_* : M^{\text{loc}} \hookrightarrow \text{Gr}(d, n)_{\widehat{\mathcal{O}_E}}$$

in a Grassmannian, where d is determined by $\iota(\mu)$.

Under the assumptions (a) and (b), we define “ $(\mathcal{G}, \{\mu\})$ -displays” which are group-theoretic generalizations of Zink’s displays [39, 40]. This is the main invention in the paper, see below. We think it has some independent interest.

We now ask for $\mathcal{O}_{E,(v)}$ -models $\mathcal{S}_K = \mathcal{S}_{K_p K^p}$ (separated schemes of finite type and flat over $\mathcal{O}_{E,(v)}$) of the Shimura variety $\text{Sh}_K(G, X)$ which are normal. In addition, we require:

1. For $K'^p \subset K^p$, there are finite étale morphisms

$$\pi_{K'_p, K_p} : \mathcal{S}_{K_p K'^p} \rightarrow \mathcal{S}_{K_p K^p}$$

which extend the natural morphism $\text{Sh}_{K_p K'^p}(G, X) \rightarrow \text{Sh}_{K_p K^p}(G, X)$.

2. The scheme $\mathcal{S}_{K_p} = \varprojlim_{K^p} \mathcal{S}_{K_p K^p}$ satisfies the “extension property” for dvrs of mixed characteristic $(0, p)$, i.e. for any such dvr R

$$\mathcal{S}_{K_p}(R[1/p]) = \mathcal{S}_{K_p}(R).$$

3. The p -adic formal schemes $\widehat{\mathcal{S}}_K = \varprojlim_n \mathcal{S}_K \otimes_{\widehat{\mathcal{O}_{E,(v)}}} \widehat{\mathcal{O}_{E,(v)}}/(p)^n$ support locally universal $(\mathcal{G}, \{\mu\})$ -displays $\mathcal{D}_K$ which are associated with $\mathcal{L}_K$. We ask that these are compatible for varying K^p , i.e. that there are compatible isomorphisms

$$\pi_{K'_p, K_p}^* \mathcal{D}_K \simeq \mathcal{D}_{K'}$$

over the system of morphisms $\pi_{K'_p, K_p}$ of (1).

We will explain below the rest of the terms in (3) including the meaning of having a $(\mathcal{G}, \{\mu\})$ -display being “associated” with the pro-étale $\mathcal{G}(\widehat{\mathbb{Z}_p})$ -cover $\mathcal{L}_K$. Our first main result is (always under the standing hypothesis on the center $Z(G)$, also $p \neq 2$):

Theorem 1.3 Assume (G, X, v, K_p) satisfy (a) and (b) above. Then there is at most one (up to unique isomorphism), pro-system of normal $\mathcal{O}_{E,(v)}$ -models $\mathcal{S}_K = \mathcal{S}_{K_p K^p}$ of the Shimura variety $\mathrm{Sh}_K(G, X)$ which satisfy (1), (2) and (3) above.

In fact, we prove a slightly more general result. (See Theorem 7.1.7 and Corollary 7.1.8.)

We call integral models $\mathcal{S}_K$ which satisfy the above, *canonical*.

Assume now that the global Shimura datum (G, X) is of Hodge type and that G splits over a tamely ramified extension of $\mathbb{Q}_p$. Then, under the assumption (a+), assumption (b) is also satisfied. In this situation, “nice” integral models $\mathcal{S}_K(G, X)$ of the Shimura variety $\mathrm{Sh}_K(G, X)$ have been constructed in [20, § 4], see [20, Theorem 4.2.7]. These integral models depend, a priori, on the choice of a suitable Hodge embedding. Our second main result is:

Theorem 1.4 (Theorem 8.2.1) Assume (G, X) is of Hodge type, G splits over a tamely ramified extension of $\mathbb{Q}_p$, and K_p is connected parahoric. The integral models $\mathcal{S}_K(G, X)$ of [20, Theorem 4.2.7] satisfy (1), (2), and (3).

Since the integral models $\mathcal{S}_K(G, X)$ of [20, Theorem 4.2.7] are normal, by combining the two results, we obtain:

Corollary 1.5 (Theorem 8.1.6) Assume (G, X) is of Hodge type, G splits over a tamely ramified extension of $\mathbb{Q}_p$, and K_p is connected parahoric. The integral models $\mathcal{S}_K(G, X)$ of [20, Theorem 4.2.7] are, up to unique isomorphism, independent of the choices in their construction.

1.6. We now explain the terms that appear in condition (3). For more details, the reader is referred to the main body of the paper.

Suppose R is a p -adic flat $\mathcal{O}_E$ -algebra. Denote by $W(R)$ the ring of (p -typical) Witt vectors with entries from R and by $\varphi : W(R) \rightarrow W(R)$ the Frobenius endomorphism.

A $(\mathcal{G}, M^{\mathrm{loc}})$ -display $\mathcal{D} = (\mathcal{P}, q, \Psi)$ over R consists of a $\mathcal{G}$ -torsor $\mathcal{P}$ over $\mathrm{Spec}(W(R))$, a $\mathcal{G}$ -equivariant morphism

$$q : \mathcal{P} \times_{\mathrm{Spec}(W(R))} \mathrm{Spec}(R) \rightarrow M^{\mathrm{loc}},$$

and a $\mathcal{G}$ -isomorphism $\Psi : \mathcal{Q} \xrightarrow{\sim} \mathcal{P}$. Here, $\mathcal{Q}$ is a $\mathcal{G}$ -torsor over $\mathrm{Spec}(W(R))$ which is the modification of $\varphi^*\mathcal{P}$ along $p = 0$, given by q (see Proposition/Construction 4.1.2).

By its construction, $\mathcal{Q}$ comes together with an isomorphism of G -torsors

$$\varphi^*\mathcal{P}_{|\mathrm{Spec}(W(R)[1/p])} \xrightarrow{\sim} \mathcal{Q}_{|\mathrm{Spec}(W(R)[1/p])}$$

over $\mathrm{Spec}(W(R)[1/p])$. Composing this isomorphism with $\Psi_{|\mathrm{Spec}(W(R)[1/p])}$ gives

$$\Phi : \varphi^*\mathcal{P}_{|\mathrm{Spec}(W(R)[1/p])} \xrightarrow{\sim} \mathcal{P}_{|\mathrm{Spec}(W(R)[1/p])}.$$

This is the “Frobenius” of the $(\mathcal{G}, M^{\mathrm{loc}})$ -display.

Since M^{loc} is determined by $(\mathcal{G}, \{\mu\})$ we often just say “ $(\mathcal{G}, \{\mu\})$ -display”. The definition of a $(\mathcal{G}, \{\mu\})$ -display resembles that of a shtuka and of its mixed characteristic variants ([35]). (In the shtuka lingo, one would say that “there is one leg along $p = 0$, bounded by μ ”.) It is also a generalization of the concept of $(\mathcal{G}, \{\mu\})$ -display due to the author and Bültel [7]. (This required the restrictive assumption that $\mathcal{G}$ is reductive over $\mathbb{Z}_p$; the extension to the parahoric case is far from obvious). When $\mathcal{G} = \text{GL}_n$ and $M^{\text{loc}} = \text{Gr}(d, n)$, it amounts to a display of height n and dimension $n - d$ in the sense of Zink (see Sect. 4.4). By using Zink’s Witt vector descent, we obtain a straightforward extension of this definition from $\text{Spf}(R)$ to non-affine p -adic formal schemes like $\mathcal{S}_K$.

If R is, in addition, a Noetherian complete local ring with perfect residue field and $p \neq 2$, there is a similar notion of a Dieudonné $(\mathcal{G}, \{\mu\})$ -display over R for which all the above objects $\mathcal{P}, q, \Psi$, are defined over Zink’s variant $\hat{W}(R)$ of the Witt ring. After applying a local Hodge embedding $\iota : \mathcal{G} \hookrightarrow \text{GL}_n$, a Dieudonné $(\mathcal{G}, \{\mu\})$ -display $\mathcal{D}$ induces a classical Dieudonné display over R . By Zink’s theory [39, 40], this gives a p -divisible group over R .

In (3), we ask that $\mathcal{D}_K = (\mathcal{P}_K, q_K, \Psi_K)$ is “associated” with $\mathcal{L}_K$. This definition (Definition 6.2.1) is modeled on the notion of “associated” used by Faltings, e.g. [12]. It requires that the $\mathbb{Z}_p$ -étale local system over the generic fiber given by the Tate module of the p -divisible group obtained from $\mathcal{D}_K$ and ι as above, agrees with the local system given by $\mathcal{L}_K$ and ι . It also requires that, after specializing at each point with values in a mixed characteristic dvr, the étale-crystalline comparison isomorphism sends the étale (i.e. Galois invariant) tensors defining $\mathcal{L}_K$ to the crystalline tensors defining $\mathcal{P}_K$.

Finally, we explain the term “locally universal” in (3):

For $\bar{x} \in \mathcal{S}_K(k)$, denote by $\hat{R}_{\bar{x}}$ the completion of the strict Henselization of the local ring of $\mathcal{S}_K$ at $\bar{x}$. Set $\mathcal{D}_{K, \bar{x}}$ for the $(\mathcal{G}, \{\mu\})$ -display over $\hat{R}_{\bar{x}}$ obtained from $\mathcal{D}_K$ by base change.

We say that the $(\mathcal{G}, \{\mu\})$ -display $\mathcal{D}_K$ is locally universal² if for all $\bar{x} \in \mathcal{S}_K(k)$, there is a “rigid” section $s_{\bar{x}}$ of the $\mathcal{G}$ -torsor $\mathcal{P}_{K, \bar{x}}$ over $\text{Spec}(W(\hat{R}_{\bar{x}}))$, so that the composition

$$q \cdot (s_{\bar{x}} \times 1) : \text{Spec}(\hat{R}_{\bar{x}}) \rightarrow \mathcal{P}_{K, \bar{x}} \times_{\text{Spec}(W(\hat{R}_{\bar{x}}))} \text{Spec}(\hat{R}_{\bar{x}}) \rightarrow M^{\text{loc}}$$

gives an isomorphism between $\hat{R}_{\bar{x}}$ and the corresponding completed local ring of the local model M^{loc} . Therefore, condition (3) fixes the “singularity type” of $\mathcal{S}_K$ at $\bar{x}$. In the above, rigid is for a version of the Gauss-Manin connection in our set-up.

Note that this somewhat unusual definition is justified by the fact that $(\mathcal{G}, \{\mu\})$ -displays are defined only over $\mathbb{Z}_p$ -flat bases, so these objects do not have deformation theory in the usual sense.

1.7. Let us discuss the proofs of the two main theorems.

We use an intermediate technical notion, the “associated system” $(\mathcal{L}_K, \{\mathcal{D}_{K, \bar{x}}\}_{\bar{x} \in \mathcal{S}_K(k)})$ (Definition 6.1.5), in which $\mathcal{D}_{K, \bar{x}}$ are Dieudonné $(\mathcal{G}, \{\mu\})$ -displays over the completions of the strict Henselizations $\hat{R}_{\bar{x}}$, as before. A $(\mathcal{G}, \{\mu\})$ -display $\mathcal{D}_K$ which is associated with $\mathcal{L}_K$, gives such an associated system by base change, as above. In fact, most of our constructions just use $(\mathcal{L}_K, \{\mathcal{D}_{K, \bar{x}}\}_{\bar{x} \in \mathcal{S}_K(k)})$. Using Tate’s theorem on

² A more correct, but also more cumbersome, term would probably be “locally formally universal”.

extending homomorphisms of p -divisible groups, we show that $(\mathcal{L}_K, \{\mathcal{D}_{K,\bar{x}}\}_{\bar{x} \in \mathcal{S}_K(k)})$ is uniquely determined by $\mathcal{L}_K$ and that the existence of a locally universal associated system is enough to characterize the integral model $\mathcal{S}_K$ uniquely. This leads to the proof of Theorem 1.3. We also show that when $\mathcal{L}_K$ comes from the Tate module of a p -divisible group with Galois invariant tensors, it can be completed to a (unique up to isomorphism) associated system. This employs comparison isomorphisms of integral p -adic Hodge theory that use work of Kisin, Scholze and others [4, 35], and of Faltings [12, 13].

To show Theorem 1.4, we first show that the integral models given in [20] support a locally universal associated system. The issue is to show that the unique associated system obtained as above is locally universal. (Recall that this imposes that the singularities of the integral model $\mathcal{S}_K$ agree with those of the local model M^{loc} .) This is done, by reexamining the proof of the main result of [20]. Then, as in work of Hamacher and Kim [16], we can also show that these models support a (global) locally universal $(\mathcal{G}, M^{\text{loc}})$ -display $\mathcal{D}_K$ associated with $\mathcal{L}_K$. It then follows, that in this case, the integral models of [20] are canonical as per the definition above.

Our point of view fits with the well-established idea, going back to Deligne, that most Shimura varieties should be moduli spaces of G -motives with level structures. As such, they should have (integral) canonical models. We can not make this precise yet. However, we consider the locally universal $\mathcal{G}$ -display as the crystalline avatar of the universal G -motive and show that its existence is enough to characterize the integral model. In fact, there should be versions of this characterization for other p -adic cohomology theories (see [2, 4, 33]). In joint work of the author with M. Rapoport [28], which developed after the first version of this paper was written, this idea is realized and a different characterization is given. This uses, instead of $\mathcal{G}$ -displays, p -adic $\mathcal{G}$ -stukas over the v -sheaf $\mathcal{S}_K^\diamond$, as defined by Scholze (see [35]). Our approach in the current paper is more “classical”, since it does not use any of the tools of modern p -adic analysis (perfectoid adic spaces, v -sheaves and diamonds, Banach-Colmez spaces, etc.) and when it applies it gives somewhat more precise information.

1.8. Finally, we give an outline of the contents of the paper. In Sect. 2 we prove some preliminary facts about rings of Witt vectors and other p -adic period rings, that are used in the constructions. We continue on to define some terminology and give some more preliminaries on torsors in Sect. 3. In Sect. 4 we give the definition of (Dieudonné) displays with $\mathcal{G}$ -structure. We also construct various relevant structures such an infinitesimal connection and give the notion of a rigid section of a $\mathcal{G}$ -display. In Sect. 5, we show how to construct, using the theory of Breuil–Kisin modules, such a display from a $\mathcal{G}(\mathbb{Z}_p)$ -valued crystalline representation of a p -adic field. We also give other similar constructions, for example a corresponding Breuil–Kisin–Fargues $\mathcal{G}$ -module. In Sect. 6, we give the definition of an associated system and show how to compare two normal schemes with the same generic fiber which both support locally universal associated systems for the same local system. We also show how to give an associated system starting from a p -divisible group whose Tate module carries suitable Galois invariant tensors. In Sect. 7, we apply this to Shimura varieties and show that systems $(\mathcal{S}_K, \mathcal{L}_K, \{\mathcal{D}_{K,\bar{x}}\}_{\bar{x} \in \mathcal{S}_K(k)})$ as above are unique. In Sect. 8, we prove

that the integral models of Shimura varieties of Hodge type constructed in [20] carry locally universal associated systems and are therefore uniquely determined.

1.10. Notations

Throughout the paper p is a prime and, as usual, we denote by $\mathbb{Z}_p$, $\mathbb{Q}_p$, the p -adic integers, resp. p -adic numbers. We fix an algebraic closure $\bar{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. If F is a finite extension of $\mathbb{Q}_p$, we will denote by $\mathcal{O}_F$ its ring of integers, by k_F its residue field and by $\tilde{F}$ the completion of the maximal unramified extension of F in $\bar{\mathbb{Q}}_p$. We often write $X \otimes_A B$ to denote the base change of a scheme X over $\mathrm{Spec}(A)$ to $\mathrm{Spec}(B)$.

2 Algebraic preliminaries

2.1.1. We begin with some preliminaries about rings of Witt vectors and other p -adic period rings. The proofs can be omitted on the first reading.

We consider the following conditions for a $\mathbb{Z}_p$ -algebra R . In what follows, we will have to assume some of these.

1. R is complete and separated for the adic topology given by a finitely generated ideal $\mathcal{A}$ that contains p , and
 2. R is formally of finite type over $W = W(k)$, where k is a perfect field of characteristic p .
- (N) R is a *normal* domain, is flat over $\mathbb{Z}_p$, and satisfies (1) and (2).
 (CN) R satisfies (N) and is a complete local ring.

By “ p -adic”, we always mean p -adically complete and separated.

2.1 Witt vectors and variants

For a $\mathbb{Z}_{(p)}$ -algebra R , we denote by

$$W(R) = \{(r_1, r_2, \dots, r_n, \dots) \mid r_i \in R\}$$

the ring of (p -typical) Witt vectors of R . Let $\varphi : W(R) \rightarrow W(R)$ and $V : W(R) \rightarrow W(R)$ denote the Frobenius and Verschiebung, respectively. Let $I_R = V(W(R)) \subset W(R)$ be the ideal of elements with $r_1 = 0$. The projection $(r_1, r_2, \dots) \mapsto r_1$ gives an isomorphism $W(R)/I_R \simeq R$. For $r \in R$, we set as usual

$$[r] = (r, 0, 0, \dots) \in W(R)$$

for the Teichmüller representative. Also denote by

$$\mathrm{gh} : W(R) \rightarrow \prod_{i \geq 1} R$$

the Witt (“ghost”) coordinates $\text{gh} = (\text{gh}_i)_i$. Recall,

$$\text{gh}_i((r_1, r_2, r_3, \dots)) = r_1^{p^{i-1}} + pr_2^{p^{i-2}} + \dots + p^{i-1}r_i.$$

2.1.2. Let R be a complete Noetherian local ring with maximal ideal $\mathfrak{m}_R$ and perfect residue field k of characteristic p . Assume that $p \geq 3$. There is a splitting $W(R) = W(k) \oplus W(\mathfrak{m}_R)$ and following Zink [39], we can consider the subring

$$\hat{W}(R) = W(k) \oplus \hat{W}(\mathfrak{m}_R) \subset W(R),$$

where $\hat{W}(\mathfrak{m}_R)$ consists of those Witt vectors with $r_n \in \mathfrak{m}_R$ for which the sequence r_n converges to 0 in the $\mathfrak{m}_R$ -topology of R . The subring $\hat{W}(R)$ is stable under φ and V . In this case, both $W(R)$ and $\hat{W}(R)$ are p -adically complete and separated local rings.

2.2. In this section, we assume that the $\mathbb{Z}_p$ -algebra R satisfies (1) and (2):

Let $\mathfrak{M} \subset W(R)$ be a maximal ideal with residue field k' . We have $I_R \subset \mathfrak{M}$, since $W(R)$ is I_R -adically complete and separated [40, Prop. 3]. Let $\mathfrak{m}_R = \mathfrak{M}/I_R \subset W(R)/I_R = R$ be the corresponding maximal ideal of R . Suppose that $\hat{R}$ is the completion of R at $\mathfrak{m}$. Then $W(\hat{R})$ is local henselian. Denote by $W(R)_{\mathfrak{M}}^h$ the henselization of the localization $W(R)_{\mathfrak{M}}$.

Lemma 2.2.1 *Assume, in addition to (1) and (2), that R is an integral domain and $\mathbb{Z}_p$ -flat. Then, the natural homomorphism $R \rightarrow \hat{R}$ induces injections*

$$W(R) \subset W(R)_{\mathfrak{M}}^h \subset W(\hat{R}).$$

Proof Since R is p -torsion free,

$$\text{gh} : W(R) \hookrightarrow \prod_{i \geq 1} R$$

is an injective ring homomorphism. If $f = (f_1, f_2, \dots) \notin \mathfrak{M}$, then $f_1 \notin \mathfrak{m}$. Since $p \in \mathfrak{m}$, we have $\text{gh}_i(f) \notin \mathfrak{m}$, for all i . In particular, $\text{gh}_i(f) \neq 0$ and so since R is a domain, f is not a zero divisor in $W(R)$. It follows that $W(R)$ is a subring of $W(R)_{\mathfrak{M}}$. We now consider $W(R) \subset W(\hat{R})$. Notice that f is invertible in $W(\hat{R})$ since $\hat{R}$ is p -adically complete and f_1 is invertible in $\hat{R}$. Hence, we have an injection $W(R)_{\mathfrak{M}} \hookrightarrow W(\hat{R})$. The ring $W(\hat{R})$ is local and henselian and $W(R)_{\mathfrak{M}} \hookrightarrow W(\hat{R})$ is a local ring homomorphism. It follows that the henselization $W(R)_{\mathfrak{M}}^h$ is contained in $W(\hat{R})$. $\square$

2.3. Here we suppose that the $\mathbb{Z}_p$ -algebra R satisfies (N).

2.3.1. We start by recalling a useful statement shown by de Jong [9, Prop. 7.3.6], in a slightly different language.

Proposition 2.3.2 *We have*

$$R = \{f \in R[1/p] \mid \forall F, \forall \xi : R \rightarrow \mathcal{O}_F, \xi(f) \in \mathcal{O}_F\},$$

where F runs over all finite extensions of $W(k)[1/p]$ and ξ over all $W(k)$ -algebra homomorphisms. $\square$

Corollary 2.3.3 *We have*

$$W(R) = (W(R)[1/p]) \cap \prod_{\xi} W(\mathcal{O}_F)$$

where F, ξ are as above.

Proof Under our assumption, $W(R)$ is $\mathbb{Z}_p$ -flat. Consider $f \in W(R)[1/p]$ so that $g = p^a f \in W(R)$ for some $a \geq 0$. Assume that $\xi(g)$ is divisible by p^a in $W(\mathcal{O}_F)$, for all $\xi : R \rightarrow \mathcal{O}_F$. We would like to show that g is divisible by p^a in $W(R)$. This will be the case when certain universal polynomials in the ghost coordinates $\text{gh}_i(g)$ which have coefficients in $\mathbb{Z}[1/p]$, take values in R . By Proposition 2.3.2, this is equivalent to asking that the same polynomials in $\text{gh}_i(\xi(g))$ take values in $\mathcal{O}_F$, for all ξ . This is true by our assumption. $\square$

Proposition 2.3.4 *Suppose R satisfies (CN). We have*

$$W(R) \cap \left(\prod_{\xi: R \rightarrow \mathcal{O}_F} \hat{W}(\mathcal{O}_F) \right) = \hat{W}(R).$$

Here the product is over all finite extensions F of $W(k)[1/p]$ and all $W(k)$ -algebra homomorphisms $\xi : R \rightarrow \mathcal{O}_F$. The intersection takes place in $\prod_{\xi: R \rightarrow \mathcal{O}_F} W(\mathcal{O}_F)$.

Proof This follows from the definitions and:

Proposition 2.3.5 *Suppose that $(f_n)_n$ is a sequence of elements of the maximal ideal $\mathfrak{m}_R$ such that, for every finite extension F of $W(k)[1/p]$ and every $W(k)$ -algebra homomorphism $\xi : R \rightarrow \mathcal{O}_F$, the sequence $(\xi(f_n))_n$ converges to 0 in the p -adic topology of F . Then $(f_n)_n$ converges to 0 in the $\mathfrak{m}_R$ -topology.*

Proof Under our assumption on R , there is a finite injective ring homomorphism

$$\varphi^* : R_0 = W(k)[[t_1, \dots, t_r]] \hookrightarrow R.$$

(See [24, p. 212], also [9, proof of Lemma 7.3.5].) We will use this to reduce the proof to the case $R = R_0$. Let d be the degree of the extension $\text{Frac}(R)/\text{Frac}(R_0)$ of fraction fields. For $f \in R$, we let $P(T; f)$ be the irreducible polynomial of f over $\text{Frac}(R_0)$; this has degree $d' \leq d$. Since R, R_0 are both normal and φ^* is integral, we see that $P(T; f)$ has coefficients in R_0 .

Assume that $(f_n)_n$ is a sequence of elements in $\mathfrak{m}_R$ that satisfies the assumption of the proposition. Fix a finite Galois extension F' of $\text{Frac}(R_0)$ that contains $\text{Frac}(R)$ and let R' be the integral closure of R_0 in F' so that $R_0 \subset R \subset R'$. Then R' is finite over R_0 [24, (31.B)]. For each n we can write

$$P(T; f_n) = \prod_{i=1}^{d'_n} (T - f_{n,i})$$

with $f_{n,i} \in R'$ and $f_{n,0} = f_n$. The elements $f_{n,i}$ are Galois conjugates of f_n . Every $\xi : R \rightarrow \mathcal{O}_F$ as in the statement of the proposition extends to $\xi' : R' \rightarrow \mathcal{O}_{F'}$, where F'/F is finite, and the valuations of $\xi'(f_{n,i})$ agree with that of $\xi(f_n)$. It follows that for every $\xi' : R' \rightarrow \mathcal{O}_{F'}$, the total sequence $(\xi'(f_{n,i}))_{n,i}$ goes to 0. Hence, the assumption of the proposition is satisfied for the sequence of the symmetric functions $(a_i(f_n))_n$ in R_0 , for each i . These give the coefficients of the $P(T; f_n)$ and f_n is a root of $T^{d-d'_n}P(T; f_n)$ which we can write

$$T^{d-d'_n}P(T; f_n) = T^d + b_{1,n}T^{d-1} + \cdots + b_{d-1,n}T + b_{d,n}, \quad b_{i,n} \in R_0.$$

Suppose now that we know that the proposition is true for R_0 . Then, we obtain that $b_{i,n}$ converges to 0 in the $\mathfrak{m}_{R_0}$ -topology. The identity $f_n^d + b_{1,n}f_n^{d-1} + \cdots + b_{d-1,n}f_n + b_{d,n} = 0$ implies that f_n^d converges to 0 in the $\mathfrak{m}_R$ -topology. For $f \in R$, consider the sequence of ideals

$$\cdots \subset I_{a+1} = (\mathfrak{m}_R^{a+1}; f) \subset I_a = (\mathfrak{m}_R^a; f) \subset \cdots$$

of R . Krull's intersection theorem implies $\bigcap_{a=0}^\infty I_a = (0)$ and so, by Chevalley's lemma, the ideals $(I_a)_a$ also define the $\mathfrak{m}_R$ -topology of R . Since $f_n^d \in \mathfrak{m}_R^a$ implies $f_n^{d-1} \in (\mathfrak{m}_R^a; f_n)$, we quickly obtain, by decreasing induction on d , that f_n converges to 0 in the $\mathfrak{m}_R$ -topology.

It remains to prove the proposition for the power series ring R_0 :

Set $Y = \text{Spec}(R_0)$, and let $h : \tilde{Y} \rightarrow Y$ be the blowup of Y at the maximal ideal $\mathfrak{m} = (p, t_1, \dots, t_r)$. The exceptional divisor E can be identified with $\mathbb{P}(\mathfrak{m}/\mathfrak{m}^2) \cong \mathbb{P}_k^r$. We argue by contradiction: Assume that $(f_n)_n$ is a sequence of elements in $\mathfrak{m}$ that satisfies the assumption of the proposition but is such that $(f_n)_n$ does not converge to 0 in the $\mathfrak{m}$ -topology. Then, by replacing $(f_n)_n$ by a subsequence, we can assume that, there is an integer $N \geq 1$, such that $f_n \in \mathfrak{m}^N - \mathfrak{m}^{N+1}$, for all n . Write

$$f_n = \sum_{i \geq N} f_{n,i}, \quad f_{n,i} = \sum_{a_0 + \cdots + a_r = i} [c_{a_0 a_1 \cdots a_r}(n)] p^{a_0} t_1^{a_1} \cdots t_r^{a_r},$$

with $c_{a_0 a_1 \cdots a_r}(n) \in k$.

Then the proper transform $Z(f_n) \subset \tilde{Y}$ of $f_n = 0$ intersects the exceptional divisor $E = \mathbb{P}_k^r$ along the hypersurface $S_n \subset \mathbb{P}_k^r = \text{Proj}_k(k[u_0, \dots, u_r])$ of degree N defined by the homogeneous equation

$$\sum_{a_0 + \cdots + a_r = N} c_{a_0 a_1 \cdots a_r}(n) u_0^{a_0} u_1^{a_1} \cdots u_r^{a_r} = 0. \quad (2.3.6)$$

Lemma 2.3.7 *After replacing $(f_n)_n$ by a subsequence, we can find a $\bar{k}$ -valued point x of the exceptional divisor $\mathbb{P}_k^r$ which does not lie on any of the proper transforms $Z(f_n)$ of $f_n = 0$, for all n .*

Proof We argue by contradiction: Assume that for any given point $x \in \mathbb{P}^r(\bar{k})$ and almost all n , $Z(f_n)$ contains x . Then, also for every finite set of points $A(m) =$

$\{x_1, \dots, x_m\}$, we have $A(m) \subset Z(f_n)$, for almost all n . Since $Z(f_n) \cap \mathbb{P}^r$ is, for each n , a hypersurface S_n of fixed degree N , when $m \gg N$ this is not possible. $\square$

Now choose x as given by the lemma and lift it to a point $\tilde{x} \in \tilde{Y}(\mathcal{O}_F)$, where F is some finite extension of $W(k)_{\mathbb{Q}}$; this induces an $\mathcal{O}_F$ -point of Y by $\tilde{Y}(\mathcal{O}_F) \rightarrow Y(\mathcal{O}_F)$ and we also denote this by $\tilde{x}$. By assumption, $\tilde{x}^*(f_n) \rightarrow 0$ in F . By our choice, the image of $\tilde{x} : \text{Spec}(\mathcal{O}_F) \rightarrow \tilde{Y}$ intersects the exceptional divisor $E = \mathbb{P}_k^r$ away from the hypersurface S_n . Using this and Eq. (2.3.6) which cuts out S_n , we obtain

$$v(\tilde{x}^*(f_{n,N})) = N \cdot \min_{i=0}^r \{\tilde{x}^*(t_i)\},$$

where v is the p -adic valuation and, for uniformity, we denote p by t_0 . Since $v(\tilde{x}^*(f_{n,i})) \geq (N+1) \cdot \min_{i=0}^r \{\tilde{x}^*(t_i)\}$ for $i \geq N+1$, we also have $v(\tilde{x}^*(f_n)) = N \cdot \min_{i=0}^r \{\tilde{x}^*(t_i)\}$, which contradicts $v(\tilde{x}^*(f_n)) \rightarrow +\infty$. $\square$

2.4 Some perfectoid rings

We assume that the $\mathbb{Z}_p$ -algebra R satisfies (CN) with $k = \bar{\mathbb{F}}_p$.

2.4.1. Fix an algebraic closure $\overline{F(R)}$ of the fraction field $F(R) = \text{Frac}(R)$. Denote by $\tilde{R}$ the union of all finite normal R -algebras R' such that:

1. $R \subset R' \subset \overline{F(R)}$, and
2. $R'[1/p]$ is finite étale over $R[1/p]$.

Note that all such R' are local and complete. We will denote by $\bar{R}$ the integral closure of R in $\overline{F(R)}$, so that $\bar{R}$ is the union of all R' as in (1).

Let us set

$$\Gamma_R = \text{Gal}(\tilde{R}[1/p]/R[1/p]),$$

which acts on $\tilde{R}$. Also denote by

$$\tilde{R}^\wedge = \varprojlim_n \tilde{R}/p^n \tilde{R}, \quad \bar{R}^\wedge = \varprojlim_n \bar{R}/p^n \bar{R},$$

the p -adic completions.

2.4.2. When $R = W = W(k)$, we denote $\bar{W}^\wedge = \tilde{W}^\wedge$ by $\mathcal{O}$.

Proposition 2.4.3 *The natural maps $\bar{R} \rightarrow \bar{R}^\wedge$, $\tilde{R} \rightarrow \tilde{R}^\wedge$, are injections and induce isomorphisms $\bar{R}/p^n \bar{R} \simeq \bar{R}^\wedge/p^n \bar{R}^\wedge$, $\tilde{R}/p^n \tilde{R} \simeq \tilde{R}^\wedge/p^n \tilde{R}^\wedge$, for all $n \geq 1$.*

Proof This is given by the argument in [5, Prop. 2.0.3] which deals with the case of $\tilde{R}$ and the case of $\bar{R}$ is similar. $\square$

Proposition 2.4.4 *Let S be $\tilde{R}^\wedge$ or $\bar{R}^\wedge$.*

- (a) S is p -adically complete and separated and is flat over $\mathbb{Z}_p$.
- (b) S is an integral perfectoid algebra (in the sense of [4, 3.2]).
- (c) S is local and strict henselian.

Proof Part (a) is also given by [5, Prop. 2.0.3]. Let us show (b) for $S = \tilde{R}^\wedge$. The argument for $\tilde{R}^\wedge$ is similar and actually simpler. We see that $\tilde{R}$ and so S , contains an element π with $\pi^p = p$. Then S is π -adically complete. Using [4, Lemma 3.10], it is now enough to show that the Frobenius $\varphi : S/\pi S \rightarrow S/pS$ is an isomorphism and that π is not a zero divisor in S . Since S is $\mathbb{Z}_p$ -flat, π is not a zero divisor. By Proposition 2.4.3, we have $\tilde{R}/p\tilde{R} \simeq S/pS$ and similarly $\tilde{R}/\pi\tilde{R} \simeq S/\pi S$. Hence, it is enough to show that $\varphi : \tilde{R}/\pi\tilde{R} \rightarrow \tilde{R}/p\tilde{R}$ is an isomorphism. Suppose now $x \in \tilde{R}$ satisfies $x^p = py$ with $y \in \tilde{R}$. Then $(x/\pi)^p = y$ and since $\tilde{R}$ is a union of normal domains, we have $x/\pi = z \in \tilde{R}$. This shows injectivity. To show surjectivity, consider $a \in R' \subset \tilde{R}$ and consider

$$R'' = R'[X]/(X^{p^2} - pX - a).$$

This is a finite R -algebra, and so also p -adically complete. It is étale over $R'[1/p]$ since, by p -adic completeness, the derivative $p(pX^{p^2-1} - 1)$ is a unit in $R''[1/p]$. Now there is $R'' \rightarrow \overline{F(R)}$ that extends $R' \subset \overline{F(R)}$ and the image b of X in $\overline{F(R)}$ is contained in a finite R' -algebra which is also étale over $R'[1/p]$. This gives $b \in \tilde{R}$ with $b^{p^2} \equiv a \pmod{p\tilde{R}}$ which implies surjectivity.

For part (c), since $S = \tilde{R}^\wedge$ is p -adically complete, it is enough to show that these properties are true for $S/pS \simeq \tilde{R}/p\tilde{R}$. We can see that $\tilde{R}$ is both local and strict henselian, and then so is the quotient $\tilde{R}/p\tilde{R}$. The argument for $\tilde{R}^\wedge$ is similar. $\square$

Theorem 2.4.5 *The action of Γ_R on $\tilde{R}$ extends to a p -adically continuous action on $\tilde{R}^\wedge$ and we have*

$$(\tilde{R}^\wedge)^{\Gamma_R} = R.$$

Proof By Faltings [13] or [5, Prop. 3.1.8], we have

$$R \subset (\tilde{R}^\wedge)^{\Gamma_R} \subset R[1/p].$$

Using this, we see that it remains to show that $\tilde{R}^\wedge \cap R[1/p] = R$, with the intersection in $\tilde{R}^\wedge[1/p]$. Suppose $f \in \tilde{R}^\wedge \cap R[1/p]$. By applying Proposition 2.3.2, we see that it is enough to show that $\xi(f) \in \mathcal{O}_F$, for all W -algebra homomorphisms $\xi : R \rightarrow \mathcal{O}_F$ with F a finite extension of $W[1/p]$. Choose such a $\xi : R \rightarrow \mathcal{O}_F$. We can extend ξ to $\bar{\xi} : \tilde{R} \rightarrow \bar{\mathcal{O}}_F = \mathcal{O}_{\bar{F}}$ and then by p -adic completion to

$$\bar{\xi}^\wedge : \tilde{R}^\wedge \rightarrow \mathcal{O}.$$

This gives $\bar{\xi}^\wedge : \tilde{R}^\wedge[1/p] \rightarrow \mathcal{O}[1/p]$. But then $\bar{\xi}^\wedge(f) \in F \cap \mathcal{O} = \mathcal{O}_F$. $\square$

2.5 Period rings

We continue with the same assumptions and notations. In particular, R satisfies (CN) with $k = \bar{\mathbb{F}}_p$.

2.5.1. We now restrict to the case $S = \tilde{R}^\wedge$. We will use the notations of [4, § 3]. Consider the tilt

$$S^\flat = \varprojlim_{\varphi} S/pS = \varprojlim_{\varphi} S$$

and similarly for $\mathcal{O}^\flat$.

Lemma 2.5.2 *The ring $S^\flat$ is local strict henselian with residue field k .*

Proof As we have seen, the rings S and $S/pS = \tilde{R}/p\tilde{R}$ are local and strict henselian with residue field k . Denote by $x \mapsto \bar{x}$ the map $S/pS \rightarrow k$. The Frobenius $S/pS \rightarrow S/pS$ is surjective and, hence, $S^\flat \rightarrow S/pS$ is surjective. If $x = (x_0, x_1, x_2, \dots) \in S^\flat$ with $x_i \in S/pS$, $x_{i+1}^p = x_i$, has x_0 a unit, then all x_i and also x are units. Hence, $S^\flat$ is local with residue field k and $(x_0, x_1, x_2, \dots) \mapsto (\bar{x}_0, \bar{x}_1, \bar{x}_2, \dots)$ is the residue field map $S^\flat \rightarrow k$. Now consider $f(T) \in S^\flat[T]$ with a simple root $\kappa = (\kappa_0, \kappa_1, \kappa_2, \dots)$ in k , with $\kappa_i = \sqrt[p^i]{\kappa_0}$. Since S/pS is local henselian, the simple root $\kappa_i \in k$ of $f_i(T)$ lifts uniquely to a root $a_i \in S/pS$. By uniqueness, we have $a_{i+1}^p = a_i$ and so $a = (a_0, a_1, \dots)$ is a root in $S^\flat$ that lifts κ . Hence, $S^\flat$ satisfies Hensel's lemma. $\square$

2.5.3. We set $A_{\text{inf}}(S) = W(S^\flat)$ for Fontaine's ring. By [4, Lemma 3.2], we have

$$A_{\text{inf}}(S) \cong \varprojlim_{\varphi} W_r(S).$$

This gives corresponding homomorphisms

$$\tilde{\theta}_r : A_{\text{inf}}(S) \rightarrow W_r(S).$$

We also have the standard homomorphism of p -adic Hodge theory

$$\theta : A_{\text{inf}}(S) \rightarrow S,$$

given by

$$\theta \left(\sum_{n \geq 0} [x_n] p^n \right) = \sum_{n \geq 0} x_n^{(0)} p^n.$$

Here, we write $x = (x^{(0)}, x^{(1)}, \dots) \in S^\flat = \varprojlim_{\varphi} S$. As in [4, § 3], the homomorphism θ lifts to

$$\theta_\infty : A_{\text{inf}}(S) \rightarrow W(S),$$

given by

$$\theta_\infty \left(\sum_{n \geq 0} [x_n] p^n \right) = \sum_{n \geq 0} [x_n^{(0)}] p^n.$$

2.5.4. In the following, α runs over all rings homomorphisms $\alpha : S \rightarrow \mathcal{O}$ which are obtained from some W -homomorphism $\tilde{R} \rightarrow \tilde{W} = \bar{W}$ by p -adic completion. There

is a corresponding $A_{\text{inf}}(S) \rightarrow A_{\text{inf}}(\mathcal{O})$ given by applying the Witt vector functor to $\alpha^b : S^b \rightarrow \mathcal{O}^b$. Note that if F is a finite extension of $W[1/p]$, then any homomorphism $\xi : R \rightarrow \mathcal{O}_F$ extends to such an $\alpha : S \rightarrow \mathcal{O}$.

Lemma 2.5.5 *a) The homomorphism $A_{\text{inf}}(S) \rightarrow \prod_{\alpha} A_{\text{inf}}(\mathcal{O})$ is injective.*

b) The ring $A_{\text{inf}}(S)$ is p -adically complete, local strict henselian and $\mathbb{Z}_p$ -flat.

Proof We first note that $\tilde{R} \rightarrow \prod_{\alpha} \tilde{W}$ is injective (reduce to the case R is a formal power series ring by an argument as in the proof of Proposition 2.3.5). Also, $\tilde{R} \cap (\prod_{\alpha} p\tilde{W}) = p\tilde{R}$, as this easily follows by Proposition 2.3.2 applied to the algebras R' . Therefore,

$$S/pS = \tilde{R}/p\tilde{R} \subset \prod_{\alpha} \tilde{W}/p\tilde{W} = \prod_{\alpha} \mathcal{O}/p\mathcal{O}$$

is injective. Hence, $S^b \rightarrow \prod_{\alpha} \mathcal{O}^b$ is injective and part (a) follows.

To show part (b), recall $A_{\text{inf}}(S) = W(S^b)$. The ring S^b is perfect, and so $pW(S^b) = I_{S^b}$, $W(S^b)/pW(S^b) = S^b$. It follows that $A_{\text{inf}}(S)$ is p -adically complete and that p is not a zero divisor. Lemma 2.5.2 now implies that $A_{\text{inf}}(S)$ is local and strict henselian. $\square$

2.5.6. Now let us fix an embedding $\tilde{W} \hookrightarrow \tilde{R}$, which induces $\mathcal{O} \hookrightarrow S$. Let

$$\epsilon = (1, \zeta_p, \zeta_{p^2}, \dots) \in \mathcal{O}^b = \varprojlim_{\varphi} \mathcal{O}$$

be a system of primitive p -th power roots of unity. Set

$$\mu = [\epsilon] - 1 \in A_{\text{inf}}(\mathcal{O}) \subset A_{\text{inf}}(S).$$

Proposition 2.5.7 *a) The element μ is not a zero divisor in $A_{\text{inf}}(S)$.*

b) Suppose that f in $A_{\text{inf}}(S)$ is such that, for every $\alpha : S \rightarrow \mathcal{O}$ obtained from $\tilde{R} \rightarrow \tilde{W}$ as above, $\alpha(f)$ is in the ideal (μ) of $A_{\text{inf}}(\mathcal{O})$. Then, f is in (μ) .

Proof Part (a) follows from [4, Prop. 3.17 (ii)]. As in the proof of loc. cit., the ghost coordinate vectors of $\tilde{\theta}_r(\mu)$ are

$$\text{gh}(\tilde{\theta}_r(\mu)) = (\zeta_{p^r} - 1, \dots, \zeta_p - 1) \in S^r,$$

and the result follows from this.

Let us show (b). Recall

$$A_{\text{inf}}(S) = \varprojlim_{\varphi} W_r(S) \subset \prod_{r \geq 1} W_r(S); \quad a \mapsto (\tilde{\theta}_r(a))_r.$$

Now suppose that $\alpha(f) = \mu \cdot b_{\alpha}$, for $b_{\alpha} \in A_{\text{inf}}(\mathcal{O})$. Apply $\tilde{\theta}_r$ to obtain

$$\alpha(\tilde{\theta}_r(f)) = \tilde{\theta}_r(\alpha(f)) = \tilde{\theta}_r(\mu) \cdot \tilde{\theta}_r(b_{\alpha}).$$

This implies that, for all $i = 1, \dots, r$, and all α ,

$$\zeta_{p^i} - 1 \mid \alpha(\mathrm{gh}_i(\tilde{\theta}_r(f)))$$

in $\mathcal{O}$. The same argument as in the proof of Lemma 2.5.5 (a) above, gives

$$S/(\zeta_{p^i} - 1)S \hookrightarrow \prod_{\alpha} \mathcal{O}/(\zeta_{p^i} - 1)\mathcal{O}.$$

This implies that $\zeta_{p^i} - 1$ (uniquely) divides $\mathrm{gh}_i(\tilde{\theta}_r(f))$ in S .

We claim that the quotients $g_{i,r} = \mathrm{gh}_i(\tilde{\theta}_r(f))/(\zeta_{p^i} - 1)$ in S are the ghost coordinates of an element γ_r of $W_r(S)$, which is then the quotient $\tilde{\theta}_r(f)/\tilde{\theta}_r(\mu)$. To check this we have to show that certain universal polynomials in the $g_{i,r}$ with coefficients in $\mathbb{Z}[1/p]$ take values in S . This holds after evaluating by $\alpha : S \rightarrow \mathcal{O}$ and so the same argument using Proposition 2.3.2 as before, shows that it is true. It follows that, for all r , $\tilde{\theta}_r(\mu)$ uniquely divides $\tilde{\theta}_r(f)$ in $W_r(S)$ and, in fact,

$$\tilde{\theta}_r(f) = \tilde{\theta}_r(\mu) \cdot \gamma_r.$$

Applying φ gives

$$\tilde{\theta}_{r-1}(f) = \tilde{\theta}_{r-1}(\mu) \cdot \varphi(\gamma_r).$$

in $W_{r-1}(S)$. Therefore, $\varphi(\gamma_r) = \gamma_{r-1}$. Hence, there is $\gamma \in A_{\mathrm{inf}}(S) = \varprojlim_{\varphi} W_r(S)$ such that $\gamma_r = \tilde{\theta}_r(\gamma)$. Then, $f = \mu \cdot \gamma$. $\square$

Corollary 2.5.8 (a) *We have $A_{\mathrm{inf}}(S) = (A_{\mathrm{inf}}(S)[1/\mu]) \cap \prod_{\alpha} A_{\mathrm{inf}}(\mathcal{O})$.*

(b) *Suppose that M_1 and M_2 are two finite free $A_{\mathrm{inf}}(S)$ -modules with $M_1[1/\mu] = M_2[1/\mu]$, and such that $\alpha^* M_1 = \alpha^* M_2$ as $A_{\mathrm{inf}}(\mathcal{O})$ -submodules of $\alpha^* M_1[1/\mu] = \alpha^* M_2[1/\mu]$, for all α . Then $M_1 = M_2$.*

Proof Part (a) follows directly from the previous proposition. Part (b) follows by applying (a) to the entries of the matrices expressing a basis of M_1 as a combination of a basis of M_2 , and vice versa. $\square$

2.5.9. As in [4, § 3], set $\xi = \mu/\varphi^{-1}(\mu)$ which is a generator of the kernel of the homomorphism $\theta : A_{\mathrm{inf}}(S) \rightarrow S$. Let $A_{\mathrm{cris}}(S)$ be the p -adic completion of the divided power envelope of $A_{\mathrm{inf}}(S)$ along (ξ) . By [35, App. to XVII], the natural homomorphism

$$A_{\mathrm{inf}}(S) \rightarrow A_{\mathrm{cris}}(S)$$

is injective. We also record:

Lemma 2.5.10 $A_{\mathrm{inf}}(S)^{\varphi=1} = \mathbb{Z}_p$.

Proof It is enough to show that $(S^\flat)^{\varphi=1} = \mathbb{F}_p$, i.e. that $(\tilde{R}/p\tilde{R})^{\varphi=1} = (S/pS)^{\varphi=1} = \mathbb{F}_p$. Now argue as in [5, 6.2.19]: Suppose $a \in \tilde{R}$ is such that $a^p = a \bmod p\tilde{R}$. Then $a \in R'$, for some R'/R finite normal with $R'[1/p]$ étale over $R[1/p]$ and $a^p = a \bmod pR'$. By Hensel's lemma for the p -adically complete R' , the equation $x^p - x = 0$ has a root b in R' which is congruent to $a \bmod pR'$. But R' is an integral domain, so any such root is one of the standard roots in $\mathbb{Z}_p$, so b is in $\mathbb{Z}_p$ and $b = a \bmod pR'$ in $\mathbb{F}_p$. $\square$

3 Shimura pairs and $\mathcal{G}$ -torsors

3.1 Shimura pairs

We first set up some notation for (integral) Shimura pairs and then define the notion of a local Hodge embedding.

3.1.1. Let G be a connected reductive algebraic group over $\mathbb{Q}_p$ and $\{\mu\}$ the $G(\bar{\mathbb{Q}}_p)$ -conjugacy class of a minuscule cocharacter $\mu : \mathbb{G}_m, \bar{\mathbb{Q}}_p \rightarrow G_{\bar{\mathbb{Q}}_p}$.

To such a pair $(G, \{\mu\})$, we associate:

- The reflex field $E \subset \bar{\mathbb{Q}}_p$. As usual, E is the field of definition of the conjugacy class $\{\mu\}$ (i.e. the finite extension of $\mathbb{Q}_p$ which corresponds to the subgroup of $\sigma \in \text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ such that $\sigma(\mu)$ is $G(\bar{\mathbb{Q}}_p)$ -conjugate to μ .)
- The G -homogeneous variety $X_\mu = X_\mu(G)$ of parabolic subgroups of G of type μ . This is a projective smooth G -variety over E with $X_\mu(\bar{\mathbb{Q}}_p) = G(\bar{\mathbb{Q}}_p)/P_\mu(\bar{\mathbb{Q}}_p)$.

3.1.2. An *integral local Shimura pair* is $(\mathcal{G}, \mathcal{M})$ where:

- $\mathcal{G}$ is a parahoric group scheme over $\mathbb{Z}_p$ with generic fiber G .
- $\mathcal{M}$ is a normal flat and projective $\mathcal{O}_E$ -scheme $\mathcal{M}$ with $\mathcal{G}$ -action which is a model of X_μ , in the sense that there is a G -equivariant isomorphism

$$\mathcal{M} \otimes_{\mathcal{O}_E} E \simeq X_\mu.$$

3.1.3. The theory of local models suggests that there should be a canonical choice of a scheme $\mathcal{M}$ as in (ii) which depends only on $(\mathcal{G}, \{\mu\})$, up to unique $\mathcal{G}$ -equivariant isomorphism.

More precisely, Scholze conjectures [35, Conjecture 21.4.1], the existence of such a scheme $M^{\text{loc}} = M^{\text{loc}}(\mathcal{G}, \{\mu\})$ (the *local model*), which has, in addition, reduced special fiber (and hence is normal), and which is uniquely characterized by its corresponding v -sheaf (see loc. cit. for details). Local models M^{loc} as in [35, Conjecture 21.4.1], have been constructed in many cases. We list some results:

1. If G splits over a tamely ramified extension of $\mathbb{Q}_p$, there is a construction of local models in [30, § 7] (which was adjusted as in [17, § 2.6] when p divides $|\pi_1(G_{\text{der}}(\bar{\mathbb{Q}}_p))|$). Conjecturally, these satisfy the conditions of Scholze's [35, Conjecture 21.4.1] ([17, Conjecture 2.16]). This has been shown in almost all cases that $(G, \{\mu\})$ is of local abelian type (see Sect. 2.1) [17, Theorem 2.15], see also [23].

2. If $(G, \{\mu\})$ is of local abelian type and p is odd, the local models M^{loc} have been constructed by Lourenço [23, 4.22, 4.24].

In our application to Shimura varieties, we would like to choose $\mathcal{M} = \mathbf{M}^{\text{loc}}$. However, it is convenient to develop the set-up for a more general $\mathcal{M}$.

3.1.4. Let $(\mathcal{G}, \mathcal{M})$ be an integral local Shimura pair. We consider the following conditions:

(H1) There is a group scheme homomorphism

$$\iota : \mathcal{G} \hookrightarrow \text{GL}_n$$

which is a closed immersion such that $\{\iota(\mu)\}$ is the conjugacy class of one of the standard minuscule cocharacters $\mu_d(a) = \text{diag}(a^{(d)}, 1^{(n-d)})$ of GL_n for some $1 \leq d \leq n-1$, and $\iota(\mathcal{G})$ contains the scalars $\mathbb{G}_m$.

Note that the corresponding GL_n -homogeneous space $X_{\mu_d}(\text{GL}_n)$ is the Grassmannian $\text{Gr}(d, n)$ of d -spaces in $\mathbb{Q}_p^n$. Under the assumption (H1), ι gives an equivariant closed embedding $X_{\mu} \subset \text{Gr}(d, n)_E$. Set $\Lambda = \mathbb{Z}_p^n$. The Grassmannian $\text{Gr}(d, n)$ has a natural model over $\mathbb{Z}_p$ which we will denote by $\text{Gr}(d, \Lambda)$.

(H2) The normalization of the Zariski closure of X_{μ} in $\text{Gr}(d, \Lambda)_{\mathcal{O}_E}$ is $\mathcal{G}$ -equivariantly isomorphic to $\mathcal{M}$. Hence, there is a $\mathcal{G}$ -equivariant finite morphism

$$\iota_* : \mathcal{M} \rightarrow \text{Gr}(d, \Lambda)_{\mathcal{O}_E}$$

which is $X_{\mu} \subset \text{Gr}(d, n)_E$ on the generic fibers.

We call an ι , that satisfies (H1) and (H2), a *integral local Hodge embedding* for the pair $(\mathcal{G}, \mathcal{M})$. When such an integral local Hodge embedding exists, we say that $(\mathcal{G}, \mathcal{M})$ is of *integral local Hodge type*.

We often need the following stronger version of (H2):

(H2*) The Zariski closure of X_{μ} in $\text{Gr}(d, \Lambda)_{\mathcal{O}_E}$ is $\mathcal{G}$ -equivariantly isomorphic to $\mathcal{M}$. Hence, $X_{\mu} \subset \text{Gr}(d, n)_E$ extends to a $\mathcal{G}$ -equivariant closed immersion

$$\iota_* : \mathcal{M} \hookrightarrow \text{Gr}(d, \Lambda)_{\mathcal{O}_E}.$$

We call an ι , that satisfies (H1) and (H2*), a *strongly integral local Hodge embedding* for the pair $(\mathcal{G}, \mathcal{M})$. When such an embedding exists, we say that $(\mathcal{G}, \mathcal{M})$ is of *strongly integral local Hodge type*.

3.1.5. This notion should also be compared to the often used weaker notion of *local Hodge type* which refers to the (rational) local Shimura pair $(G, \{\mu\})$:

We say that $(G, \{\mu\})$ is of local Hodge type if there is a group scheme homomorphism $\iota : G \hookrightarrow \text{GL}_n$ which is a closed immersion such that $\{\iota(\mu)\}$ is the conjugacy class of one of the standard minuscule cocharacters μ_d . (There is also the following related notion: $(G, \{\mu\})$ is “of abelian type” means that there is a central lift $(G_1, \{\mu_1\})$ of $(G_{\text{ad}}, \{\mu_{\text{ad}}\})$ which is of local Hodge type.)

The following statement that relates the two notions when $\mathcal{M} = \mathbf{M}^{\text{loc}}$ can be extracted from the proof of [17, Theorem 2.15]:

Proposition 3.1.6 *Suppose that $(G, \{\mu\})$ is of local Hodge type with $\iota : G \hookrightarrow \text{GL}_n$ as above such that $\mathbb{G}_m \subset \iota(G)$, and $\mathcal{G}$ is a parahoric stabilizer with $G = \mathcal{G} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.*

Assume also that p is odd, $p \nmid |\pi_1(G_{\text{der}}(\bar{\mathbb{Q}}_p))|$, and that G splits over a tamely ramified extension of $\mathbb{Q}_p$. Then $(\mathcal{G}, M^{\text{loc}})$ is of strongly integral local Hodge type. $\square$

3.2 Torsors

In this paragraph, $\mathcal{G}$ is a smooth connected affine flat group scheme over $\mathbb{Z}_p$ with generic fiber G . We will collect some general statements about $\mathcal{G}$ -torsors. We denote by $\text{Rep}_{\mathbb{Z}_p}(\mathcal{G})$ the exact tensor category of representations of $\mathcal{G}$ on finite free $\mathbb{Z}_p$ -modules, i.e. of group scheme homomorphisms $\rho : \mathcal{G} \rightarrow \text{GL}(\Lambda')$ with Λ' a finite free $\mathbb{Z}_p$ -module.

3.2.1. Suppose $\iota : \mathcal{G} \hookrightarrow \text{GL}(\Lambda) \simeq \text{GL}_n$ is a closed group scheme immersion such that $\iota(\mathcal{G})$ contains the scalars $\mathbb{G}_m$. Here Λ is a free $\mathbb{Z}_p$ -module of rank n . Denote by $\Lambda^{\otimes} = \bigoplus_{m,n \geq 0} \Lambda^{\otimes m} \otimes_{\mathbb{Z}_p} (\Lambda^{\vee})^{\otimes n}$ the total tensor algebra of Λ , where $\Lambda^{\vee} = \text{Hom}_{\mathbb{Z}_p}(\Lambda, \mathbb{Z}_p)$. By using the improved³ version of [19, Prop. 1.3.2] given in [11], we can realize $\mathcal{G}$ as the scheme theoretic stabilizer of a finite list of tensors $(s_a) \subset \Lambda^{\otimes}$: For any $\mathbb{Z}_p$ -algebra R we have

$$\mathcal{G}(R) = \{g \in \text{GL}(\Lambda \otimes_{\mathbb{Z}_p} R) \mid g \cdot (s_a \otimes 1) = (s_a \otimes 1), \forall a\}.$$

Since we assume that $\iota(\mathcal{G})$ contains the scalars $\mathbb{G}_m$ and $a \in \mathbb{G}_m$ acts on $\Lambda^{\otimes m} \otimes_{\mathbb{Z}_p} (\Lambda^{\vee})^{\otimes n}$ via a^{m-n} , we see that the s_a are contained in the part of the tensor algebra with $n = m$. In particular, we can assume that every tensor s_a is given by a $\mathbb{Z}_p$ -linear map $\Lambda^{\otimes n} \rightarrow \Lambda^{\otimes n}$, for some $n = n_a$.

3.2.2. Let A be a $\mathbb{Z}_p$ -algebra. Set $S = \text{Spec}(A)$. Suppose that $T \rightarrow S$ is a $\mathcal{G}$ -torsor. By definition, this means that T supports a (left) $\mathcal{G}$ -action $\mathcal{G} \times T \rightarrow T$ such that $\mathcal{G} \times T \xrightarrow{\sim} T \times_S T$ given by $(g, t) \mapsto (gt, t)$ is an isomorphism, and $T \rightarrow S$ is faithfully flat and quasi-compact (fpqc). By descent, T is affine, so $T = \text{Spec}(B)$ with $A \rightarrow B$ faithfully flat.

If $\rho : \mathcal{G} \rightarrow \text{GL}(\Lambda')$ is in $\text{Rep}_{\mathbb{Z}_p}(\mathcal{G})$, we can consider the vector bundle over S which is attached to T and ρ :

$$T(\rho) = T \times_{\text{Spec}(\mathbb{Z}_p)}^{\mathcal{G}} \mathbb{A}(\Lambda') = (T \times_{\text{Spec}(\mathbb{Z}_p)} \mathbb{A}(\Lambda')) / \sim$$

where $(g^{-1}t, \lambda) \sim (t, \rho(g)\lambda)$. Here, $\mathbb{A}(\Lambda')$ is the affine space $\text{Spec}(\text{Sym}_{\mathbb{Z}_p}(\Lambda^{\vee}))$ over $\text{Spec}(\mathbb{Z}_p)$. In what follows, we often abuse notation, and also denote by $T(\rho)$ the corresponding A -module of global sections of the bundle $T(\rho)$.

By [6], see also [35, 19.5.1], this construction gives an equivalence between the category of $\mathcal{G}$ -torsors $T \rightarrow S$ and the category of exact tensor functors

$$T : \text{Rep}_{\mathbb{Z}_p}(\mathcal{G}) \rightarrow \text{Bun}(S); \quad \rho \mapsto T(\rho),$$

into the category of vector bundles $\text{Bun}(S)$ on S .

Assume now that T is a $\mathcal{G}$ -torsor and $\iota : \mathcal{G} \hookrightarrow \text{GL}(\Lambda)$ is as in Sect. 3.2.

³ In the sense that one does not need symmetric and alternating tensors, as in [19, Prop. 1.3.2].

Proposition 3.2.3 *The A -module $M = T(\iota)$ is locally free of rank n and comes equipped with tensors $(m_a) \subset M^\otimes$ such that there is a $\mathcal{G}$ -equivariant isomorphism*

$$T \simeq \underline{\mathrm{Isom}}_{(m_a), (s_a)}(T(\iota), \Lambda \otimes_{\mathbb{Z}_p} A).$$

Proof This is quite standard, see for example [6, Cor. 1.3] for a similar statement. We sketch the argument: By the above, $M = T(\iota)$ is a locally free A -module of rank $n = \mathrm{rank}_{\mathbb{Z}_p}(\Lambda)$. Since the construction of $T(\rho)$ commutes with tensor operations (i.e. $\rho \mapsto T(\rho)$ gives a tensor functor) we have

$$M^\otimes \simeq T \times_{\mathrm{Spec}(\mathbb{Z}_p)}^{\mathcal{G}} \mathbb{A}(\Lambda^\otimes).$$

We can think of $s_a \in \Lambda^\otimes$ as $\mathcal{G}$ -invariant linear maps $s_a : \mathbb{Z}_p \rightarrow \Lambda^\otimes$ which give $1 \times s_a : S = \mathcal{G} \backslash T \rightarrow M^\otimes$, i.e. tensors $m_a = 1 \times s_a \in M^\otimes$. Set $T' = \underline{\mathrm{Isom}}_{(m_a), (s_a)}(M, \Lambda \otimes_{\mathbb{Z}_p} A)$ with its natural left $\mathcal{G}$ -action. The base change $T' \times_S T$ is equivariantly identified with $\mathcal{G} \times T \simeq T \times_S T$ and the proof follows. $\square$

Remark 3.2.4 Suppose that $\iota' : \mathcal{G} \hookrightarrow \mathrm{GL}(\Lambda')$ is another closed group scheme immersion that realizes $\mathcal{G}$ as the subgroup scheme that fixes $(s'_b) \subset (\Lambda')^\otimes$. It follows that there is a $\mathcal{G}$ -equivariant isomorphism

$$\underline{\mathrm{Isom}}_{(m'_b), (s'_b)}(M', \Lambda' \otimes_{\mathbb{Z}_p} A) \xrightarrow{\sim} \underline{\mathrm{Isom}}_{(m_a), (s_a)}(M, \Lambda \otimes_{\mathbb{Z}_p} A).$$

For the following, we assume in addition that A is local and henselian.

Proposition 3.2.5 *Suppose that M is a finite free A -module and let $(m_a) \subset M^\otimes$. Consider the A -scheme*

$$T = \underline{\mathrm{Isom}}_{(m_a), (s_a)}(M, \Lambda \otimes_{\mathbb{Z}_p} A)$$

which supports a natural $\mathcal{G}$ -action. Suppose that there exists a set of local $\mathbb{Z}_p$ -algebra homomorphisms $\xi : A \rightarrow R_\xi$, with $\cap_\xi \ker(\xi) = (0)$, and such that, for every $\xi : A \rightarrow R_\xi$, the base change $\xi^ T := T \times_S \mathrm{Spec}(R_\xi)$ is a $\mathcal{G}$ -torsor over $\mathrm{Spec}(R_\xi)$. Then, $T \rightarrow S$ is also a $\mathcal{G}$ -torsor.*

Proof The scheme T is affine and $T \rightarrow S$ is of finite presentation. The essential difficulty is in showing that $T \rightarrow S$ is flat but under our assumptions, this follows from [15, Thm. (4.1.2)]. The fiber of $T \rightarrow S$ over the closed point of S is not empty, hence $T \rightarrow S$ is also faithfully flat. Now the base change $T \times_S T$ admits a tautological section which gives a $\mathcal{G}$ -isomorphism $T \times_S T \simeq \mathcal{G} \times T$. This completes the proof. $\square$

We will now allow some more general $\mathbb{Z}_p$ -algebras A :

Corollary 3.2.6 *Set $A = W(R)$, where R satisfies (N). Suppose that M is a finite projective A -module, $(m_a) \subset M^\otimes$, and $T = \underline{\mathrm{Isom}}_{(m_a), (s_a)}(M, \Lambda \otimes_{\mathbb{Z}_p} A)$. Assume that for all $W(k)$ -algebra homomorphisms $\tilde{x} : R \rightarrow \mathcal{O}_F$, where F runs over all finite*

extensions of $W(k)_{\mathbb{Q}}$, the pull-back $T \otimes_A W(\mathcal{O}_F)$ is a $\mathcal{G}$ -torsor over $W(\mathcal{O}_F)$. Then T is a $\mathcal{G}$ -torsor over A .

Proof We first show the statement when R is in addition complete and local, i.e. it satisfies (CN). Then $W(R)$ is local henselian and the result follows from Proposition 3.2.5 applied to the set of homomorphisms $\xi : A = W(R) \rightarrow R_{\xi} = W(\mathcal{O}_F)$ given as $\xi = W(\tilde{x})$.

We now deal with the general case. Under our assumptions, $A = W(R)$ is flat over $\mathbb{Z}_p$. Let $\mathfrak{M} \subset W(R)$ be a maximal ideal with residue field k' . We have $I_R \subset \mathfrak{M}$, since $W(R)$ is I_R -adically complete and separated [40, Prop. 3]. Let $\mathfrak{m}_R = \mathfrak{M}/I_R \subset W(R)/I_R = R$ be the corresponding maximal ideal of R . Our assumptions on R imply that the residue field k' is a finite extension of k . Suppose that $\hat{R}$ is the completion of $R \otimes_W W(\bar{k}')$ at $\mathfrak{m}_R \otimes_W W(\bar{k}')$. Then $W(\hat{R})$ is local and strictly henselian. Denote by $W(R)_{\mathfrak{M}}^{\text{sh}}$ the strict henselization of the localization $W(R)_{\mathfrak{M}}$. By Lemma 2.2.1 we have

$$W(R)_{\mathfrak{M}}^{\text{sh}} \subset W(\hat{R}).$$

We also have

$$\hat{R} \subset \prod_{\xi: R \rightarrow \mathcal{O}_F} \mathcal{O}_F$$

where the product is over all $\xi : R \rightarrow \mathcal{O}_F$ that factor through $\hat{R}$. By Proposition 3.2.5 applied to $R_{\xi} = W(\mathcal{O}_F)$, the base change $T \otimes_{W(R)} W(R)_{\mathfrak{M}}^{\text{sh}}$ is a $\mathcal{G}$ -torsor. By descent, so is the base change $T \otimes_{W(R)} W(R)_{\mathfrak{M}}$ over $W(R)_{\mathfrak{M}}$. Since this is true for all maximal ideals $\mathfrak{M} \subset W(R)$, it follows that T is flat over $W(R)$. The result now follows as in the proof of Proposition 3.2.5. $\square$

Remark 3.2.7 When R satisfies (CN), Corollary 3.2.6 also holds with $W(R)$, $W(\mathcal{O}_F)$, replaced by $\hat{W}(R)$, $\hat{W}(\mathcal{O}_F)$ respectively.

3.2.8. Set $D = \text{Spec}(W(k)[[u]])$, $D^{\times} = \text{Spec}(W(k)[[u]]) - \{(p, u)\}$, with k perfect. We will use the following purity result of Anschütz:

Theorem 3.2.9 [1, Theorem 8.4] *Assume $\mathcal{G}$ is parahoric. Then, every $\mathcal{G}$ -torsor over $D^{\times}$ is trivial.*

Remark 3.2.10 This purity property was previously shown [20, Prop. 1.4.3] for $k = \bar{\mathbb{F}}_p$ and all parahoric group schemes $\mathcal{G}$ with $G = \mathcal{G} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ that splits over a tamely ramified extension of $\mathbb{Q}_p$ and has no factors of type E_8 [20, Prop. 1.4.3]. This is the only case needed for the proofs of Theorem 1.4 and Corollary 8.1.6. The result fails for most smooth affine group schemes over $\mathbb{Z}_p$ with reductive generic fiber.

4 Displays with $\mathcal{G}$ -structure

In this section, we define $(\mathcal{G}, \mathcal{M})$ -displays and give some basic properties. We also define and study the notion of a locally universal $(\mathcal{G}, \mathcal{M})$ -display. Recall $(\mathcal{G}, \mathcal{M})$ is an integral Shimura pair; in particular $\mathcal{G}$ is parahoric.

4.1 The construction of the modification

4.1.1. This subsection contains the main construction needed for the definition of a $(\mathcal{G}, \mathcal{M})$ -display. We assume R is a p -adic flat $\mathcal{O}_E$ -algebra. Set $A = W(R)$. (If, in addition, R is complete local Noetherian and $p \geq 3$, there is an obvious variant with $A = \hat{W}(R)$.)

Proposition/Construction 4.1.2 *Assume that $(\mathcal{G}, \mathcal{M})$ is of integral local Hodge type. There is a functor*

$$(\mathcal{P}, q) \mapsto (\mathcal{P}, \mathcal{Q}, \alpha)$$

from the groupoid of pairs $(\mathcal{P}, q)$ of a $\mathcal{G}$ -torsor $\mathcal{P}$ over $\mathrm{Spec}(A)$ together with a $\mathcal{G}$ -equivariant morphism $q : \mathcal{P} \otimes_A R \rightarrow \mathcal{M}$, to the groupoid of triples $(\mathcal{Q}', \mathcal{Q}, \alpha)$ of two $\mathcal{G}$ -torsors $\mathcal{Q}', \mathcal{Q}$ over $\mathrm{Spec}(A)$ and a $\mathcal{G}$ -equivariant isomorphism

$$\alpha : \mathcal{Q} \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A[1/p]) \xrightarrow{\sim} (\varphi^* \mathcal{Q}') \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A[1/p]) \quad (4.1.3)$$

over $\mathrm{Spec}(A[1/p])$.

We will also see that there are natural base change transformations for $R \rightarrow R'$. Also, the functor is constructed using a choice of an integral local Hodge embedding, but, up to natural isomorphism, is independent of this choice, see Remark 4.1.14.

The isomorphism α allows us to think of $\mathcal{Q}$ as a “modification” of $\varphi^* \mathcal{P}$ along $p = 0$; this modification is “bounded by $\mathcal{M}$ ”. The construction of $(\mathcal{P}, q) \mapsto (\mathcal{Q}, \alpha)$ occupies most of this subsection. The main point is the construction of a functorial map

$$\mathrm{ig}_{\mathcal{G}}(R) : \mathcal{M}(R) \rightarrow \mathrm{Gr}_{\mathcal{G}}(R)$$

(see Proposition 4.1.8), where $\mathrm{Gr}_{\mathcal{G}}(R)$ is as below.

4.1.4. If X is a scheme over $\mathrm{Spec}(A)$, we will write, for simplicity, $X[1/p]$ instead of $X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A[1/p])$.

For $A = W(R)$, we will consider the set

$$\mathrm{Gr}_{\mathcal{G}}(R) = \{(\mathcal{Q}, \alpha)\}$$

of isomorphism classes of pairs $(\mathcal{Q}, \alpha)$ of

- a $\mathcal{G}$ -torsor $\mathcal{Q}$ over $\mathrm{Spec}(A)$,
- a $\mathcal{G}$ -trivialization $\alpha : \mathcal{Q}[1/p] \xrightarrow{\sim} \mathcal{G}[1/p] \times_{\mathrm{Spec}(A[1/p])} \mathcal{Q}[1/p]$ of $\mathcal{Q}[1/p]$ over $\mathrm{Spec}(A[1/p])$.

The group $G(A[1/p])$ acts on $\mathrm{Gr}_{\mathcal{G}}(R)$ by $g \cdot (\mathcal{Q}, \alpha) = (\mathcal{Q}, g \cdot \alpha)$. Since p is not a zero divisor in A , pairs $(\mathcal{Q}, \alpha)$ as above form a discrete groupoid.

If, in addition, R is complete local with algebraically closed residue field, then $W(R)$ is local strictly henselian, and

$$\mathrm{Gr}_{\mathcal{G}}(R) \cong G(W(R)[1/p]) / \mathcal{G}(W(R)).$$

The set $\mathrm{Gr}_{\mathcal{G}}(R)$ resembles the set of R -points of an affine Grassmannian of some sort.

4.1.5. Let R be a p -adic flat $\mathbb{Z}_p$ -algebra and set $A = W(R)$. Since $\varphi \circ V = p \cdot \mathrm{id}_{W(R)}$, we have $\varphi(I_R)W(R) = pW(R)$. Hence, we obtain a ring homomorphism

$$\bar{\varphi} : R = W(R)/I_R \xrightarrow{\varphi} A/pA$$

induced by the Frobenius $\varphi : A \rightarrow A$.

4.1.6. We first discuss the case $\mathcal{G} = \mathrm{GL}(\Lambda)$ and $\mathcal{P}$ is trivial.

Set $M = \Lambda \otimes_{\mathbb{Z}_p} A$. Let $\mathcal{F} \subset \Lambda \otimes_{\mathbb{Z}_p} R = M \otimes_A R$ be the R -locally direct summand which corresponds to an R -valued point in the Grassmannian $\mathrm{Gr}(d, \Lambda)$. Set

$$M_1 = (\mathrm{id}_M \otimes \mathrm{gh}_1)^{-1}(\mathcal{F})$$

so that

$$I_R M \subset M_1 \subset M,$$

and take $\tilde{M}_1$ to be the image of the map $\varphi^* M_1 \rightarrow \varphi^* M$ induced by $M_1 \hookrightarrow M$.

The quotient $(\Lambda \otimes_{\mathbb{Z}_p} R)/\mathcal{F}$ is R -projective and $W(R)$ is I_R -adically complete. By lifting idempotents we can see that, locally on R , we can write

$$M = \Lambda \otimes_{\mathbb{Z}_p} A = L \oplus T,$$

with L and T finite projective $A = W(R)$ -modules such that $\mathcal{F}$ is the image of $L \oplus I_R T$ under $\mathrm{id}_{\Lambda} \otimes \mathrm{gh}_1 : \Lambda \otimes_{\mathbb{Z}_p} A \rightarrow \Lambda \otimes_{\mathbb{Z}_p} R$. (For more details, see the proof of [40, Lemma 2].) Then

$$\tilde{M}_1 = \varphi^*(L) \oplus p\varphi^*(T) \subset \varphi^*(M) \cong \Lambda \otimes_{\mathbb{Z}_p} A$$

so that

$$p\Lambda \otimes_{\mathbb{Z}_p} A \subset \tilde{M}_1 \subset \Lambda \otimes_{\mathbb{Z}_p} A.$$

The module $\tilde{M}_1$ has also the following description: Base change $\mathcal{F} \subset \Lambda \otimes_{\mathbb{Z}_p} R$ via $\bar{\varphi} : R \rightarrow A/pA$ to obtain an A/pA -submodule

$$\bar{\varphi}^* \mathcal{F} \subset \Lambda \otimes_{\mathbb{Z}_p} A/pA$$

which is locally an A/pA -direct summand. Then $\tilde{M}_1$ is the inverse image of $\bar{\varphi}^* \mathcal{F}$ under the reduction $\Lambda \otimes_{\mathbb{Z}_p} A \rightarrow \Lambda \otimes_{\mathbb{Z}_p} A/pA$.

The A -module $\tilde{M}_1$ gives a $\mathrm{GL}(\Lambda)$ -torsor $\mathcal{Q}_{\mathrm{GL}} = \mathrm{Isom}(\tilde{M}_1, \Lambda_A)$ over A , together with a trivialization α of $\mathcal{Q}_{\mathrm{GL}}[1/p]$ over $A[1/p]$. Sending

$$\mathcal{F} \mapsto (\mathcal{Q}_{\mathrm{GL}}, \alpha)$$

gives a functorial (in R) map

$$i_{\mathrm{GL}} : \mathrm{Gr}(d, \Lambda)(R) \rightarrow \mathrm{Gr}_{\mathrm{GL}(\Lambda)}(R).$$

This satisfies

$$i_{\mathrm{GL}}(\mathrm{gh}_1(g) \cdot x) = \varphi(g) \cdot i_{\mathrm{GL}}(x),$$

for $g \in \mathrm{GL}(\Lambda)(A)$, $x \in \mathrm{Gr}(d, \Lambda)(R)$.

4.1.7. We will now explain how to extend the construction above from $\mathrm{GL}(\Lambda)$ to a parahoric $\mathcal{G} \subset \mathrm{GL}(\Lambda)$. We assume that R is a p -adic flat $\mathcal{O}_E$ -algebra. Recall $A = W(R)$.

Proposition 4.1.8 *Suppose that $(\mathcal{G}, \mathcal{M})$ is of integral local Hodge type. There are functorial (in R) maps*

$$i_{\mathcal{G}}(R) : \mathcal{M}(R) \rightarrow \mathrm{Gr}_{\mathcal{G}}(R)$$

which satisfy

$$i_{\mathcal{G}}(\mathrm{gh}_1(g) \cdot x) = \varphi(g) \cdot i_{\mathcal{G}}(x), \quad (4.1.9)$$

for $g \in \mathcal{G}(A)$, $x \in \mathcal{M}(R)$.

Proof We choose a integral local Hodge embedding $\iota : \mathcal{G} \hookrightarrow \mathrm{GL}(\Lambda)$ which induces $\iota_* : \mathcal{M} \rightarrow \mathrm{Gr}(d, \Lambda)_{\mathcal{O}_E}$. Let $a \in \mathcal{M}(R)$ be an R -valued point of $\mathcal{M}$. It will be enough to give $i_{\mathcal{G}}(R)$ for $R = \hat{B}$, the p -adic completion of B , where $\mathrm{Spec}(B) \subset \mathcal{M}$ varies over affine charts of $\mathcal{M}$ and a the tautological point. Recall that $\mathcal{M}$ is normal, flat and proper over $\mathcal{O}_E$. So, we can assume that R satisfies (N). The image $\iota_*(a)$ is an R -valued point of the Grassmannian $\mathrm{Gr}(d, \Lambda)_{\mathcal{O}_E}$ and is given by a locally direct summand

$$\mathcal{F} = \mathcal{F}_a \subset \Lambda \otimes_{\mathbb{Z}_p} R$$

In what follows, we will omit a from the notation. The construction above for $\mathrm{GL}(\Lambda)$, applied to $\mathcal{F}$, gives $p\Lambda \otimes_{\mathbb{Z}_p} A \subset \tilde{M}_1 \subset \Lambda \otimes_{\mathbb{Z}_p} A$. Notice that $\tilde{M}_1^{\otimes} \subset \tilde{M}_1^{\otimes}[1/p] = \Lambda_A^{\otimes}[1/p]$. We have $s_a \otimes 1 \in \Lambda_A^{\otimes} \subset \Lambda_A^{\otimes}[1/p]$.

Proposition 4.1.10 (a) *The tensors $s_a \otimes 1$ belong to $\tilde{M}_1^{\otimes}$.*
 (b) *The scheme of isomorphisms that respect the tensors*

$$\mathcal{Q} := \underline{\mathrm{Isom}}_{s_a \otimes 1, s_a \otimes 1}(\tilde{M}_1, \Lambda_A)$$

is a $\mathcal{G}$ -torsor over $\mathrm{Spec}(A)$.

(c) *Since $\tilde{M}_1[1/p] = \Lambda_A[1/p]$, we have a trivialization*

$$\alpha : \mathcal{Q}[1/p] \xrightarrow{\cong} G_{A[1/p]}.$$

Proof Using purity for $\mathcal{G}$ (Theorem 3.2.9), we see that the proof of [20, Lemma 3.2.9] goes through in our situation and gives a) and b) after base-changing to $\mathcal{O}_K$, for all $\xi : R \rightarrow \mathcal{O}_K$. By Proposition 3.2.5 and Corollary 3.2.6 this now implies parts a) and b), cf. the proof of [20, Cor. 3.2.11]. (This uses that R satisfies (N), in particular that it is normal.) Part c) is easy. $\square$

The proof of Proposition 4.1.8 now follows from Proposition 4.1.10 above: Indeed, we set $i_{\mathcal{G}}(a) = (\mathcal{Q}, \alpha)$, with $\mathcal{Q}$ and α as above. This gives the desired map. $\square$

Remark 4.1.11 The maps $i_{\mathcal{G}}(R)$ are independent of the embedding ι . To see this suppose that $\iota' : \mathcal{G} \hookrightarrow \mathrm{GL}(\Lambda')$ is another integral local Hodge embedding which gives $\iota'_* : \mathcal{M} \rightarrow \mathrm{Gr}(d', \Lambda')_{\mathcal{O}_E}$. We can consider the product

$$\iota'' = \iota \times \iota' : \mathcal{G} \xrightarrow{\Delta} \mathcal{G} \times \mathcal{G} \hookrightarrow \mathrm{GL}(\Lambda) \times \mathrm{GL}(\Lambda') \subset \mathrm{GL}(\Lambda \oplus \Lambda').$$

This induces

$$\iota''_* : \mathcal{M} \xrightarrow{\Delta} \mathcal{M} \times_{\mathcal{O}_E} \mathcal{M} \rightarrow \mathrm{Gr}(d, \Lambda) \times_{\mathcal{O}_E} \mathrm{Gr}(d', \Lambda') \subset \mathrm{Gr}(d + d', \Lambda \oplus \Lambda')_{\mathcal{O}_E}.$$

Consider $a \in \mathcal{M}(R)$ and set $\mathcal{F} = \mathcal{F}_a$, $\mathcal{F}' = \mathcal{F}'_a$, $\mathcal{F}'' = \mathcal{F}''_a$, for the submodules which correspond to the points $\iota_*(a)$, $\iota'_*(a)$, $\iota''_*(a)$, in the Grassmannians. By the construction, we have $\mathcal{F}'' = \mathcal{F} \oplus \mathcal{F}'$, $\tilde{M}''_1 = \tilde{M}_1 \oplus \tilde{M}'_1$, and the projections give $\tilde{M}''_1 \rightarrow \tilde{M}_1$, $\tilde{M}'_1 \rightarrow \tilde{M}_1$. These maps induce $\mathcal{G}$ -equivariant morphisms $\mathcal{Q} \rightarrow \mathcal{Q}''$ and $\mathcal{Q}' \rightarrow \mathcal{Q}''$ which identify these $\mathcal{G}$ -torsors.

4.1.12. Proof of 4.1.2

We can now give the construction of the modification. We assume that $(\mathcal{G}, \mathcal{M})$ is of integral local Hodge type. We choose a integral local Hodge embedding $\iota : \mathcal{G} \hookrightarrow \mathrm{GL}(\Lambda)$ which induces $\iota_* : \mathcal{M} \rightarrow \mathrm{Gr}(d, \Lambda)_{\mathcal{O}_E}$. Suppose that $\mathcal{P}$ is a $\mathcal{G}$ -torsor over $A = W(R)$ given together with a $\mathcal{G}$ -equivariant morphism

$$q : \mathcal{P} \otimes_A R \rightarrow \mathcal{M}.$$

The case that $\mathcal{P}$ is a trivial $\mathcal{G}$ -torsor follows immediately from the proof of Proposition 4.1.8: If s is section of the $\mathcal{G}$ -torsor $\mathcal{P}$ then the composition $a(s) = q \cdot (s \otimes_A R)$ is an R -valued point of $\mathcal{M}$. The proof of Proposition 4.1.8 gives a pair $i_{\mathcal{G}}(a(s))$ of a new $\mathcal{G}$ -torsor $\mathcal{Q}_s$ with a trivialization α_s of $\mathcal{Q}_s[1/p]$.

Let us discuss the general case: Note that $A = W(R)$ is complete and separated in the I_R -adic topology ([40, Prop. 3]), so $(W(R), I_R)$ is a henselian pair. Hence, since $\mathcal{G}$ is smooth, $\mathcal{P}$ acquires a section over $W(R')$, where R' is an étale cover of R (cf. [7, Prop. (B.0.2)]); we can make sure that R' is also p -adic. The construction above and the equivariance property (4.1.9) combined with descent as in [40, 1.3], shows that $\mathcal{P}$ together with q , gives a $\mathcal{G}$ -torsor $\mathcal{Q}$ together with an isomorphism of G -torsors

$$\alpha : \mathcal{Q}[1/p] \xrightarrow{\sim} \varphi^* \mathcal{P}[1/p] \quad (4.1.10)$$

over $A[1/p]$.

Explicitly, if $\mathcal{P}(\iota) = M$ is the corresponding finite projective $W(R)$ -module with tensors (m_a) (see Proposition 3.2.3), then $\iota_* \cdot q$ gives

$$\mathcal{F} \subset M \otimes_{W(R)} R.$$

Set $M_1 = (\text{id}_M \otimes \text{gh}_1)^{-1}(\mathcal{F})$ so that $I_R M \subset M_1 \subset M$ and take $\tilde{M}_1$ to be the image of the map $\varphi^* M_1 \rightarrow \varphi^* M$ induced by $M_1 \hookrightarrow M$. As in Sect. 4.1, we obtain $p\varphi^*(M) \subset \tilde{M}_1 \subset \varphi^*(M)$. Then, as in Proposition 4.1.10, we have $\varphi^* m_a \in \tilde{M}_1^{\otimes}$ and

$$\mathcal{Q} = \underline{\text{Isom}}_{(\varphi^* m_a), (s_a)}(\tilde{M}_1, \Lambda_A).$$

□

Remark 4.1.14 Note that in the above, the pair $(\mathcal{Q}, \alpha : \mathcal{Q}[1/p] \xrightarrow{\sim} \varphi^* \mathcal{P}[1/p])$ only depends on $\mathcal{P}$, $\mathcal{M}$ and q and is independent of ι , up to unique isomorphism; this follows from Sect. 4.1.11. In fact, the argument gives that the functor of 4.1.2 is, up to natural isomorphism, independent of the choice of the integral local Hodge embedding ι .

Remark 4.1.15 (a) The above applies to $\mathcal{M} = M^{\text{loc}}$, where $M^{\text{loc}} = M^{\text{loc}}(\mathcal{G}, \{\mu\})$ are the local models of [30], when $\mathcal{G}(\mathbb{Z}_p)$ is connected parahoric, $p \nmid |\pi_1(G_{\text{der}}(\mathbb{Q}_p))|$, and there is a local Hodge embedding $\iota : G \hookrightarrow \text{GL}_n$ with $\mathbb{G}_m \subset \iota(G)$. This follows from Proposition 3.1.6.

(b) We conjecture that, for the local models $\mathcal{M} = M^{\text{loc}}$, the maps $i_{\mathcal{G}}$ exist in general (without assuming any Hodge type condition) and that Proposition 4.1.2 still holds:

More precisely, suppose that $M^{\text{loc}} = M^{\text{loc}}(\mathcal{G}, \{\mu\})$ is the local model over $\text{Spec}(\mathcal{O}_E)$ conjectured to exist by Scholze [35, Conjecture 21.4.1]. Then, we expect that there are canonical functorial injective maps

$$i_{\mathcal{G}, \mu}(R) : M^{\text{loc}}(R) \rightarrow \text{Gr}_{\mathcal{G}}(R),$$

for R p -adic flat over $\mathcal{O}_E$, which also satisfy the equivariance property (4.1.9).

(One can speculate that the maps $i_{\mathcal{G}, \mu}$ come from natural maps

$$M^{\text{loc}} \rightarrow \text{Gr}_{\mathcal{G}}^{\Delta},$$

where $\text{Gr}_{\mathcal{G}}^{\Delta}$ is a “prismatic affine Grassmannian” for $\mathcal{G}$.)

4.2 $(\mathcal{G}, \mathcal{M})$ -displays.

4.2.1. We now give the definition of a $(\mathcal{G}, \mathcal{M})$ -display over R , where R is a p -adic flat $\mathcal{O}_E$ -algebra. We assume that $(\mathcal{G}, \mathcal{M})$ is of integral local Hodge type.

Definition 4.2.2 A $(\mathcal{G}, \mathcal{M})$ -display over R is a triple $\mathcal{D} = (\mathcal{P}, q, \Psi)$ of:

- A $\mathcal{G}$ -torsor $\mathcal{P}$ over $W(R)$,

- a $\mathcal{G}$ -equivariant morphism $q : \mathcal{P} \otimes_{W(R)} R \rightarrow \mathcal{M}$ over $\mathcal{O}_E$,
- a $\mathcal{G}$ -isomorphism $\Psi : \mathcal{Q} \xrightarrow{\sim} \mathcal{P}$ where $\mathcal{Q}$ is the $\mathcal{G}$ -torsor over $W(R)$ which is the modification of $\varphi^*\mathcal{P}$ given by $(\mathcal{P}, q)$ in 4.1.2.

Recall that, by 4.1.2, the pair $(\mathcal{P}, q)$ gives $\mathcal{Q}$ together with an isomorphism (4.1.10)

$$\alpha : \mathcal{Q}[1/p] \xrightarrow{\sim} \varphi^*\mathcal{P}[1/p].$$

Composing α^{-1} with $\Psi[1/p] : \mathcal{Q}[1/p] \xrightarrow{\sim} \mathcal{P}[1/p]$ gives an isomorphism of $\mathcal{G}$ -torsors over $W(R)[1/p]$

$$\Phi : \varphi^*\mathcal{P}[1/p] \xrightarrow{\sim} \mathcal{P}[1/p]$$

which is also attached to the $(\mathcal{G}, \mathcal{M})$ -display $\mathcal{D} = (\mathcal{P}, q, \Psi)$.

4.2.3. Suppose $(\mathcal{G}, \{\mu\})$ is a pair of a parahoric group scheme and a conjugacy class of a minuscule cocharacter μ of $\mathcal{G}_{\mathbb{Q}_p}$. Assume Scholze's conjecture [35, Conj. 21.4.1] on the existence of the local model $M^{\text{loc}} = M^{\text{loc}}(\mathcal{G}, \{\mu\})$.

Suppose that either $(\mathcal{G}, M^{\text{loc}})$ is of integral local Hodge type, or more generally, that the conjecture of Remark 4.1.15 (b) is true for $(\mathcal{G}, \{\mu\})$. Then the construction of the modification $\mathcal{Q}$ from $(\mathcal{P}, q)$ goes through and the definition of a $(\mathcal{G}, M^{\text{loc}})$ -display makes sense. In this case, instead of “ $(\mathcal{G}, M^{\text{loc}})$ -display”, we will simply say “ $(\mathcal{G}, \{\mu\})$ -display”.

4.2.4. Assume now that $\mathfrak{X}$ is a p -adic formal scheme which is flat and formally of finite type over $\text{Spf}(\mathcal{O}_E)$. By Zink's Witt vector descent [40, §1.3, Lemma 30], there is a sheaf of rings $W(\mathcal{O}_{\mathfrak{X}})$ over $\mathfrak{X}$ such that for every open affine formal subscheme $\text{Spf}(R) \subset \mathfrak{X}$, we have $\Gamma(\text{Spf}(R), W(\mathcal{O}_{\mathfrak{X}})) = W(R)$. It now makes sense to give the natural extension of the above definition: A $(\mathcal{G}, \mathcal{M})$ -display over $\mathfrak{X}$ is a triple $\mathcal{D} = (\mathcal{P}, q, \Psi)$ with the data $\mathcal{P}, q, \Psi$, as above given over $W(\mathcal{O}_{\mathfrak{X}})$.

4.3 Dieudonné $(\mathcal{G}, \mathcal{M})$ -displays

4.3.1. We now assume that p is odd and that R is in addition complete local Noetherian. We continue to suppose that $(\mathcal{G}, \mathcal{M})$ is of integral local Hodge type.

Definition 4.3.2 A Dieudonné $(\mathcal{G}, \mathcal{M})$ -display over R is a triple $\mathcal{D} = (\mathcal{P}, q, \Psi)$ of a $\mathcal{G}$ -torsor $\mathcal{P}$ over $\hat{W}(R)$, a $\mathcal{G}$ -equivariant morphism

$$q : \mathcal{P} \otimes_{\hat{W}(R)} R \rightarrow \mathcal{M}$$

over $\mathcal{O}_E$, and a $\mathcal{G}$ -isomorphism $\Psi : \mathcal{Q} \xrightarrow{\sim} \mathcal{P}$ where $\mathcal{Q}$ is the $\mathcal{G}$ -torsor over $\hat{W}(R)$ induced by q in 4.1.2 (applied to $A = \hat{W}(R)$).

4.3.3 Note that a Dieudonné $(\mathcal{G}, \mathcal{M})$ -display over R , produces a $(\mathcal{G}, \mathcal{M})$ -display over R by base change along the inclusion $\hat{W}(R) \hookrightarrow W(R)$. Proposition 4.5.3 holds for Dieudonné $(\mathcal{G}, \mathcal{M})$ -displays over R with $W(R)$ and $W(R/\mathfrak{a}_R)$ replaced by $\hat{W}(R)$ and

$\hat{W}(R/\mathfrak{a}_R)$. Most of the notions defined for $(\mathcal{G}, \mathcal{M})$ -displays, for example, the notion of rigid section and of locally universal, have obvious analogues for Dieudonné $(\mathcal{G}, \mathcal{M})$ -displays. The obvious variant of Proposition 4.5.11 for Dieudonné displays holds. We will sometimes refer to these statements when we are really using their $\hat{W}$ -variants instead, without explicitly alerting the reader.

4.4 Relation with Zink's displays

4.4.1. In the next paragraph, we relate our notion of a (Dieudonné) $(\mathcal{G}, \mathcal{M})$ -display over R to the classical notion of Zink [39, 40]. This involves the use of the integral local Hodge embedding. To fix ideas, we only discuss Dieudonné displays and for that we assume p is odd and R is also complete local Noetherian.

4.4.2. Suppose that $\mathcal{D} = (\mathcal{P}, q, \Psi)$ is a Dieudonné $(\mathcal{G}, \mathcal{M})$ -display over R , and that $\iota : \mathcal{G} \hookrightarrow \mathrm{GL}(\Lambda)$ is an integral local Hodge embedding.

We set $M = \mathcal{P}(\iota)$ which is a finite projective $\hat{W}(R)$ -module of rank equal to $\mathrm{rank}_{\mathbb{Z}_p}(\Lambda)$. Then q composed with ι produces $\mathcal{P} \otimes_{\hat{W}(R)} R \rightarrow \mathrm{Gr}(d, \Lambda)_{\mathcal{O}_E}$. This morphism gives a locally direct summand $\mathcal{F} \subset M \otimes_{\hat{W}(R)} R$. Let us denote by M_1 the inverse image of $\mathcal{F}$ under the reduction homomorphism $M \rightarrow M \otimes_{\hat{W}(R)} R$. This is a $\hat{W}(R)$ -module with $\hat{I}_R M \subset M_1 \subset M$. As in [20, 3.1.4], we denote by $\tilde{M}_1$ the image of $\varphi^* M_1$ under the $\hat{W}(R)$ -homomorphism

$$\varphi^* M_1 \rightarrow \varphi^* M$$

which is induced by the inclusion $M_1 \hookrightarrow M$. We have

$$p\varphi^* M \subset \tilde{M}_1 \subset \varphi^* M.$$

Then, our construction of $\mathcal{Q}$ from $(\mathcal{P}, q)$ implies an identification $\tilde{M}_1 = \mathcal{Q}(\iota)$. (See Sect. 4.1, $\tilde{M}_1$ corresponds to U there.) The isomorphism Ψ gives an isomorphism

$$\Psi(\iota) : \tilde{M}_1 \xrightarrow{\sim} M.$$

We denote $\Phi_1 : \varphi^* M_1 \rightarrow \tilde{M}_1 \xrightarrow{\sim} M$ the composition. Note here that we also have $\Phi = \Phi(\iota) : \varphi^* M[1/p] \xrightarrow{\sim} M[1/p]$ given by $\varphi^* M[1/p] = \tilde{M}_1[1/p]$ composed with $\Psi(\iota)[1/p]$. In fact, we see that

$$M \subset \Phi(\varphi^* M) \subset \frac{1}{p} M.$$

We can now consider the triple (M, M_1, Φ_1) . By [20, 3.1.3, Lemma 3.1.5], this triple defines a Dieudonné display over R in the sense of Zink [39]. (Recall that we assume that R is flat over $\mathbb{Z}_p$.)

It is useful to compare the notations here to those in display theory (e.g. [39]): Φ_1 here corresponds to the linear map $F_1^\#$ induced by the semi-linear map denoted there

by $F_1 = V^{-1}$. The linear map $F^\#$ that corresponds to the (semi-linear) Frobenius F of Zink's display is given here as⁴

$$F^\# = p \cdot \Phi : \varphi^* M \rightarrow M,$$

so $\Phi = p^{-1} \cdot F^\#$, $\Phi_1 = F^\#$.

We will denote the Dieudonné display

$$(M, M_1, F_1, F)$$

by $\mathcal{D}(\iota)$, since it is derived from $\mathcal{D}$ and $\iota : \mathcal{G} \hookrightarrow \mathrm{GL}(\Lambda)$. By [39], there is a corresponding p -divisible group $\mathcal{G}_R = \mathrm{BT}(\mathcal{D}(\iota))$ over R . By [39], [21, Theorem B],

$$(M, M_1, F_1, F) = \mathbb{D}(\mathcal{G}_R)(\hat{W}(R))$$

where $\mathbb{D}(\mathcal{G}_R)$ denotes the (filtered) *covariant* Dieudonné crystal of $\mathcal{G}_R$ (the Frobenius is given by F .) Then, the tangent space of $\mathcal{G}_R$ is canonically identified with the R -module M/M_1 . Since F is determined by F_1 , we will write this as

$$(M, M_1, F_1)$$

in what follows.

4.5 Rigidity and locally universal displays

4.5.1. In this subsection, we assume until further notice that R satisfies (CN), in particular it is normal and complete local Noetherian.

We also continue to assume that $(\mathcal{G}, \mathcal{M})$ is of integral local Hodge type and that $\mathcal{D} = (\mathcal{P}, q, \Psi)$ is a $(\mathcal{G}, \mathcal{M})$ -display over R . Under our assumptions on R , the $\mathcal{G}$ -torsors $\mathcal{P}, \mathcal{Q}$ over $W(R)$ are trivial. We denote by $\mathcal{D}_0 = (\mathcal{P}_0, q_0, \Psi_0)$ the display over k obtained by reduction of $\mathcal{D} = (\mathcal{P}, q, \Psi)$ modulo $\mathfrak{m}_R$.

4.5.2. Denote by $\mathfrak{m}_R$ the maximal ideal of R . Set $\mathfrak{a}_R = \mathfrak{m}_R^2 + (\pi_E)$, where π_E is a uniformizer of $\mathcal{O}_E$. Observe that the Frobenius φ factors as

$$W(R/\mathfrak{a}_R) \rightarrow W(k) \xrightarrow{\varphi} W(k) \rightarrow W(R/\mathfrak{a}_R).$$

Proposition 4.5.3 *There is a canonical isomorphism of $\mathcal{G}$ -torsors*

$$c : \mathcal{Q} \otimes_{W(R)} W(R/\mathfrak{a}_R) \xrightarrow{\sim} \mathcal{Q}_0 \otimes_{W(k)} W(R/\mathfrak{a}_R), \quad (4.5.4)$$

where $\mathcal{Q}$, resp. $\mathcal{Q}_0$, is the $\mathcal{G}$ -torsor for the display $\mathcal{D}$, resp. $\mathcal{D}_0$, as in Definition 4.2.2.

⁴ Note $\Phi = p^{-1} \cdot F^\#$ says that the Frobenius of the classical theory is here scaled by p^{-1} , cf. [35, p. 158].

Proof Note that [20, Lemma 3.1.9] gives the corresponding statement for classical displays and $\mathcal{O}_E = \mathbb{Z}_p$. Recall that the $\mathcal{G}$ -torsor $\mathcal{Q}$ is given as in the paragraph 4.1, using the corresponding $\mathcal{F} \subset M \otimes_{W(R)} R$ and $\tilde{M}_1$. We denote by $M_0, M_{1,0}, \tilde{M}_{1,0}$, the modules associated to the display $\mathcal{D}_0$ over k obtained from $\mathcal{D}$ by base change, as above. Let us write $M = L \oplus T$, with L and T free $W(R)$ -modules, such that $\mathcal{F}$ is given by L modulo I_R . Then $M_1 = L \oplus I_R T$, so $\mathcal{F}$ gives the filtration $I_R M \subset M_1 \subset M$. Then, as in the proof of [20, Lemma 3.1.9]

$$\tilde{M}_1 \otimes_{W(R)} W(R/\mathfrak{a}_R) = \varphi^*(L) \oplus (p \otimes \varphi^*(T))$$

and $\varphi^*(L) \simeq \varphi^*(L_0) \otimes_{W(k)} W(R/\mathfrak{a}_R)$, $\varphi^*(T) \simeq \varphi^*(T_0) \otimes_{W(k)} W(R/\mathfrak{a}_R)$. Here, we write $p \otimes -$ for $p\mathbb{Z}_p \otimes_{\mathbb{Z}_p} -$, and we have $L_0 = L \otimes_{W(R)} W(k)$, $T_0 = T \otimes_{W(R)} W(k)$. This gives the isomorphism of [20, Lemma 3.1.9]

$$c : \tilde{M}_1 \otimes_{W(R)} W(R/\mathfrak{a}_R) \xrightarrow{\sim} \tilde{M}_{1,0} \otimes_{W(k)} W(R/\mathfrak{a}_R) \quad (4.5.5)$$

which is independent of the choice of the normal decomposition $L \oplus T$. Using Proposition 3.2.3 we see that it is enough to show that c preserves the tensors that correspond to s_a . As in Sect. 3.2, we can assume these are of the form $s_a : \Lambda^{\otimes n} \rightarrow \Lambda^{\otimes n}$, with $n \geq 1$. These induce $m_a : M^{\otimes n} \rightarrow M^{\otimes n}$ which induce $u_a := \varphi^*(m_a) : \tilde{M}_1^{\otimes n} \rightarrow \tilde{M}_1^{\otimes n}$. In this situation, we have to show that c is compatible with u_a in the sense that the obvious diagram

$$\begin{array}{ccc} \tilde{M}_1^{\otimes n} \otimes_{W(R)} W(R/\mathfrak{a}_R) & \xrightarrow{u_a \otimes 1} & \tilde{M}_1^{\otimes n} \otimes_{W(R)} W(R/\mathfrak{a}_R) \\ \downarrow c \simeq & & \downarrow c \simeq \\ \tilde{M}_{1,0}^{\otimes n} \otimes_{W(k)} W(R/\mathfrak{a}_R) & \xrightarrow{(u_a)_0 \otimes 1} & \tilde{M}_{1,0}^{\otimes n} \otimes_{W(k)} W(R/\mathfrak{a}_R). \end{array}$$

is commutative. We start by giving a description of $\tilde{M}_1^{\otimes n}$.

The filtration $I_R M \subset M_1 \subset M$ induces a filtration on $M^{\otimes n}$ and we are interested in the $W(R)$ -submodule

$$N := \bigoplus_{j=0}^n (L^{\otimes n-j} \oplus I_R^j T^{\otimes j}) \subset M^{\otimes n} \quad (4.5.6)$$

which is the image of $M_1^{\otimes n} \rightarrow M^{\otimes n}$ (and so is independent of the normal decomposition $M = L \oplus T$.) (Note that when $n \geq 2$, the map $M_1^{\otimes n} \rightarrow M^{\otimes n}$ is rarely injective. Also note that $I_R^m = p^{m-1} I_R$, if $m \geq 1$ ([40, (7)]), and $\varphi(I_R^m) = (p^m)$.) The image of the map $\varphi^* N \rightarrow \varphi^* M^{\otimes n}$ induced by the inclusion $N \hookrightarrow M^{\otimes n}$, is $\tilde{M}_1^{\otimes n}$.

Next, we show that $m_a : M^{\otimes n} \rightarrow M^{\otimes n}$ preserves N , i.e. restricts to $m_a : N \rightarrow N$. The rough idea is that this should hold because the point in the Grassmannian corresponding to $\mathcal{F}$ is in the closure of the G -orbit of the cocharacter μ and the tensors m_a are fixed by the group G . More precisely, we show $m_a(N) \subset N$ using “restriction to $\mathcal{O}_F$ -points” as follows: Suppose $\xi : R \rightarrow \mathcal{O}_F$ is a local $\mathcal{O}_E$ -homomorphism. Denote by $M_{\mathcal{O}_F}$ and $N_{\mathcal{O}_F}$ the corresponding $W(\mathcal{O}_F)$ -modules for the display over $\mathcal{O}_F$ obtained by base change of $\mathcal{D}$ by ξ . As in the proof of [20, Lemma 3.2.6], we

see, using that $\xi^* \mathcal{F}$ is given by a G -cocharacter conjugate to μ , that $\xi^* m_a$ preserves $N_{\mathcal{O}_F} \subset M_{\mathcal{O}_F}^{\otimes n}$. We can now deduce that $m_a : M^{\otimes n} \rightarrow M^{\otimes n}$ preserves N : It is enough to check that certain elements of $W(R)$ which are given as the coefficients of images of m_a in a basis given by the decomposition (4.5.6) lie in $I_R^m = p^{m-1} I_R$, while, by the above, we know that their images in $W(\mathcal{O}_F)$ lie in $p^{m-1} I_{\mathcal{O}_F}$, for all such $\xi : R \rightarrow \mathcal{O}_F$. But this is true by a simple extension of the argument in the proof of Lemma 2.3.3.

Now consider the commutative diagram

$$\begin{array}{ccccc} \varphi^* N \otimes_{W(R)} W(R/\mathfrak{a}_R) & \xrightarrow{\alpha} & \tilde{M}_1^{\otimes n} \otimes_{W(R)} W(R/\mathfrak{a}_R) & \longrightarrow & \varphi^* M^{\otimes n} \otimes_{W(R)} W(R/\mathfrak{a}_R) \\ \downarrow c' \simeq & & \downarrow c \simeq & & \downarrow c'' \simeq \\ \varphi^* N_0 \otimes_{W(k)} W(R/\mathfrak{a}_R) & \xrightarrow{\alpha_0} & \tilde{M}_{1,0}^{\otimes n} \otimes_{W(k)} W(R/\mathfrak{a}_R) & \longrightarrow & \varphi^* M_0^{\otimes n} \otimes_{W(k)} W(R/\mathfrak{a}_R). \end{array}$$

Here c', c'' are the canonical isomorphisms obtained by the factorization $W(R/\mathfrak{a}_R) \rightarrow W(k) \rightarrow W(k) \rightarrow W(R/\mathfrak{a}_R)$ of the Frobenius φ above. We have $\alpha(\varphi^*(m_a)) = u_a$. The fact that c is compatible with the tensor u_a now follows from the above, the functoriality of c' and the fact that α is surjective. $\square$

4.5.7. Continuing with the same assumptions, we have:

Definition 4.5.8 A section s of the $\mathcal{G}$ -torsor $\mathcal{P}$ is called rigid in the first order at $\mathfrak{m}_R$ when, under the isomorphism of Proposition 4.5.3,

$$\Psi^{-1}(s) \bmod W(\mathfrak{a}_R) = \Psi_0^{-1}(s_0) \otimes 1,$$

where, again, the subscript 0 signifies reduction modulo $\mathfrak{m}_R$.

In other words, we are asking that the diagram

$$\begin{array}{ccccc} \mathcal{Q} \otimes_{W(R)} W(R/\mathfrak{a}_R) & \xrightarrow[\simeq]{\Psi} & \mathcal{P} \otimes_{W(R)} W(R/\mathfrak{a}_R) & \xrightarrow{(s_{\mathfrak{a}_R})^{-1}} & \mathcal{G} \otimes_{\mathbb{Z}_p} W(R/\mathfrak{a}_R) \\ \downarrow c \simeq & & \downarrow \Psi_0 \cdot c \cdot \Psi^{-1} \simeq & & \downarrow \text{id} \\ \mathcal{Q}_0 \otimes_{W(k)} W(R/\mathfrak{a}_R) & \xrightarrow[\simeq]{\Psi_0} & \mathcal{P}_0 \otimes_{W(k)} W(R/\mathfrak{a}_R) & \xrightarrow{s_0^{-1}} & \mathcal{G} \otimes_{\mathbb{Z}_p} W(R/\mathfrak{a}_R) \end{array}$$

commutes. (In this, we write $s_{\mathfrak{a}_R} = s \otimes_{W(R)} W(R/\mathfrak{a}_R)$ for simplicity.)

Given any section $s : \mathcal{G} \otimes_{\mathbb{Z}_p} W(R) \rightarrow \mathbb{A}$, the composition

$$g = s_0^{-1} \cdot (\Psi_0 \cdot c \cdot \Psi^{-1}) \cdot s_{\mathfrak{a}_R} : \mathcal{G} \otimes_{\mathbb{Z}_p} W(R/\mathfrak{a}_R) \rightarrow \mathcal{G} \otimes_{\mathbb{Z}_p} W(R/\mathfrak{a}_R)$$

is given by an element $g \in \mathcal{G}(W(R/\mathfrak{a}_R))$ which reduces to the identity in $\mathcal{G}(W(k))$, i.e. with $g_0 = 1$. Since $\mathcal{G}$ is smooth, $\mathcal{G}(W(R)) \rightarrow \mathcal{G}(W(R/\mathfrak{a}_R))$ is surjective, and we can always find $h \in \mathcal{G}(W(R))$ with $h \bmod W(\mathfrak{a}_R) = g^{-1}$. Then $h \cdot s$ is rigid in the first order at $\mathfrak{m}_R$. Hence, if for example k is algebraically closed, there is always some section which is rigid in the first order at $\mathfrak{m}_R$.

Remark 4.5.9 This notion of “rigid in the first order” is comparable to a corresponding notion for Dieudonné crystals that appears in [32, Def. 3.31], see Proposition 4.5.15 (a) below. In fact, the isomorphism

$$\theta := \Psi \cdot c^{-1} \cdot \Psi_0^{-1} : \mathcal{P}_0 \otimes_{W(k)} W(R/\mathfrak{a}_R) \xrightarrow{\sim} \mathcal{P} \otimes_{W(R)} W(R/\mathfrak{a}_R)$$

should correspond, in the case of Zink displays, to the trivialization given by the crystalline structure. (Note that $(\mathfrak{m}_R/\mathfrak{a}_R)^2 = 0$.) Let us remark here that s is rigid in the first order at $\mathfrak{m}_R$ when we have $\theta(s_0) = s_{\mathfrak{a}_R}$, a condition we might think of as saying “ s is horizontal with respect to θ at the closed point of R ”.

Definition 4.5.10 A $(\mathcal{G}, \mathcal{M})$ -display $\mathcal{D}$ over R is *locally universal*, if there is a section s of $\mathcal{P}$ which is rigid in the first order, such that the composition

$$q \cdot (s \otimes 1) : \text{Spec}(R) \rightarrow \mathcal{P} \otimes_{W(R)} R \rightarrow \mathcal{M}$$

gives an isomorphism between R and the completion of the local ring of $\mathcal{M} \otimes_{\mathbb{Z}_p} W(k)$ at the image of the closed point of $\text{Spec}(R)$.

Proposition 4.5.11 Suppose that the $(\mathcal{G}, \mathcal{M})$ -display $\mathcal{D} = (\mathcal{P}, q, \Psi)$ over R is locally universal. Then $q : \mathcal{P} \otimes_{W(R)} R \rightarrow \mathcal{M}$ is formally smooth.

Proof The action morphism $m : \mathcal{G} \times_{\text{Spec}(\mathbb{Z}_p)} \mathcal{M} \rightarrow \mathcal{M}$ is smooth, since $\mathcal{G}$ is smooth. Let $s : \text{Spec}(R) \rightarrow \mathcal{P} \otimes_{W(R)} R$ be a section which is rigid in the first order and is such that $q \cdot (s \otimes 1)$ identifies R with the completion of the strict Henselization of $\mathcal{M}$. Since q is $\mathcal{G}$ -equivariant, $\mathcal{G} \times_{\text{Spec}(\mathbb{Z}_p)} \text{Spec}(R) \simeq \mathcal{P} \otimes_{W(R)} R$ given by s identifies $q : \mathcal{P} \otimes_{W(R)} R \rightarrow \mathcal{M}$ with

$$\mathcal{G} \times_{\text{Spec}(\mathbb{Z}_p)} \text{Spec}(R) \xrightarrow{\text{id}_{\mathcal{G}} \times q \cdot (s \otimes 1)} \mathcal{G} \times_{\text{Spec}(\mathbb{Z}_p)} \mathcal{M} \xrightarrow{m} \mathcal{M}.$$

This is the composition of formally smooth morphisms, so also formally smooth. $\square$

4.5.12. We return momentarily to Zink displays. We continue with the same assumptions on $\mathcal{D} = (\mathcal{P}, q, \Psi)$ and ι as in Sect. 4.4. In particular, $\mathcal{D}$ is a Dieudonné display.

Recall that (e.g. [40, 2.2], [39, Thm 3]) if (M, M_1, F_1) is a Dieudonné display, then M gives a crystal. In fact, we only need the following consequence: For $\mathfrak{a}_R = \mathfrak{m}_R^2 + (\pi_E)$ as above, there is a canonical isomorphism

$$\vartheta : M_0 \otimes_{W(k)} \hat{W}(R/\mathfrak{a}_R) \xrightarrow{\sim} M \otimes_{\hat{W}(R)} \hat{W}(R/\mathfrak{a}_R). \quad (4.5.13)$$

Using this (together with the main theorem of [39]) one can understand the deformations of the p -divisible group $\mathcal{G}_0$ given by $\mathcal{D}_0(\iota) = \mathcal{D}(\iota) \otimes_{\hat{W}(R)} W(k)$ ([39, Thm 4]):

Fix an identification $M_0 = \Lambda \otimes_{\mathbb{Z}_p} W(k)$. The p -divisible group $\mathcal{G}_{R/\mathfrak{a}_R}$ which is given by $\mathcal{D}(\iota) \otimes_{\hat{W}(R)} \hat{W}(R/\mathfrak{a}_R)$ produces a $\text{Spec}(R/\mathfrak{a}_R)$ -valued point of the Grassmannian $\text{Gr}(d, \Lambda)_k$. This point is given by the submodule of $\Lambda \otimes_{\mathbb{Z}_p} R/\mathfrak{a}_R$ which is the reduction of

$$\Lambda \otimes_{\mathbb{Z}_p} \hat{I}_{R/\mathfrak{a}_R} \subset \vartheta^{-1}(M_{1, R/\mathfrak{a}_R}) \subset \Lambda \otimes_{\mathbb{Z}_p} \hat{W}(R/\mathfrak{a}_R).$$

modulo $\hat{I}_{R/\mathfrak{a}_R}$. Conversely, every $\mathrm{Spec}(R/\mathfrak{a}_R)$ -valued point of the Grassmannian which lifts the k -point corresponding to $M_{1,0}$ comes as above from a unique deformation $\mathcal{G}_{R/\mathfrak{a}_R}$ of $\mathcal{G}_0$ over $R/\mathfrak{a}_R$. This way we can identify the tangent space $\mathfrak{T}_0\mathrm{Gr}(d, \Lambda)_k$ of $\mathrm{Gr}(d, \Lambda)_k$ at $M_{1,0}$ with the tangent space $\mathfrak{T}_0$ of the formal deformation space of the p -divisible group $\mathcal{G}_0$. Here we need a more precise statement about the deformations that lift over R and the corresponding Dieudonné $\mathcal{G}$ -displays which we will give next.

4.5.14. We continue with the same assumptions on $\mathcal{D} = (\mathcal{P}, q, \Psi)$ and ι as above. If s is a section of $\mathcal{P}$, then $s(\iota)$ is the corresponding frame, i.e. the isomorphism $s(\iota) : \Lambda \otimes_{\mathbb{Z}_p} \hat{W}(R) \xrightarrow{\sim} M = \mathcal{P}(\iota)$.

Proposition 4.5.15 (a) *A section s of $\mathcal{P}$ is rigid in the first order at $\mathfrak{m}_R$ if and only if $\vartheta(s_0(\iota)) = s_{\mathfrak{a}_R}(\iota)$, where ϑ is the map (4.5.13).*

(b) *Suppose s is a section of $\mathcal{P}$ which is rigid in the first order at $\mathfrak{m}_R$. Let*

$$\mathrm{Spec}(R/\mathfrak{a}_R) \rightarrow \mathfrak{T}_0$$

be the classifying morphism of $\mathrm{Spec}(R/\mathfrak{a}_R)$ into the tangent space $\mathfrak{T}_0$ of the deformation space of $\mathcal{G}_0$, which is given by the deformation $\mathcal{G}_{R/\mathfrak{a}_R}$. Then there is an isomorphism $\mathfrak{T}_0\mathrm{Gr}(d, n)_k \xrightarrow{\sim} \mathfrak{T}_0$ over k such that

$$\mathrm{Spec}(R/\mathfrak{a}_R) \xrightarrow{s_{\mathfrak{a}_R} \otimes 1} \mathcal{P} \otimes_{\hat{W}(R)} R/\mathfrak{a}_R \xrightarrow{q \otimes_R 1} \mathcal{M} \rightarrow \mathrm{Gr}(d, n)_{\mathcal{O}_E}$$

gives, after composing with $\mathfrak{T}_0\mathrm{Gr}(d, n)_k \xrightarrow{\sim} \mathfrak{T}_0$, the classifying morphism above.

Proof It follows directly from the definitions that if s is rigid in the first order at $\mathfrak{m}_R$, then the trivialization $s(\iota) : \Lambda \otimes_{\mathbb{Z}_p} \hat{W}(R) \xrightarrow{\sim} M$ makes $\Psi_R(\iota) : \tilde{M}_1 \xrightarrow{\sim} M$ “constant modulo $\mathfrak{a}_R$ ” in the sense of [20, Definition (3.1.11)]. (By this we mean that we take the identification $\Lambda \otimes_{\mathbb{Z}_p} \hat{W}(R) \cong M$ which is used in [20, Definition (3.1.11)] to be $s(\iota)$.) Both (a) and (b) now follow from the definition of c , the construction of the map ϑ in [40, 2.2], [39], and the argument in the proof of [20, Lemma 3.1.12]. (This lemma gives part (b) for a universal (M, M_1, F_1) .) $\square$

5 Crystalline $\mathcal{G}$ -representations

In this section, we describe “ $\mathcal{G}$ -versions” of objects of integral p -adic Hodge theory which can be attached to a $\mathcal{G}(\mathbb{Z}_p)$ -valued crystalline representation.

5.1. Fix $(\mathcal{G}, \mathcal{M})$ as in Sect. 2.1 of integral local Hodge type. Fix also an integral local Hodge embedding

$$\iota : \mathcal{G} \hookrightarrow \mathrm{GL}(\Lambda)$$

with $\mathcal{M} \rightarrow \overline{\iota_*(X_\mu)} \subset \mathrm{Gr}(g, \Lambda)_{\mathcal{O}_E}$.

Let F be a finite extension of E or of $\check{E}$ with residue field k . Let

$$\rho : \mathrm{Gal}(\bar{F}/F) \rightarrow \mathcal{G}(\mathbb{Z}_p)$$

be a Galois representation. We assume that $\iota \cdot \rho : \text{Gal}(\bar{F}/F) \rightarrow \text{GL}(\Lambda[1/p])$ is crystalline. We give three flavors of “ $\mathcal{G}$ -versions” of Frobenius modules which can be attached to ρ by integral p -adic Hodge theory.

5.2 The Breuil–Kisin $\mathcal{G}$ -module

5.2.1. Choose a uniformizer π_F of F and let $E(u) \in W(k)[u]$ be the Eisenstein polynomial with $E(\pi_F) = 0$. Choose also a compatible system of roots $\sqrt[p^n]{\pi_F}$ in $\bar{F}$. The Breuil–Kisin $\mathcal{G}$ -module attached to ρ , is by definition, a pair $(\mathcal{P}_{\text{BK}}, \varphi_{\mathcal{P}_{\text{BK}}})$ where

- $\mathcal{P}_{\text{BK}}$ is a $\mathcal{G}$ -torsor over $\mathfrak{S} = W(k)[[u]]$,
- $\varphi_{\mathcal{P}_{\text{BK}}}$ is an isomorphism of $\mathcal{G}$ -torsors

$$\varphi_{\mathcal{P}_{\text{BK}}} : \varphi^* \mathcal{P}_{\text{BK}}[1/E(u)] \xrightarrow{\sim} \mathcal{P}_{\text{BK}}[1/E(u)].$$

(Here, $\varphi : \mathfrak{S} \rightarrow \mathfrak{S}$ is the ring homomorphism which extends the Frobenius on $W(k)$ and satisfies $\varphi(u) = u^p$.)

It is constructed as follows. (It does depend on the choice of $\sqrt[p^n]{\pi_F}$, $n \geq 0$.)

As in the proof of [20, Lemma 3.3.5], we write $\mathcal{O}_{\mathcal{G}} = \varinjlim_{i \in J} \Lambda_i$ with $\Lambda_i \subset \mathcal{O}_{\mathcal{G}}$ of finite $\mathbb{Z}_p$ -rank and $\mathcal{G}$ -stable. The Galois action on Λ gives actions on Λ_i and on $\mathcal{O}_{\mathcal{G}}$. We apply the Breuil–Kisin functor

$$\mathfrak{M} : \text{Rep}_K^{\text{cris}, \circ} \rightarrow \text{Mod}_{/\mathfrak{S}}^{\varphi}$$

(see [19, § 1], [20, Theorem 3.3.2] for notations and details of its properties. This depends on the choice of $\sqrt[p^n]{\pi_F}$, $n \geq 0$, in $\bar{F}$). Let

$$\mathfrak{M}(\mathcal{O}_{\mathcal{G}}) := \varinjlim_{i \in J} \mathfrak{M}(\Lambda_i).$$

By [20, Theorem 3.3.2], the composition of $\mathfrak{M}$ with restriction to $D^{\times}$ is an exact faithful tensor functor. Hence, we obtain that $\mathfrak{M}(\mathcal{O}_{\mathcal{G}})_{|D^{\times}}$ is a sheaf of algebras over $D^{\times}$ and that

$$\mathcal{P}_{\text{BK}}^{\times} := \underline{\text{Spec}}(\mathfrak{M}(\mathcal{O}_{\mathcal{G}})_{|D^{\times}})$$

is a $\mathcal{G}$ -torsor over $D^{\times}$. Using purity, we can extend $\mathcal{P}_{\text{BK}}^{\times}$ to a $\mathcal{G}$ -torsor $\mathcal{P}_{\text{BK}}$ over $D = \text{Spec}(W(k)[[u]])$ as follows:

Let us consider the scheme

$$\mathcal{P}' := \underline{\text{Isom}}_{s_{a, \text{BK}}, s_a \otimes 1}(\mathfrak{M}(\Lambda), \Lambda \otimes_{\mathbb{Z}_p} W[[u]]) \subset \underline{\text{Hom}}(\mathfrak{M}(\Lambda), \Lambda \otimes_{\mathbb{Z}_p} W[[u]])$$

of isomorphisms taking $s_{a, \text{BK}}$ to $s_a \otimes 1$. Here, as in loc. cit.,

$$s_{a, \text{BK}} := \tilde{s}_a \in \mathfrak{M}(\Lambda)^{\otimes}$$

are the tensors obtained by applying the functor $\mathfrak{M}(-)$ to the Galois invariant tensors $s_a \in \Lambda^\otimes$. By [20, Lemma 3.3.5], the scheme $\mathcal{P}'$ is naturally a $\mathcal{G}$ -torsor over D , which, in fact, is trivial. As in the proof of [20, Lemma 3.3.5], we see that there is a natural isomorphism $\mathcal{P}'_{|D^\times} \simeq \mathcal{P}_{\text{BK}}^\times$ as $\mathcal{G}$ -torsors over $D^\times$. Hence, the $\mathcal{G}$ -torsor $\mathcal{P}'$ over D gives the desired extension; we denote it by $\mathcal{P}_{\text{BK}}$. We can see that the $\mathcal{G}$ -torsor $\mathcal{P}_{\text{BK}}$ over D is uniquely determined (up to unique isomorphism) and is independent of the choice of Λ . This follows from the fact that there is a bijection between sections of $\mathcal{P}'$ over $W[[u]]$ and sections of $\mathcal{P}'_{|D^\times} = \mathcal{P}_{\text{BK}}^\times$ over $D^\times$. The isomorphism $\varphi_{\mathcal{P}_{\text{BK}}}$ comes directly from the construction $\mathcal{P}_{\text{BK}} = \mathcal{P}'$ and is also independent of choices.

Note here that we can view the Breuil–Kisin $\mathcal{G}$ -module attached to ρ as an exact tensor functor

$$\text{Rep}_{\mathbb{Z}_p}(\mathcal{G}) \rightarrow \text{Mod}_{/\mathfrak{S}}^\varphi.$$

5.3 The Dieudonné $\mathcal{G}$ -display

5.3.1. Assume here that $\iota \cdot \rho$ has Hodge–Tate weights in $\{0, 1\}$ and that in fact, the deRham filtration on $D_{\text{dR}}(\Lambda[1/p])$ is given by a G -cocharacter conjugate to μ . Then, there is also a Dieudonné $(\mathcal{G}, \mathcal{M})$ -display

$$\mathcal{D}_\rho = (\mathcal{P}, q, \Psi)$$

over $\mathcal{O}_F$ which is attached to ρ . This is constructed as follows:

Consider the Breuil–Kisin module $\mathfrak{M} = \mathfrak{M}(\Lambda)$ attached to Λ . It comes with the Frobenius $\varphi_{\mathfrak{M}} : \varphi^*\mathfrak{M}[1/E(u)] \xrightarrow{\sim} \mathfrak{M}[1/E(u)]$. The condition on the weights implies that

$$\mathfrak{M} \subset \varphi_{\mathfrak{M}}(\varphi^*\mathfrak{M}) \subset E(u)^{-1}\mathfrak{M}.$$

Let $\mathfrak{S} = W(k)[[u]] \rightarrow \hat{W}(\mathcal{O}_F)$ be the unique Frobenius equivariant map lifting the identity on $\mathcal{O}_F$ which is given by $u \mapsto [\pi_F]$. We set

$$\mathcal{P} := \mathcal{P}_{\text{BK}} \otimes_{\mathfrak{S}, \varphi} \hat{W}(\mathcal{O}_F) = \underline{\text{Isom}}_{(s_{a,D}), (s_a \otimes 1)}(\varphi^*\mathfrak{M}(\Lambda), \Lambda \otimes_{\mathbb{Z}_p} \hat{W}(\mathcal{O}_F)).$$

Here, we set

$$s_{a,D} := \varphi^*(s_{a,\text{BK}}) \in \varphi^*\mathfrak{M}(\Lambda).$$

To obtain the rest of the data of the Dieudonné $\mathcal{G}$ -display we proceed as follows:

We can write $\mathfrak{M} = L \oplus T$, with L and T free $\mathfrak{S}$ -modules such that

$$\varphi_{\mathfrak{M}}(\varphi^*\mathfrak{M}) = L \oplus E(u)^{-1}T.$$

Denote by $\mathfrak{M}_1 \subset \varphi^*\mathfrak{M}$ the largest $\mathfrak{S}$ -submodule such that $\varphi_{\mathfrak{M}}(\mathfrak{M}_1) \subset \mathfrak{M}$. We have

$$E(u)\varphi^*\mathfrak{M} \subset \mathfrak{M}_1 \subset \varphi^*\mathfrak{M}.$$

Then $\varphi_{\mathfrak{M}}(\mathfrak{M}_1) = \mathfrak{M}$ and we have an isomorphism

$$\varphi_{\mathfrak{M}} : \mathfrak{M}_1 \xrightarrow{\sim} \mathfrak{M}.$$

The corresponding filtration

$$\overline{\mathfrak{M}}_1 \subset (\varphi^*\mathfrak{M})/E(u)(\varphi^*\mathfrak{M})$$

gives an $\mathcal{O}_F$ -valued point of a Grassmannian. Over F , this filtration is the deRham filtration of $D_{\text{dR}}(\Lambda[1/p])$ by [20, Theorem 3.3.2 (1)]. The condition that the deRham filtration on $D_{\text{dR}}(\Lambda[1/p])$ is given by a G -cocharacter conjugate to μ now implies that this point is in the closure of the G -orbit of μ , hence gives an $\mathcal{O}_F$ -point of $\mathcal{M}$. This produces a $\mathcal{G}$ -equivariant morphism

$$q : \varphi^*\mathcal{P}_{\text{BK}} \otimes_{\mathfrak{S}} \mathcal{O}_F \rightarrow \mathcal{M}.$$

Since $\mathcal{P} \otimes_{\hat{W}(\mathcal{O}_F)} \mathcal{O}_F = \varphi^*\mathcal{P}_{\text{BK}} \otimes_{\mathfrak{S}} \mathcal{O}_F$ we obtain

$$q : \mathcal{P} \otimes_{\hat{W}(\mathcal{O}_F)} \mathcal{O}_F \rightarrow \mathcal{M}.$$

This gives $\mathcal{Q}$ and Ψ is then determined by $\varphi_{\mathcal{P}_{\text{BK}}}$. To give these more explicitly, set $M = \mathfrak{M} \otimes_{\mathfrak{S}, \varphi} \hat{W}(\mathcal{O}_F) = \varphi^*\mathfrak{M} \otimes_{\mathfrak{S}} \hat{W}(\mathcal{O}_F)$ which acquires the tensors $s_{a,D} = \varphi^*(s_{a,\text{BK}}) \in M^{\otimes}$. We have

$$\mathfrak{M}_1 \otimes_{\mathfrak{S}} \hat{W}(\mathcal{O}_F) \subset M = \varphi^*\mathfrak{M} \otimes_{\mathfrak{S}} \hat{W}(\mathcal{O}_F).$$

Using that $\varphi(E([\pi_F]))/p$ is a unit in $\hat{W}(\mathcal{O}_F)$, after applying φ , we obtain a filtration

$$p(\varphi^*M) \subset \tilde{M}_1 := \varphi^*(\mathfrak{M}_1 \otimes_{\mathfrak{S}} \hat{W}(\mathcal{O}_F)) \subset \varphi^*M.$$

As in the proof of [20, Lemma 3.2.9], the tensors $\varphi^*(s_{a,D}) \in \varphi^*M^{\otimes}$ lie in $\tilde{M}_1^{\otimes}$ and

$$\mathcal{Q} = \underline{\text{Isom}}_{(\varphi^*s_{a,D}), (s_a \otimes 1)}(\tilde{M}_1, \Lambda \otimes_{\mathbb{Z}_p} \hat{W}(\mathcal{O}_F)).$$

The “divided Frobenius” $\tilde{M}_1 \xrightarrow{\sim} M$ which is obtained by pulling back $\varphi_{\mathfrak{M}} : \mathfrak{M}_1 \xrightarrow{\sim} \mathfrak{M}$ along $\varphi : \mathfrak{S} \rightarrow \hat{W}(\mathcal{O}_F)$ sends the tensors $\varphi^*(s_{a,D})$ to $s_{a,D}$ and gives the $\mathcal{G}$ -isomorphism $\Psi : \mathcal{Q} \xrightarrow{\sim} \mathcal{P}$.

5.4 The Breuil–Kisin–Fargues $\mathcal{G}$ -module

Here, we use the notations of Sects. 2.4 and 2.5. In particular, $\mathcal{O}$ is the p -adic completion of the integral closure $\bar{\mathcal{O}}_F$ of $\mathcal{O}_F$ in $\bar{F}$ and $\mathcal{O}^b$ is its tilt. For simplicity, set $A_{\text{inf}} = A_{\text{inf}}(\mathcal{O})$.

5.4.1. By definition, a (finite free) Breuil–Kisin–Fargues (BKF) module over A_{inf} is a finite free A_{inf} -module M together with an isomorphism

$$\varphi_M : (\varphi^* M)[1/\varphi(\xi)] \xrightarrow{\sim} M[1/\varphi(\xi)]$$

where ξ is a generator of the kernel of θ . (See [4, 35]).

Similarly, a Breuil–Kisin–Fargues $\mathcal{G}$ -module over A_{inf} is, by definition, a pair $(\mathcal{D}_{\text{inf}}, \varphi_{\mathcal{D}_{\text{inf}}})$, where $\mathcal{D}_{\text{inf}}$ is a $\mathcal{G}$ -torsor over A_{inf} and

$$\varphi_{\mathcal{D}_{\text{inf}}} : (\varphi^* \mathcal{D}_{\text{inf}})[1/\varphi(\xi)] \xrightarrow{\sim} \mathcal{D}_{\text{inf}}[1/\varphi(\xi)]$$

is a $\mathcal{G}$ -equivariant isomorphism.

5.4.2. Now fix a uniformizer $\pi = \pi_F$ of F and also a compatible system of roots π^{1/p^n} , for $n \geq 1$, giving an element $\pi^b = (\pi, \pi^{1/p}, \dots) \in \mathcal{O}^b$. These choices define a φ -equivariant homomorphism

$$f : \mathfrak{S} = W[[u]] \rightarrow A_{\text{inf}}$$

given by $u \mapsto [\pi^b]^p$ and which is the Frobenius on $W = W(k)$. By [4, Proposition 4.32], the association

$$\mathfrak{M} \mapsto M = \mathfrak{M} \otimes_{\mathfrak{S}} A_{\text{inf}}$$

defines an exact tensor functor from Breuil–Kisin modules over $\mathfrak{S}$ to Breuil–Kisin–Fargues (BKF) modules over A_{inf} .

We can compose the above functor with the tensor exact functor

$$\text{Rep}_{\mathbb{Z}_p}(\mathcal{G}) \rightarrow \text{Mod}_{/\mathfrak{S}}^{\varphi}$$

given by the Breuil–Kisin $\mathcal{G}$ -module $\mathcal{P}_{\text{BK}}$ over $D = \text{Spec}(\mathfrak{S})$ of Sect. 5.2. We obtain a tensor exact functor

$$\text{Rep}_{\mathbb{Z}_p}(\mathcal{G}) \rightarrow \text{Mod}_{/A_{\text{inf}}}^{\varphi}$$

to the category $\text{Mod}_{/A_{\text{inf}}}^{\varphi}$ of finite free BKF modules over A_{inf} . This functor gives a $\mathcal{G}$ -torsor $\mathcal{D}_{\text{inf}}$ over A_{inf} which admits a $\mathcal{G}$ -equivariant isomorphism

$$\varphi_{\mathcal{D}_{\text{inf}}} : (\varphi^* \mathcal{D}_{\text{inf}})[1/\varphi(\xi)] \xrightarrow{\sim} \mathcal{D}_{\text{inf}}[1/\varphi(\xi)].$$

(Here, $\varphi(\xi) = f(E(u))$ for $E(u) \in W[[u]]$ an Eisenstein polynomial for π . The element ξ generates the kernel of $\theta : A_{\text{inf}} \rightarrow \mathcal{O}$.) Hence, $(\mathcal{D}_{\text{inf}}, \varphi_{\mathcal{D}_{\text{inf}}})$ is a Breuil–Kisin–Fargues $\mathcal{G}$ -module which is attached to ρ .

More explicitly, set

$$M_{\text{inf}} := M_{\text{inf}}(\Lambda) = \mathfrak{M}(\Lambda) \otimes_{\mathfrak{S}, f} A_{\text{inf}}.$$

The tensors $s_a \in \Lambda^\otimes$ induce φ -invariant tensors $s_{a,\text{inf}} \in M_{\text{inf}}^\otimes$. These are the base changes

$$s_{a,\text{inf}} = f^*(s_{a,\text{BK}})$$

of $s_{a,\text{BK}} \in \mathfrak{M}(\Lambda)^\otimes$. We have

$$\mathcal{D}_{\text{inf}} \cong \underline{\text{Isom}}_{(s_{a,\text{inf}}), (s_a \otimes 1)}(M_{\text{inf}}, \Lambda \otimes_{\mathbb{Z}_p} A_{\text{inf}}(\mathcal{O}))$$

as $\mathcal{G}$ -torsors.

5.4.3. Assume that Λ , acted on by ρ , is isomorphic to the Galois representation on the Tate module $T = T_p(\mathcal{G})$ of a p -divisible group $\mathcal{G}$ over $\mathcal{O}_F$. We have

$$M_{\text{inf}} \cong M(\mathcal{G}). \quad (5.4.4)$$

Here, $M(\mathcal{G})$ is the BKF module associated ([35, Theorem 17.5.2]) to the base change over $\mathcal{O}$ of $\mathcal{G}$ of the p -divisible group $\mathcal{G}$. The corresponding φ -linear Frobenius $\varphi_{M_{\text{inf}}}$ satisfies

$$M_{\text{inf}} \subset \varphi_{M_{\text{inf}}}(M_{\text{inf}}) \subset \varphi(\xi)^{-1} M_{\text{inf}}.$$

(Here, again, we denote a φ -linear map and its linearization by the same symbol.)

5.4.5. Choose p -th power roots of unity giving $\epsilon = (1, \zeta_p, \zeta_{p^2}, \dots) \in \mathcal{O}^\flat$ and set $\mu = [\epsilon] - 1 \in A_{\text{inf}}$.

Let $\mathbb{Q}_p/\mathbb{Z}_p$ be the constant p -divisible group. By [35, theorem 17.5.2], there is a comparison map

$$\Lambda \cong \text{Hom}_{\mathcal{O}}(\mathbb{Q}_p/\mathbb{Z}_p, \mathcal{G}_{\mathcal{O}}) \xrightarrow{\sim} \text{Hom}_{A_{\text{inf}}, \varphi}(A_{\text{inf}}, M_{\text{inf}}) = M_{\text{inf}}^{\varphi_{M_{\text{inf}}} = 1}.$$

This induces the φ -invariant isomorphism

$$\Lambda \otimes_{\mathbb{Z}_p} A_{\text{inf}}[1/\mu] \cong M_{\text{inf}}[1/\mu] \cong \mathfrak{M}(\Lambda) \otimes_{\mathfrak{S}, f} A_{\text{inf}}[1/\mu].$$

It follows from the constructions and [4, 4.26] that under these isomorphisms the tensors $s_a \otimes 1$, $s_{a,\text{inf}}$ and $s_{a,\text{BK}} \otimes 1$ correspond.

5.4.6. The constructions of the previous paragraphs are compatible in the following sense. Assume that ρ is as in the beginning of Sect. 5.3; then Λ is the Tate module of a p -divisible group over $\mathcal{O}_F$. Fix $\pi = \pi_F$ of F and a compatible system of roots π^{1/p^n} , for $n \geq 1$, giving $\pi^\flat = (\pi, \pi^{1/p}, \dots) \in \mathcal{O}^\flat$ as above. Recall the homomorphism

$$\theta_\infty : A_{\text{inf}}(\mathcal{O}) = W(\mathcal{O}^\flat) \rightarrow W(\mathcal{O}), \quad \theta_\infty([(x^{(0)}, x^{(1)}, \dots)]) = [x^{(0)}].$$

The diagram

$$\begin{array}{ccc} \mathfrak{S} & \xrightarrow{f} & A_{\text{inf}}(\mathcal{O}) \\ \varphi \downarrow & & \downarrow \theta_{\infty} \\ \hat{W}(\mathcal{O}_F) & \longrightarrow & W(\mathcal{O}), \end{array}$$

where the bottom horizontal map is given by the inclusion, commutes. We then have isomorphisms of $\mathcal{G}$ -torsors

$$\mathcal{P} \otimes_{\hat{W}(\mathcal{O}_F)} W(\mathcal{O}) \simeq \mathcal{P}_{\text{BK}} \otimes_{\mathfrak{S}, \varphi} W(\mathcal{O}) \simeq \mathcal{D}_{\text{inf}} \otimes_{A_{\text{inf}}(\mathcal{O})} W(\mathcal{O}) \quad (5.4.7)$$

which are compatible with the Frobenius structures: This can be seen by combining results of [4, § 4], [35, § 17], and the above constructions. Similarly, we can see that both the $(\mathcal{G}, \mathcal{M})$ -display $\mathcal{D}$ and the Breuil–Kisin–Fargues $\mathcal{G}$ -module $(\mathcal{D}_{\text{inf}}, \varphi_{\mathcal{D}_{\text{inf}}})$ are, up to a canonical isomorphism, independent of the choice of π_F and its roots $\sqrt[p^n]{\pi_F}$ in $\bar{F} \subset C$.

Remark 5.4.8 The various compatibilities after (often confusing) Frobenius twists between these different objects, all attached to the same integral crystalline representation, can be explained via the theory of prisms and prismatic cohomology of Bhatt and Scholze [2]. Indeed, the BK and BKF $\mathcal{G}$ -modules should be “facets” of a single object, a prismatic Frobenius crystal with $\mathcal{G}$ -structure over $\text{Spf}(\mathcal{O}_F)$. On the other hand, the $\mathcal{G}$ -display fits somewhat less directly into this and seems to be tied more closely to p -divisible groups by using the Hodge embedding.

6 Associated systems

Here, we define the notion of an associated system and give several results. The main result says, roughly, that a pro-étale $\mathcal{G}(\mathbb{Z}_p)$ -Galois cover which is given by the Tate module of a p -divisible group over a normal base with appropriate étale tensors, can be extended uniquely to an associated system (see Theorem 6.4.1 for the precise statement). We also show how to use the existence of *locally universal* associated systems to compare formal completions of normal schemes with the same generic fiber (Proposition 6.3.1). Finally, we show that the definition of associated is independent of the choice of the local Hodge embedding (Proposition 6.5.1).

In all of Sect. 6 we assume, without further mention, that $(\mathcal{G}, \mathcal{M})$ is of *strongly* integral local Hodge type. All the Hodge embeddings $\iota : \mathcal{G} \hookrightarrow \text{GL}(\Lambda)$ we consider are strongly integral: they induce a closed immersion $\iota_* : \mathcal{M} \hookrightarrow \text{Gr}(d, \Lambda)_{\mathcal{O}_E}$.

6.1 Local systems and associated systems

Let us suppose that $\mathcal{X}$ is a flat $\mathcal{O}_E$ -scheme of finite type, which is normal and has smooth generic fiber. Suppose that we are given a Galois cover of $X = \mathcal{X}[1/p]$ with group $\mathcal{G}(\mathbb{Z}_p) = \varprojlim_n \mathcal{G}(\mathbb{Z}/p^n\mathbb{Z})$; this gives a pro-étale $\mathcal{G}(\mathbb{Z}_p)$ -cover $\mathcal{L}$ on X .

6.1.1. For any $\bar{x} \in \mathcal{X}(k)$, let $\hat{R}_{\bar{x}}$ be the completion of the strict Henselization of the local ring $R_{\bar{x}}$ of $\mathcal{X}$ at $\bar{x}$.

Suppose we have a Dieudonné $(\mathcal{G}, \mathcal{M})$ -display $\mathcal{D}_{\bar{x}} = (\mathcal{P}_{\bar{x}}, q_{\bar{x}}, \Psi_{\bar{x}})$ over $\hat{R}_{\bar{x}}$. Choose a (strongly integral) local Hodge embedding ι .

In accordance with our notations in Sects. 3.2 and 4.4, we will denote by

$$\mathcal{D}_{\bar{x}}(\iota) = (M_{\bar{x}}, M_{1,\bar{x}}, F_{1,\bar{x}})$$

the Dieudonné display over $\hat{R}_{\bar{x}}$ induced from $\mathcal{D}_{\bar{x}}$ using ι and the construction of [20, 3.1.5]. As usual, Proposition 3.2.3 gives tensors $m_{a,\bar{x}} \in M_{\bar{x}}^{\otimes}$ corresponding to $s_a \in \Lambda^{\otimes}$.

By [39], there is a corresponding p -divisible group $\mathcal{G}(\bar{x})$ over $\hat{R}_{\bar{x}}$

$$\mathcal{G}(\bar{x}) = \text{BT}(\mathcal{D}_{\bar{x}}(\iota)),$$

of height $n = \text{rank}_{\mathbb{Z}_p}(\Lambda)$. Recall (Sect. 4.4), we have a canonical isomorphism

$$\mathcal{D}_{\bar{x}}(\iota) \cong \mathbb{D}(\mathcal{G}(\bar{x}))(\hat{W}(\hat{R}_{\bar{x}})) \quad (6.1.12)$$

of Dieudonné displays, where on the right hand side, $\mathbb{D}$ denotes the evaluation of the *covariant* Dieudonné crystal.

Consider the following two conditions. The first is:

A1) There is an isomorphism $\alpha = \alpha_{\bar{x}} \in \mathbb{Z}_p$ -local systems over $\hat{R}_{\bar{x}}[1/p]$ between the local system given by the Tate module T of the p -divisible group $\mathcal{G}(\bar{x}) = \text{BT}(\mathcal{D}_{\bar{x}}(\iota))$ and the pull-back of $\mathcal{L}(\iota)$.

Before we state the second condition, we observe the following. Assuming (A1), for any $\tilde{x} \in \mathcal{X}(\mathcal{O}_F)$ that lifts $\bar{x}$, the Galois representation $\rho(x)$ obtained from $x^* \mathcal{L}$, is crystalline. By [20, Theorem 3.3.2 (2)], the isomorphism α in (A1) induces an isomorphism

$$\mathcal{D}_{\rho(x)}(\iota) \cong \mathbb{D}(\tilde{x}^* \mathcal{G}(\bar{x}))(\hat{W}(\mathcal{O}_F)). \quad (6.1.13)$$

Here, $\tilde{x}^* \mathcal{G}(\bar{x})$ is the p -divisible group over $\mathcal{O}_F$ obtained by base-changing $\mathcal{G}(\bar{x})$ by $\tilde{x}$ and $\mathcal{D}_{\rho(x)}$ is the Dieudonné $(\mathcal{G}, \mathcal{M})$ -display attached to $\rho(x)$ by Sect. 5.3. Combining (6.1.12), (6.1.13), and base change gives an isomorphism

$$\mathcal{D}_{\rho(x)}(\iota) \xrightarrow{\sim} \mathcal{D}_{\bar{x}}(\iota) \otimes_{\hat{W}(\hat{R}_{\bar{x}})} \hat{W}(\mathcal{O}_F) \quad (6.1.14)$$

of Dieudonné displays over $\mathcal{O}_F$.

We can now state the second condition (it only makes sense after we assume (A1)):

A2) For every $\tilde{x} \in \mathcal{X}(\mathcal{O}_F)$ lifting $\bar{x}$, there is an isomorphism of Dieudonné $(\mathcal{G}, \mathcal{M})$ -displays above $\mathcal{D}_{\rho(x)} \xrightarrow{\sim} \mathcal{D}_{\bar{x}} \otimes_{\hat{W}(\hat{R}_{\bar{x}})} \hat{W}(\mathcal{O}_F)$ over $\mathcal{O}_F$ which, after applying ι , induces the isomorphism (6.1.14)

$$\mathcal{D}_{\rho(x)}(\iota) \xrightarrow{\sim} \mathcal{D}_{\bar{x}}(\iota) \otimes_{\hat{W}(\hat{R}_{\bar{x}})} \hat{W}(\mathcal{O}_F).$$

More concretely, we see that condition (A2) is equivalent to the following:

A2') For every $\tilde{x} \in \mathcal{X}(\mathcal{O}_F)$ lifting $\bar{x}$, and every a , the isomorphism (6.1.14) maps the tensor $s_{a,D} = s_{a,D,\tilde{x}} \in M^\otimes$ attached to $\rho(x)$ in Sect. 5.3 to the base change $m_{a,\tilde{x}} \otimes 1 \in M_{\tilde{x}}^\otimes \otimes_{\hat{W}(\hat{R}_{\tilde{x}})} \hat{W}(\mathcal{O}_F)$ of the tensor $m_{a,\tilde{x}} \in M_{\tilde{x}}^\otimes$.

Definition 6.1.5 If (A1) and (A2) hold for $\bar{x} \in \mathcal{X}(k)$, we say that $\mathcal{L}$ and $\mathcal{D}_{\bar{x}}$ are *associated*. If (A1) and (A2) hold for all $\bar{x} \in \mathcal{X}(k)$, with $\alpha_{\bar{x}}$ the isomorphism in (A1), we call $(\mathcal{L}, \{\alpha_{\bar{x}}, \mathcal{D}_{\bar{x}}\}_{\bar{x} \in \mathcal{X}(k)})$ an *associated system*.

Definition 6.1.6 The associated system $(\mathcal{L}, \{\alpha_{\bar{x}}, \mathcal{D}_{\bar{x}}\}_{\bar{x} \in \mathcal{X}(k)})$ is *locally universal* over $\mathcal{X}$, if for every $\bar{x} \in \mathcal{X}(k)$, $\mathcal{D}_{\bar{x}}$ is locally universal in the sense of Definition 4.5.10.

The definition of “associated” uses the local Hodge embedding ι which we, for now, fix in our discussion. We will later show that it is independent of this choice, see Proposition 6.5.1. Most of the time, we will omit the notation of the isomorphisms $\alpha_{\bar{x}}$ and write $(\mathcal{L}, \{\mathcal{D}_{\bar{x}}\}_{\bar{x} \in \mathcal{X}(k)})$ for the associated system.

Proposition 6.1.7 If $\mathcal{L}$ and $\mathcal{D}_{\bar{x}}$, for $\bar{x} \in \mathcal{X}(k)$, are associated, then $\mathcal{D}_{\bar{x}}$ is, up to isomorphism, uniquely determined by $\mathcal{L}$.

Proof Suppose that $\mathcal{L}$ and $\mathcal{D}'_{\bar{x}}$ are also associated. Then $\mathcal{G}(\bar{x})[1/p] \simeq \mathcal{G}'(\bar{x})[1/p]$ as p -divisible groups over $\hat{R}_{\bar{x}}[1/p]$, since they both have the same Tate module which is given by the restriction of $\mathcal{L}(\iota)$ to $\hat{R}_{\bar{x}}[1/p]$. Tate’s theorem applied to the normal Noetherian domain $\hat{R}_{\bar{x}}$, extends this to a unique isomorphism $\beta : \mathcal{G}(\bar{x}) \xrightarrow{\sim} \mathcal{G}'(\bar{x})$. Therefore, using [39], we obtain an isomorphism of Dieudonné displays $\delta : \mathcal{D}_{\bar{x}}(\iota) \xrightarrow{\sim} \mathcal{D}'_{\bar{x}}(\iota)$. This amounts to an isomorphism

$$(M, M_1, F_1) \xrightarrow{\delta} (M', M'_1, F'_1).$$

Here both M, M' are free $\hat{W}(\hat{R}_{\bar{x}})$ -modules of rank n . The $(\mathcal{G}, \mathcal{M})$ -displays $\mathcal{D}_{\bar{x}}$ and $\mathcal{D}'_{\bar{x}}$ have corresponding $\mathcal{G}$ -torsors $\mathcal{P}, \mathcal{P}'$. By the construction of Proposition 3.2.3, these $\mathcal{G}$ -torsors are given by M, M' and tensors $m_a \in M^\otimes, m'_a \in M'^\otimes$, respectively. We would like to show that $\delta : M \rightarrow M'$ satisfies $\delta^\otimes(m_a) = m'_a$, i.e. δ lies in the $\hat{W}(\hat{R}_{\bar{x}})$ -valued points of the closed subscheme

$$\underline{\mathrm{Hom}}_{(m_a), (m'_a)}(M, M') = \mathrm{Spec}(B/I) \hookrightarrow \underline{\mathrm{Hom}}(M, M') = \mathrm{Spec}(B).$$

Here, $B \simeq \hat{W}(\hat{R}_{\bar{x}})[(t_{ij})_{1 \leq i, j \leq n}]$, non-canonically. Let us consider $f(t_{ij}) \in I$. We would like to show that $f(\delta_{ij}) = 0$ in $\hat{W}(\hat{R}_{\bar{x}})$, where $\delta_{ij} \in \hat{W}(\hat{R}_{\bar{x}})$ are the coordinates of the $\hat{W}(\hat{R}_{\bar{x}})$ -linear map δ . Condition (A2) implies that $\tilde{x}^* \delta$ respects the tensors $\tilde{x}^* m_a, \tilde{x}^* m'_a$, so $(\tilde{x})^*(f(\delta_{ij})) = 0$, for all $\tilde{x}$ lifting $\bar{x}$. This implies that $f(\delta_{ij}) = 0$, so δ respects the tensors. It now follows that δ respects the rest of the data that give the $(\mathcal{G}, \mathcal{M})$ -displays $\mathcal{D}_{\bar{x}}$ and $\mathcal{D}'_{\bar{x}}$. $\square$

6.2 Local systems and associated displays

Let $\mathcal{D}$ be a $(\mathcal{G}, \mathcal{M})$ -display over the p -adic formal scheme $\mathfrak{X} = \varprojlim_n \mathcal{X} \otimes_{\mathcal{O}_E} \mathcal{O}_E/(p)^n$.

Definition 6.2.1 We say that the $(\mathcal{G}, \mathcal{M})$ -display $\mathcal{D}$ over $\mathfrak{X}$ is associated with $\mathcal{L}$ if, for all $\bar{x} \in \mathcal{X}(k)$, there is

- a Dieudonné $(\mathcal{G}, \mathcal{M})$ -display $\mathcal{D}_{\bar{x}}$ which is associated with $\mathcal{L}$,
- an isomorphism of $(\mathcal{G}, \mathcal{M})$ -displays

$$\mathcal{D}_{\bar{x}} \otimes_{\hat{W}(\hat{R}_{\bar{x}})} W(\hat{R}_{\bar{x}}) \simeq \mathcal{D} \otimes_{W(\mathcal{O}_{\mathfrak{X}})} W(\hat{R}_{\bar{x}}).$$

Note that, then, $(\mathcal{L}, \{\mathcal{D}_{\bar{x}}\}_{\bar{x} \in \mathcal{X}(k)})$ is an associated system.

Definition 6.2.2 We say that the $(\mathcal{G}, \mathcal{M}^{\text{loc}})$ -display $\mathcal{D}$ over $\mathfrak{X}$ which is associated with $\mathcal{L}$, is locally universal over $\mathfrak{X}$, if the associated system $(\mathcal{L}, \{\mathcal{D}_{\bar{x}}\}_{\bar{x} \in \mathcal{X}(k)})$ is *locally universal* over $\mathcal{X}$.

6.3 Rigidity and uniqueness

Assume now that $\mathcal{X}$ and $\mathcal{X}'$ are two flat $\mathcal{O}_E$ -schemes of finite type, normal with the *same* smooth generic fiber $X = \mathcal{X}[1/p] = \mathcal{X}'[1/p]$. Suppose that $(\mathcal{L}, \{\mathcal{D}_{\bar{x}}\}_{\bar{x} \in \mathcal{X}(k)})$ and $(\mathcal{L}', \{\mathcal{D}'_{\bar{x}'}\}_{\bar{x}' \in \mathcal{X}'(k)})$ are locally universal associated systems on $\mathcal{X}$ and $\mathcal{X}'$ respectively, with $\mathcal{L} = \mathcal{L}'$ on X .

Denote by $\mathcal{Y}$ the normalization of the Zariski closure of the diagonal embedding of X in the product $\mathcal{X} \times_{\text{Spec}(\mathcal{O}_E)} \mathcal{X}'$. Denote by

$$\mathcal{X} \xleftarrow{\pi} \mathcal{Y} \xrightarrow{\pi'} \mathcal{X}',$$

the morphisms given by the two projections. For simplicity, we again set $\check{\mathcal{Y}} = \mathcal{Y} \otimes_{\mathcal{O}_E} \mathcal{O}_{\check{E}}$, $\check{\mathcal{X}} = \mathcal{X} \otimes_{\mathcal{O}_E} \mathcal{O}_{\check{E}}$, etc. For $\bar{y} \in \mathcal{Y}(k)$, set $\bar{x} = \pi(\bar{y})$, $\bar{x}' = \pi'(\bar{y})$.

Proposition 6.3.1 (a) *We have*

$$\pi^* \mathcal{D}_{\bar{x}} \simeq \pi'^* \mathcal{D}'_{\bar{x}'}$$

as Dieudonné $(\mathcal{G}, \mathcal{M})$ -displays on the completion $\hat{\mathcal{O}}_{\check{\mathcal{Y}}, \bar{y}}$.

(b) *The morphism π induces an isomorphism*

$$\pi^* : \hat{\mathcal{O}}_{\check{\mathcal{X}}, \bar{x}} \xrightarrow{\sim} \hat{\mathcal{O}}_{\check{\mathcal{Y}}, \bar{y}}$$

between the completions of $\check{\mathcal{Y}}$ and $\check{\mathcal{X}}$, at $\bar{y}$ and $\bar{x}$, respectively. Similarly, for π' .

Proof Part (a) follows by the argument in the proof of Proposition 6.1.7.

Let us show (b). For simplicity, set $R = \hat{\mathcal{O}}_{\check{\mathcal{X}}, \bar{x}}$, $R' = \hat{\mathcal{O}}_{\check{\mathcal{X}'}, \bar{x}'}$, $R'' = \hat{\mathcal{O}}_{\check{\mathcal{Y}}, \bar{y}}$. By the construction of $\mathcal{Y}$, we have a local homomorphism

$$R \hat{\otimes}_{\mathcal{O}_{\check{E}}} R' \rightarrow R''$$

which is finite. Write R_1 for its image:

$$R \hat{\otimes}_{\mathcal{O}_{\tilde{E}}} R' \rightarrow R_1 \hookrightarrow R''$$

Applying ι and the functor BT to $\mathcal{D}_{\tilde{x}}$, $\mathcal{D}'_{\tilde{x}}$, gives p -divisible groups $\mathcal{G}$, $\mathcal{G}'$ over R , R' respectively. By (a) we have

$$\pi^* \mathcal{G} \simeq \pi'^* \mathcal{G}' \quad (6.1.1)$$

over R'' . This isomorphism specializes to give $f_0 : \bar{x}^*(\mathcal{G}) \xrightarrow{\sim} \bar{x}'^*(\mathcal{G}')$, an isomorphism of p -divisible groups over the field k .

Let us write $T = \mathrm{Spf}(U)$ for the base change to $\mathcal{O}_{\tilde{E}}$ of the universal deformation space of a p -divisible group $\mathcal{G}_0$ over k which is isomorphic to the p -divisible groups $\bar{x}^*(\mathcal{G})$ and $\bar{x}'^*(\mathcal{G}')$ above, and fix such isomorphisms. This allows us to view f_0 as an isomorphism $f_0 : \mathcal{G}_0 \xrightarrow{\sim} \mathcal{G}_0$.

Set $\mathrm{Spf}(R) = S$, $\mathrm{Spf}(R') = S'$, $\mathrm{Spf}(R'') = S''$, and $Z = \mathrm{Spf}(R_1)$. By the locally universality condition on $\mathcal{D}_{\tilde{x}}$ and $\mathcal{D}'_{\tilde{x}}$, S and S' can both be identified with closed formal subschemes of T given by ideals I and I' of U , respectively. There is a closed formal subscheme Γ of $T \hat{\times}_{\mathcal{O}_{\tilde{E}}} T = \mathrm{Spf}(U \hat{\otimes}_{\mathcal{O}_{\tilde{E}}} U)$ prorepresenting the subfunctor of pairs of deformations of $\mathcal{G}_0$ where f_0 extends as an isomorphism. The subscheme Γ is defined by the ideal generated by $(u \otimes 1 - 1 \otimes f_0^*(u))$, $u \in U$, where $f_0^* : U \xrightarrow{\sim} U$ is the “relabelling” automorphism corresponding to f_0 . By (6.1.1), we have that $Z \subset T \hat{\times}_{\mathcal{O}_{\tilde{E}}} T$ is contained (scheme theoretically) in the “intersection”

$$\Gamma \cap (S \hat{\times}_{\mathcal{O}_{\tilde{E}}} S') = \mathrm{Spf}(U \hat{\otimes}_{\mathcal{O}_{\tilde{E}}} U / ((u \otimes 1 - 1 \otimes f_0^*(u))_{u \in U}, I \otimes U, U \otimes I').$$

The projection makes this isomorphic to $S \cap f_0^{*-1}(S')$, the formal spectrum of R/J , where we set $J := f_0^*(I')R$. From

$$\mathrm{Spf}(R_1) = Z \subset \Gamma \cap (S \hat{\times}_{\mathcal{O}_{\tilde{E}}} S') \simeq \mathrm{Spf}(R/J)$$

we have $\dim(R_1) \leq \dim(R/J)$. Since R_1 is integral of dimension equal to that of R'' and so of R , we have $\dim(R) = \dim(R_1) \leq \dim(R/J) \leq \dim(R)$. Since R is an integral domain, this implies $J = (0)$ and that R_1 , which is a quotient of R/J of the same dimension, is also isomorphic to R . Since $R \simeq R_1 \rightarrow R''$ is finite, and R , R' and R'' are normal, the birational $R_1 \rightarrow R''$ is an isomorphism; so is $R \rightarrow R''$ and, by symmetry, also $R' \rightarrow R''$. $\square$

6.4 Existence of associated systems

Suppose that $\mathcal{X}$, $\mathcal{L}$, and $\iota : \mathcal{G} \hookrightarrow \mathrm{GL}(\Lambda)$, are as in the beginning of Sect. 6.

Theorem 6.4.1 *Suppose that the étale local system $\mathcal{L}(\iota)$ is given by the Tate module of a p -divisible group $\mathcal{G}$ over $\mathcal{X}$. Then $\mathcal{L}$ is part of a unique, up to unique isomorphism, associated system $(\mathcal{L}, \{\alpha_{\tilde{x}}, \mathcal{D}_{\tilde{x}}\}_{\tilde{x} \in \mathcal{X}(k)})$ for ι .*

Proof The uniqueness part of the statement follows from Proposition 6.1.7. Our task is to construct, for each $\bar{x} \in \mathcal{X}(k)$, a $(\mathcal{G}, \mathcal{M})$ -display $\mathcal{D}_{\bar{x}}$ over $R = \hat{R}_{\bar{x}}$ that satisfies (A1) and (A2). Let

$$(M, M_1, F_1)$$

be the Dieudonné display obtained by the evaluation $M = \mathbb{D}(\mathcal{G})(\hat{W}(R))$ of the Dieudonné crystal of $\mathcal{G}$ over R . This gives a $(\mathrm{GL}_n, \mathrm{Gr}(d, \Lambda))$ -display $(\mathcal{P}_{\mathrm{GL}}, q_{\mathrm{GL}}, \Psi_{\mathrm{GL}})$. We want to upgrade this to a $(\mathcal{G}, \mathcal{M})$ -display, the main difficulty being the construction of appropriate tensors $m_a \in M^{\otimes}$. The construction occupies several paragraphs:

6.4.2. We recall the notations and results of Sects. 2.4 and 2.5, for R . In particular, we fix an algebraic closure $\bar{F}(R)$ of the fraction field $F(R)$, we denote by $\bar{R}$ the integral closure of R in $\bar{F}(R)$ and by $\tilde{R}$ the union of all finite normal R -algebras R' in $\bar{F}(R)$ such that $R'[1/p]$ is étale over $R[1/p]$. Set $\bar{R}^{\wedge}$ and $\tilde{R}^{\wedge}$ for their p -adic completions. For simplicity, we set

$$S = \tilde{R}^{\wedge}.$$

Also, we set $\mathcal{O}$ for the p -adic completion of the integral closure $\bar{\mathcal{O}}_E$ of $\mathcal{O}_E$ in the algebraic closure $\bar{E}$.

By Sect. 2.4, $\bar{R}^{\wedge}$, $S = \tilde{R}^{\wedge}$, and $\mathcal{O}$, are integral perfectoid $\mathbb{Z}_p$ -algebras in the sense of [4, 3.1], which are local Henselian and flat over $\mathbb{Z}_p$. The Galois group Γ_R acts on $\tilde{R}$, on S , and on $A_{\mathrm{inf}}(S) = W(S^{\flat})$.

6.4.3. Let $(M(\mathcal{G}) = M(\mathcal{G})(S), \varphi_{M(\mathcal{G})})$ be the (finite free) Breuil–Kisin–Fargues module over $A_{\mathrm{inf}}(S)$ attached to the base change $\mathcal{G}_S$ of $\mathcal{G}$.

By [35, Theorem 17.5.2], $M(\mathcal{G})(S)$ is the value of a functor which gives an equivalence between p -divisible groups over S and finite projective BKF modules (M, φ_M) over $A_{\mathrm{inf}}(S)$ that satisfy

$$M \subset \varphi_M(M) \subset \frac{1}{\varphi(\xi)} M.$$

By loc. cit., the equivalence is functorial in S . Therefore, $M(\mathcal{G})(S)$ supports an action of Γ_R which commutes with $\varphi_{M(\mathcal{G})(S)}$ and is semi-linear with respect to the action of Γ_R on $A_{\mathrm{inf}}(S)$. By loc. cit., we have

$$T = \mathrm{Hom}_S(\underline{\mathbb{Q}_p/\mathbb{Z}_p}, \mathcal{G}_S) \xrightarrow{\sim} \mathrm{Hom}_{A_{\mathrm{inf}}(S), \varphi}(A_{\mathrm{inf}}(S), M(\mathcal{G})) = M(\mathcal{G})^{\varphi_{M(\mathcal{G})}=1}. \quad (6.4.4)$$

This gives the comparison homomorphism

$$c : T \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}(S) \rightarrow M(\mathcal{G})(S)$$

which is φ and Galois equivariant.

Using the constructions in [35, § 17] together with Lemma 2.5.5 and Proposition 2.5.7, we see that c is injective and gives

$$T \otimes_{\mathbb{Z}_p} A_{\inf}(S) \subset M(\mathcal{G})(S) \subset T \otimes_{\mathbb{Z}_p} \frac{1}{\mu} A_{\inf}(S).$$

Therefore, we obtain a “comparison” isomorphism

$$T^{\otimes} \otimes_{\mathbb{Z}_p} A_{\inf}(S)[1/\mu] \xrightarrow{\sim} M(\mathcal{G})(S)^{\otimes}[1/\mu]. \quad (6.4.5)$$

6.4.6. Let us set:

$$M_{\inf}(S) := M(\mathcal{G})(S).$$

Let

$$s_{a,\inf} \in M_{\inf}(S)^{\otimes}[1/\mu]$$

be the tensors which correspond to $s_a \in T^{\otimes}$ under (6.4.5). We have

$$\varphi_{M_{\inf}}(s_{a,\inf}) = s_{a,\inf}.$$

We can now construct a Breuil–Kisin–Fargues $\mathcal{G}$ -module $(D_{\inf}(S), \varphi_{D_{\inf}})$ over $A_{\inf}(S)$.

Proposition 6.4.7 (a) We have $s_{a,\inf} \in M_{\inf}(S)^{\otimes}$.

(b) By (a), we can consider the $\mathcal{G}$ -scheme

$$D_{\inf}(S) := \underline{\text{Isom}}_{(s_{a,\inf}), (s_a \otimes 1)}(M_{\inf}(S), \Lambda \otimes_{\mathbb{Z}_p} A_{\inf}(S)).$$

The scheme $D_{\inf}(S)$, with its natural $\mathcal{G}$ -action, is a $\mathcal{G}$ -torsor over $A_{\inf}(S)$.

(c) There is a $\mathcal{G}$ -equivariant isomorphism

$$\varphi_{D_{\inf}} : (\varphi^* D_{\inf}(S))[1/\varphi(\xi)] \xrightarrow{\sim} D_{\inf}(S)[1/\varphi(\xi)],$$

where ξ is any generator of the kernel of $\theta : A_{\inf}(S) \rightarrow S$.

(d) Suppose $\tilde{x} : S \rightarrow \mathcal{O}$ extends a point $\bar{x} : R \rightarrow \mathcal{O}_F$ which lifts $\bar{x}$. Then the base change of $(D_{\inf}(S), \varphi_{D_{\inf}})$ by $\tilde{x} : S \rightarrow \mathcal{O}$ is isomorphic to the BKF $\mathcal{G}$ -module over $A_{\inf}(\mathcal{O})$ which is attached to $\tilde{x}^* \mathcal{L}$ by 5.4.

Proof Consider $\tilde{x} : S \rightarrow \mathcal{O}$ as in (d). Let $(M(\mathcal{G})(\mathcal{O}), \varphi_{M(\mathcal{G})(\mathcal{O})})$ be the BKF module over $A_{\inf}(\mathcal{O})$ attached to the p -divisible group $\tilde{x}^* \mathcal{G}$ over $\mathcal{O}$. By functoriality under $S \rightarrow \mathcal{O}$ of the functor of [35, Theorem 17.5.2], we have a canonical isomorphism

$$M_{\inf}(S) \otimes_{A_{\inf}(S)} A_{\inf}(\mathcal{O}) \cong M_{\inf}(\mathcal{O}) = M(\mathcal{G})(\mathcal{O})$$

respecting the Frobenius structures.

Lemma 6.4.8 *The pull-back*

$$\tilde{x}^*(s_{a,\text{inf}}) \in M_{\text{inf}}(S)^{\otimes}[1/\mu] \otimes_{A_{\text{inf}}(S)} A_{\text{inf}}(\mathcal{O}) = M_{\text{inf}}(\mathcal{O})^{\otimes}[1/\mu]$$

lies in $M_{\text{inf}}(\mathcal{O})^{\otimes}$.

Proof Recall that the tensors $s_{a,\text{inf}} \in M_{\text{inf}}(S)^{\otimes}[1/\mu]$ are defined using the comparison isomorphisms (6.4.5). The statement then follows from functoriality under $S \rightarrow \mathcal{O}$ using the fact that $(M(\mathcal{G})(\mathcal{O}), \varphi_{M(\mathcal{G})(\mathcal{O})})$ supports a $\mathcal{G}$ -BKF module structure (so, in particular, the corresponding étale tensors in $M_{\text{inf}}(\mathcal{O})^{\otimes}[1/\mu]$ extend over $A_{\text{inf}}(\mathcal{O})$, i.e. have no μ -denominators.) $\square$

Now we can proceed with the proof of the proposition. Part (a) follows from the above Lemma and Lemma 2.5.7. By Lemma 2.5.5 and Proposition 3.2.5, $D_{\text{inf}}(S)$ is a $\mathcal{G}$ -torsor, i.e. part (b) holds. The identity $\varphi_{M_{\text{inf}}}(s_{a,\text{inf}}) = s_{a,\text{inf}}$ holds in $M_{\text{inf}}(S)^{\otimes}[1/\mu]$ and so also in $M_{\text{inf}}(S)^{\otimes}$ since

$$M_{\text{inf}}(S) \subset M_{\text{inf}}(S)[1/\mu]$$

by Lemma 2.5.7. Therefore,

$$\varphi_{D_{\text{inf}}} : (\varphi^* D_{\text{inf}}(S))[1/\varphi(\xi)] \xrightarrow{\sim} D_{\text{inf}}(S)[1/\varphi(\xi)].$$

is $\mathcal{G}$ -equivariant, which is (c). Finally, (d) follows from the above and functoriality under $S \rightarrow \mathcal{O}$. $\square$

Remark 6.4.9 (a) Using these constructions and the comparison

$$T \otimes_{\mathbb{Z}_p} A_{\text{inf}}(S)[1/\mu] \simeq M(\mathcal{G})(S)[1/\mu],$$

we can see that the BKF $\mathcal{G}$ -module $(D_{\text{inf}}(S), \varphi_{D_{\text{inf}}})$ only depends, up to isomorphism, on $\mathcal{L}$. Indeed, from Sect. 5.4, this statement is true when $S = \mathcal{O}$. In general, the comparison isomorphism first implies that the $\mathcal{G}$ -torsor $D_{\text{inf}}(S)[1/\mu]$ depends, up to isomorphism, only on $\mathcal{L}$. Then, by considering restriction along $\tilde{x} : S \rightarrow \mathcal{O}$ and using Lemma 2.5.7 (b), we see that we can determine $D_{\text{inf}}(S)$ and $\varphi_{D_{\text{inf}}}$ over $A_{\text{inf}}(S)$.

(b) As usual, we may think of $D_{\text{inf}}(S)$ as an exact tensor functor

$$\text{Rep}_{\mathbb{Z}_p}(\mathcal{G}) \rightarrow \text{Mod}_{/A_{\text{inf}}(S)}^{\varphi}.$$

6.4.10. We can now complete the proof of Theorem 6.4.1. Recall that

$$\theta_{\infty} : A_{\text{inf}}(S) \rightarrow W(S)$$

factors as a composition

$$\theta_{\infty} : A_{\text{inf}}(S) \rightarrow A_{\text{cris}}(S) \rightarrow W(S).$$

By, [35, Theorem 17.5.2], the Frobenius module

$$M_{\inf}(S) \otimes_{A_{\inf}(S)} A_{\text{cris}}(S) = M(\mathcal{G})(S) \otimes_{A_{\inf}(S)} A_{\text{cris}}(S)$$

describes the covariant Dieudonné module of the base change $\mathcal{G}_S$ evaluated at the divided power thickening $A_{\text{cris}}(S) \rightarrow S$. By [21, Theorem B], this evaluation of the Dieudonné module is naturally isomorphic to $M \otimes_{\hat{W}(R)} W(S)$, with its Frobenius structure. Combining these now gives a natural isomorphism

$$M \otimes_{\hat{W}(R)} W(S) \simeq M_{\inf}(S) \otimes_{A_{\inf}(S)} W(S) \quad (6.4.11)$$

which is compatible with Frobenius and the action of Γ_R . We obtain

$$M^{\otimes} \otimes_{\hat{W}(R)} W(S) \simeq M_{\inf}(S)^{\otimes} \otimes_{A_{\inf}(S)} W(S).$$

Since the tensors $s_{a,\inf} \otimes 1 \in M_{\inf}(S)^{\otimes} \otimes_{A_{\inf}(S)} W(S)$ are Γ_R -invariant, we see that

$$s_{a,\inf} \otimes 1 \in (M^{\otimes} \otimes_{\hat{W}(R)} W(S))^{\Gamma_R} = M^{\otimes} \otimes_{\hat{W}(R)} (W(S))^{\Gamma_R}.$$

By Theorem 2.4.5, $(W(S))^{\Gamma_R} = W(R)$. Therefore,

$$s_{a,\inf} \otimes 1 \in M^{\otimes} \otimes_{\hat{W}(R)} W(R). \quad (6.4.12)$$

In fact, we also have:

Proposition 6.4.13 (a) *The tensors $m_a := s_{a,\inf} \otimes 1$ lie in $M^{\otimes}$.*

(b) *The identity*

$$\tilde{x}^*(m_a) = s_{a,D,\tilde{x}} \quad (6.4.14)$$

holds in $M^{\otimes} \otimes_{\hat{W}(R)} \hat{W}(\mathcal{O}_F) \cong M_{\mathcal{O}_F}^{\otimes}$.

In the above, $(M_{\mathcal{O}_F}, M_{1,\mathcal{O}_F}, F_{1,\mathcal{O}_F})$ is the Dieudonné display over $\mathcal{O}_F$ associated by Sect. 5.3 to the Galois representation on Λ given by $\tilde{x}^*\mathcal{L}(\iota)$, and $s_{a,D,\tilde{x}}$ are the corresponding tensors.

Proof Using the compatibility of the construction with pull-back along points $\tilde{x} : R \rightarrow \mathcal{O}_F$ we first see that the identity (6.4.14) holds in the tensor product $M^{\otimes} \otimes_{\hat{W}(R)} W(\mathcal{O}_F)$. However, the right hand side $s_{a,D,\tilde{x}}$ lies in the subset $M^{\otimes} \otimes_{\hat{W}(R)} \hat{W}(\mathcal{O}_F)$, and, hence, so is the left hand side $\tilde{x}^*(m_a)$. Proposition 2.3.4 now implies (a), and (b) also follows. $\square$

We continue with the proof of Theorem 6.4.1. The tensors $m_a \in M^{\otimes}$ allow us to define

$$\mathcal{P} := \underline{\text{Isom}}_{m_a, s_a \otimes 1}(M, \Lambda \otimes_{\mathbb{Z}_p} \hat{W}(R)).$$

By the above, $\xi^*\mathcal{P}$ is isomorphic to the $\mathcal{G}$ -torsor given in Sect. 5.3. By Corollary 3.2.6 for $A = \hat{W}(R)$, $\mathcal{P}$ is a $\mathcal{G}$ -torsor over $\hat{W}(R)$. By definition, we have $\mathcal{P}(\iota) = M$. It remains to construct q and Ψ .

The filtration $I_RM \subset M_1 \subset M$ gives a filtration of M/I_RM :

$$(0) \subset \mathrm{Fil}^1 := M_1/I_RM \subset \mathrm{Fil}^0 := M/I_RM.$$

This induces a filtration $\mathrm{Fil}^{\otimes,0}$ of $(M/I_RM)^{\otimes}$ and we have

$$m_a \otimes 1 \in \mathrm{Fil}^{\otimes,0} \subset (M/I_RM)^{\otimes}[1/p]$$

since this is true at all F -valued points. Hence, $q : \mathcal{P}_{\mathrm{GL}} \otimes_{\hat{W}(R)} R \rightarrow \mathrm{Gr}(d, \Lambda)$ restricted to $\mathcal{P} \otimes_{\hat{W}(R)} R \subset \mathcal{P}_{\mathrm{GL}} \otimes_{\hat{W}(R)} R$ lands in $X_\mu(G)$ on the generic fiber. Since $\mathcal{M}$ is the Zariski closure of $X_\mu(G)$ in $\mathrm{Gr}(d, \Lambda)_{\mathcal{O}_E}$, we obtain

$$q : \mathcal{P} \otimes_{\hat{W}(R)} R \rightarrow \mathcal{M}.$$

Recall that we use q to define the $\mathcal{G}$ -torsor $\mathcal{Q}$. From the construction, we have a $\mathcal{G}$ -equivariant closed immersion $\mathcal{Q} \subset \mathcal{Q}_{\mathrm{GL}}$. Finally, let us give Ψ : We consider $\Psi_{\mathrm{GL}} : \mathcal{Q}_{\mathrm{GL}} \xrightarrow{\sim} \mathcal{P}_{\mathrm{GL}}$. We will check that this restricts to $\Psi : \mathcal{Q} \xrightarrow{\sim} \mathcal{P}$: For this, is enough to show that the map $\Psi = \Psi_{\mathrm{GL}}(\iota) : \tilde{M}_1 \xrightarrow{\sim} M$ given by Ψ_{GL} preserves the tensors, i.e.

$$\Psi(\varphi^*(m_a)) = m_a. \quad (6.4.15)$$

This follows as in the proof of Proposition 6.1.7 by observing that the tensors are preserved after pulling back by all $\tilde{x} : R \rightarrow \mathcal{O}_F$: Indeed, we have $\Psi_{\mathcal{O}_F}(\varphi^*s_{a,D,\tilde{x}}) = s_{a,D,\tilde{x}}$. Since, by (6.4.14) we also have $\tilde{x}^*m_a = s_{a,D,\tilde{x}}$, we conclude that $\tilde{x}^*\Psi = \Psi_{\mathcal{O}_F}$ maps $\varphi^*(\tilde{x}^*m_a)$ to $\tilde{x}^*m_a$. The identity (6.4.15) now follows.

The above define the $(\mathcal{G}, \mathcal{M})$ -display $\mathcal{D}_{\tilde{x}} = (\mathcal{P}, q, \Psi) = (\mathcal{P}_{\tilde{x}}, q_{\tilde{x}}, \Psi_{\tilde{x}})$. By its construction, $\mathcal{D}_{\tilde{x}}$ satisfies (A1) and (A2). This completes the proof of Theorem 6.4.1. $\square$

6.4.16. In fact, the proof of the Theorem 6.4.1 also gives:

Proposition 6.4.17 *There is an isomorphism of $\mathcal{G}$ -torsors*

$$\mathcal{P}_{\tilde{x}} \otimes_{\hat{W}(R)} W(S) \simeq D_{\mathrm{inf}}(S) \otimes_{A_{\mathrm{inf}}(S)} W(S), \quad (6.4.18)$$

which is also compatible with the Frobenius structures. $\square$

6.5 Independence

We show that the notion of “associated” does not depend on the choice of the (strongly integral) local Hodge embedding ι . More precisely:

Proposition 6.5.1 *If $\mathcal{L}$ and $\mathcal{D}_{\tilde{x}}$ are associated for ι , i.e. satisfy (A1) and (A2) for ι , they also satisfy (A1) and (A2) for any other (strongly integral) local Hodge embedding ι' .*

Proof Assume that $\mathcal{L}$ and $\mathcal{D}_{\bar{x}}$ are associated for ι . By the uniqueness part of Theorem 6.4.1, we can assume that $\mathcal{D}_{\bar{x}}$ is obtained from $\mathcal{L}$ and ι by the construction in its proof. We will use the notations of Sects. 6.1 and 6.4: In particular, $R = \hat{R}_{\bar{x}}$ and $\mathcal{G} = \mathcal{G}(\bar{x})$ is the p -divisible group over R given by $\mathcal{D}_{\bar{x}}(\iota)$. By [22], the Tate module $T := T_p(\mathcal{G})(\bar{R})$ can be identified with the kernel of

$$F_1 - 1 : \hat{M}_1 \rightarrow \hat{M} = M \otimes_{\hat{W}(R)} \hat{W}(\bar{R}).$$

Here, we denote abusively by $\hat{W}(\bar{R})$ the p -adic completion of

$$\varinjlim_{L/F(R)} \hat{W}(R_L)$$

where L runs over finite extensions of $F(R)$ in $\overline{F(\bar{R})}$, and R_L is the normalization of R in L . There is a natural surjective homomorphism $\hat{W}(\bar{R}) \rightarrow \bar{R}^\wedge$. In the above, we set

$$\hat{M}_1 = \ker(\hat{M} \rightarrow (M/M_1) \otimes_R \bar{R}^\wedge).$$

The isomorphism

$$\text{per} : T \xrightarrow{\sim} \ker(F_1 - 1 : \hat{M}_1 \rightarrow \hat{M})$$

induces the comparison homomorphism

$$T \otimes_{\mathbb{Z}_p} \hat{W}(\bar{R}) \rightarrow M \otimes_{\hat{W}(R)} \hat{W}(\bar{R}).$$

Let us consider a second local Hodge embedding $\iota' : \mathcal{G} \hookrightarrow \text{GL}(\Lambda')$ which realizes $\mathcal{G}$ as the stabilizer of tensors (s'_b) .

As before, we have a p -divisible group $\mathcal{G}' = \mathcal{G}'(\bar{x})$ over $R = \hat{R}_{\bar{x}}$ given by $\mathcal{D}_{\bar{x}}(\iota') = (M', M'_1, F'_1)$. Denote its Tate module by $T(\mathcal{G}')$. To show (A1) for ι' , we have to give an isomorphism of $T(\mathcal{G}')$ with $T' := \mathcal{L}(\iota') = \Lambda'$.

Recall (Sect. 6.4), that we have a φ -invariant isomorphism

$$\Lambda \otimes_{\mathbb{Z}_p} A_{\text{inf}}(S)[1/\mu] \simeq M_{\text{inf}}(S)[1/\mu]$$

that sends the tensors $s_a \otimes 1$ to $s_{a,\text{inf}}$. By a standard Tannakian argument we see that this gives an isomorphism

$$\Lambda' \otimes_{\mathbb{Z}_p} A_{\text{inf}}(S)[1/\mu] \simeq M'_{\text{inf}}(S)[1/\mu] \quad (6.5.2)$$

where $M'_{\text{inf}}(S) = D_{\text{inf}}(S)(\iota')$ is obtained from the $\mathcal{G}$ -torsor $D_{\text{inf}}(S)$.

Since ι' is also a local Hodge embedding, we can see using [35, Theorem 17.5.2], that $M'_{\text{inf}}(S)$ is the BKF module $M(\mathcal{H})$ of some p -divisible group $\mathcal{H}$ over S , so

$$M'_{\text{inf}}(S) \simeq M(\mathcal{H}).$$

Lemma 6.5.3 For all $\tilde{x} : S \rightarrow \mathcal{O}$ obtained from $\tilde{x} : \tilde{R} \rightarrow \tilde{\mathcal{O}}_E$, we have:

- (a) $\tilde{x}^* \mathcal{H} \simeq \tilde{x}^* \mathcal{G}'$,
- (b) $T' \simeq T_p(\tilde{x}^* \mathcal{G}')$.

Proof Consider the Breuil–Kisin $\mathcal{G}$ -module $(\mathcal{P}_{\text{BK}}, \varphi_{\mathcal{P}_{\text{BK}}})$ attached to $\tilde{x}^* \mathcal{L}$ by Sect. 5.2. Then, $(\mathcal{P}_{\text{BK}}(\iota'), \varphi_{\mathcal{P}_{\text{BK}}}(\iota'))$ gives a “classical” Breuil–Kisin module which corresponds to a p -divisible group $\mathcal{G}'_{\tilde{x}}$ over $\mathcal{O}_F$. The construction of the Breuil–Kisin $\mathcal{G}$ -module implies that $T' \simeq \tilde{x}^* \mathcal{L}(\iota')$ is identified with the Tate module $T_p(\mathcal{G}'_{\tilde{x}})$. By the compatibility (Sect. 5.4) of the constructions in Sects. 5.4 and 5.3, Proposition 6.4.13, and the fact [35, Theorem 17.5.2] that the functor $M(-)$ gives an equivalence of categories between p -divisible groups over $\mathcal{O}$ and (suitable) BKF modules over $A_{\text{inf}}(\mathcal{O})$, the base change of $\mathcal{G}'_{\tilde{x}}$ to $\mathcal{O}$ is isomorphic to both $\tilde{x}^* \mathcal{H}$ and $\tilde{x}^* \mathcal{G}'$. This gives (a). In fact, since the Tate module of $\mathcal{G}'_{\tilde{x}}$ is identified with T' , we then obtain (b). $\square$

From (6.5.2), we obtain an injection

$$c' : \Lambda' = T' \hookrightarrow M'_{\text{inf}}(S)[1/\mu].$$

Lemma 6.5.4 The map c' gives an isomorphism

$$c' : T' \xrightarrow{\sim} M'_{\text{inf}}(S)^{\varphi_{M'}=1} = M(\mathcal{H})^{\varphi_{M(\mathcal{H})}=1}$$

which identifies T' with the Tate module of $\mathcal{H}$.

Proof For all $\tilde{x} : S \rightarrow \mathcal{O}$ given by $\tilde{x} : \tilde{R} \rightarrow \tilde{\mathcal{O}}_E$, we consider the composition

$$T' \hookrightarrow M'_{\text{inf}}(S)[1/\mu] \rightarrow M'_{\text{inf}}(\mathcal{O})[1/\mu]$$

where the second map is given by pull-back along $A_{\text{inf}}(S) \rightarrow A_{\text{inf}}(\mathcal{O})$ and functoriality. From Lemma 6.5.3 and its proof, we see that this composition is identified with the comparison isomorphism for $\tilde{x}^* \mathcal{H}$ and so its image is contained in $M'_{\text{inf}}(\mathcal{O})$, in fact in $M'_{\text{inf}}(\mathcal{O})^{\varphi_{M'}=1}$. It follows from Corollary 2.5.8 (a) that the image of c' is contained in $M'_{\text{inf}}(S)$. Hence, we obtain:

$$c' : T' = \Lambda' \hookrightarrow M'_{\text{inf}}(S)^{\varphi_{M'}=1}.$$

Now, for all such $\tilde{x} : S \rightarrow \mathcal{O}$, consider

$$T' \hookrightarrow M'_{\text{inf}}(S)^{\varphi_{M'}=1} \rightarrow M'_{\text{inf}}(\mathcal{O})^{\varphi_{M'}=1}.$$

As above, this composition is identified with the comparison map for $\tilde{x}^* \mathcal{H}$ and is therefore an isomorphism. However,

$$(M'_{\text{inf}}(S))^{\varphi_{M'}=1} \simeq (M(\mathcal{H}))^{\varphi_{M(\mathcal{H})}=1}$$

is the Tate module of $\mathcal{H}$, a finite free $\mathbb{Z}_p$ -module of rank equal to $\text{rank}_{\mathbb{Z}_p}(T')$. Therefore

$$c' : T' \xrightarrow{\sim} M'_{\text{inf}}(S)^{\varphi_{M'}=1} = M(\mathcal{H})^{\varphi_{M(\mathcal{H})}=1}$$

is an isomorphism and it identifies T' with the Tate module of $\mathcal{H}$ as desired. $\square$

Lemma 6.5.5 *The natural homomorphism*

$$g : A_{\text{inf}}(S) \xrightarrow{\theta_{\infty}} W(S) = W(\bar{R}^{\wedge}) \rightarrow W(\bar{R}^{\wedge})$$

factors through $\hat{W}(\bar{R})$ as a composition

$$A_{\text{inf}}(S) \hookrightarrow A_{\text{cris}}(S) \rightarrow \hat{W}(\bar{R}) \rightarrow W(\bar{R}^{\wedge}).$$

Proof The diagram

$$\begin{array}{ccc} A_{\text{inf}}(S) & \xrightarrow{\theta_{\infty}} & W(S) \\ \downarrow & & \downarrow \\ A_{\text{inf}}(\bar{R}^{\wedge}) & \xrightarrow{\theta_{\infty}} & W(\bar{R}^{\wedge}) \end{array}$$

with vertical arrows given by $S \rightarrow \bar{R}^{\wedge}$, is commutative. Hence, the composition g is equal to

$$A_{\text{inf}}(S) \rightarrow A_{\text{inf}}(\bar{R}^{\wedge}) \xrightarrow{\theta_{\infty}} W(\bar{R}^{\wedge}).$$

We want to show $\theta_{\infty} : A_{\text{inf}}(\bar{R}^{\wedge}) \rightarrow W(\bar{R}^{\wedge})$ factors through $\hat{W}(\bar{R})$. We can argue as in the proof of [22, Lemma 6.1]: Note that, for each $n \geq 1$, all elements a of the kernel of $\hat{W}(\bar{R})/p^n \rightarrow \bar{R}^{\wedge}/p\bar{R}^{\wedge}$ satisfy $a^{p^n} = 0$. Therefore, by the universal property of the Witt vectors (e.g. [21, Lemma 1.4]), this gives

$$A_{\text{inf}}(\bar{R}^{\wedge}) = W((\bar{R}^{\wedge})^b) \rightarrow \hat{W}(\bar{R}) \rightarrow \bar{R}^{\wedge}/p\bar{R}^{\wedge}.$$

This lifts $\theta_{\bar{R}^{\wedge}} : A_{\text{inf}}(\bar{R}^{\wedge}) \rightarrow \bar{R}^{\wedge}$. Now since $\hat{W}(\bar{R}) \rightarrow \bar{R}^{\wedge}$ is a divided power extension of p -adic rings, the map $\theta_{\bar{R}^{\wedge}}$ factors

$$A_{\text{inf}}(\bar{R}^{\wedge}) \rightarrow A_{\text{cris}}(\bar{R}^{\wedge}) \rightarrow \hat{W}(\bar{R}) \rightarrow \bar{R}^{\wedge}$$

and using this we can conclude the proof. $\square$

As in (6.4.11), (6.4.18), we can use the above lemma to obtain an isomorphism

$$M'_{\text{inf}}(S) \otimes_{A_{\text{inf}}(S)} \hat{W}(\bar{R}) \simeq M' \otimes_{\hat{W}(\bar{R})} \hat{W}(\bar{R}) \quad (6.5.6)$$

which respects the Frobenius and Galois structures. This combined with the above gives

$$T' \simeq M'_{\text{inf}}(S)^{\varphi_{M'}=1} \rightarrow \hat{M}'_1{}^{F_1=1} \simeq T(\mathcal{G}')$$

where both source and target are finite free $\mathbb{Z}_p$ -modules of the same rank as that of T' . By pulling back via all $\tilde{x} : \tilde{R} \rightarrow \tilde{\mathcal{O}}_E$ and using Lemma 6.5.3, we see that this map is an isomorphism. Therefore, we have

$$T' \simeq T(\mathcal{G}')$$

which shows (A1). For (A2), it is enough to show that the tensors m'_b in $\mathcal{D}_{\tilde{x}}(t')^{\otimes}$ restrict via $\tilde{x} : R \rightarrow \mathcal{O}_F$ to the corresponding tensors $s'_{b, \text{D}, \tilde{x}}$ in $\mathcal{D}_{\rho(x)}(t')^{\otimes}$. This now follows from the above construction and (6.5.6). $\square$

7 Canonical integral models

7.1. We now consider Shimura varieties and their arithmetic models. Under certain assumptions, we give a definition of a “canonical” integral model.

7.1.1. Let G be a connected reductive group over $\mathbb{Q}$ and X a conjugacy class of maps of algebraic groups over $\mathbb{R}$

$$h : \mathbb{S} = \text{Res}_{\mathbb{C}/\mathbb{R}} G_m \rightarrow G_{\mathbb{R}},$$

such that (G, X) is a Shimura datum ([10] §2.1.)

For any $\mathbb{C}$ -algebra R , we have $R \otimes_{\mathbb{R}} \mathbb{C} = R \times c^*(R)$ where c denotes complex conjugation, and we denote by $\mu = \mu_h$ the cocharacter given on R -points by $R^{\times} \rightarrow (R \times c^*(R))^{\times} = (R \otimes_{\mathbb{R}} \mathbb{C})^{\times} = \mathbb{S}(R) \xrightarrow{h} G_{\mathbb{C}}(R)$.

Let $\mathbb{A}_f$ denote the finite adeles over $\mathbb{Q}$, and $\mathbb{A}_f^p \subset \mathbb{A}_f$ the subgroup of adeles with trivial component at p . Let $K = K_p K^p \subset G(\mathbb{A}_f)$ where $K_p \subset G(\mathbb{Q}_p)$, and $K^p \subset G(\mathbb{A}_f^p)$ are compact open subgroups.

If K^p is sufficiently small then the Shimura variety

$$\text{Sh}_K(G, X)_{\mathbb{C}} = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f) / K$$

has a natural structure of an algebraic variety over $\mathbb{C}$. This has a canonical model $\text{Sh}_K(G, X)$ over the reflex field; a number field $E = E(G, X)$ which is the minimal field of definition of the conjugacy class of μ_h . (See, for example, [27].) We will always assume in the following that K^p is sufficiently small; in particular, the quotient above exists as an algebraic variety. We will also assume that the center $Z(G)$ has the same $\mathbb{Q}$ -split rank as $\mathbb{R}$ -split rank. (This condition is automatic for Shimura varieties of Hodge type.)

Now choose a place v of E over p , given by an embedding $\bar{\mathbb{Q}} \rightarrow \bar{\mathbb{Q}}_p$. We denote by $E = E_v / \mathbb{Q}_p$ the local reflex field and by $\{\mu\}$ the $G(\bar{\mathbb{Q}}_p)$ -conjugacy class -which is

defined over E - of the minuscule cocharacter μ_h . We denote by $\mathcal{O}_{E,(v)}$ the localization of the ring of integers $\mathcal{O}_E$ at v .

Let $\mathcal{G}$ be a parahoric group scheme over $\mathbb{Z}_p$ with generic fiber $G_{\mathbb{Q}_p}$ and take $K_p = \mathcal{G}(\mathbb{Z}_p) \subset G(\mathbb{Q}_p)$.

We consider the system of covers $\mathrm{Sh}_{K'}(G, X) \rightarrow \mathrm{Sh}_K(G, X)$ where $K' = K'_p K^p \subset K = K_p K^p$, with K'_p running over all compact open subgroups of $K_p = \mathcal{G}(\mathbb{Z}_p)$. Using the condition on the center of G , we see that this gives a pro-étale $\mathcal{G}(\mathbb{Z}_p)$ -cover $\mathcal{L}_K$ over $\mathrm{Sh}_K(G, X)$ (eg. see [25, III], [27, Thm 5.2.6]).

7.1.2. Assume $p > 2$, $(G, \{\mu\})$ is of local Hodge type and $M^{\mathrm{loc}} = M^{\mathrm{loc}}(\mathcal{G}, \{\mu\})$ is a local model as in [35, Conjecture 21.4.1] (see the discussion in paragraph 2.1). Assume also that the pair $(\mathcal{G}, M^{\mathrm{loc}})$ is of *strongly integral* local Hodge type.

Suppose that for all sufficiently small K^p we have $\mathcal{O}_{E,(v)}$ -models $\mathcal{S}_K = \mathcal{S}_{K_p K^p}$ (schemes of finite type, separated, and flat over $\mathcal{O}_{E,(v)}$) of the Shimura variety $\mathrm{Sh}_K(G, X)$ which are normal. We consider the conditions:

1. For $K'^p \subset K^p$, there are finite étale morphisms

$$\pi_{K'_p, K_p} : \mathcal{S}_{K_p K'^p} \rightarrow \mathcal{S}_{K_p K^p}$$

which extend the natural $\mathrm{Sh}_{K_p K'^p}(G, X) \rightarrow \mathrm{Sh}_{K_p K^p}(G, X)$.

2. The scheme $\mathcal{S}_{K_p} = \varprojlim_{K^p} \mathcal{S}_{K_p K^p}$ satisfies the “extension property” for dvrs of mixed characteristic $(\bar{0}, p)$:

$$\mathcal{S}_{K_p}(R[1/p]) = \mathcal{S}_{K_p}(R),$$

for any such dvr R .

3. The p -adic formal schemes $\hat{\mathcal{S}}_K = \varprojlim_n \mathcal{S}_K \otimes_{\mathcal{O}_{E,(v)}} \mathcal{O}_{E,(v)}/(p)^n$ support locally universal $(\mathcal{G}, M^{\mathrm{loc}})$ -displays $\mathcal{D}_K$ which are associated with $\mathcal{L}_K$. We ask that these are compatible for varying K^p , i.e. that there are compatible isomorphisms

$$\pi_{K'_p, K_p}^* \mathcal{D}_K \simeq \mathcal{D}_{K'}.$$

Instead of (3) we can also consider the condition:

- (3*) The schemes $\mathcal{S}_K$ support locally universal associated systems

$$\hat{\mathcal{D}}_K := (\mathcal{L}_K, \{\mathcal{D}_{\bar{x}}\}_{\bar{x} \in \mathcal{A}_K(k)}),$$

where $\mathcal{D}_{\bar{x}}$ are Dieudonné $(\mathcal{G}, M^{\mathrm{loc}})$ -displays.

Note that (3) implies (3*); this follows from the definitions.

Theorem 7.1.7 below makes the following definition reasonable.

Definition 7.1.3 A projective system of $\mathcal{O}_{E,(v)}$ -models $\mathcal{S}_K$ of the Shimura varieties $\mathrm{Sh}_K(G, X)$, for $K = K_p K^p$ with K_p fixed as above, is *canonical*, if the models are normal and satisfy the conditions (1), (2), (3) above.

We conjecture that, under the hypotheses above, such canonical models always exist. In the next section, we show this for Shimura varieties of Hodge type at tamely ramified primes.

Remark 7.1.4 We could also consider a notion of a “ $\mathcal{M}$ -canonical model”, where $(\mathcal{G}, \mathcal{M})$ is a more general integral local Shimura pair, i.e. with $\mathcal{M}$ not $\mathbf{M}^{\text{loc}}$. However, it is not clear if such added generality is very useful here.

Remark 7.1.5 Property (3) implies the existence of a *local model diagram*:

$$\begin{array}{ccc} & \widetilde{\mathcal{S}}_K & \\ \pi_K \swarrow & & \searrow q_K \\ \widehat{\mathcal{S}}_K & & \mathbf{M}^{\text{loc}}, \end{array} \quad (7.1.6)$$

where π_K is a $\mathcal{G}$ -torsor and q_K is $\mathcal{G}$ -equivariant and smooth.

Indeed, let $\mathcal{D}_K = (\mathcal{P}_K, q_K, \Psi_K)$ be the “universal” $(\mathcal{G}, \mathbf{M}^{\text{loc}})$ -display over $\widehat{\mathcal{S}}_K$ as in (3). We set

$$\widetilde{\mathcal{S}}_K := \mathcal{P}_K \otimes_{W(\mathcal{O}_{\widehat{\mathcal{S}}_K})} \mathcal{O}_{\widehat{\mathcal{S}}_K}.$$

This is a $\mathcal{G}$ -torsor over $\widehat{\mathcal{S}}_K$ and gives π_K . The morphism q_K is obtained directly from the display datum. The smoothness of q_K follows from the local universality condition by Proposition 4.5.11.

Theorem 7.1.7 Fix $K_p = \mathcal{G}(\mathbb{Z}_p)$ as above. Suppose that $\mathcal{S}_K, \mathcal{S}'_K$ are $\mathcal{O}_{E,(v)}$ -models of the Shimura variety $\text{Sh}_K(G, X)$ for $K = K_p K^p$ that satisfy (1), (2) and (3*). Then there are isomorphisms $\mathcal{S}_K \simeq \mathcal{S}'_K$ giving the identity on the generic fibers and which are compatible with the data in (1) and (3*).

Since condition (3) implies (3*), this immediately gives:

Corollary 7.1.8 Fix $K_p = \mathcal{G}(\mathbb{Z}_p)$ as above. Suppose that $\mathcal{S}_K, \mathcal{S}'_K$ are canonical $\mathcal{O}_{E,(v)}$ -models of the Shimura variety $\text{Sh}_K(G, X)$ for $K = K_p K^p$. Then there are isomorphisms $\mathcal{S}_K \simeq \mathcal{S}'_K$ giving the identity on the generic fibers and which are compatible with the data in (1). $\square$

Proof Let us denote by $\mathcal{S}''_K$ the normalization of the Zariski closure of the diagonal embedding of $\text{Sh}_K(G, X)$ in $\mathcal{S}_K \times_{\mathcal{O}_{E,(v)}} \mathcal{S}'_K$. This is a third $\mathcal{O}_{E,(v)}$ -model of the Shimura variety $\text{Sh}_K(G, X)$ which is also normal. We can easily see that $\mathcal{S}''_K$, for varying K^p , come equipped with data as in (1) and that (2) is satisfied. Denote by

$$\pi_K : \mathcal{S}''_K \rightarrow \mathcal{S}_K, \quad \pi'_K : \mathcal{S}''_K \rightarrow \mathcal{S}'_K$$

the morphisms induced by the projections. Both of these morphisms are the identity on the generic fiber and so they are birational. Using condition (3*), we see that by

Proposition 6.3.1, π_K and π'_K give isomorphisms between the completions of the strict Henselizations at geometric closed points of the special fiber. It follows that the fibers of π_K and of π'_K over all such points are zero-dimensional. Hence, π_K and π'_K are quasi-finite. The desired result now quickly follow from this, Zariski's main theorem and the following:

Proposition 7.1.9 *The morphisms π_K and π'_K are proper.*

Proof It is enough to prove that π_K is proper, the properness of π'_K then given by symmetry. We can also base change to the completion $\mathcal{O}_{\tilde{E}}$ of the maximal unramified extension of $\mathcal{O}_{E,(v)}$; for simplicity we will omit this base change from the notation. We apply the Nagata compactification theorem ([8, Thm. 4.1]) to π_K . This provides a proper morphism $\tilde{\pi}_K : \mathcal{T} \rightarrow \mathcal{S}_K$ and an open immersion $j : \mathcal{S}_K'' \hookrightarrow \mathcal{T}$ with $\pi_K = \tilde{\pi}_K \cdot j$. By replacing $\mathcal{T}$ by the scheme theoretic closure of j , we can assume that $j(\mathcal{S}_K'')$ is dense in $\mathcal{T}$. Since $\pi_K[1/p]$ is an isomorphism and hence proper, $j[1/p]$ is also proper. Hence, $j[1/p]$ is an isomorphism as a proper open immersion with dense image. Since $\mathcal{T}$ is the closure of its generic fiber by construction, it follows that $\mathcal{T}$ is flat over $\mathcal{O}_E$ and that the “boundary”, $\mathcal{T} - j(\mathcal{S}_K'')$, if non-empty, is supported on the special fiber of $\mathcal{T} \rightarrow \text{Spec}(\mathcal{O}_E)$.

If $\mathcal{T} - j(\mathcal{S}_K'') \neq \emptyset$, there is a k -valued point $\bar{t}$ of $\mathcal{T} - j(\mathcal{S}_K'')$. By flatness, $\bar{t}$ lifts to $\tilde{t} \in \mathcal{T}(\mathcal{O}_F)$, for some finite extension $F/\tilde{E}$. Set $t = x$ for the corresponding F -valued point of the Shimura variety $\mathcal{T}[1/p] = \mathcal{S}_K[1/p] = \mathcal{S}_K''[1/p]$. This extends to $\tilde{x} := \tilde{\pi}_K(\tilde{t}) \in \mathcal{S}_K(\mathcal{O}_F)$. Since $\mathcal{O}_F$ is strictly henselian, the point $\tilde{x}$ lifts to a point

$$\tilde{z} \in \mathcal{S}_{K_p}(\mathcal{O}_F) = \varprojlim_{K^p} \mathcal{S}_{K_p K^p}(\mathcal{O}_F).$$

By the dvr extension property for $\mathcal{S}_{K_p}''$, this also gives a point $\tilde{z}'' \in \mathcal{S}_{K_p}''(\mathcal{O}_F)$. This maps to a point $\tilde{x}'' \in \mathcal{S}_K''(\mathcal{O}_F)$ which agrees with $x \in \mathcal{S}_K(F)$ on the generic fiber. Since $\tilde{\pi}_K : \mathcal{T} \rightarrow \mathcal{S}_K$ is separated, this implies that $\tilde{t}$ lies on $j(\mathcal{S}_K'')$, which is a contradiction. We conclude that j is an isomorphism and so π_K is proper. $\square$

8 Shimura varieties at tame parahoric primes

8.1. We now concentrate our attention to Shimura varieties of Hodge type at tame primes where the level is parahoric ([20]).

8.1.1. Fix a $\mathbb{Q}$ -vector space V with a perfect alternating pairing ψ . For any $\mathbb{Q}$ -algebra R , we write $V_R = V \otimes_{\mathbb{Q}} R$. Let $\text{GSp} = \text{GSp}(V, \psi)$ be the corresponding group of symplectic similitudes, and let $S^{\pm}$ be the Siegel double space, defined as the set of maps $h : S \rightarrow \text{GSp}_{\mathbb{R}}$ such that

1. The $\mathbb{C}^{\times}$ -action on $V_{\mathbb{R}}$ gives rise to a Hodge structure

$$V_{\mathbb{C}} \cong V^{-1,0} \oplus V^{0,-1}$$

of type $(-1, 0), (0, -1)$.

2. $(x, y) \mapsto \psi(x, h(i)y)$ is (positive or negative) definite on $V_{\mathbb{R}}$.

8.1.2. Let (G, X) be a Shimura datum and $K = K_p K^p \subset G(\mathbb{A}_f)$ with $K_p \subset G(\mathbb{Q}_p)$ and $K^p \subset G(\mathbb{A}_f^p)$ as above, where p is an odd prime. We assume:

1. (G, X) is of Hodge type: There is a symplectic faithful representation $\rho : G \hookrightarrow \mathrm{GSp}(V, \psi)$ inducing an embedding of Shimura data

$$(G, X) \hookrightarrow (\mathrm{GSp}(V, \psi), S^\pm).$$

2. G splits over a tamely ramified extension of $\mathbb{Q}_p$.
3. $K_p = \mathcal{G}(\mathbb{Z}_p)$ is a parahoric stabilizer, so $\mathcal{G}$ is the Bruhat-Tits stabilizer group scheme $\mathcal{G}_x$ of a point x in the extended Bruhat-Tits building of $G(\mathbb{Q}_p)$ and $\mathcal{G}$ is connected, i.e. we have $\mathcal{G} = \mathcal{G}_x = \mathcal{G}_x^\circ$.
4. $p \nmid \pi_1(G_{\mathrm{der}}(\bar{\mathbb{Q}}_p))$.

We now fix a place v of the reflex field E over p and let $E = E_v/\mathbb{Q}_p$ and $\{\mu\}$ be as in Sect. 7 above. Associated with $(\mathcal{G}, \{\mu\})$ and x , we have the local model

$$M^{\mathrm{loc}} = M^{\mathrm{loc}}(\mathcal{G}, \{\mu\}) = M_x^{\mathrm{loc}}(G, \{\mu\}).$$

(Under the current assumptions, we can appeal to [30] for the construction of M^{loc} . This satisfies Scholze's characterization by [17, Theorem 2.15].)

In [20, 2.3.1, 2.3.15, 2.3.16], it is shown that under the assumptions (1)-(4) above, there is a (possibly different) Hodge embedding

$$\iota : (G, X) \hookrightarrow (\mathrm{GSp}(V, \psi), S^\pm)$$

and a $\mathbb{Z}_p$ -lattice $\Lambda \subset V_{\mathbb{Q}_p}$ such that $\Lambda \subset \Lambda^\vee$ and

- (a) There is a group scheme homomorphism which is a closed immersion

$$\iota : \mathcal{G} \hookrightarrow \mathrm{GL}(\Lambda),$$

such that $\iota(\mathcal{G})$ contains the scalars $\mathbb{G}_m$, and which extends

$$G_{\mathbb{Q}_p} \hookrightarrow \mathrm{GSp}(V_{\mathbb{Q}_p}, \psi_{\mathbb{Q}_p}) \subset \mathrm{GL}(V_{\mathbb{Q}_p}).$$

- (b) There is a corresponding equivariant closed immersion

$$\iota_* : M^{\mathrm{loc}} \hookrightarrow \mathrm{Gr}(g, \Lambda)_{\mathcal{O}_E}.$$

(So ι is a strongly integral local Hodge embedding for $(\mathcal{G}, M^{\mathrm{loc}})$.)

Here, $\dim_{\mathbb{Q}_p}(V) = 2g$ and $\mathrm{Gr}(g, \Lambda)$ is the Grassmannian over $\mathbb{Z}_p$.

8.1.3. Let $V_{\mathbb{Z}_{(p)}} = \Lambda \cap V$, and fix a $\mathbb{Z}$ -lattice $V_{\mathbb{Z}} \subset V$ such that $V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} = V_{\mathbb{Z}_{(p)}}$ and $V_{\mathbb{Z}} \subset V_{\mathbb{Z}}^\vee$. Consider the Zariski closure $G_{\mathbb{Z}_{(p)}}$ of G in $\mathrm{GL}(V_{\mathbb{Z}_{(p)}})$; then $G_{\mathbb{Z}_{(p)}} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p \cong \mathcal{G}$. Fix a finite set of tensors $(s_a) \subset V_{\mathbb{Z}_{(p)}}^\otimes$ whose stabilizer is $G_{\mathbb{Z}_{(p)}}$. Such a set exists by [19, Lemma 1.3.2] and [11].

Set $K_p = \mathcal{G}(\mathbb{Z}_p)$, and $K^b_p = \mathrm{GSp}(V_{\mathbb{Q}_p}) \cap \mathrm{GL}(\Lambda)$. We set $K = K_p K^p$ and similarly for K^b . By [19, Lemma 2.1.2], for any compact open subgroup $K^p \subset G(\mathbb{A}_f^p)$ there exists $K^{b,p} \subset \mathrm{GSp}(\mathbb{A}_f^p)$ such that ι induces an embedding over E

$$\mathrm{Sh}_K(G, X) \hookrightarrow \mathrm{Sh}_{K^b}(\mathrm{GSp}(V, \psi), S^\pm) \otimes_{\mathbb{Q}} E.$$

The choice of lattice $V_{\mathbb{Z}}$ gives an interpretation of the Shimura variety $\mathrm{Sh}_{K^b}(\mathrm{GSp}, S^\pm)$ as a moduli scheme of polarized abelian varieties with $K^{b,p}$ -level structure, and hence an integral model $\mathcal{A}_{K^b} = \mathcal{S}_{K^b}(\mathrm{GSp}, S^\pm)$ over $\mathbb{Z}_{(p)}$ (see [19, 20]).

We denote by $\mathcal{S}_K^-(G, X)$ the (reduced) closure of $\mathrm{Sh}_K(G, X)$ in the $\mathcal{O}_{E, (v)}$ -scheme $\mathcal{S}_{K^b}(\mathrm{GSp}, S^\pm) \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}_{E, (v)}$, and by $\mathcal{S}_K(G, X)$, the normalization of the closure $\mathcal{S}_K(G, X)^-$. For simplicity, we set

$$\mathcal{S}_K := \mathcal{S}_K(G, X)$$

when there is no danger of confusion.

Theorem 8.1.4 *Assume that p is odd and that the Shimura data (G, X) and the level subgroup K satisfy the assumptions (1)–(4) of Sect. 8. Then, the $\mathcal{O}_{E, (v)}$ -models $\mathcal{S}_K(G, X)$ support locally universal associated systems*

$$\mathcal{D}_K = (\mathcal{L}_K, \{\mathcal{D}_{\bar{x}}\}_{\bar{x} \in \mathcal{S}_K(k)}),$$

where $\mathcal{D}_{\bar{x}}$ are Dieudonné $(\mathcal{G}, M^{\mathrm{loc}})$ -displays.

Proof Recall the pro-étale $\mathcal{G}(\mathbb{Z}_p)$ -cover $\mathcal{L}_K$ over $\mathrm{Sh}_K(G, X)$ given as in Sect. 7 above. Let $h : A \rightarrow \mathcal{S}_K$ denote the restriction of the universal abelian scheme via $\mathcal{S}_K \rightarrow \mathcal{S}_{K^b}(\mathrm{GSp}, S^\pm)$. Then the $\mathbb{Z}_p$ -local system $\mathcal{L}_K(\iota)$ is isomorphic to the local system given by the Tate module of the p -divisible group $A[p^\infty]$ of the universal abelian scheme over $\mathcal{S}_K$. The tensors $s_a \in \Lambda^\otimes$ give corresponding global sections $s_{a, \text{ét}}$ of $\mathcal{L}_K(\iota)^\otimes$ over $\mathrm{Sh}_K(G, X)$. Theorem 6.4.1 implies that $\mathcal{L}_K$ extends to an associated system $(\mathcal{L}_K, \{\mathcal{D}_{\bar{x}}\}_{\bar{x} \in \mathcal{S}_K(k)})$, where $\mathcal{D}_{\bar{x}}$ are Dieudonné $(\mathcal{G}, M^{\mathrm{loc}})$ -displays. It remains to show: $\square$

Proposition 8.1.5 *For every $\bar{x} \in \mathcal{S}_K(k)$, the Dieudonné $(\mathcal{G}, M^{\mathrm{loc}})$ -display $\mathcal{D}_{\bar{x}}$ over $R = \hat{\mathcal{O}}_{\mathcal{S}_K, \bar{x}}$ is locally universal.*

Proof Set $\mathcal{D}_{\bar{x}} = (\mathcal{P}, q, \Psi)$. Choose a section s of $\mathcal{P}$ over $\hat{W}(R)$ which is rigid in the first order at m_R . Then the corresponding section $\mathrm{Spec}(\hat{W}(R)) \rightarrow \mathcal{P} \subset \mathcal{P}_{\mathrm{GL}}$ is rigid in the first order for the GL-display $\mathcal{D}_{\bar{x}}(\iota) = (\mathcal{P}_{\mathrm{GL}}, q_{\mathrm{GL}}, \Psi_{\mathrm{GL}})$ induced by ι and $\mathcal{D}_{\bar{x}}$. We have a morphism

$$q \cdot (s \otimes 1) : \mathrm{Spec}(R) \rightarrow M^{\mathrm{loc}} \subset \mathrm{Gr}(g, \Lambda)_{\mathcal{O}_{\tilde{E}}}.$$

We also have the morphism

$$i : \mathrm{Spec}(R) = \mathrm{Spec}(\hat{\mathcal{O}}_{\mathcal{S}_K, \bar{x}}) \rightarrow \mathcal{A}_{K^b} \otimes_{\mathbb{Z}_p} \mathcal{O}_{\tilde{E}}.$$

induced by the Hodge embedding. By [20, Prop. 4.2.2] and its proof, i is a closed immersion. Since $\mathcal{D}_{\bar{x}} = (\mathcal{P}, q, \Psi)$ is associated with $\mathcal{L}_K$ (for ι), the p -divisible group that corresponds to $\mathcal{D}_{\bar{x}}(\iota)$ is the p -divisible group obtained by pulling back the (versal) p -divisible group of the universal abelian scheme via i . By Proposition 4.5.15 we obtain that the morphism $\mathrm{Spec}(R) \rightarrow M^{\mathrm{loc}} \subset \mathrm{Gr}(g, \Lambda)_{\mathcal{O}_E}$ induces a surjection on cotangent spaces. It follows that $\mathrm{Spec}(R) \rightarrow M^{\mathrm{loc}}$ also induces a surjection

$$\hat{\mathcal{O}}_{M^{\mathrm{loc}}, \bar{y}} \rightarrow R = \hat{\mathcal{O}}_{\mathcal{S}_K, \bar{x}}$$

where $\bar{y} = (q \cdot (s \otimes 1))(\bar{x})$. This surjection between complete local normal rings of the same dimension has to be an isomorphism. This completes the proof. $\square$

By combining Theorems 8.1.4 and 7.1.8 we now obtain:

Theorem 8.1.6 *Assume that p is odd and that the Shimura data (G, X) and the level subgroup K satisfy the assumptions (1)–(4) of Sect. 8. Suppose v is a place of E over p . Then the $\mathcal{O}_{E, (v)}$ -scheme $\mathcal{S}_K(G, X)$ of [20] is independent of the choices of Hodge embedding $\rho : (G, X) \hookrightarrow (\mathrm{GSp}(V, \psi), S^{\pm})$, lattice $V_{\mathbb{Z}} \subset V$ and tensors (s_a) , used in its construction.* $\square$

8.2. Finally, we show:

Theorem 8.2.1 *Assume that p is odd and that the Shimura data (G, X) and the level subgroup K satisfy the assumptions (1)–(4) of Sect. 8. Then, the $\mathcal{O}_{E, (v)}$ -models $\mathcal{S}_K(G, X)$ of [20] are canonical, in the sense of Definition 7.1.3.*

Proof We already know that $\mathcal{S}_K = \mathcal{S}_K(G, X)$ supports a locally universal associated system by Theorem 8.1.4. We need to “upgrade” this and show there is also an associated $(\mathcal{G}, M^{\mathrm{loc}})$ -display $\mathcal{D}_K$ as in Definition 6.2.1. Write $\hat{\mathcal{S}}_K$ for the formal scheme obtained as the p -adic completion of $\mathcal{S}_K$. The Dieudonné crystal $\mathbb{D}_K := \mathbb{D}(A[p^{\infty}]) (W(\mathcal{O}_{\hat{\mathcal{S}}_K}))$ of the universal p -divisible group over $\mathcal{S}_K$ gives a GL-display over $\hat{\mathcal{S}}_K$. By work of Hamacher and Kim [16, 3.3], there are Frobenius invariant tensors $s_{a, \mathrm{univ}} \in \mathbb{D}_K^{\otimes}$ which have the following property: For every $\bar{x} \in \mathcal{S}_K(k)$, the base change isomorphism

$$\mathbb{D}_K \otimes_{W(\mathcal{O}_{\hat{\mathcal{S}}_K})} W(\hat{R}_{\bar{x}}) \simeq M_{\bar{x}} \otimes_{\hat{W}(\hat{R}_{\bar{x}})} W(\hat{R}_{\bar{x}}) \quad (8.2.2)$$

maps $s_{a, \mathrm{univ}}$ to $m_a \otimes 1$. Here, we write $\mathcal{D}_{\bar{x}}(\iota) = (M_{\bar{x}}, M_{1, \bar{x}}, F_{1, \bar{x}})$ and we recall that $m_a \in M_{\bar{x}}^{\otimes}$ are the tensors which are associated with $s_{a, \mathrm{ét}} \in \mathcal{L}_K(\iota)^{\otimes}$ and are given by the $\mathcal{G}$ -torsor $\mathcal{P}_{\bar{x}}$ of the Dieudonné $(\mathcal{G}, M^{\mathrm{loc}})$ -display $\mathcal{D}_{\bar{x}} = (\mathcal{P}_{\bar{x}}, q_{\bar{x}}, \Psi_{\bar{x}})$ (see the proof of Theorem 6.4.1). We can now use this to give a $(\mathcal{G}, M^{\mathrm{loc}})$ -display $\mathcal{D}_K = (\mathcal{P}_K, q_K, \Psi_K)$ over $\hat{\mathcal{S}}_K$ as follows: First set

$$\mathcal{P}_K = \underline{\mathrm{Isom}}_{(s_{a, \mathrm{univ}}), (s_a \otimes 1)}(\mathbb{D}_K, \Lambda \otimes_{\mathbb{Z}_p} W(\mathcal{O}_{\hat{\mathcal{S}}_K})).$$

Consider an open affine formal subscheme $\mathrm{Spf}(R) \subset \hat{\mathcal{S}}_K$. Then R satisfies condition (N). Since (8.2.2) above respects the tensors, $\mathcal{P}_K \otimes_{W(\mathcal{O}_{\hat{\mathcal{S}}_K})} W(\hat{R}_{\bar{x}}) \simeq \mathcal{P}_{\bar{x}}$, for all

$\bar{x} \in \operatorname{Spec}(R/p)(k)$. Hence, for example by Corollary 3.2.6, $\mathcal{P}_K \otimes_{W(\mathcal{O}_{\mathcal{F}_K})} W(R)$ is a $\mathcal{G}$ -torsor over $\operatorname{Spec}(W(R))$. Therefore, $\mathcal{P}_K$ is also a $\mathcal{G}$ -torsor. It remains to give q_K and Ψ_K . Recall that, under our assumptions, [20, Theorem 4.2.7] gives a (“local model”) diagram

$$\begin{array}{ccc} & \widetilde{\mathcal{F}}_K & \\ \pi_K \swarrow & & \searrow q_K \\ \mathcal{F}_K & & M^{\text{loc}}, \end{array}$$

in which the left arrow is a $\mathcal{G}$ -torsor and the right arrow is smooth and $\mathcal{G}$ -equivariant. The $\mathcal{G}$ -torsor $\widetilde{\mathcal{F}}_K \rightarrow \mathcal{F}_K$ is given as

$$\widetilde{\mathcal{F}}_K := \underline{\operatorname{Isom}}_{(s_a, \text{DR}), (s_a)}(H_{\text{DR}}^1(A)^\vee, \Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathcal{F}_K}).$$

Since by [16, Cor. 3.3.4] the comparison

$$\mathbb{D}_K \otimes_{W(\mathcal{O}_{\mathcal{F}_K})} \mathcal{O}_{\mathcal{F}_K} \cong H_{\text{DR}}^1(A)^\vee$$

takes $s_{a, \text{univ}} \otimes 1$ to $s_{a, \text{DR}}$, we have

$$\mathcal{P}_K \otimes_{W(\mathcal{O}_{\mathcal{F}_K})} \mathcal{O}_{\mathcal{F}_K} \cong \widetilde{\mathcal{F}}_K \xrightarrow{q_K} M^{\text{loc}}$$

which gives the desired q_K . Finally, we can give Ψ_K using the Frobenius structure on $\mathbb{D}_K$ following the dictionary in Sect. 4.4. By similar arguments as above, this respects the tensors and so it gives an isomorphism of $\mathcal{G}$ -torsors. Then $\mathcal{D}_K$ gives the desired $(\mathcal{G}, M^{\text{loc}})$ -display which satisfies the requirements of Sect. 7. The result follows. $\square$

Remark 8.2.3 We expect that the above results (Theorems 8.1.6, 8.2.1), can be extended so that they also apply to the integral models $\mathcal{S}_K(G, X)$ constructed in [18]. In the set-up of [18], assumption (2) of Sect. 8 is weakened to allow for some wildly ramified groups with $G_{\text{ad}} \otimes_{\mathbb{Q}} \mathbb{Q}_p \simeq \prod_{i=1}^m \operatorname{Res}_{F_i/\mathbb{Q}_p}(H_i)$, where each H_i splits over a tamely ramified extension of F_i .

Acknowledgements We thank M. Rapoport, P. Scholze, and the referee, for useful suggestions and corrections, and V. Drinfeld for interesting discussions.

References

1. Anschütz, J.: Extending torsors on the punctured $\operatorname{Spec}(A_{\text{inf}})$. Preprint, [arXiv:1804.06356](https://arxiv.org/abs/1804.06356)
2. Bhatt, B., Scholze, P.: Prisms and Prismatic Cohomology. Preprint, [arXiv:1905.08229](https://arxiv.org/abs/1905.08229)
3. Bhatt, B., Scholze, P.: Projectivity of the Witt vector affine Grassmannian. *Invent. Math.* **209**, 329–423 (2017)
4. Bhatt, B., Morrow, M., Scholze, P.: Integral p -adic Hodge theory. *Publ. Math. IHES* **128**, 219–397 (2018)
5. Brinon, O.: Représentations p -adiques cristallines et de de Rham dans le cas relatif. *Mém. Soc. Math. Fr. (N.S.)* No. 112, pp. vi+159 (2008)
6. Broshi, M.: G -torsors over a Dedekind scheme. *J. Pure Appl. Algebra* **217**(1), 11–19 (2013)

7. Bültel, O., Pappas, G.: (G, μ) -displays and Rapoport-Zink spaces. *J. Inst. Math. Jussieu* **19**(4), 1211–1257 (2020)
8. Conrad, B.: Deligne's notes on Nagata compactifications. *J. Ramanujan Math. Soc.* **22**(3), 205–257 (2007)
9. de Jong, A.J.: Crystalline Dieudonné module theory via formal and rigid geometry. *Publ. Math. IHES* **82**, 5–96 (1995)
10. Deligne, P.: Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques, Automorphic forms, representations and L -functions, Part 2, *Proc. Sympos. Pure Math.*, XXXIII, Amer. Math. Soc., Providence, R.I., pp. 247–289 (1979)
11. Deligne, P.: Letter to M. Kisin. August 26, (2011)
12. Faltings, G.: Crystalline Cohomology and p -Adic Galois-Representations. *Algebraic Analysis, Geometry, and Number Theory* (Baltimore, MD, 1989), 25–80, p. 1989. Johns Hopkins Univ. Press, Baltimore (1988)
13. Faltings, G.: p -adic Hodge theory. *J. Am. Math. Soc.* **1**(1), 255–299 (1988)
14. Faltings, G.: Almost étale extensions. *Cohomologies p -adiques et applications arithmétiques*, II. *Astérisque* No. 279, 185–270 (2002)
15. Gruson, L., Raynaud, M.: Critères de platitude et de projectivité. Techniques de platification d'un module. *Invent. Math.* **13**, 1–89 (1971)
16. Hamacher, P., Kim, W.: l -adic étale cohomology of Shimura varieties of Hodge type with non-trivial coefficients. *Math. Ann.* **375**(3–4), 973–1044 (2019)
17. He, X., Pappas, G., Rapoport, M.: Good and semi-stable reductions of Shimura varieties. *J. l'École Polytech. Math.* **7**, 497–571 (2020)
18. Kisin, M., Zhou, R.: Independence of ℓ for Frobenius conjugacy classes attached to abelian varieties. Preprint, [arXiv:2103.09945](https://arxiv.org/abs/2103.09945)
19. Kisin, M.: Integral models for Shimura varieties of abelian type. *J. Am. Math. Soc.* **23**(4), 967–1012 (2010)
20. Kisin, M., Pappas, G.: Integral models of Shimura varieties with parahoric level structure. *Publ. Math. IHES* **128**, 121–218 (2018)
21. Lau, E.: Relations between Dieudonné displays and crystalline Dieudonné theory. *Algebra Num. Theory* **8**(9), 2201–2262 (2014)
22. Lau, E.: Displayed equations for Galois representations. *Nagoya Math. J.* **235**, 86–114 (2019)
23. Lourenço, J.N.P.: Thesis, Bonn (2020)
24. Matsumura, H.: *Commutative Ring Theory*. Cambridge Studies in Advanced Mathematics, 8, 2nd edn. Cambridge University Press, Cambridge, pp. xiv+320 (1989)
25. Milne, J.: Canonical models of (mixed) Shimura varieties and automorphic vector bundles. In: *Automorphic Forms, Shimura Varieties, and L-Functions*, vol. I, pp. 283–414. Ann Arbor, MI (1988). *Perspect. Math.*, vol. 10. Academic Press, Boston (1990)
26. Milne, J.: The Points on a Shimura Variety Modulo a Prime of Good Reduction, *The Zeta Functions of Picard Modular Surfaces*. Univ. Montreal, Montreal, pp. 151–253 (1992)
27. Milne, J.: Introduction to Shimura varieties. Harmonic analysis, the trace formula and Shimura varieties. *Clay Math. Proc.* **4**, 265–378 (2005)
28. Pappas, G., Rapoport, M.: p -adic shtukas and the theory of global and local Shimura varieties. Preprint, [arXiv:2106.08270](https://arxiv.org/abs/2106.08270)
29. Pappas, G.: Arithmetic models for Shimura varieties. In: *Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018*. Vol. II. Invited lectures. World Sci. Publ., Hackensack, pp. 377–398 (2018)
30. Pappas, G., Zhu, X.: Local models of Shimura varieties and a conjecture of Kottwitz. *Invent. math.* **194**, 147–254 (2013)
31. Rapoport, M.: A guide to the reduction modulo p of Shimura varieties. *Astérisque* **298**, 271–318 (2005)
32. Rapoport, M., Zink, T.: *Period Spaces for p -Divisible Groups*. Ann. of Math. Studies 141. Princeton University Press, Princeton (1996)
33. Scholze, P.: p -adic geometry. In: *Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018*. Vol. I. Plenary lectures. World Sci. Publ., Hackensack, pp. 899–933 (2018)
34. Scholze, P., Weinstein, J.: Moduli of p -divisible groups. *Camb. J. Math.* **1**(2), 145–237 (2013)
35. Scholze, P., Weinstein, J.: *Berkeley Lectures on p -adic Geometry*, Annals of Math. Studies. Number 207. Princeton University Press, Princeton (2020)

36. Tits, J.: Reductive groups over local fields, Automorphic forms, representations and L-functions. In: Proc. Sympos. Pure Math., XXXIII, part 1, Amer. Math. Soc., Providence, pp. 29–69 (1979)
37. Vasiu, A.: Integral canonical models of Shimura varieties of preabelian type. *Asian J. Math.* **3**(2), 401–518 (1999)
38. Vasiu, A., Zink, T.: Purity results for p -divisible groups and abelian schemes over regular bases of mixed characteristic. *Doc. Math.* **15**, 571–599 (2010)
39. Zink, T.: A Dieudonné theory for p -divisible groups. In: Class Field Theory—Its Centenary and Prospect (Tokyo, 1998), Adv. Stud. Pure Math., 30, Math. Soc. Japan, Tokyo, pp. 139–160 (2001)
40. Zink, T.: The display of a formal p -divisible group. *Cohomologies p -adiques et applications arithmétiques. I.* *Astérisque* No. **278**, 127–248 (2002)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.