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Review

Space–Time Physics in Background-Independent Theories of Quantum Gravity

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Abstract: Background independence is often emphasized as an important property of a quantum theory of gravity that takes seriously the geometrical nature of general relativity. In a background-independent formulation, quantum gravity should determine not only the dynamics of space–time but also its geometry, which may have equally important implications for claims of potential physical observations. One of the leading candidates for background-independent quantum gravity is loop quantum gravity. By combining and interpreting several recent results, it is shown here how the canonical nature of this theory makes it possible to perform a complete space–time analysis in various models that have been proposed in this setting. In spite of the background-independent starting point, all these models turned out to be non-geometrical and even inconsistent to varying degrees, unless strong modifications of Riemannian geometry are taken into account. This outcome leads to several implications for potential observations as well as lessons for other background-independent approaches.

Keywords: background independence; space–time physics; geometry; loop quantum gravity; covariance

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1. Introduction

A key feature of general relativity is its ability to determine both the dynamics and the structure of space–time. A complete quantum theory of gravity should therefore refrain from presupposing space–time structure; only then can it be considered a proper quantization of the theory. As a conclusion, space–time structure must be derived after quantization for a subsequent physical analysis, and the result may be modified compared with the familiar Riemannian structure. Depending on the quantization procedure, it may even happen that no consistent space–time structure exists for its solutions. A detailed analysis is then required to see whether the theory can be considered a valid candidate for quantum gravity, even if it is formally consistent, judged by non-geometrical standards such as conditions commonly imposed on quantizations of gauge theories. These questions are highly non-trivial in any approach. A detailed analysis is now available in models of loop quantum gravity, but it remains preliminary owing to the tentative nature of physical models of space–time in this theory.

Loop quantum gravity is often advertised as a background-independent approach to quantum gravity. This characterization suggests that the theory might indeed be free of pre-supposed space–time structures. In practice, however, the rather involved nature of methods suitable for derivations of space–time structures, combined with the canonical treatment used in the more successful realizations of loop quantum gravity, has for some time obscured the role and nature of space–time in this theory. In fact, several long-standing doubts exist as to the possibility of covariance in models of loop quantum gravity. For instance, the "bounce" idea, used in a majority of cosmological and black-hole models in this setting, is largely based on calculations available for the dynamics in homogeneous cosmological models, introducing formal properties of discreteness or boundedness seen in the kinematics of the full theory of loop quantum gravity. Since it remains unknown whether there is space–time dynamics consistent with the kinematics of the full theory, there is no guarantee that kinematical ingredients exported to homogeneous models of quantum cosmology can give rise to a meaningful structure of space–time and some sense of general covariance.

More specifically, kinematical features that apply spatial discreteness work as a cut-off which, if it is a fixed scale, is hard to reconcile with the transformations required for covariance. If one accepts the possibility that quantum gravity may well lead to non-classical space–time structures that require a modified and perhaps weakened version of general covariance, consistency requires a detailed demonstration of how one can avoid various low-energy problems that may then trickle down from the Planck regime, as pointed out in [1,2]. Moreover, in such a situation, it is important to determine how a modified space–time structure can be described in meaningful terms, for instance by addressing the question of whether such a theory can still be considered geometrical and whether there is an extended range of parameters (such as $\hbar$) in which effective line elements may still be available.

A consideration of space–time structure in bounce models also raises the question of how exactly singularity theorems are evaded. In models of loop quantum cosmology, bounce solutions are obtained without modifying matter Hamiltonians. The standard energy conditions therefore remain satisfied, obscuring the possibility of avoided singularities often claimed in this setting. Since singularity theorems make statements about boundaries of space–time and use the general properties of Riemannian geometry such as the Ricci curvature and the geodesic deviation equation, they depend on and require a consistent form of space–time structure. Unfortunately, however, bounce models of loop quantum gravity are often accompanied by poorly justified and contradictory statements about space–time. For instance, standard line elements are commonly used to express modified gravitational dynamics in tractable form, implicitly presupposing that space–time remains Riemannian. However, then, singularity theorems should be applicable to the resulting modified solutions since the behavior of matter energy is assumed to remain unchanged, making it impossible to evade singularities by a bounce. (The behavior of singularities may depend on a possibly modified relationship between stress–energy and Ricci curvature even if one maintains positive-energy conditions. However, simple bounce models based on modified Friedmann equations do not provide such a relationship because their space–time structure remains unclear.) The fact that this contradiction has gone unnoticed for several years in this field serves to highlight the challenging nature of questions about space–time in loop quantum gravity.

Independently of bounce claims, results about space–time structure in models of loop quantum gravity have been accumulating in recent years. This review presents a summary, highlighting the similarities between different ways in which covariance can be and often is violated. By now, all the high-profile claims made in the last decade in the context of loop quantum gravity, including [3–6], have been shown to rest upon inconsistent assumptions about space–time structure and covariance. It is therefore of interest to combine and compare the various ways in which covariance can be violated in order to arrive at a general perspective. (Some of these models have already been presented in an overview form in [7]. The focus of this previous review was on implications for models of black holes, while the present one emphasizes the role of these results for general aspects of background independence and the viability of quantum gravity. Moreover, it presents further comparisons between the different results).

A discrete fundamental theory is not expected to respect all the properties that we are used to from classical space–time. Some violation of classical covariance may therefore be allowed. Nevertheless, because covariance does not only describe a property of classical space–time but also implies that all consistency conditions are met for gravity as a gauge theory, the requirement of general covariance cannot just be abandoned without suitable replacements. One task to be completed for a consistent theory of quantum gravity is to find suitable middle ground between completely broken covariance and the strictly classical notion of general covariance. Considerations of covariance therefore remain important even if one believes that quantum gravity may completely change the structure of space–time in its fundamental formulation.

The examples of violations of covariance discussed here do not directly apply to fundamental quantum gravity but rather to models used for phenomenological studies of cosmology or black holes. In this context, the question of covariance is even more pressing because a general (but often implicit) strategy in this context is to use well-understood Riemannian geometry to analyze potential modifications in the dynamical equations of quantum gravity. Since these modifications may easily affect space–time structure as well, any implicit assumptions about space–time must be uncovered and analyzed before an analysis can be considered meaningful. In this phenomenological context, the question of space–time structure is not as challenging as it is at the fundamental level, but it is still relevant. The task is to show that a certain geometrical structure applies to solutions of an effective description of quantum gravity not only in the strict classical limit where $\hbar = 0$ but also within some finite range of the expansion parameter, given for instance by $\rho/\rho_{\rm P}$ in a cosmological model with energy density ρ relative to the Planck density.

The studies [3–6] of interest here implicitly assume that space–time structure remains unmodified even in the presence of modified dynamics, and sometimes even all the way to the Planck scale [3,6]. This strong assumption is implemented by inserting solutions of modified equations in a standard line element, without checking whether the modified solutions obey gauge transformations compatible with coordinate transformations such that an invariant line element results. Such a line element is crucial in these studies because it enables the formulation of new claims of potential physical effects that make these studies interesting and publishable in high-profile journals. The same ingredient makes these studies vulnerable to violations of covariance, as reviewed in detail in the following sections.

The concluding section of this review points out general properties of covariance in models of loop quantum gravity that may be useful for other approaches. It is generally expected that quantum gravity leads to new geometrical features at large curvature that can no longer be described by a classical form of space–time with its common sense of covariance. Loop quantum gravity is only one approach in which a specific example of discreteness or other non-classical geometrical effects is being explored. The general question to be addressed is then whether quantum gravity at large curvature remains a geometrical theory in the sense that its solutions can still be described in terms of space–time with a certain generalized meaning compared with our classical notion.

2. Models of Loop Quantum Gravity

In order to set up our analysis, we should first introduce the general form of modifications implemented in models of loop quantum gravity (see [8] for more details). It is sufficient to illustrate these modifications by recalling the basics of loop quantum cosmology for spatially flat, isotropic models.

2.1. Holonomy Modifications and Space–Time Structure

The classical dynamics of the scale factor a can be expressed by a canonical pair (q, p) where $q = \dot{a}$ (a proper-time derivative) and $|p| = a^2$, subject to the Friedmann constraint:

$$-\frac{q^2}{|p|} + \frac{8\pi G}{3}\rho = 0 \tag{1}$$

with the energy density ρ. Kinematical aspects of loop quantization suggest the replacement, or "holonomy modification":

$$\frac{q^2}{|p|} \mapsto \frac{\sin(\ell q/\sqrt{|p|})^2}{\ell^2} \tag{2}$$

where ℓ is a suitable, possibly running length scale, such as the Planck length $\ell_{\rm P}$ in simple cases.

Taken in isolation, holonomy modifications imply non-singular behavior in isotropic models with a modified Friedmann constraint:

$$\frac{\sin(\ell q/\sqrt{|p|})^2}{\ell^2} = \frac{8\pi G}{3}\rho \tag{3}$$

because the energy density of any solution to this equation must be bounded (assuming that ℓ is constant, as commonly done in this context). However, this equation includes only one type of expected quantum corrections. In addition, a complete effective description of some underlying dynamics of quantum gravity (of any kind) should also include the remnants of higher-curvature terms in an isotropic model. Higher-curvature terms, just like holonomy modifications, require a given length scale, which we may assume to equal ℓ if holonomy modifications and higher-curvature terms are derived from a single quantum theory of gravity. It is easy to see that higher-curvature terms are not described by (3) because they generically imply higher time derivatives and therefore extend the phase space by additional momenta.

The Equation (3) is therefore incomplete from the viewpoint of effective theory. Nevertheless, it may be useful because it determines at least one type of quantum corrections. However, knowing that there are additional terms not included in (3) that also depend on ℓ, we cannot trust the full function $\sin^2(\ell q/\sqrt{|p|})/\ell^2$ but should rather expand:

$$\frac{\sin(\ell q/\sqrt{|p|})^2}{\ell^2} \sim \frac{q^2}{|p|}\left(1 - \frac{1}{3}\ell^2\frac{q^2}{|p|} + \cdots\right) \tag{4}$$

and only include leading-order terms. If $\ell \sim \ell_{\rm P}$, these leading corrections are of the order $\ell_{\rm P}^2 q^2/|p| \sim \rho/\rho_{\rm P}$, which is the same as the order expected for higher-curvature terms. Even the leading corrections in (3) should therefore not be considered to be definitely certain and considered with caution. Interpreting the full series expansion or its sum to the sine function as an indication of bounded densities is unjustified in the absence of information about higher-curvature terms.

Higher-curvature terms are also of interest from the point of view of space–time structure. We already used the fact that they generically include higher time derivatives, but the specific appearance of such terms is not arbitrary and is instead guided by requirements of general covariance. In loop quantum cosmology, the form of quantum corrections that may appear in addition to holonomy modifications can therefore be determined only if there is good control on space–time structure in this setting.

Isotropic and homogeneous models are not sufficient for an analysis of space–time structure and covariance because these questions rely on how spatial and temporal dependencies are related in differential equations and their solutions. At least one spatial direction of inhomogeneity should then be included in suitable models, in addition to the non-trivial time dependence already described by models such as (3). While such (midisuperspace) models have been considered in loop quantum gravity for some time, their application to the question of covariance is rather new and has led to several surprising results.

2.2. Three Examples and One Theorem

We will review three examples of the proposed methods to describe inhomogeneity in models of loop quantum gravity and the reasons why they turn out to violate covariance in ways that render them inconsistent. The first example, the dressed-metric approach for cosmological inhomogeneity [9], has been used several times as a crucial ingredient in cosmological model building, leading to claims of observational testability that, given the underlying problems with space–time structure, turn out to be unfounded. (Similar arguments regarding violations of covariance apply to the hybrid approach to inhomogeneity in loop quantum cosmology [10]). The remaining two examples, given by partial

Abelianizations of constraints in spherically symmetric models [4] as well as a misleadingly named "covariant polymerization" [11] in related studies apply to proposed scenarios for quantum black holes. (The proposal of [11] was intended to justify modified equations used for a study of critical collapse in [5]).

In addition, we will describe a detailed no-go theorem based on a minisuperspace description of the static Schwarzschild exterior by a homogeneous time-like slicing, as originally proposed for a different purpose in [6].

3. Dressed-Metric Approach

In classical gravity, as is well known, it is possible to describe cosmological inhomogeneity in the early universe as a coupled system of two independent sets of degrees of freedom, given by inhomogeneous perturbations evolving on a homogeneous background with THE choice of a time coordinate (such as proper time or conformal time). In a discussion of possibly modified dynamics and space–time structure, it is important to remember that these two ingredients, background and perturbations, have rather different properties related to covariance.

3.1. Background and Perturbations

The dynamics of any homogeneous background can be modified without violating covariance because there is a single constraint, (3), which is always consistent with itself in any modified form: because $\{C, C\} = 0$ for any Poisson bracket, Hamilton's equations generated by a constraint C are guaranteed to preserve the constraint equation $C = 0$ imposed on initial values.

Applied to the Friedmann constraint C, we generate equations of motion:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \{f, NC\} \tag{5}$$

for any phase-space function f, with respect to a time coordinate t indirectly determined by the lapse function $N > 0$. The generic time derivative, applied to solutions of the constraint $C = 0$, can be rewritten as

$$\frac{1}{N}\frac{\mathrm{d}f}{\mathrm{d}t} = \{f, C\} = \frac{\mathrm{d}f}{\mathrm{d}\tau} \tag{6}$$

introducing proper time τ in the last step by the usual definition $\mathrm{d}\tau = N\mathrm{d}t$.

All allowed choices of time coordinates (monotonically related to τ) can therefore be described by a single line element:

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + \tilde{a}(\tau)^2\mathrm{d}\sigma^2 \tag{7}$$

where $\tilde{a}(\tau)$ denotes the scale factor subject to potentially modified dynamics, and $\mathrm{d}\sigma^2$ is a standard isotropic spatial line element. Because the definition of τ implies that the line element is correctly transformed to:

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + \tilde{a}(t)^2\mathrm{d}\sigma^2 \tag{8}$$

for any other time coordinate t, there is a suitable way to describe any modified homogeneous dynamics, subject to a single constraint, by a space–time geometry that is invariant with respect to the full coordinate changes allowed by the symmetry, given by reparameterizations of time.

Coordinate changes are more involved in the case of spatial inhomogeneity because several independent coordinates may be related by transformations. In the canomical language of constraints, the presence of a multitude of independent ones, one Hamiltonian constraint per spatial point as well as diffeomorphism constraints, which implies that a modification of one or more constraints no longer implies the consistency of their Hamiltonian flows with respect to the other constraints. Since the relevant constraints implement space–time transformations, a dedicated space–time analysis then becomes important.

For small, perturbative inhomogeneity, there is a standard way to describe curvature perturbations in terms of combinations of metric and matter fields that are invariant with respect to small coordinate changes [12]. However, compared with the reparameterizations of time relevant for the background, it is much harder to derive a suitable invariant line element extending (8) in a way that is consistent with Hamilton's equations generated by modified constraints for perturbative inhomogeneity. In fact, the standard derivations of curvature perturbations [12,13] as well as the canonical version given in [14] assume that space–time is of its classical form, for instance by directly working with the coordinate substitutions in a line element. A modified or quantum treatment then cannot take it for granted that the form of these curvature perturbations remains unchanged, because the space–time structure itself may be modified in quantum gravity.

The dressed-metric approach proceeds by quantizing standard curvature perturbations on a modified background, leading to wave equations for perturbations on a modified background line element $\mathrm{d}s^2 = \tilde{g}_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta$ of the form (8). The approach therefore implicitly assumes that space–time structure remains classical even while the dynamics of at least the background are modified. Upon closer inspection, this assumption turns out to be unjustified.

3.2. *The Metric's New Clothes*

As already pointed out in [15], Bardeen variables or curvature perturbations are "gauge invariant" under small coordinate changes, but not necessarily under all coordinate changes relevant for a given cosmological situation. In particular, in cosmological models of perturbative inhomogeneity, we also need invariance under potentially large background transformations of time, such as transforming from proper time to conformal time.

Small coordinate changes of perturbations and large reparameterizations of background time are not independent of each other. Algebraically, they form a semidirect product rather than a direct one, as shown in [16]. The non-trivial interplay between these transformations can be deduced from vector-field commutators such as:

$$\left[f(t)\frac{\partial}{\partial t}, \xi^\alpha \frac{\partial}{\partial x^\alpha}\right] = f\dot{\xi}^\alpha \frac{\partial}{\partial x^\alpha} - \dot{f}\xi^0 \frac{\partial}{\partial t} \tag{9}$$

which in general are not zero (in contrast to what a direct product would imply) but rather form a small inhomogeneous transformation. This interplay is a general property of perturbations in Riemannian geometry, as encoded in line elements suitable for perturbative inhomogeneity.

The applicability of standard line elements requires the precise algebra of coordinate transformations to be modeled by gauge transformations in a canonical formulation of any gravity theory. However, while the dressed-metric approach assumes the availability of standard line elements with the usual coordinate dependence (but possibly modified metric coefficients), it violates the algebraic condition by its independent treatment of background and perturbations: quantizing the background separately from the perturbations evolving on it implicitly assumes a direct product of coordinate changes. Writing a line element $\mathrm{d}s^2 = \tilde{g}_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta$ based on modified metric components $\tilde{g}_{\alpha\beta}$ in a dressed-metric model is therefore meaningless.

3.3. *Effective Line Element*

Because a line element $\mathrm{d}s^2 = g_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta$ is defined as the square of an infinitesimal distance, it can be meaningful as a description of geometry only if it is independent of coordinate choices that affect $\mathrm{d}x^\alpha$ as well as $g_{\alpha\beta}$. For $\mathrm{d}s^2$ to be invariant, the metric coefficients $g_{\alpha\beta}$ must be subject to the standard tensor-transformation law:

$$g_{\alpha'\beta'} = \frac{\partial x^\alpha}{\partial x^{\alpha'}} \frac{\partial x^\beta}{\partial x^{\beta'}} g_{\alpha\beta} \tag{10}$$

if coordinates x^α are transformed to $x^{\alpha'}$.

Canonical quantization in its usual form, as applied in models of loop quantum gravity, does not modify space–time coordinates x^α and their transformations, but it may alter the equations of motion (with respect to these coordinates) for the spatial metric q_{ij} in the generic canonical line element:

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + q_{ij}(\mathrm{d}x^i + M^i\mathrm{d}t)(\mathrm{d}x^j + M^j\mathrm{d}t)\,. \tag{11}$$

Modifications of the remaining components, the lapse function N and shift vector M^i, are also determined by canonical equations, although more indirectly because N and M^i do not have unconstrained momenta. In the presence of modifications, altered equations for q_{ij}, N and M^i must remain consistent with coordinate transformations if an effective line element $\mathrm{d}s^2$ is to be meaningful.

A crucial ingredient in a canonical analysis of covariance is therefore given by the transformations of N and M^i, in addition to the more obvious transformations of q_{ij}. The full set of canonical transformations makes use of the specific properties of the constraints of the theory. At this point, the analysis of geometrical properties relevant for effective line elements benefits from a discussion of hypersurface deformations in space–time, which are generated from the constraints. While properties of hypersurface deformations constitute some of the classic results in canonical general relativity [17–21], they do not appear to be widely known. What follows is a construction of hypersurface deformations based on elementary properties of special relativity.

3.3.1. Hypersurface Deformations

In special relativity, an observer moving at speed v assigns new coordinates to events in space–time according to a Lorentz transformation:

$$x' = \frac{x - vt}{\sqrt{1-v^2}} \quad , \quad t' = \frac{t - vx}{\sqrt{1-v^2}}\,. \tag{12}$$

Interpreting this transformation as a linear deformation of axes in a space–time diagram, as shown in Figure 1, the set of all Poincaré transformations can be geometrically represented by linear hypersurface deformations with respect to lapse functions $N(\mathbf{x}) = \mathbf{\Delta t} + \mathbf{v}\cdot\mathbf{x}$ (deformations in the normal direction of a spatial slice) and shift vector fields $\mathbf{M}(\mathbf{x}) = \mathbf{\Delta x} + \mathbf{R}\mathbf{x}$ (tangential deformations within a spatial slice). The parameters in these expressions for linear lapse functions and shift vector fields determine a time translation Δt, a boost velocity $\mathbf{v}$, a spatial shift $\Delta\mathbf{x}$ and a spatial rotation matrix $\mathbf{R}$.

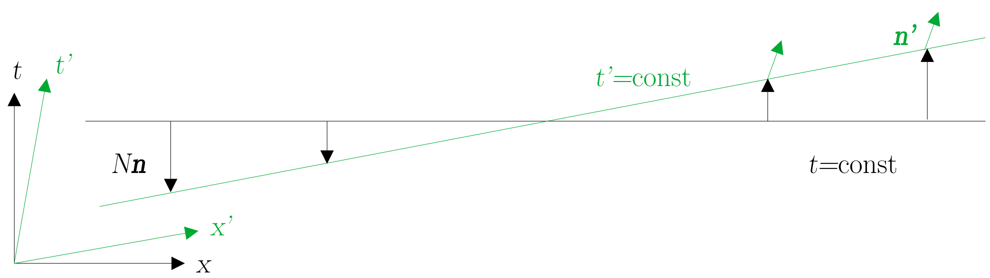

Figure 1. A Lorentz transformation in Minkowski space–time, shown in the traditional way by means of axes as well as in terms of linear normal deformations of a spatial slice. A slice t = const in the original coordinate system was transformed to a new spatial slice t' = const by a linear deformation with position-dependent displacement $N(x) = N_0 + vx$ along the unit normal vector field $\mathbf{n}$.

We extend these considerations to general relativity by replacing the restricted set of translations, rotations and Lorentz boosts with arbitrary non-linear coordinate changes. Correspondingly, hypersurfaces are subject to non-linear deformations [17]. Infinitesimal hypersurface deformations in Riemannian space–time, split into "temporal" deformations

$T(N)$ in a normal direction and "spatial" deformations $S(\mathbf{M})$ in tangential directions, can be shown to obey the commutators:

$$[S(\mathbf{M_1}), \mathbf{S}(\mathbf{M_2})] = S((\mathbf{M_1} \cdot \nabla)\mathbf{M_2} - (\mathbf{M_2} \cdot \nabla)\mathbf{M_1}) \quad (13)$$
$$[T(N), S(\mathbf{M})] = -T(\mathbf{M} \cdot \nabla \mathbf{N}) \quad (14)$$
$$[T(N_1), T(N_2)] = S(N_1 \nabla N_2 - N_2 \nabla N_1) \quad (15)$$

when they are applied in two alternative orderings. A visualization is shown in Figure 2. The brackets (13)–(15) represent general covariance in canonical form. While specific expressions for S and T can vary depending on the gravitational theory, such as different higher-curvature actions [22], the brackets remain the same as long as the underlying geometry of space–time is Riemannian. Conversely, deviations of the brackets from their Riemannian form can be used to detect non-classical space–time structures in modified canonical gravity. The algebraic nature of the brackets makes it possible to analyze gravitational theories without presupposing specific geometrical formulations of space–time, constituting a major strength of the canonical approach.

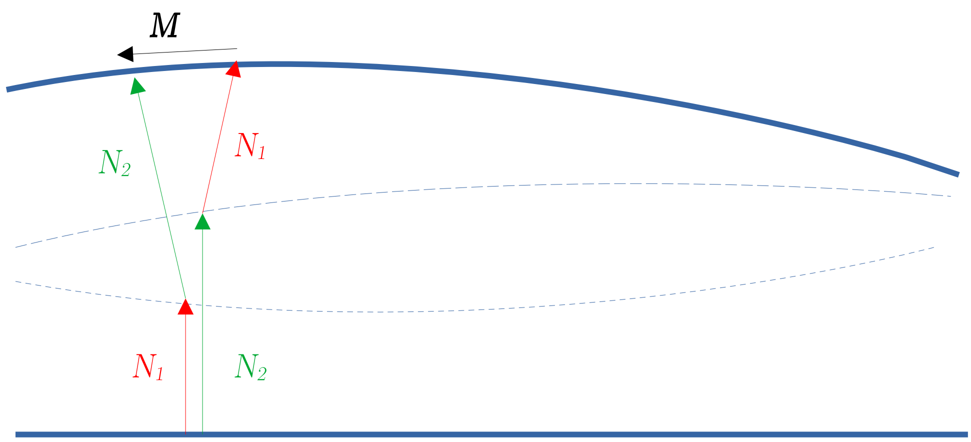

Figure 2. Two non-linear normal deformations, one with a lapse function N_1 and one with a lapse function N_2, applied in two different orderings, show the commutator (15) given by a spatial displacement **M**.

Figure 3 represents the commutator of an infinitesimal time translation and an infinitesimal normal deformation. This picture can be interpreted as a version of the vector-field commutator (9) of a background transformation and a small perturbative transformation. The non-zero result of (9) corresponds to the presence of a spatial shift on the right-hand side of Figure 3. Even though there is no immediate time dependence of the canonical data on which a background vector field as in (9) would act, the semidirect product of background and perturbative transformations is clear. In canonical language, the failure of the dressed-metric approach to realize the correct semidirect product means that there is no common $T(N)$ for background and perturbations in this setting. The non-existence of consistent temporal deformations signals the break-down of space–time and covariance.

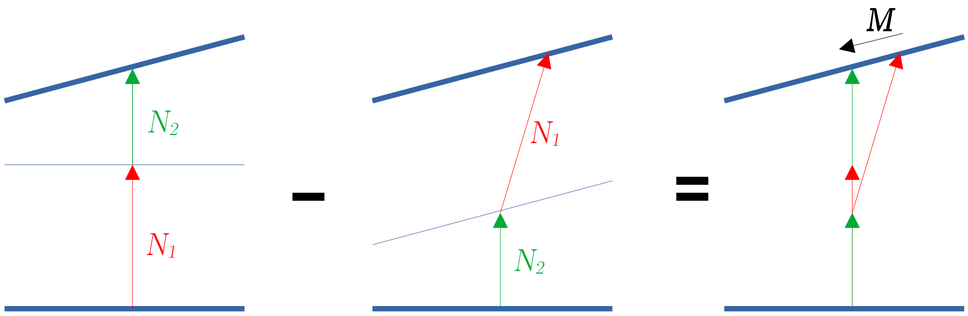

Figure 3. Semidirect product of time reparameterizations and inhomogeneous transformations as in (9), represented in the picture of hypersurface deformations: The commutator of two such normal deformations produces a non-zero spatial shift **M**.

3.3.2. Structure Functions

The brackets of hypersurface deformations have structure functions because the gradient in (15) requires the use of the spatial metric, and therefore depend on the geometry described by these brackets. A canonical realization of these brackets is given by the Hamiltonian and diffeomorphism constraints, $H[N]$ and $D[M^i]$, of a given gravity theory. Written in the form:

$$\{D[M_1^i], D[M_2^j]\} = D[[M_1, M_2]^i] \quad (16)$$
$$\{H[N], D[M^i]\} = -H[M_1^i \nabla_i N] \quad (17)$$
$$\{H[N_1], H[N_2]\} = D[q^{ij}(N_1\nabla_j N_2 - N_2\nabla_j N_1)]\,, \quad (18)$$

they make the appearance of structure functions explicit, depending on the inverse spatial metric q^{ij}. Formally, we may write the constraint brackets as $\{C_A, C_B\} = F^D_{AB} C_D$ with indices A, B and D that combine spatial positions with the type of constraint (Hamiltonian or a component of the diffeomorphism constraint). The coefficients F^D_{AB} are not constants but phase-space functions.

The presence of structure functions causes long-standing problems in the quantization of canonical gravity [23,24]: upon quantization, q^{ij} as well as D and H are turned into operators. Maintaining closed brackets therefore requires specific ordering, regularization, or other choices. Even if the brackets can remain closed under certain conditions, quantized structure functions may be quantum corrected. A question relevant for covariance is then that of whether a meaningful interpretation of the generators as hypersurface deformations in space–time still exists.

As shown in [25], a meaningful space–time interpretation does exist at least in some cases of modified structure functions. To see this, it is necessary to construct a space–time line element that is consistent with the modified gauge transformations generated by (18) with the quantum-corrected structure functions. If these functions are modified, so are the versions of hypersurface deformations they represent, and therefore the objects q_{ij}, N and M^i in which the brackets are formulated, do not directly define the components of a meaningful line element because this notion is based on classical space–time with standard hypersurface deformations. However, in some cases, suitable redefinitions of the canonical fields are available that can serve this purpose.

A derivation of proper effective line elements is based on the general property of Hamiltonian and diffeomorphism constraints as generators of evolution equations, giving the time derivative:

$$\dot{f} = \mathcal{L}_t f = \{f, H[N] + D[M^i]\} \quad (19)$$

of any phase-space function f with respect to the time-evolution vector field $t^\alpha = Nn^\alpha + M^\alpha$. (The space–time vector field M^α is the push-forward of the spatial vector field M^i by the embedding map of a spatial slice in space–time). In addition, the constraints generate gauge transformations:

$$\delta_\epsilon f = \{f, H[\epsilon] + D[\epsilon^i]\} \tag{20}$$

which would correspond to coordinate changes generated by the vector field $\xi^\alpha = \epsilon n^\alpha + \epsilon^\alpha$ if structure functions were unmodified.

In all cases—modified and unmodified structure functions—evolution equations and gauge transformations must be consistent with each other: a gauge-transformed f must evolve according to the general Equation (19) with the same generators H and D as the original f, but possibly with a new time-evolution vector field. Since the direction of the time-evolution vector field within a given theory is determined by lapse and shift, this consistency condition can be used to derive gauge transformations for N and M^i. Together with the gauge transformations of q_{ij}, directly determined by (20) because q_{ij} are phase-space functions, all components of a candidate space–time line element can therefore be unambiguously transformed.

For generic structure functions F^D_{AB}, evolution and gauge transformations are consistent with each other, provided the multipliers $(N^A) = (N, M^i)$ gauge transform according to [26]:

$$\delta_\epsilon N^A = \dot{\epsilon}^A + N^B \epsilon^C F^A_{BC}\,. \tag{21}$$

Unlike in the case of $\delta_\epsilon q_{ij} = \{q_{ij}, H[\epsilon] + D[\epsilon^i]\}$, the structure functions appear explicitly in (21). Structure functions, and their possible modifications, are therefore directly relevant for space–time structure and the existence of meaningful effective line elements:

$$\mathrm{d}s^2 = -\tilde{N}^2 \mathrm{d}t^2 + \tilde{q}_{ij}(\mathrm{d}x^i + \tilde{M}^i \mathrm{d}t)(\mathrm{d}x^j + \tilde{M}^j \mathrm{d}t) \tag{22}$$

which may require field redefinitions of $\tilde{N}$, $\tilde{M}^i$ as well as $\tilde{q}_{ij}$ if the structure functions F^A_{BC} are modified.

3.4. Lessons from Hypersurface Deformations

In canonical models of modified gravity, control on space–time structure requires full expressions for the Hamiltonian constraint $H[N]$ and the diffeomorphism constraint $D[M^i]$ with closed brackets. This condition is violated in the dressed-metric approach (as well as in hybrid loop quantum cosmology) because the independent treatment of remnant coordinate freedom in background and perturbations, the former through deparameterization and the latter by using curvature perturbations, precludes the construction of joint constraints for both sets of degrees of freedom. The common assumption that space–time in this setting can still be described by a line element, presupposing a Riemannian structure of space–time, is therefore unjustified. Detailed discussions of the underlying modifications of contributions to the Hamiltonian constraint from background and perturbations show that the implicit assumption of unmodified brackets, and thus Riemannian structures, is inconsistent [16].

For a consistent space–time structure, the gauge behavior of the classical theory must remain intact, even while it may be modified and subject to quantum effects. In general, this condition requires anomaly freedom, such that the same number of physical degrees of freedom as in classical gravity is realized in a modified version. If this condition is violated, the modified theory cannot have the correct classical limit owing to a discontinuity in the number of degrees of freedom. An anomalous modification or quantization of gravity does not permit a semiclassical or effective treatment by line elements in any form because it is incompatible with the gauge structure of space–time.

A formal statement of the condition that the gauge behavior remains intact is the existence of closed Poisson brackets of $H[N]$ and $D[M^i]$ for all relevant N and M^i, depending on whether one considers the full theory or a restricted version such as a midisuperspace model. This condition allows for possible quantum corrections in the structure functions of the gauge algebra, given in the case of gravity by the inverse spatial metric q^{ij} as it appears in:

$$\{H[N_1], H[N_2]\} = D[\beta(q,p)q^{ij}(N_1\nabla_j N_2 - N_2\nabla_j N_1)] \tag{23}$$

with a possible modification function $\beta(q,p)$ on phase space. We have the classical space–time structure if $\beta = \pm 1$, giving two possible choices of the signature of a classical four-dimensional metric, where $\beta = 1$ for Lorentzian-signature space–time and $\beta = -1$ for 4-dimensional Euclidean-signature space. (In each case, the name only refers to the signature and does not imply flatness).

We have a consistent non-classical space–time structure if the brackets are closed such that $\beta \neq \pm 1$. The modification function β determines the structure functions of hypersurface-deformation brackets in the modified theory. Modified structure functions, in turn, show via (21) how lapse and shift transform and whether it is possible to find suitable field redefinitions of these fields that can be used in a proper effective line element as discussed in detail in [25].

As we saw in the present section, suitable transformations of lapse and shift as components of the space–time metric require knowledge of the structure functions of $H[N]$ and $D[M^i]$. If the brackets do not close, as in the dressed-metric approach, there are no meaningful transformations of lapse and shift and it is impossible to construct a valid structure of space–time. Such a structure exists only in anomaly-free modifications of the constraints. However, the condition of anomaly-freedom is not sufficient if it does not imply a clear modification of the structure function of hypersurface-deformation brackets, for instance in cases in which the constrained system is reformulated before it is modified or quantized. An example for such an approach is given by a partial Abelianization of the constraints [4], to which we turn next.

4. Spherical Symmetry

An instructive set of examples is given by spherically symmetric space–time geometries with the line element:

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + L^2(\mathrm{d}x + M\mathrm{d}t)^2 + S^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2) \tag{24}$$

where N, L, M and S are functions of t and x. Together with the momenta p_L and p_S of L and S, respectively, the components L and S of the spatial metric in classical general relativity are subject to the Hamiltonian constraint:

$$H[N] = \int_{-\infty}^{\infty} N\left(-\frac{p_L p_S}{S} + \frac{L p_L^2}{2S^2} + \frac{(S')^2}{2L} + \frac{SS''}{L} - \frac{SS'L'}{L^2} - \frac{L}{4}\right)\mathrm{d}x \tag{25}$$

and the diffeomorphism constraint:

$$D[\epsilon] = \int_{-\infty}^{\infty} \epsilon\left(p_S S' - L p_L'\right)\mathrm{d}x\,. \tag{26}$$

The relevant bracket with a stucture function is given by

$$\{H[N_1], H[N_2]\} = D[L^{-2}(N_1 N_2' - N_2 N_1')\,. \tag{27}$$

4.1. Reformulating the Constrained System

In [4], a reformulation of the constraints has been suggested that can remove the structure function and even partially Abelianize the brackets. Instead of $H[N]$, this reformulation uses the linear combination:

$$H[2PS'/L] + D[2Pp_L/(SL)] = \int_{-\infty}^{\infty} P\frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{p_L^2}{S} + \frac{S(S')^2}{L^2} - S\right)\mathrm{d}x \tag{28}$$

of Hamiltonian and diffeomorphism constraints. Specifically, the combination replaces $H[N]$ with a new constraint whose integrand (except for the multiplier P) is a complete derivative. Imposing (28) as a constraint therefore requires that the parenthesis in this expression equals a constant, C_0. The same condition can be expressed by the alternative constraint:

$$C[Q] = \int_{-\infty}^{\infty} Q\left(-\frac{p_L^2}{S} + \frac{S(S')^2}{L^2} - S - C_0\right) \mathrm{d}x\,. \tag{29}$$

(The constant can be related to boundary values). Because $C[Q]$ depends neither on p_S nor on spatial derivatives of L, it is easy to see that two such constraints always have a vanishing Poisson bracket, unlike two Hamiltonian constraints. Together with the original diffeomorphism constraint, we have the brackets:

$$\{C[Q], D[\epsilon]\} = -C[(\epsilon Q)'] \quad , \quad \{C[Q_1], C[Q_2]\} = 0 \tag{30}$$

free of structure functions. Therefore, it may be expected that using the reformulated constraints greatly simplifies the quantization procedure or the derivation of viable modifications.

However, the reformulation has made use of metric-dependent coefficients S'/L and $p_L/(SL)$ in (28). In general, it is not clear whether these coefficients will be subject to quantum corrections, in which case it may be difficult or impossible to reconstruct valid hypersurface-deformation brackets with the correct classical limit from a quantization or modification of the system (30). The non-trivial nature of this question has been shown in [27] and the related [28], where examples were presented in which (30) can easily be modified while no hypersurface-deformation brackets can be reconstructed at all or only in modified form.

For instance, the modification:

$$C_f[Q] = \int_{-\infty}^{\infty} Q\left(-\frac{f(p_L)^2}{S} + \frac{S(S')^2}{L^2} - S - C_0\right) \mathrm{d}x \tag{31}$$

with a free function $f(p_L)$, such as $\sin(\ell p_L)/\ell$ where ℓ is a suitable length scale analogous to (3), and an unchanged $D[\epsilon]$ maintains the brackets (30) and is therefore anomaly-free in the reformulated system. By reverting the steps undertaken in (28), it can be seen that (31) corresponds to the modified Hamiltonian constraint:

$$H_f[N] = \int_{-\infty}^{\infty} N\left(-\frac{p_S}{S}\frac{\mathrm{d}f(p_L)}{\mathrm{d}p_L} + \frac{Lf(p_L)}{2S^2} + \frac{(S')^2}{2L} + \frac{SS''}{L} - \frac{SS'L'}{L^2} - \frac{L}{4}\right) \mathrm{d}x\,. \tag{32}$$

This modification of the Hamiltonian constraint, which has already been found in [29], also turns out to be anomaly-free, but with a modified bracket:

$$\{H_f[N_1], H_f[N_2]\} = D[\beta(p_L)L^{-2}(N_1N_2' - N_2N_1')] \tag{33}$$

where:

$$\beta(p_L) = \frac{1}{2}\frac{\mathrm{d}^2 f}{\mathrm{d}p_L^2}\,. \tag{34}$$

The modified structure function is an example of signature change because β is negative around any local maximum of f.

If spherically symmetric gravity is coupled to a scalar field, the partial Abelianization of [4] is still available and can be modified as in (31). However, in this case, there is no consistent set of hypersurface-deformation generators [27]. Therefore, the modified theory is formally consistent but not geometrical: its solutions cannot be described by Riemannian geometry or effective line elements, even after a field redefinition. This problem poses a significant challenge to loop quantization because an application to vacuum models would only be too restrictive. Moreover, the problem is broader because polarized Gowdy models, which can also be partially Abelianized, do not admit a consistent set of modified hypersurface-deformation brackets [28]. To date, therefore, midisuperspace models with local physical degrees of freedom cannot be geometrically described in the presence of loop modifications.

4.2. Non-Bijective Canonical Transformation

To circumvent this problem, ref. [11] proposed a modification of spherically symmetric gravity based on a non-bijective canonical transformation:

$$p_L = \frac{\sin(\ell \tilde{p}_L)}{\ell} \quad , \quad L = \frac{\tilde{L}}{\cos(\ell \tilde{p}_L)} \,. \tag{35}$$

The transformation can be applied to the Abelianized constraint $C[Q]$ or to the Hamiltonian constraint by inserting $p_L(\tilde{p}_L)$ and $L(\tilde{L}, \tilde{p}_L)$ in their classical expressions. (The diffeomorphism constraint is not modified by this transformation.) Terms depending on p_L in $C[Q]$ are then modified as before in (31) with a specific version of $f(p_L)$, and there are new modifications in the L-term. As postulated in [11], this procedure, based on a canonical transformation, might be able to preserve the covariance of the classical theory even in the presence of a scalar field, and yet allow room for new quantum effects because of the non-bijective nature of the canonical transformation.

Unfortunately, this hope remains unfulfilled precisely because the transformation is not bijective [30]. In particular, the bijective nature breaks down at hypersurfaces defined by $\ell p_L = \pm 1$ or $\ell \tilde{p}_L = (n + 1/2)\pi$, and p_L as well as $\tilde{p}_L$ are spatial scalars but not space–time scalars. Therefore, while the transformation preserves symmetries of the classical theory when it can be restricted to regions of phase space in which it is bijective, these regions themselves are defined in terms that are not space–time covariant. The resulting theory is not covariant.

For the same reason, $\tilde{p}_L$ not being a space–time scalar, the variable $\tilde{L}$ introduced by the canonical transformation does not have the same behavior as $L = \tilde{L}/\cos(\ell \tilde{p}_L)$ under space–time transformations. As a consequence, $\tilde{L}$ cannot be used in a space–time line element based on $\tilde{L}^2 dx^2$. A meaningful effective line element is obtained only after a suitable field redefinition that leads to a function of $\tilde{L}$ with the correct transformation properties. Since we already know that $\tilde{L}$ was derived from such a function, L, the field redefinition simply sends us back from $\tilde{L}$ to L in regions in which the canonical transformation is invertible, undoing the modification of the theory in such regions. (More systematically, such a field redefinition can be derived using the methods introduced in [31].) In these regions, exact classical solutions without any modifications are produced, but different regions are connected along hypersurfaces (again, given by $\ell p_L = \pm 1$ or $\ell \tilde{p}_L = (n + 1/2)\pi$) that are not covariantly defined. Since these hypersurfaces refer to fixed values of certain components of extrinsic curvature, their positions in space–time depend on choices of coordinates and spatial slicings.

In particular, slicings with large $p_L \sim 1/\ell$ exist even in flat space–time, and therefore violations of covariance in this model cannot be considered a "large-curvature effect". These violations can occur at a low space–time curvature (in an invariant meaning), and therefore the model cannot be considered a permissible model of quantum gravity that would have non-standard geometrical features only at the Planck scale. The model could be permissible only if it were combined with a mechanism that somehow prevents one from choosing slicings that lead to large extrinsic curvature p_L. However, preventing such slicings (or any slicing) from being allowed requires violations of covariance that are hard to reconcile with the application of line elements, even if they were only used in low-curvature regions.

4.3. Bijective Canonical Transformation

As discussed in more detail in [30], the application of canonical transformations makes an analysis of space–time structure rather non-trivial even if the transformation is bijective. A bijective canonical transformation from (L, p_L) to some $(\tilde{L}, \tilde{p}_L)$ may well be such that all possible values of p_L are mapped to a finite range of $\tilde{p}_L$. One could then conclude that the transformed theory resolves singularities if $\tilde{p}_L$, interpreted as some curvature expression in the new theory, remains bounded. However, the new theory was obtained by applying a bijective canonical transformation that cannot modify the physics of classical spherically symmetric models.

The answer to this conundrum relies on effective line elements. For a transformation with a significantly modified $\tilde{p}_L$ to be canonical, $\tilde{L}$ must also be modified compared with L. Then, the structure function in (27) is modified when expressed in terms of $\tilde{L}$ instead of L, and solutions of the transformed theory cannot be directly interpreted in terms of a line element where $\tilde{L}$ directly takes the place of L. An effective line element, derived again as in [31]), requires the undoing of the canonical transformation for a valid coefficient of $\mathrm{d}x^2$, sending us back to the classical theory in its geometrical interpretation.

Models of loop quantum gravity are not obtained by bijective canonical transformations and could lead to new physics. However, the example of a bijective canonical transformation demonstrates that predictions can only be reliable if a proper effective line element is derived. Unfortunately, this task is rarely performed in phenomenological studies of models of loop quantum gravity. In several proposals, as in the dressed-metric approach, it is even impossible to construct an effective line element because they do not amount to consistent modifications of the crucial bracket (27) that determines the structure of space–time.

5. Homogeneity in Schwarzschild Space–Time

It is well known that a spatially homogeneous geometry of Kantowski–Sachs type [32], with the line element:

$$\mathrm{d}s^2 = -N(t)^2\mathrm{d}t^2 + a(t)^2\mathrm{d}x^2 + b(t)^2\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2\right) \tag{36}$$

is realized in the Schwarzschild interior—in the (almost) standard version:

$$\mathrm{d}s^2 = -(1-2M/r)\mathrm{d}\tilde{t}^2 + \frac{\mathrm{d}r^2}{2M/r-1} + r^2\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2\right) \tag{37}$$

of the Schwarzschild line element, $\tilde{t}$ is a time coordinate only for $r > 2M$, outside of the horizon. For $r < 2M$, the coordinate r may be used as time while $\tilde{t}$ contributes to a positive, space-like part of the line element. Indicating the modified roles of the coordinates in the notation, we define $t = r$ and $x = \tilde{t}$ for $r < 2M$, such that the line element turns into:

$$\mathrm{d}s^2 = -\frac{\mathrm{d}t^2}{2M/t-1} + (2M/t-1)\mathrm{d}x^2 + t^2\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2\right) \tag{38}$$

for $t < 2M$. A suitable identification of $N(t)$, $a(t)$ and $b(t)$ shows that this line element is of the general form (36).

The coordinates t and x determine a homogeneous space-like slicing in the interior of Schwarzschild space–time. It is therefore possible to apply minisuperspace quantizations to the interior region. However, such models do not show how a modified quantum interior may be connected to an inhomogeneous exterior, and they do not reveal properties of space–time structure (let alone physical processes such as occasionally hypothesized explosions of black holes).

5.1. Time-Like Homogeneity of Exterior Static Solutions

A complex canonical transformation $A = ia$ and $p_A = -ip_a$ together with $n = iN$ in (36) implies a Kantowski–Sachs line element of the form:

$$\mathrm{d}s^2 = n(t)^2\mathrm{d}t^2 - A(t)^2\mathrm{d}x^2 + b(t)^2\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2\right). \tag{39}$$

The complex transformation has the same effect as crossing the horizon in the Schwarzschild geometry: it flips the roles of t and x as time and space coordinates. Defining $X = t$ and $T = x$, the transformed line element (40) takes the form:

$$ds^2 = -A(X)^2 dT^2 + n(X)^2 dX^2 + b(X)^2 \left(d\vartheta^2 + \sin^2 \vartheta d\varphi^2 \right). \quad (40)$$

The exterior Schwarzschild line element:

$$ds^2 = -(1 - 2M/X) dT^2 + \frac{dX^2}{1 - 2M/X} + X^2 \left(d\vartheta^2 + \sin^2 \vartheta d\varphi^2 \right) \quad (41)$$

with $X > 2M$ is now of this general form. In particular, the coordinates T and X determines a homogeneous time-like slicing in the exterior. Methods of minisuperspace quantization can therefore be applied even to inhomogeneous geometries [6], possibly leading to modified space–time structures.

Symmetries of individual space–time solutions such as homogeneity, as opposed to general covariance which relates different solutions of the underlying partial differential equations, are built into the setup of the model. Therefore, they are preserved by minisuperspace quantization. Time-like homogeneity then remains intact for any modified dynamics in this setting. As shown in Figure 4, time-like homogeneity with the given number of degrees of freedom, in turn, implies the existence of a static spherically symmetric configuration if the resulting theory is covariant and slicing-independent (described by a meaningful line element). Since the black-hole analysis of [6] is based on line elements and refers to notions of Riemannian geometry, such as horizons, curvature scalars or Penrose diagrams, slicing independence is one of the ingredients of the construction and does not need to be assumed independently. It must therefore be possible to formulate the same physics claimed in [6] for a homogeneous time-like slicing also within a covariant spherically symmetric theory, restricted to static solutions.

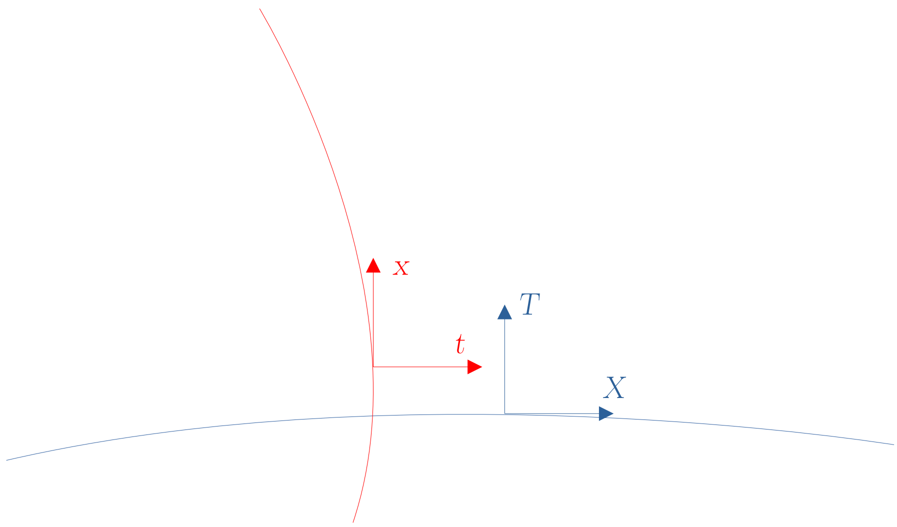

Figure 4. A homogeneous time-like slicing with coordinates (t, x) and an inhomogeneous space-like slicing with coordinates (T, X), both in the same static spherically symmetric space–time.

Covariant versions of spherically symmetric gravity models and their static solutions are under good control, thanks to work on dilaton gravity [33,34] and its generalizations [35,36]. It is therefore possible to check whether a proposed modification of the homogeneous time-like slicing has a chance of corresponding to a covariant theory.

5.2. Line Elements

Time-like homogeneity with modified dynamics leads to a formal line element:

$$\mathrm{d}s^2 = \tilde{n}(t)^2\mathrm{d}t^2 - \tilde{A}(t)^2\mathrm{d}x^2 + \tilde{b}(t)^2\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2\right) \tag{42}$$

if solutions $\tilde{n}$, $\tilde{A}$ and $\tilde{b}$ are simply inserted in the classical line element. Since properties of space–time transformations have not been checked at this point, there is no guarantee that (42) presents a proper effective line element.

Assuming that the Kantowski–Sachs-like (42) is a proper line element that describes a slicing-independent theory, it is equivalent to the Schwarzschild-like:

$$\mathrm{d}s^2 = -K(X)^2\mathrm{d}T^2 + L(X)^2\mathrm{d}X^2 + S(X)^2\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2\right) \tag{43}$$

where $X = t$, $T = x$ and:

$$\tilde{A} = K \quad , \quad \tilde{b} = S \quad , \quad \tilde{n} = L\,. \tag{44}$$

By construction, the coefficients in (43) depend on X but not on T. The line element therefore presents a static solution in a spherically symmetric model, subject to some modified dynamics because K, L and S are only Schwarzschild-like but not exactly of Schwarzschild form if the dynamics of the underlying homogeneous model is modified.

If the assumption of covariance, made implicitly in [6] is justified, (43) must be a solution of a $1+1$-dimensional gravity model in terms of time and space coordinates (T, X). Such theories are under strong control: all local covariant theories of this midisuperspace form are known as generalized dilaton gravity models [35]. (Their equivalence to Horndeski theories in $1+1$ dimensions has been shown in [36]). While several-free functions exist in this general setting to specify the dynamics, for instance through an action, as they can only depend on the variable analogous to our field S. Loop quantum cosmology applied to the homogeneous time-like slicing, however, implies modifications that do not fulfill this condition: such minisuperspace modifications depend non-linearly on momenta $p_{\tilde{A}}$ and $p_{\tilde{b}}$, which are linear combinations of $\mathrm{d}\tilde{A}/\mathrm{d}t$ and $\mathrm{d}\tilde{b}/\mathrm{d}t$ that, according to (44), are translated to $\partial K/\partial X$ and $\partial S/\partial X$ in the spherically symmetric slicing. Therefore, no holonomy modified dynamics of Kantowski–Sachs-style models can be part of a covariant space–time theory [37].

6. Conclusions

We discussed the main constructions that were supposed to circumvent difficulties in earlier applications of loop quantization to inhomogeneous models. Instead of solving older problems, however, these constructions led to no-go results for covariance in models of loop quantum gravity. A complete understanding of covariance in any given model is important not only to demonstrate its consistency, but also to evaluate possible observational implications of the underlying theory. For instance, if one neglects the identification of suitable space–time structures for a model of modified or quantum gravity, one could be led to posing initial conditions at an inadmissible place where there is, in fact, no meaningful version of time. A detailed space–time analysis may well show other regions in which initial values could reliably be posed, however, the altered location, perhaps at a different range of curvature values, would affect implied phenomenological effects. Addressing such questions requires an understanding of different ways in which covariance can be violated, which we compare in the next subsection. The final two subsections will discuss general implications for loop quantum gravity and a brief outlook on covariance in other approaches.

6.1. Comparison of Different Violations of Covariance

The examples reviewed in the preceding sections show different ways in which covariance can be violated in models of loop quantum gravity. The dressed-metric approach, just as hybrid loop quantum cosmology, is based on the incorrect assumption that background and perturbations can be quantized or modified independently in an inhomogeneous model. This assumption ignores a crucial feature of space–time and covariance, according to which background and perturbative transformations form a semidirect product but not a direct one as an independent treatment would require. The fundamental nature of this property implies that covariance is completely broken in these models, which are therefore inconsistent as a description of (quantum) space–time.

As usual, one may expect that space–time is non-classical at large curvature and may exhibit properties different from classical space–time. However, this expectation does not redeem quantum models that violate covariance unless they can demonstrate that the classical properties are recovered in a suitable classical limit. Moreover, the dressed-metric and the hybrid approach both refer to features of classical space–time, such as line elements or curvature perturbations, even close to the Planck curvature.

The inconsistency of these approaches is rooted not so much in possible modifications of classical space–time properties near the Planck curvature, but rather in the unquestioned (and often implicit) application of classical space–time ingredients for an analysis in this regime. For a model to be consistent, such an assumption must be justified, but this crucial step has not been attempted in the dressed-metric and hybrid approaches. There is therefore reason to doubt the validity of these constructions and their implications.

The technical observation that a key property of classical space–time is violated, given by the semidirect-product nature of transformation, serves as a concrete property that turns this doubt into a proof that the models are inconsistent, not only in the Planck regime but to any order in a semiclassical expansion by $\hbar$ or $\ell_{\rm P}$. Consistency is recovered only in the strict limit of $\hbar \to 0$, just because we happen to know that the classical theory is covariant and has solutions that can be described by line elements. In such modifications, there is a strong discontinuity at $\hbar = 0$ in geometrical structures, seen as an $\hbar$-dependent family of modifications. In practice, this discontinuity translates into low-curvature physical problems, as discovered in the case of black-hole models of loop quantum gravity in [38,39].

Similarly, the original attempt in [6] to describe the inhomogeneous Schwarzschild exterior by homogeneous models, using time-like slicings in a static geometry, was based on an untested assumption that is true in classical space–time but may be violated in the presence of quantum modifications. The description of inhomogeneity in this case is different from the preceding example because it is non-perturbative in a space-like slicing. Here, homogeneous and inhomogeneous configurations do not appear as background and perturbations, but rather as models of a single space–time geometry using two different slicings.

Classically, any slicing gives an equivalent description of the full geometry, but this does not need to be the case once equations have been modified, in contrast to what has implicitly been assumed in [6]. The good control on covariant local theories for spherically symmetric dynamics makes it possible to test and invalidate this assumption. Again, it is the application of line elements in [6] even in the presence of quantum modifications that makes it possible to demonstrate inconsistency. It is not necessary to assume additional classical features in the inconsistency proof, beyond properties that have already been used in [6], explicitly or implicitly.

Models that work directly with spherically symmetric inhomogeneity usually tread more carefully because the appearance of first-class constraints is explicit. A consistent quantization or modification then requires that the first-class nature be preserved, i.e., that there are no anomalies, in order to prevent spurious degrees of freedom or over-constraining the theory. However, even in an anomaly-free modification, the structure of space–time and geometry may remain unclear without further analysis. Here, our remaining two examples are relevant, given by different modifications implemented for reformulated, partially Abelianized constraints and modification through a non-bijective canonical transformation, respectively. These modifications are anomaly-free and therefore consistent in a formal sense used for general constrained systems. Nevertheless, they turn out to violate covariance in different ways, even though the papers in which they have been proposed go on and analyze their solutions by standard line elements.

6.2. Covariance Crisis of Loop Quantum Gravity

As we just saw, a crucial ingredient of proofs of inconsistency and non-covariance in models of loop quantum gravity focuses on the application of line elements used routinely to evaluate solutions of modified equations in canonical gravity. Since modifications of canonical equations need not preserve covariance, even if they may remain formally consistent and anomaly-free, line elements are rendered meaningless. It might therefore be possible to evade some of the no-go results by foregoing line elements or related and more advanced methods, such as Penrose diagrams. In principle, a physical analysis would still be possible, at least in the anomaly-free case, by expressing solutions of anomaly-free modified equations in terms of suitable canonical observables.

However, this option is rarely exercised in interesting models because of the complicated nature of deriving strict observables, compared with the simple procedure of modifying coefficients in a formal line element. Furthermore, if such an analysis could be performed, it would not be clear in which sense solutions of the modified theory could still be considered geometrical, even when quantum modifications are very small, or more practically, how one would define the horizon of a black hole or curvature perturbations for cosmology in the absence of geometry. The important covariant form of general relativity and its geometrical nature would be a mere accident of the classical theory, rather than a fundamental property of gravity that could be extended to even the tiniest of corrections. While requiring a geometrical nature for quantum gravity may be largely a matter of taste, it also has practical implications because most of the gravitational methods and definitions that we know and understand are based on geometry.

A few additional ways might remain to solve these deep problems. First, in the context of Section 5, non-local effects might help because they would evade the strong control on possible covariant theories with spherical symmetry. However, the underlying analysis of minisuperspace dynamics in [6] implicitly assumes locality because there is a single momentum for each classical metric or triad component. If one were to try non-locality in order to solve the covariance problem in models of loop quantum gravity, the entire formalism used until now would have to change, even in minisuperspace models. Moreover, non-locality is often pathological and there is no indication that loop quantization could lead to more controlled situations.

Secondly, one may try to understand non-Riemannian space–time structures as they would be implied by modified hypersurface-deformation brackets ($\beta \neq \pm 1$). In some (but not all) cases, these modified geometries can be described by an effective Riemannian line element after suitable field redefinitions. At present, such models, recently analyzed in [25,40–42], are the only well-defined descriptions of geometries that may incorporate quantum modifications. If suitable field redefinitions exist, strict effective line elements are available, but in the presence of holonomy modifications they generically imply a signature change at high curvature.

There has been progress in constructing anomaly-free versions of the Hamiltonian constraint directly at the operator level in various versions of loop quantum gravity [43–48]. These constructions do not directly refer to symmetry-reduced models but, for now, implement restrictions of general ingredients such as the spatial dimension, the local gauge group, or the signature of gravity. In this approach, progress is usually made by reformulating the constraints, simplifying their brackets in a way that is conceptually similar to partial Abelianizations discussed in Section 4. As in this case, the successful construction of anomaly-free reformulated constraints does not immediately reveal whether they describe a consistent structure of space–time or covariance.

6.3. Lessons for Other Approaches

Background independence implies that space–time structure must be derived in some way and cannot be presupposed. We should not simply assume that inserting modified solutions in classical-type line elements is consistent. As a consequence, quantum gravity may not be "geometrical" as we understand it from general relativity. In the main body of this paper, we discussed how the canonical nature of loop quantum gravity gives access to powerful space–time methods, based on algebra, that can be used to rule out many models that might otherwise look reasonable.

It is not easy to see whether there may be possible analogs of our results in alternative approaches to quantum gravity if they are not canonical. Nevertheless, we are able to draw several lessons of general form. First, non-canonical theories do not directly aim to quantize generators of hypersurface deformations, but it should still be of interest to construct them and consider their properties in order to facilitate a space–time analysis. Instead of using these generators, covariance is often expressed in terms of coordinate choices or embeddings of discrete structures, but these ingredients do not directly refer to the actual degrees of freedom of gravity. Moreover, the explicit application of these space–time ingredients reduces the freedom in formulating suitable modifications of space–time structures if they are called for by modified dynamics.

Secondly, the no-go results we encountered are very general. In particular, they do not require a specific form of modifications but only qualitative features related to discreteness, such as bounded modification functions with local maxima. They should therefore be expected to be largely independent of the specific approach. Even though they were derived for canonical quantum gravity, the no-go results can be applied to any modified cosmological dynamics that can be presented in canonical form, even if it has been derived from a non-canonical approach. It would be interesting to see how other approaches might be able to circumvent our no-go theorems, for instance by requiring new quantum degrees of freedom or specific non-local behaviors. (For an example of non-local effects derived for effective actions, see [49]).

Finally, hypersurface-deformation generators make it possible to analyze different space–time structures because they express geometrical properties through algebra. It is easier to control possible modifications or deformations of algebras (or algebroids), compared with geometrical structures. The strong algebraic background of canonical gravity is therefore the main reason why it is possible to analyze space–time structures in detail with canonical methods. Non-canonical approaches are often viewed as preferable because they can provide a direct four-dimensional space–time picture, at least heuristically. However, this proximity to the standard four-dimensional formulation of classical gravity also implies that hidden assumptions about the underlying geometry may easily and unwittingly be incorporated in a specific approach. As shown in the present paper, even canonical approaches are not immune to such hidden assumptions, but they also provide strong methods to spot and test unjustified assumptions.

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