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Abelianized Structures in Spherically Symmetric Hypersurface Deformations

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Abstract: In canonical gravity, general covariance is implemented by hypersurface-deformation symmetries on the phase space. The different versions of hypersurface deformations required for full covariance have complicated interplays with one another, governed by non-Abelian brackets with structure functions. For spherically symmetric space-times, it is possible to identify a certain Abelian substructure within general hypersurface deformations, which suggests a simplified realization as a Lie algebra. The generators of this substructure can be quantized more easily than full hypersurface deformations, but the symmetries they generate do not directly correspond to hypersurface deformations. The availability of consistent quantizations therefore does not guarantee general covariance or a meaningful quantum notion thereof. In addition to placing the Abelian substructure within the full context of spherically symmetric hypersurface deformation, this paper points out several subtleties relevant for attempted applications in quantized space-time structures. In particular, it follows that recent constructions by Gambini, Olmedo, and Pullin in an Abelianized setting fail to address the covariance crisis of loop quantum gravity.

Keywords: canonical gravity; covariance; black holes



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1. Introduction

Canonical gravity describes the 4-dimensional, generally covariant structure of space-time by canonical fields defined on the slices of a spatial foliation. The evolution of these fields in time as well as transformations between different foliations are described by the geometrical structure of hypersurface deformations. In a canonical theory, these transformations are generated by certain phase-space functions, the diffeomorphism and Hamiltonian constraints. In spherically symmetric models, which will be considered here, the full set of constraints can be written as $D[M]$ and $H[N]$ with arbitrary spatial functions M (of density weight -1) and N . The constraint equations $D[M] = 0$ and $H[N] = 0$, valid for any M and N , restrict the phase-space degrees of freedom, given by the spatial metric and its momentum related to extrinsic curvature.

At the same time, the constraints generate (i) time evolution,

$$\mathcal{L}_{t(N,M)}f = \{f, H[N] + D[M]\} \quad (1)$$

for a phase-space function f along a time-evolution vector field $t^a = Nn^a + Ms^a$ in space-time with the unit normal n^a to a spatial slice and the tangent vector field $s^a = (\partial/\partial x)^a$ within the radial manifold (with coordinate x) of a spatial slice, and (ii) gauge transformations

$$\delta_{\tilde{\zeta}(\eta,\epsilon)}f = \{f, H[\eta] + D[\epsilon]\} \quad (2)$$

along a space-time vector field

$$\tilde{\zeta}^a = \eta n^a + \epsilon s^a \quad (3)$$

where ϵ , like M , has density weight -1 .

The reference to normal and tangential directions relative to a foliation implies crucial differences between the mathematical formulation of hypersurface deformations in canonical gravity and the more common formulation of general covariance in terms of space-time tensors. In space-time, vector components ζ^a transform, by definition, in such a way that $\zeta^a \partial / \partial x^a$ determines a unique direction independent of coordinate choices. Similarly, the spatial vector $\epsilon s^a = \epsilon \partial / \partial x$ defines a coordinate-independent direction because a scalar of density weight -1 in one dimension transforms like a 1-form dual to $\partial / \partial x$. The normal deformation, however, cannot be introduced in this way because the canonical setting does not provide a time coordinate or the corresponding $\partial / \partial t$. Moreover, even if such a coordinate could be introduced by hand, for instance by using t merely as a parameter as it also appears in Hamilton's equations, it would be impossible to endow η with a density weight -1 in the time direction because, canonically, there is no time manifold. The only alternative is given by the procedure that has been used since [1,2] and formalized in [3]: The normalization of n^a as a unit vector (with respect to the space-time metric, which is available in the canonical setting through the spatial metric on a slice as well as lapse N and shift M) associates a unique normal displacement to any given function η (without density weight).

The normal can be made unit only by reference to the metric, which provides some of the canonical degrees of freedom. The geometrical meaning of normal hypersurface deformations and their commutators depend on the spatial metric, resulting in structure functions in the canonical bracket relations. As a consequence, the canonical symmetries do not form a Lie algebra. This property is responsible for several complications well-known in attempts of canonical quantizations of the theory, starting with [4]. It also makes it harder to develop suitable mathematical structures for transformations generated by the constraints, in particular in an off-shell manner when one does not insist on solving the constraint equations. In [3], for instance, it was shown that a direct composition of transformations generated by the constraints is meaningful in the sense of path independence (a notion introduced in there) only on-shell.

The full structure of transformations is nevertheless required for general covariance to be implemented properly in the solutions of a canonical theory of gravity, in particular one that has been quantized, modified or deformed by new physical effects. While the restricted on-shell behavior may be easier to handle, the off-shell structure is important to make sure that the theory has a well-defined space-time structure, independently of the dynamics. Only in this case can the theory be considered a geometrical effective theory of some deeper and as yet unknown quantum space-time, just as different dynamical versions of gravity given by higher-curvature effective actions make use of the same Riemannian form of space-time. Because of its importance for covariance and the classification of meaningful effective theories, we will review the structure of hypersurface deformations in the beginning of our first section below, combining classic results from gravitational physics with more recent mathematical developments [5,6].

We will focus on aspects of hypersurface deformations of importance for a suggested simplification of the hypersurface-deformation brackets in spherically symmetric models, given by a partial Abelianization [7], but our statements will apply also to a variety of other reformulations that rely on phase-space dependent lapse and shift. Analyzing a partial Abelianization in the context of hypersurface deformations, we will show that this construction captures only a certain subset of these transformations and, upon modification or quantization, does not guarantee that invariance under hypersurface deformations or general covariance are still realized. This conclusion may be surprising because, at first sight, a partial Abelianization appears to implement the same number of symmetry generators as standard hypersurface deformations and uses only a linear redefinition of the generators. However, the coefficients of these linear redefinitions are phase-space dependent, complicating their mathematical description [5,6]. (Heuristically, phase-space dependent linear redefinitions of the generators introduce new structure functions or modify existing ones.) It is then a non-trivial question whether the redefinitions can be inverted. If they cannot be

inverted, the redefined theory is not invariant under full hypersurface deformations and its solutions violate general covariance. An additional construction is therefore needed in a partially Abelianized model (or other reformulations of standard hypersurface deformations) in order to recover all space-time transformations. As shown by explicit examples, this is not always possible if the generators have been modified by quantum corrections.

A recent paper [8] claims that it may be possible to realize general covariance in partial Abelianizations of spherically symmetric models with different types of quantum modifications, such as a spatial discretization. The claim is not accompanied by a successful reconstruction of hypersurface deformations and instead relies on a technical and so far incomplete case-by-case study of quantities that should be invariant in a covariant theory. Using our results about general hypersurface deformation structures, we will explain why the covariance claims of [8] cannot hold.

2. Hypersurface Deformations

Space-time vector fields with their standard Lie bracket generate the Lie algebra of diffeomorphisms. Similarly, the transformations generated by the canonical constraints form an algebraic structure. They are labeled by the components η and ϵ of a vector field ζ used in (3) in a basis (n^a, s^a) adapted to a spatial foliation, rather than a coordinate basis. Their commutators

$$\begin{aligned} & \delta_{\zeta_2}(\delta_{\zeta_1}f) - \delta_{\zeta_1}(\delta_{\zeta_2}f) \\ &= \{ \{f, H[\eta_1] + D[\epsilon_1]\}, H[\eta_2] + D[\epsilon_2] \} - \{ \{f, H[\eta_2] + D[\epsilon_2]\}, H[\eta_1] + D[\epsilon_1] \} \\ &= \{f, \{H[\eta_1] + D[\epsilon_1], H[\eta_2] + D[\epsilon_2]\} \} \end{aligned} \quad (4)$$

are determined by Poisson brackets $\{H[\eta_1] + D[\epsilon_1], H[\eta_2] + D[\epsilon_2]\}$ of the constraints (using the Jacobi identity). Because the unit normal n^a is normalized by using the space-time metric, including the spatial components q_{ab} on a slice, the brackets of two canonical gauge transformations [1,2,9] turn out to depend on the metric. In spherically symmetric models, in which the radial part of the metric is determined by a single function, q (of density weight 2), we have

$$\{H[\eta_1] + D[\epsilon_1], H[\eta_2] + D[\epsilon_2]\} = H[\epsilon_1\eta_2' - \epsilon_2\eta_1'] + D[\epsilon_1\epsilon_2' - \epsilon_2\epsilon_1' + q^{-1}(\eta_1\eta_2' - \eta_2\eta_1')]. \quad (5)$$

In general, the metric components are spatial functions independent of the components η and ϵ that label different gauge transformations. Unlike the Lie bracket of two space-time vector fields, the bracket of two pairs δ_{ζ_i} , $i = 1, 2$, implied by the Poisson bracket (5) does not form a Lie algebra because coefficients determined by spatial fields q_{ab} or q cannot be considered structure constants.

2.1. Algebroids

Instead, the brackets have structure functions or, in a suitable mathematical formulation, form the higher algebraic structure of an L_∞ -algebroid rather than a Lie algebra [10–12]. An L_∞ -algebroid is defined as a vector bundle over a base manifold M with fiber F and bracket relations on bundle sections together with suitable anchor maps that map bundle sections to objects in the tangent bundle of M . A Lie algebroid [13], for instance, has a Lie bracket $[\cdot, \cdot]$ on its sections and an anchor ρ that maps (as a homomorphism) bundle sections to vector fields on the base manifold, such that the Lie bracket of vector fields is compatible with the algebroid bracket. The anchor map also appears in the Leibniz rule

$$[s_1, fs_2] = f[s_1, s_2] + s_2\mathcal{L}_{\rho(s_1)}f \quad (6)$$

where s_1 and s_2 are sections and f is a function on the base manifold. The anchor brings abstract algebraic relations on bundle sections in correspondence with geometrical transformations as vector fields on the base manifold. While an anchor that maps any section to the zero vector field is always consistent with the Lie-algebroid axioms (in which case the Lie algebroid is a bundle of Lie algebras given by the fibers), non-trivial transformations on the

base require a larger image of the anchor. A Lie algebroid with a non-trivial anchor generalizes bundles of Lie algebras. Yet more generally, and in particular in the case of structure functions, the brackets of bundle sections obey the axioms of an L_∞ -algebra, a generalized form of a Lie algebra in which the Jacobi identity is not required to hold strictly.

The introduction of the base manifold makes it possible to formalize brackets with structure functions in terms of an L_∞ -algebroid. In particular for gravity, the base manifold is (a suitable extension [6]) of the canonical phase space, given by the spatial metrics and momenta related to extrinsic curvature. The fibers are parameterized by the components η and ϵ of a gauge transformation. A section is then an assignment of spatial functions η and ϵ to any metric (or a pair of a metric and its momentum). In this way, the q -dependent structure function in (5) finds a natural home as a bracket of sections over the space of metrics (and momenta).

Constant sections, given by pairs of η and ϵ that are functions on space but do not depend on the phase-space degrees of freedom, have a bracket, implied by (4), that can be realized as a special case of sections of a Lie algebroid [5]. General, non-constant sections of this Lie algebroid have a bracket that may differ from what hypersurface deformations would suggest. Non-constant sections over phase space, discussed in more detail in [6], either violate some of the Lie-algebra relations on sections (in the controlled way of a specific L_∞ -structure, as it follows from a BV-BFV extension of general relativity [14,15]) or require a base manifold that extends the phase space of canonical gravity in a way that is not smooth. (The latter can be formulated by using the notion of a Lie-Rinehart algebra [16] in which functions on the base manifold are replaced with a suitable commutative algebra.

Phase-space dependent functions η and ϵ are also important for physics. They are often considered in specific gravitational applications, as in the simple case of cosmological evolution written in conformal time where the lapse function equals the scale factor, a metric component. More importantly for our purposes, the partial Abelianization of [7] relies on an application of phase-space dependent ϵ and η . Hypersurface deformations with such non-constant sections form a Lie algebroid only on-shell [6] when the constraints are solved. The partial Abelianization is therefore able to describe the solution space to all constraints and its covariance transformations, but it is not guaranteed that it correctly captures off-shell transformations which are relevant for general covariance.

Since the standard derivation of the brackets (5) assumes that η and ϵ are not phase-space dependent, the general brackets must be extended by additional terms that, heuristically, result from Poisson brackets of constraints with phase-space dependent η and ϵ . (A complete derivation is based on the BV-BFV analysis of [14,15]). The Poisson bracket of two diffeomorphism constraints, for instance, can still be written in the compact form

$$\{D[\epsilon_1], D[\epsilon_2]\} = D[\epsilon_2 \epsilon'_1 - \epsilon_1 \epsilon'_2] \quad (7)$$

but with an application of the chain rule in the derivatives. Similarly, the mixed Poisson bracket of a Hamiltonian and a diffeomorphism constraint in general form reads

$$\{H[\eta], D[\epsilon]\} = H[-\epsilon \eta'] + D[\eta \mathcal{L}_n \epsilon] \quad (8)$$

where the normal derivative $\mathcal{L}_n$ of a spatial function is defined by the Poisson bracket with the Hamiltonian constraint, $\eta_1 \mathcal{L}_n \eta_2 = \{H[\eta_1], \eta_2\}$. For two Hamiltonian constraints, we have the Poisson bracket

$$\{H[\eta_1], H[\eta_2]\} = D[q^{-1}(\eta_1 \eta'_2 - \eta_2 \eta'_1)] + H[\eta_1 \mathcal{L}_n \eta_2 - \eta_2 \mathcal{L}_n \eta_1]. \quad (9)$$

In general, the extra terms implied by phase-space dependent η and ϵ , such as those in $\epsilon' = \partial_x \epsilon + (\partial_x q_i)(\partial_{q_i} \epsilon) + (\partial_x k_i)(\partial_{k_i} \epsilon)$ summing over the two independent components q_i , $i = 1, 2$, of a spherically symmetric spatial metric as well as two components k_i of extrinsic curvature, introduce further structure functions, such as $\partial_x q_i$ and $\partial_x k_i$, that depend on the metric as well as its momenta.

While these Poisson brackets illustrate the additional complications encountered with phase-space dependent ϵ and η , they do not immediately show the algebraic nature of general non-constant sections of hypersurface deformations. In particular, Poisson brackets do not directly mirror relevant L_∞ -structures. In our following discussion, we will not need the full algebraic structure and instead perform a comparison of different versions of constant and non-constant sections in gravitational applications.

2.2. Partial Abelianization

As noticed in [7], certain linear combinations of $H[\eta]$ and $D[\epsilon]$ have vanishing Poisson brackets in spherically symmetric models. In order to specify these combinations, we have to refer to explicit variables that determine the spatial metric and its momenta. Following Refs. [17–19], this is conveniently done in triad variables (E^x, E^φ) such that the spatial metric is given by the line element

$$ds^2 = \frac{(E^\varphi)^2}{E^x} dx^2 + E^x (d\vartheta^2 + \sin^2 \vartheta d\varphi^2) \quad (10)$$

in standard spherical coordinates. (For our purposes, it is sufficient to assume $E^x > 0$, fixing the orientation of the triad.) The triad components are canonically conjugate (up to constant factors) to components of extrinsic curvature, (K_x, K_φ) , such that

$$\{K_x(x), E^x(y)\} = 2G\delta(x, y) \quad , \quad \{K_\varphi(x), E^\varphi(y)\} = G\delta(x, y) \quad (11)$$

with Newton's constant G . (We keep a factor of two in the first relation. As implicitly done in [7,8], this factor can easily be eliminated by a rescaling of K_x . Since this procedure would not affect the main equations and conclusions shown below, we do not make use of this rescaling and instead keep the original components of extrinsic curvature).

The delta functions disappear in Poisson brackets of integrated (smeared) expressions, resulting in well-defined brackets. In particular, the diffeomorphism constraint

$$D[M] = \frac{1}{G} \int dx M(x) \left(-\frac{1}{2} (E^x)' K_x + K_\varphi' E^\varphi \right), \quad (12)$$

and Hamiltonian constraint

$$H[N] = \frac{-1}{2G} \int dx N(x) \left(|E^x|^{-1/2} E^\varphi K_\varphi^2 + 2|E^x|^{1/2} K_\varphi K_x + |E^x|^{-1/2} (1 - \Gamma_\varphi^2) E^\varphi + 2\Gamma_\varphi' |E^x|^{1/2} \right) \quad (13)$$

where $\Gamma_\varphi = -(E^x)' / (2E^\varphi)$ have Poisson brackets

$$\{D[M_1], D[M_2]\} = D[M_1 M_2'] \quad (14)$$

$$\{H[N], D[M]\} = -H[MN'] \quad (15)$$

$$\{H[N_1], H[N_2]\} = D[E^x (E^\varphi)^{-2} (N_1 N_2' - N_2 N_1')] \quad (16)$$

(for spatial functions M_i and N_i , $i = 1, 2$, that do not depend on the phase-space variables) of the correct form for hypersurface deformations in spherically symmetric space-times.

Simple algebra and integration by parts shows that the linear combinations

$$C[L] = H[(E^x)' (E^\varphi)^{-1} \int E^\varphi L dx] - 2D[K_\varphi \sqrt{E^x} (E^\varphi)^{-1} \int E^\varphi L dx], \quad (17)$$

where $\int E^\varphi L dx$ is understood as a function of x obtained by integrating $E^\varphi L$ from a fixed starting point up to x , have zero Poisson brackets with one another for different L :

$$\{C[L_1], C[L_2]\} = 0 \quad (18)$$

for all functions L_1 and L_2 on a spatial slice. To see this, it is sufficient to notice that the combination eliminates any dependence on K_x and on spatial derivatives of E^φ . The anti-

symmetric nature of the Poisson bracket then implies that it must vanish. Explicitly, the new combination of constraints takes the form

$$C[L] = -\frac{1}{G} \int dx L(x) E^\varphi \left(\sqrt{|E^x|} (1 + K_\varphi^2 - \Gamma_\varphi^2) + \text{const.} \right). \quad (19)$$

A free constant appears because a constant $\int E^\varphi L dx$ implies a non-vanishing lapse function in (17), and therefore a non-trivial constraint, but corresponds to a vanishing $E^\varphi L$ in (19). The new constraint $C[L]$ therefore constrains one degree of freedom less than the original $H[N]$. The free constant in (19) can be determined through boundary conditions, which would also restrict the lapse functions allowed in gauge transformations.

At first sight, it seems that the partial Abelianization eliminates structure functions from the brackets and may simplify quantization and the preservation of symmetries and therefore covariance. However, the importance of metric-dependent structure functions in the standard brackets, which make sure that deformations are defined with respect to a unit normal that is in fact normalized, raises the question of whether an elimination of these structure functions and their metric dependence by redefined generators can still capture the full picture of general covariance. To answer this question, it is instructive to place the partial Abelianization of the brackets in the context of the hypersurface-deformation structure. Several features of the full mathematical construction are then relevant.

First, the integration of $E^\varphi L$ required to define $C[L]$ as a combination of $H[N]$ and $D[M]$ may seem unusual, but while this means that the relevant N and M are non-local in space, they are local within both the fiber (spatial functions N and M) and the base (the gravitational phase space with independent functions E^x , E^φ , K_x and K_φ or a suitable extension) that may be used to construct a corresponding L_∞ -algebroid. The combination (17) therefore defines an admissible set of sections.

Secondly, while the section defined by (17) makes use of phase-space dependent N and M in the Hamiltonian and diffeomorphism constraints, which are therefore not constant over the base manifold, an Abelian bracket (18) is obtained only for functions L_1 and L_2 that do not have the full phase-space dependence allowed for general sections. In particular, if L_1 or L_2 are allowed to depend on $(E^\varphi)'$ or K_x , the bracket $\{C[L_1], C[L_2]\}$ no longer vanishes, and it can then have structure functions. Partial Abelianization is therefore obtained for a restricted class of sections, defined such that L does not depend on $(E^\varphi)'$ and K_x (while it may still have an unrestricted spatial dependence). If L does not depend on $(E^\varphi)'$ and K_x but on the other independent phase-space variables, K_φ as well as E^x or on E^φ but not its derivatives, the bracket $\{C[L_1], C[L_2]\}$ remains zero, but there are then structure functions in the bracket of $C[L]$ with the diffeomorphism constraint, analogously to (8). Therefore, structure functions are eliminated from the brackets only for a restricted class of sections. This observation raises the question whether full covariance can still be realized.

A restriction to constant sections over the base manifold is not unusual, for certain purposes. A similar assumption is made in the standard form (14)–(16) of hypersurface-deformation brackets, in which case the original N and M are often assumed to be constant over the base (while their spatial dependence remains unrestricted). There is, however, a crucial difference between assuming constant N and M over the base and assuming constant L over the base: In the former case, allowing for non-constant sections produces additional terms in the brackets, shown in (7)–(9), that follow directly from an application of the product rule of Poisson brackets. The partial Abelianization, however, relies on cancellations between different structure functions in the original brackets that are no longer realized once non-constant sections with phase-space dependent L are allowed.

In particular, allowing for phase-space dependent L and M in the $(D[M], C[L])$ system makes the transformation from (N, M) to (M, L) invertible. It is then possible to write the original $H[N]$ as a combination of $D[M]$ and $C[L]$ in the partial Abelianization, regaining the full non-Abelian brackets with metric-dependent structure functions. Restricting the system to phase-space independent L , by contrast, implies that the transformation from the original hypersurface-deformation structure to the brackets of $D[M]$ and $C[L]$ is not

invertible. It is then unclear whether hypersurface deformations and general covariance can be recovered from a partial Abelianization, in particular if the latter has been modified by quantum corrections.

2.3. Modified Deformations

It has been known for some time [20–22] that spherically symmetric hypersurface deformations can be modified consistently, maintaining closed brackets while modifying the structure functions. The dependence on K_φ in (13) can be generalized to

$$H[N] = \frac{-1}{2G} \int dx N(x) \left(|E^x|^{-1/2} E^\varphi f_1(K_\varphi) + 2|E^x|^{1/2} f_2(K_\varphi) K_x + |E^x|^{-1/2} (1 - \Gamma_\varphi^2) E^\varphi + 2\Gamma'_\varphi |E^x|^{1/2} \right) \quad (20)$$

where f_1 and f_2 are functions of K_φ related by

$$f_2(K_\varphi) = \frac{1}{2} \frac{df_1(K_\varphi)}{dK_\varphi}. \quad (21)$$

If this equation is satisfied, the bracket of two Hamiltonian constraints is still closed,

$$\{H[N_1], H[N_2]\} = D[\beta(K_\varphi) E^x (E^\varphi)^{-2} (N_1 N'_2 - N_2 N'_1)] \quad (22)$$

for phase-space independent N_1 and N_2 . In this bracket, $D[M]$ is the unmodified diffeomorphism constraint, but the structure function is multiplied by a new factor of

$$\beta(K_\varphi) = \frac{df_2(K_\varphi)}{dK_\varphi} = \frac{1}{2} \frac{d^2 f_1(K_\varphi)}{dK_\varphi^2}. \quad (23)$$

Additional terms in the bracket for non-constant sections follow immediately from the product rule for Poisson brackets.

Similarly, the Abelianized constraint $C[L]$ can be generalized in its dependence on K_φ , using the same function f_1 as before:

$$C[L] = -\frac{1}{G} \int dx L(x) E^\varphi \left(\sqrt{|E^x|} \left(1 + f_1(K_\varphi) - \Gamma_\varphi^2 \right) + \text{const.} \right). \quad (24)$$

Its brackets remain Abelian for phase-space independent L . There is no obvious term in $C[L]$ where the second function f_2 might appear or the important consistency condition (21). It therefore seems easier to modify (or quantize) the constraint $C[L]$ compared with $H[N]$. However, for full hypersurface deformations and covariance to be realized in the modified setting, we still have to make sure that the transformation from (N, M) to (L, M) can be inverted. As shown in [23], this is possible only if we also modify the transformation (17) to

$$C[L] = H[(E^x)'(E^\varphi)^{-1} \int E^\varphi L dx] - 2D[f_2(K_\varphi) \sqrt{E^x} (E^\varphi)^{-1} \int E^\varphi L dx] \quad (25)$$

where f_2 obeys the same consistency condition with f_1 , (21), as derived from the modified Hamiltonian constraint. The partial Abelianization and the original form of hypersurface deformations therefore imply equivalent results, provided one makes sure that the transformation of sections can be inverted. Only then can access to full hypersurface deformations and covariance be realized.

3. Non-Covariant Modifications of Abelianized Brackets

A recent paper [8] by Gambini, Olmedo and Pullin (GOP) argues that general covariance can be realized in modified versions of spherically symmetric models, for which a partial Abelianization of the brackets plays a crucial role: As the abstract claims, “We show explicitly that the resulting space-times, obtained from Dirac observables of the quantum theory, are covariant in the usual sense of the way—they preserve the quantum line element—for any gauge that is stationary (in the exterior, if there is a horizon). The con-

struction depends crucially on the details of the Abelianized quantization considered, the satisfaction of the quantum constraints and the recovery of standard general relativity in the classical limit and suggests that more informal polymerization constructions of possible semi-classical approximations to the theory can indeed have covariance problems.”

These claims raise several questions. For instance, how can the construction depend “crucially on the details of the Abelianized quantization considered” if a partial Abelianization is either completely equivalent to the non-Abelian original version of hypersurface deformations (if the transformation is made sure to be invertible) or gives access to only a subset of hypersurface deformations (if the transformation is not invertible owing to a restriction to a subset of sections)?

A closer inspection of technical calculations performed by GOP shows that spherically symmetric hypersurface deformations are, in fact, violated in the construction. GOP use two different kinds of modifications, a generalized dependence of $C[L]$ on K_φ of the form (24), and a spatial discretization of phase-space functions and their derivatives. Because the authors use a certain combination of solutions to the constraints and gauge-fixing conditions, it turns out that only the latter modification survives in the final expressions for line elements that are supposed to be invariant.

However, also the former (a generalized dependence on K_φ) is relevant because, as we have seen, the correct form of a modification must appear in two different places, in the constraint $C[L]$ and in the transformation back to unrestricted hypersurface deformations. These two appearances are clear but somewhat implicit in [8]: The modified $C[L]$ is implied by the modified solutions in Equation (14) in [8] (or, equivalently, (21) there, referring to the preprint version) where $f_1(K_\varphi) = \sin^2(\rho K_\varphi)/\rho^2$ with a spatial function ρ . The modified transformation back to unrestricted hypersurface deformations is implied by Equation (20) in [8] which in our notation amounts to replacing K_φ in (17) with $\sqrt{f_1(K_\varphi)}$. Using the same function $f_1(K_\varphi)$ is crucial for the constructions in [8] because the partial gauge fixing employed there replaces $\sqrt{f_1(K_\varphi)}$ with a fixed function on space (rather than phase space). The same gauge-fixing function is then used in both places, in the constraint $C[L]$ or its solutions and in the transformation back to unrestricted hypersurface deformations from which a line element can be constructed. However, this construction, which is equivalent to assuming $f_2(K_\varphi) = \sqrt{f_1(K_\varphi)}$ in (25), violates the condition (21) required for unrestricted hypersurface deformations to follow for the modified constraint. (For the specific $f_1(K_\varphi)$ considered by GOP, f_2 should have an additional cosine factor, or equivalently have a doubled argument of the sine function.) The constructions of [8] therefore violate hypersurface deformations.

How can GOP then claim to have performed crucial steps toward demonstrating general covariance in this setting? Unfortunately, much of the constructions are obscured by an application of incompletely defined mixtures of gauge fixings and idiosyncratic notions of observables. Here, it suffices to highlight only a few of the shortcomings found in the GOP analysis. (For more details, see [24].) Continuing with the replacement of $\sqrt{f_1(K_\varphi)}$ by a gauge-fixing function that depends only on space, GOP replace any appearance of $\sqrt{f_1(K_\varphi)}$ with gauge-fixing functions (on space) derived from the classical solutions for K_φ in two specific slicings. Implicitly, the authors simply remove the modification in this way because they indirectly equate $\sqrt{f_1(K_\varphi)}$ with K_φ , mediated by the gauge-fixing function. As a result, they do not test how non-classical $f_1(K_\varphi)$ can be consistent with covariance. It is also problematic that this step in a rather careless gauge-fixing procedure replaces a phase-space function K_φ that does not Poisson commute with the constraints with a spatial function that does obey this commutation property. The procedure turns a K_φ -dependent expression for E^φ , obtained by solving $C[L] = 0$, into a function that Poisson commutes with $C[L]$. GOP then call the result a Dirac observable, even though E^φ is not gauge invariant.

After replacing K_φ with a spatial function, the resulting expression for E^φ still does not Poisson commute with the diffeomorphism constraint and is therefore not a Dirac observable, even if K_φ could meaningfully be replaced. The same expression for E^φ also depends on E^x , which is not a spatial invariant. Indeed, unlike $C[L]$, the diffeomorphism constraint (12) depends on K_x and therefore does not Poisson commute with E^x . GOP arrive at their conclusion about E^φ being a Dirac observable by misidentifying E^x as a Dirac observable because the (loop) quantization procedure they use establishes a correspondence between an operator $\hat{E}^x$ and labels of a spherically symmetric spin network state [17,25] that are unchanged by the spatial shifts of a finite diffeomorphism. However, having a correspondence between a classical object, E^x , that is not a Dirac observable and a quantum operator, $\hat{E}^x$, that is a Dirac observable may indicate that the theory fails to have the correct classical limit. Since this way of imposing the diffeomorphism constraint is directly inherited from more general constructions in the full theory of loop quantum gravity [26,27], the issues revealed by our analysis of [8] might hint at deeper problems within the kinematics of loop quantum gravity.

4. Conclusions

Our discussion of phase-space dependent coefficients in hypersurface deformations has clarified a previously puzzling issue of partial Abelianizations in spherically symmetric models: Is it possible for partial Abelianizations to simplify the construction of quantum modifications of hypersurface deformation generators and, at the same time, retain full access to all transformations required for general covariance? We have shown that the answer is negative. A simplified construction of modified generators is based on the absence of structure functions in partially Abelianized brackets obtained for a specific choice of phase-space dependent gauge generators (lapse and shift functions). However, the partial Abelianization is maintained only if the new generators are then restricted to be phase-space independent. This condition renders the transformation from hypersurface-deformation brackets to partially Abelian brackets non-invertible. Access to unrestricted hypersurface deformations and general covariance is therefore lost in a partially Abelianized setting. Consistent modifications of the partially Abelian brackets then do not necessarily imply consistent realizations of general covariance.

A recent paper [8] by Gambini, Olmedo and Pullin has implicitly recognized this shortcoming and instead proposed to test general covariance in a tedious case-by-case study of presumed invariants, beginning with a discretized version of the line element. We have pointed out a specific place (the choice of modification functions f_1 and f_2) where hypersurface deformations are treated inconsistently in these constructions, which may perhaps lead to improved versions of the transformations considered by GOP. However, correcting this inconsistency requires an analysis of unrestricted hypersurface deformations even in the partially Abelianized setting, making sure that the transformation between these two versions of the brackets can be inverted. It is therefore impossible to analyze covariance in isolation from general hypersurface deformations, as proposed by GOP. No-go results [28] for covariance in models of loop quantum gravity, partially based on various analyses of modified hypersurface deformations, therefore cannot be evaded by the constructions of GOP.

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